Biextensions of 1-motives in Voevodsky's category of motives

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BIEXTENSIONS OF 1-MOTIVES IN VOEVODSKY'S CATEGORY OF MOTIVES

CRISTIANA BERTOLIN AND CARLO MAZZA

Abstract. Let k be a perfect field. In this paper we prove that biextensions of 1-motives define multilinear morphisms between 1-motives in Voevodsky's triangulated category $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, \mathbb{Q})$ of effective geometrical motives over k with rational coefficients.

Contents

Introduction

Let k be a perfect field. In [O] Orgogozo constructs a fully faithful functor

$$\mathcal{O} : \mathcal{D}^b(1-\mathrm{Isomot}(k)) \longrightarrow \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, \mathbb{Q}) \tag{0.1}$$

from the bounded derived category of the category $1-\mathrm{Isomot}(k)$ of 1-motives over k defined modulo isogenies to Voevodsky's triangulated category $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, \mathbb{Q})$ of effective geometrical motives over k with rational coefficients. If M_i (for $i = 1, 2, 3$) is a 1-motive defined over k modulo isogenies, in this paper we prove that the group of isomorphism classes of biextensions of (M_1, M_2) by M_3 is isomorphic to the group of morphisms of the category $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, \mathbb{Q})$ from the tensor product $\mathcal{O}(M_1) \otimes_{tr} \mathcal{O}(M_2)$ to $\mathcal{O}(M_3)$:

Theorem 0.1. *Let M_i (for $i = 1, 2, 3$) be a 1-motive defined over a perfect field k. Then*

$$\mathrm{Biext}^1(M_1, M_2; M_3) \otimes \mathbb{Q} \cong \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k,\mathbb{Q})}(\mathcal{O}(M_1) \otimes_{tr} \mathcal{O}(M_2), \mathcal{O}(M_3)).$$

This isomorphism answers a question raised by Barbieri-Viale and Kahn in [BK1] Remark 7.1.3 2). In loc. cit. Proposition 7.1.2 e) they prove the above theorem in the case where M_3 is a semi-abelian variety. Our proof is a generalization of theirs.

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If k is a field of characteristic 0 embeddable in $\mathbb{C}$, by [D] (10.1.3) we have a fully faithful functor

$$\mathrm{T} : 1-\mathrm{Mot}(k) \longrightarrow \mathcal{MR}(k) \tag{0.2}$$

from the category $1-\mathrm{Mot}(k)$ of 1-motives over k to the Tannakian category $\mathcal{MR}(k)$ of mixed realizations over k (see [J] I 2.1), which attaches to each 1-motive its Hodge realization for any embedding $k \hookrightarrow \mathbb{C}$, its de Rham realization, its ℓ-adic realizations for any prime number ℓ, and its comparison isomorphisms. According to [B1] Theorem 4.5.1, if M_i (for $i = 1, 2, 3$) is a 1-motive defined over k modulo isogenies, the group of isomorphism classes of biextensions of (M_1, M_2) by M_3 is isomorphic to the group of morphisms of the category $\mathcal{MR}(k)$ from the tensor product $\mathrm{T}(M_1) \otimes \mathrm{T}(M_2)$ of the realizations of M_1 and M_2 to the realization $\mathrm{T}(M_3)$ of M_3. Putting together this result with Theorem 0.1, we get the following isomorphisms

$$\begin{aligned} \mathrm{Biext}^1(M_1, M_2; M_3) \otimes \mathbb{Q} &\cong \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k;\mathbb{Q})}(\mathcal{O}(M_1) \otimes_{tr} \mathcal{O}(M_2), \mathcal{O}(M_3)) \\ &\cong \mathrm{Hom}_{\mathcal{MR}(k)}\big(\mathrm{T}(M_1) \otimes \mathrm{T}(M_2), \mathrm{T}(M_3)\big). \end{aligned} \tag{0.3}$$

These isomorphisms fit into the following context: in [H] Huber constructs a functor

$$\mathcal{H} : \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, \mathbb{Q}) \longrightarrow \mathcal{D}(\mathcal{MR}(k))$$

from Voevodsky's category $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, \mathbb{Q})$ to the triangulated category $\mathcal{D}(\mathcal{MR}(k))$ of mixed realizations over k, which respects the tensor structures. Extending the functor T (0.2) to the derived category $\mathcal{D}^b(1-\mathrm{Isomot}(k))$, we obtain the following diagram

$$\begin{array}{ccc} \mathcal{D}^b(1-\mathrm{Isomot}(k)) & \xrightarrow{\ \mathrm{T}\ } & \mathcal{D}(\mathcal{MR}(k)) \\ {\scriptstyle \mathcal{O}}\big\downarrow & \nearrow{\scriptstyle \mathcal{H}} & \\ \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, \mathbb{Q}) & & \end{array} \tag{0.4}$$

The isomorphisms (0.3) mean that biextensions of 1-motives define in a compatible way bilinear morphisms between 1-motives in each category involved in the above diagram. Barbieri-Viale and Kahn informed the authors that in [BK2] they have proved the commutativity of the diagram (0.4) in an axiomatic setting. If $k = \mathbb{C}$, they can prove its commutativity without assuming axioms.

We finish generalizing Theorem 0.1 to multilinear morphisms between 1-motives.

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Notation

If C is an additive category, we denote by $C \otimes \mathbb{Q}$ the associated $\mathbb{Q}$-linear category which is universal for functors from C to a $\mathbb{Q}$-linear category. Explicitly, the category $C \otimes \mathbb{Q}$ has the same objects as the category C, but the sets of arrows of $C \otimes \mathbb{Q}$ are the sets of arrows of C tensored with $\mathbb{Q}$, i.e. $\mathrm{Hom}_{C\otimes\mathbb{Q}}(-,-) = \mathrm{Hom}_C(-,-) \otimes_{\mathbb{Z}} \mathbb{Q}$.

We give a quick review of Voevodsky's category of motives (see [V]). Denote by $\mathrm{Sm}(k)$ the category of smooth varieties over a field k. Let $A = \mathbb{Z}$ or $\mathbb{Q}$ be the coefficient ring. Let $\mathrm{SmCor}(k, A)$ be the category whose objects are smooth varieties over k and whose morphisms are finite correspondences with coefficients in A. It is an additive category.

The **triangulated category** $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k,A)$ **of effective geometrical motives over** k is the pseudo-abelian envelope of the localization of the homotopy category $\mathcal{H}^b(\mathrm{SmCor}(k,A))$ of bounded complexes over $\mathrm{SmCor}(k,A)$ with respect to the thick subcategory generated by the complexes $X \times_k \mathbb{A}^1_k \to X$ and $U \cap V \to U \oplus V \to X$ for any smooth variety X and any Zariski-covering $X = U \cup V$.

The **category of Nisnevich sheaves on** $\mathrm{Sm}(k)$, $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}(k))$, is the category of abelian sheaves on $\mathrm{Sm}(k)$ for the Nisnevich topology.

A presheaf with transfers on $\mathrm{Sm}(k)$ is an additive contravariant functor from $\mathrm{SmCor}(k,A)$ to the category of abelian groups. It is called a Nisnevich sheaf with transfers if the corresponding presheaf of abelian groups on $\mathrm{Sm}(k)$ is a sheaf for the Nisnevich topology. Denote by $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{SmCor}(k,A))$ the **category of Nisnevich sheaves with transfers**. By [V] Theorem 3.1.4 it is an abelian category.

A presheaf with transfers F is called homotopy invariant if for any smooth variety X the natural map $F(X) \to F(X \times_k \mathbb{A}^1_k)$ induced by the projection $X \times_k \mathbb{A}^1_k \to X$ is an isomorphism. A Nisnevich sheaf with transfers is called homotopy invariant if it is homotopy invariant as a presheaf with transferts.

The **category** $\mathrm{DM}^{\mathrm{eff}}_{-}(k,A)$ **of effective motivic complexes** is the full subcategory of the derived category $\mathcal{D}^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{SmCor}(k,A)))$ of complexes of Nisnevich sheaves with transfers bounded from the above, which consists of complexes with homotopy invariant cohomology sheaves.

There exists a functor $L : \mathrm{SmCor}(k,A) \to \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{SmCor}(k,A))$ which associates to each smooth variety X a Nisnevich sheaf with transfers given by $L(X)(U) = c(U,X)_A$, where $c(U,X)_A$ is the free A-module generated by prime correspondences from U to X. This functor extends to complexes furnishing a functor

$$L : \mathcal{H}^b(\mathrm{SmCor}(k,A)) \longrightarrow \mathcal{D}^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{SmCor}(k,A))).$$

There exists also a functor $C_* : \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{SmCor}(k,A)) \to \mathrm{DM}^{\mathrm{eff}}_{-}(k,A)$ which associates to each Nisnevich sheaf with transfers F the effective motivic complex $C_*(F)$ given by $C_n(F)(U) = F(U \times \Delta^n)$ where Δ^* is the standard cosimplicial object. This functor extends to a functor

$$\mathbf{R}C_* : \mathcal{D}^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{SmCor}(k,A))) \longrightarrow \mathrm{DM}^{\mathrm{eff}}_{-}(k,A) \tag{0.5}$$

which is left adjoint to the natural embedding. Moreover, this functor identifies the category $\mathrm{DM}^{\mathrm{eff}}_{-}(k,A)$ with the localization of $\mathcal{D}^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{SmCor}(k,A)))$ with respect to the localizing subcategory generated by complexes of the form $L(X \times_k \mathbb{A}^1_k) \to L(X)$ for any smooth variety X (see [V] Proposition 3.2.3).

If X and Y are two smooth varieties over k, the equality

$$L(X) \otimes L(Y) = L(X \times_k Y) \tag{0.6}$$

defines a tensor structure on the category $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{SmCor}(k,A))$, which extends to the derived category $\mathcal{D}^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{SmCor}(k,A)))$. The tensor structure on $\mathrm{DM}^{\mathrm{eff}}_{-}(k,A)$, that we denote by $\otimes_{tr}$, is the descent with respect to the projection $\mathbf{R}C_*$ (0.5) of the tensor structure on $\mathcal{D}^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{SmCor}(k,A)))$.

If we assume k to be a perfect field, by [V] Proposition 3.2.6 there exists a functor

$$i : \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k,A) \to \mathrm{DM}^{\mathrm{eff}}_{-}(k,A) \tag{0.7}$$

which is a full embedding with dense image and which makes the following diagram commutative

$$\begin{array}{ccc} \mathcal{H}^b(\mathrm{SmCor}(k,A)) & \stackrel{L}{\to} & \mathcal{D}^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{SmCor}(k,A))) \\ \downarrow & & \downarrow \mathbf{R}C_* \\ \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k,A) & \stackrel{i}{\dashrightarrow} & \mathrm{DM}^{\mathrm{eff}}_{-}(k,A). \end{array}$$

Remark 0.2. For Voevodsky's theory of motives with rational coefficients, the étale topology gives the same motivic answer as the Nisnevich topology: if we construct the category of effective motivic complexes using the étale topology instead of the Nisnevich topology, we get a triangulated category $\mathrm{DM}^{\mathrm{eff}}_{-,\acute{e}t}(k,A)$ which is equivalent as triangulated category to the category $\mathrm{DM}^{\mathrm{eff}}_{-}(k,A)$ if we assume $A = \mathbb{Q}$ (see [V] Proposition 3.3.2).

1. 1-motives in Voevodsky's category

A **1-motive** $M = (X, A, T, G, u)$ over a field k (see [D] §10) consists of

- a group scheme X over k, which is locally for the étale topology, a constant group scheme defined by a finitely generated free $\mathbb{Z}$-module,
- an extention G of an abelian k-variety A by a k-torus T,
- a morphism $u : X \longrightarrow G$ of commutative k-group schemes.

A 1-motive $M = (X, A, T, G, u)$ can be viewed also as a length 1 complex $[X \stackrel{u}{\to} G]$ of commutative k-group schemes. In this paper, as a complex we shall put X in degree 0 and G in degree 1. A morphism of 1-motives is a morphism of complexes of commutative k-group schemes. Denote by $1-\mathrm{Mot}(k)$ the category of 1-motives over k. It is an additive category but it isn't an abelian category.

Denote by $1-\mathrm{Isomot}(k)$ the $\mathbb{Q}$-linear category $1-\mathrm{Mot}(k) \otimes \mathbb{Q}$ associated to the category of 1-motives over k. The objects of $1-\mathrm{Isomot}(k)$ are called 1-isomotifs and the morphisms of $1-\mathrm{Mot}(k)$ which become isomorphisms in $1-\mathrm{Isomot}(k)$ are the isogenies between 1-motives, i.e. the morphisms of complexes $[X \to G] \to [X' \to G']$ such that $X \to X'$ is injective with finite cokernel, and $G \to G'$ is surjective with finite kernel. The category $1-\mathrm{Isomot}(k)$ is an abelian category (see [O] Lemma 3.2.2).

Assume now k to be a perfect field. The two main ingredients which furnish the link between 1-motives and Voevodsky's motives are:

1. any commutative k-group scheme represents a Nisnevich sheaf with transfers, i.e. an object of $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{SmCor}(k,A))$ ([O] Lemma 3.1.2),
2. if A (resp. T, resp. X) is an abelian k-variety (resp. a k-torus, resp. a group scheme over k, which is locally for the étale topology, a constant group scheme defined by a finitely generated free $\mathbb{Z}$-module), then the Nisnevich sheaf with transfers that it represents is homotopy invariant ([O] Lemma 3.3.1).

Since we can view 1-motives as complexes of smooth varieties over k, we have a functor from the category of 1-motives to the category $\mathcal{C}(\mathrm{Sm}(k))$ of complexes over $\mathrm{Sm}(k)$. According to (1), this functor factorizes through the category of complexes over $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{SmCor}(k,A))$:

$$1-\mathrm{Mot}(k) \longrightarrow \mathcal{C}(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{SmCor}(k,A)))$$

If we tensor with $\mathbb{Q}$, we get an additive exact functor between abelian categories

$$1 - \mathrm{Isomot}(k) \longrightarrow \mathcal{C}(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{SmCor}(k,A)) \otimes \mathbb{Q}).$$

Taking the associated bounded derived categories, we obtain a triangulated functor

$$\mathcal{D}^b(1 - \mathrm{Isomot}(k)) \longrightarrow \mathcal{D}^b(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{SmCor}(k,A)) \otimes \mathbb{Q}).$$

Finally, according to (2) this last functor factorizes through the triangulated functor

$$\mathcal{O} : \mathcal{D}^b(1 - \mathrm{Isomot}(k)) \longrightarrow \mathrm{DM}^{\mathrm{eff}}_{-}(k,A) \otimes \mathbb{Q}.$$

By [O] Proposition 3.3.3 this triangulated functor is fully faithful, and by loc. cit. Theorem 3.4.1 it factorizes through the thick subcategory $d_1\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k,\mathbb{Q})$ of $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k,\mathbb{Q})$ generated by smooth varieties of dimension ≤ 1 over k and it induces an equivalence of triangulated categories, that we denote again by $\mathcal{O}$,

$$\mathcal{O} : \mathcal{D}^b(1 - \mathrm{Isomot}(k)) \longrightarrow d_1\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k,\mathbb{Q}).$$

In order to simplify notation, if M is a 1-motive, we denote again by M its image in $d_1\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k,\mathbb{Q})$ through the above equivalence of categories and also its image in $\mathrm{DM}^{\mathrm{eff}}_{-}(k,A)$ through the full embedding (0.7).

For the proof of Theorem 0.1, we will need the following

Proposition 1.1. *Let M_i (for $i = 1,2,3$) be a 1-motive defined over k. The forgetful triangulated functors*

$$\mathrm{DM}^{\mathrm{eff}}_{-}(k,A) \overset{a}{\hookrightarrow} \mathcal{D}^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{SmCor}(k,A))) \overset{b}{\hookrightarrow} \mathcal{D}^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}(k)))$$

induce an isomorphism

$$\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{-}(k,A)}(M_1 \otimes_{tr} M_2, M_3) \cong \mathrm{Hom}_{\mathcal{D}^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}(k)))}(M_1 \overset{\mathbb{L}}{\otimes} M_2, M_3).$$

Proof. The forgetful functor a admits as left adjoint the functor $\mathbf{R}C_*$ (0.5). The forgetful functor b from the category of Nisnevich sheaves with transfers to the category of Nisnevich sheaves admits as left adjoint the free sheaf with transfers functor

$$\Phi : \mathcal{D}^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}(k))) \longrightarrow \mathcal{D}^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{SmCor}(k,A))) \tag{1.1}$$

([V] Remark 1 page 202). If X is a smooth variety over k, let $\mathbb{Z}(X)$ be the sheafification with respect to the Nisnevisch topology of the presheaf $U \mapsto \mathbb{Z}[\mathrm{Hom}_{\mathrm{Sm}(k)}(U,X)]$. Clearly $\Phi(\mathbb{Z}(X))$ is the Nisnevich sheaf with transfers $L(X)$. If Y is another smooth variety over k, we have that $\mathbb{Z}(X) \otimes \mathbb{Z}(Y) = \mathbb{Z}(X \times_k Y)$ (see [MVW] Lemma 12.14) and so by formula (0.6) we get

$$\Phi(\mathbb{Z}(X) \otimes \mathbb{Z}(Y)) = \Phi(\mathbb{Z}(X)) \otimes_{tr} \Phi(\mathbb{Z}(Y)).$$

The tensor structure on $\mathrm{DM}^{\mathrm{eff}}_{-}(k,A)$ is the descent of the tensor structure on $\mathcal{D}^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{SmCor}(k,A)))$ with respect to $\mathbf{R}C_*$ and therefore

$$\mathbf{R}C_* \circ \Phi(\mathbb{Z}(X) \otimes \mathbb{Z}(Y)) = \mathbf{R}C_* \circ \Phi(\mathbb{Z}(X)) \otimes_{tr} \mathbf{R}C_* \circ \Phi(\mathbb{Z}(Y)).$$

Using this equality and the fact that the composite $\mathbf{R}C_* \circ \Phi$ is the left adjoint of $b \circ a$, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}(k)))}(M_1 \overset{\mathbb{L}}{\otimes} M_2, M_3) &\cong \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{-}(k,A)}(\mathbf{R}C_* \circ \Phi(M_1 \overset{\mathbb{L}}{\otimes} M_2), M_3) \\
&\cong \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{-}(k,A)}(\mathbf{R}C_* \circ \Phi(M_1) \otimes_{tr} \mathbf{R}C_* \circ \Phi(M_2), M_3).
\end{aligned}$$

Since 1-motives are complexes of homotopy invariant Nisnevich sheaves with transferts, the counit arrows $\mathbf{R}C_* \circ \Phi(M_i) \to M_i$ (for $i = 1, 2$) are isomorphisms and so we can conclude. □

2. Bilinear morphisms between 1-motives

Let $K_i = [A_i \stackrel{u_i}{\to} B_i]$ (for $i = 1, 2, 3$) be a length 1 complex of abelian sheaves (over any topos $\mathbf{T}$) with A_i in degree 1 and B_i in degree 0. A **biextension** $(\mathcal{B}, \Psi_1, \Psi_2, \lambda)$ **of** (K_1, K_2) **by** K_3 consists of

(1) a biextension of $\mathcal{B}$ of (B_1, B_2) by B_3;
(2) a trivialization Ψ_1 (resp. Ψ_2) of the biextension $(u_1, id_{B_2})^*\mathcal{B}$ of (A_1, B_2) by B_3 (resp. of the biextension $(id_{B_1}, u_2)^*\mathcal{B}$ of (B_1, A_2) by B_3) obtained as pull-back of $\mathcal{B}$ via $(u_1, id_{B_2}) : A_1 \times B_2 \to B_1 \times B_2$ (resp. via $(id_{B_1}, u_2) : B_1 \times A_2 \to B_1 \times B_2$). These two trivializations have to coincide over $A_1 \times A_2$;
(3) a morphism $\lambda : A_1 \otimes A_2 \to A_3$ such that the composite $A_1 \otimes A_2 \xrightarrow{\lambda} A_3 \xrightarrow{u_3} B_3$ is compatible with the restriction over $A_1 \times A_2$ of the trivializations Ψ_1 and Ψ_2.

We denote by $\mathbf{Biext}(K_1, K_2; K_3)$ the category of biextensions of (K_1, K_2) by K_3. The Baer sum of extensions defines a group law for the objects of the category $\mathbf{Biext}(K_1, K_2; K_3)$, which is therefore a Picard category (see [SGA7] Exposé VII 2.4, 2.5 and 2.6). Let $\mathrm{Biext}^0(K_1, K_2; K_3)$ be the group of automorphisms of any biextension of (K_1, K_2) by K_3, and let $\mathrm{Biext}^1(K_1, K_2; K_3)$ be the group of isomorphism classes of biextensions of (K_1, K_2) by K_3.

According to the main result of [B2], we have the following homological interpretation of the groups $\mathrm{Biext}^i(K_1, K_2; K_3)$:

$$\mathrm{Biext}^i(K_1, K_2; K_3) \cong \mathrm{Ext}^i(K_1 \overset{\mathbb{L}}{\otimes} K_2, K_3) \qquad (i = 0, 1) \tag{2.1}$$

Since we can view 1-motives as complexes of commutative S-group schemes of length 1, all the above definitions apply to 1-motives.

Remark 2.1. The homological interpretation (2.1) of biextensions computed in [B2] is done for chain complexes $K_i = [A_i \xrightarrow{u_i} B_i]$ with A_i in degree 1 and B_i in degree 0. In this paper 1-motives are considered as cochain complexes $M_i = [X_i \stackrel{u_i}{\to} G_i]$ with X in degree 0 and G in degree 1. Therefore after switching from homological notation to cohomological notation, the homological interpretation of the group $\mathrm{Biext}^1(M_1, M_2; M_3)$ can be stated as follow:

$$\mathrm{Biext}^1(M_1, M_2; M_3) \cong \mathrm{Ext}^1(M_1[1] \overset{\mathbb{L}}{\otimes} M_2[1], M_3[1])$$

where the shift functor $[i]$ on a cochain complex C^* acts as $(C^*[i])^j = C^{i+j}$.

Proof of Theorem 0.1 By proposition 1.1, we have that

$$\begin{aligned} \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k,\mathbb{Q})}(M_1 \otimes_{tr} M_2, M_3) &\cong \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{-}(k,A)\otimes\mathbb{Q}}(M_1 \otimes_{tr} M_2, M_3) \\ &\cong \mathrm{Hom}_{\mathcal{D}^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}(k)))}(M_1 \overset{\mathbb{L}}{\otimes} M_2, M_3) \otimes \mathbb{Q}. \end{aligned}$$

On the other hand, according to the remark 2.1 we have the following homological interpretation of the group $\mathrm{Biext}^1(M_1, M_2; M_3)$:

$$\mathrm{Biext}^1(M_1, M_2; M_3) \cong \mathrm{Ext}^1(M_1[1] \overset{\mathbb{L}}{\otimes} M_2[1], M_3[1]) \cong \mathrm{Hom}_{\mathcal{D}^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}(k)))}(M_1 \overset{\mathbb{L}}{\otimes} M_2, M_3)$$

and so we can conclude.

3. Multilinear morphisms between 1-motives

1-motives are endowed with an increasing filtration, called the weight filtration. Explicitly, the weight filtration W_* on a 1-motive $M=[X \stackrel{u}{\rightarrow} G]$ is

$$\begin{aligned} \mathrm{W}_i(M) &= M \text{ for each } i \geq 0, \\ \mathrm{W}_{-1}(M) &= [0 \longrightarrow G], \\ \mathrm{W}_{-2}(M) &= [0 \longrightarrow Y(1)], \\ \mathrm{W}_j(M) &= 0 \text{ for each } j \leq -3. \end{aligned}$$

Defining $\mathrm{Gr}_i^{\mathrm{W}} = \mathrm{W}_i/\mathrm{W}_{i+1}$, we have $\mathrm{Gr}_0^{\mathrm{W}}(M)=[X\rightarrow 0], \mathrm{Gr}_{-1}^{\mathrm{W}}(M)=[0\rightarrow A]$ and $\mathrm{Gr}_{-2}^{\mathrm{W}}(M)=[0\rightarrow Y(1)]$. Hence locally constant group schemes, abelian varieties and tori are the pure 1-motives underlying M of weights 0,-1,-2 respectively.

The main property of morphisms of motives is that they have to respect the weight filtration, i.e. any morphism $f: A \rightarrow B$ of motives satisfies the following equality

$$f(A) \cap \mathrm{W}_i(B) = f(\mathrm{W}_i(A)) \qquad \forall\, i \in \mathbb{Z}.$$

Assume M and $M_1,\ldots,M_l$ to be 1-motives over a perfect field k and consider a morphism

$$F: \otimes_{j=1}^{l} M_j \rightarrow M.$$

Because morphisms of motives have to respect the weight filtration, the only non trivial components of the morphism F are the components of the morphism

$$\otimes_{j=1}^{l} M_j \Big/ \mathrm{W}_{-3}(\otimes_{j=1}^{l} M_j) \longrightarrow M.$$

Using [B1] Lemma 3.1.3 with $i=-3$, we can write explicitly this last morphism in the following way

$$\sum_{\substack{\iota_1<\iota_2 \text{ and } \nu_1<\cdots<\nu_{l-2} \\ \iota_1,\iota_2 \notin \{\nu_1,\ldots,\nu_{l-2}\}}} X_{\nu_1} \otimes \cdots \otimes X_{\nu_{l-2}} \otimes (M_{\iota_1} \otimes M_{\iota_2} / \mathrm{W}_{-3}(M_{\iota_1} \otimes M_{\iota_2})) \longrightarrow M.$$

To have the morphism

$$X_{\nu_1} \otimes \cdots \otimes X_{\nu_{l-2}} \otimes (M_{\iota_1} \otimes M_{\iota_2} / \mathrm{W}_{-3}(M_{\iota_1} \otimes M_{\iota_2})) \longrightarrow M$$

is equivalent to have the morphism

$$M_{\iota_1} \otimes M_{\iota_2} / \mathrm{W}_{-3}(M_{\iota_1} \otimes M_{\iota_2}) \longrightarrow X_{\nu_1}^{\vee} \otimes \cdots \otimes X_{\nu_{l-2}}^{\vee} \otimes M$$

where $X_{\nu_n}^{\vee}$ is the k-group scheme $\underline{\mathrm{Hom}}(X_{\nu_n},\mathbb{Z})$ for $n=1,\ldots,l-2$. But as observed in [B1] §1.1 "to tensor a motive by a motive of weight zero" means to take a certain number of copies of this motive, and so applying Theorem 0.1 we get

Theorem 3.1. *Let M and $M_1,\ldots,M_l$ be 1-motives over a perfect field k. Then,*

$$\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k,\mathbb{Q})}(M_1 \otimes_{tr} M_2 \otimes_{tr} \cdots \otimes_{tr} M_l, M) \cong$$

$$\sum \mathrm{Biext}^1(M_{\iota_1}, M_{\iota_2}; X_{\nu_1}^{\vee} \otimes \cdots \otimes X_{\nu_{l-2}}^{\vee} \otimes M) \otimes \mathbb{Q}$$

where the sum is taken over all the $(l-2)$-uplets $\{\nu_1,\ldots,\nu_{l-i+1}\}$ and all the 2-uplets $\{\iota_1,\iota_2\}$ of $\{1,\cdots,l\}$ such that $\{\nu_1,\ldots,\nu_{l-2}\}\cap\{\iota_1,\iota_2\}=\emptyset$ and $\nu_1<\cdots<\nu_{l-2}$, $\iota_1<\iota_2$.

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NWF-I Mathematik, Universität Regensburg, D-93040 Regensburg
E-mail address: cristiana.bertolin@mathematik.uni-regensburg.de

Dip. di Matematica, Università di Genova, Via Dodecaneso 35, I-16133 Genova
E-mail address: mazza@dima.unige.it