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EXTENSIONS AND BIEXTENSIONS OF LOCALLY CONSTANT GROUP SCHEMES, TORI AND ABELIAN SCHEMES

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Abstract. Let S be a scheme. We compute explicitly the group of homomorphisms, the S-sheaf of homomorphisms, the group of extensions, and the S-sheaf of extensions involving locally constant S-group schemes, abelian S-schemes, and S-tori. Using the obtained results, we study the categories of biextensions involving these geometrical objets. In particular, we prove that if G_i (for $i = 1, 2, 3$) is an extension of an abelian S-scheme A_i by an S-torus T_i, the category of biextensions of (G_1, G_2) by G_3 is equivalent to the category of biextensions of the underlying abelian S-schemes (A_1, A_2) by the underlying S-torus T_3.

Introduction

The notion of biextension was introduced by D. Mumford [Mu69] in the context of formal groups in order to express the relations between the formal groups associated to an abelian scheme and the dual abelian scheme. Successively in the Exposés VII and VIII of [SGA7], Grothendieck studies in a systematic way the notion of biextension in the more general setting of abelian sheaves over any topos. In particular he investigates the case of biextensions of commutative group schemes by the multiplicative group $\mathbb{G}_m$.

The aim of this paper is to study biextensions involving locally constant group schemes, tori and abelian schemes defined over an arbitrary base scheme S. We treat all possible cases - biextensions of abelian schemes by abelians schemes, biextensions of tori and abelian schemes by locally constant schemes, ... - and in order to have a self-contained work, we recall briefly the cases which already exist in the literature. Working in the topos $\mathbf{T}_{\mathrm{fppf}}$ associated to the site of locally of finite presentation S-schemes endowed with the fppf topology, our main results are (respectively Theorems 2.3.1, 2.4.6, 2.5.2):

Theorem A: *Let A be an abelian S-scheme, let T be an S-torus and let P be a divisible commutative S-group scheme locally of finite presentation over S, with connected fibres. The category* $\mathbf{Biext}(P, T; A)$ *of biextensions of (P, T) by A and the category* $\mathbf{Biext}(T, P; A)$ *of biextensions of (T, P) by A are equivalent to the trivial category.*

Theorem B: *Let A_i (for $i = 1, 2, 3$) be an abelian S-scheme. The category* $\mathbf{Biext}(A_1, A_2; A_3)$ *of biextensions of (A_1, A_2) by A_3 is equivalent to the trivial category.*

Theorem C: *Let S be a scheme. Let G_i (for $i = 1, 2, 3$) be a commutative extension of an abelian S-scheme A_i by an S-torus T_i. The category of biextensions*

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of (G_1, G_2) *by* G_3 *is equivalent to the category of biextensions of the underlying abelian* S*-schemes* (A_1, A_2) *by the underlying* S*-torus* T_3*.*
The reason of the choice of the topos $\mathbf{T}_{\mathrm{fppf}}$ is that for the fppf topology the tori and the abelian schemes are divisible groups.

The theory of motives leads us to the investigation of biextensions involving locally constant group schemes, tori and abelian schemes defined over an arbitrary base scheme S. In fact in [B] we introduce the notion of biextension of 1-motives by 1-motives and we define bilinear morphisms between 1-motives as isomorphism classes of such biextensions. We then check that our definition is compatible with the realizations of 1-motives, i.e. that the group of isomorphism classes of biextensions of 1-motives is a group of bilinear morphisms in an appropriate category of mixed realizations. In this context **Theorem A**, **B** and **C** mean that biextensions satisfy the main property of morphisms of motives, i.e. they respect the weight filtration W_* on motives. For example, if A_i (for $i = 1, 2, 3$) is an abelian S-scheme, **Theorem B** says that there are no nonzero bilinear morphisms from $A_1 \times A_2$ to A_3. But this is exactly what is predicted by Grothendieck's philosophy of motives: since morphisms of motives have to respect the weight filtration W_*, it is not possible to have a morphism from the motive $A_1 \otimes A_2$ of weight -2 to the motive A_3 of weight -1.

Before we investigate biextensions, we have to understand extensions and homomorphisms: working over an arbitrary base scheme S, we start computing explicitly the group of homomorphisms, the S-sheaf of homomorphisms, the group of extensions and the S-sheaf of extensions involving locally constant S-group schemes, abelian S-schemes and S-tori. There are a lot of results in the literature about homomorphisms between such geometric objects but very few about their extensions. This relies on the fact that with extensions of group schemes we go outside the category of schemes: extensions of S-group schemes are a priori only algebraic spaces over S and this makes things immediately more complicated since in the literature there are only few results about algebraic spaces that are relevant to the study of extensions considered in this paper. This will be the most difficult point in the study of extensions of an S-torus T by an abelian S-scheme A, which begins with the proof that for such extensions it is equivalent to be a scheme or to be of finite order locally over S (Proposition 1.2.5). Using the fact that these extensions are representable by schemes (Theorem 1.2.6), we can then conclude that the extensions of S-tori by abelian S-schemes are of finite order locally over S, i.e. the S-sheaf $\underline{\mathrm{Ext}}^1(T, A)$ is a torsion sheaf (Corollary 1.2.7). Moreover we show that this S-sheaf is in fact an S-group scheme which is separated and étale over S (Corollary 1.2.8). With extensions of abelian S-schemes by abelian S-schemes we don't have problems of representability by schemes since it is a classical result that abelian algebraic spaces are abelian schemes. The main difficulty with these extensions is that they are not of finite order locally over S (it is true only if we assume the base scheme S to be integral and geometrically unibranched, see Proposition 1.3.5) and we cannot say much about them.

Because of the homological interpretation of biextensions furnished by Grothendieck in [SGA7] Exposé VII 3.6.5 and (3.7.4), our results on biextensions are essentially consequences of the results on extensions obtained in the first part of this paper. Through a homological "dévissage", the fact that the S-sheaf $\underline{\mathrm{Ext}}^1(T, A)$

is a separated and étale S-group scheme implies **Theorem A**. The proof of **Theorem B** is done again trough a homological "dévissage" but it is not based on results of the first part of this paper since, as we have already said, extensions of abelian schemes by abelian schemes are not of finite order locally over S if S fails to be integral and geometrically unibranched! Concerning **Theorem C**, in [SGA7] Exposé VIII (3.6.1) Grothendieck proves that in the topos $\mathbf{T}_{\mathrm{fppf}}$, the category of biextensions of (G_1, G_2) by $\mathbb{G}_m$ is equivalent to the category of biextensions of the underlying abelian S-schemes (A_1, A_2) by $\mathbb{G}_m$. Therefore the proof of **Theorem C** reduces to verifying that the category of biextensions of (G_1, G_2) by G_3 is equivalent to the category of biextensions of (G_1, G_2) by the underlying torus T_3. Because of the homological interpretation of biextensions, the proof of this last equivalence of categories is an easy consequence of **Theorem A** and **B**.

This article is a shortened version of the first two chapters of author's Habilitationsschrift [B].

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Notation

In this paper S is an arbitrary scheme.

Let s a point of S, with residue field $k(s)$. We denote by $\overline{s}$ a geometric point over s and by $\overline{k(s)}$ its residue field.

If P, Q are S-group schemes, we write P_Q for the fibred product $P \times_S Q$ of P and Q over S, viewed as scheme over Q. In particular, if s is a point of S, we denote by $P_s = P \times_S \mathrm{Spec}\,(k(s))$ the fibre of P over s.

An **abelian S-scheme** is an S-group scheme which is smooth, proper over S and with connected fibres. An **S-torus** is an S-group scheme which is locally isomorphic for the fpqc topology (equivalently for the étale topology) to a S-group scheme of the kind $\mathbb{G}_m^r$ with r an integer bigger or equal to 0. The **character group** $\underline{\mathrm{Hom}}(T, \mathbb{G}_m)$ and the **cocharacter group** $\underline{\mathrm{Hom}}(\mathbb{G}_m, T)$ of a S-torus T are S-group schemes which are **locally** for the étale topology **constant group schemes defined by finitely generated free $\mathbb{Z}$-modules**. These S-group schemes are just locally of finite presentation over S (not of finite presentation), and so in some later considerations it will be necessary to allow S-group schemes that are merely locally of finite presentation over S. Sometimes it will be more convenient to denote by $Y(1)$ a torus with cocharacter group $Y = \underline{\mathrm{Hom}}(\mathbb{G}_m, Y(1))$ and character group $Y^\vee = \underline{\mathrm{Hom}}(Y(1), \mathbb{G}_m)$.

An S-group scheme is **divisible** if for each integer $n, n \neq 0$, the "multiplication by n" on it is an epimorphism for the fppf topology.

An **isogeny** $f : P \to Q$ between S-group schemes is a morphism of S-group schemes which is finite, faithfully flat and of finite presentation.

An **algebraic space over** S is a functor $X : (S-Schemes)^\circ \to (Sets)$ satisfying the following conditions:

(1) X is a sheaf for the étale topology,
(2) X is locally representable: there exists an S-scheme U, and a map $U \to X$ which is representable by étale surjective maps,
(3) X is quasi-separated over S.

1. Homomorphisms and extensions

Let S be a scheme.

We are interested only in commutative extensions of commutative S-group schemes. We consider such extensions in the category of abelian S-sheaves for the fppf site over S.

Lemma 1.0.1. *Any commutative extension of commutative S-group schemes is an algebraic space over S which is a group object.*

Proof. Since by [SGA7] Exposé VII 1.2 a commutative extension of commutative S-group schemes is a torsor endowed with a group law, this Lemma is a consequence of [LM-B] Corollary 10.4.2. □

1.1. **The case in which locally constant group schemes are involved.** We start by studying the homomorphisms from a group scheme with connected fibres to an separated and unramified group scheme:

Lemma 1.1.1. *Let X be a commutative S-group scheme separated and unramified over S, and let P be a commutative S-group scheme locally of finite presentation over S, with connected fibres. Then*

$$\mathrm{Hom}(P, X) = 0.$$

Proof. Let $f : P \to X$ be an S-homomorphism and consider the S-morphism $(f, \epsilon \circ p)_S : P \to X \times_S X$, where $\epsilon : S \to X$ is the unit section of X and $p : P \to S$ is the structural morphism of P. According to [EGAIV] 4 Proposition 17.4.6, the inverse image $(f, \epsilon \circ p)_S^{-1}(\Delta_{X/S})$ of the diagonal $\Delta_{X/S}$ of X is an open and closed subscheme of P whose restriction over each point of S is not empty. Since the fibres of P are connected, this inverse image $(f, \epsilon \circ p)_S^{-1}(\Delta_{X/S})$ is equal to P and so f is trivial. □

Concerning extensions of group schemes by locally constant group schemes, by [SGA3] Exposé X 5.5 we have that

Lemma 1.1.2. *Let X be an S-group scheme which is locally for the étale topology a constant group scheme defined by a finitely generated free $\mathbb{Z}$-module. Any X-torsor is a scheme. In particular, any extension of an S-group scheme by X is a scheme.*

In [SGA7] Exposé VIII Proposition 3.4 Grothendieck shows that there are no non-trivial extensions of smooth S-group schemes with connected fibres by S-group schemes which are locally for the étale topology constant group schemes defined by finitely generated free $\mathbb{Z}$-modules. Here we prove more in general that

Proposition 1.1.3. *Let X be commutative S-group scheme separated, unramified over S and with constant geometric fibres defined by finitely generated free $\mathbb{Z}$-modules, and let P be a smooth commutative S-group scheme with connected fibres. Then the category $\mathbf{Ext}(P, X)$ of extensions of P by X is the trivial category. More precisely,*

$$\mathrm{Hom}(P, X) = 0 \qquad \textit{and} \qquad \mathrm{Ext}^1(P, X) = 0.$$

In particular, both S-sheaves $\underline{\mathrm{Hom}}(P, X)$ and $\underline{\mathrm{Ext}}^1(P, X)$ are trivial.

Proof. Let E be an extension of P by X. We first show that this extension E is an S-group scheme. Assume S to be affine and Noetherian. According to Lemma 1.0.1 the extension E is an algebraic space over S which is a group object. By hypothesis the X-torsor E is separated, unramified (hence locally of finite presentation, locally quasi-finite) over P and so according to [A68] Theorem 3.3 (or [K] II Corollary 6.16) E is a scheme, which is separated and unramified over P.
By Lemma 1.1.1 it is enough to prove that the extension E is trivial locally for the Zariski topology, i.e. we can suppose S to be the spectrum of a local Artin ring. Then by [SGA3] Exposé $\mathrm{VI_A}$ 2.3 and Proposition 2.4, the connected component of the identity of the extension E exists. We denote it by E°. The scheme P is of finite presentation over the affine Noetherian scheme S. Therefore P is also Noetherian and in particular it is a finite disjoint union of its connected components. Restricting over one of these components we can suppose P connected. Since the structural morphism $p : E \to P$ of the torsor E over P is separated and unramified, according to [EGAIV] 4 Proposition (17.4.9), to the connected component E° corresponds a section $s : P \to E$ of p which is an isomorphism from P to E° and therefore the extension E is trivial. □

According to [SGA7] Exposé VII 1.4 for any commutative S-group scheme P the group $\mathrm{Ext}^1(\mathbb{Z}, P)$ is isomorphic to the group of isomorphism classes of P-torsors with respect to the fppf topology. Remark that if P is smooth over S, P-torsors for the fppf topology are the same as P-torsors for the étale topology. We have the following lemma

Lemma 1.1.4. *Let X be a S-group scheme which is locally for the étale topology a constant group scheme defined by a finitely generated free $\mathbb{Z}$-module and let P be any commutative S-group scheme. The S-sheaf $\underline{\mathrm{Ext}}^1(X, P)$ is trivial.*

1.2. **The case in which tori are involved.** We start investigating homomorphisms between an S-torus T and an abelian S-scheme A. In [SGA7] Exposé VII 1.3.8 Grothendieck proves that

$$\underline{\mathrm{Hom}}(A, T) = 0.$$

Lemma 1.2.1. *Let A be an abelian S-scheme and let T be an S-torus. Then*

$$\mathrm{Hom}(T, A) = 0.$$

In particular, the S-sheaf $\underline{\mathrm{Hom}}(T, A)$ is trivial.

Proof. Let $f : T \to A$ be a morphism from T to A. Consider the restriction $f_i : T[q^i] \to A[q^i]$ of f to the points of order q^i with q an integer bigger than 1 and invertible over S, and $i > 0$. For each i, f_i is a morphism of finite and étale S-schemes which factors set-theoretically through the unit section of $A[q^i]$, and so it is trivial. Since the family $(T[q^i])_i$ is schematically dense (see [SGA3] Exposé IX §4), we can conclude. □

We now investigate the extensions involving abelians schemes and tori. Since S-tori are affine over S, according to [SGA1] Exposé VIII 2.1 we have that

Lemma 1.2.2. *Any torsor under an S-torus is a scheme. In particular, any extension of an S-group scheme by an S-torus is a scheme.*

It is a classical result that over a separably closed field extensions of tori by tori are trivial. This is no longer true over an arbitrary scheme S. Nevertheless, in [SGA7] Exposé VIII Proposition 3.3.1 Grothendieck proves that if T_i (for $i = 1, 2$) is a torus over an arbitrary scheme S, then

$$\underline{\mathrm{Ext}}^1(T_1, T_2) = 0.$$

Over an algebraically closed field k the extensions of an abelian k-variety A by the k-torus $\mathbb{G}_m$ are far from trivial: they are parametrized by the k-rational points of the dual abelian variety of A. More in general, if A is an abelian scheme over an arbitrary scheme S, we have that

$$\underline{\mathrm{Ext}}^1(A, \mathbb{G}_m) = A^*.$$

where A^* is the dual abelian scheme of A ([O] Chapter I §5).

In the literature there are only few results about extensions of group schemes by abelian schemes (see [S60] 7.4 Corollary 1). The most technical difficulty in studying such extensions is that we are going outside the category of schemes. In fact an extension of a group scheme by an abelian scheme is in particular a torsor under this abelian scheme and in [R] XIII 3.2 Raynaud gives an example of a torsor under an abelian S-scheme which is not representable by a scheme. Nevertheless according to Lemma 1.0.1 such extensions of S-group schemes by abelian S-schemes are algebraic spaces over S.

We focus our attention on extensions of S-tori by abelian S-schemes. Working over an arbitrary base scheme S we prove that for such extensions it is equivalent to be a scheme or to be of finite order locally over S for the Zariski topology. Then we show that these extensions are in fact representable by schemes. Hence we can conclude that the extensions of S-tori by abelian S-schemes are of finite order locally over S. We proceed in this way because it turns out to be easier to prove that these extensions are schemes than to prove that they are of finite order locally over S, even though these two facts are equivalent in the end.

We start our investigation about extensions of tori by abelian schemes working over an algebraically closed field.

Proposition 1.2.3. *Let k be an algebraically closed field. An extension E of a k-torus T by an abelian k-variety A is a connected smooth k-algebraic group. Moreover, if we denote by A' and T' respectively the abelian k-variety and the k-torus given by Chevalley's decomposition (*[Ro] *Theorem 16) of E, we have that*

- *the torus T' is the maximal torus of the extension E and it is isogenous to the torus T;*
- $\dim T' = \dim T$ *and* $\dim A' = \dim A$.

Moreover, E is of finite order.

Proof. By Lemma 1.0.1 and by [A69] Lemma 4.2, the extension E is a connected smooth k-algebraic group.

Since the quotient E/T' is an abelian variety, the torus T' is the maximal torus of E and so by [SGA3] Exposé XII Theorem 6.6 (d), via the surjective morphism $E \to T$, the torus T' goes onto T. Moreover the kernel of this surjective morphism $T' \to T$ is $T' \cap A$ which is finite. Therefore the morphism $T' \to T$ is an isogeny. This implies that $\dim T' = \dim T$ and that $\dim A' = \dim A$.

The existence of an isogeny between the two tori T' and T has a geometric implication: the extension E is of finite order. In fact, let n be a positive integer

which annihilates the kernel of this isogeny. By definition of push-down (see [S59] Chapter VII §1.1), the image T'' of the torus T' in the push-down n_*E of E via the multiplication by n on A, is isomorphic to the torus T and this isomorphism between T'' and T furnishes the section which splits the extension n_*E. □

Now we go back to the general case: let S be an arbitrary scheme.

Lemma 1.2.4. *An extension of an S-torus by an abelian S-scheme is an algebraic space over S which is a group object and which is smooth (in particular flat), separated, of finite presentation over S and with connected fibres.*

Proof. Let E be an extension of an S-torus T by an abelian S-scheme A. By Lemma 1.0.1, the extension E is an algebraic space over S which is a group object. Clearly it has connected fibres. Since the abelian scheme A is proper and smooth over S, the A-torsor E is proper and smooth over T and so E is smooth, separated and of finite presentation over S. □

By Proposition 1.2.3, over each geometric point $\overline{s}$ of S the fibre $E_{\overline{s}}$ is a connected smooth algebraic group over the algebraically closed field $\overline{k(s)}$, and so we can generalized to the algebraic space E the notion of abelian and reductive rank introduced by Grothendieck in [SGA3] Exposé X page 121:

- the **abelian rank of** E **at the point** s, denoted by $\rho_{\mathrm{ab}}(s)$, is the dimension of the abelian $\overline{k(s)}$-variety appearing in Chevalley's decomposition of $E_{\overline{s}}$.
- the **reductive rank of** E **at the point** s, denoted by $\rho_{\mathrm{r}}(s)$, is the dimension of the maxinal tori of $E_{\overline{s}}$, i.e. the dimension of the $\overline{k(s)}$-torus appearing in Chevalley's decomposition of $E_{\overline{s}}$.

Proposition 1.2.5. *Let E be an extension of an S-torus by an abelian S-scheme. The following conditions are equivalent:*

(i)**:** *E is a scheme,*
(ii)**:** *E is of finite order locally over S for the Zariski topology. If S is quasi-compact, then the extension E is globally of finite order.*

Proof. Let E be an extension of an S-torus T by an abelian S-scheme A.
$(i) \Rightarrow (ii)$: Assume E to be an S-scheme. Let U be a quasi-compact open subset of S. We have to show that the extension E_U of T_U by A_U is of finite order. Denote by $\mathcal{T}$ (resp. by $\mathcal{MT}$) the functor of sub-tori (resp. the functor of maximal sub-tori) of E_U (see [SGA3] Exposé XV §8 for the definition of these functors). Since E is a commutative group scheme the functor of sub-tori coincide with the functor of central sub-tori. In Lemma 1.2.4 we have showed that the extension E_U is smooth, separated and of finite presentation over U. Moreover by Proposition 1.2.3 its abelian rank and its reductive rank are locally constant functions over U. Therefore according to [SGA3] Exposé XV Corollary 8.11 and corollary 8.17, the functor $\mathcal{T}$ is representable by an étale and separated U-scheme and the functor $\mathcal{MT}$ is representable by an open and closed sub-scheme of $\mathcal{T}$. In particular $\mathcal{MT}$ is étale over $\mathcal{T}$ and so over U. Since over each geometric point of U, the extension E_U admits a unique maximal torus, by [EGAIV] 4 Corollary 17.9.5 $\mathcal{MT}$ is isomorphic to U, which implies that there is a unique maximal sub-torus T' of E_U.
Over each geometric point of U, we have an epimorphism from the torus T' to the torus T_U. The kernel of this epimorphism is the scheme $T' \cap A_U$ which is flat and finite over U. Therefore $T' \to T_U$ is an isogeny. Now let n be a positive integer

which annihilates the kernel of this isogeny. By definition of push-down, the image T'' of the torus T' in the push-down n_*E_U of E via the multiplication by n on A_U, is isomorphic to the torus T_U and this isomorphism between T'' and T_U furnishes the section which splits the extension n_*E_U.
$(ii) \Rightarrow (i)$: If the extension E of T by A is trivial, then E is isomorphic to the product $A \times T$ and so it is an S-scheme. If the extension E of T by A is of order n, the extension n_*E is trivial and hence, as we have just seen, it is an S-scheme. Consider the short exact sequence given by the multiplication by n

$$0 \longrightarrow E[n] \longrightarrow E \longrightarrow n_*E \longrightarrow 0.$$

The kernel $E[n]$ of the multiplication by n is an $A[n]$-torsor and so it is finite over S. By [A68] Theorem 3.3 $E[n]$ is then a scheme. Since the S-scheme $E[n]$ is affine over S, the $E[n]$-torsor E is affine over the S-scheme n_*E and therefore according to [SGA1] VIII Theorem 2.1 E is an S-scheme. □

Theorem 1.2.6. *Any extension of an S-torus by an abelian S-scheme is a scheme.*

Proof. Let E be an extension of an S-torus T by an abelian S-scheme A. Since the question is local over S, we start doing two reduction steps:

- by Lemma 1.2.4 the algebraic space E is of finite presentation over S and so we can suppose S to be an affine Noetherian scheme;
- by [A68] Theorem 3.2 an algebraic space E is a scheme if and only if the reduced algebraic space E_{red} associated to E is one. Hence we may assume S to be reduced.

Let s to be a maximal point of S, i.e. a generic point of an irreducible component of S. According to Proposition 1.2.3, the fibre E_s over s admits a maximal torus, which extends to a maximal torus of E_U, with U an open non-empty subset of S containing s. Doing the same thing at a maximal point of the complement of U in S, by Noetherian recurrence on S, there exists a finite covering of S by locally closed sub-schemes S_i, for $i = 1, \ldots, r$, such that for each i the restriction E_i of the extension E over S_i admits a maximal torus Z_i. As we have observed in the proof of Proposition 1.2.5 $(i) \Rightarrow (ii)$, this torus Z_i is isogenous to the restriction T_i of the torus T over S_i and if n_i is a positive integer which annihilates the kernel of this isogeny, the push-down $(n_i)_*E_i$ of the extension E_i via the multiplication by n_i on the restriction of A over S_i, is trivial since the image of the torus Z_i in this push-down $(n_i)_*E_i$ is isomorphic to T_i.
Let n be the least common multiple of $n_1, \ldots, n_r$. If the extension n_*E is a scheme, then E is also one via the argument given in the proof of Proposition 1.2.5 $(ii) \Rightarrow (i)$. Therefore modulo multiplication by n, we can assume for each i that the extension E_i is trivial and the torus Z_i is isomorphic to T_i. In particular, the tori Z_i are locally closed subspaces of the extension E.
Denote by Z the finite union of the tori Z_i for $i = 1, \ldots, r$. Since the tori Z_i are locally closed in E and E is Noetherian, the set Z is a globally constructible set of the underlying topological space of E. Set-theoretically Z is characterized by the following property: if x is a point of E over the point s of S, then x is a point of Z if and only if x is a point of the maximal torus of the $\overline{k(s)}$-algebraic group $E_{\overline{s}}$. This characterization of Z shows that Z commutes with base extensions, i.e. if S' is a Noetherian S-scheme, if E' is the algebraic space obtained from E by base extension $S' \to S$, and if Z' is the constructible set of E' constructed as Z in E, then

Z' is the inverse image of Z by the canonical projection $pr : E' = E \times_S S' \to E$.
Now we will prove that this globally constructible set Z is a closed subset of the algebraic space E. Using [LM-B] Corollary (5.9.3) we can generalize to algebraic spaces the characterization of closed constructible subsets of a scheme given in [EGAI] Chapter I Corollary (7.3.2): a constructible subset of a Noetherian algebraic space is closed if and only if it is stable under specialization. In order to prove that Z is closed under specialization we use the valuative criterion of specialization [LM-B] Proposition (7.2.1): if x is a point of Z and y is a specialization of x, i.e. $y \in \overline{\{x\}}$, there is an S-scheme L, spectrum of a discrete valuation ring, and an S-morphism $\phi : L \to E$ which sends the generic point of L to x and the closed point of L to y. Therefore, Z is closed if and only if the image via ϕ of the closed point of L is a point of Z. According to [An] Chap. IV Theorem 4.B the extension $E_L = E \times_S L$, which is obtained by base extension through the canonical morphism $L \to S$, is an L-scheme. Denote by Z_L the inverse image of Z by the canonical projection $E_L = E \times_S L \to E$. Using the set-theoretical characterization of Z_L, we observe that Z_L is the maximal torus of the L-group scheme E_L (see definition [SGA3] II Exposé XV 6.1) and hence it is closed. The S-morphism $\phi : L \to E$ furnishes a section $\phi_L : L \to E_L$ of the L-scheme E_L, which sends the generic point of L to Z_L. But since Z_L is closed, also the closed point of L maps to Z_L via ϕ_L. Therefore the morphism ϕ, which is the composite of the section ϕ_L with the canonical projection $E_L \to E$, sends the closed point of L to Z. This concludes the proof that Z is a closed subset of E.
We endow Z with the reduced induced closed algebraic subspace structure. The algebraic subspace Z is then proper over T.
If s is a point of S, according to Proposition 1.2.3 the maximal torus of $E_{\overline{s}}$ is isogenous to $T_{\overline{s}}$. The set-theoretical characterization of Z implies then that for each point y of T over s, the fibre Z_y over y is a scheme consisting of only one point which has the same residue field of the point y. Hence

- by [LM-B] Corollary (A.2.1), the morphism $Z \to T$ is finite. In particular, according to [A68] Theorem 3.3 the algebraic subspace Z is a scheme.
- by [EGAI] Chapter I Proposition (3.7.1) (c) and Corollary (3.6.3) respectively, the morphism $Z \to T$ is a universal homeomorphism.

Denote by $\overline{E}$ the A-torsor over Z obtained as pull-back of the A-torsor E via the morphism $Z \to T$. The closed immersion $Z \to E$ trivialized this A-torsor $\overline{E}$, which is therefore a scheme. Because of the structure of A-torsor, the morphism $\overline{E} \to E$ is a finite universal homeomorphism. In particular, if U is an affine open subset of $\overline{E}$, there exists a Zariski open subset V of E such that U is the inverse image of V via this universal homeomorphism. The restriction $U \to V$ of $\overline{E} \to E$ is again a finite universal homeomorphism and so, by Chevalley's Theorem [K] III 4.1 the Zariski open subset V is affine. In this way we get an open affine covering of E, which is therefore a scheme. □

Corollary 1.2.7. *Let E be an extension of an S-torus T by an abelian S-scheme A. Then E is of finite order locally over S for the Zariski topology. If S is quasi-compact, E is globally of finite order. In particular, the S-sheaf $\underline{\mathrm{Ext}}^1(T, A)$ is a torsion sheaf.*

Corollary 1.2.8. *Let A be an abelian S-scheme and let T be an S-torus. The S-sheaf $\underline{\mathrm{Ext}}^1(T, A)$ is an S-group scheme which is separated and étale over S.*

Proof. Denote by $\underline{\mathrm{Ext}}^1(T,A)[n]$ the kernel of the multiplication by n on the sheaf $\underline{\mathrm{Ext}}^1(T,A)$ for each integer n bigger than 0. According to the above Corollary, the S-sheaf $\underline{\mathrm{Ext}}^1(T,A)$ is a torsion sheaf and so it is the inductive limit of the family of sheaves $\{\underline{\mathrm{Ext}}^1(T,A)[n]\}_n$:

$$\underline{\mathrm{Ext}}^1(T,A) = \varinjlim \underline{\mathrm{Ext}}^1(T,A)[n]. \tag{1.2.1}$$

Consider the long exact sequence associated to the multiplication by n on T:

$$\begin{aligned} 0 \to \underline{\mathrm{Hom}}(T,A) \to \underline{\mathrm{Hom}}(T,A) \to \underline{\mathrm{Hom}}(T[n],A) \xrightarrow{d} \underline{\mathrm{Ext}}^1(T,A) \\ \xrightarrow{n^*} \underline{\mathrm{Ext}}^1(T,A) \to \underline{\mathrm{Ext}}^1(T[n],A) \end{aligned}$$

where $T[n]$ is the kernel of the multiplication by n on T, d is the connecting morphism, and n^* is the pull-back of the extensions via the multiplication by n on T. By Lemma 1.2.1, the connecting morphism d is injective. Therefore the S-sheaf $\underline{\mathrm{Hom}}(T[n],A)$ is isomorphic to the S-sheaf $\underline{\mathrm{Ext}}^1(T,A)[n]$ and the inductive limit (1.2.1) can be rewritten in the following way:

$$\underline{\mathrm{Ext}}^1(T,A) = \varinjlim \underline{\mathrm{Hom}}(T[n],A) \tag{1.2.2}$$

where the inductive system is determined by the morphisms between the kernels $T[n]$ given by the multiplication by integers: $m : T[mn] \to T[n]$. Since the multiplication by a nonzero integer on a torus is an epimorphism for the fppf topology, the morphisms $\underline{\mathrm{Hom}}(T[n],A) \to \underline{\mathrm{Hom}}(T[mn],A)$ of the inductive system are monomorphisms for the fppf topology. According to [EGAI] Chp. I (2.4.3) and Chp. 0 Proposition 4.5.4, in order to prove that the inductive limit (1.2.2) is a scheme, we have to verify that

- (a)**:** for each integer n, the S-sheaf $\underline{\mathrm{Hom}}(T[n],A)$ is a scheme;
- (b)**:** for each integer m, each monomorphism $\underline{\mathrm{Hom}}(T[n],A) \to \underline{\mathrm{Hom}}(T[mn],A)$ of the inductive system is an open immersion.

Condition (a) is a consequence of [FGA] Exposé 221 4.c.
Before to prove (b), we show that for each n the scheme $\underline{\mathrm{Hom}}(T[n],A[n])$ is étale over S. In order to show this, it is enough to prove that the scheme $\underline{\mathrm{Hom}}(A[n]^*,T[n]^*)$, where $A[n]^*$ (resp. $T[n]^*$) is the Cartier dual of $A[n]$ (resp. $T[n]$), is étale over S. Consider the S-sheaf of morphisms $\underline{\mathrm{Mor}}(A[n]^*,T[n]^*)$. Since the question is local over S, we can suppose $T[n]^*$ to be a constant group scheme and so for any S-scheme S', to have a S'-morphism from $A[n]^*_{S'}$ to $T[n]^*_{S'}$ is equivalent to have a partition of $A[n]^*_{S'}$ in a finite number of open and closed subsets. By [EGAIV] 4 Lemma (18.5.3) the S-sheaf $\underline{\mathrm{Mor}}(A[n]^*,T[n]^*)$ is therefore an S-scheme which is affine, étale and of finite presentation over S. Consider now the morphism

$$\mathcal{H} : \underline{\mathrm{Mor}}(A[n]^*,T[n]^*) \longrightarrow \underline{\mathrm{Mor}}(A[n]^* \times A[n]^*,T[n]^*)$$

which sends a morphism f to the morphism $(x,y) \mapsto f(x)f(y)f(xy)^{-1}$. Since the scheme $\underline{\mathrm{Mor}}(A[n]^* \times A[n]^*,T[n]^*)$ is étale, its unit section $\epsilon : S \to \underline{\mathrm{Mor}}(A[n]^* \times A[n]^*,T[n]^*)$ is an open immersion. Therefore the kernel $\mathcal{H}^{-1}(\epsilon(S))$ of $\mathcal{H}$, which is $\underline{\mathrm{Hom}}(A[n]^*,T[n]^*)$, is an open subset of $\underline{\mathrm{Mor}}(A[n]^*,T[n]^*)$ and so we can conclude that the scheme $\underline{\mathrm{Hom}}(A[n]^*,T[n]^*)$ is étale over S. Condition (b) is now a consequence of the fact that étale monomorphisms are open immersions. $\square$

1.3. **The case in which only abelian schemes are involved.** We first study the S-sheaf of homomorphisms between two abelian schemes.

Proposition 1.3.1. *Let A and B be two abelian S-schemes. Then the S-sheaf $\underline{\mathrm{Hom}}(A,B)$ is an S-group scheme, which has constant geometric fibres defined by finitely generated free $\mathbb{Z}$-modules, and which is separated, unramified and essentially proper over S.*

Proof. We can suppose the base scheme S to be affine and Noetherian. By [A69] Corollary 6.2 (see also correctum [A74] Appendix 1) the Hilbert functor $\mathrm{Hilb}_{(A\times_S B)/S}$ of $A\times_S B$ is an algebraic space over S which is locally of finite presentation and separated over S. The S-sheaf $\underline{\mathrm{Hom}}(A,B)$ is then an algebraic subspace of $\mathrm{Hilb}_{(A\times_S B)/S}$ which is locally of finite presentation and separated over S. Moreover it has a group structure and it has constant geometric fibres defined by finitely generated free $\mathbb{Z}$-modules. Since its fibres are discrete, the structural morphism $\underline{\mathrm{Hom}}(A,B)\to S$ is locally quasi-finite and so by [A68] Theorem 3.3 $\underline{\mathrm{Hom}}(A,B)$ is a scheme.
In order to show that the scheme $\underline{\mathrm{Hom}}(A,B)$ is unramified over S, it is enough to show that if S is the spectrum of a local Artin ring R with residue field k and if $f:A\to B$ is an S-morphism whose restriction over $\mathrm{Spec}\,(k)$ is trivial, then f is trivial. Consider the restriction $f_i:A[q^i]\to B[q^i]$ of f to the points of order q^i with q an integer bigger than 1 which is coprime with the characteristic of k, and $i>0$. For each i, f_i is a morphism of finite and étale S-schemes which is trivial over $\mathrm{Spec}\,(k)$, and so it is trivial. Since the family $(A[q^i])_i$ is schematically dense (see [SGA3] Exposé IX §4), we can conclude.
Finally, $\underline{\mathrm{Hom}}(A,B)$ is essentially proper because of the Neronian property of abelian schemes over a discrete valuation ring. □

In order to investigate extensions of abelian S-schemes by abelian S-schemes, we start working over a field:

Proposition 1.3.2. *Over a field,*

(1) *an extension of an abelian variety by an abelian variety is an abelian variety.*
(2) *an extension of an abelian variety by an abelian variety is of finite order.*

Proof. Let E be an extension of an abelian k-variety A by an abelian k-variety B. By Lemma 1.0.1, the extension E is an algebraic space over k which is a group object. Clearly E is connected, proper and smooth over k and so it is an abelian variety.
By Poincaré's complete reducibility theorem, there exists an abelian k-sub-variety A' of E such that $E=B\cdot A'$, $B\cap A'$ is finite and $\dim E=\dim B+\dim A'$. We have a natural surjection $A'\to A'/B\cap A'=E/B=A$. The kernel of this surjection is $B\cap A'$ which is finite, and the surjection $A'\to A$ is an isogeny. This isogeny $A'\to A$ has a geometric consequence: the extension E is of finite order. In fact let n be a positive integer n which annihilates the kernel of this isogeny. By definition of push-down (see [S59] Chapter VII §1.1), the image A'' of the abelian variety A' in the push-down n_*E of E via the multiplication by n on B, is isomorphic to A and this isomorphism between A'' and A furnishes the section which split the extension n_*E. □

Now we go back to the general case: let S be an arbitrary scheme.

Lemma 1.3.3. *An extension of an abelian S-scheme by an abelian S-scheme is an abelian S-scheme.*

Proof. Let E be an extension of an abelian S-scheme A by an abelian S-scheme B. By Lemma 1.0.1, the extension E is an algebraic space over S which is a group object. Clearly it has connected fibres. Moreover the B-torsor E is proper and smooth over A and so E is proper and smooth over S. Therefore by [FC] Theorem 1.9 the extension E is an abelian S-scheme. □

Therefore working with extensions of abelian S-schemes by abelian S-schemes we don't go outside the category of schemes. The difficulties with these extensions lie on the fact that it is not true that they are of finite order, not even of finite order locally over S (see Remark below). Nevertheless, if we assume the base scheme S to be integral and geometrically unibranched we will show that these extensions are of finite order locally over S. Recall that a scheme S is **unibranched at the point s** if the local ring $\mathcal{O}_{S,s}$ at the point s is unibranched, i.e. the ring $(\mathcal{O}_{S,s})_{\mathrm{red}} = \mathcal{O}_{S,s}/\mathcal{N}ilradical$ is integral and the integral closure of $(\mathcal{O}_{S,s})_{\mathrm{red}}$ is a local ring. A scheme S is **geometrically unibranched at the point s** if the local ring $\mathcal{O}_{S,s}$ at the point s is unibranched and the residue field of the local ring, integral closure of $(\mathcal{O}_{S,s})_{\mathrm{red}}$, is a radiciel extension of the residue field of $\mathcal{O}_{S,s}$. Finally a scheme S is **geometrically unibranched** if it is geometrically unibranched at each point s of S.

Remark 1.3.4. *(Extension of abelian schemes by abelian schemes which are of infinite order)* Let X be the scheme obtained from infinitely many copies of the projective line $\mathbb{P}^1_k$ over a field k identifying the point 0 of a copy of $\mathbb{P}^1_k$ with the point at the infinity of the following copy of $\mathbb{P}^1_k$ and so on... The group $\mathbb{Z}$ acts on X sending one copy of $\mathbb{P}^1_k$ in the following one. The quotient $S = X/\mathbb{Z}$ is a scheme having a double point with distinct tangent lines.
Let E be an elliptic curve over S. Consider the automorphism of $E \times_S E$ of infinite order

$$\begin{array}{rcl} \mu : E \times_S E & \longrightarrow & E \times_S E \\ (a,b) & \longmapsto & (a+b,b). \end{array}$$

Consider also the trivial extension $E \times_S E$ of E by E over S. Since the automorphism μ respects exact sequences, we can use it in order to twist the cocycle defining the trivial extension, getting in this way an extension $\widetilde{E \times_S E}$ of E by E over S which is of infinite order. Over X, the extension $\widetilde{E \times_S E}$ becomes the trivial extension.

Proposition 1.3.5. *Assume S to be integral and geometrically unibranched. Let E be an extension of an abelian S-scheme by an abelian S-scheme. Then E is of finite order locally over S for the Zariski topology. If S is integral, geometrically unibranched and quasi-compact, E is globally of finite order.*

Proof. Let $S = \mathrm{Spec}\,(R)$ be an affine, integral and geometrically unibranched scheme. We can write R as inductive limit of $\mathbb{Z}$-algebras R_i of finite type. The R_i are integral but not necessarily geometrically unibranched. Denote by η_i the generic point of $S_i = \mathrm{Spec}\,(R_i)$. Let E be an extension of an abelian S-scheme A by an abelian S-scheme B and let $p : E \to A$ be the structural surjection of E over A. Consider the descent E_i of the extension E over S_i for i big enough. According Proposition 1.3.2 the extension E_{i,η_i} is of finite order, say of order n. Hence there

exists an open non-empty subset U_i of S_i and an U_i-morphism $q_i : A_{i,U_i} \to E_{i,U_i}$ such that the composite of q_i with the restriction of p_i over U_i is the multiplication by n (here A_i and p_i denote respectively the descent of A and p over S_i). Going back to S, there exists an open non-empty subset U of S and an U-morphism $q : A_U \to E_U$ such that the composite of q with the restriction $p_U : E_U \to A_U$ of p over U is the multiplication by n: $p_U \circ q = n$. According to Propositions 1.3.1, the S-group scheme $\underline{\mathrm{Hom}}(A, E)$ is unramified over S and it satisfies the valuative criterion for properness. Since S is integral and geometrically unibranched, by [EGAIV] 4 remark (18.10.20) the U-morphism q extends to an S-morphism $q : A \to E$ whose composition with the projection p is the multiplication by n. By definition of pull-back, q defines a section of the pull-back n^*E of E via the multiplication by n on the abelian scheme A, which means that the extension E is of order n. □

Let A and B abelian S-schemes. The S-sheaf $\underline{\mathrm{Ext}}^1(A, B)$ is a complicated object: it is not representable by an S-scheme since it does not commute with adic-limits (see condition (F_3) in [Mr]), and it is not a torsion sheaf as we have observed in remark 1.3.4. Only if we assume the base scheme S to be integral and geometrically unibranched, using [EGAIV] 4 Theorem (18.10.1) we get that in the small étale site, the S-sheaf $\underline{\mathrm{Ext}}^1(A, B)$ is a torsion sheaf.

1.4. **The case of extensions of abelian schemes by tori.** Through several "dévissages", using the results of the previous sections we can study the group of homomorphisms, the S-sheaf of homomorphisms, the group of extensions and the S-sheaf of extensions involving extensions of abelian schemes by tori.

Proposition 1.4.1. *Let S be a scheme. Let G_i (for $i = 1, 2$) be a commutative extension of an abelian S-scheme A_i by an S-torus T_i. The groups* $\mathrm{Hom}(G_1, G_2)$ *and* $\mathrm{Ext}^1(G_1, G_2)$ *live respectively in the following diagrams whose columns and rows are exact:*

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \mathrm{Hom}(A_1, A_2) \\
 & & & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & \to & \mathrm{Hom}(G_1, T_2) & \to & \mathrm{Hom}(T_1, T_2) & \to & \mathrm{Ext}^1(A_1, T_2) \\
 & & \downarrow & & \downarrow & & \cong & & \downarrow \\
0 & \to & \mathrm{Hom}(A_1, G_2) & \to & \mathrm{Hom}(G_1, G_2) & \to & \mathrm{Hom}(T_1, G_2) & \to & \mathrm{Ext}^1(A_1, G_2) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \mathrm{Hom}(A_1, A_2) & \cong & \mathrm{Hom}(G_1, A_2) & \to & 0 & \to & \mathrm{Ext}^1(A_1, A_2) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Hom}(T_1, T_2) & \to & \mathrm{Ext}^1(A_1, T_2) & \to & \mathrm{Ext}^1(G_1, T_2) & \to & \mathrm{Ext}^1(T_1, T_2) & &
\end{array}$$

$$\begin{array}{ccccccc}
0 & \to & \mathrm{Hom}(T_1, T_2) & \cong & \mathrm{Hom}(T_1, G_2) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Hom}(A_1, A_2) & \to & \mathrm{Ext}^1(A_1, T_2) & \to & \mathrm{Ext}^1(A_1, G_2) & \to & \mathrm{Ext}^1(A_1, A_2) \\
\cong & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Hom}(G_1, A_2) & \to & \mathrm{Ext}^1(G_1, T_2) & \to & \mathrm{Ext}^1(G_1, G_2) & \to & \mathrm{Ext}^1(G_1, A_2) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \mathrm{Ext}^1(T_1, T_2) & \to & \mathrm{Ext}^1(T_1, G_2) & \to & \mathrm{Ext}^1(T_1, A_2)
\end{array}$$

If S is the spectrum of an algebraically closed field, we have that

- the map $\mathrm{Ext}^1(A_1, T_2) \to \mathrm{Ext}^1(G_1, T_2)$ is surjective;

- the group $\mathrm{Ext}^1(T_1, G_2)$ is a subgroup of $\mathrm{Ext}^1(T_1, A_2)$. In particular the extensions of T_1 by G_2 are of finite order;
- the extensions of G_1 by A_2 are of finite order.

For the S-sheaf of homomorphisms and the S-sheaf of extensions involving extensions of abelian schemes by tori the situation is quite similar: we get two diagrams which are analogous to the ones of Proposition 1.4.1 and in particular as before we have that

- the S-sheaf $\underline{\mathrm{Hom}}(G_1, T_2)$ is a sub-sheaf of $\underline{\mathrm{Hom}}(T_1, T_2)$;
- the S-sheaf $\underline{\mathrm{Hom}}(G_1, A_2)$ is isomorphic to the S-sheaf $\underline{\mathrm{Hom}}(A_1, A_2)$;
- the S-sheaf $\underline{\mathrm{Hom}}(T_1, G_2)$ is isomorphic to the S-sheaf $\underline{\mathrm{Hom}}(T_1, T_2)$;
- the S-sheaf $\underline{\mathrm{Hom}}(A_1, G_2)$ is a sub-sheaf of $\underline{\mathrm{Hom}}(A_1, A_2)$.

What is different with respect to the case of groups is that without putting any hypothesis on S we have also that

- the map $\underline{\mathrm{Ext}}^1(A_1, T_2) \to \underline{\mathrm{Ext}}^1(G_1, T_2)$ is surjective;
- the S-sheaf $\underline{\mathrm{Ext}}^1(T_1, G_2)$ is a sub-sheaf of $\underline{\mathrm{Ext}}^1(T_1, A_2)$, and so in particular it is a torsion sheaf.

If the base scheme S is integral and geometrically unibrached, in the small étale site the S-sheaf $\underline{\mathrm{Ext}}^1(A_1, A_2)$ is a torsion sheaf and so

- the S-sheaf $\underline{\mathrm{Ext}}^1(G_1, A_2)$ is a torsion sheaf in the small étale site.

2. Biextensions

Let S be a scheme.

Let P, Q and G be commutative S-group schemes. A **biextension of** (P, Q) **by** G is a $G_{P\times Q}$-torsor B, endowed with a structure of commutative extension of Q_P by G_P and a structure of commutative extension of P_Q by G_Q, which are compatible one with another (for the definition of compatible extensions see [SGA7] Exposé VII Definition 2.1). Let B_i (for $i = 1, 2$) be a biextension of (P_i, Q_i) by G_i, with P_i, Q_i and G_i commutative S-group schemes. A **morphism of biextensions** $B_1 \to B_2$ is a system (F, f_P, f_Q, f_G) where $f_P : P_1 \to P_2,\ f_Q : Q_1 \to Q_2,\ f_G : G_1 \to G_2$ are morphisms of S-group schemes, and $F : B_1 \to B_2$ is a morphism between the underlying sheaves of B_1 and B_2, such that F is contemporaneously a morphism of extensions associated to the morphisms

$$f_P \times f_Q : Q_{1\,P_1} \longrightarrow Q_{2\,P_2}, \qquad f_P \times f_G : G_{1\,P_1} \longrightarrow G_{2\,P_2}$$

and a morphism of extensions associated to the morphisms

$$f_P \times f_Q : P_{1\,Q_1} \longrightarrow P_{2\,Q_2}, \qquad f_Q \times f_G : G_{1\,Q_1} \longrightarrow G_{2\,Q_2}.$$

We denote by $\mathbf{Biext}(P, Q; G)$ the category of biextensions of (P, Q) by G. Let $\mathrm{Biext}^0(P, Q; G)$ be the group of automorphisms of any biextension of (P, Q) by G, and let $\mathrm{Biext}^1(P, Q; G)$ be the group of isomorphism classes of biextensions of (P, Q) by G. By [SGA7] Exposé VII 2.7, the categories $\mathbf{Biext}(P, Q; G)$ and $\mathbf{Biext}(Q, P; G)$ are equivalent and so all what we prove for one of these categories is automatically proved also for the other.

Formalizing and generalizing Mumford's work [Mu69] on biextensions, in [SGA7] Exposé VII 3.6.5 and (3.7.4) Grothendieck points out the following homological

interpretation of the groups $\mathrm{Biext}^0(P,Q;G)$ and $\mathrm{Biext}^1(P,Q;G)$:

$$\begin{aligned}\mathrm{Biext}^0(P,Q;G) &\cong \mathrm{Ext}^0(P \overset{\mathbb{L}}{\otimes} Q, G) \cong \mathrm{Hom}(P\otimes Q, G)\\ \mathrm{Biext}^1(P,Q;G) &\cong \mathrm{Ext}^1(P \overset{\mathbb{L}}{\otimes} Q, G)\end{aligned} \tag{2.0.1}$$

where $\mathrm{Hom}(P\otimes Q,G)$ is the group of bilinear morphisms from $P\times Q$ to G, $P \overset{\mathbb{L}}{\otimes} Q$ is the derived functor of the functor $Q \to P\otimes Q$ in the derived category of abelian sheaves. Using this homological interpretation of biextensions, in [SGA7] Exposé VIII (1.1.4) he gets the exact sequence of 5 terms

$$\begin{aligned}0 \to \mathrm{Ext}^1(P, \underline{\mathrm{Hom}}(Q,G)) \to \mathrm{Biext}^1(P,Q;G) \to \mathrm{Hom}(P, \underline{\mathrm{Ext}}^1(Q,G))\\ \to \mathrm{Ext}^2(P, \underline{\mathrm{Hom}}(Q,G)) \to \mathrm{Ext}^2(P, \mathrm{R}\underline{\mathrm{Hom}}(Q,G)).\end{aligned} \tag{2.0.2}$$

We finish observing that since a biextension is in particular a torsor, by [LM-B] Corollary 10.4.2 we have

Lemma 2.0.2. *Any biextension of commutative S-group schemes is an algebraic space over S.*

2.1. Biextensions involving locally constant group schemes.

Proposition 2.1.1. *Let X be an S-group scheme which is locally for the étale topology a constant group scheme defined by a finitely generated free $\mathbb{Z}$-module, let P be a smooth commutative S-group scheme with connected fibres, and let Q be a commutative S-group scheme. Then,*

$$\mathbf{Biext}(P,Q;X) = \mathbf{Biext}(Q,P;X) = 0.$$

Proof. Since the S-sheaf $\underline{\mathrm{Hom}}(P,X)$ is trivial,

- the category $\mathbf{Biext}(Q,P;X)$ is rigid, and
- by the exact sequence of 5 terms (2.0.2), there exists the canonical isomorphism
$$\mathrm{Biext}^1(Q,P;X) \cong \mathrm{Hom}(Q, \underline{\mathrm{Ext}}^1(P,X)).$$

Hence using again Proposition 1.1.3 we can conclude. □

Let X be an S-group scheme which is locally for the étale topology a constant group scheme defined by a finitely generated free $\mathbb{Z}$-module, and let P be a commutative S-group scheme. The tensor product $P\otimes X$ is an algebraic space over S with the following property: there exist an étale surjective morphism $S' \to S$ for which $(P\otimes X)_{S'}$ is isomorphic to a finite product of copies of the S'-group scheme $P_{S'}$. In particular, if P is an S-group scheme which is locally for the étale topology a constant group scheme defined by a finitely generated free $\mathbb{Z}$-module (resp. an S-torus, resp. an abelian S-scheme, resp. an extension of an abelian S-scheme by an S-torus), so is $P\otimes X$.

Proposition 2.1.2. *Let X be an S-group scheme which is locally for the étale topology a constant group scheme defined by a finitely generated free $\mathbb{Z}$-module, and let P_i (for $i=1,2$) be a commutative S-group scheme. Then,*

$$\mathbf{Biext}(X,P_1;P_2) \cong \mathbf{Ext}(X\otimes P_1, P_2) \cong \mathbf{Biext}(P_1,X;P_2).$$

Proof. Consequence of [SGA7] Exposé VIII 1.2. □

Corollary 2.1.3. *Let X and Y be S-group schemes which are locally for the étale topology constant group schemes defined by finitely generated free $\mathbb{Z}$-modules and let P be a commutative S-group scheme. Then,*

$$\mathbf{Biext}(X, Y; P) \cong \mathbf{Ext}(X \otimes Y, P).$$

Moreover, the objects of these categories are trivial locally over S for the fppf topology.

Proof. Consequence of the above Proposition and of Lemma 1.1.4. □

2.2. **Biextensions of group schemes by tori.** Biextensions of group schemes by tori have been studied by Grothendieck in [SGA7] Exposé VIII §3. We recall here some of his results.

Proposition 2.2.1. *Let X be an S-group scheme which is locally for the étale topology a constant group scheme defined by a finitely generated free $\mathbb{Z}$-module, let A be an abelian S-scheme, and let $Y(1)$ and $Y'(1)$ be S-tori. Then,*

(1) $\mathbf{Biext}(X, Y'(1); Y(1)) \cong \mathbf{Ext}(X \otimes Y'(1), Y(1)) \cong \mathbf{Biext}(Y'(1), X; Y(1))$.
Moreover, the objects of these categories are trivial locally over S for the fppf topology.

(2) $\mathbf{Biext}(X, A; Y(1)) \cong \mathbf{Ext}(A, X^\vee \otimes Y(1)) \cong \mathbf{Biext}(A, X; Y(1))$.
These categories are rigid and for the objects we have the canonical isomorphisms

$$\begin{aligned} \mathrm{Biext}^1(A, X; Y(1)) &\cong \mathrm{Ext}^1(A, \underline{\mathrm{Hom}}(X, Y(1))) \\ \mathrm{Biext}^1(A, X; Y(1)) &\cong \mathrm{Hom}(X, \underline{\mathrm{Ext}}^1(A, Y(1))) = \mathrm{Hom}(X, Y \otimes A^*), \end{aligned}$$

where A^* is the dual abelian scheme $\underline{\mathrm{Ext}}^1(A, \mathbb{Z}(1))$ of A.

(3) $\mathbf{Biext}(X, A; Y(1)) \cong \mathbf{Ext}(X \otimes A, Y(1)) \cong \mathbf{Biext}(A, X; Y(1))$.
In particular,

$$\mathbf{Biext}(X, A; Y(1)) \cong \mathbf{Ext}(X \otimes A, Y(1)) \cong \mathbf{Ext}(A, X^\vee \otimes Y(1)).$$

Proof. (1) Consequence of Proposition 2.1.2. The last assertion is due to [SGA7] Exposé VIII Proposition 3.3.1.
(2) [SGA7] Exposé VIII Proposition 3.7.
(3) Consequence of Proposition 2.1.2 and of assertion 2. □

Proposition 2.2.2. *Let $Y_1(1)$ and $Y_2(1)$ be S-tori and let P be a commutative S-group scheme. Then,*

$$\mathbf{Biext}(P, Y_1(1); Y_2(1)) \cong \mathbf{Ext}(P, Y_1^\vee \otimes Y_2) \cong \mathbf{Biext}(Y_1(1), P; Y_2(1)). \tag{2.2.1}$$

where $Y_1^\vee$ is the character group of $Y_1(1)$ and Y_2 is the cocharacter group of $Y_2(1)$. Moreover,

(1) *If P is a smooth commutative S-group scheme with connected fibres,*

$$\mathbf{Biext}(P, Y_1(1); Y_2(1)) = \mathbf{Biext}(Y_1(1), P; Y_2(1)) = 0.$$

(2) *If P is an S-group scheme X which is locally for the étale topology a constant group scheme defined by a finitely generated free $\mathbb{Z}$-module,*

$$\mathbf{Biext}(X, Y_1(1); Y_2(1)) \cong \mathbf{Ext}(X \otimes Y_1(1), Y_2(1)) \cong \mathbf{Ext}(X, Y_1^\vee \otimes Y_2).$$

Proof. By [SGA7] Exposé VIII Proposition 3.3.1 and paragraph 1.5, we have that

$$\mathbf{Biext}(P, Y_1(1); Y_2(1)) \cong \mathbf{Ext}(P, \underline{\mathrm{Hom}}(Y_1(1), Y_2(1))).$$

Using the canonical isomorphism $\underline{\mathrm{Hom}}(Y_1(1), Y_2(1)) \cong Y_1^\vee \otimes Y_2$, we get the equivalence of categories (2.2.1). Assertion (1) is a consequence of Proposition 1.1.3 and of (2.2.1). Assertion (2) is a consequence of Proposition 2.1.2 and of (2.2.1). □

The group of isomorphism classes of biextensions of abelian schemes by tori is isomorphic to the group of homomorphisms between abelian schemes. In fact,

Proposition 2.2.3. *Let A_i (for $i = 1, 2$) abelian S-schemes and let $Y(1)$ be a S-torus. The category $\mathbf{Biext}(A_1, A_2; Y(1))$ of biextensions of (A_1, A_2) by $Y(1)$ is rigid and for the objects we have the following canonical isomorphism*

$$\mathrm{Biext}^1(A_1, A_2; Y(1)) \cong \mathrm{Hom}(A_1, \underline{\mathrm{Ext}}^1(A_2, Y(1))) = \mathrm{Hom}(A_1, Y \otimes A_2^*).$$

where A_2^* is the dual abelian scheme $\underline{\mathrm{Ext}}^1(A_2, \mathbb{Z}(1))$ of A_2.

Proof. Since the S-sheaf $\underline{\mathrm{Hom}}(A, Y(1))$ is trivial,

- the categories $\mathbf{Biext}(A_1, A_2; Y(1))$ is rigid, and
- by the exact sequence of 5 terms (2.0.2), there exists the canonical isomorphism

$$\mathrm{Biext}^1(A_1, A_2; Y(1)) \cong \mathrm{Hom}(A_1, \underline{\mathrm{Ext}}^1(A_2, Y(1))).$$

□

2.3. **Biextensions of group schemes by an abelian scheme.** Using the fact that the extensions of tori by abelian schemes are of finite order locally over S (see Proposition 1.2.5 and Theorem 1.2.6), we prove that there are no non-trivial biextensions of tori and divisible group schemes by abelian schemes.

Theorem 2.3.1. *Let A be an abelian S-scheme, let $Y(1)$ be an S-torus and let P be a divisible commutative S-group scheme locally of finite presentation over S, with connected fibres. Then,*

$$\mathbf{Biext}(P, Y(1); A) = \mathbf{Biext}(Y(1), P; A) = 0$$

Proof. Since the S-sheaf $\underline{\mathrm{Hom}}(Y(1), A)$ is trivial,

- the categories $\mathbf{Biext}(P, Y(1); A)$ is rigid, and
- by the exact sequence of 5 terms (2.0.2), there exists the canonical isomorphism

$$\mathrm{Biext}^1(P, Y(1); A) \cong \mathrm{Hom}(P, \underline{\mathrm{Ext}}^1(Y(1), A)).$$

According to Corollary 1.2.8, the S-sheaf $\underline{\mathrm{Ext}}^1(Y(1), A)$ is an S-group scheme which is separated and étale over S. Since P has connected fibres, by Lemma 1.1.1 we can conclude. □

There is a more down-to-earth proof of the above theorem: let B be a biextension of $(P, Y(1))$ by A and let $\Phi : P \times Y(1) \times Y(1) \to A$ and $\Psi : P \times P \times Y(1) \to A$ be the two co-cycles defining it. Since by Lemma 1.2.1 the group $\mathrm{Biext}^0(P, Y(1); A)$ is zero, in order to prove the triviality of the biextension B we can work locally on the base and hence Corollary 1.2.7 implies that it exists a positive integer n such that $n\Phi = 0$. The pull-back $(id \times n)^* B$ of B via $id \times n : P \times Y(1) \to P \times Y(1)$ is defined by the trivial co-cycle $n\Phi : P \times Y(1) \times Y(1) \to A$ and by the co-cycle

$\Psi : P \times P \times Y(1) \to A$. But these 2 co-cycles are related one with another by the relation

$$n\Phi(p_1+p_2,q_1,q_2) - n\Phi(p_1,q_1,q_2) - n\Phi(p_2,q_1,q_2) = \\ \Psi(p_1,p_2,q_1+q_2) - \Psi(p_1,p_2,q_1) - \Psi(p_1,p_2,q_2)$$

for any p_i point of P and any q_i point of $Y(1)$ (for $i = 1,2$). Therefore by Lemma 1.2.1 also the co-cycle Ψ is trivial. This means that the pull-back $(id\times n)^*B$ of B is trivial and so by the following Lemma, which is a consequence of [SGA7] Exposé VII Proposition 3.8.8, we can conclude.

Lemma 2.3.2. *Let S be a scheme and let P, Q and G be three commutative S-group schemes. Assume P and Q divisible. Then for integers n and m, $n \neq 0, m \neq 0$, the kernel of the map*

$$\begin{array}{rcl} \Upsilon_{(n\times m)} : \mathrm{Biext}^1(P,Q;G) & \longrightarrow & \mathrm{Biext}^1(P,Q;G) \\ B & \longmapsto & (n\times m)^*B \end{array}$$

sending each element of $\mathrm{Biext}^1(P,Q;G)$ *to its pull-back via* $n\times m : P\times Q \to P\times Q$ *is trivial.*

We finish describing biextensions of locally constant group schemes and abelian schemes by abelian schemes and biextensions of locally constant schemes and tori by abelian schemes.

Proposition 2.3.3. *Let X be an S-group scheme which is locally for the étale topology a constant group scheme defined by a finitely generated free $\mathbb{Z}$-module, let $Y(1)$ be an S-torus and let A and A' be abelian S-schemes. Then,*

(1) $\mathbf{Biext}(X,A';A) \cong \mathbf{Ext}(X\otimes A',A) \cong \mathbf{Biext}(A',X;A)$.
Moreover, if the base scheme S is integral and geometrically unibranched, their objects are of finite order locally over S for the Zariski topology.

(2) $\mathbf{Biext}(X,Y(1);A) \cong \mathbf{Ext}(X\otimes Y(1),A) \cong \mathbf{Biext}(Y(1),X;A)$.
Moreover, these categories are rigid and their objects are of finite order locally over S for the Zariski topology.

Proof. (1) Consequence of Proposition 2.1.2 and of Proposition 1.3.5
(2) The first assertion is a consequence of Proposition 2.1.2. Lemma 1.2.1 implies that the categories $\mathbf{Biext}(X,Y(1);A)$ and $\mathbf{Ext}(X\otimes Y(1),A)$ are rigid. The last assertion is a consequence of Proposition 1.2.7. □

2.4. **Biextensions of abelian schemes by abelian schemes.** Even though there are extensions of abelian schemes by abelian schemes which are of infinite order (see Remark 1.3.4), in this section we show that there are no non-trivial biextensions of abelian S-schemes by abelian S-schemes. In order to do this we use the exact sequence of 5 terms (2.0.2) applied to tree abelian S-schemes A, B and C

$$\begin{array}{c} 0 \to \mathrm{Ext}^1(A,\underline{\mathrm{Hom}}(B,C)) \to \mathrm{Biext}^1(A,B;C) \to \mathrm{Hom}(A,\underline{\mathrm{Ext}}^1(B,C)) \\ \to \mathrm{Ext}^2(A,\underline{\mathrm{Hom}}(B,C)) \to \mathrm{Ext}^2(A,\mathrm{R}\underline{\mathrm{Hom}}(B,C)), \end{array} \tag{2.4.1}$$

and we check that the first term $\mathrm{Ext}^1(A,\underline{\mathrm{Hom}}(B,C))$ and the third term $\mathrm{Hom}(A,\underline{\mathrm{Ext}}^1(B,C))$ are both trivial. The triviality of the first term is a consequence of Propositions 1.3.1 and 1.1.3. Before we prove the triviality of the third term $\mathrm{Hom}(A,\underline{\mathrm{Ext}}^1(B,C))$ we need some Lemmas:

Lemma 2.4.1. *Let A, B and C be three abelian S-schemes. Then the group of morphisms from the abelian scheme A to the S-sheaf $\underline{\mathrm{Ext}}^1(B,C)$ is torsion-free.*

Proof. Consequence of the fact that the abelian scheme A is a divisible group scheme for the fppf topology. □

If $f : X \to S$ is a morphism of schemes and $\mathcal{G}$ is an abelian étale X-sheaf we have the Leray spectral sequence

$$\mathrm{H}^p(S, \mathrm{R}^q f_* \mathcal{G}) \Rightarrow \mathrm{H}^{p+q}(X, \mathcal{G}). \tag{2.4.2}$$

Recall also that a local ring $\mathcal{O}$, with residue field k, is **henselian** if for each étale and separated morphism $S \to \mathrm{Spec}\,(\mathcal{O})$, the set of $\mathrm{Spec}\,(\mathcal{O})$-sections of S is in one to one correspondence with the set of $\mathrm{Spec}\,(k)$-sections of $S \times_{\mathrm{Spec}\,(\mathcal{O})} \mathrm{Spec}\,(k)$. A local ring is **strictly local** if it is an henselian ring and its residue field is separably closed.

Lemma 2.4.2. *Let S and X be Noetherian schemes and let $f : X \to S$ be a morphism which is faithfully flat over S. Denote by η the scheme of generic points of X and $i : \eta \to X$ the canonical morphism. Let $\mathcal{H}$ be an abelian étale sheaf over η. Then*

- (a)**:** *the sheaves $\mathrm{R}^j i_*(\mathcal{H})$ and $\mathrm{R}^j (fi)_*(\mathcal{H})$ are torsion sheaves for each $j \geq 1$;*
- (b)**:** *the sheaves $\mathrm{R}^j f_*(i_*(\mathcal{H}))$ are torsion sheaves for each $j \geq 1$.*

Proof. (a) We can assume X (resp. S) to be the spectrum of a strictly local ring. We get assertion (a) by applying the Leray spectral sequence (2.4.2) to the sheaf $\mathcal{H}$ and to the morphisms $i : \eta \to X$ and $fi : \eta \to S$ respectively, since $\mathrm{H}^j(\eta, \mathcal{H})$ is a torsion group for each $j \geq 1$.
(b) Consequence of (a). □

In the next proofs we need a very naive notion of constructible étale sheaf of $\mathbb{Z}$-modules and we are not really working in the framework of constructibility as in SGA4 but rather in a mild variant adapted to our needs. Assume S to be a Noetherian scheme. Here an étale S-sheaf $\mathcal{F}$ of $\mathbb{Z}$-modules is said **constructible** if it exists a finite partition S_i of S, with S_i locally closed, such that $\mathcal{F}_{|S_i}$ is a locally constant sheaf whose fibres are finitely generated $\mathbb{Z}$-modules. This notion is stable by inverse image via morphisms of finite type and by direct image via finite morphisms. In fact,

- If $T \to S$ is a finite morphism of schemes, the function $s \mapsto$ separable degree of $T \times k(s)$ is constructible.
- If S is a normal Noetherian irreducible scheme with fraction field K and if T is the normalisation of S in some finite étale Galois extension L of K, with Galois group G, then the class of conjugacy of the inertia group at a point t of T, varies in a constructible way. (One needs such a fact to prove the constructibility of the direct image of a sheaf concentrated on the generic point).

Lemma 2.4.3. *Let S be a Noetherian scheme and let $f : X \to S$ be a morphism which is proper, smooth and with geometrically connected fibres. Moreover let $\mathcal{F}$ be an abelian étale constructible S-sheaf of $\mathbb{Z}$-modules. Denote by $\mathcal{K}$ the inverse image $f^*\mathcal{F}$ of $\mathcal{F}$ by the morphism f. Then*

- (a)**:** *the canonical morphism $\mathcal{F} \to f_*(\mathcal{K})$ is an isomorphism,*
- (b)**:** *the S-sheaves $\mathrm{R}^j f_*(\mathcal{K})$ are torsion sheaves for each $j \geq 1$.*

Proof. Remark that for the proof of this Lemma we cannot use the proper base change Theorem since $\mathcal{F}$ may not be a torsion sheaf.
(a) It suffices to compare the global sections over a strictly local base. This assertion is then a consequence of the fact that f has a trivial Stein factorization.
(b) We can suppose S to be the spectrum of a strictly local Noetherian ring. We prove (b) by induction on the dimension of the support of $\mathcal{F}$. Denote by ζ the scheme of generic points of S and by $k : \zeta \to S$ the canonical morphism. Consider the canonical morphism $u : \mathcal{F} \to k_*(k^*\mathcal{F})$. The sheaves $k_*(k^*\mathcal{F}), \mathrm{Ker}(u)$ and $\mathrm{Coker}(u)$ are constructible sheaves over S. By induction, we can assume that assertion (b) is true for the sheaves $\mathrm{Ker}(u)$ and $\mathrm{Coker}(u)$. Therefore it is enough to prove this assertion for a sheaf $\mathcal{F}$ of the kind $k_*(\mathcal{G})$ where $\mathcal{G}$ is a sheaf over ζ.
Denote by η the scheme of generic points of X and by $i : \eta \to X$ the canonical morphism. Since f is smooth, by [SGA4] Exposé XVI Corollary 1.2 the sheaf $f^*(\mathcal{F}) = f^*k_*(\mathcal{G})$ is the sheaf $i_*(\mathcal{H})$, where $\mathcal{H}$ the inverse image of $\mathcal{G}$ by the natural morphism $\eta \to \zeta$. Therefore we are reduced to prove the assertion (b) for a sheaf over X of the kind $i_*(\mathcal{H})$, but this is exactly Lemma 2.4.2. □

Lemma 2.4.4. *Let S be a Noetherian scheme and let G be an S-group scheme unramified over S. Denote by G' the maximal étale open sub-scheme of G. Denote by $G_{ét}$ (resp. G_{fppf}) the étale sheaf (resp. fppf sheaf) underlying G on the small site over S. Idem for G'.*

- (a)**:** *We have that $G'_{ét} = G_{ét}, G'_{fppf} = G_{fppf}$ and the canonical applications $\mathrm{H}^i(S_{ét}, G_{ét}) \to \mathrm{H}^i(S_{fppf}, G_{fppf})$ are bijections;*
- (b)**:** *Moreover, if G is separated over S and for each geometric point $\overline{s}$ of S $G_{\overline{s}}$ is of finite type, then $G_{ét}$ is a constructible sheaf of $\mathbb{Z}$-modules over S.*

Proof. Remark that G' is in fact a group sub-scheme of G since it contains the unit section.
(a) Consider a section of G above S. We have to show that it factors through G'. Since G is unramified over S, this section is an open immersion and so G is flat over the points of this section. Again by hypothesis of unramifiedness, G is flat at a point g if and only if G is étale at g. This implies that G is étale over the points of this section which factors therefore through G'. Hence on the small site over S, G and G' furnish the same étale sheaf (resp. fppf sheaf). Since G' is smooth over S, the last assertion is a consequence of [G68] Theorem 11.7.
(b) The hypothesis on G are also satisfied by G', and so we can assume $G = G'$. We are therefore reduce to show that for any integral sub-scheme T of S, G'_T is a locally constant scheme, whose fibres are finitely generated $\mathbb{Z}$-modules, over a non-empty open subset of T. Denote by t the generic point of T. By hypothesis on the geometric fibres of G, there exists a finite étale extension $t' \to t$ which splits the fibre of G over t. We can extend the morphism $t' \to t$ to a finite étale morphism $T' \to T$ such that $G_{T'}$ admits a constant group sub-scheme H' which has the same fibre as G over the generic point t'. The scheme H' is then an open group sub-scheme of $G_{T'}$ which satisfies the valuative criterion for properness over T' and therefore it is closed (since G is separated over S.) Hence $H' = G_{T'}$. □

We will apply the two above Lemmas to a S-group scheme G of the kind $\underline{\mathrm{Hom}}(B, C)$, where B and C are two abelian S-schemes. Observe that in general the fibres of G don't vary in a constructible way as the one of G', i.e. $G_{ét}$ is constructible, but not G in general.

Proposition 2.4.5. *Let A, B and C be three abelian S-schemes. Then,*

$$\mathrm{Hom}(A, \underline{\mathrm{Ext}}^1(B, C)) = 0.$$

Proof. Since the question is local over S, by [EGAIV] 4 Proposition 18.8.18 and [EGAIV] 3 Theorem 8.8.2 we can suppose S to be the spectrum of a strictly local Noetherian ring $\mathcal{O}$, with maximal ideal $\mathcal{M}$ and residue field k. For each $n \geq 1$, denote by S_n the S-scheme $\mathrm{Spec}\,(\mathcal{O}/\mathcal{M}^n)$.
In order to study the group $\mathrm{Hom}(A, \underline{\mathrm{Ext}}^1(B, C))$ we used the exact sequence of 4 terms

$$\begin{aligned} 0 \to \mathrm{H}^1(S, \underline{\mathrm{Hom}}(B, C)) \to \mathrm{Ext}^1(B, C) \to \mathrm{H}^0(S, \underline{\mathrm{Ext}}^1(B, C)) \\ \to \mathrm{H}^2(S, \underline{\mathrm{Hom}}(B, C)) \end{aligned} \tag{2.4.3}$$

associated to the spectral sequence $\mathrm{H}^p(S, \underline{\mathrm{Ext}}^q(B, C)) \Rightarrow \mathrm{Ext}^*(B, C)$, where the cohomology groups are computed with respect to the fppf topology.
Let u be a morphism from A to $\underline{\mathrm{Ext}}^1(B, C)$. We can consider the morphism $u : A \to \underline{\mathrm{Ext}}^1(B, C)$ as a section of $\underline{\mathrm{Ext}}^1(B, C)$ over A satisfying some additive properties and the property that its restriction to the unit section of A is trivial. The exact sequence (2.4.3) implies that the obstruction to represent u as a global extension of B_A by C_A lies in the cohomology group $\mathrm{H}^2(A, \underline{\mathrm{Hom}}(B_A, C_A))$. By Lemma 2.4.4 we can compute this cohomology group with respect to the étale topology and we can reduce to a constructible sheaf. Since S is the spectrum of a strictly local ring, the Leray spectral sequence (2.4.2) implies that $\mathrm{H}^2(A, \underline{\mathrm{Hom}}(B_A, C_A))$ is the group $\mathrm{H}^0(S, \mathrm{R}^2 f_* \underline{\mathrm{Hom}}(B_A, C_A))$, where $f : A \to S$ is the structural morphism of A. But then according to Lemma 2.4.3 $\mathrm{H}^2(A, \underline{\mathrm{Hom}}(B_A, C_A))$ is a torsion group annihilated by an integer, says N. Since by Lemma 2.4.1 it is enough to prove that $Nu = 0$, we can then suppose that u comes from a global extension E of B_A by C_A. The extension E is an abelian A-scheme (see Lemma 1.3.3). Since the restriction of u over the unit section of A is trivial, also the extension E is trivializable over the unit section of A.
Now we will prove that for $n \geq 1$ the restriction E_n of the extension E over the S_n-scheme A_{S_n} is trivializable. In order to simplify the notations, we denote by A_n the abelian S_n-scheme $A \times_S S_n$ and by B_{A_n} (resp. C_{A_n}) the abelian A_n-scheme $B_A \times_A A_n$ (resp. $C_A \times_A A_n$).

- We start with the case $n = 1$. Since A_1 is smooth over $S_1 = \mathrm{Spec}\,(k)$, A_1 is integral and geometrically unibranched. By Proposition 1.3.5 we have that the restriction E_1 of E over A_1 is of finite order, which implies that the restriction of u over A_1 is a torsion element and therefore by Lemma 2.4.1 it is trivial. Applying the exact sequence (2.4.3) to $S = A_1$, we observe that this restriction of u over A_1 comes from an element of $\mathrm{H}^1(A_1, \underline{\mathrm{Hom}}(B_{A_1}, C_{A_1}))$. But the group $\mathrm{H}^1(A_1, \underline{\mathrm{Hom}}(B_{A_1}, C_{A_1}))$ is trivial: in fact let P be a $\underline{\mathrm{Hom}}(B_{A_1}, C_{A_1})$-torsor over A_1. If η denote the generic point of A_1, the group $\mathrm{H}^1(\eta, \underline{\mathrm{Hom}}(B_{A_1}, C_{A_1})_\eta)$ is trivial (recall that k is separably closed) and so P has a section over the generic point η. Since A_1 is integral and geometrically unibranched, by [EGAIV] 4 remark (18.10.20) this section over the generic point extends over the whole abelian variety A_1, i.e. P is a trivial torsor. Therefore we can conclude that the extension E_1 corresponding to the restriction of u over A_1 is trivializable.

- For each n the restriction E_n of E over A_n is trivializable over the unit section of A_n, that we identify with S_n. We have the following diagram

$$\begin{array}{ccccccccc} E_1 & \to & \cdots & \to & E_n & \to & E_{n+1} & \to & \cdots \\ \downarrow & & & & \downarrow & & \downarrow & & \\ A_1 & \to & \cdots & \to & A_n & \to & A_{n+1} & \to & \cdots \end{array}$$

  By [I] Proposition 3.2 (b) the liftings of the abelian scheme E_n are classified by the group $\mathrm{Ext}^1(E_1, \mathrm{Lie}\,(E_1)^\vee \otimes \mathcal{M}^n/\mathcal{M}^{n+1})$. According to the theory of universal vectorial extensions, we know that this group is isomorphic to the group

$$\mathrm{Hom}(\omega_{E_1}, \mathrm{Lie}\,(E_1)^\vee \otimes \mathcal{M}^n/\mathcal{M}^{n+1}) \tag{2.4.4}$$

  where ω_{E_1} is the vector group $\mathrm{Hom}(\mathrm{Ext}^1(E_1, \mathbb{G}_a), \mathbb{G}_a)$. Therefore the liftings of E_n are classified by morphisms between locally free $\mathcal{O}_{A_1}$-modules of finite rank. Since the k-abelian variety is projective, these morphisms are defines by constants. Moreover the restriction of E_n over the unit section of A_n is trivializable and so these constants are trivial, i.e. the group (2.4.4) is trivial. This implies that there exists a unique way to lift the extension E_n and therefore since the trivial way is one way to do it, the extension E_n is trivializable.

We know that the restriction of the extension E over the unit section of A and over A_n for $n \geq 1$ is trivializable. We want to show that this global extension E of B_A by C_A is zero as section of $\underline{\mathrm{Ext}}^1(B, C)$ over A. Consider the subsheaf $\underline{\mathrm{F}}$ of $\underline{\mathrm{Hom}}(B_A, E)$ consisting of those morphisms from the abelian scheme B_A to the abelian scheme E which are the identity once composed with the projection $E \to B_A$: in other words, $\underline{\mathrm{F}}$ is the A-sheaf of the trivializations of the extension E. According to Proposition 1.3.1, this sheaf $\underline{\mathrm{F}}$ is an A-scheme locally of finite presentation, separated and non-ramified over A. The scheme $\underline{\mathrm{Hom}}(B_A, C_A)$ acts freely and transitively on $\underline{\mathrm{F}}$. We want to show that $\underline{\mathrm{F}}$ is in fact a $\underline{\mathrm{Hom}}(B_A, C_A)$-torsor over the abelian scheme A. We have therefore to prove that the set of points of $\underline{\mathrm{F}}$ where the structural morphism $\underline{\mathrm{F}} \to A$ is étale is sent surjectively to A. According to [EGAIV] 4 Theorem (17.6.1) it is enough to check that the set of points of $\underline{\mathrm{F}}$ where $\underline{\mathrm{F}} \to A$ is flat is sent surjectively to A. Choose a trivialization σ of the extension E over the unit section of the abelian scheme A. Since the extension E_n over A_n is trivializable, there exists a section of the scheme $\underline{\mathrm{F}}$ over A_n. Modulo translation, we can suppose that these section coincides with the restriction of σ over the unit section of the abelian scheme A_n. In this way we get a well-defined section τ_n the scheme $\underline{\mathrm{F}}$ over A_n. In other words, we have used the trivialization σ in order to find a "compatible" family of trivialization $\{\tau_n\}_n$. In particular $\underline{\mathrm{F}} \times_A A_n \to A_n$ is étale along τ_n. Taking the limit over n, we get that the structural morphism $\underline{\mathrm{F}} \to A$ is étale along τ_n for each n. Hence if we denote by U the open subset of $\underline{\mathrm{F}}$ where $\underline{\mathrm{F}}$ is étale over A, we have that U contains τ_n for each n. Since the restriction of the structural morphism $\underline{\mathrm{F}} \to A$ to U is locally of finite presentation and flat, by [EGAIV] 2 Theorem (2.4.6) it is universally open and so the image of U in A is an open subset V of A. The open subset U contains τ_1 and therefore the open subset V of A contains the closed fibre A_1. But the abelian scheme A is proper (hence universally closed) over S and so V is equal to A. This finishes the proof that $\underline{\mathrm{F}}$ is a $\underline{\mathrm{Hom}}(B_A, C_A)$-torsor.

The torsor $\underline{\mathrm{F}}$ is an element of $\mathrm{H}^1(A, \underline{\mathrm{Hom}}(B_A, C_A))$ whose image in $\mathrm{Ext}^1(B_A, C_A)$

via the exact sequence (2.4.3) is the global extension E. Therefore the image of this extension E in $\mathrm{H}^0(A, \underline{\mathrm{Ext}}^1(B_A, C_A))$ via the exact sequence (2.4.3) is zero, i.e. this extension E of B_A by C_A is zero as section of $\underline{\mathrm{Ext}}^1(B, C)$ over A. But this section was represented by the morphism $u : A \to \underline{\mathrm{Ext}}^1(B, C)$ and so u is trivial. □

Theorem 2.4.6. *Let A_i (for $i = 1, 2, 3$) be an abelian S-scheme. Then,*

$$\mathbf{Biext}(A_1, A_2; A_3) = 0.$$

Proof. Since

$$\mathrm{Biext}^0(A_1, A_2; A_3) \cong \mathrm{Hom}(A_1 \otimes A_2, A_3) \cong \mathrm{Hom}(A_1, \underline{\mathrm{Hom}}(A_2, A_3)),$$

Proposition 1.3.1 and Lemma 1.1.1 imply that the category $\mathbf{Biext}(A_1, A_2; A_3)$ is rigid. Using Propositions 1.3.1, 1.1.3 and 2.4.5, from the exact sequence of 5 terms (2.4.1) we get that the group $\mathrm{Biext}^1(A_1, A_2; A_3)$ is trivial. □

2.5. **Biextensions of extensions of abelian schemes by tori.** Through several "dévissages", using Theorems 2.3.1 and 2.4.6 we prove now the main theorem of this paper. We start with a first "dévissage":

Proposition 2.5.1. *Let A be an abelian S-scheme and let G_i (for $i = 1, 2$) be a commutative extension of an abelian S-scheme A_i by an S-torus $Y_i(1)$. Then,*

$$\mathbf{Biext}(G_1, G_2; A) = 0$$

Proof. According to the homological interpretation (2.0.1) of the groups Biext^i (for $i = 0, 1$), from the short exact sequence $0 \to Y_2(1) \to G_2 \to A_2 \to 0$, we have the two long exact sequences

$$\begin{aligned} &0 \to \mathrm{Biext}^0(A_1, A_2; A) \to \mathrm{Biext}^0(A_1, G_2; A) \to \mathrm{Biext}^0(A_1, Y_2(1); A) \to \\ &\to \mathrm{Biext}^1(A_1, A_2; A) \to \mathrm{Biext}^1(A_1, G_2; A) \to \mathrm{Biext}^1(A_1, Y_2(1); A) \to ... \end{aligned}$$

$$\begin{aligned} &0 \to \mathrm{Biext}^0(Y_1(1), A_2; A) \to \mathrm{Biext}^0(Y_1(1), G_2; A) \to \mathrm{Biext}^0(Y_1(1), Y_2(1); A) \to \\ &\to \mathrm{Biext}^1(Y_1(1), A_2; A) \to \mathrm{Biext}^1(Y_1(1), G_2; A) \to \mathrm{Biext}^1(Y_1(1), Y_2(1); A) \to ... \end{aligned}$$

By Theorem 2.3.1 and Theorem 2.4.6, these long exact sequences furnish the relations

$$\mathbf{Biext}(A_1, G_2; A) = 0 \qquad \mathbf{Biext}(Y_1(1), G_2; A) = 0.$$

On the other hand, from the exact sequence $0 \to Y_1(1) \to G_1 \to A_1 \to 0$, we get the long exact sequence

$$\begin{aligned} &0 \to \mathrm{Biext}^0(A_1, G_2; A) \to \mathrm{Biext}^0(G_1, G_2; A) \to \mathrm{Biext}^0(Y_1(1), G_2; A) \to \\ &\to \mathrm{Biext}^1(A_1, G_2; A) \to \mathrm{Biext}^1(G_1, G_2; A) \to \mathrm{Biext}^1(Y_1(1), G_2; A) \to ..... \end{aligned}$$

Using this long exact sequence and the above relations, we can conclude. □

Theorem 2.5.2. *Let S be a scheme. Let G_i (for $i = 1, 2, 3$) be a commutative extension of an abelian S-scheme A_i by an S-torus $Y_i(1)$. In the topos $\mathbf{T}_{\mathrm{fppf}}$, the category of biextensions of (G_1, G_2) by G_3 is equivalent to the category of biextensions of the underlying abelian S-schemes (A_1, A_2) by the underlying S-torus $Y_3(1)$:*

$$\mathbf{Biext}(G_1, G_2; G_3) \cong \mathbf{Biext}(A_1, A_2; Y_3(1))$$

In particular, for $i = 0, 1$, we have the isomorphisms

$$\mathrm{Biext}^i(G_1, G_2; G_3) \cong \mathrm{Biext}^i(A_1, A_2; Y_3(1)).$$

Proof. We will prove the following equivalences of categories:

$$\begin{array}{c}\mathbf{Biext}(G_1,G_2;Y_3(1))\cong\mathbf{Biext}(A_1,A_2;Y_3(1))\\ \mathbf{Biext}(G_1,G_2;G_3)\cong\mathbf{Biext}(G_1,G_2;Y_3(1))\end{array}\tag{2.5.1}$$

By [SGA3] Exposé X Corollary 4.5, we can suppose that the tori are split (if necessary we localize over S for the étale topology) and therefore we can assume that $Y_3(1)$ is $\mathbb{G}_m^{\mathrm{rk}Y_3}$. Since the categories $\mathbf{Biext}(G_1,G_2;\mathbb{G}_m)$ and $\mathbf{Biext}(A_1,A_2;\mathbb{G}_m)$ are additive in the variable $\mathbb{G}_m$ (cf. [SGA7] I Exposé VII (2.4.2)), for the first equivalence of categories of (2.5.1) it is enough to prove that

$$\mathbf{Biext}(G_1,G_2;\mathbb{G}_m)\cong\mathbf{Biext}(A_1,A_2;\mathbb{G}_m)$$

and this is done in [SGA7] Exposé VIII (3.6.1).
By the homological interpretation (2.0.1) of the groups Biext^i (for $i=0,1$), from the short exact sequence $0\to Y_3(1)\to G_3\to A_3\to 0$ we have the long exact sequence

$$\begin{array}{c}0\to\mathrm{Biext}^0(G_1,G_2;Y_3(1))\to\mathrm{Biext}^0(G_1,G_2;G_3)\to\mathrm{Biext}^0(G_1,G_2;A_3)\to\\ \to\mathrm{Biext}^1(G_1,G_2;Y_3(1))\to\mathrm{Biext}^1(G_1,G_2;G_3)\to\mathrm{Biext}^1(G_1,G_2;A_3)\to .....\end{array}$$

Using Proposition 2.5.1, we get the second equivalence of categories of (2.5.1). □

Remark 2.5.3. The above Theorem says essentially that each biextension $\mathcal{B}$ of (G_1,G_2) by G_3 comes from a biextension B of the underlying abelian S-schemes (A_1,A_2) by the underlying S-torus $Y_3(1)$. If for $i=1,2,3$, we denote by $\pi_i:G_i\to A_i$ the projection of G_i over A_i and by $\iota_i:Y_i(1)\to G_i$ the inclusion of $Y_i(1)$ in G_i, we can describe explicitly the biextension $\mathcal{B}$ of (G_1,G_2) by G_3 in term of the corresponding biextension B of (A_1,A_2) by $Y_3(1)$ as follow: $\mathcal{B}$ is the push-down by ι_3 of the biextension of (G_1,G_2) by $Y_3(1)$ which is the pull-back by (π_1,π_2) of B, i.e. $\mathcal{B}=\iota_{3\,*}(\pi_1,\pi_2)^*B$.

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