UNIVERSITÀ DEGLI STUDI DI TORINO

DIPARTIMENTO DI MATEMATICA GIUSEPPE PEANO

SCUOLA DI DOTTORATO IN
SCIENZE DELLA NATURA E TECNOLOGIE INNOVATIVE

Dottorato di Ricerca in Matematica Pura ed Applicata - XXX ciclo

Hopf Structures and Duality

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Anni Accademici: 2014 - 2017

Settore scientifico-disciplinare di afferenza: MAT/02 - Algebra

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HOPF STRUCTURES AND DUALITY

Tesi di Dottorato

Università di Torino

2017

To my parents, for supporting me.
To my brother, for withstanding me.

To Ilaria, for the time shared,
for the days to come.
I-lúmenyar ná elya, herinya.

Acknowledgements

First and foremost, my deepest gratitude (and even more) goes to Ilaria for her tireless support, love and patience, both past and future. The time (our time) she gave me and that I spent pursuing this achievement is a present that has no price.

Secondly, it goes to Alessandro Ardizzoni for following me up constantly, for sharing with me his knowledge and introducing me to his friends, for always finding time to discuss and work together. More than just a supervisor, he has been and still is a master and a friend. Equally, it goes to Laiachi El Kaoutit and Dragoş Ştefan for the many mathematical discussions, for the hours we spent racking our brains and for the time we enjoyed together in Torino, Ceuta and Bucharest. To Laiachi, almost my second supervisor, it goes a particular thank for following me up since the master studies and bestowing on me a wealth of advice and help. Their friendship means a great deal to me, as well as the many coffees and beers taken together, that since ancient times always rinse down great ideas. It is an honour for me to learn from all of them.

Thirdly, it goes to my parents and to my brother, who pushed me to pursue this path and always encouraged me to be at my best.

Moreover, I would like to thank my office mates and friends at the department of Mathematics, Vincenzo Antonelli, Alessandro Audrito, Cristina Bertone, Filippo Calderoni, Gabriele Cora, Lorenza D'Elia, Alessandro Duca, Andrea Guidolin, Mattia Puja, Giorgio Tortone, Stefano Vita and many many others for the time we enjoyed together.

Last but not least, I wish to thank the referees and members of the jury, Daniel Bulacu, Margherita Roggero and Joost Vercruysse, for their willingness and the careful reading of this thesis.

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Notational conventions

The following conventions hold generally throughout this thesis. Nevertheless, we will make them explicit as much as possible.

Algebras and coalgebras. We will reserve the symbol $\Bbbk$ for denoting a fixed commutative and unital ring. Sometimes a field, but it will be explicitly mentioned. Algebras (in the sense of monoids in monoidal categories) will be denoted with capital letters $A, A', B, B', \ldots$ or $R, R', S, S', \ldots$ and will be usually non-commutative. When it will be the time to distinguish between commutative and non-commutative algebras, we will use $A, A', B, B', \ldots$ to denote the commutative ones and $R, R', S, S', \ldots$ to denote the non-commutative ones. Coalgebras will be denoted by capital letters $C, C', D, D', \ldots$. We will use $M, M', N, N', P, P', \ldots$ both for modules and comodule indifferently, unless they are over $\Bbbk$, in which case we would prefer to use $V, V', W, W', \ldots$.

Categories and functors. We will denote categories with calligraphic capital letters such as $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ and we will keep the calligraphic capital M with decorations, i.e. $\mathcal{M}, \mathcal{M}', \ldots$, for the monoidal ones. For A, B algebras in a monoidal category $\mathcal{M}$, with ${}_A\mathcal{M}, \mathcal{M}_B$ and ${}_A\mathcal{M}_B$ we will mean the categories of left, right and bimodules over them. Analogously with C, D coalgebras and ${}^C\mathcal{M}, \mathcal{M}^D$ and ${}^C\mathcal{M}^D$ the categories of left, right and bicomodules. The notation $\mathfrak{M}$ will be reserved for the category of modules over $\Bbbk$. Functors will be denoted by calligraphic capital letters such as $\mathcal{F}, \mathcal{F}', \mathcal{G}, \mathcal{G}', \ldots$ and natural transformations with Greek small letters $\eta, \eta', \theta, \theta', \ldots$. The symbol $\boldsymbol{\omega}$ will be reserved for a distinguished forgetful-like functor in the Tannaka reconstruction context, while α, λ and ρ will be kept for the constraints of a monoidal category. For bicategories and bifunctor we will use script capital characters such as $\mathscr{C}, \mathscr{D}, \ldots$ and $\mathscr{F}, \mathscr{G}, \ldots$.

Objects and morphisms. Objects in a generic category $\mathcal{C}$ will be denoted with capital letters such as $X, X', Y, Y', Z, Z', \ldots$ and arrows with small letters like $f, f', g, g', h, h', \ldots$. To say that X is an object of a category $\mathcal{C}$, by a slight abuse of notation we will write $X \in \mathcal{C}$. Given two objects X, Y in $\mathcal{C}$, a morphism f between them will be denoted indifferently by $f : X \to Y$ or $X \xrightarrow{f} Y$ and the collection of all morphisms between X and Y will be denoted by $\mathsf{Hom}_{\mathcal{C}}(X, Y)$ or simply $\mathsf{Hom}(X, Y)$ (if the category is clear from the context). In the particular case of (co)linear morphisms between (co)modules we will write ${}_A\mathsf{Hom}(M, N)$, $\mathsf{Hom}_B(M, N)$, ${}_A\mathsf{Hom}_B(M, N)$, ${}^C\mathsf{Hom}(P, Q)$, $\mathsf{Hom}^D(P, Q)$ and ${}^C\mathsf{Hom}^D(P, Q)$. For an object X in $\mathcal{C}$ we will often use X to mean its identity morphism as well, which otherwise will be denoted by Id_X.

Whenever it may happen that these conventions will not be observed, it will be explicitly stated.

Introduction

This thesis concerns the study of algebraic objects whose structure resembles more or less closely that of a Hopf algebra and of some crucial aspects of "duality" between them. It contains (part of) the fruits of a three years work with A. Ardizzoni and L. El Kaoutit that already appeared as published papers or submitted drafts, plus some original ideas arisen while revising the material.

A Hopf algebra over a field $\Bbbk$ is, naively speaking, a $\Bbbk$-algebra H whose left modules form a right closed monoidal category in the nicest way one can think of, that is to say, the tensor product $M \otimes_{\Bbbk} N$ and the set of morphisms $\mathsf{Hom}_{\Bbbk}(M, N)$ of the underlying spaces of any two left H-modules M and N is still a left H-module and the constraints are the same of the category of vector spaces. This property is encoded in the existence of three additional structure maps that graft onto the algebra structure: a coassociative comultiplication $\Delta : H \to H \otimes_{\Bbbk} H$, a counit $\varepsilon : H \to \Bbbk$ for it and an antipode $S : H \to H$ (without the antipode, these are known under the name of bialgebras). However, this is somehow a reductive description of what a Hopf algebra is.

The notion of Hopf algebra has been abstracted from the work of the topologists in the 1940s dealing with compact Lie groups and their homogeneous spaces. Namely, the first example of such a structure was observed by H. Hopf [Hf] in 1941 and it was the homology of a connected Lie group. These primitive versions were subject to more restrictive conditions such as commutativity or the existence of a grading and it was in P. Cartier seminar [Crt] in 1955 that the previous restrictions were removed, arriving at a first formal definition (under the name *hyperalgebras*). The term *algébre de Hopf* was coined by A. Borel in 1953, honoring the pioneering work of H. Hopf, and the definition that we use nowadays appeared firstly in a paper by B. Kostant [Kt] in 1966.

Since then, Hopf algebras experienced a first period of great success in algebraic geometry thanks to the work of P. Cartier, M. Demazure, J. Dieudonné, P. Gabriel, A. Grothendieck, G. Hochschild, B. Kostant, J. Milnor, J. Moore and many others. Here they appear for example as universal enveloping algebras of Lie algebras, algebras of regular functions on affine algebraic groups or algebras of representative functions on compact Lie groups. They turned out to be useful tools in other fields of Mathematics as well, like group theory (the group ring of an abstract group is a Hopf algebra), Galois theory and separable field extensions (in relation with the so-called Hopf-Galois theory), graded ring theory (one can cook up a Hopf algebra out of a graded ring and there is a strong relationship between graded modules and Hopf modules).

In 1969, with the appearance of Sweedler's book [Sw], Hopf algebras started to be studied from a strictly algebraic point of view, thus becoming a distinct branch of mathematics, and by the end of the 1980s research in this field received a strong boost from the connections with quantum mechanics that showed up following the appearance of the paper *Quantum groups* by V. Drinfel'd [Dr1]. New examples and interactions with other areas of mathematics such as non-commutative geometry, knot theory, conformal field theory, category theory, combinatorics, quantum statistical mechanics arose. For example, by adopting a categorical viewpoint, a quantum space is a representable functor on the category of (non-commutative) algebras. If the representing object of the quantum space is a Hopf algebra, then the quantum space is called a quantum group. Hopf algebras have been therefore accepted as the natural analogue, from the point of view of non-commutative (algebraic) geometry, of the classical notion of group.

Due to this wide range of applications, it was inevitable that generalizations and extensions of Hopf algebras should arise to satisfy the different needs that might show up. Let us mention a couple of them that will be treated in this thesis.

On the one hand, the appearance of groupoids in geometry to describe generalised symmetries called for an algebraic counterpart as Hopf algebras are for groups. This led to the introduction first of commutative Hopf algebroids (see e.g. [Hv, Rav]) and then to their non-commutative analogues (probably the first purely non-commutative one appeared in [L]. An equivalent definition has been given in [BS3]. See [Bö1] for an account on the subject). Roughly speaking, these can be considered as non-commutative Hopf algebras over non-commutative base rings.

On the other hand, in 1989 Drinfel'd [Dr2] introduced the notion of quasi-bialgebra and of (quasitriangular) quasi-Hopf algebra in connection with his work on Knizhnik-Zamolodchikov equations and the semiclassical limit of Wess-Zumino-Witten models. Few years later, S. Majid introduced in [Mj3] the dual notion of coquasi-bialgebra in order to prove a Tannaka-Kreĭn type theorem for quasi-Hopf algebras and in [Mj2] he connected (co)quasi-bialgebras with Topological Quantum Field Theories, as internal symmetry quasi-quantum groups. Quasi-bialgebras and coquasi-bialgebras can be thought of as bialgebras in which we dropped coassociativity of the comultiplication or associativity of the multiplication, respectively.

Many other variants of the original definition arose along the time, such as $\times_A$-bialgebras and Hopf algebras [Tk, Sc3], weak bialgebras and Hopf algebras [BS1, BNS], Hopf quasigroups [KM], multiplier and weak multiplier Hopf algebras [VD, VDW], Hopf monads [BV] and so on, but these are out of the purposes of this work and hence we will not dwell on them any more. Given the popping up of structures that resemble a Hopf algebra, we decided to use the term "*Hopf structures*" to refer generally to objects that are "mutations" (in a suitable sense) of Hopf algebras.

As we claimed at the very beginning of this introduction, saying that a Hopf algebra is an algebra whose category of left modules is a right closed monoidal category, with the same monoidal structure and right internal homs of the category of vector spaces, is exhaustive but a little bit restrictive, because there are many other important properties that characterize these rich algebraic structures. For example, a Hopf algebra H is also a coalgebra whose category of comodules with finite-dimensional underlying vector space is rigid monoidal [U], or it is a bialgebra whose Hopf modules (objects that are H-modules and H-comodules at the same time and such that the two structures are compatible) satisfy a certain Structure Theorem (namely, every Hopf module M is free, in the sense that it can be decomposed as $M \cong M^{\mathrm{co}H} \otimes H$, where $M^{\mathrm{co}H}$ is a suitable subspace of M. A first, more restrictive form of this theorem appeared originally in [LS]).

Another distinguished feature of Hopf algebras (already of bialgebras in fact) is that they are algebras whose dual vector space is still an algebra in a natural way. To say it in a more expressive way, if one converts the axioms of a Hopf algebra into commutative diagrams, then these are self-dual in the sense that reversing all the arrows and interchanging comultiplication and counit with multiplication and unit give the same set of axioms. Given a finite-dimensional Hopf algebra H, this means that there is a (dual) Hopf algebra structure on the dual vector space H^*. In the infinite-dimensional case this is not true, of course, but there exists a suitable notion of *finite dual* H° (concretely, a carefully chosen subspace of H^*) which is still a (dual) Hopf algebra (see e.g. [Sw, Chapter VI]).

Thus, whenever a new Hopf structure enters the picture, a very natural question that arises is which nice properties of Hopf algebras pass to the new gadget or what one should ask to let them pass. The central example in this sense that led our steps along the last few years is that of (co)quasi-bialgebras. We will recall in §1.7 that, by their own definition, the category of (co)modules over a (co)quasi-bialgebra is a monoidal category with the same tensor product and unit object of vector spaces, but different constraints. It has been shown, in [AP1] for coquasi-bialgebras and afterwards in [Sa2] for the quasi case, that the Structure Theorem for (co)quasi-Hopf bi(co)modules (i.e. every (co)quasi-Hopf bi(co)module is free) is equivalent to the existence of a certain linear endomorphism that has been baptised *preantipode*. This proved to be a better-behaved analogue of the antipode with respect to the previously introduced (co)quasi-antipodes (see [Dr2] and [Mj1]), since the existence of these latter implies the Structure Theorem, but is not equivalent to it (in fact, in [Sc5] an example is shown of a coquasi-bialgebra with preantipode but without coquasi-antipode). In §2.2 we will present a renewed version (with respect to [Sa2]) of the Structure Theorem for quasi-Hopf bimodules over a quasi-bialgebra A, now written in terms of a suitable hom-tensor adjunction which descends from the closure of the category of A-modules shown in §2.1, and its connection with the preantipode (we point out however that the dual version for coquasi-bialgebras of this new formulation seems to be less trivial than one may expect). In §2.3 instead we are going to prove a Tannaka-Kreĭn reconstruction theorem for coquasi-bialgebras with preantipode in the spirit of [U], recovering in particular that these can be characterized as those coalgebras whose category of finite-dimensional comodules is rigid monoidal (in fact, this was proven firstly by P. Schauenburg in [Sc8, Theorem 2.6] but following a totally different approach). This time, in turn, it is the dual property for quasi-bialgebras that appears to be more problematic.

In spite of the fact that the axioms of a quasi-bialgebra (with preantipode) are dual, in a categorical sense, to those of a coquasi-bialgebra (with preantipode), there is an apparent lack of duality between the properties they satisfy. As a further evidence, we mention the fact that even if it is known a coquasi-bialgebra with preantipode which is not a coquasi-Hopf algebra, such an example is still missing in the quasi-bialgebra framework. Prompted by these considerations, we applied ourselves to building up a duality (i.e. a contravariant adjunction) between quasi and coquasi-bialgebras. The outcome of our efforts underlies §2.4, where we construct a duality between the category of quasi-bialgebras and a suitable subcategory of that of coquasi-bialgebras, by shaping a finite dual that resembles the one that exists for Hopf algebras.

It was during this joint work that L. El Kaoutit and the author started to face another problem connected with "duality". Recall that the integration problem for Lie algebras asks for when a Lie algebra $\mathfrak{g}$ is the Lie algebra of a Lie group G. The celebrated Lie's third theorem asserts that any finite-dimensional Lie algebra can be integrated to a connected and simply connected Lie group. The corresponding integration problem for Lie algebroids instead is highly non-trivial and it has a negative answer in general [Mk, MM1]. However, M. Kapranov [Kp] proved that any (real) Lie algebroid can be integrated to a formal groupoid (this is, roughly speaking, a groupoid object in the category of formal ind-schemes, which are inductive limits of presheaves of schemes with some additional conditions). We don't want to go into the details of this work or these notions (the interested reader may refer to [AGV, KV, Kp] for further details). What is of interest for us is that the formal groupoid constructed by Kapranov is the (formal spectrum of the) full linear dual $U_A(L)^*$ of the universal enveloping algebra $U_A(L)$ of the Lie-Rinehart algebra $(A = \mathcal{C}^\infty(\mathcal{M}), L = \Gamma(\mathcal{L}))$ associated to the Lie algebroid $\mathcal{L} \to \mathcal{M}$ and the reason why this called our attention is because we had at hand another "groupoid" that we may naturally associate to $\mathcal{L}$ (more precisely, to $U_A(L)$) and that we think may shed new light on the problem.

Namely, let $\mathcal{M}$ be a smooth connected real manifold and denote by $A = \mathcal{C}^\infty(\mathcal{M})$ its $\mathbb{R}$-algebra of smooth functions. For a given Lie algebroid $\mathcal{L}$ with anchor map $\omega : \mathcal{L} \to T\mathcal{M}$ (see Example 3.2.8), we consider the category $\mathsf{Rep}_{\mathcal{M}}(\mathcal{L})$ consisting of those vector bundles $\mathcal{E}$ with a $\mathcal{L}$-action. That is, an A-module morphism $\varrho_{-} : \Gamma(\mathcal{L}) \to \mathsf{End}_{\mathbb{R}}(\Gamma(\mathcal{E}))$ which is a Lie algebra map satisfying $\varrho_X(fs) = f\varrho_X(s) + \Gamma(\omega)_X(f)s$, for any section $s \in \Gamma(\mathcal{E})$ and any function $f \in A$. The category $\mathsf{Rep}_{\mathcal{M}}(\mathcal{L})$ turns out to be a (not necessarily abelian) symmetric rigid monoidal category, which is endowed with a fiber functor $\boldsymbol{\omega} : \mathsf{Rep}_{\mathcal{M}}(\mathcal{L}) \to \mathsf{proj}(A)$ to the category of finitely generated and projective A-modules (see for example [Cr, §1.4]).

The Tannaka reconstruction process shows then that the pair $(\mathsf{Rep}_{\mathcal{M}}(\mathcal{L}), \boldsymbol{\omega})$ leads to a (universal) commutative Hopf algebroid which we may denote by $(A, {}^{\circ}U)$ and then, via the completion procedure of §3.2.1, to a complete commutative Hopf algebroid $(A, \widehat{{}^{\circ}U})$, where A is considered as a discrete topological ring. Since the category $\mathsf{Rep}_{\mathcal{M}}(\mathcal{L})$ may be identified with the category of (left) modules over the universal enveloping Hopf algebroid $(A, U = U_A(L))$ of the Lie-Rinehart algebra $(A, L = \Gamma(\mathcal{L}))$ such that the underlying A-module is finitely generated and projective, it follows from a left-handed counterpart of [EKG, §4.2] that $(A, {}^{\circ}U)$ is the finite dual Hopf algebroid (in the sense of [EKG]) of (A, U). Moreover, under some favourable conditions, the representations of the groupoid ${}^{\circ}U_A(L)$ corresponds to those of the starting Lie algebroid and the completion $\widehat{{}^{\circ}U_A(L)}$ is closely connected with Kapranov's formal groupoid $U_A(L)^*$, as we will investigate in §3.3.2. We conclude by pointing out that a rigorous treatment of these constructions requires to resort to notions such as the complete tensor product and the completion functor and that the formalism of bicategories seems to provide us with a suitable framework where to deal with these notions. In fact, the completion functor turns out to be a 2-functor between the bicategory of filtered bimodules on the one side and the one of complete bimodules on the other.

Outline of the thesis. The first chapter is devoted to fix a bit of terminology and to collect some notions and results from algebra and category theory that are needed all along the thesis. Sections from 1.1 to 1.3 present the general framework in which we set most of our results: that of monoidal categories. Sections 1.4 and 1.5 are mainly used as preliminaries for Section 1.6, in which we report on a bicategorical construction that will allow us to give a personal presentation of the theory of filtered and complete bimodules in Section 3.1. Finally, in Section 1.7 we revise the definitions and main properties of quasi and coquasi-bialgebras and in Section 1.8 we give a very brief account on the Tannaka-Kreĭn reconstruction procedure. It will be explicitly used in 2.3 and it will implicit underlie the finite dual Hopf algebroid construction of §3.3.2.1.

The second chapter contains the outcomes of our research on (co)quasi-bialgebras and preantipodes. In Section 2.1 we give a small contribution to the general theory of quasi-bialgebras by showing that their categories of representations are both left and right closed. Then we begin with the origins of the notion of preantipode in the quasi-bialgebra setting in Section 2.2. These are strictly connected with the so-called Structure Theorem for quasi-Hopf bimodules, which is presented in a revised, original version in §2.2.2 by taking advantage of the closure property. In Section 2.3 we shift the attention to the dual case and we provide a Tannaka-Kreĭn reconstruction theorem for coquasi-bialgebras with preantipode which extends Ulbrich's renowned result for Hopf algebras [U]. We conclude the second chapter by establishing a duality between the category of quasi-bialgebras and a proper subcategory of the one of coquasi-bialgebras. This clarifies once for all that, despite the fact that they are dual constructions from a categorical point of view, the connections between quasi and coquasi-bialgebras cannot be reduced to a mere dualization (by simply reversing the structure arrows) and results on one side cannot be directly obtained from their analogues on the other by means of a general duality principle.

In the third chapter we deal mainly with the completion procedure for filtered bimodules, but applied to the theory of Hopf algebroids and their linear and finite duals. In details, we first revise some notions on the linear topology of filtered rings and modules and their completions from a bicategorical point of view, in Section 3.1. Then (§3.2.1) we apply these tools to let complete commutative Hopf algebroids enter into the picture and to extend the completion functor to the commutative Hopf algebroid setting. The introduction of this notion finds a justification in the subsequent Section 3.3. Indeed, given a (right) cocommutative Hopf algebroid U endowed with what we are going to call an "admissible filtration", we recognize (mimicking [Kp]) that its full (right) linear dual U^* comes endowed with a structure of complete commutative Hopf algebroid (see §3.3.1). It is noteworthy to mention that the construction of an antipode in this case is not an easy task and an additional assumption is in fact required, which however is always fulfilled for universal enveloping Hopf algebroids of Lie-Rinehart algebras for example. Furthermore, after recalling in §3.3.2.1 the construction of the finite dual commutative Hopf algebroid U° of U, we realize that U° can be naturally equipped with a structure of filtered commutative Hopf algebroid in such a way that the canonical morphism $\zeta : U^\circ \to U^*$ connecting it with the full linear dual is filtered as well. Thus, our extension of the completion functor allows us to introduce the main morphism of complete commutative Hopf algebroids $\widehat{\zeta} : \widehat{U^\circ} \to U^*$ and to study conditions under which this turns out to be a filtered isomorphism. We conclude the chapter by showing with a concrete example that asking when $\widehat{\zeta}$ is a filtered isomorphism is not a trivial question. It is worthy, due and necessary to point out that, even if one application of this theory is in the Lie algebroid framework as showed in the foregoing (which is, historically, a "left-handed world": left representations, left bialgebroids, and so on), in this chapter we decided to keep working in the right-handed context (right modules, right bialgebroids, right duals, as in the paper [ES1]) for a matter of consistency with our previous work and with our main source for the finite dual construction (i.e. [EKG]). Moreover, the material of this third chapter serves also as preliminary results for a project of L. El Kaoutit and the author aimed at studying the connections between the ring of all linear differential operators over a commutative algebra (eventually with some additional properties) and its module of infinite jets (see [Kr] for a brief exposition) in which, up to the present moment, we need to work on the right-hand side. Namely, when $\mathcal{L} = T\mathcal{M}$ is the tangent bundle with its obvious Lie algebroid structure, it can be shown that its (right) universal enveloping Hopf algebroid U can be suitably identified with the algebra $\mathrm{Diff}(A)$ of all differential operators on A. On the other hand, one can extend the duality between differential operators of order l and l-jets of smooth functions on $\mathcal{M}$ to an isomorphism of complete Hopf algebroids between the convolution algebra $\mathrm{Diff}(A)^*$ and the algebra of infinite jets $\mathcal{J}(A) := \widehat{A \otimes_{\mathbb{R}} A}$. One of our aims in the aforementioned project is to study under which conditions all the morphisms in the commutative diagram

$$
\begin{tikzcd}
\widehat{\mathrm{Diff}(A)^\circ} \arrow[rr, "\widehat{\zeta}"] & & \mathrm{Diff}(A)^* \\
& \mathcal{J}(A) \arrow[ul, "\widehat{\eta}"] \arrow[ur, "\cong", "\widehat{\vartheta}"'] &
\end{tikzcd}
$$

of complete Hopf algebroids are isomorphisms, where the algebra maps $\vartheta : A \otimes_{\mathbb{R}} A \to \mathrm{Diff}(A)^*$ and $\eta : A \otimes_{\mathbb{R}} A \to \mathrm{Diff}(A)^\circ$ define the source and the target of $\mathrm{Diff}(A)^*$ and $\mathrm{Diff}(A)^\circ$ respectively.

The Appendices contain some technical results on tensor algebras and particular finitely generated and projective filtered (bi)modules that are needed in the exposition but that go out of the purposes of this thesis.

To conclude, a few words are in order to explain our approach. Even if we are aware of the wide range of applicability of the theory we deal with, for example in quantum physics, topology, differential and algebraic geometry, we have consciously decided to approach it from an algebraic and categorical point of view (at least, up to this moment), which consequently reflects on the kind of results, examples and applications that we will provide all along this work. Nevertheless, although most of them keep a (categorical) algebraic flavour, we tried sometimes to suggest possible uses in other fields of the tools we developed and to connect our work with other areas, as for example we do with the integration problem for Lie algebroids.

Chapter 1

Preliminaries from category theory

In this first chapter we collect some notions and results from algebra and category theory that will be needed in the sequel. We start by revising some basics of category theory and, in particular, the notion of a monoidal category and of algebras and coalgebras therein (§§1.1-1.3). Then we recall how to perform, from a categorical point of view, the tensor product over an algebra and we report briefly on the construction of the bicategory of bimodules and algebras over a monoidal category (§§1.4-1.6). Finally we recall what a (co)quasi-bialgebra is, some basic properties of them and we give a short review of the Tannaka-Kreĭn reconstruction procedure (§§1.7-1.8). Apart from some remarks in §1.6 and §1.8, such as Theorem 1.6.6 or Lemma 1.8.5, the most part of the content of this chapter is already well-known.

1.1 Broad recalls on categories

As far as we are concerned in this thesis, a category $\mathcal{C}$ is understood to be a *locally small category* (resoundingly speaking, one may say Set-enriched). That is, it consists of the following data:

1. a class $\mathcal{O}b(\mathcal{C})$ or $\mathcal{C}_0$ whose elements are called objects of $\mathcal{C}$,
2. for each ordered pair (X, Y) of objects of $\mathcal{C}$, a set $\mathsf{Hom}_{\mathcal{C}}(X, Y)$ (also denoted by $\mathcal{C}(X, Y)$ or simply by $\mathsf{Hom}(X, Y)$, if the category is clear from the context) consisting of the arrows or morphisms from X to Y,
3. for each ordered triple (X, Y, Z) of objects of $\mathcal{C}$, a composition map $\circ : \mathsf{Hom}_{\mathcal{C}}(X, Y) \times \mathsf{Hom}_{\mathcal{C}}(Y, Z) \to \mathsf{Hom}_{\mathcal{C}}(X, Z)$,

subject to the following conditions:

1. if $(X, Y) \neq (X', Y')$ then $\mathsf{Hom}_{\mathcal{C}}(X, Y) \cap \mathsf{Hom}_{\mathcal{C}}(X', Y') = \emptyset$,
2. the composition $\circ$ is associative,
3. for every object X there exists a distinguished morphism Id_X or simply X (if no confusion may arise) in $\mathsf{Hom}_{\mathcal{C}}(X, X)$ which behaves like a left and right neutral element for the composition, i.e. for every $f \in \mathsf{Hom}_{\mathcal{C}}(X, Y)$
$$f \circ \mathsf{Id}_X = f = \mathsf{Id}_Y \circ f.$$

Example 1.1.1. Some very well-known categories that will appear in this thesis are the category Set of sets, the category $\mathsf{Mod}_{\Bbbk}$ (often denoted by $\mathfrak{M}$) of modules over a commutative ring $\Bbbk$ and its variant version $\mathsf{Vect}_{\Bbbk}$ (also denoted by $\mathfrak{M}$) of $\Bbbk$-vector spaces when $\Bbbk$ is a field, the category Top of topological spaces, the category ${}_R\mathfrak{M}_S$ of (R, S)-bimodules for R, S two $\Bbbk$-algebras.

Given two categories $\mathcal{C}$ and $\mathcal{D}$, the *product* $\mathcal{C} \times \mathcal{D}$ of $\mathcal{C}$ and $\mathcal{D}$ is the category whose objects are pairs (X, Y) composed by an object X of $\mathcal{C}$ and an object Y of $\mathcal{D}$ and whose arrows are pairs of arrows $(f, g) : (X, Y) \to (X', Y')$ such that $f : X \to X'$ is in $\mathcal{C}$ and $g : Y \to Y'$ is in $\mathcal{D}$.

If $\mathcal{C}$ is any category and $\mathcal{C}^{\mathrm{op}}$ is its opposite category, then we will write X^{op} for the object X of $\mathcal{C}$ as seen in $\mathcal{C}^{\mathrm{op}}$ and $f^{\mathrm{op}} : Y^{\mathrm{op}} \to X^{\mathrm{op}}$ for the arrow $f : X \to Y$ of $\mathcal{C}$ as seen in $\mathcal{C}^{\mathrm{op}}$.

A category $\mathcal{C}$ is said to be *small* if its class of objects $\mathcal{O}b(\mathcal{C})$ is a proper set. It is said to be *essentially small* if it is equivalent to a small category (sometimes, these are also called *skeletally small* categories, as every category is equivalent to (anyone of) its skeletons, [ML, p. 91]).

Example 1.1.2. If $\Bbbk$ is a field, then the category $\mathfrak{M}_f$ of finite-dimensional vector spaces over $\Bbbk$ is an example of an essentially small category. Indeed, the class of isomorphism classes of finite-dimensional vector spaces is in bijection with the set of natural numbers $\mathbb{N}$.

Example 1.1.3. If $\Bbbk$ is a commutative ring, then the category $\mathfrak{M}_f$ of finitely generated and projective (fgp in short) $\Bbbk$-modules is essentially small. Indeed, up to isomorphism, every fgp $\Bbbk$-module is a submodule of a free $\Bbbk$-module of the form $\Bbbk^n$ ($n \in \mathbb{N}$). Thus, the isomorphism classes of fgp $\Bbbk$-modules are in bijection with the sets of isomorphism classes of submodules[(1)] of the $\Bbbk^n$'s for n varying in $\mathbb{N}$, which is again a set.

By a functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ between two categories $\mathcal{C}$ and $\mathcal{D}$ we will always mean a covariant functor, that means that it preserves the composition: $\mathcal{F}(g \circ h) = \mathcal{F}(g) \circ \mathcal{F}(h)$ for all g, h composable morphisms in $\mathcal{C}$. In case $\mathcal{F}$ satisfies $\mathcal{F}(g \circ h) = \mathcal{F}(h) \circ \mathcal{F}(g)$, then we will write explicitly that we have a functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}^{\mathrm{op}}$ or we will refer to it as a contravariant functor. In particular, by a *contravariant adjunction* or *duality* between $\mathcal{C}$ and $\mathcal{D}$ we mean a pair of adjoint functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}^{\mathrm{op}}$ and $\mathcal{G} : \mathcal{D}^{\mathrm{op}} \to \mathcal{C}$, that is to say, for all C in $\mathcal{C}$ and D in $\mathcal{D}$ we have a natural bijection

$$\mathsf{Hom}_{\mathcal{D}^{\mathrm{op}}}\left(\mathcal{F}(C), D\right) \cong \mathsf{Hom}_{\mathcal{C}}\left(C, \mathcal{G}(D)\right).$$

As a matter of notation, for any two functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ we will write $\mathcal{F} \dashv \mathcal{G}$ to mean that $\mathcal{F}$ is left adjoint to $\mathcal{G}$.

Example 1.1.4. Let $\Bbbk$ denote a commutative ring as usual. One of the better known examples of duality is the one induced by the dual module functor $(-)^* : \mathfrak{M} \to \mathfrak{M}^{\mathrm{op}}$ that sends every $\Bbbk$-module V to $V^* = \mathsf{Hom}_{\Bbbk}(V, \Bbbk)$. This is obviously a contravariant functor which is (left) adjoint to the functor $(-)^* : \mathfrak{M}^{\mathrm{op}} \to \mathfrak{M}$. If we write $V^{**} := (V^*)^*$, then both the unit and the counit are given by the canonical map

$$\chi_V : V \to V^{**}, \quad \big(v \mapsto [f \mapsto f(v)]\big).$$

As a matter of notation, if $\mathcal{F} : \mathcal{B} \to \mathcal{C}$, $\mathcal{G}, \mathcal{H} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{K} : \mathcal{D} \to \mathcal{E}$ are functors and if $\eta : \mathcal{G} \to \mathcal{H}$ is a natural transformation, we adopt the following conventions for X in $\mathcal{B}$ and Y in $\mathcal{C}$

$$(\eta\mathcal{F})_X := \eta_{\mathcal{F}(X)} : \mathcal{G}(\mathcal{F}(X)) \to \mathcal{H}(\mathcal{F}(X)) \qquad \text{and} \qquad (\mathcal{K}\eta)_Y := \mathcal{K}(\eta_Y) : \mathcal{K}(\mathcal{G}(Y)) \to \mathcal{K}(\mathcal{H}(Y)).$$

Let $\mathcal{J}$ be a small category. We may call it a *scheme*[(2)], sometimes. Its objects will be denoted by small letters $i, j, k, \ldots$ or $\alpha, \beta, \gamma, \ldots$. Let $\mathcal{C}$ be a category and $\mathbb{D} : \mathcal{J} \to \mathcal{C}$ a functor. We may call it a *diagram* over the scheme $\mathcal{J}$. The image of the object i via $\mathbb{D}$ will be denoted by $\mathbb{D}_i$. If there is a morphism between i and j in $\mathcal{J}$, its image via $\mathbb{D}$ will be usually denoted by $f_{i,j} : \mathbb{D}_i \to \mathbb{D}_j$. Often we will simply write that $\{\mathbb{D}_i, f_{i,j} : \mathbb{D}_i \to \mathbb{D}_j\}_{\mathcal{J}}$ is a diagram over $\mathcal{J}$ to make the notation clearer. A *(natural) source* for $\mathbb{D}$ is a pair $\left(X, (g_i)_{i \in \mathcal{J}}\right)$ consisting of an object X in $\mathcal{C}$ and a family of morphisms $g_i : X \to \mathbb{D}_i$ with domain X such that $g_j = f_{i,j} \circ g_i$ (in [ML, §III.4] these are called *cones to the base* $\mathbb{D}$ or *cocones*). We will eventually use the more compact notation $\left(X \xrightarrow{g_i} \mathbb{D}_i \right)_{\mathcal{J}}$ and often we will simply say that X is a source, omitting the structure maps. Its dual notion is that of a *(natural) sink*, that is a pair $\left(Y, (h_i)_{i \in \mathcal{J}}\right)$ consisting of an object Y in $\mathcal{C}$ and a family

(1) The category of modules over a commutative ring $\Bbbk$ is *wellpowered*, meaning that the class of subobjects of any object is in fact a proper set (see e.g. [ML, p. 126]).

(2) Terminology and notations have been borrowed from [AHS].

of morphisms $h_i : \mathbb{D}_i \to Y$ with codomain Y such that $h_j \circ f_{i,j} = h_i$. A *limit* of the functor $\mathbb{D}$ is a source $\mathcal{L} = \left(L \xrightarrow{\sigma_i} \mathbb{D}_i \right)_{\mathcal{J}}$ universal with the property that for any other source $\left(X \xrightarrow{g_i} \mathbb{D}_i \right)_{\mathcal{J}}$ there exists a unique morphism $\phi : X \to L$ in $\mathcal{C}$ such that $\sigma_i \circ \phi = g_i$ for all $i \in \mathcal{J}$.

A functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is said to

- *preserve limits* (over a scheme $\mathcal{J}$) if for every limit $\mathcal{L} = \left(L \xrightarrow{\sigma_j} \mathbb{D}_j \right)_{\mathcal{J}}$ of a diagram $\mathbb{D} : \mathcal{J} \to \mathcal{C}$ in $\mathcal{C}$, $\mathcal{F}(\mathcal{L}) = \left(\mathcal{F}(L) \xrightarrow{\mathcal{F}(\sigma_j)} \mathcal{F}(\mathbb{D}_j) \right)_{\mathcal{J}}$ is a limit of the diagram $\mathcal{F}\mathbb{D} : \mathcal{J} \to \mathcal{D}$ in $\mathcal{D}$;
- *detect limits* provided that a diagram $\mathbb{D} : \mathcal{J} \to \mathcal{C}$ has a limit whenever $\mathcal{F} \circ \mathbb{D}$ has one;
- *reflect limits* provided that for each diagram $\mathbb{D} : \mathcal{J} \to \mathcal{C}$ a source $\mathcal{S} = \left(S \xrightarrow{\sigma_j} \mathbb{D}_j \right)_{\mathcal{J}}$ in $\mathcal{C}$ is a limit of $\mathbb{D}$ whenever $\mathcal{F}(\mathcal{S})$ is a limit of $\mathcal{F}\mathbb{D}$ in $\mathcal{D}$;
- *lift limits (uniquely)* provided that for every diagram $\mathbb{D} : \mathcal{J} \to \mathcal{C}$ and every limit $\mathcal{L}'$ of $\mathcal{F}\mathbb{D}$ in $\mathcal{D}$ there exists a (unique) limit $\mathcal{L}$ of $\mathbb{D}$ in $\mathcal{C}$ such that $\mathcal{F}(\mathcal{L}) = \mathcal{L}'$;
- *create limits* provided that for every diagram $\mathbb{D} : \mathcal{J} \to \mathcal{C}$ and every limit $\mathcal{L}'$ of $\mathcal{F}\mathbb{D}$ in $\mathcal{D}$ there exists a unique source $\mathcal{S} = \left(S \xrightarrow{\sigma_j} \mathbb{D}_j \right)_{\mathcal{J}}$ in $\mathcal{C}$ such that $\mathcal{F}(\mathcal{S}) = \mathcal{L}'$ and $\mathcal{S}$ is a limit of $\mathbb{D}$ in $\mathcal{C}$.

Dually, one has the notions of colimit and of functors that preserve, detect, reflect, lift and create colimits. For the connections that exist between these notions, we refer to [AHS, §III.13].

Example 1.1.5. The forgetful functor $\mathcal{U} : \mathsf{Top} \to \mathsf{Set}$ lifts limits uniquely (it is enough to endow the limit L of the underlying sets with the initial topology that makes continuous the structure maps σ_j) and preserves them (because Set is complete and $\mathcal{U}$ lifts limits), but does not create them (since every topology τ on L finer than the initial topology makes of $\mathcal{L}$ a source for $\mathbb{D}$ such that $\mathcal{U}((L, \tau)) = L$). As a consequence, $\mathcal{U}$ cannot reflect limits, because a functor that lifts limits uniquely and reflects them has to create them as well. On the other hand, the forgetful functor $\mathcal{U} : \mathsf{Vect}_{\Bbbk} \to \mathsf{Set}$ creates and preserves limits, since there is a unique way to endow the limit L of the underlying sets with a structure of vector space in such a way that $\mathcal{L} = \left(L \xrightarrow{\sigma_j} \mathbb{D}_j \right)_{\mathcal{J}}$ becomes a source for $\mathbb{D}$ in $\mathsf{Vect}_{\Bbbk}$ and it automatically becomes a limit as well.

1.2 Monoidal categories

A *monoidal category* $(\mathcal{M}, \otimes, \mathbb{I}, \alpha, \lambda, \rho)$ is a category $\mathcal{M}$ endowed with a functor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, the *tensor product*, with a distinguished object $\mathbb{I}$, the *unit*, and with three natural isomorphisms

$$\begin{aligned} \alpha &: \otimes(\otimes \times \mathsf{Id}_{\mathcal{M}}) \to \otimes(\mathsf{Id}_{\mathcal{M}} \times \otimes) \quad (\textit{associativity constraint}) \\ \lambda &: \otimes(\mathbb{I} \times \mathsf{Id}_{\mathcal{M}}) \to \mathsf{Id}_{\mathcal{M}} \quad (\textit{left unit constraint}) \\ \rho &: \otimes(\mathsf{Id}_{\mathcal{M}} \times \mathbb{I}) \to \mathsf{Id}_{\mathcal{M}} \quad (\textit{right unit constraint}) \end{aligned}$$

that satisfy the *Pentagon* and the *Triangle Axioms*, i.e. such that the following diagrams commute

$$\begin{tikzcd}
 & (X \otimes Y) \otimes (Z \otimes W) \arrow[dr, "\alpha_{X,Y,Z\otimes W}"] & \\
((X \otimes Y) \otimes Z) \otimes W \arrow[ur, "\alpha_{X\otimes Y,Z,W}"] \arrow[d, "\alpha_{X,Y,Z}\otimes W"'] & & X \otimes (Y \otimes (Z \otimes W)) \\
(X \otimes (Y \otimes Z)) \otimes W \arrow[dr, "\alpha_{X,Y\otimes Z,W}"'] & & \\
 & X \otimes ((Y \otimes Z) \otimes W) \arrow[uur, "X\otimes\alpha_{Y,Z,W}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(X \otimes \mathbb{I}) \otimes Y \arrow[dr, "\rho_X \otimes Y"] \arrow[dd, "\alpha_{X,\mathbb{I},Y}"'] & \\
 & X \otimes Y \\
X \otimes (\mathbb{I} \otimes Y) \arrow[ur, "X \otimes \lambda_Y"'] &
\end{tikzcd}
\tag{1.1}$$

for all X,Y,Z,W objects in $\mathcal{M}$.

Example 1.2.1. A key example is the monoidal category $(\mathfrak{M},\otimes,\Bbbk,a,l,r)$ of modules over a commutative ring $\Bbbk$, where $\otimes=\otimes_{\Bbbk}$ and a, l and r are the obvious isomorphisms (we will often omit the constraints when they are the trivial ones).

Other well-known examples are the category of bimodules $({}_A\mathfrak{M}_A,\otimes_A,A)$ over a $\Bbbk$-algebra A, the category of sets $(\mathsf{Set},\times,*)$ with the Cartesian product and the singleton, the category $(\mathfrak{M}_G,\otimes,\Bbbk)$ of G-graded $\Bbbk$-modules $V=\bigoplus_{g\in G}V_g$ for G a monoid, where $\Bbbk_g=0$ for all $g\neq e$ and

$$(V\otimes W)_g=\bigoplus_{\substack{x,y\in G\\ xy=g}}V_x\otimes W_y.$$

Given two monoidal categories $(\mathcal{M},\otimes,\mathbb{I},\alpha,\lambda,\rho)$ and $(\mathcal{M}',\otimes',\mathbb{I}',\alpha',\lambda',\rho')$, a *lax tensor functor* $(\mathcal{F},\varphi_0,\varphi)$ between $\mathcal{M}$ and $\mathcal{M}'$ is a functor $\mathcal{F}:\mathcal{M}\to\mathcal{M}'$ together with a morphism $\varphi_0:\mathbb{I}'\to\mathcal{F}(\mathbb{I})$ and a family of morphisms $\varphi_{X,Y}:\mathcal{F}(X)\otimes'\mathcal{F}(Y)\to\mathcal{F}(X\otimes Y)$ for X,Y objects in $\mathcal{M}$, which are natural in both entrances. A lax tensor functor $\mathcal{F}$ is said to be *neutral* if the following diagrams are commutative

$$\begin{tikzcd}
\mathbb{I}'\otimes'\mathcal{F}(X) \arrow[r, "\lambda'_{\mathcal{F}(X)}"] \arrow[d, "\varphi_0\otimes'\mathcal{F}(X)"'] & \mathcal{F}(X) \\
\mathcal{F}(\mathbb{I})\otimes'\mathcal{F}(X) \arrow[r, "\varphi_{\mathbb{I},X}"'] & \mathcal{F}(\mathbb{I}\otimes X) \arrow[u, "\mathcal{F}(\lambda_X)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}(X)\otimes'\mathbb{I}' \arrow[r, "\rho'_{\mathcal{F}(X)}"] \arrow[d, "\mathcal{F}(X)\otimes'\varphi_0"'] & \mathcal{F}(X) \\
\mathcal{F}(X)\otimes'\mathcal{F}(\mathbb{I}) \arrow[r, "\varphi_{X,\mathbb{I}}"'] & \mathcal{F}(X\otimes\mathbb{I}) \arrow[u, "\mathcal{F}(\rho_X)"']
\end{tikzcd}
\tag{1.2}$$

(this unusual terminology has been borrowed from [Sc4]) and $\mathcal{F}$ is said to be *lax monoidal* if

$$\begin{tikzcd}
(\mathcal{F}(X)\otimes'\mathcal{F}(Y))\otimes'\mathcal{F}(Z) \arrow[r, "\varphi_{X\otimes Y,Z}\circ\left(\varphi_{X,Y}\otimes'\mathcal{F}(Z)\right)"] \arrow[d, "\alpha'_{\mathcal{F}(X),\mathcal{F}(Y),\mathcal{F}(Z)}"'] & \mathcal{F}((X\otimes Y)\otimes Z) \arrow[d, "\mathcal{F}\left(\alpha_{X,Y,Z}\right)"] \\
\mathcal{F}(X)\otimes'(\mathcal{F}(Y)\otimes'\mathcal{F}(Z)) \arrow[r, "\varphi_{X,Y\otimes Z}\circ\left(\mathcal{F}(X)\otimes'\varphi_{Y,Z}\right)"'] & \mathcal{F}(X\otimes(Y\otimes Z))
\end{tikzcd}
\tag{1.3}$$

commutes as well. Furthermore, it is said to be *strict* if φ_0 and φ are the identities.

Dually one may introduce the notions of *colax tensor functor*, *neutral colax tensor functor* and *colax monoidal functor*. A (co)lax tensor functor whose structure morphisms φ_0 and φ are isomorphisms is called a *tensor functor*. A tensor functor which satisfies (1.2) is called a *neutral tensor functor*. If it satisfies also (1.3) then it is a *monoidal functor*[(3)]. A monoidal functor $(\mathcal{F},\varphi_0,\varphi)$ such that $\mathcal{F}$ is an equivalence of categories is called a *monoidal equivalence*[(4)].

A *tensor natural transformation* $\eta\colon(\mathcal{F},\varphi_0,\varphi)\to(\mathcal{G},\psi_0,\psi)$ between tensor functors from $(\mathcal{M},\otimes,\mathbb{I},\alpha,\lambda,\rho)$ to $(\mathcal{M}',\otimes',\mathbb{I}',\alpha',\lambda',\rho')$ is a natural transformation $\eta\colon\mathcal{F}\to\mathcal{G}$ such that the following diagrams commute for each couple (X,Y) of objects in $\mathcal{M}$

$$\begin{tikzcd}
 & \mathcal{F}(\mathbb{I}) \arrow[dd, "\eta_{\mathbb{I}}"] \\
\mathbb{I}' \arrow[ur, "\varphi_0"] \arrow[dr, "\psi_0"'] & \\
 & \mathcal{G}(\mathbb{I})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}(X)\otimes'\mathcal{F}(Y) \arrow[r, "\varphi_{X,Y}"] \arrow[d, "\eta_X\otimes'\eta_Y"'] & \mathcal{F}(X\otimes Y) \arrow[d, "\eta_{X\otimes Y}"] \\
\mathcal{G}(X)\otimes'\mathcal{G}(Y) \arrow[r, "\psi_{X,Y}"'] & \mathcal{G}(X\otimes Y)
\end{tikzcd}
\tag{1.4}$$

(3) In [AMa, Definition 3.5], these are called *strong* monoidal functors.

(4) The terminology is justified by the fact that the quasi-inverse of a monoidal functor becomes monoidal as well, see [Riv, Proposition 4.4.2].

(see also [AMa, Definition 3.8]). A *tensor natural isomorphism* is a tensor natural transformation that is also a natural isomorphism. A *tensor equivalence* between monoidal categories is a tensor functor $\mathcal{F}$ from $\mathcal{M}$ to $\mathcal{M}'$ such that there exist another tensor functor $\mathcal{G}\colon \mathcal{M}' \to \mathcal{M}$ and tensor natural isomorphisms $\eta\colon \mathsf{Id}_{\mathcal{M}} \to \mathcal{G}\mathcal{F}$ and $\epsilon\colon \mathcal{F}\mathcal{G} \to \mathsf{Id}_{\mathcal{M}'}$. If, moreover, both composition are actually identity functors, then it is called an *isomorphism* of tensor categories.

Let $(\mathcal{M}, \otimes, \mathbb{I}, \alpha, \lambda, \rho)$ be a monoidal category. If the endofunctor $X \otimes - : Y \mapsto X \otimes Y$ (resp. $- \otimes X : Y \mapsto Y \otimes X$) has a right adjoint for every X in $\mathcal{M}$, then $\mathcal{M}$ is called a *left-closed* (resp. *right-closed*) monoidal category. A *dual object* of X in $\mathcal{M}$ is a triple $(X^\star, \mathrm{ev}_X, \mathrm{db}_X)$ in which $X^\star$ is an object in $\mathcal{M}$ and $\mathrm{ev}_X : X \otimes X^\star \to \mathbb{I}$ and $\mathrm{db}_X : \mathbb{I} \to X^\star \otimes X$ are morphisms in $\mathcal{M}$, called *evaluation* and *dual basis* respectively, that satisfy

$$\lambda_X \circ (\mathrm{ev}_X \otimes X) \circ \alpha^{-1}_{X,X^\star,X} \circ (X \otimes \mathrm{db}_X) \circ \rho^{-1}_X = \mathrm{id}_X, \tag{1.5}$$

$$\rho_{X^\star} \circ (X^\star \otimes \mathrm{ev}_X) \circ \alpha_{X^\star,X,X^\star} \circ (\mathrm{db}_X \otimes X^\star) \circ \lambda^{-1}_{X^\star} = \mathrm{id}_{X^\star}. \tag{1.6}$$

An object which admits a right dual object is said to be *right rigid* (or *dualizable*). If every object in $\mathcal{M}$ is right rigid, then we say that $\mathcal{M}$ is *right rigid*. If $f : X \to Y$ is a morphism between right rigid objects in a monoidal category $\mathcal{M}$ then its *dual map* (or *transpose*) is given by the composition

$$f^\star := \rho_{X^\star} \circ (X^\star \otimes \mathrm{ev}_Y) \circ (X^\star \otimes (f \otimes Y^\star)) \circ \alpha_{X^\star,X,Y^\star} \circ (\mathrm{db}_X \otimes Y^\star) \circ \lambda^{-1}_{Y^\star}.$$

We will often refer to right dual objects simply as *right duals* or just *duals*. Dually one defines *left duals* and *left rigid* monoidal categories. A right rigid monoidal category is left-closed with right adjoint to $X \otimes -$ the functor $X^\star \otimes -$ and conversely a left rigid one is right-closed with right adjoint $- \otimes X^\star$.

Example 1.2.2. The classical example of dualizable objects is provided by finitely generated and projective (fgp for short) $\Bbbk$-modules. Given V a fgp module, there exist elements $\{e_i \mid i = 1, \ldots, d\}$ in V and elements $\{e^i \mid i = 1, \ldots, d\}$ in $V^* = \mathsf{Hom}_{\Bbbk}(V, \Bbbk)$ such that $v = \sum_{i=1}^d e^i(v) e_i$ for all $v \in V$. The evaluation map is $\mathrm{ev}_V(v \otimes f) = f(v)$ for $v \in V$ and $f \in V^*$. The dual basis is given by $\mathrm{db}_V(1_{\Bbbk}) = \sum_{i=1}^d e^i \otimes e_i$. For all $u \in V$, $f \in V^*$, relations (1.5) and (1.6) amounts to the well-known $v = \sum_{i=1}^d e^i(v)\, e_i$ and $f = \sum_{i=1}^d f(e_i)\, e^i$.

Once chosen a right dual object $X^\star$ for every object X in a right rigid monoidal category $\mathcal{M}$, we have that the assignment $(-)^\star : \mathcal{M}^{\mathrm{op}} \to \mathcal{M}$ defines a functor and $\mathrm{ev} : (-) \otimes (-)^\star \to \mathbb{I}$ and $\mathrm{db} : \mathbb{I} \to (-)^\star \otimes (-)$ define dinatural transformations[(5)], i.e. for every X, Y and $f : X \to Y$ in $\mathcal{M}$ we have

$$(f^\star \otimes Y) \circ \mathrm{db}_Y = (X^\star \otimes f) \circ \mathrm{db}_X \qquad \text{and} \qquad \mathrm{ev}_X \circ (X \otimes f^\star) = \mathrm{ev}_Y \circ (f \otimes Y^\star).$$

If we have a different choice $(-)^\vee : \mathcal{M}^{\mathrm{op}} \to \mathcal{M}$ of right dual objects, then we will write $\mathrm{ev}^{(\star)}$ and $\mathrm{db}^{(\star)}$ to mean the evaluation and dual basis maps associated with the dual $(-)^\star$ and $\mathrm{ev}^{(\vee)}$ and $\mathrm{db}^{(\vee)}$ to mean those associated with $(-)^\vee$. We know (see e.g. [Mj1, §9.3]) that for every X in $\mathcal{M}$, its right dual is unique up to isomorphism whenever it exists, i.e. we have an isomorphism $\kappa_X : X^\star \to X^\vee$ in $\mathcal{M}$ given by the composition

$$\kappa_X := \rho_{X^\vee} \circ \left(X^\vee \otimes \mathrm{ev}^{(\star)}_X\right) \circ \alpha_{X^\vee,X,X^\star} \circ \left(\mathrm{db}^{(\vee)}_X \otimes X^\star\right) \circ \lambda^{-1}_{X^\star}. \tag{1.7}$$

Lemma 1.2.3. *The isomorphism $\kappa_X : X^\star \to X^\vee$ is natural in X and the dinatural transformations $\mathrm{ev}^{(\star)}$, $\mathrm{db}^{(\star)}$, $\mathrm{ev}^{(\vee)}$ and $\mathrm{db}^{(\vee)}$ satisfy*

$$(\kappa \otimes \mathsf{Id}) \circ \mathrm{db}^{(\star)} = \mathrm{db}^{(\vee)} \qquad \textit{and} \qquad \mathrm{ev}^{(\vee)} \circ (\mathsf{Id} \otimes \kappa) = \mathrm{ev}^{(\star)}. \tag{1.8}$$

(5) More precisely, these should be referred to as *wedges*, since they are dinatural transformations to a constant functor. However, we avoided this in order to spare the proliferation of terminology. For the definition of dinatural transformations and wedges we refer to [ML, §9.4].

A *braiding* for a monoidal category $\mathcal{M}$ is a family of isomorphisms $\tau_{X,Y} : X \otimes Y \to Y \otimes X$ for X, Y in $\mathcal{M}$ which is natural in both arguments and makes the following diagrams commutative

$$
\begin{tikzcd}[column sep=small]
& (Y \otimes X) \otimes Z \arrow[dr, "\alpha_{Y,X,Z}"] & \\
(X \otimes Y) \otimes Z \arrow[ur, "\tau_{X,Y\otimes Z}"] \arrow[d, "\alpha_{X,Y,Z}"'] & & Y \otimes (X \otimes Z) \arrow[d, "Y\otimes\tau_{X,Z}"] \\
X \otimes (Y \otimes Z) \arrow[dr, "\tau_{X,Y\otimes Z}"'] & & Y \otimes (Z \otimes X) \\
& (Y \otimes Z) \otimes X \arrow[ur, "\alpha_{Y,Z,X}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& X \otimes (Y \otimes Z) \arrow[dr, "\tau^{-1}_{Y\otimes Z,X}"] & \\
(X \otimes Y) \otimes Z \arrow[ur, "\alpha_{X,Y,Z}"] \arrow[d, "\tau^{-1}_{Y,X}\otimes Z"'] & & (Y \otimes Z) \otimes X \arrow[d, "\alpha_{Y,Z,X}"] \\
(Y \otimes X) \otimes Z \arrow[dr, "\alpha_{Y,X,Z}"'] & & Y \otimes (Z \otimes X) \\
& Y \otimes (X \otimes Z) \arrow[ur, "Y\otimes\tau^{-1}_{Z,X}"'] &
\end{tikzcd}
$$

A braiding is called a *symmetry* if $\tau_{Y,X} = \tau^{-1}_{X,Y}$. A *braided* (resp. *symmetric*) monoidal category is a monoidal category with a chosen braiding (resp. symmetry).

Remark 1.2.4. An observation is in order before concluding the section. Assume that a (strict) monoidal category $(\mathcal{M}, \otimes, \mathbb{I})$ is given and that $\mathcal{M}$ is also additive. Then it turns out that the set $\mathsf{End}_{\mathcal{M}}(\mathbb{I})$ is a $\mathbb{Z}$-algebra with the composition. In such a case, taken two objects X, Y in $\mathcal{M}$ the set $\mathsf{Hom}_{\mathcal{M}}(X,Y)$ becomes an $\mathsf{End}_{\mathcal{M}}(\mathbb{I})$-bimodule in a natural way, namely

$$f \cdot \phi \cdot g = f \otimes \phi \otimes g \tag{1.9}$$

for all $f, g \in \mathsf{End}_{\mathcal{M}}(\mathbb{I})$ and all $\phi \in \mathsf{Hom}_{\mathcal{M}}(X,Y)$. However, in general this do not convert $\mathcal{M}$ into a $\mathsf{End}_{\mathcal{M}}(\mathbb{I})$-linear category unless the left and right actions of (1.9) coincide. When this happens, the category $\mathcal{M}$ is called a *Penrose category*. This is the case, for example, when $\mathcal{M}$ is symmetric or braided (see e.g. [Brg, Remarque on page 5825]).

1.3 Algebras and coalgebras, modules and comodules

The notions of algebra, module over an algebra, coalgebra and comodule over a coalgebra can be introduced in the general setting of monoidal categories (see e.g. [AMa, §1.2] and [Pa3]). Namely, an *associative monoid* or *algebra* in a monoidal category $(\mathcal{M}, \otimes, \mathbb{I}, \alpha, \lambda, \rho)$ is a triple (A, m, u) where A is an object in $\mathcal{M}$, $m : A \otimes A \to A$ and $u : \mathbb{I} \to A$ are morphisms in $\mathcal{M}$ and they satisfy the associativity and unitality axioms, i.e. the following diagrams are commutative

$$
\begin{tikzcd}
(A \otimes A) \otimes A \arrow[rr, "\alpha_{A,A,A}"] \arrow[d, "m\otimes A"'] & & A \otimes (A \otimes A) \arrow[d, "A\otimes m"] \\
A \otimes A \arrow[r, "m"'] & A & A \otimes A \arrow[l, "m"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{I} \otimes A \arrow[r, "u\otimes A"] \arrow[dr, "\lambda_A"'] & A \otimes A \arrow[d, "m"] & A \otimes \mathbb{I} \arrow[l, "A\otimes u"'] \arrow[dl, "\rho_A"] \\
& A &
\end{tikzcd}
$$

Henceforth, all monoids will be assumed to be associative, unless differently specified. Given (A, m_A, u_A) and (B, m_B, u_B) two monoids, a *morphism of monoids* is a morphism $f : A \to B$ in $\mathcal{M}$ such that $m_B \circ (f \otimes f) = f \circ m_A$ and $f \circ u_A = u_B$. Given an monoid (A, m, u) in a monoidal category $(\mathcal{M}, \otimes, \mathbb{I}, \alpha, \lambda, \rho)$, a *left A-module* is a pair (M, μ) where $\mu : A \otimes M \to M$ makes the following diagrams commutative

$$
\begin{tikzcd}
(A \otimes A) \otimes M \arrow[rr, "\alpha_{A,A,M}"] \arrow[d, "m\otimes M"'] & & A \otimes (A \otimes M) \arrow[d, "A\otimes\mu"] \\
A \otimes M \arrow[r, "\mu"'] & M & A \otimes M \arrow[l, "\mu"]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes M \arrow[d, "\mu"'] & \mathbb{I} \otimes M \arrow[l, "u\otimes M"'] \arrow[dl, "\lambda_M"] \\
M &
\end{tikzcd}
$$

Given two left A-modules (M, μ_M) and (N, μ_N), a *morphism of A-modules* (or *A-linear morphism*) is an arrow $f : M \to N$ in $\mathcal{M}$ such that $\mu_N \circ (A \otimes f) = f \circ \mu_M$. Symmetrically, one may define the notions of *right A-module* and *right A-linear morphism*. Given (A, m_A, u_A) and (B, m_B, u_B) two monoids in $\mathcal{M}$, an (A,B)-*bimodule* is a triple $(M, \mu_{A,M}, \mu_{M,B})$ where $(M, \mu_{A,M})$ is a left A-module, $(M, \mu_{M,B})$ is a right B-module and the two structures are compatible, in the sense that the following diagram commutes

$$\begin{tikzcd} (A \otimes M) \otimes B \arrow[rr, "\alpha_{A,M,B}"] \arrow[d, "\mu_{A,M} \otimes B"'] & & A \otimes (M \otimes B) \arrow[d, "A \otimes \mu_{M,B}"] \\ M \otimes B \arrow[r, "\mu_{M,B}"'] & M & A \otimes M \arrow[l, "\mu_{A,M}"] \end{tikzcd} \tag{1.10}$$

A *morphism of (A,B)-bimodules* is a morphism in $\mathcal{M}$ which is left A-linear and right B-linear at the same time.

Example 1.3.1. Let us collect here some common examples of monoids in monoidal categories.

- We already mentioned in Example 1.2.1 that the category Set of small sets is monoidal with tensor product $\times$ and unit object $\{*\}$, the singleton. One can easily see that monoids in Set are ordinary monoids.

- The category Top is again monoidal with the Cartesian product $\times$ (endowed with the product topology) and the singleton $\{*\}$. Monoids in Top are topological monoids: ordinary monoids with a topology on the underlying set with respect to which the binary operation is continuous.

- Consider a $\Bbbk$-algebra A. A monoid in $({}_A\mathfrak{M}_A, \otimes_A, A)$ is usually called an *A-ring*, that is to say, an A-bimodule with an associative and unital A-bilinear multiplication $m : T \otimes_A T \to T$ with unit $u : A \to T$ or, equivalently, a $\Bbbk$-algebra T together with a $\Bbbk$-algebra extension $A \to T$.

- For G a monoid, the category of G-graded vector spaces $(\mathfrak{M}_G, \otimes, \Bbbk)$ admits G-graded $\Bbbk$-algebras as monoids, that is to say, $\Bbbk$-algebras endowed with a G-graduation which is compatible with the multiplication in the sense that $A_g \cdot A_h \subseteq A_{gh}$ for all $g, h \in G$.

- If, for a moment, we allow non-Set-enriched categories to enter the picture, then one may see strict monoidal categories[(6)] as monoids in the monoidal category of categories.

Example 1.3.2. Let $(\mathfrak{M}, \otimes, \Bbbk)$ be the monoidal category of $\Bbbk$-modules for $\Bbbk$ a commutative ring. Monoids in $\mathfrak{M}$ are simply $\Bbbk$-algebras. Pick A, B two $\Bbbk$-algebras and M, N two objects in $\mathfrak{M}$. If M is an (A,B)-bimodule with actions denoted by simple juxtaposition, then $\mathsf{Hom}_\Bbbk(M,N)$ is a (B,A)-bimodule and $\mathsf{Hom}_\Bbbk(N,M)$ is an (A,B)-bimodule as follows

$$(b \rightharpoonup f \leftharpoonup a)(m) = f(amb) \qquad \text{and} \qquad (a \cdot g \cdot b)(n) = ag(n)b \tag{1.11}$$

respectively, for all $a, b \in A$, $m \in M$, $n \in N$, $f \in \mathsf{Hom}_\Bbbk(M,N)$ and $g \in \mathsf{Hom}_\Bbbk(N,M)$.

Dually, i.e. by reversing all arrows, one may define the notions of comonoid, comonoid morphism, comodule, colinear morphism and bicomodule. Namely, a *coassociative comonoid* or *coalgebra* in a monoidal category $(\mathcal{M}, \otimes, \mathbb{I}, \alpha, \lambda, \rho)$ is a triple (C, Δ, ε) where C is an object in $\mathcal{M}$, $\Delta : C \to C \otimes C$ and $\varepsilon : C \to \mathbb{I}$ are morphisms in $\mathcal{M}$ and they satisfy the coassociativity and counitality axioms, i.e. the following diagrams are commutative

$$\begin{tikzcd} (C \otimes C) \otimes C \arrow[rr, "\alpha_{C,C,C}"] & & C \otimes (C \otimes C) \\ C \otimes C \arrow[u, "\Delta \otimes C"] & C \arrow[l, "\Delta"] \arrow[r, "\Delta"'] & C \otimes C \arrow[u, "C \otimes \Delta"'] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{I} \otimes C & C \otimes C \arrow[l, "\varepsilon \otimes C"'] \arrow[r, "C \otimes \varepsilon"] & C \otimes \mathbb{I} \\ & C \arrow[ul, "\lambda_C^{-1}"] \arrow[u, "\Delta"] \arrow[ur, "\rho_C^{-1}"'] & \end{tikzcd}$$

As for monoids, from now on all comonoids will be assumed to be coassociative, unless differently specified. Given $(D, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ two comonoids, a *morphism of comonoids* is a

(6) Monoidal categories for which the constraints are the identity morphisms.

morphism $f : C \to D$ in $\mathcal{M}$ such that $(f \otimes f) \circ \Delta_C = \Delta_D \circ f$ and $\varepsilon_D \circ f = \varepsilon_C$. Given a comonoid (C, Δ, ε) in a monoidal category $(\mathcal{M}, \otimes, \mathbb{I}, \alpha, \lambda, \rho)$, a *left C-comodule* is a pair (M, ρ) where $\rho : M \to C \otimes M$ makes the following diagrams commutative

$$\begin{tikzcd} C \otimes M \arrow[d, "\Delta \otimes M"'] & M \arrow[l, "\rho"'] \arrow[r, "\rho"] & C \otimes M \arrow[d, "C \otimes \rho"] \\ (C \otimes C) \otimes M \arrow[rr, "\alpha_{C,C,M}"] & & C \otimes (C \otimes M) \end{tikzcd} \qquad \begin{tikzcd} C \otimes M \arrow[r, "\varepsilon \otimes M"] & \mathbb{I} \otimes M \\ M \arrow[u, "\rho"] \arrow[ur, "\lambda_M^{-1}"'] & \end{tikzcd}$$

Given two left C-comodules (M, ρ_M) and (N, ρ_N), a *morphism of C-comodules* (or *C-colinear morphism*) is an arrow $f : M \to N$ in $\mathcal{M}$ such that $\rho_N \circ f = (C \otimes f) \circ \rho_M$. Symmetrically, one may define the notions of *right C-comodule* and *right C-colinear morphism*. Given $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ two comonoids in $\mathcal{M}$, an *(C, D)-bicomodule* is a triple $(M, \rho_{C,M}, \rho_{M,D})$ where $(M, \rho_{C,M})$ is a left C-comodule, $(M, \rho_{M,D})$ is a right D-comodule and the two structures are compatible, in the sense that the following diagram commutes

$$\begin{tikzcd} M \otimes D \arrow[d, "\rho_{C,M} \otimes D"'] & M \arrow[l, "\rho_{M,D}"'] \arrow[r, "\rho_{C,M}"] & C \otimes M \arrow[d, "C \otimes \rho_{M,D}"] \\ (C \otimes M) \otimes D \arrow[rr, "\alpha_{C,M,D}"] & & C \otimes (M \otimes D) \end{tikzcd} \tag{1.12}$$

A *morphism of (C, D)-bicomodules* is a morphism in $\mathcal{M}$ which is left C-colinear and right D-colinear at the same time.

As a consequence, given a monoid A in $\mathcal{M}$ one can define the categories ${}_A\mathcal{M}$, $\mathcal{M}_A$ and ${}_A\mathcal{M}_A$ of left, right and two-sided modules over A respectively. Similarly, given a comonoid C in $\mathcal{M}$, one can define the categories of C-comodules ${}^C\mathcal{M}, \mathcal{M}^C, {}^C\mathcal{M}^C$.

Example 1.3.3. Let $(\mathcal{M}, \otimes, \mathbb{I}, \alpha, \lambda, \rho)$ be a monoidal category. It can be proven that $\lambda_{\mathbb{I}} = \rho_{\mathbb{I}}$. Setting $m_{\mathbb{I}} := \rho_{\mathbb{I}}$ and $u_{\mathbb{I}} := \mathsf{Id}_{\mathbb{I}}$ endows $\mathbb{I}$ with a structure of monoid in $\mathcal{M}$. Every object M in $\mathcal{M}$ turns out to be a bimodule over $\mathbb{I}$ with λ_M and ρ_M. Therefore, we may identify $\mathcal{M}$ with ${}_{\mathbb{I}}\mathcal{M}_{\mathbb{I}}$. Analogously one may see that $\Delta_{\mathbb{I}} := \rho_{\mathbb{I}}^{-1}$ and $\varepsilon_{\mathbb{I}} := \mathsf{Id}_{\mathbb{I}}$ endows $\mathbb{I}$ with a structure of comonoid in $\mathcal{M}$ and that every object M in $\mathcal{M}$ is a bicomodule over $\mathbb{I}$ with λ_M^{-1} and ρ_M^{-1}. Therefore, we may identify $\mathcal{M}$ with ${}^{\mathbb{I}}\mathcal{M}^{\mathbb{I}}$ as well.

Pick an object V in $\mathcal{M}$. If (A, m, u) is an monoid in $\mathcal{M}$, $(M, \mu_{A,M})$ a left A-module and $(N, \mu_{N,A})$ a right A-module then we have functors $M \otimes - : \mathcal{M} \to {}_A\mathcal{M}$ and $- \otimes N : \mathcal{M} \to \mathcal{M}_A$ where $V \otimes N$ is a right A-module with action

$$(V \otimes N) \otimes A \xrightarrow{\alpha_{V,N,A}} V \otimes (N \otimes A) \xrightarrow{V \otimes \mu_{N,A}} V \otimes N$$

and symmetrically $M \otimes V$ is a left A-module with

$$A \otimes (M \otimes V) \xrightarrow{\alpha^{-1}_{A,M,V}} (A \otimes M) \otimes V \xrightarrow{\mu_{A,M} \otimes V} M \otimes V.$$

The distinguished functors $A \otimes - : \mathcal{M} \to {}_A\mathcal{M}$ and $- \otimes A : \mathcal{M} \to \mathcal{M}_A$ turn out to be left adjoints to the respective forgetful functors.

Dually for (C, Δ, ε) a comonoid, $(M, \rho_{C,M})$ a left C-comodule and $(N, \rho_{N,C})$ a right C-comodule we have functors $M \otimes - : \mathcal{M} \to {}^C\mathcal{M}$ and $- \otimes N : \mathcal{M} \to \mathcal{M}^C$ where $V \otimes N$ is a right C-comodule with coaction

$$V \otimes N \xrightarrow{V \otimes \rho_{N,C}} V \otimes (N \otimes C) \xrightarrow{\alpha^{-1}_{V,N,C}} (V \otimes N) \otimes C$$

and symmetrically $M \otimes V$ is a left C-comodule with

$$M \otimes V \xrightarrow{\rho_{C,M} \otimes V} (C \otimes M) \otimes V \xrightarrow{\alpha_{C,M,V}} C \otimes (M \otimes V)\,.$$

The distinguished functors $C \otimes - : \mathcal{M} \to {}^C\mathcal{M}$ and $- \otimes C : \mathcal{M} \to \mathcal{M}^C$ turn out to be right adjoints to the respective forgetful functors. Let $N \in \mathcal{M}^C$ and $V \in \mathcal{M}$. Since this will be useful later on, we highlight that the adjunction in the second case is given by

$$\begin{array}{rcl} \mathsf{Hom}\,(N,V) & \longleftrightarrow & \mathsf{Hom}^C\,(N, V \otimes C) \\ f & \longmapsto & (f \otimes C) \circ \rho_N \\ (V \otimes \varepsilon) \circ g & \longleftarrow\!\shortmid & g \end{array} \tag{1.13}$$

Remark 1.3.4. The bimodule condition (1.10) amounts to say that $\mu_{M,B}$ is a morphism of left A-modules or, equivalently, that $\mu_{A,N}$ is a morphism of right B-modules. Analogously for the bicomodule condition (1.12). Moreover, for M a left A-module and V, W objects in $\mathcal{M}$ we have that $\alpha_{M,V,W}$ and ρ_M are morphisms of left A-modules. Indeed,

$$\begin{aligned} &\mu_{A,M\otimes(V\otimes W)} \circ (A \otimes \alpha_{M,V,W}) = (\mu_{A,M} \otimes (V \otimes W)) \circ \alpha^{-1}_{A,M,V\otimes W} \circ (A \otimes \alpha_{M,V,W}) \\ &= (\mu_{A,M} \otimes (V \otimes W)) \circ \alpha_{A\otimes M,V,W} \circ \left(\alpha^{-1}_{A,M,V} \otimes W\right) \circ \alpha^{-1}_{A,M\otimes V,W} \\ &= \alpha_{M,V,W} \circ ((\mu_{A,M} \otimes V) \otimes W) \circ \left(\alpha^{-1}_{A,M,V} \otimes W\right) \circ \alpha^{-1}_{A,M\otimes V,W} \\ &= \alpha_{M,V,W} \circ (\mu_{A,M\otimes V} \otimes W) \circ \alpha^{-1}_{A,M\otimes V,W} = \alpha_{M,V,W} \circ \mu_{A,(M\otimes V)\otimes W} \end{aligned}$$

and

$$\rho_M \circ \mu_{A,M\otimes\mathbb{I}} = \rho_M \circ (\mu_{A,M} \otimes \mathbb{I}) \circ \alpha^{-1}_{A,M,\mathbb{I}} = \mu_{A,M} \circ \rho_{A\otimes M} \circ \alpha^{-1}_{A,M,\mathbb{I}} = \mu_{A,M} \circ (A \otimes \rho_M).$$

Analogously, if N is a right B-module and V, W are objects in $\mathcal{M}$ then $\alpha_{V,W,N}$ and λ_N are right B-linear. The same happens dually for comonoids and comodules over comonoids.

Given A a monoid and C a comonoid in a monoidal category $\mathcal{M}$ and given M in $\mathcal{M}$ we will sometimes denote by ${}_AM$ (or ${}_\bullet M$, if the monoid is clear from the context) the object M itself with a left A-module structure on it and by ${}^CM$ (or ${}^\bullet M$, if the comonoid is clear from the context) the same object M but with a left C-comodule structure on it. An analogous notation will be used for right (co)actions.

Remark 1.3.5. It is worthy to highlight the following: even if comonoids and comodules in a monoidal category are just dual notions of monoids and modules, in the sense that we may obtain the definition of the former ones by simply reversing the arrows in the definition of the latter ones, they do not necessarily enjoy dual properties.

Let us clarify this somehow informal sentence with two examples. Let $(\mathfrak{M}, \otimes, \Bbbk)$ be the monoidal category of modules over a commutative ring $\Bbbk$ and let C be a $\Bbbk$-coalgebra and A be a $\Bbbk$-algebra.

1. The category $\mathfrak{M}_A$ of right A-modules is a Grothendieck category. In particular, it has enough injectives. However, in general $\mathfrak{M}^C$ has not enough projectives ([DNR, Theorem 3.2.3]).

2. Comodules in $\mathfrak{M}^C$ satisfy the so-called *Fundamental Theorem of Comodules* ([DNR, Theorem 2.1.7]), i.e. every comodule is the inductive limit of its finite-dimensional subcomodules. However, it is not true that every module in $\mathfrak{M}_A$ is the projective limit of its finite-dimensional quotients. Take for example $\Bbbk = \mathbb{C}$ and $A = \mathbb{C}[X]$. Then the finite-dimensional quotients of $\mathbb{C}[X]$ are of the form $\mathbb{C}[X]/\langle p(X)\rangle$ for $p(X)$ a non-zero polynomial. The projective limit $\varprojlim\,(\mathbb{C}[X]/\langle p(X)\rangle)$ in the category of $\mathbb{C}[X]$-modules may be identified with $\prod_{k\in\mathbb{C}} \mathbb{C}[[X-k]]$ (a kind of "profinite poynomials") which is far away from being isomorphic to $\mathbb{C}[X]$.

We conclude this section with a few words of explanation on the reason why we introduced the two terminologies of algebras and monoids instead of choosing one. First of all, both of them have their *raison d'être*: monoids in the monoidal category of sets are ordinary monoids and algebras in the monoidal category of modules over a commutative ring $\Bbbk$ are ordinary $\Bbbk$-algebras. For this reason, both terminologies are used quite often in the literature, depending also on which "base category" one is planning to work. Secondly, in many of the references followed writing this thesis (and mainly in those of the author) the term "algebra" has been preferred over the term "monoid", but sometimes it is useful to have a synonym to increase the clarity of the exposition: we will see an example of this idea in the next section and in Chapter 3. Nevertheless, as long as no confusion may arise, we will still prefer to use the term "algebra".

1.4 (Co)monads and (co)limits

A *monad* on a category $\mathcal{C}$ is a triple $(T, \mathfrak{m}, \mathfrak{u})$ where $T : \mathcal{C} \to \mathcal{C}$ is a functor and $\mathfrak{u} : \mathsf{Id}_{\mathcal{C}} \to T$, $\mathfrak{m} : TT \to T$ are natural transformations such that the following diagrams commute

$$\begin{tikzcd} TTT \arrow[r, "\mathfrak{m}T"] \arrow[d, "T\mathfrak{m}"'] & TT \arrow[d, "\mathfrak{m}"] \\ TT \arrow[r, "\mathfrak{m}"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\mathfrak{u}T"] \arrow[dr, equal] & TT \arrow[d, "\mathfrak{m}"] & T \arrow[l, "T\mathfrak{u}"'] \arrow[dl, equal] \\ & T & \end{tikzcd}$$

Dually, a *comonad* on a category $\mathcal{C}$ is a triple $(L, \boldsymbol{\delta}, \boldsymbol{\varepsilon})$ where $L : \mathcal{C} \to \mathcal{C}$ is a functor and $\boldsymbol{\varepsilon} : L \to \mathsf{Id}_{\mathcal{C}}$, $\boldsymbol{\delta} : L \to LL$ are natural transformations such that the following diagrams commute

$$\begin{tikzcd} L \arrow[r, "\boldsymbol{\delta}"] \arrow[d, "\boldsymbol{\delta}"'] & LL \arrow[d, "\boldsymbol{\delta}L"] \\ LL \arrow[r, "L\boldsymbol{\delta}"'] & LLL \end{tikzcd} \qquad \begin{tikzcd} L \arrow[dr, equal] & LL \arrow[l, "L\boldsymbol{\varepsilon}"'] \arrow[r, "\boldsymbol{\varepsilon}L"] & L \arrow[dl, equal] \\ & L \arrow[u, "\boldsymbol{\delta}"] & \end{tikzcd}$$

Example 1.4.1. For a monoid (A, m, u) in a monoidal category $\mathcal{M}$, the functor $A \otimes - : \mathcal{M} \to \mathcal{M}$ is a monad on $\mathcal{M}$ with $\mathfrak{m}_X : A \otimes (A \otimes X) \to A \otimes X$ given by $\mathfrak{m}_X = (m \otimes X) \circ \alpha^{-1}_{A,A,X}$ and $\mathfrak{u}_X : X \to A \otimes X$ given by $\mathfrak{u}_X = (u \otimes X) \circ \lambda_X$ for all X in $\mathcal{M}$. Dually, for a comonoid (C, Δ, ε) in $\mathcal{M}$, the functor $C \otimes - : \mathcal{M} \to \mathcal{M}$ is a comonad with $\boldsymbol{\delta}_X = \alpha_{C,C,X} \circ (\Delta \otimes X)$ and $\boldsymbol{\varepsilon}_X = \lambda_X \circ (\varepsilon \otimes X)$.

In general, every adjunction $(\mathcal{F}, \mathcal{G}, \eta, \epsilon)$ with left adjoint $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and right adjoint $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ gives rise to a monad $(\mathcal{G}\mathcal{F}, \mathcal{G}\epsilon\mathcal{F}, \eta)$ and a comonad $(\mathcal{F}\mathcal{G}, \mathcal{F}\eta\mathcal{G}, \epsilon)$.

Given a monad $\mathbb{T} = (T, \mathfrak{m}, \mathfrak{u})$ on $\mathcal{C}$, an *algebra*[7] for the monad $\mathbb{T}$ is a pair (M, μ) where M is an object in $\mathcal{C}$ and $\mu : T(M) \to M$ is a morphism in $\mathcal{C}$ such that the following diagrams commute

$$\begin{tikzcd} TT(M) \arrow[r, "T\mu"] \arrow[d, "\mathfrak{m}_M"'] & T(M) \arrow[d, "\mu"] \\ T(M) \arrow[r, "\mu"'] & M \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\mathfrak{u}_M"] \arrow[dr, equal] & T(M) \arrow[d, "\mu"] \\ & M \end{tikzcd}$$

A *morphism of* $\mathbb{T}$*-algebras* between (M, μ) and (N, ν) is an arrow $f : M \to N$ in $\mathcal{C}$ such that $\nu \circ T(f) = f \circ \mu$. Algebras for a monad $\mathbb{T}$ and their morphisms form a category, denoted by $\mathcal{C}^{\mathbb{T}}$ and called the *Eilenberg-Moore category* of the monad. It comes together with a natural forgetful functor $\mathcal{U}^{\mathbb{T}} : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$, which is right adjoint to the free $\mathbb{T}$-algebra functor $\mathcal{F}^{\mathbb{T}} : \mathcal{C} \to \mathcal{C}^{\mathbb{T}}$ sending every object X in $\mathcal{C}$ to $(T(X), \mathfrak{m}_X)$. Notice that the monad induced by this adjunction is $\mathbb{T}$ itself (see e.g. [ML, Theorem VI.2.1] or [Brx2, Proposition 4.2.2]).

Dually, given a comonad $\mathbb{L} = (L, \boldsymbol{\delta}, \boldsymbol{\varepsilon})$ on $\mathcal{C}$, a *coalgebra* for the comonad $\mathbb{L}$ is a pair (N, ρ) where N is an object in $\mathcal{C}$ and $\rho : N \to L(N)$ is a morphism in $\mathcal{C}$ such that the following diagrams commute

$$\begin{tikzcd} N \arrow[r, "\rho"] \arrow[d, "\rho"'] & L(N) \arrow[d, "L\rho"] \\ L(N) \arrow[r, "\boldsymbol{\delta}_N"'] & LL(N) \end{tikzcd} \qquad \begin{tikzcd} N \arrow[dr, equal] & L(N) \arrow[l, "\boldsymbol{\varepsilon}_N"'] \\ & N \arrow[u, "\rho"'] \end{tikzcd}$$

[7] This is a classical terminology in the literature and hence the use of the word "monoid" instead of "algebra" is justified in this context to avoid confusion.

A *morphism of* $\mathbb{L}$*-coalgebras* between (N,ρ) and (P,σ) is an arrow $f : N \to P$ in $\mathcal{C}$ such that $\sigma \circ f = L(f) \circ \rho$. Coalgebras for a comonad $\mathbb{L}$ and their morphisms form a category, denoted by $\mathcal{C}_{\mathbb{L}}$ and called the *Eilenberg-Moore category* of the comonad. It comes together with a natural forgetful functor $\mathcal{U}_{\mathbb{L}} : \mathcal{C}_{\mathbb{L}} \to \mathcal{C}$, which is left adjoint to the free $\mathbb{L}$-coalgebra functor $\mathcal{F}_{\mathbb{L}} : \mathcal{C} \to \mathcal{C}_{\mathbb{L}}$ sending every object X in $\mathcal{C}$ to $(L(X), \boldsymbol{\delta}_X)$. Notice that the comonad induced by this adjunction is $\mathbb{L}$ itself.

Proposition 1.4.2 ([Brx2, Propositions 4.3.1 and 4.3.2])**.** *Let $\mathbb{T} = (T, \mathfrak{m}, \mathfrak{u})$ be a monad on a category $\mathcal{C}$. The forgetful functor $\mathcal{U}^{\mathbb{T}}$ creates all limits and it creates all those colimits which are preserved by T and TT. In particular, if $\mathcal{C}$ has some type of colimits preserved by T, then $\mathcal{C}^{\mathbb{T}}$ has the same type of colimits and these are preserved by $\mathcal{U}^{\mathbb{T}}$.*

Proposition 1.4.3 (dual to Proposition 1.4.2)**.** *Let $\mathbb{L} = (L, \boldsymbol{\delta}, \boldsymbol{\varepsilon})$ be a comonad on a category $\mathcal{C}$. The forgetful functor $\mathcal{U}_{\mathbb{L}}$ creates all colimits and it creates all those limits which are preserved by L and LL. In particular, if $\mathcal{C}$ has some type of limits preserved by L, then $\mathcal{C}_{\mathbb{L}}$ has the same type of limits and these are preserved by $\mathcal{U}_{\mathbb{L}}$.*

Example 1.4.4. Given a monoidal category $\mathcal{M}$ and a monoid A and a comonoid C in $\mathcal{M}$, we have that

- the algebras over the monad $A \otimes -$ are exactly the left A-modules and the associated Eilenberg-Moore category is ${}_A\mathcal{M}$;
- the coalgebras over the comonad $C \otimes -$ are exactly the left C-comodules and the associated Eilenberg-Moore category is ${}^C\mathcal{M}$;
- the left-right symmetric versions of the previous claims hold.

From Proposition 1.4.2 it follows that ${}_A\mathcal{M}$ admits all limits that exists in $\mathcal{M}$. In fact, these can be computed in $\mathcal{M}$ and they come with a natural left A-module structure. For colimits this is true for those which are preserved by $A \otimes -$ and $(A \otimes A) \otimes -$. In particular, if $\mathcal{M}$ has some type of colimits which are preserved by $A \otimes -$, then ${}_A\mathcal{M}$ has the same type of colimits and these are preserved by the forgetful functor. Dually, from Proposition 1.4.3 it follows that ${}^C\mathcal{M}$ admits all colimits that exists in $\mathcal{M}$, while for limits this is true for those which are preserved by $C \otimes -$ and $(C \otimes C) \otimes -$. In particular, if $\mathcal{M}$ has some type of limits which are preserved by $C \otimes -$, then ${}^C\mathcal{M}$ has the same type of limits and these are preserved by the forgetful functor.

A remarkable fact is the following. Let A, B monoids in $\mathcal{M}$. As we already observed in Remark 1.3.4, both $\alpha_{M,B,B}$ and ρ_M are left A-linear for all M in ${}_A\mathcal{M}$, which means that even if B is not a monoid in ${}_A\mathcal{M}$, $- \otimes B$ is still a monad on ${}_A\mathcal{M}$. The algebras for the monad $- \otimes B : {}_A\mathcal{M} \to {}_A\mathcal{M}$ are exactly the (A, B)-bimodules. The same happens for $A \otimes -$ on $\mathcal{M}_B$ and, dually, for comonoids C, D in $\mathcal{M}$ and comodules.

1.5 Tensor product over an algebra

Let as usual $(\mathcal{M}, \otimes, \mathbb{I}, \alpha, \lambda, \rho)$ be a monoidal category. From this section on, we will drop the terms "monoid" and "comonoid" up to Chapter 3 and we will use "algebra" and "coalgebra" instead.

Recall from [ARV, Definition 3.1] that a *reflexive pair* (of morphisms) in a category is a pair of parallel split epimorphisms having a common section. As a matter of terminology, a *reflexive coequalizer* is the coequalizer of a reflexive pair of morphisms.

Lemma 1.5.1. *Assume that $\mathcal{M}$ admits all reflexive coequalizers and that the endofunctors $A \otimes -$ and $- \otimes A$ of $\mathcal{M}$ preserve them for every algebra A in $\mathcal{M}$. Then for all algebras A, B in $\mathcal{M}$ we have that ${}_A\mathcal{M}$, $\mathcal{M}_B$ and ${}_A\mathcal{M}_B$ admit all reflexive coequalizers.*

Proof. This is true in general for every type $\mathfrak{P}$ of colimits which are preserved by tensoring by an algebra. For A and B algebras in $\mathcal{M}$, the claims on ${}_A\mathcal{M}$ and $\mathcal{M}_B$ follow from Example 1.4.4. To prove that the same holds for ${}_A\mathcal{M}_B$ as well, it is enough to show that $- \otimes B : {}_A\mathcal{M} \to {}_A\mathcal{M}$

preserves all colimits of type $\mathfrak{P}$ (we already noticed that it is still a monad). Thus, consider a diagram $\mathbb{D} : \mathcal{J} \to {}_A\mathcal{M}, i \mapsto (\mathbb{D}_i, \mu_{A,i})$ that admits a colimit $(X, \mu_{A,X})$ of type $\mathfrak{P}$ with structure maps $\tau_i : X \to \mathbb{D}_i$ for all i in $\mathcal{J}$. If we denote by $\mathcal{U} : {}_A\mathcal{M} \to \mathcal{M}$ the forgetful functor from A-modules to $\mathcal{M}$ then we have that X is the colimit of the diagram $\mathcal{U}\mathbb{D} : \mathcal{J} \to \mathcal{M}$ in $\mathcal{M}$ and hence $X \otimes B$ is the colimit of the diagram $\mathcal{U}\mathbb{D} \otimes B$ by assumption on $- \otimes B$. By the hypothesis on $A \otimes -$, $X \otimes B$ comes endowed with a left A-module structure which makes of it the colimit of the diagram $\mathbb{D} \otimes B$ in ${}_A\mathcal{M}$. Therefore, the monad $- \otimes B$ on ${}_A\mathcal{M}$ preserves all colimits of type $\mathfrak{P}$, which then exist in the associated Eilenberg-Moore category ${}_A\mathcal{M}_B$. □

Remark 1.5.2. It may be worthy to notice that the A-module structure on $X \otimes B$ in the proof of Lemma 1.5.1 is given by the unique morphism $\mu_{A,X\otimes B}$ in $\mathcal{M}$ such that

$$\mu_{A,X\otimes B} \circ (A \otimes (\tau_i \otimes B)) = (\tau_i \otimes B) \circ (\mu_{A,i} \otimes B) \circ \alpha^{-1}_{A,\mathbb{D}_i,B},$$

whence it coincides with the left A-action on $X \otimes B$ induced by the left A-module structure on X, i.e. $\mu_{A,X\otimes B} = (\mu_{A,X} \otimes B) \circ \alpha^{-1}_{A,X,B}$.

From now on, we assume more or less implicitly that the monoidal category $\mathcal{M}$ admits all reflexive coequalizers and that for every algebra A in $\mathcal{M}$ the monads $A \otimes -$ and $- \otimes A$ preserve them. Let A, B, C be algebras in $\mathcal{M}$ and M, N objects in ${}_A\mathcal{M}_B$ and ${}_B\mathcal{M}_C$ respectively. The arrows

$$\chi_{M,N} : (M \otimes B) \otimes N \xrightarrow{\mu_{M,B}\otimes N} M \otimes N,$$

$$\gamma_{M,N} : (M \otimes B) \otimes N \xrightarrow{\alpha_{M,B,N}} M \otimes (B \otimes N) \xrightarrow{M\otimes\mu_{B,N}} M \otimes N.$$

in $\mathcal{M}$ form a reflexive pair since

$$\sigma_{M,N} : M \otimes N \xrightarrow{\rho^{-1}_M\otimes N} (M \otimes \mathbb{I}) \otimes N \xrightarrow{(M\otimes u_B)\otimes N} (M \otimes B) \otimes N$$

is a common section. Consider their coequalizer

$$(M \otimes B) \otimes N \underset{\gamma_{M,N}}{\overset{\chi_{M,N}}{\rightrightarrows}} M \otimes N \xrightarrow{\omega_{M,N}} M \otimes_B N.$$

Remark 1.5.3. Keeping in mind Lemma 1.5.1, notice that the arrows $\chi_{M,N}$ and $\gamma_{M,N}$ are arrows in $\mathcal{M}_C$, ${}_A\mathcal{M}$ and ${}_A\mathcal{M}_C$ as well. Since these categories admit all reflexive coequalizers, we may perform $M \otimes_B N$ in each category. However, since the functors $- \otimes C : \mathcal{M} \to \mathcal{M}$ and $A \otimes - : \mathcal{M} \to \mathcal{M}$ preserve reflexive coequalizers by hypothesis, all these $M \otimes_B N$ coincide up to isomorphism in $\mathcal{M}$ and we will often identify them.

Definition 1.5.4. Given A, B, C algebras in $\mathcal{M}$ and M, N objects in ${}_A\mathcal{M}_B$ and ${}_B\mathcal{M}_C$ respectively, we define the *tensor product over* B of M and N to be the reflexive coequalizer

$$(M \otimes B) \otimes N \underset{\gamma_{M,N}}{\overset{\chi_{M,N}}{\rightrightarrows}} M \otimes N \xrightarrow{\omega_{M,N}} M \otimes_B N.$$

in the category ${}_A\mathcal{M}_C$.

Remark 1.5.5. The left A-module structure on $M \otimes_B N$ is the unique morphism $\mu_{A,M\otimes_B N} : A \otimes (M \otimes_B N) \to M \otimes_B N$ in $\mathcal{M}$ such that

$$\mu_{A,M\otimes_B N} \circ (A \otimes \omega_{M,N}) = \omega_{M,N} \circ (\mu_{A,M} \otimes N) \circ \alpha^{-1}_{A,M,N}.$$

Analogously for the right C-module structure. In particular, $\omega_{M,N}$ is a morphism of (A, C)-bimodules. Moreover $\omega_{M,N}$ is an epimorphism in ${}_A\mathcal{M}_C$, due to the universal property it satisfies (in fact, for every coequalizer (Q, q) the arrow q is an epimorphism)[8].

[8] Actually, $\omega_{M,N}$ is an epimorphism in $\mathcal{M}$, ${}_A\mathcal{M}$ and $\mathcal{M}_C$ as well for the same reason.

Example 1.5.6. Let $(\mathcal{M}, \otimes, \mathbb{I}, \alpha, \lambda, \rho)$ be a monoidal category. We already noticed in Example 1.3.3 that $\mathbb{I}$ is an algebra in $\mathcal{M}$ and that given M, N in $\mathcal{M}$ we may consider them as $\mathbb{I}$-bimodules. Therefore we may want to construct $M \otimes_{\mathbb{I}} N$. Since $\chi_{M,N} = \rho_M \otimes N = (M \otimes \lambda_N) \circ \alpha_{M,\mathbb{I},N} = \gamma_{M,N}$ it follows that $M \otimes N = M \otimes_{\mathbb{I}} N$, without any further assumption on $\mathcal{M}$. That is, $\mathcal{M}$ always admits the split coequalizer of $\chi_{M,N}$ and $\gamma_{M,N}$ in this particular case.

1.6 The bicategory of bimodules and algebras

We recall from [Be] that a *bicategory* $\mathcal{S}$ is determined by the following data:

1. A class $\mathcal{S}_0$ of *objects* or *vertices* of $\mathcal{S}$, also called 0-*cells*.

2. For each pair of objects A, B in $\mathcal{S}_0$, a category $\mathcal{S}(A, B)$ whose objects are called *edges* or *arrows* or 1-*cells* of $\mathcal{S}$ and whose arrows are called 2-*cells* of $\mathcal{S}$. These are referred to as the categories of $\{1, 2\}$-*cells*. The composition of 2-cells is usually called *vertical composition*.

3. For each triple A, B, C of objects in $\mathcal{S}_0$, a *(horizontal) composition functor*
$$\diamond_{A,B,C} : \mathcal{S}(A, B) \times \mathcal{S}(B, C) \to \mathcal{S}(A, C).$$
When the objects will be clear from the context, it will be simply denoted by $\diamond$ and we will write $X \diamond Y$ and $f \diamond g$ for (X, Y) objects and (f, g) arrows in $\mathcal{S}(A, B) \times \mathcal{S}(B, C)$.

4. For each object A in $\mathcal{S}_0$, a distinguished object $\mathbb{I}_A$ in $\mathcal{S}(A, A)$ which is called the *identity* 1-*cell*. The identity map of $\mathbb{I}_A$ in $\mathcal{S}(A, A)$ is called the *identity* 2-*cell*.

5. For each quadruple A, B, C, D of objects in $\mathcal{S}_0$, a natural isomorphism
$$\alpha_{A,B,C,D} : \diamond_{A,C,D} \circ (\diamond_{A,B,C} \times \mathsf{Id}_{\mathcal{S}(C,D)}) \to \diamond_{A,B,D} \circ (\mathsf{Id}_{\mathcal{S}(A,B)} \times \diamond_{B,C,D})$$
called the *associativity isomorphism*. For (X, Y, Z) and object in $\mathcal{S}(A, B) \times \mathcal{S}(B, C) \times \mathcal{S}(C, D)$, the associated component of $\alpha_{A,B,C,D}$ will be abbreviated in
$$\alpha_{X,Y,Z} : (X \diamond Y) \diamond Z \to X \diamond (Y \diamond Z).$$

6. For each pair A, B of objects in $\mathcal{S}_0$, two natural isomorphisms
$$\lambda_{A,B} : \diamond_{A,A,B} \circ (\mathbb{I}_A \times \mathsf{Id}_{\mathcal{S}(A,B)}) \to \mathsf{Id}_{\mathcal{S}(A,B)} \quad \text{and} \quad \rho_{A,B} : \diamond_{A,B,B} \circ (\mathsf{Id}_{\mathcal{S}(A,B)} \times \mathbb{I}_B) \to \mathsf{Id}_{\mathcal{S}(A,B)}.$$
If X is an object in $\mathcal{S}(A, B)$, the associated components of $\lambda_{A,B}$ and $\rho_{A,B}$ will be abbreviated in
$$\lambda_X : \mathbb{I}_A \diamond X \to X \quad \text{and} \quad \rho_X : X \diamond \mathbb{I}_B \to X.$$

The families of isomorphisms $\alpha_{A,B,C,D}$, $\lambda_{A,B}$ and $\rho_{A,B}$ are furthemore required to satisfy the following axioms

A.C. *Associativity coherence*. If (X, Y, Z, W) is an object in $\mathcal{S}(A, B) \times \mathcal{S}(B, C) \times \mathcal{S}(C, D) \times \mathcal{S}(D, E)$, the following diagram commutes

$$\begin{tikzcd}
 & (X \diamond Y) \diamond (Z \diamond W) \arrow[dr, "\alpha_{X,Y,Z \diamond W}"] & \\
((X \diamond Y) \diamond Z) \diamond W \arrow[ur, "\alpha_{X \diamond Y, Z, W}"] \arrow[d, "\alpha_{X,Y,Z} \diamond W"'] & & X \diamond (Y \diamond (Z \diamond W)) \\
(X \diamond (Y \diamond Z)) \diamond W \arrow[dr, "\alpha_{X, Y \diamond Z, W}"'] & & \\
 & X \diamond ((Y \diamond Z) \diamond W) \arrow[uur, "X \diamond \alpha_{Y,Z,W}"'] &
\end{tikzcd}$$

I.C. *Identity coherence.* If (X, Y) is an object in $\mathcal{S}(A,B) \times \mathcal{S}(B,C)$, the following diagram commutes

$$\begin{tikzcd}
(X \diamond \mathbb{I}_B) \diamond Y \arrow[dd, "\alpha_{X,\mathbb{I}_B,Y}"'] \arrow[dr, "\rho_X \diamond Y"] & \\
& X \diamond Y \\
X \diamond (\mathbb{I}_B \diamond Y) \arrow[ur, "X \diamond \lambda_Y"'] &
\end{tikzcd}$$

Given two bicategories $\mathcal{S} = (\mathcal{S}_0, \diamond, \mathbb{I}, \alpha, \lambda, \rho)$ and $\mathcal{S}' = (\mathcal{S}'_0, \diamond', \mathbb{I}', \alpha', \lambda', \rho')$ a *(lax) bifunctor* or *(lax) morphism of bicategories* $(\mathscr{F}, \psi_0, \psi)$ from $\mathcal{S}$ to $\mathcal{S}'$ is determined by the following data:

1. An assignment $\mathscr{F} : \mathcal{S}_0 \to \mathcal{S}'_0$.
2. A family of functors $\mathscr{F}_{A,B} : \mathcal{S}(A,B) \to \mathcal{S}'(\mathscr{F}(A), \mathscr{F}(B))$.
3. For each object A in $\mathcal{S}_0$, a 2-cell $\psi_0(A) : \mathbb{I}'_{\mathscr{F}(A)} \to \mathscr{F}(\mathbb{I}_A)$ of $\mathcal{S}'$.
4. A family of natural transformations

$$\psi_{A,B,C} : \diamond'_{\mathscr{F}(A),\mathscr{F}(B),\mathscr{F}(C)} \circ (\mathscr{F}_{A,B} \times \mathscr{F}_{B,C}) \to \mathscr{F}_{A,C} \circ \diamond_{A,B,C}$$

If (X, Y) is an object in $\mathcal{S}(A,B) \times \mathcal{S}(B,C)$, the associated component of $\psi_{A,B,C}$ will be denoted for simplicity by

$$\psi_{X,Y} : \mathscr{F}(X) \diamond' \mathscr{F}(Y) \to \mathscr{F}(X \diamond Y).$$

These data are required to satisfy the following coherence axioms

M.1 If (X, Y, Z) is an object in $\mathcal{S}(A,B) \times \mathcal{S}(B,C) \times \mathcal{S}(C,D)$, the following diagram is commutative

$$\begin{tikzcd}
(\mathscr{F}(X) \diamond' \mathscr{F}(Y)) \diamond' \mathscr{F}(Z) \arrow[r, "\psi_{X \diamond Y, Z} \circ \left(\psi_{X,Y} \diamond' \mathscr{F}(Z)\right)"] \arrow[d, "\alpha'_{\mathscr{F}(X),\mathscr{F}(Y),\mathscr{F}(Z)}"'] & \mathscr{F}((X \diamond Y) \diamond Z) \arrow[d, "\mathscr{F}\left(\alpha_{X,Y,Z}\right)"] \\
\mathscr{F}(X) \diamond' (\mathscr{F}(Y) \diamond' \mathscr{F}(Z)) \arrow[r, "\psi_{X, Y \diamond Z} \circ \left(\mathscr{F}(X) \diamond' \psi_{Y,Z}\right)"'] & \mathscr{F}(X \diamond (Y \diamond Z))
\end{tikzcd}$$

M.2 If X is an object in $\mathcal{S}(A,B)$, the following diagrams are commutative

$$\begin{tikzcd}
\mathbb{I}'_{\mathscr{F}(A)} \diamond' \mathscr{F}(X) \arrow[r, "\lambda'_{\mathscr{F}(X)}"] \arrow[d, "\psi_0(A) \diamond' \mathscr{F}(X)"'] & \mathscr{F}(X) \\
\mathscr{F}(\mathbb{I}_A) \diamond' \mathscr{F}(X) \arrow[r, "\psi_{\mathbb{I}_A, X}"'] & \mathscr{F}(\mathbb{I}_A \diamond X) \arrow[u, "\mathscr{F}(\lambda_X)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{F}(X) \diamond' \mathbb{I}'_{\mathscr{F}(B)} \arrow[r, "\rho'_{\mathscr{F}(X)}"] \arrow[d, "\mathscr{F}(X) \diamond' \psi_0(B)"'] & \mathscr{F}(X) \\
\mathscr{F}(X) \diamond' \mathscr{F}(\mathbb{I}_B) \arrow[r, "\psi_{X, \mathbb{I}_B}"'] & \mathscr{F}(X \diamond \mathbb{I}_B) \arrow[u, "\mathscr{F}(\rho_X)"']
\end{tikzcd}$$

One may also define *colax* bifunctors, i.e. bifunctors as above but with the structure maps reversed

$$\psi_{A,B,C} : \mathscr{F}_{A,C} \circ \diamond_{A,B,C} \to \diamond'_{\mathscr{F}(A),\mathscr{F}(B),\mathscr{F}(C)} \circ (\mathscr{F}_{A,B} \times \mathscr{F}_{B,C}),$$
$$\psi_0(A) : \mathscr{F}(\mathbb{I}_A) \to \mathbb{I}'_{\mathscr{F}(A)}.$$

Example 1.6.1. The most widely known example of a bicategory is the bicategory of rings, bimodules and bimodule homomorphisms, where the horizontal compositions are given by tensor products over the rings. By admitting also non-locally small categories to enter into the picture, another familiar example is the bicategory of categories, functors and natural transformations.

Remark 1.6.2. If $\mathcal{S}$ is a bicategory, then for every object A in $\mathcal{S}_0$ the category $\mathcal{S}(A,A)$ is monoidal with tensor product $\diamond_{A,A,A}$, unit object $\mathbb{I}_A$ and constraints $\alpha_{A,A,A,A}$, $\lambda_{A,A}$ and $\rho_{A,A}$. In particular, a monoidal category can be seen as a bicategory with only one 0-cell. Moreover, given a bifunctor $\mathscr{F}$ between bicategories $\mathcal{S}$ and $\mathcal{S}'$, the functors $\mathscr{F}_{A,A}$ are monoidal functors with structure maps $\psi_0(A)$ and $\psi_{A,A,A}$. This is also the reason why we are using the same notation for the constraints in a monoidal category and in a bicategory.

The construction of the bicategory of bimodules and algebras on a monoidal category with coequalizers $\mathcal{M}$ such that the tensor product preserves coequalizers has already been performed by G. Böhm and K. Szlachány in [BS2, Appendix] under the name of *bicategory of internal bimodules*. We observe here that this construction can be weakened by just requiring that $\mathcal{M}$ admits all reflexive coequalizers and that the tensor product preserves them. Nevertheless, we will leave the proofs of the results we are going to mention to the interested reader, since they would follow closely the ideas in [BS2] and since for the applications we have in mind the former construction is enough. A minor original contribution, which does not appear in [BS2], comes from Theorem 1.6.6, which states that any monoidal functor between suitable monoidal categories as above induces a bifunctor between the bicategories of internal bimodules.

Throughout this section we will always assume that $(\mathcal{M},\otimes,\mathbb{I},\alpha,\lambda,\rho)$ is a monoidal category which admits all reflexive coequalizers and that for every algebra A in $\mathcal{M}$ the endofunctors $A\otimes -$ and $-\otimes A$ preserve them.

Lemma 1.6.3. *Under the standing hypotheses of the section, for A,B,C algebras in $\mathcal{M}$ the tensor product $\otimes_B$ introduced in Definition 1.5.4 induces a functor*

$$-\otimes_B - : {}_A\mathcal{M}_B \times {}_B\mathcal{M}_C \to {}_A\mathcal{M}_C$$

and the canonical morphism $\omega_{M,N} : M\otimes N \to M\otimes_B N$ in ${}_A\mathcal{M}_C$ becomes natural in both components.

Corollary 1.6.4. *Let A,B,C,D be algebras and ${}_AM_B, {}_BN_C, {}_CP_D$ be bimodules in $\mathcal{M}$. Then*

$$\omega_{M\otimes_B N,P}\circ(\omega_{M,N}\otimes P) = (\omega_{M,N}\otimes_C P)\circ\omega_{M\otimes N,P}.$$

Theorem 1.6.5. *Let $(\mathcal{M},\otimes,\mathbb{I},\alpha,\lambda,\rho)$ be a monoidal category such that $\mathcal{M}$ admits all reflexive coequalizers and such that for every object X in $\mathcal{M}$ both the functors $X\otimes -$ and $-\otimes X$ from $\mathcal{M}$ to itself preserve reflexive coequalizers. Then there exists a bicategory $\mathscr{B}im_{\mathcal{M}}$ whose 0-cells are algebras in $\mathcal{M}$ and whose categories of $\{1,2\}$-cells are the categories of bimodules over these algebras. The composition functors are provided by the tensor products $\otimes_B$ and the identity arrows by the algebras themselves. The associativity isomorphisms are induced by the associativity constraint α and the left and right identities by the left and right actions.*

Theorem 1.6.6. *Let $(\mathcal{M},\otimes,\mathbb{I},\alpha,\lambda,\rho)$ and $(\mathcal{M}',\otimes',\mathbb{I}',\alpha',\lambda',\rho')$ be two monoidal categories that satisfy the hypotheses of Theorem 1.6.5 and let $(\mathcal{F},\varphi_0,\varphi)$ be a lax monoidal functor from $\mathcal{M}$ to $\mathcal{M}'$. Then $\mathcal{F}$ lifts to a bifunctor $\mathscr{F}$ from $\mathscr{B}im_{\mathcal{M}}$ to $\mathscr{B}im_{\mathcal{M}'}$ in a natural way. Namely, $\mathscr{F}(A)=\mathcal{F}(A)$ and $\mathscr{F}({}_AM_B) = {}_{\mathcal{F}(A)}\mathcal{F}(M)_{\mathcal{F}(B)}$ for all algebras A,B and all bimodules ${}_AM_B$. The family of natural transformations $\psi_{M,N} : \mathscr{F}(M)\otimes_{\mathscr{F}(B)}\mathscr{F}(N)\to\mathscr{F}(M\otimes_B N)$ for A,B,C running through the algebras in $\mathcal{M}$ and ${}_AM_B, {}_BN_C$ bimodules as denoted is uniquely determined by the condition*

$$\psi_{M,N}\circ\omega_{\mathcal{F}(M),\mathcal{F}(N)} = \mathcal{F}(\omega_{M,N})\circ\varphi_{M,N}. \tag{1.14}$$

Since the proof of Theorem 1.6.6 amounts to check that the above definitions give rise to a bifunctor and it would have been technical and, in our opinion, not particularly interesting, it is omitted.

Remark 1.6.7. The dual construction holds as well. Namely, if $(\mathcal{M},\otimes,\mathbb{I},\alpha,\lambda,\rho)$ is a monoidal category such that $\mathcal{M}$ admits all reflexive equalizers and if $-\otimes X$ and $X\otimes -$ preserve them for every object X in $\mathcal{M}$, then there exists a bicategory $\mathscr{B}icom_{\mathcal{M}}$ whose 0-cells are coalgebras in $\mathcal{M}$ and whose categories of $\{1,2\}$-cells are the categories of bicomodules over these. The composition functors are provided by the cotensor products $\square_C$ for C running through the coalgebras in $\mathcal{M}$.

The cotensor product $\square_C$ is defined for ${}^BM^C$ and ${}^CN^D$ bicomodules as the following equalizer in the category of (B,D)-bicomodules

$$0 \longrightarrow M\square_C N \xrightarrow{\omega_{M,N}} M\otimes N \underset{\alpha^{-1}_{M,C,N}\circ\left(M\otimes\rho_{C,N}\right)}{\overset{\rho_{M,C}\otimes N}{\rightrightarrows}} (M\otimes C)\otimes N.$$

The identity arrows are given by the coalgebras themselves. The associativity isomorphisms are induced by the associativity constraint α and the left and right identities by the left and right coactions. Moreover, any colax monoidal functor $(\mathcal{F},\varphi_0,\varphi)$ from $\mathcal{M}$ to $\mathcal{M}'$ can be lifted to a colax bifunctor $\mathscr{F}$ from $\mathscr{Bicom}_{\mathcal{M}}$ to $\mathscr{Bicom}_{\mathcal{M}'}$. Namely, $\mathscr{F}(C)=\mathcal{F}(C)$ and $\mathscr{F}\left({}^CM^D\right)={}^{\mathcal{F}(C)}\mathcal{F}(M)^{\mathcal{F}(D)}$ for all coalgebras C,D and all bicomodules ${}^CM^D$. The family of natural transformations $\psi_{M,N}:\mathscr{F}(M\square_C N)\to\mathscr{F}(M)\square_{\mathscr{F}(C)}\mathscr{F}(N)$ for B,C,D running through the coalgebras in $\mathcal{M}$ and ${}^BM^C,{}^CN^D$ bicomodules as denoted is uniquely determined by the condition

$$\omega_{\mathcal{F}(M),\mathcal{F}(N)}\circ\psi_{M,N}=\varphi_{M,N}\circ\mathcal{F}(\omega_{M,N}).$$

Example 1.6.8. Let $\mathfrak{M}$ be the category of vector spaces over a field $\Bbbk$. This is a complete and cocomplete monoidal category and for every vector space V both functors $V\otimes -$ and $-\otimes V$ are continuous and cocontinuous. Consider the lax monoidal functor $(-)^*:\mathfrak{M}^{\mathrm{op}}\to\mathfrak{M}$. In view of Theorems 1.6.5 and 1.6.6 $(-)^*$ extends to a bifunctor $(-)^*:\mathscr{Bicom}_{\Bbbk}{}^{\mathrm{op}}\to\mathscr{Bim}_{\Bbbk}$ which sends every $\Bbbk$-coalgebra C to its *convolution algebra* C^* with structures

$$\Bbbk \xrightarrow{\varepsilon_C^*} C^* \qquad\text{and}\qquad C^*\otimes C^* \xrightarrow{\varphi_{C,C}} (C\otimes C)^* \xrightarrow{\Delta_C^*} C^* \tag{1.15}$$

and every (C,D)-bicomodule N to the (C^*,D^*)-bimodule N^*, whose actions are given by

$$C^*\otimes N^* \xrightarrow{\varphi_{C,N}} (C\otimes N)^* \xrightarrow{\left(\rho_N^l\right)^*} N^* \qquad\text{and}\qquad N^*\otimes D^* \xrightarrow{\varphi_{N,D}} (N\otimes D)^* \xrightarrow{\left(\rho_N^r\right)^*} N^*.$$

Here by $\mathscr{Bicom}_{\Bbbk}{}^{\mathrm{op}}$ we mean the bicategory given as follows. The 0-cells are $\Bbbk$-coalgebras. For every pair C,D of $\Bbbk$-coalgebras the category ${}^C\left(\mathscr{Bicom}_{\Bbbk}{}^{\mathrm{op}}\right)^D$ is the opposite category $\left({}^C\mathfrak{M}^D\right)^{\mathrm{op}}$ of (C,D)-bicomodules. The horizontal composition is given by the cotensor product. The identity arrows are the coalgebras. The associativity and unitality isomorphisms are the opposites of the inverses of those in $\mathscr{Bicom}_{\Bbbk}$. Notice that if we consider instead the colax monoidal functor $(-)^*:\mathfrak{M}\to\mathfrak{M}^{\mathrm{op}}$, then this can be lifted to a colax bifunctor $(-)^*:\mathscr{Bicom}_{\Bbbk}\to\mathscr{Bim}_{\Bbbk}{}^{\mathrm{op}}$.

1.7 Quasi-bialgebras and coquasi-bialgebras

Let $\Bbbk$ be a commutative ring with identity and denote by $(\mathfrak{M},\otimes,\Bbbk)$ the monoidal category of $\Bbbk$-modules as in Example 1.2.1. A $\Bbbk$-algebra is simply an algebra in $\mathfrak{M}$. It is well-known that $\Bbbk$-bialgebras can be characterized as those $\Bbbk$-algebras whose category of (right or left) modules is monoidal with strict monoidal forgetful functor to $\mathfrak{M}$ (this characterization goes back to [Pa1, §2, Propositon 6], see also [V, Theorem 5.21]). The following definition of a quasi-bialgebra over $\Bbbk$ comes out as a natural extension of this characterization of bialgebras.

Definition 1.7.1 (see [V, §5.5.1.1])**.** A $\Bbbk$-algebra A is a *quasi-bialgebra* if the category of A-modules $\mathfrak{M}_A$ is monoidal and the forgetful functor $\mathcal{U}:\mathfrak{M}_A\to\mathfrak{M}$ is a tensor functor.

As one may expect, this definition turns out to be equivalent to the original one given by Drinfel'd in [Dr2], in the sense of the subsequent result.

Lemma 1.7.2 ([ABM, Theorem 1])**.** *A $\Bbbk$-algebra A is a quasi-bialgebra if and only if there exist two algebra morphisms, the comultiplication $\Delta:A\to A\otimes A$ and the counit $\varepsilon:A\to\Bbbk$, and three distinguished invertible elements $\Phi\in A\otimes A\otimes A$ and $l,r\in A$ such that Φ,l,r satisfy*

$$(A\otimes A\otimes\Delta)(\Phi)(\Delta\otimes A\otimes A)(\Phi)=(1\otimes\Phi)(A\otimes\Delta\otimes A)(\Phi)(\Phi\otimes 1), \tag{1.16a}$$

$$(A \otimes \varepsilon \otimes A)(\Phi) = r^{-1} \otimes l \tag{1.16b}$$

and Δ is coassociative and counital with counit ε up to conjugation by Φ, l and r, that is,

$$\Phi(\Delta \otimes A)(\Delta(a)) = (A \otimes \Delta)(\Delta(a))\Phi, \tag{1.17a}$$
$$(\varepsilon \otimes A)(\Delta(a)) = lal^{-1}, \qquad (A \otimes \varepsilon)(\Delta(a)) = rar^{-1} \tag{1.17b}$$

for all $a \in A$. In such a case, the tensor product of two A-modules is given by the ordinary tensor product $\otimes$ with diagonal action, i.e. $(m \otimes n) \cdot a = \sum m \cdot a_1 \otimes n \cdot a_2$ for all $m \in M$, $n \in N$ A-modules and for all $a \in A$. The unit is $\Bbbk$ via the trivial action, i.e. $1 \cdot a = \varepsilon(a)$. The associativity and unit constraints are given by

$$\alpha_{M,N,P}\,(m \otimes n \otimes p) = (m \otimes n \otimes p) \cdot \Phi^{-1}, \tag{1.18a}$$
$$\lambda_M(1 \otimes m) = m \cdot l \qquad \text{and} \qquad \rho_M(m \otimes 1) = m \cdot r. \tag{1.18b}$$

As a matter of notation, we will write $\Phi = \sum \Phi^1 \otimes \Phi^2 \otimes \Phi^3 = \sum \Psi^1 \otimes \Psi^2 \otimes \Psi^3 = \ldots$ and $\Phi^{-1} = \sum \varphi^1 \otimes \varphi^2 \otimes \varphi^3 = \sum \psi^1 \otimes \psi^2 \otimes \psi^3 = \ldots$. We will also say that Δ is *quasi-coassociative* and we will refer to Φ as the *reassociator*(9) of the quasi-bialgebra.

Remark 1.7.3. In Definition 1.7.1 we considered monoidal structures on $\mathfrak{M}_A$ such that the underlying functor $(\mathcal{U}, \varphi_0, \varphi) : \mathfrak{M}_A \to \mathfrak{M}$ is tensor, but it is not restrictive to focus only on those for which $\mathcal{U} : \mathfrak{M}_A \to \mathfrak{M}$ is in fact strictly tensor. Indeed, if we are given a monoidal structure $(\mathfrak{M}_A, \boxtimes, \mathbb{I}, \boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\rho})$ on $\mathfrak{M}_A$ such that $\mathcal{U} : \mathfrak{M}_A \to \mathfrak{M}$ is a tensor functor, then we can always construct a new tensor product $\otimes$ on $\mathfrak{M}_A$ such that the identity $(\mathfrak{M}_A, \boxtimes, \mathbb{I}, \boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\rho}) \to (\mathfrak{M}_A, \otimes, \Bbbk, \alpha, \lambda, \rho)$ is a monoidal equivalence and the underlying functor $\mathcal{U} : (\mathfrak{M}_A, \otimes, \Bbbk, \alpha, \lambda, \rho) \to (\mathfrak{M}, \otimes, \Bbbk)$ is a strict tensor functor. One just takes $M \otimes N$ for $M, N \in \mathfrak{M}_A$ to be $\mathcal{U}(M) \otimes \mathcal{U}(N)$ endowed with the A-module structure that makes φ an A-linear morphism and $\Bbbk$ endowed with the A-module structure that makes φ_0 and A-linear morphism as well (compare with [Sc2, Remark 5.3]).

Definition 1.7.4. A *morphism of quasi-bialgebras*

$$f : (A, m, u, \Delta, \varepsilon, \Phi, l, r) \to (A', m', u', \Delta', \varepsilon', \Phi', l', r')$$

is a morphism of algebras $f : (A, m, u) \to (A', m', u')$ such that

$$\Delta' \circ f = (f \otimes f) \circ \Delta, \quad \varepsilon' \circ f = \varepsilon, \quad (f \otimes f \otimes f)(\Phi) = \Phi', \quad f(l) = l', \quad f(r) = r'. \tag{1.19}$$

The category of quasi-bialgebras and their morphisms will be denoted by $\mathsf{QBialg}_{\Bbbk}$.

Remark 1.7.5. Even if the results to come may be presented in the full generality for quasi-bialgebras of the form $(A, m, u, \Delta, \varepsilon, \Phi, l, r)$, it is well-known (see e.g. [Ks]) that one can restrict to the simpler case in which $r = 1 = l$. We recall here briefly how, for the sake of completeness.

If $(A, m, u, \Delta, \varepsilon, \Phi, l, r)$ is a quasi-bialgebra and $F \in A \otimes A$ is an invertible element, then we may define a new comultiplication via

$$\Delta_F(a) := F\Delta(a)F^{-1} \tag{1.20}$$

for all $a \in A$ and three distinguished elements

$$\Phi_F := (1 \otimes F)(A \otimes \Delta)(F)\Phi(\Delta \otimes A)(F^{-1})(F^{-1} \otimes 1), \tag{1.21a}$$
$$l_F := (\varepsilon \otimes A)(F)l, \qquad r_F := (A \otimes \varepsilon)(F)r. \tag{1.21b}$$

It turns out that $(A, m, u, \Delta_F, \varepsilon, \Phi_F, l_F, r_F)$ is still a quasi-bialgebra, which we denote by A_F (see [Ks, Proposition XV.3.2] and [Dr2, Remark, p. 1422]). Two quasi-bialgebras $(A, m, u, \Delta, \varepsilon, \Phi, l, r)$ and $(A', m', u', \Delta', \varepsilon', \Phi', l', r')$ are said to be *twist equivalent* if there exists an invertible element $F \in A' \otimes A'$ and an isomorphism of quasi-bialgebras $\varphi\colon A \to A'_F$.

(9) In [Ks, page 369] this is called the Drinfel'd associator.

Given a quasi-bialgebra $(A, m, u, \Delta, \varepsilon, \Phi, l, r)$, one may observe that $\varepsilon(r) = \varepsilon(l)$ and then consider the invertible element $F := \varepsilon(r)(r^{-1} \otimes l^{-1}) \in A \otimes A$. It is clear that A is twist equivalent to $(A_F, m, u, \Delta_F, \varepsilon, \Phi_F, l_F, r_F)$ where

$$\begin{aligned} l_F &= (\varepsilon \otimes A)(F)l = \varepsilon(r^{-1})\varepsilon(r)l^{-1}l = 1, \\ r_F &= (A \otimes \varepsilon)(F)r = r^{-1}\varepsilon(l^{-1})\varepsilon(l)r = 1. \end{aligned}$$

Summing up, every quasi-bialgebra $(A, m, u, \Delta, \varepsilon, \Phi, l, r)$ is twist equivalent to a quasi-bialgebra $(A', m', u', \Delta', \varepsilon', \Phi', l', r')$ such that $r' = l' = 1$.

From a categorical point of view, the property of being twist equivalent for two quasi-bialgebras reflects on their categories of modules as follows. Let $(A, m, u, \Delta, \varepsilon, \Phi, l, r)$ be a quasi-bialgebra and let $F \in A \otimes A$ be an invertible element. Then the triple

$$(R, \varphi_0, \varphi_2) = (\mathsf{Id}_{\mathfrak{M}_A}, \mathsf{Id}_{\Bbbk}, \varphi_2) \colon (\mathfrak{M}_A, \otimes, \Bbbk, \alpha, \lambda, \rho) \to (\mathfrak{M}_{A_F}, \otimes, \Bbbk, \alpha_F, \lambda_F, \rho_F),$$

where

$$\begin{gathered} \varphi_2(M, N) \colon R(M) \otimes R(N) \to R(M \otimes N), \quad (m \otimes n \mapsto (m \otimes n) \cdot F), \\ \alpha_F \colon (M \otimes N) \otimes P \to M \otimes (N \otimes P), \quad ((m \otimes n) \otimes p \mapsto (m \otimes (n \otimes p)) \cdot \Phi_F^{-1}), \\ \lambda_F \colon \Bbbk \otimes M \to M, \quad (1 \otimes m \mapsto m \cdot l_F), \qquad \rho_F \colon M \otimes \Bbbk \to M, \quad (m \otimes 1 \mapsto m \cdot r_F) \end{gathered}$$

defines a monoidal isomorphism (i.e. a monoidal functor that is also an isomorphism of categories). As a consequence, twist equivalent quasi-bialgebras have isomorphic categories of modules.

In particular, it follows that for any quasi-bialgebra $(A, m, u, \Delta, \varepsilon, \Phi, l, r)$ there exists an isomorphism of monoidal categories between $(\mathfrak{M}_A, \otimes, \Bbbk, \alpha, \lambda, \rho)$ and $(\mathfrak{M}_{A_F}, \otimes, \Bbbk, \alpha_F)$ where the unit constraints in the latter are the same constraints of $(\mathfrak{M}, \otimes, \Bbbk)$ and $F = \varepsilon(r)(r^{-1} \otimes l^{-1})$.

To conclude the remark, observe also that if $(A, m, u, \Delta, \varepsilon, \Phi)$ and $(A', m', u', \Delta', \varepsilon', \Phi')$ are twist equivalent quasi-bialgebras such that $l = 1 = r$ and $l' = 1 = r'$, then we have that

$$(\varepsilon \otimes A)(F)l = l_F = \varphi(l') = 1 = \varphi(r') = r_F = (A \otimes \varepsilon)(F)r$$

where $F \in A \otimes A$ is invertible and $\varphi \colon A' \to A_F$ is an isomorphism of quasi-bialgebras. Thus F satisfies $(A \otimes \varepsilon)(F) = 1 = (\varepsilon \otimes A)(F)$. An invertible element F of $A \otimes A$ such that

$$(A \otimes \varepsilon)(F) = 1 = (\varepsilon \otimes A)(F) \tag{1.22}$$

is said to be a *gauge transformation*. Let $(B, m, u, \Delta, \varepsilon)$ be an ordinary bialgebra and consider $\Phi = 1 \otimes 1 \otimes 1$. Then $(B, m, u, \Delta, \varepsilon, \Phi)$ is trivially a quasi-bialgebra. For any gauge transformation F on B, it turns out that B_F is a quasi-bialgebra but in general it is not a bialgebra. Indeed

$$\Phi_F = (1 \otimes F)(A \otimes \Delta)(F)(\Delta \otimes A)(F^{-1})(F^{-1} \otimes 1) \neq 1 \otimes 1 \otimes 1$$

and Δ_F is not coassociative. In such cases, B_F is a non trivial example of quasi-bialgebra.

Henceforth, for the sake of simplicity, we will only consider quasi-bialgebras A with $l = 1 = r$ or, equivalently, monoidal structures on $\mathfrak{M}_A$ such that the forgetful functor is a neutral tensor functor. In particular, relations (1.16b) and (1.17b) now rewrites for all $a \in A$

$$(A \otimes \varepsilon \otimes A)(\Phi) = 1 \otimes 1, \quad \text{and} \tag{1.23a}$$

$$(\varepsilon \otimes A)(\Delta(a)) = a, \qquad (A \otimes \varepsilon)(\Delta(a)) = a. \tag{1.23b}$$

Example 1.7.6 (see [Dr2, Remark 4, page 1424])**.** Let $(A, m, u, \Delta, \varepsilon, \Phi)$ be a quasi-bialgebra. The opposite algebra structure on A induces a new quasi-bialgebra structure on it which is given by $A^{\mathrm{op}} := (A, m^{\mathrm{op}}, u, \Delta, \varepsilon, \Phi^{-1})$. The co-opposite comultiplication $\Delta^{\mathrm{cop}} : x \mapsto \sum x_2 \otimes x_1$ induces another one given by $A^{\mathrm{cop}} := (A, m, u, \Delta^{\mathrm{cop}}, \varepsilon, \sum \varphi^3 \otimes \varphi^2 \otimes \varphi^1)$. Finally, if we combine the two we get a third quasi-bialgebra $A^{\mathrm{op,cop}} := (A, m^{\mathrm{op}}, u, \Delta^{\mathrm{cop}}, \varepsilon, \sum \Phi^3 \otimes \Phi^2 \otimes \Phi^1)$.

Remark 1.7.7. Almost in the same way in which the axioms of quasi-bialgebra are (necessary and) sufficient to have that the category $\mathfrak{M}_A$ of right modules over A is monoidal with neutral tensor underlying functor (see Lemma 1.7.2), the same happens for the monoidal categories $({}_A\mathfrak{M}, \otimes, \Bbbk, {}_A\alpha, l, r)$ and $({}_A\mathfrak{M}_A, \otimes, \Bbbk, {}_A\alpha_A, l, r)$, where

$$({}_A\alpha)_{M,N,P}((m \otimes n) \otimes p) := \Phi \cdot (m \otimes (n \otimes p)),$$
$$({}_A\alpha_A)_{M,N,P}((m \otimes n) \otimes p) := \Phi \cdot (m \otimes (n \otimes p)) \cdot \Phi^{-1},$$

for all $m \in M$, $n \in N$, $p \in P$ and all M, N, P A-(bi)modules.

Dually to quasi-bialgebras, we have the notion of coquasi-bialgebras. These can be described as those coalgebras whose category of comodules is monoidal with (neutral) tensor underlying functor (see e.g. [Sc4, §2.3]), but for the sake of brevity we will give here the algebraic definition directly. To this aim, recall that if C is a $\Bbbk$-coalgebra and A a $\Bbbk$-algebra then $\mathsf{Hom}_{\Bbbk}(C, A)$ turns out to be a monoid with the so-called *convolution product*

$$(f * g)(c) = (m_A \circ (f \otimes g) \circ \Delta_C)(c) = \sum f(c_1)g(c_2) \tag{1.24}$$

for all $f, g \in \mathsf{Hom}_{\Bbbk}(C, A)$ and $c \in C$ (see also (1.15)).

Definition 1.7.8 (cf. [Mj3, §1] and [Sc1, §2])**.** A *coquasi-bialgebra* $(H, m, u, \Delta, \varepsilon, \omega)$ (also named *dual quasi-bialgebra*) is a coassociative and counital coalgebra (H, Δ, ε) endowed with a multiplication $m : H \otimes H \to H$ and a unit $u : \Bbbk \to H$, which are coalgebra morphisms, and with a convolution invertible linear map $\omega : H \otimes H \otimes H \to \Bbbk$. These are required to satisfy

$$\big(\omega \circ (H \otimes H \otimes m)\big) * \big(\omega \circ (m \otimes H \otimes H)\big) = (\varepsilon \otimes \omega) * \big(\omega \circ (H \otimes m \otimes H)\big) * (\omega \otimes \varepsilon), \tag{1.25a}$$
$$\omega(h \otimes 1_H \otimes k) = \varepsilon(h)\varepsilon(k) \tag{1.25b}$$
$$\big(m \circ (H \otimes m)\big) * \omega = \omega * \big(m \circ (m \otimes H)\big), \tag{1.25c}$$
$$m(1_H \otimes h) = h, \qquad m(h \otimes 1_H) = h, \tag{1.25d}$$

for all $h, k \in H$, where $*$ denotes the convolution product[(10)] and $1_H := u(1_{\Bbbk})$. We say that m is *quasi-associative* and we refer to ω as the *reassociator* of the coquasi-bialgebra. A *morphism of coquasi-bialgebras*

$$f : (H, m, u, \Delta, \varepsilon, \omega) \to (H', m', u', \Delta', \varepsilon', \omega')$$

is a coalgebra homomorphism $f : (H, \Delta, \varepsilon) \to (H', \Delta', \varepsilon')$ such that

$$m' \circ (f \otimes f) = f \circ m, \qquad f \circ u = u', \qquad \omega' \circ (f \otimes f \otimes f) = \omega. \tag{1.26}$$

The category of coquasi-bialgebras and their morphisms will be denoted by $\mathsf{CQBialg}_{\Bbbk}$.

Remark 1.7.9. As we mentioned, for H a coquasi-bialgebra the categories ${}^H\mathfrak{M}$, $\mathfrak{M}^H$ and ${}^H\mathfrak{M}^H$ of left, right and bicomodules over H respectively are monoidal categories with neutral tensor underlying functor. In particular, we recall that the associativity constraints are given by

$$\begin{gathered}{}^H\alpha_{X,Y,Z}((x \otimes y) \otimes z) = \omega^{-1}(x_{-1} \otimes y_{-1} \otimes z_{-1})x_0 \otimes (y_0 \otimes z_0), \\ \alpha^H_{X,Y,Z}((x \otimes y) \otimes z) = x_0 \otimes (y_0 \otimes z_0)\omega(x_1 \otimes y_1 \otimes z_1), \\ {}^H\alpha^H_{X,Y,Z}((x \otimes y) \otimes z) = \omega^{-1}(x_{-1} \otimes y_{-1} \otimes z_{-1})x_0 \otimes (y_0 \otimes z_0)\omega(x_1 \otimes y_1 \otimes z_1),\end{gathered} \tag{1.27}$$

for all X, Y, Z suitable (bi)comodules and all $x \in X, y \in Y, z \in Z$. Notice that every morphism of coquasi-bialgebras $f : H \to H'$ induces a strict monoidal functor ${}^f\mathfrak{M} : {}^H\mathfrak{M} \to {}^{H'}\mathfrak{M}$, which is given by the assignments

$${}^f\mathfrak{M}(X, \rho_X : X \to H \otimes X) = (X, (f \otimes X) \circ \rho_X) \qquad \text{and} \qquad {}^f\mathfrak{M}(\gamma : X \to Y) = \gamma.$$

The same happens, dually, for quasi-bialgebra morphisms.

(10) In formula (1.25c) the product $*$ stands for e.g. $\Big(\big(m \circ (H \otimes m)\big) * \omega\Big)(x \otimes y \otimes z) = \sum x_1(y_1z_1)\omega(x_2 \otimes y_2 \otimes z_2)$ for all $x, y, z \in H$, which resembles closely the classical convolution product of linear morphisms from a coalgebra to an algebra. This is why, by a slightly abuse of notation and terminology, we referred to it as a convolution product and we used the same symbol.

For the sake of simplifying the exposition, we will often leave the structure maps out when referring to quasi and coquasi-bialgebras, i.e. we will simply say that A is a quasi-bialgebra or that H is a coquasi-bialgebra, without explicitly mentioning Δ, ε, m, u, ω or Φ.

Remark 1.7.10. In spite of the fact that quasi and coquasi-bialgebras are dual notions from a categorical point of view, to construct a duality between them seems to be not an easy task. This argument will be treated more deeply in Section 2.4, where a duality between quasi-bialgebras and a suitable subcategory of coquasi-bialgebras will be provided. As a consequence, some results which hold in one framework can be dualized to the other and some others no. We will exhibit a positive example in Section 2.2. For the moment, we observe that if A is a finite-dimensional quasi-bialgebra over a field $\Bbbk$, then $A^* = \mathsf{Hom}_{\Bbbk}(A, \Bbbk)$ is a coquasi-bialgebra and conversely. The structure maps can be obtained easily by duality.

1.8 The Tannaka-Kreĭn reconstruction principle

Tannaka-Kreĭn reconstruction process traces its line back to the early works of T. Tannaka [Tn] and M. G. Kreĭn [Kn] and originally it addressed questions such as if it is possible to recover a (compact topological) group from its finite-dimensional representations (the Reconstruction Problem) or which $\Bbbk$-linear categories are (equivalent to) categories of representations of a group (the Recognition Problem). The extension of the duality obtained by them to affine group schemes and the introduction of the notions of Tannakian categories and fibre functors goes back to Grothendieck's school and more precisely to the works of N. Saavedra Rivano [Riv], P. Deligne and J. S. Milne [DM, Dl]. Nowadays there exist many extensions and variations of these results, by changing the categories involved or the properties of the "forgetful" functors, which allow one to reconstruct more general objects such as coalgebras, bialgebras, (non-commutative) Hopf algebras and (some of) their generalizations. We recall here shortly the main steps of the Tannaka-Kreĭn reconstruction as we see (and we plan to use) it in this thesis and, in particular, we will discuss its application to coquasi-bialgebras. We refer the reader to [Sc7] for an account on the subject. Our approach will follow closely [Mj3], [Sc7] and [U]. See also [DM, HR, JS, St, V] for further details.

Broadly speaking, Tannaka-Kreĭn reconstruction is concerned with constructing an object H in a suitably chosen category $\mathcal{A}$ with certain specific properties, out of the datum of a functor $\boldsymbol{\omega} : \mathcal{M} \to \mathcal{A}$ (which can be considered as some kind of "fibre" or "forgetful" or "underlying" functor) from another category $\mathcal{M}$ with some additional properties that are preserved by $\boldsymbol{\omega}$. Moreover, this object has to be the best one we may construct, in the sense that it has to satisfy a certain universal property (we will try to make this rather informal sentences more formal in what follows).

To be more precise, let $\mathcal{A}$ be a fixed braided monoidal category which is cocomplete and such that the tensor product is compatible with arbitrary colimits in each argument. Denote by $\mathcal{A}_0$ its subcategory of (left or right) dualizable objects. Consider another category $\mathcal{M}$ with a functor $\boldsymbol{\omega} : \mathcal{M} \to \mathcal{A}_0$. Out of these data, one can construct a coalgebra H in $\mathcal{A}$ such that every object $\boldsymbol{\omega}(X)$ becomes a H-comodule and H is universal in the sense that for any other coalgebra C such that $\boldsymbol{\omega}(X)$ is a C-comodule for every object X in $\mathcal{M}$ and $\boldsymbol{\omega}(f)$ is a C-colinear morphism for every arrow f in $\mathcal{M}$, there exists a unique coalgebra morphism $\varphi : H \to C$ such that the C-comodule structure on $\boldsymbol{\omega}(X)$ is induced by the H-comodule one via φ.

Moreover, the "reconstructed" coalgebra reflects many of the properties of the starting datum $\boldsymbol{\omega} : \mathcal{M} \to \mathcal{A}_0$. For example, if $\mathcal{M}$ and $\boldsymbol{\omega}$ are monoidal, then H becomes a bialgebra. If $\mathcal{M}$ is also rigid, then H becomes a Hopf algebra. If $\mathcal{M}$ is monoidal and $\boldsymbol{\omega}$ is only a neutral tensor functor, we recall in this section how H becomes a coquasi-bialgebra.

Example 1.8.1. There are different ways to construct the above universal object and in what follows we will see some examples, but before going into the details of the procedure we are interested in, let us sketch some classical cases of Tannaka-Kreĭn reconstruction.

Assume that $\Bbbk$ is a field, that $\mathcal{M}$ is a rigid abelian $\Bbbk$-linear (i.e. enriched in vector spaces over $\Bbbk$) symmetric monoidal category such that $\mathsf{End}(\mathbb{I}) = \Bbbk$, and assume that $\boldsymbol{\omega} : \mathcal{M} \to \mathfrak{M}_f$ (the category of finite-dimensional vector spaces over $\Bbbk$) is an exact faithful $\Bbbk$-linear (i.e. the maps between hom-sets induced by ω are $\Bbbk$-linear) symmetric monoidal functor. In such a case $\mathcal{M}$ is said to be a *neutral Tannakian category* and ω is called a *fibre functor* [DM, Definition 2.19]. For every commutative $\Bbbk$-algebra A, consider the (monoidal) functor $\omega \otimes A : (\mathcal{M}, \otimes, \mathbb{I}, \alpha, \lambda, \rho) \to (\mathfrak{M}_A, \otimes_A, A)$ which sends every X to $\omega(X) \otimes A$. It turns out that the functor $\mathsf{CAlg}_\Bbbk \to \mathsf{Grp}$ which associates any commutative $\Bbbk$-algebra A with the group $\mathsf{Aut}^{\otimes}(\omega \otimes A)$ of monoidal natural automorphisms of $\omega \otimes A$ is representable and it is represented by a commutative Hopf algebra H_ω (i.e. an affine group scheme). Furthermore, ω factors through a $\Bbbk$-linear equivalence between $\mathcal{M}$ and the category $\mathfrak{M}_f^H$ of H-comodules whose underlying vector space is finite-dimensional [DM, Theorem 2.11].

1) If $\mathcal{M}$ is already the category of finite-dimensional (right) comodules over a commutative Hopf algebra A (equivalently, the category of representations of the affine group scheme represented by A) then $\mathcal{M}$ is a neutral Tannakian category and the forgetful functor $\mathcal{M} \to \mathfrak{M}_f$ is a fibre functor. In this case, $H_\omega \cong A$ as Hopf algebras.

2) Let $(\Bbbk, (-)')$ be a differential field (i.e. a field $\Bbbk$ with a derivation $(-)' : \Bbbk \to \Bbbk$). Assume that its field of constants $\mathfrak{c}$ (those $c \in \Bbbk$ such that $c' = 0$) does not coincides with $\Bbbk$ and it is algebraically closed of characteristic 0. The category $\mathsf{Diff}_\Bbbk$ of differential modules over $\Bbbk$ is defined as the category whose objects are pairs (M, ∂_M) composed by a finite-dimensional $\Bbbk$-vector space M and an additive map $\partial_M : M \to M$ (the differential) such that $\partial_M(km) = k'm + k\partial_M(m)$ and whose morphisms are $\Bbbk$-linear morphisms $f : M \to N$ that commute with the differentials. It turns out that $\mathsf{Diff}_\Bbbk$ is a neutral Tannakian category with a fibre functor to finite-dimensional $\mathfrak{c}$-vector spaces and hence it is represented by an affine group scheme over $\mathfrak{c}$ (see [Dl] and [SP, §§10.1, B.23]). The latter is called the *universal Galois group* of $\mathsf{Diff}_\Bbbk$ and, in fact, it can be seen as the group of $\Bbbk$-linear automorphisms of a special differential ring UnivR, called the *universal Picard-Vessiot ring* of $\mathsf{Diff}_\Bbbk$. This procedure can be performed for every subcategory $\mathcal{C}$ of $\mathsf{Diff}_\Bbbk$ which is still Tannakian. Given a differential module (M, ∂), of particular interest is the full subcategory $\{\{M\}\}$ of $\mathsf{Diff}_\Bbbk$ whose objects are the sub-quotients (i.e. quotients P/Q with $Q \subseteq P \subseteq M$) of finite direct sums of $M \otimes \cdots \otimes M \otimes M^* \otimes \cdots \otimes M^*$, that is, the tensor product of n copies of M and of m copies of M^*. This turns out to be a neutral Tannakian category whose associated group scheme is exactly what is called the *differential Galois group* of (M, ∂) [SP, Theorem 2.33].

Let us pick up now the threads of our argument. For our purposes, we are interested in case $\mathcal{A} = \mathfrak{M}$, the monoidal category of modules over a commutative ring $\Bbbk$, and $\mathcal{A}_0 = \mathfrak{M}_f$, its full subcategory of finitely generated and projective ones. Let $\mathcal{M}$ be an essentially small category equipped with a functor $\omega : \mathcal{M} \to \mathfrak{M}_f$ from $\mathcal{M}$ into the category of finitely generated and projective $\Bbbk$-modules. For every V in $\mathfrak{M}$, denote by $\mathsf{Nat}(\omega, V \otimes \omega)$ the set of natural transformations between ω seen as a functor from $\mathcal{M}$ to $\mathfrak{M}$ and the functor $V \otimes \omega : \mathcal{M} \to \mathfrak{M}$ mapping every object X in $\mathcal{M}$ to the object $V \otimes \omega(X)$ in $\mathfrak{M}$. Then the functor $\mathsf{Nat}(\omega, - \otimes \omega) : \mathfrak{M} \to \mathsf{Set}$ is represented by a coalgebra H_ω, also denoted by $\underline{\mathrm{coend}}(\omega)$ or by $\int^X \omega(X)^* \otimes \omega(X)$, which is called the *coendomorphism coalgebra of* ω (or of $(\mathcal{M}, \omega)$ if some confusion may arise). That means that we have a natural isomorphism

$$\vartheta : \mathsf{Hom}_\Bbbk(H_\omega, -) \cong \mathsf{Nat}(\omega, - \otimes \omega). \tag{1.28}$$

Since ω is fixed, we may write H instead of H_ω. As a vector space, H is defined to be the coend of the functor $\omega \otimes \omega^* = \otimes(\omega \times \omega^*)$ from $\mathcal{M} \times \mathcal{M}^{\mathrm{op}}$ to $\mathfrak{M}$ (see e.g. [ML, §IX.6] for details about the coend construction). It is endowed with a coalgebra structure as follows. Consider the natural transformation $\delta^H := \vartheta_H(\mathsf{Id}_H) : \omega \to H \otimes \omega$, which we will denote by δ when no confusion may arise. The comultiplication Δ and the counit ε are the unique linear maps such that $\vartheta_{H \otimes H}(\Delta) = (H \otimes \delta) \circ \delta$ and $\vartheta_\Bbbk(\varepsilon) = \mathsf{Id}_\omega$. Naturality of ϑ implies that for all V in $\mathfrak{M}$ and $f \in \mathsf{Hom}_\Bbbk(H, V)$

$$\vartheta_V(f) = (\vartheta_V \circ \mathsf{Hom}_\Bbbk(H, f))(\mathsf{Id}_H) = (\mathsf{Nat}(\omega, f \otimes \omega) \circ \vartheta_H)(\mathsf{Id}_H) = (f \otimes \omega) \circ \delta. \tag{1.29}$$

Therefore, $\vartheta_V(f)_X = (f \otimes \omega(X)) \circ \delta_X$ for every X in $\mathcal{M}$. Furthermore, from

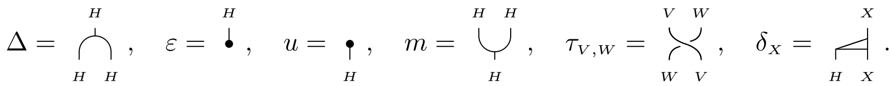
$$(\Delta \otimes \omega(X)) \circ \delta_X = \vartheta_{H \otimes H}(\Delta)_X = (H \otimes \delta_X) \circ \delta_X \tag{1.30a}$$

$$\text{and} \quad (\varepsilon \otimes \boldsymbol{\omega}(X)) \circ \delta_X = \vartheta_{\Bbbk}(\varepsilon)_X = \mathsf{Id}_{\boldsymbol{\omega}(X)} \tag{1.30b}$$

we deduce that every $\Bbbk$-module $\boldsymbol{\omega}(X)$ is automatically endowed with an H-comodule structure $(\boldsymbol{\omega}(X), \delta_X)$. Thus $\boldsymbol{\omega}$ factorizes through the obvious forgetful functor ${}^H\mathcal{U} : {}^H\mathfrak{M} \to \mathfrak{M}$, i.e. there is a functor $\boldsymbol{\omega}^H : \mathcal{M} \to {}^H\mathfrak{M}$ such that ${}^H\mathcal{U} \circ \boldsymbol{\omega}^H = \boldsymbol{\omega}$. Moreover, H enjoys the following universal property: if C is another coalgebra and if $\mathcal{V} : \mathcal{M} \to {}^C\mathfrak{M}$ is another functor such that ${}^C\mathcal{U} \circ \mathcal{V} = \boldsymbol{\omega}$, then there is a unique morphism of coalgebras $\epsilon : H \to C$, given as the image of the C-coaction $\delta^C \in \mathsf{Nat}(\boldsymbol{\omega}, C \otimes \boldsymbol{\omega})$ in $\mathsf{Hom}_{\Bbbk}(H, C)$ via (1.28), that induces a functor ${}^{\epsilon}\mathfrak{M} : {}^H\mathfrak{M} \to {}^C\mathfrak{M}$ such that ${}^{\epsilon}\mathfrak{M} \circ \boldsymbol{\omega}^H = \mathcal{V}$.

Example 1.8.2. Assume that $\Bbbk$ is a field, that $\mathcal{M}$ is already the category ${}^C\mathfrak{M}_f$ of finite-dimensional left comodules over a $\Bbbk$-coalgebra C and that $\boldsymbol{\omega}$ is already the forgetful functor $\mathcal{U}$. Then the canonical morphism $\epsilon : H \to C$ turns out to be an isomorphism of coalgebras (see [Sc7, Lemma 2.2.1]). We point out, however, that this is not always the case if $\Bbbk$ is not a field.

Notation 1.8.3. In this thesis we will perform some computations in terms of braided diagrams in the category of $\Bbbk$-modules, since we believe that they may increase its readability. We will adopt the following notations



where for every pair of objects V, W in $\mathfrak{M}$, $\tau_{V,W} : V \otimes W \to W \otimes V$ denotes the natural transformation acting as $\tau_{V,W}(v \otimes w) = w \otimes v$. We will also omit the functor $\boldsymbol{\omega}$ in braided diagrams.

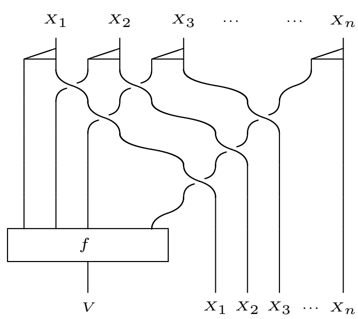
Assume now that $(\mathcal{M}, \boxtimes, \mathbb{I}, \alpha, \lambda, \rho)$ is monoidal[11] and that $\boldsymbol{\omega}$ is a tensor functor. This means that in $\mathfrak{M}$ we have a family of isomorphisms $\varphi_{X,Y} : \boldsymbol{\omega}(X) \otimes \boldsymbol{\omega}(Y) \to \boldsymbol{\omega}(X \boxtimes Y)$ which is natural in both components and an isomorphism $\varphi_0 : \Bbbk \to \boldsymbol{\omega}(\mathbb{I})$ compatible with the left and right unit constraints, where the compatibility is expressed by the commutativity of the diagrams (1.2). Following [Mj3], H can be endowed with a coquasi-bialgebra structure as follows. The functors $\mathsf{Nat}(\boldsymbol{\omega}^n, - \otimes \boldsymbol{\omega}^n)$ for $n \geq 2$ are representable, where $\boldsymbol{\omega}^n : \mathcal{M}^n \to \mathfrak{M}_f$ maps every n-uple of objects $(X_1, \ldots, X_n)$ to the tensor product $\boldsymbol{\omega}(X_1) \otimes \cdots \otimes \boldsymbol{\omega}(X_n)$. They are represented by the n-fold tensor product $H^{\otimes n}$, which means that we have a family of isomorphisms

$$\vartheta_V^n : \mathsf{Hom}_{\Bbbk}(H^{\otimes n}, V) \cong \mathsf{Nat}(\boldsymbol{\omega}^n, V \otimes \boldsymbol{\omega}^n)$$

for all $n \geq 1$ and for all V in $\mathfrak{M}$, which in natural in V. As a matter of notation and if no confusion may arise, given an arrow $f : X \to Y$ and other two objects X and Y in a category $\mathcal{M}$ with tensor product $\boxtimes$, we will eventually write simply f instead of $X \boxtimes f \boxtimes Y$, leaving the identity morphisms out and understanding that f has to be applied to the suitable tensorand. With this convention we have explicitly

$$\vartheta_V^n(f)_{X_1,\ldots,X_n} = (f \otimes \boldsymbol{\omega}^n(X_1, \ldots, X_n)) \circ \tau_{\boldsymbol{\omega}^{n-1}(X_1,\ldots,X_{n-1}),H} \circ \cdots \circ \tau_{\boldsymbol{\omega}(X_1),H} \circ (\delta_{X_1} \otimes \cdots \otimes \delta_{X_n})$$

(see [Mj3] and [Sc7, Lemma 2.3.6]). Graphically, it looks like

$\vartheta_V^n(f)_{X_1,\ldots,X_n} =$ 

[11] We denoted the monoidal structure in $\mathcal{M}$ by $\boxtimes$ in order to avoid confusion with $\otimes$, the tensor product over $\Bbbk$.

As a consequence of the representability of the functors $\mathsf{Nat}\left(\boldsymbol{\omega}^n, - \otimes \boldsymbol{\omega}^n\right)$, we can define the multiplication $m : H \otimes H \to H$ as the unique map such that

$$(H \otimes \varphi_{X,Y}) \circ \vartheta_H^2(m)_{X,Y} = \delta_{X \boxtimes Y} \circ \varphi_{X,Y} \tag{1.31}$$

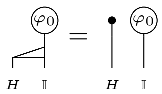
for all X, Y in $\mathcal{M}$. It is a coalgebra morphism (see [Mj3, Lemma 2.4]). In turn, the reassociator $\omega \in (H \otimes H \otimes H)^*$ is the unique map such that for all X, Y, Z in $\mathcal{M}$

$$\varphi_{X \boxtimes Y, Z} \circ (\varphi_{X,Y} \otimes \boldsymbol{\omega}(Z)) \circ \vartheta_{\Bbbk}^3(\omega)_{X,Y,Z} = \boldsymbol{\omega}\left(\alpha_{X,Y,Z}^{-1}\right) \circ \varphi_{X, Y \boxtimes Z} \circ (\boldsymbol{\omega}(X) \otimes \varphi_{Y,Z}). \tag{1.32}$$

The unit of H is the unique morphism $u : \Bbbk \to H$ such that

$$(H \otimes \varphi_0) \circ r_H^{-1} \circ u = \delta_{\mathbb{I}} \circ \varphi_0. \tag{1.33}$$

Graphically, this becomes



(1.34)

Observe that, since unitality is unrelated with the coherence condition (1.3) with the associativity constraint, the unit is the same that has been constructed from ordinary monoidal functors in [U, page 255] and in [Sc7, Corollary 2.3.7], for example.

Remark 1.8.4. If C is a coalgebra endowed with two morphisms of coalgebras $\mu : C \otimes C \to C$ and $\eta : \Bbbk \to C$, then the tensor product $M \otimes N$ of two left C-comodules (M, δ_M), (N, δ_N) is still a C-comodule with $\mathrm{coact}_{M \otimes N} = (\mu \otimes M \otimes N) \circ \tau_{M,H} \circ (\delta_M \otimes \delta_N)$, and $\Bbbk$ is a C-comodule with $\mathrm{coact}_{\Bbbk} = r_C^{-1} \circ \eta$. By definition of m, for every X, Y in $\mathcal{M}$ we have

$$\vartheta_H^2(m)_{X,Y} = (m \otimes \boldsymbol{\omega}(X) \otimes \boldsymbol{\omega}(Y)) \circ \tau_{\boldsymbol{\omega}(X),H} \circ (\delta_X \otimes \delta_Y) = \mathrm{coact}_{\boldsymbol{\omega}(X) \otimes \boldsymbol{\omega}(Y)}.$$

Therefore, relation (1.31) becomes $\delta_{X \boxtimes Y} \circ \varphi_{X,Y} = (H \otimes \varphi_{X,Y}) \circ \mathrm{coact}_{\boldsymbol{\omega}(X) \otimes \boldsymbol{\omega}(Y)}$ so that all the morphisms $\varphi_{X,Y} : \boldsymbol{\omega}(X) \otimes \boldsymbol{\omega}(Y) \to \boldsymbol{\omega}(X \boxtimes Y)$ turn out to be comodule morphisms. Moreover, from (1.33) it follows that φ_0 is H-colinear as well, as

$$\delta_{\mathbb{I}} \circ \varphi_0 = (H \otimes \varphi_0) \circ \mathrm{coact}_{\Bbbk}. \tag{1.35}$$

Thus, we see that the comodule structures on the tensor product $X \boxtimes Y$ and on the unit object $\mathbb{I}$ are compatible with the monoidal structure defined on the category ${}^H\mathfrak{M}$, in the sense that the functor $\boldsymbol{\omega}^H : \mathcal{M} \to {}^H\mathfrak{M}$ is a tensor functor. Furthermore, by definition of ω, for all $x \in \boldsymbol{\omega}(X)$, $y \in \boldsymbol{\omega}(Y)$ and $z \in \boldsymbol{\omega}(Z)$,

$$\vartheta_{\Bbbk}^3(\omega)_{X,Y,Z}(x \otimes (y \otimes z)) = \sum \omega(x_{-1} \otimes y_{-1} \otimes z_{-1})(x_0 \otimes y_0) \otimes z_0.$$

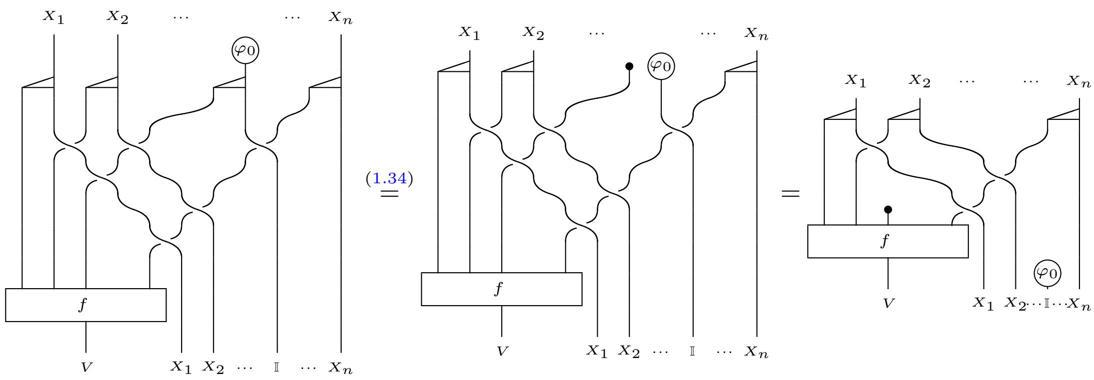
Once proven that H is a coquasi-bialgebra, this is exactly the (inverse of the) associativity constraint in the category of left H-comodules (cf. relation (1.27)). Relation (1.32) encodes then the fact that the functor $\boldsymbol{\omega}^H : \mathcal{M} \to {}^H\mathfrak{M}$ assigning to every object X in $\mathcal{M}$ the H-comodule $(\boldsymbol{\omega}(X), \delta_X)$ is a monoidal functor, differently from $\boldsymbol{\omega} : \mathcal{M} \to \mathfrak{M}$.

Lemma 1.8.5. *The reassociator ω and the multiplication m are unital, in the sense that conditions (1.25b) and (1.25d) are satisfied*[12].

Proof. Denote temporarily by $H^{(i)}$ the i-th tensorand in $H^{\otimes n}$ for some $n \geq 1$ and some $1 \leq i \leq n$ and by $\varphi_0^{(i)}$ the morphism φ_0 applied in the i-th position of a tensor product. The following

[12] In [Mj3] there's no explicit reference to the unitality of the multiplication or of the reassociator. This is the reason why we added this lemma.

computation



shows that if $f : H^{\otimes n+1} \to V$ is an arrow in $\mathfrak{M}$ for V a $\Bbbk$-module, then

$$\vartheta_V^{n+1}(f)_{X_1,\ldots,\mathbb{I},X_i,\ldots,X_n} \circ \varphi_0^{(i)} = \varphi_0^{(i+1)} \circ \vartheta_V^n\left(f \circ u_{H^{(i)}}\right)_{X_1,\ldots,X_n}. \tag{1.36}$$

By omitting the constraints l and r, the relations in (1.2) become

$$\boldsymbol{\omega}(\lambda_X) \circ \varphi_{\mathbb{I},X} \circ (\varphi_0 \otimes \boldsymbol{\omega}(X)) = \mathsf{Id}_{\boldsymbol{\omega}(X)} = \boldsymbol{\omega}(\rho_X) \circ \varphi_{X,\mathbb{I}} \circ (\boldsymbol{\omega}(X) \otimes \varphi_0),$$

so that Equation (1.36) can be rewritten as

$$\boldsymbol{\omega}(\lambda_{X_i}) \circ \varphi_{\mathbb{I},X_i} \circ \vartheta_V^{n+1}(f)_{X_1,\ldots,\mathbb{I},X_i,\ldots,X_n} = \vartheta_V^n\left(f \circ u_{H^{(i)}}\right)_{X_1,\ldots,X_n} \circ \boldsymbol{\omega}(\lambda_{X_i}) \circ \varphi_{\mathbb{I},X_i} \quad \text{or} \tag{1.37a}$$

$$\boldsymbol{\omega}\left(\rho_{X_{i-1}}\right) \circ \varphi_{X_{i-1},\mathbb{I}} \circ \vartheta_V^{n+1}(f)_{X_1,\ldots,X_{i-1},\mathbb{I},\ldots,X_n} = \vartheta_V^n\left(f \circ u_{H^{(i)}}\right)_{X_1,\ldots,X_n} \circ \boldsymbol{\omega}(\lambda_{X_i}) \circ \varphi_{\mathbb{I},X_i} \tag{1.37b}$$

(notice that Equation (1.37b) holds only if $1 < i < n$, but as we will see this is the only case we are interested in). By naturality of $\vartheta_H(\mathsf{Id}_H)$, one checks that

$$\begin{aligned}\vartheta_H(m \circ (u \otimes H))_X \circ \boldsymbol{\omega}(\lambda_X) &\overset{(1.37a)}{=} (H \otimes \boldsymbol{\omega}(\lambda_X)) \circ (H \otimes \varphi_{\mathbb{I},X}) \circ \vartheta_H^2(m)_{\mathbb{I},X} \circ \varphi_{\mathbb{I},X}^{-1} \\ &\overset{(1.31)}{=} (H \otimes \boldsymbol{\omega}(\lambda_X)) \circ \vartheta_H(\mathsf{Id}_H)_{\mathbb{I}\boxtimes X} = \vartheta_H(\mathsf{Id}_H)_X \circ \boldsymbol{\omega}(\lambda_X)\end{aligned}$$

for all X in $\mathcal{M}$, which means that $m \circ (u \otimes H) = \mathsf{Id}_H$ as desired. Unitality on the other side may be checked similarly. Let us conclude with the unitality of ω. As above, we may compute

$$\begin{aligned}&\vartheta_\Bbbk^2(\omega \circ (H \otimes u \otimes H))_{X,Y} \\ &\overset{(1.37b)}{=} (\boldsymbol{\omega}(\rho_X) \otimes \boldsymbol{\omega}(Y)) \circ (\varphi_{X,\mathbb{I}} \otimes \boldsymbol{\omega}(Y)) \circ \vartheta_\Bbbk^3(\omega)_{X,\mathbb{I},Y} \circ \left(\boldsymbol{\omega}(X) \otimes \varphi_{\mathbb{I},Y}^{-1}\right) \circ \left(\boldsymbol{\omega}(X) \otimes \boldsymbol{\omega}\left(\lambda_Y^{-1}\right)\right) \\ &\overset{(1.32)}{=} (\boldsymbol{\omega}(\rho_X) \otimes \boldsymbol{\omega}(Y)) \circ \varphi_{X\boxtimes\mathbb{I},Y}^{-1} \circ \boldsymbol{\omega}\left(\alpha_{X,\mathbb{I},Y}^{-1}\right) \circ \varphi_{X,\mathbb{I}\boxtimes Y} \circ \left(\boldsymbol{\omega}(X) \otimes \boldsymbol{\omega}\left(\lambda_Y^{-1}\right)\right) \\ &= \varphi_{X,Y}^{-1} \circ \boldsymbol{\omega}(\rho_X \boxtimes Y) \circ \boldsymbol{\omega}\left(\alpha_{X,\mathbb{I},Y}^{-1}\right) \circ \boldsymbol{\omega}\left(X \boxtimes \lambda_Y^{-1}\right) \circ \varphi_{X,Y} \\ &\overset{(1.1)}{=} \mathsf{Id}_{\boldsymbol{\omega}(X)\otimes\boldsymbol{\omega}(Y)} = \vartheta_\Bbbk^2(m_\Bbbk \circ (\varepsilon \otimes \varepsilon))\end{aligned}$$

for all X, Y in $\mathcal{M}$. Thus, $\omega(x \otimes 1_H \otimes y) = \varepsilon(x)\varepsilon(y)$ for all $x, y \in H$. $\square$

Summing up, we have the following theorem (compare with [Mj3, Theorem 2.2]).

Theorem 1.8.6 (Reconstruction Theorem for coquasi-bialgebras). *Let $(\mathcal{M}, \boxtimes, \mathbb{I}, \alpha, \lambda, \rho)$ be an essentially small monoidal category and let $(\boldsymbol{\omega}, \varphi_0, \varphi) : \mathcal{M} \to \mathfrak{M}_f$ be a tensor functor. Then there is a coquasi-bialgebra H, unique up to isomorphism, universal with the property that $\boldsymbol{\omega}$ factorizes as a monoidal functor $\boldsymbol{\omega}^H : \mathcal{M} \to {}^H\mathfrak{M}$ followed by the forgetful functor. Universal means that if H' is another coquasi-bialgebra and $\boldsymbol{\omega}^{H'} : \mathcal{M} \to {}^{H'}\mathfrak{M}$ a functor as above then there is a unique* *map of coquasi-bialgebras $\epsilon : H \to H'$ inducing a functor ${}^{\epsilon}\mathfrak{M} : {}^{H}\mathfrak{M} \to {}^{H'}\mathfrak{M}$ such that the following commutes*

$$\begin{tikzcd} & \mathcal{M} \arrow[dl, "\omega^H"'] \arrow[dr, "\omega^{H'}"] & \\ {}^{H}\mathfrak{M} \arrow[rr, "{}^{\epsilon}\mathfrak{M}"'] & & {}^{H'}\mathfrak{M} \end{tikzcd}$$

Example 1.8.7. Again let $\Bbbk$ be a field and let B be a coquasi-bialgebra over $\Bbbk$. Consider the monoidal category of finite-dimensional left B-comodules ${}^{B}\mathfrak{M}_f$ together with the forgetful functor $\mathcal{U} : {}^{B}\mathfrak{M}_f \to \mathfrak{M}_f$, which is a tensor functor. As in Example 1.8.2, we may choose B itself as a representing object for $\mathsf{Nat}\,(\mathcal{U}, - \otimes \mathcal{U})$ and in this way we recover its comultiplication and counit. Moreover, since the original m, u and ω of B satisfy (1.31), (1.32) and (1.33) respectively, we recover these as well as the structure maps provided by Theorem 1.8.6.

Remark 1.8.8. Before concluding, it is important for us to summarily mention the work of Bruguières [Brg], in which he develops a slightly different Tannakian theory with respect to the one presented here. For $\Bbbk$ a commutative ring and B a non-commutative $\Bbbk$-algebra, he considers categories $\mathcal{M}$ together with a functor $\boldsymbol{\omega} : \mathcal{M} \to \mathsf{proj}(B)$ landing into the category $\mathsf{proj}(B)$ of finitely generated and projective right B-modules. Then he proves that the functor $\mathsf{Nat}(\boldsymbol{\omega}, \boldsymbol{\omega} \otimes_B -) : {}_B\mathfrak{M}_B \to \mathsf{Set}$ is representable and it is represented by a B-bimodule $L_{\Bbbk}(\boldsymbol{\omega})$ that naturally inherits a B-coring structure [Brg, §4]. In case B is commutative, if $\mathcal{M}$ is a (left and right rigid) monoidal category and $\boldsymbol{\omega}$ is a monoidal functor, then $L_{\Bbbk}(\boldsymbol{\omega})$ becomes a $(B \otimes B)$-algebra and a (Hopf) bialgebroid over B in the sense of [Brg, §7]. Moreover, this process characterizes transitive Hopf algebroids in the sense of [Brg, Theorem 7.1]. The Tannaka reconstruction process in [EKG] and previously mentioned in the introduction draws its inspiration from this work.

References

Most of the results in this chapter can be found in textbooks on (categorical) algebra or Hopf algebras. For Sections from 1.1 up to 1.5 we referred to [AHS, AMa, Brx2, EM, ML, Pa3, PP], for example. For Section 1.6 we looked at [Be] and in writing Sections 1.7 and 1.8 we found inspiration in [Ks, Sc7, St].

Chapter 2

Preantipodes for quasi and coquasi-bialgebras

This second chapter is devoted to the study of (co)quasi-bialgebras and preantipodes. Broadly speaking, preantipodes are distinguished endomorphisms that characterize those (co)quasi-bialgebras whose (co)quasi-Hopf bi(co)modules satisfy a suitable structure theorem (see [AP1] and [Sa2]). From a categorical point of view, we already saw in §1.7 that quasi and coquasi-bialgebras are dual notions, meaning that we may recover the definition of one of these by simply reversing the arrows in the definition of the other, and the same happens with the definition of preantipodes for them.

Nevertheless, there seems to be a lack of duality between the properties they satisfy. In the first part of the chapter (§§2.1-2.2) we will see that the Structure Theorem for quasi-Hopf bimodules over a quasi-bialgebra can be rephrased in terms of a suitable hom-tensor adjunction, which in turn descends from the the fact that the category of left modules over a quasi-bialgebra is right-closed (see Lemma 2.1.2 and Theorem 2.2.7). This new formulation does not seem to have a (immediate, at least) counterpart for coquasi-bialgebras. On the other hand, in the second part of the chapter (§2.3) we will prove a Tannaka-Kreĭn reconstruction-type theorem for coquasi-bialgebras with preantipode (see Theorems 2.3.13 and 2.3.20) which extends Ulbrich's result [U] for Hopf algebras. However, now it is the dual version for quasi-bialgebras that appears to be much more problematic.

In the last part of the chapter (§2.4) we will show that a duality (i.e. a contravariant adjunction) in fact exists, but between the category of quasi-bialgebras and a proper subcategory of the one of coquasi-bialgebras (see Theorem 2.4.18). This may justify the lack of duality that we observed and it may explain why results on one side cannot be directly obtained from their analogues on the other by means of a general duality principle.

2.1 Closure of the category of modules over a quasi-bialgebra

Our first task will be that of showing that the category of left modules over a quasi-bialgebra A is right-closed. Even if this may sound trivial and it could be already known (an analogous result for modules over a $\times_R$-bialgebra appeared previously in [Sc3, Proposition 3.3], for example) and even if the final purpose we have in mind is to use this result to provide us with a left adjoint to the free quasi-Hopf bimodule functor $-\otimes A$ (see the forthcoming Section 2.2), we consider it interesting in its own and we decided that it could deserve a dedicated section.

Remark 2.1.1. Notice that the existence of a right adjoint to the functor $-\otimes N : {}_A\mathfrak{M} \to {}_A\mathfrak{M}$ can be derived from the dual version of the Freyd's Special Adjoint Functor Theorem (see [ML, Corollary V.8] for the classical version). Indeed, the category of left A-modules is small cocomplete, it admits A as a generator and it is co-wellpowered[(1)] since every epimorphism $e : M \to P$ is

[(1)] Recall that two epimorphisms $e_1 : X \to Y_1$ and $e_2 : X \to Y_2$ in a category $\mathcal{C}$ are said to be *equivalent* if there is an isomorphism $v : Y_1 \to Y_2$ such that $v \circ e_1 = e_2$. A category $\mathcal{C}$ is said to be *co-wellpowered* if for each object X of equivalent to a canonical projection $\pi_N : M \to M/N$, for $N \subseteq M$ a submodule (take $N = \mathsf{ker}(e)$). Since $- \otimes N$ preserves small colimits, the dual SAFT ensures that it admits a right adjoint. Nevertheless, what we are going to do here is to exhibit an explicit right adjoint, which we will need later on in Section 2.2.

Lemma 2.1.2. *Let A be a quasi-bialgebra. Then the category ${}_A\mathfrak{M}$ of left A-modules is left and right-closed. Namely, we have bijections*

$$ {}_A\mathsf{Hom}\left(M \otimes N, P\right) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} {}_A\mathsf{Hom}\left(M, {}_A\mathsf{Hom}\left(A \otimes N, P\right)\right), \tag{2.1}$$

$$ {}_A\mathsf{Hom}\left(N \otimes M, P\right) \underset{\psi'}{\overset{\varphi'}{\rightleftarrows}} {}_A\mathsf{Hom}\left(M, {}_A\mathsf{Hom}\left(N \otimes A, P\right)\right), $$

natural in M and P, given explicitly by

$$ \begin{aligned} \varphi(f)(m) &: a \otimes n \mapsto f(a \cdot m \otimes n), & \psi(g) &: m \otimes n \mapsto g(m)(1 \otimes n), \\ \varphi'(f)(m) &: n \otimes a \mapsto f(n \otimes a \cdot m), & \psi'(g) &: n \otimes m \mapsto g(m)(n \otimes 1), \end{aligned} $$

where the left A-module structures on ${}_A\mathsf{Hom}\left(N \otimes A, P\right)$ and ${}_A\mathsf{Hom}\left(A \otimes N, P\right)$ are induced by the right A-module structure on A itself.

Proof. Let M, N, P be left A-modules. To make the exposition clearer, we will denote them via ${}_\bullet M$, ${}_\bullet N$ and ${}_\bullet P$ to underline the given actions. We are claiming that there is a bijection

$$ {}_A\mathsf{Hom}\left({}_\bullet M \otimes {}_\bullet N, {}_\bullet P\right) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} {}_A\mathsf{Hom}\left({}_\bullet M, {}_A\mathsf{Hom}\left({}_\bullet A \otimes {}_\bullet N, {}_\bullet P\right)\right) $$

natural in M and P. Consider a generic $f \in {}_A\mathsf{Hom}\left({}_\bullet M \otimes {}_\bullet N, {}_\bullet P\right)$. For all $m \in M$, $n \in N$ and $a, b, c \in A$ we have that

$$ \begin{aligned} \big(\varphi(f)(c \cdot m)\big)\big(b \cdot (a \otimes n)\big) &= \sum f\Big(\big(b_1 a \cdot (c \cdot m)\big) \otimes (b_2 \cdot n)\Big) = f\Big(b \cdot \big((ac \cdot m) \otimes n\big)\Big) \\ &= b \cdot \big(\varphi(f)(m)\big)\,(ac \otimes n) = b \cdot \big(c \cdot \varphi(f)(m)\big)\,(a \otimes n)\,. \end{aligned} $$

Taking $c = 1$ gives the left A-linearity of $\varphi(f)(m)$ while taking $b = 1$ gives the left A-linearity of $\varphi(f)$, whence φ is well-defined. On the other hand, for all $g \in {}_A\mathsf{Hom}\left({}_\bullet M, {}_A\mathsf{Hom}\left({}_\bullet A \otimes {}_\bullet N, {}_\bullet P\right)\right)$, $m \in M$, $n \in N$ and $a \in A$ we have also

$$ \begin{aligned} \psi(g)(a \cdot (m \otimes n)) &= \sum g(a_1 \cdot m)(1 \otimes (a_2 \cdot n)) = \sum (a_1 \cdot g(m))\,(1 \otimes (a_2 \cdot n)) \\ &= \sum g(m)(a_1 \otimes (a_2 \cdot n)) = g(m)(a \cdot (1 \otimes n)) = a \cdot g(m)(1 \otimes n) = a \cdot \psi(g)(m \otimes n), \end{aligned} $$

which implies that $\psi(g)$ is left A-linear and ψ is well-defined as well. To check the naturality in M and P consider two left A-linear morphisms $h : M' \to M$ and $l : P \to P'$. Then

$$ \begin{aligned} \Big(\big({}_A\mathsf{Hom}\left(A \otimes N, l\right) \circ \varphi(f) \circ h\big)(m)\Big)(a \otimes n) &= \big(l \circ \varphi(f)(h(m))\big)(a \otimes n) = l\Big(\big(\varphi(f)(h(m))\big)(a \otimes n)\Big) \\ &= l\big(f(a \cdot h(m) \otimes n)\big) = l\Big(f\big(h(a \cdot m) \otimes n\big)\Big) \\ &= \Big(\varphi\big(l \circ f \circ (h \otimes N)\big)(m)\Big)(a \otimes n) \end{aligned} $$

To conclude, it is enough to check that φ and ψ are inverses each other. To this aim, we may compute directly

$$ \left(\varphi\psi(g)(m)\right)(a \otimes n) = \psi(g)(a \cdot m \otimes n) = g(a \cdot m)(1 \otimes n) = g(m)(a \otimes n), $$

$\mathcal{C}$ there is a set of epimorphisms $\{e_i : X \to Y_i \mid i \in I\}$ such that each epimorphism $e : X \to Y$ is equivalent to some e_i (see [Ad, Definition p.191]).

$$(\psi\varphi(f))\,(m\otimes n) = (\varphi(f)(m))\,(1\otimes n) = f(m\otimes n)$$

for all $m \in N$, $n \in N$, $a \in A$, $f \in {}_A\mathsf{Hom}\left({}_\bullet M \otimes {}_\bullet N, {}_\bullet P\right)$ and $g \in {}_A\mathsf{Hom}\left({}_\bullet M, {}_A\mathsf{Hom}\left({}_\bullet A \otimes {}_\bullet N, {}_\bullet P\right)\right)$. Therefore, the first claim holds. The second claim can be proved analogously or may be deduced as follows. Consider the $\Bbbk$-modules $(N\otimes A)\otimes_A M$ and $N\otimes(A\otimes_A M)$ endowed with the A-actions

$$\begin{aligned} a\cdot\big((n\otimes b)\otimes_A m\big) &:= \sum\left((a_1\cdot n)\otimes a_2 b\right)\otimes_A m, \\ a\cdot\big(n\otimes(b\otimes_A m)\big) &:= \sum(a_1\cdot n)\otimes(a_2 b\otimes_A m)\,, \end{aligned}$$

for all $a, b \in A$, $m \in M$ and $n \in N$. The canonical isomorphism $(N\otimes A)\otimes_A M \cong N\otimes(A\otimes_A M)$ ("the identity") is a morphism in ${}_A\mathfrak{M}$. Therefore, by the classical hom-tensor adjunction (see [Pa3, §3] for a very general approach), we have a chain of natural isomorphisms

$$\begin{aligned} {}_A\mathsf{Hom}\left({}_\bullet N\otimes{}_\bullet M, {}_\bullet P\right) &\cong {}_A\mathsf{Hom}\left({}_\bullet N\otimes({}_\bullet A\otimes_A M), {}_\bullet P\right) \cong {}_A\mathsf{Hom}\left(({}_\bullet N\otimes{}_\bullet A)\otimes_A M, {}_\bullet P\right) \\ &\cong {}_A\mathsf{Hom}\left({}_\bullet M, {}_A\mathsf{Hom}\left({}_\bullet N\otimes{}_\bullet A, {}_\bullet P\right)\right) \end{aligned}$$

whose composition gives exactly φ' and ψ'. $\square$

Remark 2.1.3. Let A be a quasi-bialgebra. A right-handed version of Lemma 2.1.2 holds as well. Namely, the category $\mathfrak{M}_A$ of right A-modules is right and left-closed in the sense that we have bijections

$$\mathsf{Hom}_A\left(M\otimes N, P\right) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathsf{Hom}_A\left(M, \mathsf{Hom}_A\left(A\otimes N, P\right)\right)\,,$$

$$\mathsf{Hom}_A\left(N\otimes M, P\right) \underset{\psi'}{\overset{\varphi'}{\rightleftarrows}} \mathsf{Hom}_A\left(M, \mathsf{Hom}_A\left(N\otimes A, P\right)\right)\,,$$

natural in M and P, given explicitly by

$$\begin{aligned} \varphi(f)(m) &: a\otimes n \mapsto f(m\cdot a\otimes n), & \psi(g) &: m\otimes n \mapsto g(m)(1\otimes n), \\ \varphi'(f)(m) &: n\otimes a \mapsto f(n\otimes m\cdot a), & \psi'(g) &: n\otimes m \mapsto g(m)(n\otimes 1), \end{aligned}$$

where the right A-module structure on $\mathsf{Hom}_A\left(A\otimes N, P\right)$ and $\mathsf{Hom}_A\left(N\otimes A, P\right)$ is induced by the left A-module structure on A itself. This follows from Lemma 2.1.2 and the fact that we have an obvious monoidal functor ${}_{A^{\mathrm{op}}}\mathfrak{M} \to \mathfrak{M}_A$ which is also an isomorphism of categories, where A^{op} is the quasi-bialgebra of Example 1.7.6.

2.2 Preantipodes for quasi-bialgebras

The structure theorem for Hopf modules (see e.g. [BW, §15.5]) allows one to characterize Hopf algebras as those bialgebras H such that every Hopf module M satisfy $M \cong M^{\mathrm{co}H}\otimes H$, where $M^{\mathrm{co}H} = \{m \in M \mid \rho(m) = m\otimes 1\}$ is the space of H-coinvariant elements. It has been proven by Hausser and Nill in [HN] that quasi-Hopf bimodules over a quasi-Hopf algebra satisfy an analogue of the structure theorem (see also [BC]) and by Schauenburg in [Sc8] that the same is true for coquasi-Hopf bicomodules over a coquasi-Hopf algebra. Nevertheless, these do not characterize (co)quasi-Hopf algebras as one may expect. Instead, it has been shown by Ardizzoni and Pavarin in [AP1] that a better analogue of the notion of an antipode for a coquasi-bialgebra, at least from this point of view, is that of a preantipode. They proved that the structure theorem for coquasi-Hopf bicomodules is in fact equivalent to the existence of such a distinguished linear endomorphism for the coquasi-bialgebra. In this section we present the counterpart of the notion of a preantipode in the framework of quasi-bialgebras and we show how Hausser and Nill's structure theorem turns out to be equivalent to the existence of a preantipode for the quasi-bialgebra under consideration. It will become clear soon that this is not a mere dualization of the results in [AP1], in the first place because the dual of a coquasi-bialgebra is not, in general, a quasi-bialgebra. In addition, we will provide a new formulation of the structure theorem which involves a right adjoint to the free quasi-Hopf bimodule functor, instead of a left one. This will rely on the closure results from Section 2.1. The main source for this section is [Sa2].

2.2.1 The category of quasi-Hopf bimodules over a quasi-bialgebra

Henceforth let us fix a quasi-bialgebra $(A, m, u, \Delta, \varepsilon, \Phi)$. Notice that A is not a coalgebra in $\mathfrak{M}$ and, in general, neither in ${}_A\mathfrak{M}$ nor in $\mathfrak{M}_A$. However, (A, m, m) as an (A, A)-bimodule, endowed with Δ and ε, becomes a coalgebra in ${}_A\mathfrak{M}_A$ and so we can consider the category of the so-called *(right) quasi-Hopf bimodules*

$$ {}_A\mathfrak{M}_A^A := \left({}_A\mathfrak{M}_A\right)^A . $$

Coassociativity and counitality of a coaction are expressed explicitly by

$$ \sum m_0\varepsilon(m_1) = m, \qquad \text{and} \qquad \sum(m_0 \otimes m_{1,1} \otimes m_{1,2}) \cdot \Phi = \sum \Phi \cdot (m_{0,0} \otimes m_{0,1} \otimes m_1) \tag{2.2} $$

for all $m \in M$, $M \in {}_A\mathfrak{M}_A^A$ (cf. [HN, Definition 3.1]). Notice that (see [Sc6, Lemma and Definition 3.2]) this is a monoidal category with tensor product $\otimes_A$ and unit object A itself. The constraints are the same of the category of A-bimodules, that is to say, the "identity" morphisms. For M, N in ${}_A\mathfrak{M}_A^A$, their tensor product $M \otimes_A N$ is endowed with the diagonal actions and the coaction

$$ \sum (m \otimes_A n)_0 \otimes (m \otimes_A n)_1 = \sum (m_0 \otimes_A n_0) \otimes m_1 n_1 . $$

In view of the fact that A is a coalgebra in the monoidal category ${}_A\mathfrak{M}_A$ and of what we saw in §1.3, we have an adjunction $(\mathcal{U}, \mathcal{T})$ between ${}_A\mathfrak{M}_A^A$ and ${}_A\mathfrak{M}_A$ with right adjoint given by

$$ \mathcal{T} : {}_A\mathfrak{M}_A \to {}_A\mathfrak{M}_A^A; \quad {}_\bullet M_\bullet \mapsto {}_\bullet M_\bullet \otimes {}_\bullet A_\bullet^\bullet, $$

(as before, the full dots denote the given actions and coaction). Explicitly, if we denote by $\rho_{M\otimes A}$ the coaction on $M \otimes A$, we have

$$ \begin{aligned} a \cdot (m \otimes b) \cdot c &= \sum a_1 \cdot m \cdot c_1 \otimes a_2 b c_2, \\ \rho_{M\otimes A}(m \otimes a) &= \sum \Phi^{-1} \cdot ((m \otimes a_1) \otimes a_2) \cdot \Phi \end{aligned} \tag{2.3} $$

for all $a, b, c \in A$ and $m \in M$. The left adjoint $\mathcal{U}$ is the underlying functor.

We may compose this with another well-known adjunction $(\mathcal{L}, \mathcal{R})$ between ${}_A\mathfrak{M}_A$ and ${}_A\mathfrak{M}$, where $\mathcal{L} : {}_A\mathfrak{M}_A \to {}_A\mathfrak{M}$ and $\mathcal{R}\colon {}_A\mathfrak{M} \to {}_A\mathfrak{M}_A$ are given by

$$ \mathcal{L}\left({}_\bullet M_\bullet\right) := {}_\bullet M_\bullet \otimes_A {}_\circ\Bbbk \qquad \text{and} \qquad \mathcal{R}\left({}_\bullet V\right) := \mathrm{Hom}_\Bbbk\left(\Bbbk, {}_\bullet V\right) \cong {}_\bullet V_\circ $$

and $\Bbbk$ is an A-(bi)module via the algebra map $\varepsilon : A \to \Bbbk$ (we will always denote the action via ε by an empty dot $\circ$ and we will often refer to this as the trivial action). Notice that if M is an A-bimodule then

$$ M \otimes_A \Bbbk \cong \frac{M}{MA^+} =: \overline{M}, $$

as left A-modules, where $A^+ := \mathsf{ker}(\varepsilon)$ is the augmentation ideal of A. Define

$$ \mathcal{F} := \mathcal{L}\mathcal{U} : {}_A\mathfrak{M}_A^A \to {}_A\mathfrak{M} \qquad \text{and} \qquad \mathcal{G} := \mathcal{T}\mathcal{R} : {}_A\mathfrak{M} \to {}_A\mathfrak{M}_A^A $$

so that, for every $M \in {}_A\mathfrak{M}_A^A$ and $N \in {}_A\mathfrak{M}$ we have that $\mathcal{F}\left({}_\bullet M_\bullet^\bullet\right) = {}_\bullet\overline{M}$ with structure given by $a \cdot \overline{m} = \overline{a \cdot m}$ for all $a \in A, m \in M$ and $\mathcal{G}({}_\bullet N) = {}_\bullet N_\circ \otimes {}_\bullet A_\bullet^\bullet$ with structures given by

$$ \begin{aligned} a \cdot (n \otimes b) \cdot c &= \sum a_1 \cdot n \otimes a_2 b c, \\ \rho(n \otimes b) &= \sum \Phi^{-1} \cdot ((n \otimes b_1) \otimes b_2) \end{aligned} \tag{2.4} $$

for all $a, b, c \in A, n \in N$. The content of the subsequent Proposition 2.2.1 is essentially the same of [Sc6, Proposition 3.6].

Proposition 2.2.1. *Let $(A, m, u, \Delta, \varepsilon, \Phi)$ be a quasi-bialgebra. The functor $\mathcal{G} : {}_A\mathfrak{M} \to {}_A\mathfrak{M}_A^A$ sending every N to $N \otimes A$ is a monoidal functor and it is right adjoint to the functor $\mathcal{F} : {}_A\mathfrak{M}_A^A \to {}_A\mathfrak{M}$ given (on objects) by $M \mapsto M \otimes_A \Bbbk = \overline{M}$. The unit η and the counit ϵ of this adjunction are*

$$\eta_M : M \to \overline{M} \otimes A; \quad \left(m \mapsto \sum \overline{m_0} \otimes m_1\right), \tag{2.5}$$

$$\epsilon_N : \overline{N \otimes A} \to N; \quad \left(\overline{n \otimes a} \mapsto n\varepsilon(a)\right), \tag{2.6}$$

for all $M \in {}_A\mathfrak{M}_A^A$, $N \in {}_A\mathfrak{M}$. Moreover, ϵ_N is always an isomorphism with inverse $\epsilon_N^{-1} : n \mapsto \overline{n \otimes 1}$.

Remark 2.2.2. Being the left adjoint of a monoidal functor, the functor $\mathcal{F}$ is naturally colax monoidal (see [AMa, Proposition 3.84]). Furthermore, since the counit of the adjunction $(\mathcal{F}, \mathcal{G}, \eta, \epsilon)$ is a natural isomorphism, the right adjoint $\mathcal{G}$ is full and faithful (see [ML, Theorem IV.3.1]). It follows then that the adjunction is *monadic*, in light of [AGM, Proposition 2.5]. This means that if we consider the monad $\mathbb{T}$ on ${}_A\mathfrak{M}_A^A$ associated with $(\mathcal{F}, \mathcal{G}, \eta, \epsilon)$, then the *comparison functor* $\mathcal{K} : {}_A\mathfrak{M} \to \left({}_A\mathfrak{M}_A^A\right)^{\mathbb{T}}; N \mapsto (N \otimes A, \epsilon_N \otimes A)$ is an equivalence of categories, where the latter is the Eilenberg-Moore category of the monad (see §1.4).

Apart from the adjunction $(\mathcal{F}, \mathcal{G})$ of Proposition 2.2.1, there is another distinguished adjunction connecting ${}_A\mathfrak{M}$ and ${}_A\mathfrak{M}_A^A$. Recall from Lemma 2.1.2 that we have a bijection

$${}_A\mathsf{Hom}\left({}_\bullet M \otimes {}_\bullet N, {}_\bullet P\right) \cong {}_A\mathsf{Hom}\left({}_\bullet M, {}_A\mathsf{Hom}\left({}_\bullet A \otimes {}_\bullet N, {}_\bullet P\right)\right).$$

This in turn induces a bijection

$${}_A\mathsf{Hom}_A^A\left({}_\bullet M \otimes {}_\bullet N_\bullet^\bullet, {}_\bullet P_\bullet^\bullet\right) \cong {}_A\mathsf{Hom}\left({}_\bullet M, {}_A\mathsf{Hom}_A^A\left({}_\bullet A \otimes {}_\bullet N_\bullet^\bullet, {}_\bullet P_\bullet^\bullet\right)\right)$$

which encodes the fact that for a quasi-Hopf A-bimodule N the functor $- \otimes N \colon {}_A\mathfrak{M} \to {}_A\mathfrak{M}_A^A$ is left adjoint to the functor ${}_A\mathsf{Hom}_A^A(A \otimes N, -)$. We recall that the left A-module structure on ${}_A\mathsf{Hom}_A^A(A \otimes N, P)$ is given by the right action of A on itself via multiplication. If we consider the distinguished case $N = A$, we get the following result.

Proposition 2.2.3 (see also [BW, §3.10])**.** *Let $(A, m, u, \Delta, \varepsilon, \Phi)$ be a quasi-bialgebra. The functor $\mathcal{H} : {}_A\mathfrak{M}_A^A \to {}_A\mathfrak{M}$ given (on objects) by the assignment $M \mapsto {}_A\mathsf{Hom}_A^A\left({}_\bullet A \otimes {}_\bullet A_\bullet^\bullet, {}_\bullet M_\bullet^\bullet\right)$ is right adjoint to the functor $\mathcal{G}$. The unit γ and the counit θ of this adjunction are given by*

$$\gamma_N : N \to {}_A\mathsf{Hom}_A^A(A \otimes A, N \otimes A); \quad \Big(n \mapsto [a \otimes b \mapsto a \cdot n \otimes b]\Big), \tag{2.7}$$

$$\theta_M : {}_A\mathsf{Hom}_A^A(A \otimes A, M) \otimes A \to M; \quad \Big(f \otimes a \mapsto \big(f(1 \otimes a) = f(1 \otimes 1) \cdot a\big)\Big), \tag{2.8}$$

for all objects M in ${}_A\mathfrak{M}_A^A$ and N in ${}_A\mathfrak{M}$. Moreover, γ_N is always an isomorphism with inverse $\gamma_N^{-1} : f \mapsto (N \otimes \varepsilon)(f(1 \otimes 1))$.

Proof. We only prove the last claim. From Remark 2.2.2 we know that $\mathcal{G}$ is a fully faithful functor and hence, by the dual statement of [ML, Theorem IV.3.1], the unit of the adjunction $(\mathcal{G}, \mathcal{H})$ is a natural isomorphism. To check that the inverse is really the stated one, proceed as follows. Since ${}_\bullet A_\bullet$ is a coalgebra in ${}_A\mathfrak{M}_A$, we may resort to the adjunction (1.13) to say that for every $f \in {}_A\mathsf{Hom}_A^A(A \otimes A, N \otimes A)$

$$f = \Big(\big((N \otimes \varepsilon) \circ f\big) \otimes A\Big) \circ \rho_{{}_\bullet A \otimes {}_\bullet A_\bullet^\bullet} = \Big(\big((N \otimes \varepsilon) \circ f\big) \otimes A\Big) \circ \left({}_A\alpha_A\right)_{A,A,A}^{-1} \circ (A \otimes \Delta). \tag{2.9}$$

Moreover, observe that for every $a \in A$

$$\begin{aligned} a \cdot (N \otimes \varepsilon)\big(f(1 \otimes 1)\big) &= (N \otimes \varepsilon)\big(a \cdot f(1 \otimes 1)\big) = \sum (N \otimes \varepsilon)\big(f(a_1 \otimes a_2)\big) \\ &= \sum (N \otimes \varepsilon)\big(f(a_1 \otimes 1) \cdot a_2\big) = \sum (N \otimes \varepsilon)\big(f(a_1 \otimes 1)\varepsilon(a_2)\big) = (N \otimes \varepsilon)\big(f(a \otimes 1)\big). \end{aligned}$$

Therefore, the following direct computation

$$
\begin{aligned}
f(a \otimes b) &= f(a \otimes 1) \cdot b \overset{(2.9)}{=} \Big(\big(((N \otimes \varepsilon) \circ f) \otimes A \big) \circ ({}_A\alpha_A)^{-1}_{A,A,A} \circ (A \otimes \Delta) \Big) (a \otimes 1) \cdot b \\
&= \Big((N \otimes \varepsilon) \big(f\left(\varphi^1 a \varepsilon(\Phi^1) \otimes \varphi^2 \Phi^2 \right) \big) \otimes \varphi^3 \Phi^3 \Big) \cdot b \\
&\overset{(1.23a)}{=} \Big((N \otimes \varepsilon) \big(f(\varphi^1 a \otimes 1) \cdot \varphi^2 \big) \Big) \otimes \varphi^3 b = \Big((N \otimes \varepsilon) \big(f(\varphi^1 a \otimes 1) \varepsilon(\varphi^2) \big) \Big) \otimes \varphi^3 b \\
&\overset{(1.23a)}{=} \Big((N \otimes \varepsilon) \big(f(a \otimes 1) \big) \Big) \otimes b = a \cdot ((N \otimes \varepsilon)(f(1 \otimes 1))) \otimes b
\end{aligned}
$$

allows one to check easily that γ_N^{-1} and γ_N are inverses each other. □

2.2.2 The preantipode for a quasi-bialgebra and a revised Structure Theorem for quasi-Hopf bimodules

As we saw in the previous Subsection 2.2.1, we have a chain of adjunctions

$$
\overline{(-)} \quad \dashv \quad (- \otimes A) \quad \dashv \quad {}_A\mathsf{Hom}_A^A (A \otimes A, -) \,. \tag{2.10}
$$

In [Sa2, Theorem 4] it has been shown that, under a suitable hypothesis on the quasi-bialgebra A, the leftmost adjunction becomes an adjoint equivalence (i.e. the counit is a natural isomorphism), converting in this way the rightmost one in an equivalence as well. The stated suitable hypothesis is the existence of a distinguished linear endomorphism S of A called *preantipode*. This subsection is devoted to recall the definition and the properties of the preantipode and to see which consequences this has with respect to the rightmost adjunction in (2.10).

Definition 2.2.4 ([Sa2, Definition 1])**.** A *preantipode* for a quasi-bialgebra $(A, m, u, \Delta, \varepsilon, \Phi)$ is a $\Bbbk$-linear map $S : A \to A$ that satisfies

$$
\sum a_1 S(b a_2) = \varepsilon(a) S(b), \tag{2.11a}
$$

$$
\sum S(a_1 b) a_2 = \varepsilon(a) S(b), \tag{2.11b}
$$

$$
\sum \Phi^1 S(\Phi^2) \Phi^3 = 1, \tag{2.11c}
$$

for all $a, b \in A$.

Remark 2.2.5 ([Sa2, Remark 3])**.** By evaluating 2.11a and 2.11b at $b = 1$, we have that

$$
(\mathsf{Id}_A * S)(a) = \varepsilon(a) S(1) = (S * \mathsf{Id}_A)(a) \tag{2.12}
$$

for all $a \in A$, where $*$ denotes the convolution product. In particular, if $\Phi = 1 \otimes 1 \otimes 1$ (i.e. A is an ordinary bialgebra) then $S(1) = 1$ and so A is an ordinary Hopf algebra. In general S does not satisfy $S(1) = 1$ (an example will be provided later on). However, applying ε on both sides of 2.11c gives $\varepsilon\left(S(1)\right) = 1$ and applying ε again on both sides of the leftmost equality in 2.12 gives that the preantipode preserves the counit $\varepsilon \circ S = \varepsilon$.

Example 2.2.6.

1. If $(H, m, u, \Delta, \varepsilon, S)$ is an ordinary Hopf algebra then $(H, m, u, \Delta, \varepsilon, \Phi, S)$ with $\Phi = 1 \otimes 1 \otimes 1$ is a quasi-bialgebra with preantipode.
2. If $(H, m, u, \Delta, \varepsilon, \Phi, s, \alpha, \beta)$ is a quasi-Hopf algebra with quasi-antipode (s, α, β) (see [Dr2, Definition on page 1424]) then $(H, m, u, \Delta, \varepsilon, \Phi, S)$ with $S(a) = \beta s(a) \alpha$ for all $a \in H$ is a quasi-bialgebra with preantipode (see [Sa2, Theorem 6]). It is important to point out that in the finite-dimensional case the converse is true as well, that is to say, for a finite-dimensional quasi-bialgebra H to admit a preantipode is equivalent to admitting a quasi-antipode (see [Sc8, Theorem 3.1]). Notice however that it is not clear how to write explicitly the quasi-antipode

by means of the preantipode in such a case, because the proof of the aforementioned result is not a constructive one. Since this question has already been addressed extensively elsewhere and we have no new contributions in this direction, we do not dwell further on the matter and we refer the reader to [Sa2, §4.1] for a more detailed analysis. We would just conclude by recalling that, in spite of the fact that in the finite-dimensional case the two notions are equivalent, we believe that not all quasi-bialgebras with preantipode come from quasi-Hopf algebras as above, even if an example in this sense is still missing.

3. Let $C_2 := \langle g \rangle$ be the cyclic group of order 2 with generator g and let $\Bbbk$ be a field with $\mathrm{char}(\Bbbk) \neq 2$. The group bialgebra $H(2) := \Bbbk C_2$ can be endowed with the quasi-bialgebra structure induced by $\Phi := 1 \otimes 1 \otimes 1 - 2p \otimes p \otimes p$, where $p = \frac{1}{2}(1-g)$. It is a quasi-Hopf algebra with $\beta = 1$, $\alpha = g$ and $s = \mathsf{Id}_{H(2)}$, whence it is a quasi-bialgebra with preantipode where $S(x) = xg$. We recall that $H(2)$ is not twist equivalent to any Hopf algebra ([EG]).

4. As it happens for Hopf algebras, a quasi-bialgebra $(A, m, u, \Delta, \varepsilon, \Phi)$ admits a preantipode if and only if the quasi-bialgebra $A^{\mathrm{op,cop}} = (A, m^{\mathrm{op}}, u, \Delta^{\mathrm{cop}}, \varepsilon, \sum \Phi^3 \otimes \Phi^2 \otimes \Phi^1)$ of Example 1.7.6 admits a preantipode.

In the same way in which antipodes characterize those bialgebras whose Hopf modules satisfy a suitable structure theorem, preantipodes characterize those quasi-bialgebras for which the adjunctions (2.10) are in fact equivalences (being an equivalence for the leftmost one encodes the so-called Structure Theorem for quasi-Hopf bimodules as it appears in [HN] or [Sa2], for example).

Theorem 2.2.7 (Revised Structure Theorem for quasi-Hopf bimodules)**.** *Let $(A, m, u, \Delta, \varepsilon, \Phi)$ be a quasi-bialgebra. The following assertions are equivalent:*

(a) The functor $\mathcal{G}$ is a monoidal equivalence of monoidal categories.

(b) The adjunction $(\mathcal{F}, \mathcal{G}, \eta, \epsilon)$ is an adjoint equivalence of categories.

(c) The adjunction $(\mathcal{G}, \mathcal{H}, \gamma, \theta)$ is an adjoint equivalence of categories.

(d) The following component of the unit η is bijective

$$\hat{\eta}_A : {}_{\circ}A_{\bullet} \otimes {}_{\bullet}A_{\bullet}^{\bullet} \to \frac{{}_{\circ}A_{\bullet} \otimes {}_{\bullet}A_{\bullet}^{\bullet}}{\left({}_{\circ}A_{\bullet} \otimes {}_{\bullet}A_{\bullet}^{\bullet}\right) A^{+}} \otimes {}_{\bullet}A_{\bullet}^{\bullet}; \quad \left(a \otimes b \mapsto \sum \overline{a\Phi^1 \otimes b_1 \Phi^2} \otimes b_2 \Phi^3\right). \tag{2.13}$$

(e) The following component of the counit θ is bijective

$$\hat{\theta}_A : {}_{A}\mathsf{Hom}_{A}^{A}\left({}_{\bullet}A_{\circ} \otimes {}_{\bullet}A_{\bullet}^{\bullet}, {}_{\circ}A_{\bullet} \otimes {}_{\bullet}A_{\bullet}^{\bullet}\right) \otimes {}_{\bullet}A_{\bullet}^{\bullet} \to {}_{\circ}A_{\bullet} \otimes {}_{\bullet}A_{\bullet}^{\bullet}; \quad \left(f \otimes a \mapsto f(1 \otimes 1) \cdot a\right). \tag{2.14}$$

(f) There exists a preantipode S.

Proof. We only prove the equivalences *(a)* $\Leftrightarrow$ *(b)* $\Leftrightarrow$ *(c)* and *(d)* $\Leftrightarrow$ *(e)*. First of all, observe that since $\mathcal{G}$ is a monoidal functor, both *(b)* and *(c)* implies *(a)*. Conversely, if $\mathcal{G}$ is a monoidal equivalence then there exists a quasi-inverse $\mathcal{G}'$ for $\mathcal{G}$ and two natural isomorphisms $\alpha : \mathcal{G}'\mathcal{G} \cong \mathsf{Id}$ and $\beta : \mathsf{Id} \cong \mathcal{G}\mathcal{G}'$. Naturality of η and the adjunction equations imply that

$$\mathcal{G}\epsilon_{\mathcal{G}'} \circ \mathcal{G}\mathcal{F}\beta \circ \eta = \mathcal{G}\epsilon_{\mathcal{G}'} \circ \eta_{\mathcal{G}\mathcal{G}'} \circ \beta = \beta,$$

so that $\eta = \mathcal{G}\mathcal{F}\beta^{-1} \circ \mathcal{G}\epsilon_{\mathcal{G}'}^{-1} \circ \beta$ and it becomes a natural isomorphism. This proves that *(a)* $\Rightarrow$ *(b)*; the other one is analogous. To conclude, observe that the naturality of η and θ entails that

$$\mathcal{G}\mathcal{F}\theta \circ \eta\mathcal{G}\mathcal{H} = \eta \circ \theta = \theta\mathcal{G}\mathcal{F} \circ \mathcal{G}\mathcal{H}\eta \tag{2.15}$$

where $\eta\mathcal{G}\mathcal{H}$ and $\theta\mathcal{G}\mathcal{F}$ are natural isomorphisms with inverses $\mathcal{G}\epsilon\mathcal{H}$ and $\mathcal{G}\gamma\mathcal{F}$ respectively. Therefore η is a natural isomorphism if and only if θ is and, in particular, $\hat{\eta}_A$ is an isomorphism if and only if $\hat{\theta}_A$ is. This concludes the proof. □

Remark 2.2.8. The chain of implications *(b)* $\Rightarrow$ *(d)* $\Rightarrow$ *(f)* $\Rightarrow$ *(b)* is contained in [Sa2, §3]. We will not prove them again, but we recall briefly some formulas involved in the proof. Having a preantipode S for A allows us to write down explicitly the inverse of η_M

$$\eta_M^{-1}\left(\overline{m}\otimes a\right)=\sum\Phi^1\cdot m_0\cdot S\left(\Phi^2 m_1\right)\Phi^3 a.$$

Conversely, if η is a natural isomorphism we can define

$$S(a):=\left(A\otimes\varepsilon\right)\left(\hat{\eta}_A^{-1}\left(\overline{1\otimes a}\otimes 1\right)\right).$$

As a consequence, if A admits a preantipode and if we consider the component at M in ${}_A\mathfrak{M}_A^A$ of the natural transformations in Equation (2.15) then

$$\theta_M^{-1}=\left({}_A\mathsf{Hom}_A^A\left(A\otimes A,\eta_M^{-1}\right)\otimes A\right)\circ\left(\gamma_{\overline{M}}\otimes A\right)\circ\eta_M,$$

so that, for all $m\in M$,

$$\theta_M^{-1}(m)=\sum\left(\eta_M^{-1}\circ\gamma_{\overline{M}}\left(\overline{m_0}\right)\right)\otimes m_1. \tag{2.16}$$

Conversely, if we know that θ is a natural isomorphism, then we may define

$$\eta_M^{-1}=\theta_M\circ\left(\epsilon_{\mathcal{H}(M)}\otimes A\right)\circ\left(\overline{\theta_M^{-1}}\otimes A\right),$$

from which is follows that for all $a\in A$

$$\begin{aligned}S(a)&=\left(A\otimes\varepsilon\right)\left(\hat{\eta}_A^{-1}\left(\overline{1\otimes a}\otimes 1\right)\right)=\left(A\otimes\varepsilon\right)\left(\hat{\theta}_A\left(\epsilon_{\mathcal{H}(M)}\left(\overline{\hat{\theta}_A^{-1}\left(1\otimes a\right)}\right)\otimes 1\right)\right)\\&\overset{(*)}{=}\left(A\otimes\varepsilon\right)\left(\hat{\theta}_A\left(a\cdot\epsilon_{\mathcal{H}(M)}\left(\overline{\hat{\theta}_A^{-1}\left(1\otimes 1\right)}\right)\otimes 1\right)\right)\\&\overset{(2.8)}{=}\left(A\otimes\varepsilon\right)\left(a\cdot\epsilon_{\mathcal{H}(M)}\left(\overline{\hat{\theta}_A^{-1}\left(1\otimes 1\right)}\right)\left(1\otimes 1\right)\right)\\&=\left(A\otimes\varepsilon\right)\left(\epsilon_{\mathcal{H}(M)}\left(\overline{\hat{\theta}_A^{-1}\left(1\otimes 1\right)}\right)\left(a\otimes 1\right)\right)\end{aligned}$$

where in $(*)$ we used the A-linearity of θ^{-1} and ϵ.

As one may expect, once we know that either $(\mathcal{F},\mathcal{G})$ or $(\mathcal{G},\mathcal{H})$ is an adjoint equivalence then we have that $\mathcal{F}\cong\mathcal{H}$.

Corollary 2.2.9. *Let $(A,m,u,\Delta,\varepsilon,\Phi)$ be a quasi-bialgebra. Consider the A-linear morphism*

$$\left(\sigma_M:=\overline{\theta_M}\circ\epsilon_{\mathcal{H}(M)}^{-1}\right):{}_A\mathsf{Hom}_A^A\left(A\otimes A,M\right)\to\frac{M}{MA^+};\qquad\left(f\mapsto\overline{f(1\otimes 1)}\right)$$

which is natural in $M\in{}_A\mathfrak{M}_A^A$. If A admits a preantipode S, then each component of σ is an isomorphism.

Remark 2.2.10. By applying $\epsilon_{\mathcal{H}(M)}$ to both sides of Relation (2.16) one gets that

$$\epsilon_{\mathcal{H}(M)}\left(\overline{\theta_M^{-1}(m)}\right)=\eta_M^{-1}\circ\gamma_{\overline{M}}\left(\overline{m}\right). \tag{2.17}$$

Therefore one may consider ${}_A\mathsf{Hom}_A^A\left(A\otimes A,\eta_M^{-1}\right)\circ\gamma_{\overline{M}}:\overline{M}\to{}_A\mathsf{Hom}_A^A\left(A\otimes A,M\right)$ as explicit inverse of $\overline{\theta_M}\circ\epsilon_{\mathcal{H}(M)}^{-1}$. In fact, the assignments

$$\begin{array}{ccc}M/MA^+ & \longleftrightarrow & {}_A\mathsf{Hom}_A^A\left(A\otimes A,M\right)\\ \overline{m} & \longmapsto & \left[(a\otimes b)\mapsto\left(\sum\Phi^1 a_1\cdot m_0\cdot S(\Phi^2 a_2 m_1)\Phi^3 b\right)\right]\\ \overline{f(1\otimes 1)} & \longleftarrow\!\shortmid & [f\colon A\otimes A\to M]\end{array}$$

give the desired bijection.

As it happens for the antipode of a Hopf algebra, when a preantipode exists, it is unique ([Sa2, Theorem 5]). Moreover, if we deform a quasi-bialgebra with preantipode $(A, m, u, \Delta, \varepsilon, \Phi, S)$ by using a gauge transformation $F \in A \otimes A$, we get still a quasi-bialgebra with preantipode $(A, m, u, \Delta_F, \varepsilon, \Phi_F, S_F)$ where, for all $a \in A$,

$$S_F(a) := \sum F^1 S(f^1 a F^2) f^2 \tag{2.18}$$

([Sa2, Proposition 5]). Therefore, quasi-bialgebras with preantipode form a class closed under gauge transformations. The proofs are omitted and can be found in [Sa2]. We only mention that uniqueness of the preantipode relies on the uniqueness of the inverse of the unit η.

We conclude this section by spending a few words on the notion of coinvariant elements for a quasi-Hopf bimodule. Let A be a quasi-bialgebra and M in ${}_A\mathfrak{M}_A^A$. In light of [BW, §§14.8 and 14.9], we may define the space of *coinvariant elements* (or simply *coinvariants*) $M^{\overline{\mathrm{co}A}}$ of M to be the image of the $\Bbbk$-linear morphism

$$\varsigma : {}_A\mathsf{Hom}_A^A(A \otimes A, M) \to M; \quad f \mapsto f(1 \otimes 1).$$

Notice that the composition of the canonical projection $M \to \overline{M}$ with this morphism gives the canonical A-linear map of Corollary 2.2.9. Our aim now is to show that when A admits a preantipode, then $M^{\overline{\mathrm{co}A}}$ coincides with the space $M^{\mathrm{co}A} := \{\sum \Phi^1 \cdot m_0 \cdot S(\Phi^2 m_1)\Phi^3 \mid m \in M\}$, as defined in [Sa2, Definition 2].

Lemma 2.2.11. *Let A be a quasi-bialgebra and let M be a quasi-Hopf A-bimodule. If A admits a preantipode, then $\varsigma\left({}_A\mathsf{Hom}_A^A(A \otimes A, M)\right) = M^{\mathrm{co}A} \subseteq M$.*

Proof. For every $f \in {}_A\mathsf{Hom}_A^A(A \otimes A, M)$ and for all $a \in A$, colinearity of f implies that

$$\begin{aligned}
&\sum \Phi^1 \cdot f(a \otimes 1)_0 \cdot S(\Phi^2 f(a \otimes 1)_1)\Phi^3 \overset{(2.4)}{=} \sum \Phi^1 f(\varphi^1 a \otimes \varphi^2) S(\Phi^2 \varphi^3)\Phi^3 \\
&= f\left(\sum \Phi_1^1 \varphi^1 a \otimes \Phi_2^1 \varphi^2 S(\Phi^2 \varphi^3)\Phi^3\right) \overset{(1.16a)}{=} f\left(\sum \varphi^1 \Psi^1 a \otimes \varphi^2 \Phi^1 \Psi_1^2 S(\varphi_1^3 \Phi^2 \Psi_2^2)\varphi_2^3 \Phi^3 \Psi^3\right) \\
&\overset{(2.11)}{=} f(a \otimes 1),
\end{aligned}$$

so that $f(1 \otimes 1) \in M^{\mathrm{co}A}$. From [Sa2, Proposition 4] we know that the assignment $\upsilon : M^{\mathrm{co}A} \to \overline{M}$ sending $m \mapsto \overline{m}$ is an isomorphism of $\Bbbk$-modules with inverse $\tilde{\tau} : \overline{m} \mapsto \sum \Phi^1 \cdot m_0 \cdot S(\Phi^2 m_1)\Phi^3$. Thus, since $\upsilon\big(\varsigma(f)\big) = \sigma(f)$ for every $f \in {}_A\mathsf{Hom}_A^A(A \otimes A, M)$, it follows that the corestriction of ς to $M^{\mathrm{co}A}$ becomes an isomorphism, which is also A-linear if we endow $M^{\mathrm{co}A}$ with the A-module structure given by $a \blacktriangleright m := \sum \Phi^1 a_1 \cdot m_0 \cdot S(\Phi^2 a_2 m_1)\Phi^3$ ([Sa2, Proposition 1]). $\square$

2.2.3 A relation for the preantipode of a quasi-bialgebra

It is well-known that the antipode of a Hopf algebra is anti-multiplicative. Here we provide a relation for the preantipode of a quasi-bialgebra that resembles anti-multiplicativity. It will be needed in §2.3.3, but we think it may be interesting on its own.

Let $(A, m, u, \Delta, \varepsilon, \Phi, S)$ be a quasi-bialgebra with preantipode and consider the A-actions on $\mathsf{End}(A) = \mathsf{Hom}_\Bbbk(A, A)$ defined by $(f \leftharpoonup a)(b) = f(ab)$ and $(a \rightharpoonup f)(b) = f(ba)$ for all $a, b \in A$ and for all $f \in \mathsf{End}(A)$ (see Example 1.3.2). Define the elements

$$\begin{aligned}
p &:= \sum \varphi^1 \otimes \varphi^2 \left(\varphi^3 \rightharpoonup S\right) \quad \in \quad A \otimes \mathsf{End}(A), \\
q &:= \sum \left(S \leftharpoonup \varphi^1\right) \varphi^2 \otimes \varphi^3 \quad \in \quad \mathsf{End}(A) \otimes A,
\end{aligned} \tag{2.19}$$

where $(x\,(y \rightharpoonup f))\,(a) = xf(ay)$ and $((f \leftharpoonup x)\,y)\,(a) = f(xa)y$ for all $a, x, y \in A$ and for all $f \in \mathsf{End}(A)$. Let us introduce the following notation for shortness:

$$p := \sum p^1 \otimes p^2 \qquad \text{and} \qquad q := \sum q^1 \otimes q^2.$$

Lemma 2.2.12. *With the foregoing notation we have that for every $a \in A$*

$$\begin{aligned}\sum p^1 \otimes p^2(a) &= \sum \varphi_1^1\psi^1 \otimes \varphi_2^1\psi^2\Phi^1 S\left(a\varphi^2\psi_1^3\Phi^2\right)\varphi^3\psi_2^3\Phi^3,\\ \sum q^1(a) \otimes q^2 &= \sum \Phi^1\varphi_1^1\psi^1 S\left(\Phi^2\varphi_2^1\psi^2 a\right)\Phi^3\varphi^2\psi_1^3 \otimes \varphi^3\psi_2^3.\end{aligned} \tag{2.20}$$

Moreover, the following relations hold for every $a, b \in A$

$$\sum p^1 a \otimes p^2(b) = \sum a_{11}p^1 \otimes a_{12}p^2(ba_2), \tag{2.21}$$

$$\sum q^1(a) \otimes bq^2 = \sum q^1(b_1a)b_{21} \otimes q^2b_{22}. \tag{2.22}$$

Proof. In view of relation (1.16a), the reassociator Φ satisfies

$$\sum \varphi_1^1\psi^1 \otimes \varphi_2^1\psi^2\Phi^1 \otimes \varphi^2\psi_1^3\Phi^2 \otimes \varphi^3\psi_2^3\Phi^3 = \sum \varphi^1\psi^1 \otimes \varphi^2\psi_1^2 \otimes \varphi^3\psi_2^2 \otimes \psi^3.$$

Applying $(A \otimes m) \circ (A \otimes A \otimes m) \circ (A \otimes A \otimes (S \leftharpoonup a) \otimes A)$ to both sides we get

$$\begin{aligned}\sum \varphi_1^1\psi^1\otimes\varphi_2^1\psi^2\Phi^1 S\left(a\varphi^2\psi_1^3\Phi^2\right)\varphi^3\psi_2^3\Phi^3 &= \sum \varphi^1\psi^1 \otimes \varphi^2\psi_1^2 S\left(a\varphi^3\psi_2^2\right)\psi^3\\ &\overset{(2.11a)}{=} \sum \varphi^1 \otimes \varphi^2 S\left(a\varphi^3\right) = \sum p^1 \otimes p^2(a),\end{aligned}$$

which is the first identity in (2.20). The second one is proved analogously. Let us check that (2.21) holds as well ((2.22) is proved similarly). We compute

$$\begin{aligned}\sum p^1 a \otimes p^2(b) &\overset{(2.19)}{=} \sum \varphi^1 a \otimes \varphi^2 S(b\varphi^3) \overset{(2.11a)}{=} \sum \varphi^1 a_1 \otimes \varphi^2 a_{21} S(b\varphi^3 a_{22})\\ &\overset{(1.17a)}{=} \sum a_{11}\varphi^1 \otimes a_{12}\varphi^2 S(ba_2\varphi^3) = \sum a_{11}p^1 \otimes a_{12}p^2(ba_2).\end{aligned}$$ □

Lemma 2.2.13. *Let $(A, m, u, \Delta, \varepsilon, \Phi, S)$ be a quasi-bialgebra with preantipode and let p, q be defined as above. For all $a \in A$ we have that*

$$S(a) = \sum q^1(1)S\left(p^1aq^2\right)p^2(1) = \sum S\left(\varphi^1\right)\varphi^2 S\left(\psi^1 a\varphi^3\right)\psi^2 S\left(\psi^3\right).$$

Proof. Keeping in mind that Φ^{-1} is counital, i.e. that it satisfies

$$(\varepsilon \otimes A \otimes A)\left(\Phi^{-1}\right) = 1 \otimes 1 = (A \otimes \varepsilon \otimes A)\left(\Phi^{-1}\right) = 1 \otimes 1 = (A \otimes A \otimes \varepsilon)\left(\Phi^{-1}\right),$$

we may compute directly

$$\begin{aligned}&\sum S\left(\varphi^1\right)\varphi^2 S\left(\psi^1 a\varphi^3\right)\psi^2 S\left(\psi^3\right) = \sum q^1(1)S\left(p^1aq^2\right)p^2(1)\\ &\overset{(2.20)}{=} \sum \Phi^1\varphi_1^1\psi^1 S\left(\Phi^2\varphi_2^1\psi^2\right)\Phi^3\varphi^2\psi_1^3 S\left(\gamma_1^1\phi^1 a\varphi^3\psi_2^3\right)\gamma_2^1\phi^2\Psi^1 S\left(\gamma^2\phi_1^3\Psi^2\right)\gamma^3\phi_2^3\Psi^3\\ &\overset{(2.11)}{=} \sum \Phi^1\varphi_1^1 S\left(\Phi^2\varphi_2^1\right)\Phi^3\varphi^2 S\left(\phi^1 a\varphi^3\right)\phi^2\Psi^1 S\left(\phi_1^3\Psi^2\right)\phi_2^3\Psi^3\\ &\overset{(2.11)}{=} \sum \Phi^1 S\left(\Phi^2\right)\Phi^3 S\left(a\right)\Psi^1 S\left(\Psi^2\right)\Psi^3 \overset{(2.11c)}{=} S(a).\end{aligned}$$ □

Proposition 2.2.14. *Let $(A, m, u, \Delta, \varepsilon, \Phi, S)$ be a quasi-bialgebra with preantipode. For all $a, b \in A$ we have*

$$S\left(ab\right) = \sum S\left(\varphi^1 b\right)\varphi^2 S\left(\psi^1\varphi^3\right)\psi^2 S\left(a\psi^3\right). \tag{2.23}$$

Proof. We know from Lemma 2.2.13 that $S(a) = \sum q^1(1)S\left(p^1aq^2\right)p^2(1)$. Relation (2.23) is proved directly by applying it to $S\left(ab\right)$:

$$\begin{aligned}S\left(ab\right) &= \sum q^1(1)S\left(p^1abq^2\right)p^2(1) \overset{(2.21)}{=} \sum q^1(1)S\left(a_{11}p^1bq^2\right)a_{12}p^2(a_2)\\ &\overset{(2.11b)}{=} \sum q^1(1)S\left(p^1bq^2\right)p^2(a) \overset{(2.22)}{=} \sum q^1(b_1)b_{21}S\left(p^1q^2b_{22}\right)p^2(a)\\ &\overset{(2.11a)}{=} \sum q^1(b)S\left(p^1q^2\right)p^2(a) = \sum S\left(\varphi^1 b\right)\varphi^2 S\left(\psi^1\varphi^3\right)\psi^2 S\left(a\psi^3\right).\end{aligned}$$ □

In case $\Phi = 1 \otimes 1 \otimes 1$ or, equivalently, if A is an ordinary Hopf algebra (see Remark 2.2.5), formula (2.23) turns out to be the anti-multiplicativity of the antipode: $S(ab) = S(b)S(a)$ for all $a, b \in A$. Thus, it can be considered in general as an anti-multiplicativity of the preantipode.

2.3 Coquasi-bialgebras with preantipode and rigid monoidal categories

It is well-known that every rigid monoidal category together with a monoidal functor to the category $\mathfrak{M}_f$ of finitely generated and projective $\Bbbk$-modules gives rise to a Hopf algebra (see e.g. [U]). It has been shown by Majid in [Mj3] that every monoidal category $\mathcal{M}$ together with a neutral tensor functor $\boldsymbol{\omega} : \mathcal{M} \to \mathfrak{M}_f$ gives rise to a coquasi-bialgebra H (see Section 1.8 for a brief review on the construction). However, without further assumptions (as the existence of a natural isomorphism between $\boldsymbol{\omega}(-^\star)$ and $\boldsymbol{\omega}(-)^*$, as in [Ha]), there seemed to be no way to relate the rigidity of $\mathcal{M}$ with a richer structure on H.

Our aim in this section is to show how this rigidity is related with the existence of a preantipode on H. We will do this without any further assumption on $\mathcal{M}$, apart from rigidity.

As a by-product, we will recover the fact that the existence of a preantipode for a coquasi-bialgebra over a field is related with the category of its finite-dimensional comodules being rigid, as can be inferred from [Sc8, Theorem 2.6] and [AP1, Theorem 3.9]. Explicitly, coquasi-bialgebras with preantipode can be characterized as those coquasi-bialgebras whose category of corepresentations with finitely generated and projective underlying $\Bbbk$-module is rigid.

Before proceeding, recall that if H is a coquasi-bialgebra (see Definition 1.7.8), then a *preantipode* for H ([AP1, Definition 3.6]) is a $\Bbbk$-linear map $S : H \to H$ such that, for all $h \in H$,

$$\begin{aligned}\sum S(h_1)_1 h_2 \otimes S(h_1)_2 &= 1_H \otimes S(h),\\ \sum S(h_2)_1 \otimes h_1 S(h_2)_2 &= S(h) \otimes 1_H,\\ \sum \omega(h_1 \otimes S(h_2) \otimes h_3) &= \varepsilon(h).\end{aligned} \tag{2.24}$$

Remark 2.3.1. Notice moreover that if H is a coquasi-bialgebra with a preantipode S, then

$$\sum h_1 S(h_2) = \varepsilon S(h) 1_H = \sum S(h_1) h_2$$

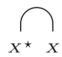
for all $h \in H$, [AP1, Remark 3.7]. In particular, if $\varepsilon S(h) = \varepsilon(h)$ then S is an ordinary antipode.

2.3.1 The natural transformation ∇

Henceforth and unless stated otherwise, we assume that $\mathcal{M}$ is an essentially small right rigid monoidal category endowed with a neutral tensor functor $(\boldsymbol{\omega}, \varphi_0, \varphi) : \mathcal{M} \to \mathfrak{M}_f$ and that a choice $(-)^\star$ of dual objects has been performed. We will denote by $\boldsymbol{\omega}^\star : \mathcal{M}^{\mathrm{op}} \to \mathfrak{M}$ the functor given by $\boldsymbol{\omega}^\star = \boldsymbol{\omega} \circ (-)^\star$, which sends every object X in $\mathcal{M}$ to $\boldsymbol{\omega}(X^\star)$. Let us consider the following maps

$$\mathrm{ev}_{\boldsymbol{\omega}(X)} := \varphi_0^{-1} \circ \boldsymbol{\omega}\left(\mathrm{ev}_X\right) \circ \varphi_{X,X^\star} \qquad \text{and} \qquad \mathrm{db}_{\boldsymbol{\omega}(X)} := \varphi_{X^\star,X}^{-1} \circ \boldsymbol{\omega}\left(\mathrm{db}_X\right) \circ \varphi_0, \tag{2.25}$$

which we will represent simply as $\mathrm{ev}_{\boldsymbol{\omega}(X)} =$ 

and $\mathrm{db}_{\boldsymbol{\omega}(X)} =$ 

.

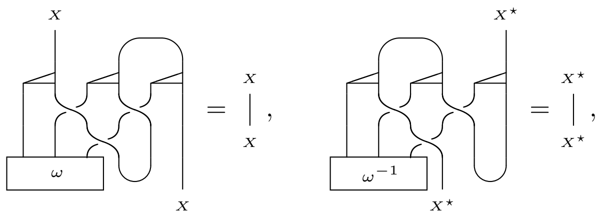
These do not endow $\boldsymbol{\omega}(X^\star)$ with a structure of right dual object of $\boldsymbol{\omega}(X)$ in the category $\mathfrak{M}$ because the functor $\boldsymbol{\omega} : \mathcal{M} \to \mathfrak{M}$ does not satisfy the associativity condition (1.3). Nevertheless, we have the following result, whose proof follows easily from the definitions and the dinaturality of ev and db.

Lemma 2.3.2. *The assignments* $\mathrm{ev}_{\boldsymbol{\omega}(X)}$ *and* $\mathrm{db}_{\boldsymbol{\omega}(X)}$ *defined in* (2.25) *give rise to dinatural transformations* $\mathrm{ev}_{\boldsymbol{\omega}(-)} : \boldsymbol{\omega} \otimes \boldsymbol{\omega}^\star \to \Bbbk$ *and* $\mathrm{db}_{\boldsymbol{\omega}(-)} : \Bbbk \to \boldsymbol{\omega}^\star \otimes \boldsymbol{\omega}$.

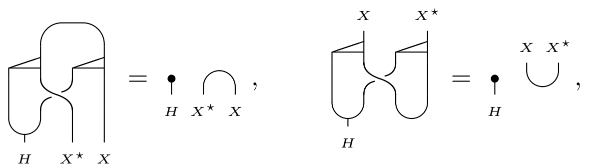
Remark 2.3.3. Recall that if $(\mathcal{F}, \phi_0, \phi) : \mathcal{M} \to \mathcal{N}$ is a monoidal functor between monoidal categories and if X in $\mathcal{M}$ has a right dual $(X^\star, \mathrm{ev}_X, \mathrm{db}_X)$, then $\mathcal{F}(X)$ is right rigid with dual object $\mathcal{F}(X^\star)$ and structure maps

$$\mathrm{ev}_{\mathcal{F}(X)} = \phi_0^{-1} \circ \mathcal{F}\left(\mathrm{ev}_X\right) \circ \phi_{X,X^\star} \qquad \text{and} \qquad \mathrm{db}_{\mathcal{F}(X)} = \phi_{X^\star,X}^{-1} \circ \mathcal{F}\left(\mathrm{db}_X\right) \circ \phi_0$$

(cf. e.g. [St, page 86]). Therefore, even if $\boldsymbol{\omega}(X^\star)$ is not a right dual of $\boldsymbol{\omega}(X)$ in $\mathfrak{M}$, $(\boldsymbol{\omega}(X^\star), \delta_{X^\star})$ is a right dual of $(\boldsymbol{\omega}(X), \delta_X)$ in ${}^H\mathfrak{M}$ because $\boldsymbol{\omega}^H : \mathcal{M} \to {}^H\mathfrak{M}$ is monoidal. Evaluation and coevaluation maps are the same given in (2.25) and they are morphisms of comodules. In particular,



(2.26)



(2.27)

where (2.26) encodes relations (1.5) and (1.6).

Let us pick an object X in $\mathcal{M}$. As a matter of notation, we are going to write

$$\mathrm{db}_{\boldsymbol{\omega}(X)}(1_\Bbbk) = \sum_t \lambda^t \otimes x^t \in \boldsymbol{\omega}(X^\star) \otimes \boldsymbol{\omega}(X)$$

and also $\gamma(x) := \mathrm{ev}_{\boldsymbol{\omega}(X)}(x \otimes \gamma)$ for all $x \in \boldsymbol{\omega}(X)$ and $\gamma \in \boldsymbol{\omega}(X^\star)$. Since $\boldsymbol{\omega}(X^\star)$ is dual of $\boldsymbol{\omega}(X)$ in ${}^H\mathfrak{M}$, with these conventions we may explicitly write (2.26) as follows

$$y = \sum_t \omega\left(y_{-1} \otimes \lambda^t_{-1} \otimes x^t_{-1}\right) \lambda^t_0(y_0)\, x^t_0, \qquad \gamma = \sum_t \omega^{-1}\left(\lambda^t_{-1} \otimes x^t_{-1} \otimes \gamma_{-1}\right) \gamma_0\left(x^t_0\right) \lambda^t_0,$$

for all $y \in \boldsymbol{\omega}(X)$ and for all $\gamma \in \boldsymbol{\omega}(X^\star)$.

Lemma 2.3.4. *We have natural transformations $\nu : \boldsymbol{\omega}(-^\star) \to \boldsymbol{\omega}(-)^*$ and $\nu' : \boldsymbol{\omega}(-)^* \to \boldsymbol{\omega}(-^\star)$ such that $\nu_X(\gamma)(x) = \gamma(x)$ and $\nu'_X(f) = \sum_t \lambda^t f(x^t)$ for all $x \in \boldsymbol{\omega}(X)$, $\gamma \in \boldsymbol{\omega}(X^\star)$ and $f \in \boldsymbol{\omega}(X)^*$, where $(-)^*$ denotes the linear dual.*

Proof. The morphism $\nu_X : \boldsymbol{\omega}(X^\star) \to \boldsymbol{\omega}(X)^*$ is defined as the composition

$$\boldsymbol{\omega}(X^\star) \xrightarrow{\mathrm{db}^\Bbbk_{\boldsymbol{\omega}(X)} \otimes \boldsymbol{\omega}(X^\star)} \boldsymbol{\omega}(X)^* \otimes \boldsymbol{\omega}(X) \otimes \boldsymbol{\omega}(X^\star) \xrightarrow{\boldsymbol{\omega}(X)^* \otimes \mathrm{ev}_{\boldsymbol{\omega}(X)}} \boldsymbol{\omega}(X)^*,$$

where $\mathrm{db}^\Bbbk_{\boldsymbol{\omega}(X)} : \Bbbk \to \boldsymbol{\omega}(X)^* \otimes \boldsymbol{\omega}(X)$ is the (twisted version of the) ordinary dual basis map for finitely generated and projective $\Bbbk$-modules. The morphism the other way around, $\nu'_X : \boldsymbol{\omega}(X)^* \to \boldsymbol{\omega}(X^\star)$, is given analogously as the composition

$$\boldsymbol{\omega}(X)^* \xrightarrow{\mathrm{db}_{\boldsymbol{\omega}(X)} \otimes \boldsymbol{\omega}(X)^*} \boldsymbol{\omega}(X^\star) \otimes \boldsymbol{\omega}(X) \otimes \boldsymbol{\omega}(X)^* \xrightarrow{\boldsymbol{\omega}(X^\star) \otimes \mathrm{ev}^\Bbbk_{\boldsymbol{\omega}(X)}} \boldsymbol{\omega}(X^\star).$$

Naturality in X of both maps is a straightforward computation. □

Remark 2.3.5. Notice that these are not inverses each other in general. Moreover, since $\boldsymbol{\omega}(X)^*$ does not have a natural structure of left H-comodule (it is a *right* H-comodule in fact), the linear maps ν_X and ν'_X cannot be seen as maps in ${}^H\mathfrak{M}$.

Lemma 2.3.6. *Let X be an object in $\mathcal{M}$ and V in $\mathfrak{M}$. We have linear morphisms*

$$\phi_{X,V} : \boldsymbol{\omega}\left(X^{\star}\right) \otimes V \to \mathsf{Hom}_{\Bbbk}\left(\boldsymbol{\omega}\left(X\right), V\right), \quad \psi_{X,V} : \mathsf{Hom}_{\Bbbk}\left(\boldsymbol{\omega}\left(X^{\star}\right), V\right) \to V \otimes \boldsymbol{\omega}\left(X\right)$$

which are natural both in X and in V. Explicitly, for every generator $\gamma \otimes v \in \boldsymbol{\omega}\left(X^{\star}\right) \otimes V$, every $x \in \boldsymbol{\omega}(X)$ and every $f : \boldsymbol{\omega}\left(X^{\star}\right) \to V$ we have

$$\phi_{X,V}\left(\gamma \otimes v\right)\left(x\right) = \gamma\left(x\right) v \qquad \textit{and} \qquad \psi_{X,V}\left(f\right) := \sum_{t} f\left(\lambda^{t}\right) \otimes x^{t}.$$

Proof. Recall that since $\boldsymbol{\omega}(X)$ is finitely generated and projective, we have isomorphisms

$$\boldsymbol{\omega}\left(X\right)^{*} \otimes V \cong \mathsf{Hom}_{\Bbbk}\left(\boldsymbol{\omega}\left(X\right), V\right) \qquad \text{and} \qquad \mathsf{Hom}_{\Bbbk}\left(\boldsymbol{\omega}\left(X\right)^{*}, V\right) \cong V \otimes \boldsymbol{\omega}\left(X\right)$$

natural in V and X (in fact, they are the same up to a twist and the isomorphism $\boldsymbol{\omega}(X)^{**} \cong \boldsymbol{\omega}(X)$). If we pre-compose them with $\nu_X \otimes V$ and $\mathsf{Hom}_{\Bbbk}\left(\nu'_X, V\right)$ respectively, we find the natural transformations of the statement. $\square$

Lemma 2.3.7. *For every $V \in \mathfrak{M}$ we have natural bijections in* Set

$$\begin{aligned} \Phi_V &: \mathsf{Nat}\left(\boldsymbol{\omega}^{\star}, \mathsf{Hom}_{\Bbbk}\left(\boldsymbol{\omega}, V\right)\right) \to \mathsf{Dinat}\left(\boldsymbol{\omega}^{\star} \otimes \boldsymbol{\omega}, V\right), \\ \Psi_V &: \mathsf{Dinat}\left(\boldsymbol{\omega}^{\star} \otimes \boldsymbol{\omega}, V\right) \to \mathsf{Nat}\left(\boldsymbol{\omega}, \mathsf{Hom}_{\Bbbk}\left(\boldsymbol{\omega}^{\star}, V\right)\right). \end{aligned}$$

Proof. First of all, let us show that the statement makes sense, that is that the objects we are working with are in fact sets. Recall from [ML, IX.6] that a coend $\int^X \boldsymbol{\omega}\left(X^{\star}\right) \otimes \boldsymbol{\omega}\left(X\right)$ of the functor $\boldsymbol{\omega}^{\star} \otimes \boldsymbol{\omega} : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathfrak{M}$ is a dinatural transformation $\zeta : \boldsymbol{\omega}^{\star} \otimes \boldsymbol{\omega} \to \int^X \boldsymbol{\omega}\left(X^{\star}\right) \otimes \boldsymbol{\omega}\left(X\right)$, universal among dinatural transformations from $\boldsymbol{\omega}^{\star} \otimes \boldsymbol{\omega}$ to a constant. Since $\mathfrak{M}$ is cocomplete and $\mathcal{M}$ is essentially small, the coend $\int^X \boldsymbol{\omega}\left(X^{\star}\right) \otimes \boldsymbol{\omega}\left(X\right)$ exists and we have a bijective correspondence

$$\mathsf{Dinat}\left(\boldsymbol{\omega}^{\star} \otimes \boldsymbol{\omega}, V\right) \cong \mathsf{Hom}_{\Bbbk}\left(\int^{X} \boldsymbol{\omega}\left(X^{\star}\right) \otimes \boldsymbol{\omega}\left(X\right), V\right)$$

for every V. This implies that $\mathsf{Dinat}\left(\boldsymbol{\omega}^{\star} \otimes \boldsymbol{\omega}, V\right)$ is in fact a set. Once proven the existence of the bijective correspondences Φ_V and Ψ_V, we will have that both $\mathsf{Nat}\left(\boldsymbol{\omega}^{\star}, \mathsf{Hom}_{\Bbbk}\left(\boldsymbol{\omega}, V\right)\right)$ and $\mathsf{Nat}\left(\boldsymbol{\omega}, \mathsf{Hom}_{\Bbbk}\left(\boldsymbol{\omega}^{\star}, V\right)\right)$ are sets as well. In turn, the bijections are explicitly given by

$$\begin{aligned} \Phi_V\left(\nu\right)_X\left(\gamma \otimes x\right) &= \nu_X\left(\gamma\right)\left(x\right), & \Phi_V^{-1}\left(\delta\right)_X\left(\gamma\right)(x) &= \delta_X(\gamma \otimes x), \\ \Psi_V\left(\delta\right)_X\left(x\right)\left(\gamma\right) &= \delta_X\left(\gamma \otimes x\right), & \Psi_V^{-1}\left(\mu\right)_X\left(\gamma \otimes x\right) &= \mu_X(x)(\gamma), \end{aligned}$$

for every ν in $\mathsf{Nat}\left(\boldsymbol{\omega}^{\star}, \mathsf{Hom}_{\Bbbk}\left(\boldsymbol{\omega}, V\right)\right)$, δ in $\mathsf{Dinat}\left(\boldsymbol{\omega}^{\star} \otimes \boldsymbol{\omega}, V\right)$, X in $\mathcal{M}$, $x \in \boldsymbol{\omega}\left(X\right)$ and $\gamma \in \boldsymbol{\omega}\left(X^{\star}\right)$. Since checking that Φ_V^{-1} and Ψ_V^{-1} are in fact inverses of Φ_V and Ψ_V respectively and that Φ_V and Ψ_V are natural in V is analogous to the classical hom-tensor adjunction case, we will skip it. $\square$

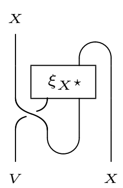
As a consequence, we may consider the chain of natural transformations

$$\begin{tikzcd}[column sep=tiny] \mathsf{Nat}\left(\boldsymbol{\omega}^{\star}, - \otimes \boldsymbol{\omega}^{\star}\right) \arrow[dr, "(\phi\circ\tau)\circ-"'] & & \mathsf{Nat}\left(\boldsymbol{\omega}, \mathsf{Hom}_{\Bbbk}\left(\boldsymbol{\omega}^{\star}, -\right)\right) \arrow[dr, "\psi\circ-"] & \\ & \mathsf{Nat}\left(\boldsymbol{\omega}^{\star}, \mathsf{Hom}_{\Bbbk}\left(\boldsymbol{\omega}, -\right)\right) \arrow[ur, "\Psi\circ\Phi"] & & \mathsf{Nat}\left(\boldsymbol{\omega}, - \otimes \boldsymbol{\omega}\right) \end{tikzcd}$$

whose composition induces a natural transformation $\nabla^{\boldsymbol{\omega}} : \mathsf{Nat}(\boldsymbol{\omega}, - \otimes \boldsymbol{\omega}) \to \mathsf{Nat}(\boldsymbol{\omega}, - \otimes \boldsymbol{\omega})$ given by

$$\nabla^{\boldsymbol{\omega}}_V(\xi)_X = \Big(V \otimes \mathrm{ev}_{\boldsymbol{\omega}(X)} \otimes \boldsymbol{\omega}(X)\Big) \circ \tau_{\boldsymbol{\omega}(X),V} \circ \Big(\boldsymbol{\omega}(X) \otimes \xi_{X^{\star}} \otimes \boldsymbol{\omega}(X)\Big) \circ \Big(\boldsymbol{\omega}(X) \otimes \mathrm{db}_{\boldsymbol{\omega}(X)}\Big) \tag{2.28}$$

for all V in $\mathfrak{M}$, $\xi \in \mathsf{Nat}(\boldsymbol{\omega}, V \otimes \boldsymbol{\omega})$ and X in $\mathcal{M}$. Graphically, this may be represented by

$\nabla^{\boldsymbol{\omega}}_V(\xi)_X =$ 

. (2.29)

Proposition 2.3.8. *Let $\mathcal{M}$ and $\mathcal{N}$ be essentially small right rigid monoidal categories. Let $(\mathcal{V},\psi_0,\psi)$, $\mathcal{V}:\mathcal{N}\to\mathfrak{M}_f$, be a neutral tensor functor and let $(\mathcal{G},\zeta_0,\zeta)$, $\mathcal{G}:\mathcal{M}\to\mathcal{N}$, be a monoidal one. For all $V\in\mathfrak{M}$ and $\xi\in\mathsf{Nat}(\mathcal{V},V\otimes\mathcal{V})$ we have*

$$\nabla^{\mathcal{V}}_V(\xi)\mathcal{G}=\nabla^{\mathcal{V}\mathcal{G}}_V(\xi\mathcal{G}). \tag{2.30}$$

Proof. Assume that we are given a choice of right duals $(-)^\star$ in $\mathcal{M}$ and $(-)^\vee$ in $\mathcal{N}$. Since $\mathcal{G}$ is monoidal we have a natural isomorphism $\kappa_X:\mathcal{G}(X^\star)\to\mathcal{G}(X)^\vee$ as in (1.7). Note that the composition $\mathcal{V}\mathcal{G}$ is still a neutral tensor functor with structure isomorphisms $\phi=(\mathcal{V}\zeta)\circ\psi(\mathcal{G}\times\mathcal{G})$ and $\phi_0=\mathcal{V}(\zeta_0)\circ\psi_0$. We will need the following relations, which descend from (1.8),

$$(\mathcal{V}\kappa\otimes\mathcal{V}\mathcal{G})\circ\mathrm{db}(\mathcal{V}\mathcal{G})=(\mathrm{db}\mathcal{V})\mathcal{G}\quad\text{ and }\quad\mathrm{ev}(\mathcal{V}\mathcal{G})=(\mathrm{ev}\mathcal{V})\,\mathcal{G}\circ(\mathcal{V}\mathcal{G}\otimes\mathcal{V}\kappa)\,.$$

That is, for every object X in $\mathcal{M}$ we have

$$(\mathcal{V}\left(\kappa_X\right)\otimes\mathcal{V}\mathcal{G}(X))\circ\mathrm{db}_{\mathcal{V}\mathcal{G}(X)}=\mathrm{db}_{\mathcal{V}(\mathcal{G}(X))},\qquad \mathrm{ev}_{\mathcal{V}\mathcal{G}(X)}=\mathrm{ev}_{\mathcal{V}(\mathcal{G}(X))}\circ(\mathcal{V}\mathcal{G}(X)\otimes\mathcal{V}\left(\kappa_X\right)). \tag{2.31}$$

As a consequence, for every $\xi\in\mathsf{Nat}(\mathcal{V},V\otimes\mathcal{V})$ we can compute directly (with the same convention adopted in §1.8 for tensoring with identity morphisms)

$$\begin{aligned}\nabla^{\mathcal{V}}_V(\xi)_{\mathcal{G}(X)} &\overset{(2.28)}{=} (V\otimes\mathrm{ev}_{\mathcal{V}(\mathcal{G}(X))}\otimes\mathcal{V}\mathcal{G}\left(X\right))\circ\tau_{\mathcal{V}\mathcal{G}(X),V}\circ\xi_{\mathcal{G}(X)^\vee}\circ(\mathcal{V}\mathcal{G}\left(X\right)\otimes\mathrm{db}_{\mathcal{V}(\mathcal{G}(X))})\\ &\overset{(2.31)}{=} (V\otimes\mathrm{ev}_{\mathcal{V}(\mathcal{G}(X))}\otimes\mathcal{V}\mathcal{G}\left(X\right))\circ\tau_{\mathcal{V}\mathcal{G}(X),V}\circ\xi_{\mathcal{G}(X)^\vee}\circ\mathcal{V}\left(\kappa_X\right)\circ(\mathcal{V}\mathcal{G}\left(X\right)\otimes\mathrm{db}_{\mathcal{V}\mathcal{G}(X)})\\ &\overset{(*)}{=} (V\otimes\mathrm{ev}_{\mathcal{V}(\mathcal{G}(X))}\otimes\mathcal{V}\mathcal{G}\left(X\right))\circ\tau_{\mathcal{V}\mathcal{G}(X),V}\circ\mathcal{V}\left(\kappa_X\right)\circ\xi_{\mathcal{G}(X^\star)}\circ(\mathcal{V}\mathcal{G}\left(X\right)\otimes\mathrm{db}_{\mathcal{V}\mathcal{G}(X)})\\ &\overset{(2.31)}{=} (V\otimes\mathrm{ev}_{\mathcal{V}\mathcal{G}(X)}\otimes\mathcal{V}\mathcal{G}\left(X\right))\circ\tau_{\mathcal{V}\mathcal{G}(X),V}\circ\xi_{\mathcal{G}(X^\star)}\circ(\mathcal{V}\mathcal{G}\left(X\right)\otimes\mathrm{db}_{\mathcal{V}\mathcal{G}(X)})\overset{(2.28)}{=}\nabla^{\mathcal{V}\mathcal{G}}_V\left(\xi\mathcal{G}\right)_X\end{aligned}$$

where in $(*)$ we used the naturality of ξ. ☐

Corollary 2.3.9. *Assume that $\mathcal{M}$ is an essentially small right rigid monoidal category and that $\boldsymbol{\omega}:\mathcal{M}\to\mathfrak{M}_f$ is a neutral tensor functor. The natural transformation $\nabla^{\boldsymbol{\omega}}$ of (2.28) does not depend on the choice of the dual objects.*

Proof. It is enough to take $\mathcal{N}=\mathcal{M}$ and $\mathcal{G}=\mathsf{Id}_{\mathcal{M}}$ in the proof of Proposition 2.3.8. ☐

Remark 2.3.10. Mimiking [Sc7] we may consider a category $\mathfrak{C}$ whose objects are pairs $(\mathcal{M},\boldsymbol{\omega})$ where $\mathcal{M}$ is an essentially small right rigid monoidal category and $\boldsymbol{\omega}:\mathcal{M}\to\mathfrak{M}_f$ is a neutral tensor functor. Morphisms in $\mathfrak{C}$ between two objects $(\mathcal{M},\boldsymbol{\omega})$ and $(\mathcal{N},\boldsymbol{\omega}')$ are given by monoidal functors $\mathcal{G}:\mathcal{M}\to\mathcal{N}$ such that $\boldsymbol{\omega}'\mathcal{G}=\boldsymbol{\omega}$ as tensor functors. It follows from Proposition 2.3.8 that the transformation $\nabla^{\sim}$ introduced above is a natural transformation between the functor $\mathsf{Nat}(\sim,-\otimes\sim):\mathfrak{C}\to\mathsf{Funct}(\mathfrak{M},\mathsf{Set})$ sending $(\mathcal{M},\boldsymbol{\omega})$ to $\mathsf{Nat}(\boldsymbol{\omega},-\otimes\boldsymbol{\omega})$ and itself.

2.3.2 Rigidity and the preantipode

In this subsection we show how to provide a preantipode for the coendomorphism coquasi-bialgebra of a right rigid monoidal category $\mathcal{M}$ with a neutral tensor functor $\boldsymbol{\omega}:\mathcal{M}\to\mathfrak{M}_f$. The key information is the existence of the natural transformation $\nabla^{\boldsymbol{\omega}}$. In fact, since H represents the functor $\mathsf{Nat}(\boldsymbol{\omega},-\otimes\boldsymbol{\omega}):\mathfrak{M}\to\mathsf{Set}$ and in light of Yoneda Lemma we have

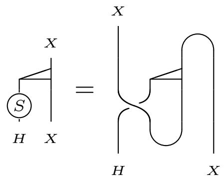
$$\mathsf{Nat}\Big(\mathsf{Nat}(\boldsymbol{\omega},-\otimes\boldsymbol{\omega}),\mathsf{Nat}(\boldsymbol{\omega},-\otimes\boldsymbol{\omega})\Big)\cong\mathsf{Nat}\Big(\mathsf{Hom}\left(H,-\right),\mathsf{Nat}(\boldsymbol{\omega},-\otimes\boldsymbol{\omega})\Big)\cong\mathsf{Nat}(\boldsymbol{\omega},H\otimes\boldsymbol{\omega})$$

and hence there exists a unique natural transformation in $\mathsf{Nat}(\boldsymbol{\omega},H\otimes\boldsymbol{\omega})$ which corresponds to $\nabla^{\boldsymbol{\omega}}$ and it is exactly $\nabla^{\boldsymbol{\omega}}_H(\delta)$. Its component at X is

$$\nabla^{\boldsymbol{\omega}}_H(\delta)_X=\Big(H\otimes\mathrm{ev}_{\boldsymbol{\omega}(X)}\otimes\boldsymbol{\omega}(X)\Big)\circ\tau_{\boldsymbol{\omega}(X),H}\circ\Big(\boldsymbol{\omega}(X)\otimes\delta_{X^\star}\otimes\boldsymbol{\omega}(X)\Big)\circ\Big(\boldsymbol{\omega}(X)\otimes\mathrm{db}_{\boldsymbol{\omega}(X)}\Big). \tag{2.32}$$

Moreover, there exists a unique linear endomorphism S of H such that

$$\vartheta_H(S)_X = \quad = \quad = \nabla_H^{\boldsymbol{\omega}}(\delta)_X. \tag{2.33}$$

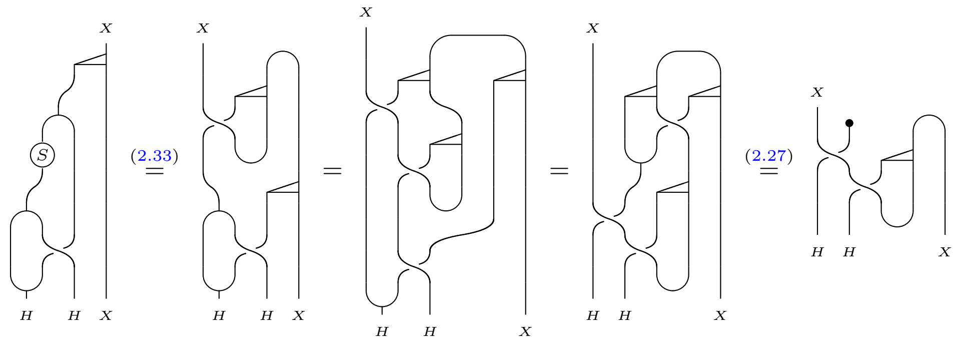


Notice that for all $g : H \to V$ in $\mathfrak{M}$ we have

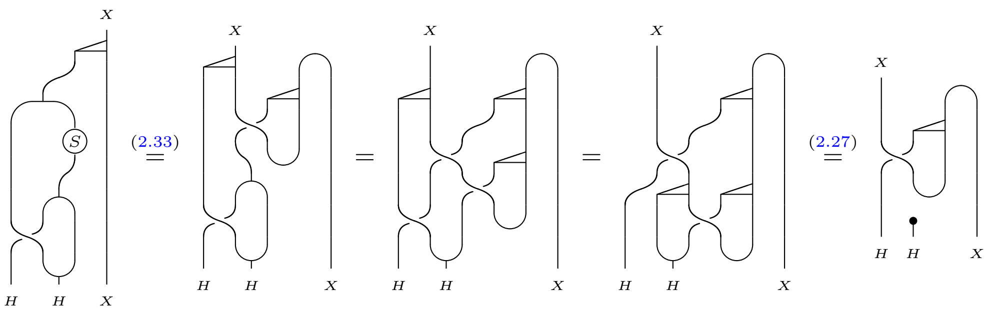
$$\vartheta_V(g \circ S) = \nabla_V^{\boldsymbol{\omega}}((g \otimes \boldsymbol{\omega}) \circ \delta). \tag{2.34}$$

Lemma 2.3.11. *The unique S satisfying $\vartheta_H(S) = \nabla_H^{\boldsymbol{\omega}}(\delta)$ is a preantipode for H.*

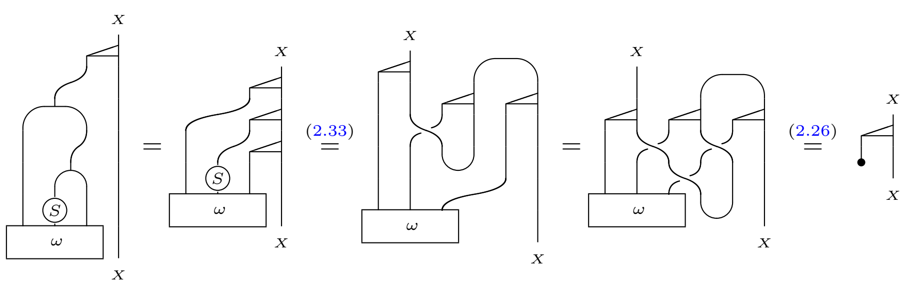
Proof. In a nutshell, the result follows from the fact that for all X in $\mathcal{M}$, $\boldsymbol{\omega}(X^\star)$ is a right dual for $\boldsymbol{\omega}(X)$ in ${}^H\mathfrak{M}_f$. In details, since $\mathrm{db}_{\boldsymbol{\omega}(X)}$ is H-colinear, it follows that



that is, for every $h \in H$ we have $\sum S(h_1)_1 h_2 \otimes S(h_1)_2 = 1_H \otimes S(h)$. Now, since $\mathrm{ev}_{\boldsymbol{\omega}(X)}$ is H-colinear as well, we have also

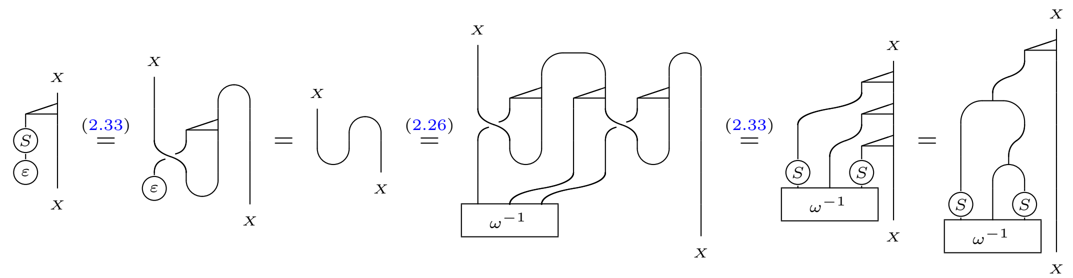


that is, for every $h \in H$ we have $\sum S(h_2)_1 \otimes h_1 S(h_2)_2 = S(h) \otimes 1_H$. Finally



so that $\sum \omega(h_1 \otimes S(h_2) \otimes h_3) = \varepsilon(h)$ for all $h \in H$ and this concludes the proof. $\square$

Remark 2.3.12. Between the distinguished natural transformations in $\mathsf{Nat}(\boldsymbol{\omega},\boldsymbol{\omega})$ that one may consider, there is also $(\mathrm{ev}_{\boldsymbol{\omega}(X)} \otimes \boldsymbol{\omega}(X)) \circ (\boldsymbol{\omega}(X) \otimes \mathrm{db}_{\boldsymbol{\omega}(X)})$. This however does not endow H with a new structure map. Instead, it can be checked that



whence $\omega^{-1}(S(h_1) \otimes h_2 \otimes S(h_3)) = \varepsilon S(h)$ for all $h \in H$ as in [AP2, Lemma 2.14].

Summing up, we proved the following.

Theorem 2.3.13 (Reconstruction Theorem for coquasi-bialgebras with preantipode)**.** *Let $\mathcal{M}$ be an essentially small right rigid monoidal category together with a neutral tensor functor $\boldsymbol{\omega} : \mathcal{M} \to \mathfrak{M}_f$. Then there exists a preantipode S for the coendomorphism coquasi-bialgebra H of $(\mathcal{M},\boldsymbol{\omega})$.*

This is a weaker result with respect to, for example, Theorem 1.8.6, since we are missing the fact that the unique morphism induced by the universal property of H preserves preantipodes. We will prove this in the particular case in which $\Bbbk$ is a field in Subsection 2.3.3 to come. For the moment, let us conclude this subsection by drawing some inferences from the results we got so far.

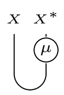
Recall that a *coquasi-Hopf algebra* $(H, m, u, \Delta, \varepsilon, \omega, s, \alpha, \beta)$ is a coquasi-bialgebra H endowed with a *coquasi-antipode* (s, α, β), that is a coalgebra anti-homomorphism $s : H \to H$ and two maps α, β in H^*, such that, for all $h \in H$

$$\sum h_1\beta(h_2)s(h_3) = \beta(h)1_H, \quad \sum s(h_1)\alpha(h_2)h_3 = \alpha(h)1_H,$$
$$\sum \omega(h_1 \otimes \beta(h_2)s(h_3)\alpha(h_4) \otimes h_5) = \varepsilon(h),$$
$$\sum \omega^{-1}(s(h_1) \otimes \alpha(h_2)h_3\beta(h_4) \otimes s(h_5)) = \varepsilon(h).$$

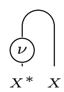
Remark 2.3.14 (A Reconstruction Theorem for Coquasi-Hopf Algebras)**.** This result can be considered as the dual version of [Ha, Lemma 4]. Let $\mathcal{M}$ be an essentially small right rigid monoidal category and let $\boldsymbol{\omega} : \mathcal{M} \to \mathfrak{M}_f$ be a tensor functor. Consider the associated coendomorphism coquasi-bialgebra H. Assume in addition that we have a natural isomorphism $\nu_X : \boldsymbol{\omega}(X^\star) \to \boldsymbol{\omega}(X)^*$ in $\mathfrak{M}_f$. We may endow $\boldsymbol{\omega}(X)^*$ with an H-comodule structure given by $\rho_{\boldsymbol{\omega}(X)^*} := (H \otimes \nu_X) \circ \delta_{X^\star} \circ \nu_X^{-1}$. To simplify the exposition, we will denote it by δ_{X^*}, even if this notation does not strictly make sense. With this coaction, $\boldsymbol{\omega}(X)^*$ becomes a right dual object of $\boldsymbol{\omega}(X)$ in ${}^H\mathfrak{M}_f$ with evaluation and dual basis maps given by

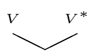
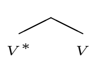
$$\mathrm{ev}^{(*)}_{\boldsymbol{\omega}(X)} = \mathrm{ev}_{\boldsymbol{\omega}(X)} \circ (\boldsymbol{\omega}(X) \otimes \nu_X^{-1}) \qquad \text{and} \qquad \mathrm{db}^{(*)}_{\boldsymbol{\omega}(X)} = (\nu_X \otimes \boldsymbol{\omega}(X)) \circ \mathrm{db}_{\boldsymbol{\omega}(X)}.$$

If we denote ν_X simply by ν and its inverse by μ then we may represent these graphically as



and



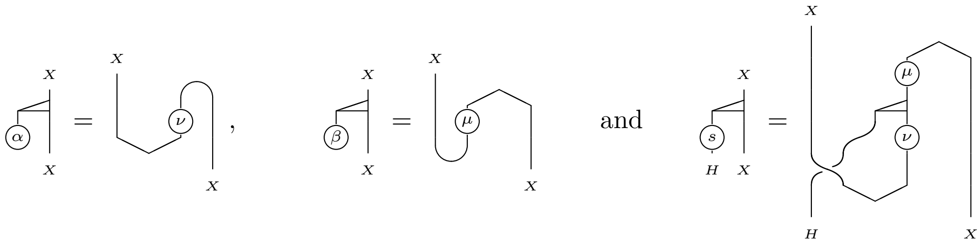
Let $\mathcal{V} : {}^H\mathfrak{M}_f \to \mathfrak{M}_f$ be the forgetful functor and let $\mathrm{ev}^{(\Bbbk)}_V : V \otimes V^* \to \Bbbk$ and $\mathrm{db}^{(\Bbbk)}_V : \Bbbk \to V^* \otimes V$ be the ordinary evaluation and dual basis for finitely generated and projective $\Bbbk$-modules as in Example 1.2.2. Graphically,  and  respectively for every V in $\mathfrak{M}_f$. There exist unique linear morphisms $\alpha, \beta \in H^*$ and $s : H \to H$ such that

$$(\alpha \otimes \boldsymbol{\omega}(X)) \circ \delta_X = \left(\mathrm{ev}^{(\Bbbk)}_{\boldsymbol{\omega}(X)} \otimes \boldsymbol{\omega}(X)\right) \circ \left(\boldsymbol{\omega}(X) \otimes \mathrm{db}^{(*)}_{\boldsymbol{\omega}(X)}\right),$$

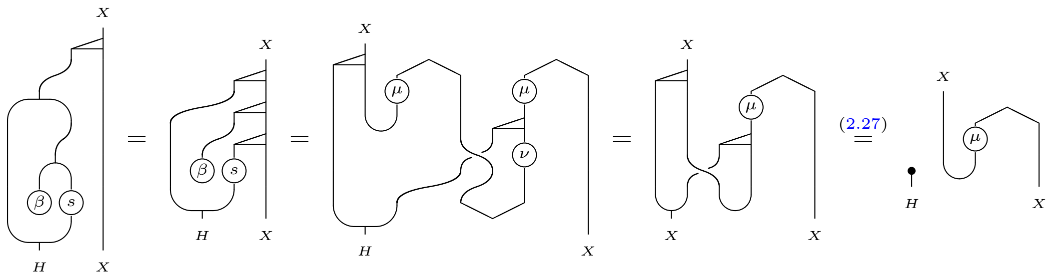
$$(\beta \otimes \boldsymbol{\omega}(X)) \circ \delta_X = \left(\mathrm{ev}^{(*)}_{\boldsymbol{\omega}(X)} \otimes \boldsymbol{\omega}(X)\right) \circ \left(\boldsymbol{\omega}(X) \otimes \mathrm{db}^{(\Bbbk)}_{\boldsymbol{\omega}(X)}\right),$$

$$(s \otimes \boldsymbol{\omega}(X)) \circ \delta_X = \left(H \otimes \mathrm{ev}^{(\Bbbk)}_{\boldsymbol{\omega}(X)} \otimes \boldsymbol{\omega}(X)\right) \circ \tau_{\boldsymbol{\omega}(X),H} \circ \delta_{X^*} \circ \left(\boldsymbol{\omega}(X) \otimes \mathrm{db}^{(\Bbbk)}_{\boldsymbol{\omega}(X)}\right),$$

that is,



It turns out that H is a coquasi-Hopf algebra with coquasi-antipode (s, α, β). Indeed, for example



so that $\sum x_1 \beta(x_2) s(x_3) = \beta(x) 1_H$ for all $x \in H$. A posteriori, the following relations hold

$$\tau_{\boldsymbol{\omega}(X)^*,H} \circ \left(\boldsymbol{\omega}(X)^* \otimes H \otimes \mathrm{ev}^{(\Bbbk)}_{\boldsymbol{\omega}(X)}\right) \circ s \circ \delta_X \circ (\mathrm{db}^{(\Bbbk)}_{\boldsymbol{\omega}(X)} \otimes \boldsymbol{\omega}(X)^*) = \delta_{X^*}, \tag{2.35a}$$

$$(\boldsymbol{\omega}(X)^* \otimes \alpha \otimes \boldsymbol{\omega}(X)) \circ (\boldsymbol{\omega}(X)^* \otimes \delta_X) \circ \mathrm{db}^{(\Bbbk)}_{\boldsymbol{\omega}(X)} = \mathrm{db}^{(*)}_{\boldsymbol{\omega}(X)}, \tag{2.35b}$$

$$\mathrm{ev}^{(\Bbbk)}_{\boldsymbol{\omega}(X)} \circ (\beta \otimes \boldsymbol{\omega}(X) \otimes \boldsymbol{\omega}(X)^*) \circ (\delta_X \otimes \boldsymbol{\omega}(X)^*) = \mathrm{ev}^{(*)}_{\boldsymbol{\omega}(X)}. \tag{2.35c}$$

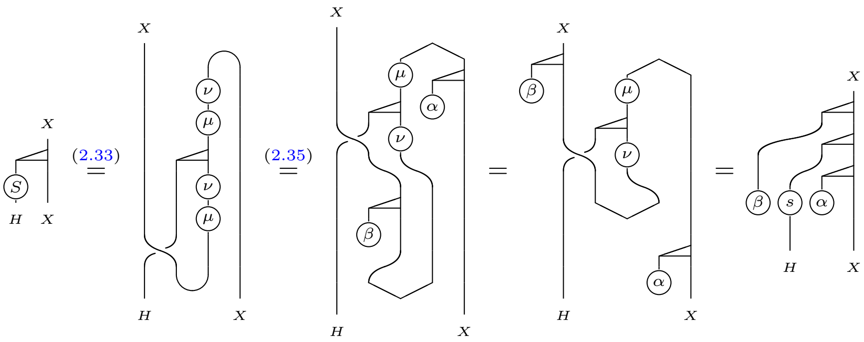
The two examples to come retrieve the well-known result of Ulbrich about Hopf algebras and rigid monoidal categories and the fact that any (coquasi-)Hopf algebra is in particular a coquasi-bialgebra with preantipode respectively.

Example 2.3.15 (Reconstruction Theorem for Hopf Algebras, see [U, page 255, Theorem])**.** Let $\mathcal{M}$ be an essentially small right rigid monoidal category and let $\boldsymbol{\omega} : \mathcal{M} \to \mathfrak{M}_f$ be a monoidal functor. Then the coendomorphism coquasi-bialgebra H provided by Theorem 1.8.6 is a bialgebra (i.e. $\omega = \varepsilon \otimes \varepsilon \otimes \varepsilon$) and the preantipode provided by Theorem 2.3.13 satisfies

$$\varepsilon S(h) = \sum \omega(h_1 \otimes S(h_2) \otimes h_3) \overset{(2.24)}{=} \varepsilon(h),$$

that is, it is an ordinary antipode (see Remark 2.3.1) and H is a Hopf algebra.

Example 2.3.16 ([AP1, Theorem 3.10])**.** Let H be a coquasi-Hopf algebra with coquasi-antipode (s, α, β). It is known that the converse of Remark 2.3.14 holds true, in the sense that the category of finite-dimensional left H-comodules ${}^H\mathfrak{M}_f$ is a right rigid monoidal category. In details, the dual of (V, ρ_V) in ${}^H\mathfrak{M}_f$ is given by its dual vector space V^* with structure maps given as in (2.35) (cf. [Sc5, page 334]). If we consider the forgetful functor $\mathcal{U} : {}^H\mathfrak{M}_f \to \mathfrak{M}_f$, it is a tensor functor, so that we may apply Theorem 2.3.13 as well as Remark 2.3.14 with $\nu_V = \mathsf{Id}_{V^*}$ for all (V, ρ_V) in ${}^H\mathfrak{M}_f$. The outcome is a coquasi-bialgebra with preantipode structure on H, where ω is the former one and S is uniquely given by



Therefore, $S = \beta * s * \alpha$ where $*$ is (essentially) the convolution product.

2.3.3 Coquasi-bialgebras over a field

In this subsection $\Bbbk$ is assumed to be a field and all quasi and coquasi-bialgebras are assumed to be over $\Bbbk$, as well as all modules and comodules. Some results may still old for $\Bbbk$ a commutative ring, but, up to our knowledge, the main tools used here hold only under that stronger hypothesis.

In the case of coquasi-bialgebras over a field, even before showing that the morphism S of Equation (2.33) was a preantipode for the coendomorphism coquasi-bialgebra, two important consequences might be drawn from its uniqueness: the uniqueness of the preantipode for any coquasi-bialgebra and the fact that any morphism of coquasi-bialgebras automatically preserves preantipodes, as it happens in the Hopf algebra case.

Remark 2.3.17. Let B be a coquasi-bialgebra with a preantipode S_B. Denote by $\mathcal{U} : {}^B\mathfrak{M}_f \to \mathfrak{M}_f$ the forgetful functor and by $\rho \in \mathsf{Nat}\,(\mathcal{U}, B \otimes \mathcal{U})$ the natural coaction of the B-comodules in ${}^B\mathfrak{M}_f$. It can be checked, directly or by deducing it from [Sc8], that for V in ${}^B\mathfrak{M}_f$, a right dual of V is given by $V^\star = (V^* \otimes B)^{\mathrm{co}B}$ with coaction $\rho_{V^\star}\left(\sum_t f^t \otimes b^t\right) = \sum_t b_1^t \otimes (f^t \otimes b_2^t)$. Evaluation and dual basis maps are given by

$$\mathrm{ev}_V\left(u \otimes \sum_t (f^t \otimes b^t)\right) = \sum_t f^t(u)\varepsilon(b^t), \qquad \mathrm{db}_V(1_\Bbbk) = \sum_{i=1}^d \left(v_0^i \otimes S_B(v_1^i)\right) \otimes v_i,$$

for all $\sum_t f^t \otimes b^t \in V^\star$, $u \in V$, where $\sum_{i=1}^d v^i \otimes v_i \in V^* \otimes V$ is a dual basis for V as a finite-dimensional vector space and $d = \dim_\Bbbk(V)$. In particular, ${}^B\mathfrak{M}_f$ is right rigid.

Lemma 2.3.18. *If a preantipode for a coquasi-bialgebra B exists, then it is unique.*

Proof. Since ${}^B\mathfrak{M}_f$ is right rigid, $\mathcal{U} : {}^B\mathfrak{M}_f \to \mathfrak{M}_f$ is a tensor functor and B is a representing object for $\mathsf{Nat}(\mathcal{U}, - \otimes \mathcal{U})$, we have the natural transformation $\nabla_B^{\mathcal{U}}(\rho) \in \mathsf{Nat}\,(\mathcal{U}, B \otimes \mathcal{U})$ as in (2.32) (in fact, $\rho = \vartheta_B(\mathsf{Id}_B)$). In view of Remark 2.3.17, we may compute explicitly for all V in ${}^B\mathfrak{M}_f$ and $y \in \mathcal{U}(V)$

$$\begin{aligned}\nabla_B^{\mathcal{U}}(\rho)_V(y) &\overset{(2.32)}{=} ((B \otimes \mathrm{ev}_V \otimes V) \circ \tau_{V,B} \circ \rho_{V^\star})\left(\sum_{i=1}^d y \otimes \left(v_0^i \otimes S_B\left(v_1^i\right)\right) \otimes v_i\right) \\ &= \sum_{i=1}^d S_B\left(v_1^i\right)_1 v_0^i\,(y)\,\varepsilon\left(S_B\left(v_1^i\right)_2\right) \otimes v_i = \sum_{i=1}^d S_B\left(v_0^i\,(y)\,v_1^i\right) \otimes v_i \\ &= \sum_{i=1}^d S_B(y_{-1} v^i\,(y_0)) \otimes v_i = \sum S_B\,(y_{-1}) \otimes y_0,\end{aligned}$$

so that $\nabla_B^{\mathcal{U}}(\rho) = (S_B \otimes \mathcal{U}) \circ \rho$. This means that S_B satisfies condition (2.33) and so it follows that $S_B = S$, the unique linear endomorphism induced on B by $\nabla_B^{\mathcal{U}}(\rho)$. $\square$

Lemma 2.3.19. *Let $g : A \to B$ be a coquasi-bialgebra morphism between coquasi-bialgebras A and B with preantipodes S_A and S_B respectively. Then $g \circ S_A = S_B \circ g$.*

Proof. Since g is a coquasi-bialgebra morphism, it induces a strict monoidal functor ${}^{g}\mathfrak{M} : {}^{A}\mathfrak{M} \to {}^{B}\mathfrak{M}$, which in turn restricts to a strict monoidal functor $\mathcal{G} : {}^{A}\mathfrak{M}_f \to {}^{B}\mathfrak{M}_f$ such that $\mathcal{V}\mathcal{G} = \mathcal{U}$, where $\mathcal{U} : {}^{A}\mathfrak{M}_f \to \mathfrak{M}_f$ and $\mathcal{V} : {}^{B}\mathfrak{M}_f \to \mathfrak{M}_f$ are the forgetful functors. Observe that, in particular, this implies that $(g \otimes \mathcal{U}(X)) \circ \rho^A_X = \rho^B_{\mathcal{G}(X)}$ for every X in ${}^{A}\mathfrak{M}_f$. Let us denote by $\vartheta : \mathsf{Hom}_{\Bbbk}(A, -) \to \mathsf{Nat}\,(\mathcal{U}, - \otimes \mathcal{U})$ as well the natural isomorphism such that $\vartheta_V(f) = (f \otimes \mathcal{U}) \circ \rho^A$ for all V in $\mathfrak{M}$ and $f \in \mathsf{Hom}_{\Bbbk}(A, V)$. We want to show that $\vartheta_B(g \circ S_A) = \vartheta_B(S_B \circ g)$. Omitting the constraints a, r, l, for all X in ${}^{A}\mathfrak{M}_f$ we may compute

$$\begin{aligned}\vartheta_B(g \circ S_A)_X &\overset{(2.34)}{=} \nabla^{\mathcal{U}}_B\left((g \otimes \mathcal{U}) \circ \rho^A\right)_X = \nabla^{\mathcal{V}\mathcal{G}}_B\left(\rho^B \mathcal{G}\right)_X \overset{(2.30)}{=} \nabla^{\mathcal{V}}_B\left(\rho^B\right)_{\mathcal{G}(X)} \\ &= (S_B \otimes \mathcal{V}\mathcal{G}(X))\,\rho^B_{\mathcal{G}(X)} = (S_B \otimes \mathcal{U}(X)) \circ (g \otimes \mathcal{U}(X)) \circ \rho^A_X = \vartheta_B\left(S_B \circ g\right)_X.\end{aligned}$$

Hence $g \circ S_A = S_B \circ g$ as claimed. $\square$

We may now prove the stronger version of the Reconstruction Theorem for coquasi-bialgebras with preantipode that we mentioned after Theorem 2.3.13.

Theorem 2.3.20. *Let $\mathcal{M}$ be an essentially small right rigid monoidal category together with a neutral tensor functor $\boldsymbol{\omega} : \mathcal{M} \to \mathfrak{M}_f$. Then there exists a preantipode S for the coendomorphism coquasi-bialgebra H of $(\mathcal{M}, \boldsymbol{\omega})$. Furthermore, if B is another coquasi-bialgebra with preantipode such that $\boldsymbol{\omega}$ factorizes through a monoidal functor $\mathcal{G} : \mathcal{M} \to {}^{B}\mathfrak{M}$ followed by the forgetful functor, then the unique coquasi-bialgebra morphism $\epsilon : H \to B$ provided by Theorem 1.8.6 preserves the preantipodes.*

Proof. The existence of a preantipode S for H has already been established in Theorem 2.3.13. If B is another coquasi-bialgebra with preantipode as in the statement, then Majid's Theorem 1.8.6 implies that there exists a unique map of coquasi-bialgebras $\epsilon : H \to B$ inducing a functor ${}^{\epsilon}\mathfrak{M} : {}^{H}\mathfrak{M} \to {}^{B}\mathfrak{M}$ such that ${}^{\epsilon}\mathfrak{M}\,\boldsymbol{\omega}^H = \mathcal{G}$. In view of Lemma 2.3.19, the unique morphism ϵ preserves the preantipodes. $\square$

Corollary 2.3.21. *Let C be a $\Bbbk$-coalgebra. Then C is a coquasi-bialgebra with preantipode if and only if the category of finite-dimensional left comodules ${}^{C}\mathfrak{M}_f$ is a right rigid monoidal category and the forgetful functor $\mathcal{U} : {}^{C}\mathfrak{M}_f \to \mathfrak{M}_f$ is a neutral tensor functor.*

As a final application of the theory we developed, let us show that the finite dual coalgebra of a quasi-bialgebra with preantipode is a coquasi-bialgebra with preantipode. The proof of this fact relies on Lemma 2.3.22, which can be deduced from [Ab, Chapter 3, §1.2].

Recall from [Sw, Chapter VI] that, given an algebra A, the vector space

$$A^\circ := \{f \in A^* \mid \ker(f) \supseteq I \text{ for a finite-codimensional ideal } I \subseteq A\} \tag{2.36}$$

can be endowed with a coalgebra structure such that $\varepsilon_\circ(f) = f(1_A)$ and $\Delta(f) = \sum f_1 \otimes f_2$ is uniquely determined by the relation $\sum f_1(a) f_2(b) = f(ab)$ for all $a, b \in A$. This is called the *finite dual coalgebra* of the algebra A.

Lemma 2.3.22. *Let A be an algebra and A° be its finite dual coalgebra. We have an isomorphism $\mathcal{L} : {}^{A^\circ}\mathfrak{M}_f \to {}_f\mathfrak{M}_A$ between the category of finite-dimensional left A°-comodules and that of finite-dimensional right A-modules that satisfies $\mathcal{V}\mathcal{L} = \mathcal{U}$, where $\mathcal{V} : {}_f\mathfrak{M}_A \to \mathfrak{M}_f$ and $\mathcal{U} : {}^{A^\circ}\mathfrak{M}_f \to \mathfrak{M}_f$ are the obvious forgetful functors.*

For the sake of completeness, let us recall that $\mathcal{L}$ associates every left A°-comodule (N, ρ_N) with the right A-module (N, μ^ρ_N) where the action is given by $\mu^\rho_N(n \otimes a) = \sum n_{-1}(a) n_0$. For every M in $\mathfrak{M}_A$ and every $m \in M$, set $\mu_m(a) := \mu_M(m \otimes a)$ for all $a \in A$. Then the inverse functor $\mathcal{R} : {}_f\mathfrak{M}_A \to {}^{A^\circ}\mathfrak{M}_f$ assigns to every finite-dimensional right A-module (M, μ_M), the left A°-comodule (M, ρ^μ_M) with coaction

$$\rho^\mu_M(m) = \sum_{i=1}^{d} \left(e^i \circ \mu_m\right) \otimes e_i \tag{2.37}$$

where $\sum_{i=1}^{d} e^i \otimes e_i \in M^* \otimes M$ is a dual basis for M as a vector space ($d = \dim_{\Bbbk}(M)$). Notice that $\mathcal{U}\mathcal{R} = \mathcal{V}$ as well.

Lemma 2.3.23. *Let $(A, m, u, \Delta, \varepsilon, \Phi, S)$ be a quasi-bialgebra with preantipode. The category of finite-dimensional right A-modules ${}_f\mathfrak{M}_A$ is a right rigid monoidal category with neutral tensor forgetful functor $\mathcal{V} : {}_f\mathfrak{M}_A \to \mathfrak{M}_f$.*

Proof. We already know from Lemma 1.7.2 that the category of right A-modules $\mathfrak{M}_A$ is a monoidal category with a neutral tensor underlying functor. One may check directly that a dual object of a finite-dimensional right A-module M is given by

$$M^\star := \frac{A \otimes M^*}{A^+ (A \otimes M^*)}$$

where $A^+ := \mathsf{ker}\,(\varepsilon)$ and M^* is the $\Bbbk$-linear dual of M. The A-module structure on $M^\star$ is given by $\overline{a \otimes f} \cdot x = \overline{ax \otimes f}$ for all $a, x \in A$ and $f \in M^*$ and the evaluation and dual basis maps by

$$\mathrm{ev}_M\left(m \otimes \overline{a \otimes f}\right) = f\left(m \cdot S(a)\right) \qquad \text{and} \qquad \mathrm{db}_M\left(1_{\Bbbk}\right) = \sum_{i=1}^{d} \overline{1_A \otimes e^i} \otimes e_i$$

for all $m \in M$, $f \in M^*$ and $a \in A$ and where $\sum_{i=1}^{d} e^i \otimes e_i \in M^* \otimes M$ is a dual basis of M as a finite-dimensional vector space and $d = \dim_{\Bbbk}(M)$. ☐

Remark 2.3.24. As an alternative, one may mimic [Sc8] and prove that the free (left) quasi-Hopf bimodule functor $A \otimes - : (\mathfrak{M}_A, \otimes, \Bbbk, \alpha_A) \to ({}^A_A\mathfrak{M}_A, \otimes_A, A)$ is a monoidal functor and that for every $M \in {}_f\mathfrak{M}_A$ the quasi-Hopf A-bimodule ${}^{\bullet}_{\bullet}A_{\bullet} \otimes_{\bullet} (M^*) \cong {}_A\mathsf{Hom}\,(A \otimes M, A)$ is the right dual of $A \otimes M$. Therefore, by the left-handed version of Theorem 2.2.7, if A admits a preantipode then $A \otimes -$ becomes an equivalence which is also a monoidal functor and so its quasi-inverse $\overline{(-)}$ becomes monoidal as well. Thus it sends rigid objects to rigid objects and the $M^\star$ of above is exactly the right dual object obtained in this way.

Proposition 2.3.25. *Assume that $(A, m, u, \Delta, \varepsilon, \Phi, S)$ is a quasi-bialgebra with preantipode. Let $(A^\circ, \Delta_\circ, \varepsilon_\circ)$ be its finite dual coalgebra. Then A° can be endowed with a structure of a coquasi-bialgebra with preantipode.*

Proof. Denote by $\mathcal{V} : {}_f\mathfrak{M}_A \to \mathfrak{M}_f$ and $\mathcal{U} : {}^{A^\circ}\mathfrak{M}_f \to \mathfrak{M}_f$ the forgetful functors. As a consequence of Lemma 2.3.22, we have a chain of natural isomorphism

$$\mathrm{Nat}\,(\mathcal{V}, - \otimes \mathcal{V}) \cong \mathrm{Nat}\,(\mathcal{U}, - \otimes \mathcal{U}) \cong \mathsf{Hom}_{\Bbbk}\,(A^\circ, -)$$

which allows us to consider A° itself as a representing object for $\mathrm{Nat}\,(\mathcal{V}, - \otimes \mathcal{V})$. If we consider then the category of finite-dimensional right A-modules ${}_f\mathfrak{M}_A$ as a right rigid monoidal category together with a neutral tensor forgetful functor $\mathcal{V} : {}_f\mathfrak{M}_A \to \mathfrak{M}_f$, then A° can be endowed with a structure of a coquasi-bialgebra with preantipode in view of Theorem 2.3.20. ☐

Remark 2.3.26. It is worthy to point out that the corestriction $\mathcal{V}^{A^\circ} : {}_f\mathfrak{M}_A \to {}^{A^\circ}\mathfrak{M}_f$ of the functor $\mathcal{V}^{A^\circ} : {}_f\mathfrak{M}_A \to {}^{A^\circ}\mathfrak{M}$ provided by Theorem 2.3.20 coincides with the functor $\mathcal{R}$, which becomes a strict monoidal functor.

Remark 2.3.27. If we want to know explicitly the coquasi-bialgebra structure on A° we may proceed as follows. First of all observe that the neutral tensor structure on $\mathcal{V} : {}_f\mathfrak{M}_A \to \mathfrak{M}_f$ is the strict one: $\varphi_{M,N} = \mathrm{id}_{M\otimes N}$ and $\varphi_0 = \mathrm{id}_\Bbbk$. Secondly, for every object M in ${}_f\mathfrak{M}_A$ the natural transformation $\rho_M : \mathcal{V}(M) \to A^\circ \otimes \mathcal{V}(M)$ is given by the coaction (2.37). Let us denote by $\sum_{i=1}^{d_M} e_M^i \otimes e_i^M \in M^* \otimes M$ a dual basis for M as a vector space, for all M in ${}_f\mathfrak{M}_A$. If we denote by $\mu^{M\otimes N}$ the A-action on the tensor product, then

$$\rho_{M\otimes N}(x) = \sum_{i,j} \left(\left(e_M^i \otimes e_N^j \right) \circ \mu_x^{M\otimes N} \right) \otimes \left(e_i^M \otimes e_j^N \right)$$

for all $x \in M \otimes N$, where we considered $M^* \otimes N^*$ injected in $(M\otimes N)^*$. Furthermore, it is well-known from the associative case that the convolution product $*$, given by $(f*g)(a) = \sum f(a_1)g(a_2)$ for all $f, g \in A^*$ and $a \in A$, restricts to a morphism $* : A^\circ \otimes A^\circ \to A^\circ$. It is also clear that $\varepsilon \in A^\circ$. To show that they are the multiplication and the unit induced on A°, denote by μ^M and μ^N the A-actions on M and N respectively and compute for $\sum_{i=1}^t m_i \otimes n_i \in M \otimes N$

$$(A^\circ \otimes \varphi_{M,N}) \left(\vartheta^2_{A^\circ}(*)_{M,N} \left(\sum_{i=1}^t m_i \otimes n_i \right) \right) = \sum_{i,h,k} \left((e_M^h \circ \mu_{m_i}^M) * (e_N^k \circ \mu_{n_i}^N) \right) \otimes (e_h^M \otimes e_k^N).$$

Since for every $a \in A$, $f \in M^*$, $g \in N^*$ and $x = \sum_{i=1}^t m_i \otimes n_i \in M \otimes N$ we have

$$\sum_{i=1}^t \left(\left(f \circ \mu_{m_i}^M \right) * \left(g \circ \mu_{n_i}^N \right) \right)(a) = \sum_{i=1}^t \left(f \circ \mu_{m_i}^M \right)(a_1) \left(g \circ \mu_{n_i}^N \right)(a_2) = (f \otimes g)\, \mu_x^{M\otimes N}(a),$$

we conclude that $(A^\circ \otimes \varphi_{M,N}) \circ \vartheta^2_{A^\circ}(*)_{M,N} = \rho_{M\otimes N} \circ \varphi_{M,N}$ and by uniqueness of the morphism $A^\circ \otimes A^\circ \to A^\circ$ satisfying this relation we have that the multiplication induced on A° is exactly $*$. Moreover, if we compute

$$r_{A^\circ}\left(\rho_\Bbbk (1_\Bbbk) \right) = r_{A^\circ}(\varepsilon \otimes 1_\Bbbk) = \varepsilon,$$

then we recover that the unit of the multiplication $*$ is ε, in view of (1.33) and the fact that $\varphi_0 = \mathrm{id}_\Bbbk$. Consider also the assignment

$$\omega : A^\circ \otimes A^\circ \otimes A^\circ \to \Bbbk; \quad \omega(f \otimes g \otimes h) = \sum f(\Phi^1)\, g(\Phi^2)\, h(\Phi^3).$$

For every M, N, P in ${}_f\mathfrak{M}_A$ and all $m \in M$, $n \in N$, $p \in P$, it satisfies

$$\begin{aligned}
&\varphi_{M\otimes N,P}\left((\varphi_{M,N} \otimes \mathcal{V}(P)) \left(\vartheta^3_\Bbbk(\omega)_{M,N,P}(m \otimes n \otimes p) \right) \right) \\
&= \sum_{i,j,k} \omega\left(\left(e_M^i \circ \mu_m^M \right) \otimes \left(e_N^j \circ \mu_n^N \right) \otimes \left(e_P^k \circ \mu_p^P \right) \right) e_i^M \otimes e_j^N \otimes e_k^P \\
&= \sum m \cdot \Phi^1 \otimes n \cdot \Phi^2 \otimes p \cdot \Phi^3,
\end{aligned}$$

whence $\varphi_{M\otimes N,P} \circ (\varphi_{M,N} \otimes \mathcal{V}(P)) \circ \vartheta^3_\Bbbk(\omega)_{M,N,P} = \mathcal{V}(\alpha^{-1}_{M,N,P}) \circ \varphi_{M,N\otimes P} \circ (\mathcal{V}(M) \otimes \varphi_{N,P})$ and so ω is in fact the induced reassociator. The preantipode can be constructed explicitly as well. Consider the transpose $S^* : A^* \to A^*$. Let us show firstly that S^* factors through a linear map $S^\circ : A^\circ \to A^\circ$. The proof relies on formula (2.23) from Subsection 2.2.3. Pick $f \in A^\circ$ and compute

$$\begin{aligned}
S^*(f)(ab) &= f(S(ab)) \overset{(2.23)}{=} \sum f\left(S(\varphi^1 b)\, \varphi^2 S(\psi^1 \varphi^3)\, \psi^2 S(a\psi^3) \right) \\
&= \sum f_1 S(\varphi^1 b)\, f_2\left(\varphi^2 S(\psi^1 \varphi^3)\, \psi^2 \right) f_3 S(a\psi^3) \\
&= \left(\sum (\psi^3 \rightharpoonup f_3 S) \otimes f_2\left(\varphi^2 S(\psi^1 \varphi^3)\, \psi^2 \right) (f_1 S \leftharpoonup \varphi^1) \right)(a \otimes b).
\end{aligned}$$

Since this implies that $m^*(S^*(f)) \in A^* \otimes A^*$, in view of [Sw, Proposition 6.0.3] we have that $S^*(f) \in A^\circ$. Let us prove now that S° satisfies the relation $\vartheta_{A^\circ}(S^\circ) = \nabla^{\mathcal{V}}_{A^\circ}(\rho)$. For all M in ${}_f\mathfrak{M}_A$

and all $m \in M$ we need to show that

$$\sum S^{\circ}(m_{-1}) \otimes m_0 = \sum_{i=1}^{d_M} \left(\overline{1_A \otimes e^i}\right)_0 (m) \left(\overline{1_A \otimes e^i}\right)_{-1} \otimes e_i. \tag{2.38}$$

Since $M^\star$ is finite-dimensional, we may fix a dual basis $\sum_{j=1}^{d_{M^\star}} \gamma^j \otimes \gamma_j \in (M^\star)^* \otimes M^\star$ of $M^\star$ as an object in $\mathfrak{M}_f$ and then, in light of (2.37), the right-hand member of (2.38) can be rewritten as

$$\sum_{i=1}^{d_M} \sum_{j=1}^{d_{M^\star}} \gamma_j(m) \left(\gamma^j \circ \mu^{M^\star}_{\overline{1_A \otimes e^i}}\right) \otimes e_i.$$

Let us focus on $\sum_{j=1}^{d_{M^\star}} \gamma_j(m) \left(\gamma^j \circ \mu^{M^\star}_{\overline{1_A \otimes e^i}}\right) \in A^{\circ}$. For all $a \in A$,

$$\sum_{j=1}^{d_{M^\star}} \gamma_j(m) \left(\gamma^j \circ \mu^{M^\star}_{\overline{1_A \otimes e^i}}\right)(a) = \sum_{j=1}^{d_{M^\star}} \gamma_j(m)\gamma^j\left(\overline{a \otimes e^i}\right) = \left(\overline{a \otimes e^i}\right)(m) = e^i(m \cdot S(a))$$

and since $e^i(m \cdot S(a)) = S^{\circ}\left(e^i \circ \mu^M_m\right)(a)$, we have

$$\sum_{i=1}^{d_M} \sum_{j=1}^{d_{M^\star}} \gamma_j(m) \left(\gamma^j \circ \mu^{M^\star}_{\overline{1_A \otimes e^i}}\right) \otimes e_i = \sum_i S^{\circ}\left(e^i \circ \mu^M_m\right) \otimes e_i \stackrel{(2.37)}{=} \sum S^{\circ}(m_{-1}) \otimes m_0.$$

We can conclude then that relation (2.38) is satisfied, as desired.

2.4 Duality between quasi-bialgebras and coquasi-bialgebras

It is clear from the definitions that quasi-bialgebras and coquasi-bialgebras are dual notions, in the sense that the definition of the latter ones can be obtained from the one of the formers by reversing the structure arrows. Nevertheless, constructing a duality between them (i.e. a contravariant adjunction) seems not to be an easy task, even working over a field instead of a commutative ring. We concluded the previous section (see Proposition 2.3.25) by showing that the finite dual functor $(-)^{\circ} : \mathsf{Alg}_{\Bbbk} \to \mathsf{Coalg}_{\Bbbk}{}^{\mathrm{op}}$ restricts to a functor $(-)^{\circ} : \mathsf{QBialg}_{\Bbbk} \to \mathsf{CQBialg}_{\Bbbk}{}^{\mathrm{op}}$ from the category of quasi-bialgebras (with preantipode) to the one of coquasi-bialgebras (with preantipode). In this section we will construct a contravariant functor going 'almost' the other way around (the sense of this sentence will be made more precise in what follows) which, together with $(-)^{\circ}$, defines a duality between quasi-bialgebras and a suitable subcategory of the category of coquasi-bialgebras.

In this section we keep on assuming $\Bbbk$ to be a field. By a *non-associative algebra* we mean a unital but not necessarily associative algebra over $\Bbbk$, i.e. a vector space A endowed with two linear maps $m : A \otimes A \to A$, $a \otimes b \mapsto ab$ (the multiplication) and $u : \Bbbk \to A$, $k \mapsto k1_A$ (the unit) such that $a1_A = a = 1_A a$, for every $a \in A$ (see e.g. [Bk1, page 428]). From a categorical point of view, these can be considered as magmas in the monoidal category of $\Bbbk$-vector spaces, but coherently with the choices performed in §1.5 we preferred the "algebraic" terminology. A morphism of non-associative algebras is simply a multiplicative and unital $\Bbbk$-linear map. The category of non-associative algebras is denoted by $\mathsf{NAlg}_{\Bbbk}$. A similar terminology is used to refer to *non-coassociative coalgebras*. Their category is denoted by $\mathsf{NCoalg}_{\Bbbk}$. In order to underline that we are working with vector spaces, we will use the notation $\mathsf{Vect}_{\Bbbk}$ for the category of $\Bbbk$-vector spaces, instead of the usual $\mathfrak{M}$.

2.4.1 The construction of the finite dual of a non-associative algebra and examples.

We start by recalling how it is possible to construct a non-coassociative coalgebra starting from a non-associative algebra in such a way that, in the associative case, this construction hands back the classical finite dual coalgebra (2.36) (see [Ab, Mo, Sw] for the classical construction and [AGW] for the case of coalgebras over commutative rings). In a nutshell, this coalgebra will be the largest coalgebra inside the linear dual of the underlying vector space of the initial algebra. To illustrate our techniques, we include two basic examples concerning alternative and Jordan algebras.

Given two vector spaces V and W, we can consider the canonical injection

$$\varphi_{V,W} : V^* \otimes W^* \to (V \otimes W)^*, \quad \Big(f \otimes g \mapsto \big[v \otimes w \mapsto f(v)g(w)\big]\Big). \tag{2.39}$$

This morphism makes of $(-)^* : \mathsf{Vect}_{\Bbbk}{}^{\mathrm{op}} \to \mathsf{Vect}_{\Bbbk}$ a lax monoidal functor, where $\mathsf{Vect}_{\Bbbk}{}^{\mathrm{op}}$ is monoidal with tensor product $V^{\mathrm{op}} \otimes W^{\mathrm{op}} := (V \otimes W)^{\mathrm{op}}$ and unit object $\Bbbk^{\mathrm{op}} = \Bbbk$. Moreover, recall from Example 1.1.4 that $(-)^* : \mathsf{Vect}_{\Bbbk}{}^{\mathrm{op}} \to \mathsf{Vect}_{\Bbbk}$ is right adjoint to the functor $(-)^* : \mathsf{Vect}_{\Bbbk} \to \mathsf{Vect}_{\Bbbk}{}^{\mathrm{op}}$, which is colax monoidal with the same family of natural transformations. Unit and counit of this adjunction are given by the same map $\chi_V : V \to V^{**}$ such that $\chi_V(v)(f) = f(v)$ for all $v \in V$, $f \in V^*$ and V in $\mathsf{Vect}_{\Bbbk}$. In particular, by the uniqueness property in [AMa, Proposition 3.84] we have that

$$(\chi_V \otimes \chi_W)^* \circ (\varphi_{V^*,W^*})^* \circ \chi_{V^* \otimes W^*} = \varphi_{V,W} \tag{2.40}$$

for all V, W in $\mathsf{Vect}_{\Bbbk}$ (this relation may be also checked by a direct computation).

Now, let (A, m, u) be a non-associative algebra. Mimicking [Mi, page 13], a subspace $V \subseteq A^*$ is called *good* in case $m^*(V) \subseteq \varphi_{A,A}(V \otimes V)$, where $m^* : A^* \to (A \otimes A)^*$ is the transpose of the multiplication m.

Example 2.4.1. Let I be an ideal of A, namely a vector subspace of A stable under both left and right A-actions. For every $a \in A$, we have $aI \subseteq I$ and $Ia \subseteq I$, so that the multiplication of A factors through $\overline{m} : A/I \otimes A/I \to A/I$ (see [Bk1, page 430]). Set $\pi : A \to A/I$ and assume that A/I is finite-dimensional as a vector space. In such a case we say that I is a *finite-codimensional ideal* of A. Set $V = (A/I)^*$, which we identify with a subspace of A^* via $\pi^* : V \to A^*$. Since

$$(m^* \circ \pi^*)(f) = \big(\varphi_{A,A} \circ (\pi^* \otimes \pi^*) \circ \varphi^{-1}_{A/I,A/I} \circ \overline{m}\big)(f)$$

for every $f \in V$, it follows that V is a good subspace of A^*.

Let $\mathcal{G}$ denote the set of all good subspaces of A^* and set

$$A^\bullet := \sum_{V \in \mathcal{G}} V. \tag{2.41}$$

By the same proof of [Mi, Proposition, page 13], one gets that $A^\bullet$ is a good subspace of A^* and hence it is the maximal good subspace of A^*. Given two non-associative algebras A and B and a linear map $f : A \to B$ such that $f^*(B^\bullet) \subseteq A^\bullet$, then we can consider the linear map $f^\bullet : B^\bullet \to A^\bullet$, $h \mapsto f^*(h)$, which is uniquely determined by the commutativity of the following diagram

$$\begin{tikzcd} B^\bullet \arrow[r, dashed, "f^\bullet"] \arrow[d, hook, "j_B"'] & A^\bullet \arrow[d, hook, "j_A"] \\ B^* \arrow[r, "f^*"] & A^* \end{tikzcd} \tag{2.42}$$

where the vertical arrows are the canonical injections.

If we consider a good subspace $V \subseteq A^*$, then we may define a unique map $\Delta_V : V \to V \otimes V$ that satisfies, for every $f \in V$,

$$\varphi_{A,A}(\Delta_V(f)) = m^*(f).$$

In particular, for every $f \in A^\bullet, a, b \in A$, $\Delta_{A^\bullet}(f) = \sum f_1 \otimes f_2$ is uniquely determined by

$$f(ab) = m^*(f)(a \otimes b) = \varphi_{A,A}(\Delta_{A^\bullet}(f))(a \otimes b) = \sum f_1(a) f_2(b). \tag{2.43}$$

The non-(co)unital counterpart of the content of this section may be found in [ACM]. Parts of the subsequent lemma find their analogues for associative algebras in [Sw, Lemma 6.0.1] and for Lie algebras in [Mi, pages 14-15]. The proof is omitted.

Lemma 2.4.2. *For every pair of non-associative algebras* (A, m, u) *and* (B, m', u') *and for any morphism* $f : A \to B$*, denote with* $f^* : B^* \to A^*$ *the dual map. Then* $m^* : A^* \to (A \otimes A)^*$ *induces a map* $\Delta_{A^\bullet} := m^* : A^\bullet \to A^\bullet \otimes A^\bullet$ *satisfying* (2.43) *and* $u^* : A^* \to \Bbbk^* \cong \Bbbk : f \mapsto f(1)$ *restricts to a map* $\varepsilon_{A^\bullet} := u^* : A^\bullet \to \Bbbk$ *such that* $(A^\bullet, \Delta_{A^\bullet}, \varepsilon_{A^\bullet})$ *becomes a non-coassociative coalgebra.*

Remark 2.4.3. Let A be an object in $\mathsf{NAlg}_\Bbbk$ and set

$$A^\circ = \Big\{ g \in A^* \mid \mathsf{ker}(g) \text{ contains a finite-codimensional ideal of } A \Big\}, \tag{2.44}$$

where an ideal I of A is of finite codimension if A/I is a finite-dimensional vector space. For any $f \in A^\circ$, there exists a finite-codimensional ideal I such that $f(I) = 0$. Then f belongs to the space $(A/I)^*$, which is identified with a good subspace of A^* as in Example 2.4.1. By equation (2.41), this means that $f \in A^\bullet$. We have so proved that $A^\circ \subseteq A^\bullet$. This fact can also be seen as a consequence of [ACM, Theorem (2.6)] which asserts that $A^\circ = \mathrm{Loc}(A^\bullet)$, where the latter denotes the sum of all locally finite subcoalgebras of $A^\bullet$ (recall that a non-coassociative coalgebra C is named *locally finite* if and only if any $x \in C$ lies in some finite-dimensional subcoalgebra $D \subset C$).

Moreover, for any A in $\mathsf{Alg}_\Bbbk$ the finite dual $A^\bullet$ coincides with A°. Indeed, by the foregoing $A^\circ \subseteq A^\bullet$. Conversely, if $V \subseteq A^*$ is any good subspace then for every $v \in V$, $m^*(v) \in \varphi_{A,A}(A^* \otimes A^*)$. In view of [Sw, Proposition 6.0.3], $v \in A^\circ$ and hence $V \subseteq A^\circ$. Thus $A^\bullet \subseteq A^\circ$ so that $A^\bullet = A^\circ$.

Nevertheless, in general A° is strictly contained in $A^\bullet$. To show this take $A = C^*$ for a non-coassociative coalgebra C that is not locally finite. We will see in the proof of Lemma 2.4.6 that C injects into $C^{*\bullet}$, hence $A^\bullet$ cannot be locally finite. At the same time, $A^\circ = \mathrm{Loc}(A^\bullet)$ so that it is locally finite and hence it cannot coincide with $A^\bullet$. We now provide an example of a coalgebra which is not locally finite. Explicitly, consider $C = \Bbbk[X]$ the vector space of polynomials of any degree in the indeterminate X endowed with the comultiplication given by

$$\Delta(1) = 1 \otimes 1, \quad \Delta(X) = X \otimes 1 + 1 \otimes X, \quad \Delta(X^n) = X^n \otimes 1 + 1 \otimes X^n + X^{n+1} \otimes X + X \otimes X^{n+1},$$

for all $n \geq 2$ and the counit given by $\varepsilon(X^n) = \delta_{n,0}$ for all $n \geq 0$. It is easy to check that (C, Δ, ε) belongs to $\mathsf{NCoalg}_\Bbbk$. Note that factoring out by the coideal $\Bbbk 1$ and denoting by x_n the class of X^{n+1} in the quotient yields the Lie coalgebra E considered in [Mi, page 9]. As for E, one sees that X^2 does not lie in any finite-dimensional subcoalgebra of C. Thus C is not locally finite.

We now provide two examples of finite dual coalgebras of non-associative algebras.

Example 2.4.4 (Coalternative coalgebras)**.** Assume $\mathsf{char}(\Bbbk) \neq 2$. Let A be an *alternative algebra* (see e.g. [My, page 9]), that is a non-associative algebra over $\Bbbk$ which satisfies the following identities for every $x, y \in A$

$$x(xy) = x^2 y \quad \text{and} \quad xy^2 = (xy)y. \tag{2.45}$$

Replacing x by $x + z$ one sees that these are equivalent to the identities

$$(xy)z - x(yz) = y(xz) - (yx)z \quad \text{and} \quad (xy)z - x(yz) = x(zy) - (xz)y$$

respectively, for every $x, y, z \in A$. Denote by $\tau : V \otimes W \to W \otimes V$ the natural flip map as in Notation 1.8.3 and set $\tau_1 = \tau \otimes \mathsf{Id}$ and $\tau_2 = \mathsf{Id} \otimes \tau$. Consider the finite dual coalgebra $C = A^\bullet$ as in Lemma 2.4.2. Then the comultiplication of C satisfies the identity

$$\Big(\mathsf{Id} + \tau_1\Big) \circ \Big((\Delta \otimes C) - (C \otimes \Delta)\Big) \circ \Delta = 0 \tag{2.46a}$$

$$\Big(\mathsf{Id} + \tau_2\Big) \circ \Big((\Delta \otimes C) - (C \otimes \Delta)\Big) \circ \Delta = 0 \tag{2.46b}$$

which, over elements, says that for any function $f \in C$ we have

$$\sum f_{1,1} \otimes f_{1,2} \otimes f_2 - \sum f_1 \otimes f_{2,1} \otimes f_{2,2} = \sum f_{2,1} \otimes f_1 \otimes f_{2,2} - \sum f_{1,2} \otimes f_{1,1} \otimes f_2,$$
$$\sum f_{1,1} \otimes f_{1,2} \otimes f_2 - \sum f_1 \otimes f_{2,1} \otimes f_{2,2} = \sum f_1 \otimes f_{2,2} \otimes f_{2,1} - \sum f_{1,1} \otimes f_2 \otimes f_{1,2},$$

A coalgebra C which satisfies the identities (2.46) is called a *coalternative coalgebra*.

Example 2.4.5 (Jordan coalgebras)**.** Assume $\mathsf{char}(\Bbbk) \notin \{2,3\}$. Let A be a *Jordan algebra* (see e.g. [My, page 2]), that is a non-associative algebra over $\Bbbk$, which satisfies the following identities

$$xy = yx, \quad x^2(yx) = (x^2y)x$$

for every $x, y \in A$. The second equality above comes out to be equivalent to

$$((xy)z)t + ((xt)z)y + ((ty)z)x = (xy)(zt) + (xt)(zy) + (ty)(zx)$$

for all $x, y, z, t \in A$. Set as above $\tau_1 = \tau \otimes \mathsf{Id} \otimes \mathsf{Id}$, $\tau_2 = \mathsf{Id} \otimes \tau \otimes \mathsf{Id}$ and $\tau_3 = \mathsf{Id} \otimes \mathsf{Id} \otimes \tau$. Following [ACM, Example (3) page 4709], we can consider the finite dual coalgebra $C = A^\bullet$ as in Lemma 2.4.2. Then the comultiplication of C is cocommutative and satisfies the identity

$$\Big[\mathsf{Id} + (\tau_3 \circ \tau_2 \circ \tau_3) + (\tau_3 \circ \tau_2 \circ \tau_1 \circ \tau_2 \circ \tau_3)\Big] \circ \Big[(\Delta \otimes C \otimes C) - (C \otimes C \otimes \Delta)\Big] \circ (\Delta \otimes C) \circ \Delta = 0, \quad (2.47)$$

which, over elements, says that for any function $f \in C$ we have

$$\begin{aligned}\sum f_{1,1,1} \otimes f_{1,1,2} \otimes f_{1,2} \otimes f_2 + \sum f_{1,1,1} \otimes f_2 \otimes f_{1,2} \otimes f_{1,1,2} + \sum f_2 \otimes f_{1,1,2} \otimes f_{1,2} \otimes f_{1,1,1} \\ = \sum f_{1,1} \otimes f_{1,2} \otimes f_{2,1} \otimes f_{2,2} + \sum f_{1,1} \otimes f_{2,2} \otimes f_{2,1} \otimes f_{1,2} + \sum f_{2,2} \otimes f_{1,2} \otimes f_{2,1} \otimes f_{1,1}.\end{aligned}$$

A cocommutative coalgebra C which satisfies the identity (2.47) is called a *Jordan coalgebra*.

2.4.2 Split coquasi-bialgebras and a duality with quasi-bialgebras

The assignment $A \mapsto A^\bullet$ from $\mathsf{NAlg}_\Bbbk$ to $\mathsf{NCoalg}_\Bbbk$ introduced in the previous Subsection 2.4.1 is our candidate for an adjoint functor to $(-)^\circ : \mathsf{QBialg}_\Bbbk \to \mathsf{CQBialg}_\Bbbk$. In fact, we are going to show that there exists a full subcategory of the category of coquasi-bialgebras, whose objects we called *split coquasi-bialgebras*, such that $(-)^\bullet$ and $(-)^\circ$ induce a duality between them and quasi-bialgebras.

To this aim, recall that $(-)^* : \mathsf{Coalg}_\Bbbk \to \mathsf{Alg}_\Bbbk$ defines a contravariant functor between the category of coalgebras and that of algebras (see [Sw, Theorem 6.0.5]) that easily extends to a contravariant functor $(-)^* : \mathsf{NCoalg}_\Bbbk \to \mathsf{NAlg}_\Bbbk$. In [ACM, page 4700] it is claimed that the assignment $A \mapsto A^\bullet$ induces a contravariant functor $(-)^\bullet : \mathsf{NAlg}_\Bbbk \to \mathsf{NCoalg}_\Bbbk$ such that $(-)^\bullet$ is right adjoint to $(-)^*$ (we just point out that their (co)algebras have no (co)unit). Such an adjunction extends (in a suitable sense) the usual contravariant adjunction between algebras and coalgebras as it appears in [Sw, Theorem 6.0.5]. For the sake of completeness and as reference for the sequel, we detail here the proof of this claim.

Lemma 2.4.6. *Let $f : A \to B$ be a morphism of non-associative algebras and $f^* : B^* \to A^*$ its linear dual map. We have $f^*(B^\bullet) \subseteq A^\bullet$, so that f^* induces a map $f^\bullet : B^\bullet \to A^\bullet$ which comes out to be a morphism in $\mathsf{NCoalg}_\Bbbk$. Therefore the assignments $A \mapsto A^\bullet$ and $f \mapsto f^\bullet$ establish a functor*

$$(-)^\bullet : {\mathsf{NAlg}_\Bbbk}^{\mathrm{op}} \to \mathsf{NCoalg}_\Bbbk.$$

Moreover, for (A, m, u) a non-associative algebra and (C, Δ, ε) a non-coassociative coalgebra, we have a bijection natural in A and C

$$\Phi_{(A,C)} : \mathsf{NAlg}_\Bbbk(A, C^*) \to \mathsf{NCoalg}_\Bbbk(C, A^\bullet) \quad (2.48)$$

so that the functor $(-)^\bullet : {\mathsf{NAlg}_\Bbbk}^{\mathrm{op}} \to \mathsf{NCoalg}_\Bbbk$ is right adjoint to $(-)^* : \mathsf{NCoalg}_\Bbbk \to {\mathsf{NAlg}_\Bbbk}^{\mathrm{op}}$. Both the unit η and the counit ϵ of this adjunction are induced by the natural transformation $\chi_V : V \to V^{**}$, $v \mapsto \mathrm{ev}_v$ of Example 1.1.4.

Proof. Denote by $j_A : A^\bullet \to A^*$ the inclusion of the finite dual of a non-associative algebra A into its ordinary dual (which is natural in A). The multiplicativity of f, the definition of $\Delta_{B^\bullet}$ and the naturality of $\varphi_{-,-}$ together entail the commutativity of Diagram (2.49) below.

$$
\begin{tikzcd}
 & A^* \arrow[rr, "m_A^*"] & & (A\otimes A)^* \\
B^* \arrow[ur, "f^*"] \arrow[rr, "m_B^*"'] & & (B\otimes B)^* \arrow[ur, "(f\otimes f)^*"] & A^*\otimes A^* \arrow[u, "\varphi_{A,A}"'] \\
B^\bullet \arrow[u, hook, "j_B"] \arrow[r, "\Delta_{B^\bullet}"'] & B^\bullet\otimes B^\bullet \arrow[r, hook, "j_B\otimes j_B"'] & B^*\otimes B^* \arrow[u, "\varphi_{B,B}"'] \arrow[ur, "f^*\otimes f^*"'] &
\end{tikzcd}
\tag{2.49}
$$

In particular, $m_A^*\left(f^*(g)\right) \in \varphi_{A,A}\left(f^*\left(B^\bullet\right) \otimes f^*\left(B^\bullet\right)\right)$ for every $g \in B^\bullet$, so that $f^*\left(B^\bullet\right)$ is a good subspace of A^*. Furthermore, the commutativity of all the other quads in Diagram (2.50) below implies the commutativity of the one at the bottom, which encodes the comultiplicativity of $f^\bullet$

$$
\begin{tikzcd}
B^* \arrow[rrr, "m_B^*"] \arrow[dr, "f^*"'] & & & (B\otimes B)^* \arrow[dl, "(f\otimes f)^*"] \\
 & A^* \arrow[r, "m_A^*"] & (A\otimes A)^* & \\
 & A^\bullet \arrow[u, hook, "j_A"] \arrow[r, "\Delta_{A^\bullet}"'] & A^\bullet\otimes A^\bullet \arrow[u, "\varphi_{A,A}\circ(j_A\otimes j_A)"'] & \\
B^\bullet \arrow[uuu, hook, "j_B"] \arrow[ur, "f^\bullet"] \arrow[rrr, "\Delta_{B^\bullet}"'] & & & B^\bullet\otimes B^\bullet \arrow[ul, "f^\bullet\otimes f^\bullet"'] \arrow[uuu, "\varphi_{B,B}\circ(j_B\otimes j_B)"']
\end{tikzcd}
\tag{2.50}
$$

Moreover, $f^\bullet$ is counital since $\varepsilon_{A^\bullet} \circ f^\bullet = {u_A}^\bullet \circ f^\bullet = (f \circ u_A)^\bullet = {u_B}^\bullet = \varepsilon_{B^\bullet}$. By Lemma 2.4.2 it follows that $(-)^\bullet$ actually defines a contravariant functor from $\mathsf{NAlg}_\Bbbk$ to $\mathsf{NCoalg}_\Bbbk$.

Let us check now that if C is a coalgebra then $\chi_C(C) \subseteq C^{*\bullet}$ (compare with [Mi, Note on page 15]). This follows once proved that $\chi_C(C)$ is a good subspace of C^{**}. To this aim, for $c \in C$ and $\phi, \psi \in C^*$ we compute

$$
\begin{aligned}
(m_{C^*})^*(\chi_C(c))(\phi\otimes\psi) &= \chi_C(c)\left(m_{C^*}(\phi\otimes\psi)\right) = m_{C^*}(\phi\otimes\psi)(c) = \sum \phi(c_1)\psi(c_2) \\
&= \sum \chi_C(c_1)(\phi)\chi_C(c_2)(\psi) = \varphi_{C^*,C^*}\left(\sum \chi_C(c_1)\otimes\chi_C(c_2)\right)(\phi\otimes\psi)
\end{aligned}
$$

so that $(m_{C^*})^*(\chi_C(c)) = \varphi_{C^*,C^*}\left(\sum \chi_C(c_1)\otimes\chi_C(c_2)\right)$ for all $c \in C$. Thus, it follows that

$$
(m_{C^*})^* \circ \chi_C = \varphi_{C^*,C^*} \circ (\chi_C \otimes \chi_C) \circ \Delta \tag{2.51}
$$

and $(m_{C^*})^*(\chi_C(C)) \subseteq \varphi_{C^*,C^*}(\chi_C(C) \otimes \chi_C(C))$, so that $\chi_C(C)$ is good by definition. In particular, we have shown that for any non-coassociative coalgebra C, χ_C induces a linear map $\eta_C : C \to C^{*\bullet}$ that is still natural in C and it satisfies

$$
j_{C^*} \circ \eta_C = \chi_C. \tag{2.52}
$$

Let us check that η_C is in fact a comultiplicative and counital map. Denote by $\Delta_{C^{*\bullet}}$ the comultiplication of $C^{*\bullet}$. This is the only map that satisfies $\varphi_{C^*,C^*} \circ (j_{C^*} \otimes j_{C^*}) \circ \Delta_{C^{*\bullet}} = (m_{C^*})^* \circ j_{C^*}$. As a consequence:

$$
\begin{aligned}
\varphi_{C^*,C^*}\circ(j_{C^*}\otimes j_{C^*}) \circ \Delta_{C^{*\bullet}} \circ \eta_C &= (m_{C^*})^* \circ j_{C^*} \circ \eta_C \overset{(2.52)}{=} (m_{C^*})^* \circ \chi_C \\
&\overset{(2.51)}{=} \varphi_{C^*,C^*} \circ (\chi_C \otimes \chi_C) \circ \Delta \overset{(2.52)}{=} \varphi_{C^*,C^*} \circ (j_{C^*} \otimes j_{C^*}) \circ (\eta_C \otimes \eta_C) \circ \Delta
\end{aligned}
$$

and, by injectivity of φ_{C^*,C^*} and of j_{C^*}, we have that $\Delta_{C^{*\bullet}} \circ \eta_C = (\eta_C \otimes \eta_C) \circ \Delta$. Moreover,

$$\varepsilon_{C^{*\bullet}}(\eta_C(c)) \overset{(\dagger)}{=} (u^{\bullet}_{C^*}(\eta_C(c)))(1_{\Bbbk}) = (\eta_C(c))(u_{C^*}(1_{\Bbbk})) = (\eta_C(c))(\varepsilon_C) = \varepsilon_C(c)$$

for any $c \in C$, where in $(\dagger)$ we identified $\Bbbk^*$ with $\Bbbk$. Hence, η_C is comultiplicative and counital.

On the other hand, the injection $j_A : A^\bullet \hookrightarrow A^*$ induces a map $\epsilon_A : A \to A^{\bullet *}$ given by

$$\epsilon_A := {j_A}^* \circ \chi_A. \tag{2.53}$$

We claim that this is an algebra morphism. For all $a, b \in A$ and for any $f \in A^\bullet$ we compute

$$\begin{aligned} m_{A^{\bullet *}}(\epsilon_A(a) \otimes \epsilon_A(b))(f) &= \varphi_{A^\bullet,A^\bullet}(\epsilon_A(a) \otimes \epsilon_A(b))\,(\Delta_{A^\bullet}(f)) = \sum \epsilon_A(a)\,(f_1)\,\epsilon_A(b)\,(f_2) \\ &= \sum f_1(a) f_2(b) \overset{(2.43)}{=} f(ab) = \epsilon_A(ab)(f) \end{aligned}$$

so that it is multiplicative. Moreover, $\epsilon_A(u(1))(f) = f(1_A) = \varepsilon_{A^\bullet}(f) = (\varepsilon_{A^\bullet})^*(1)(f) = u_{A^{\bullet *}}(1)(f)$ for all $f \in A^\bullet$, whence the unitality of ϵ_A.

Let us check finally that η and ϵ satisfy the conditions to be the unit and the counit of the adjunction, respectively. By a direct computation

$$j_A \circ {\epsilon_A}^\bullet \circ \eta_{A^\bullet} \overset{(2.42)}{=} {\epsilon_A}^* \circ j_{A^{\bullet *}} \circ \eta_{A^\bullet} \overset{(2.52)}{=} {\epsilon_A}^* \circ \chi_{A^\bullet} \overset{(2.53)}{=} {\chi_A}^* \circ {j_A}^{**} \circ \chi_{A^\bullet} \overset{(*)}{=} {\chi_A}^* \circ \chi_{A^*} \circ j_A = j_A$$

where in $(*)$ we used the naturality of j and the last equality follows from the fact the $(-)^*$ is adjoint to itself at the level of vector spaces. Therefore, by injectivity of j_A, we have that ${\epsilon_A}^\bullet \circ \eta_{A^\bullet} = \mathsf{Id}_{A^\bullet}$. For the other composition, let us compute

$${\eta_C}^* \circ \epsilon_{C^*} \overset{(2.53)}{=} {\eta_C}^* \circ {j_{C^*}}^* \circ \chi_{C^*} \overset{(2.52)}{=} {\chi_C}^* \circ \chi_{C^*} = \mathsf{Id}_{C^*}$$

and this concludes the proof. □

Remark 2.4.7. By adapting this construction to the categories $\mathsf{Lie}_{\Bbbk}$ and $\mathsf{LieCo}_{\Bbbk}$ of Lie algebras and Lie coalgebras, one recovers Michaelis' result [Mi, Theorem on page 15].

Next we study how the functor $(-)^\bullet$ behaves with respect to the tensor product of two algebras. To this aim, notice that both the category $\mathsf{NCoalg}_{\Bbbk}$ and the category $\mathsf{NAlg}_{\Bbbk}$ are still monoidal with tensor product $\otimes$ and unit object $\Bbbk$ (the (co)algebra structures on the tensor products are given componentwise). Moreover, since the canonical map $\varphi_{C,D} : C^* \otimes D^* \to (C \otimes D)^*$ is multiplicative and unital for every pair C, D in $\mathsf{NCoalg}_{\Bbbk}$, the functor $(-)^* : \mathsf{NCoalg}_{\Bbbk} \to {\mathsf{NAlg}_{\Bbbk}}^{\mathrm{op}}$ is colax monoidal and so $(-)^\bullet : {\mathsf{NAlg}_{\Bbbk}}^{\mathrm{op}} \to \mathsf{NCoalg}_{\Bbbk}$, being right adjoint to $(-)^*$, becomes lax monoidal (as one may expect by having a look at [AMa, Proposition 3.84], for example). However, a bit more can be said in this context.

Proposition 2.4.8. *Let A, B be in $\mathsf{NAlg}_{\Bbbk}$. The canonical injection $\varphi_{A,B} : A^* \otimes B^* \to (A \otimes B)^*$ of equation (2.39) induces the natural isomorphism in $\mathsf{NCoalg}_{\Bbbk}$*

$$\varphi'_{A,B} := \left(\left((\epsilon_A \otimes \epsilon_B)^\bullet \circ (\varphi_{A^\bullet,B^\bullet})^\bullet \circ \eta_{(A^\bullet \otimes B^\bullet)} \right) : A^\bullet \otimes B^\bullet \xrightarrow{\cong} (A \otimes B)^\bullet \right). \tag{2.54}$$

Proof. As we observed, $\varphi_{C,D} : C^* \otimes D^* \to (C \otimes D)^*$ is a morphism in $\mathsf{NAlg}_{\Bbbk}$ for any C, D in $\mathsf{NCoalg}_{\Bbbk}$. Thus the morphism defined in equation (2.54) is well-defined. Moreover, notice that

$$\begin{aligned} & j_{A \otimes B} \circ (\epsilon_A \otimes \epsilon_B)^\bullet \circ (\varphi_{A^\bullet,B^\bullet})^\bullet \circ \eta_{(A^\bullet \otimes B^\bullet)} \overset{(*)}{=} (\epsilon_A \otimes \epsilon_B)^* \circ (\varphi_{A^\bullet,B^\bullet})^* \circ j_{(A^\bullet \otimes B^\bullet)^*} \circ \eta_{(A^\bullet \otimes B^\bullet)} \\ & \overset{(2.52)}{=} (\epsilon_A \otimes \epsilon_B)^* \circ (\varphi_{A^\bullet,B^\bullet})^* \circ \chi_{(A^\bullet \otimes B^\bullet)} \overset{(2.53)}{=} (\chi_A \otimes \chi_B)^* \circ ({j_A}^* \otimes {j_B}^*)^* \circ (\varphi_{A^\bullet,B^\bullet})^* \circ \chi_{(A^\bullet \otimes B^\bullet)} \\ & \overset{(**)}{=} (\chi_A \otimes \chi_B)^* \circ (\varphi_{A^*,B^*})^* \circ \chi_{(A^* \otimes B^*)} \circ (j_A \otimes j_B) \overset{(2.40)}{=} \varphi_{A,B} \circ (j_A \otimes j_B) \end{aligned}$$

where in $(*)$ we used the naturality of j and in $(**)$ that of φ and χ. Thus

$$j_{A\otimes B} \circ \varphi'_{A,B} = \varphi_{A,B} \circ (j_A \otimes j_B) \tag{2.55}$$

and in particular $\varphi'_{A,B}$ is injective. It remains to find an inverse for $\varphi'_{A,B}$. To this aim, consider the algebra morphisms

$$i_A : A \to A \otimes B : a \mapsto a \otimes 1 \qquad \text{and} \qquad i_B : B \to A \otimes B : b \mapsto 1 \otimes b.$$

Then the composition $\psi_{A,B} := ({i_A}^\bullet \otimes {i_B}^\bullet) \circ \Delta_{(A\otimes B)^\bullet}$ satisfies

$$\begin{aligned} j_{A\otimes B} \circ \varphi'_{A,B} \circ \psi_{A,B} &\overset{(2.55)}{=} \varphi_{A,B} \circ (j_A \otimes j_B) \circ ({i_A}^\bullet \otimes {i_B}^\bullet) \circ \Delta_{(A\otimes B)^\bullet} \\ &= (i_A \otimes i_B)^* \circ \varphi_{A\otimes B, A\otimes B} \circ (j_{A\otimes B} \otimes j_{A\otimes B}) \circ \Delta_{(A\otimes B)^\bullet} \\ &= (i_A \otimes i_B)^* \circ {m_{A\otimes B}}^* \circ j_{A\otimes B} = j_{A\otimes B}. \end{aligned}$$

This means that $\varphi'_{A,B}$ is also surjective and hence, *a fortiori*, an isomorphism with inverse $\psi_{A,B}$. $\square$

Remark 2.4.9. Notice that relation (2.55) simply says that if $f \in A^\bullet \subseteq A^*$ and $g \in B^\bullet \subseteq B^*$ then $\varphi'_{A,B}(f \otimes g)(a \otimes b) = f(a)g(b)$ for all $a \in A$ and $b \in B$. We will use this fact more or less implicitly in what follows.

As a consequence of Proposition 2.4.8, we may extend the adjunction $((-)^*, (-)^\bullet)$ to the categories of coalgebras with multiplication and unit and algebras with comultiplication and counit, denoted by $\mathsf{NAlg}\,(\mathsf{Coalg}_\Bbbk)$ and $\mathsf{NCoalg}\,(\mathsf{Alg}_\Bbbk)$ respectively. Let us recall from [Mj3, Preliminaries], [Ks, Definition XV.1.1 and Proposition XV.1.2], the following definitions.

Definition 2.4.10. A *coalgebra with multiplication and unit* is a datum $(C, \Delta, \varepsilon, m, u)$ where the triple (C, Δ, ε) defines an object in $\mathsf{Coalg}_\Bbbk$ and the maps $m\colon C \otimes C \to C$ and $u\colon \Bbbk \to C$ are morphisms in $\mathsf{Coalg}_\Bbbk$ such that m is unital with unit u.

In other words this is a *not necessarily associative algebra* or *magma* inside the monoidal category of coassociative and counital coalgebras or, equivalently, a coalgebra in the monoidal category of non-associative algebras. A morphism of coalgebras with multiplication and unit is a linear map which is compatible with both structures, that is, it is a morphism of coalgebras which is multiplicative and unital. The category so obtained will be denoted by $\mathsf{NAlg}\,(\mathsf{Coalg}_\Bbbk)$. Dualizing Definition 2.4.10 leads to the construction of the category $\mathsf{NCoalg}\,(\mathsf{Alg}_\Bbbk)$ of *algebras with comultiplication and counit*, whose objects are denoted by $(A, m, u, \Delta, \varepsilon)$. Thus, an object in $\mathsf{NCoalg}\,(\mathsf{Alg}_\Bbbk)$ is a *not necessarily coassociative coalgebra* or *comagma* inside the monoidal category of associative and unital algebras or, equivalently, an algebra in the monoidal category of non-coassociative coalgebras. Notice that even if it is clear that any quasi-bialgebra is in particular an algebra with comultiplication and counit, it is not true that any object in $\mathsf{NCoalg}\,(\mathsf{Alg}_\Bbbk)$ can be equipped with a reassociator, as the subsequent Example 2.4.11 shows.

Example 2.4.11. Let C be in $\mathsf{NCoalg}_\Bbbk$ and consider the tensor algebra $T(C)$. By the universal property of the tensor algebra, the comultiplication and the counit of C induce a comultiplication and a counit on $T = T(C)$ respectively that make it into an object in $\mathsf{NCoalg}\,(\mathsf{Alg}_\Bbbk)$. Suppose that T is in $\mathsf{QBialg}_\Bbbk$. Then it admits a reassociator $\Phi \in T^{\otimes 3}$ but, in view of Corollary A.4, $\Phi \in \Bbbk \cdot 1 \otimes 1 \otimes 1$. By (1.23a), $\Phi = 1 \otimes 1 \otimes 1$ which means that T is in $\mathsf{Bialg}_\Bbbk$. This forces C to be in $\mathsf{Coalg}_\Bbbk$. Therefore, if we consider C in $\mathsf{NCoalg}_\Bbbk$ but not in $\mathsf{Coalg}_\Bbbk$, then $T(C)$ is in $\mathsf{NCoalg}\,(\mathsf{Alg}_\Bbbk)$ but not in $\mathsf{QBialg}_\Bbbk$.

We now give a pair of explicit examples of a C as in Example 2.4.11.

Example 2.4.12. Consider the non-associative algebra A constructed as follows. As a vector space $A = \Bbbk e \oplus \Bbbk x \oplus \Bbbk y$, with multiplication table given as follows

$\cdot$	e	x	y
e	e	x	y
x	x	y	x
y	y	x	x

This is a unital, commutative but not associative algebra. Let us take its ordinary linear dual $C = A^* = \Bbbk E \oplus \Bbbk X \oplus \Bbbk Y$ where $\{E, X, Y\}$ is the dual basis. It comes out to be a counital, cocommutative but not coassociative coalgebra. The induced comultiplication and counit are given by

$$\Delta(X) = X \otimes Y + Y \otimes X + Y \otimes Y + X \otimes E + E \otimes X, \qquad \varepsilon(X) = 0,$$
$$\Delta(Y) = X \otimes X + Y \otimes E + E \otimes Y, \qquad \varepsilon(Y) = 0,$$
$$\Delta(E) = E \otimes E, \qquad \varepsilon(E) = 1.$$

Example 2.4.13 (Octonion coalgebra)**.** Consider the algebra of octonions $\mathbb{O}$ (see [Ba, §2]). This is the 8-dimensional real vector space $\mathbb{O} := \mathrm{Span}_{\mathbb{R}} \{e_i \mid i = 0, \ldots, 7\}$ with multiplication table given on the basis by

$\cdot$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	e_2	$-e_4$	$-e_0$	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	e_3	$-e_7$	$-e_5$	$-e_0$	e_6	e_2	$-e_4$	e_1
e_4	e_4	e_2	$-e_1$	$-e_6$	$-e_0$	e_7	e_3	$-e_5$
e_5	e_5	$-e_6$	e_3	$-e_2$	$-e_7$	$-e_0$	e_1	e_4
e_6	e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	$-e_0$	e_2
e_7	e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	$-e_0$

This is a unital ($1_{\mathbb{O}} = e_0$), almost anti-commutative ($e_i \cdot e_j = -e_j \cdot e_i$ for all $i \neq j$ non-zero) and non-associative algebra ($(e_1 \cdot e_2) \cdot e_3 = -e_6$, while $e_1 \cdot (e_2 \cdot e_3) = e_6$) over $\mathbb{R}$. Let us take its ordinary linear dual $\mathbb{O}^c := \mathbb{O}^* = \bigoplus_{i=0}^{7} \mathbb{R}E_i$ where $\{E_i \mid i = 0, \ldots, 7\}$ is the dual basis. It comes out to be a counital but not coassociative coalgebra. The induced comultiplication and counit are given by

$$\Delta(E_0) = E_0 \otimes E_0, \qquad \varepsilon(E_0) = 1, \qquad \varepsilon(E_i) = 0 \quad \text{for} \quad i \neq 0,$$
$$\Delta(E_1) = E_0 \otimes E_1 + E_1 \otimes E_0 + E_2 \otimes E_4 - E_4 \otimes E_2 + E_3 \otimes E_7 - E_7 \otimes E_3 + E_5 \otimes E_6 - E_6 \otimes E_5,$$
$$\Delta(E_2) = E_0 \otimes E_2 + E_2 \otimes E_0 + E_4 \otimes E_1 - E_1 \otimes E_4 + E_3 \otimes E_5 - E_5 \otimes E_3 + E_6 \otimes E_7 - E_7 \otimes E_6,$$
$$\Delta(E_3) = E_0 \otimes E_3 + E_3 \otimes E_0 + E_7 \otimes E_1 - E_1 \otimes E_7 + E_5 \otimes E_2 - E_2 \otimes E_5 + E_4 \otimes E_6 - E_6 \otimes E_4,$$
$$\Delta(E_4) = E_0 \otimes E_4 + E_4 \otimes E_0 + E_1 \otimes E_2 - E_2 \otimes E_1 + E_6 \otimes E_3 - E_3 \otimes E_6 + E_5 \otimes E_7 - E_7 \otimes E_5,$$
$$\Delta(E_5) = E_0 \otimes E_5 + E_5 \otimes E_0 + E_6 \otimes E_1 - E_1 \otimes E_6 + E_2 \otimes E_3 - E_3 \otimes E_2 + E_7 \otimes E_4 - E_4 \otimes E_7,$$
$$\Delta(E_6) = E_0 \otimes E_6 + E_6 \otimes E_0 + E_1 \otimes E_5 - E_5 \otimes E_1 + E_3 \otimes E_4 - E_4 \otimes E_3 + E_7 \otimes E_2 - E_2 \otimes E_7,$$
$$\Delta(E_7) = E_0 \otimes E_7 + E_7 \otimes E_0 + E_1 \otimes E_3 - E_3 \otimes E_1 + E_2 \otimes E_6 - E_6 \otimes E_2 + E_4 \otimes E_5 - E_5 \otimes E_4.$$

To see explicitly why $\mathbb{O}^c$ is not coassociative, it is enough to consider the elements $(\Delta \otimes \mathbb{O}^c)(\Delta(E_6))$ and $(\mathbb{O}^c \otimes \Delta)(\Delta(E_6))$ as seen in $(\mathbb{O} \otimes \mathbb{O} \otimes \mathbb{O})^*$. Indeed,

$$(\Delta \otimes \mathbb{O}^c)(\Delta(E_6))(e_1 \otimes e_2 \otimes e_3) = E_6((e_1 \cdot e_2) \cdot e_3) = -1$$
$$(\mathbb{O}^c \otimes \Delta)(\Delta(E_6))(e_1 \otimes e_2 \otimes e_3) = E_6(e_1 \cdot (e_2 \cdot e_3)) = 1.$$

Mimicking what has already been done for coalternative coalgebras and Jordan coalgebras, we may call $\mathbb{O}^c$ the *octonion coalgebra*.

As we have mentioned in Remark 2.4.3 (see also the references quoted therein), there is a contravariant functor

$$(-)^\circ : \mathsf{Alg}_\Bbbk \to \mathsf{Coalg}_\Bbbk, \tag{2.56}$$

which in fact is the restriction of the functor $(-)^\bullet$ to the associative framework[(2)]. In particular, Proposition 2.4.8 holds for $(-)^\circ$, providing us for a natural isomorphism of coalgebras that we

(2) Even thought we could use the same notation for these two functors without ambiguity, we preferred to keep different symbols, in order to distinguish between the associative and the non-associative case.

denote by $\varphi'_{A,B} : A^\circ \otimes B^\circ \cong (A \otimes B)^\circ$ as well. We will use this isomorphism freely in what follows, eventually referring directly to (2.54) instead of to this observation. Moreover, by simply forgetting (co)associativity, one may adapt the process in [Sw, §6.2] to prove the following lemma.

Lemma 2.4.14. *The functor $(-)^\circ$ is lifted to a functor $(-)^\circ : \mathsf{NCoalg}(\mathsf{Alg}_\Bbbk) \to \mathsf{NAlg}(\mathsf{Coalg}_\Bbbk)$. That is, we have a commutative diagram*

$$\begin{tikzcd} \mathsf{NCoalg}(\mathsf{Alg}_\Bbbk) \arrow[r, dashed, "(-)^\circ"] \arrow[d] & \mathsf{NAlg}(\mathsf{Coalg}_\Bbbk) \arrow[d] \\ \mathsf{Alg}_\Bbbk \arrow[r, "(-)^\circ"] & \mathsf{Coalg}_\Bbbk \end{tikzcd}$$

where the vertical functors are the forgetful ones.

In fact, the same thing happens more generally to the functor $(-)^\bullet$.

Theorem 2.4.15. *Let $(C, \Delta, \varepsilon, m, u)$ be a coalgebra with multiplication and unit. Then $C^\bullet$ becomes an algebra with comultiplication and counit so that $(-)^\bullet$ establishes a contravariant functor*

$$(-)^\bullet : \mathsf{NAlg}(\mathsf{Coalg}_\Bbbk) \to \mathsf{NCoalg}(\mathsf{Alg}_\Bbbk)$$

acting on morphism as prescribed in Lemma 2.4.6. Furthermore there is a bijection, natural in $A \in \mathsf{NCoalg}(\mathsf{Alg}_\Bbbk)$ and $C \in \mathsf{NAlg}(\mathsf{Coalg}_\Bbbk)$,

$$\mathsf{NCoalg}(\mathsf{Alg}_\Bbbk)\left(A, C^\bullet\right) \cong \mathsf{NAlg}(\mathsf{Coalg}_\Bbbk)\left(C, A^\circ\right) \tag{2.57}$$

from which it descends that the functor $(-)^\bullet : \mathsf{NAlg}(\mathsf{Coalg}_\Bbbk)^{\mathrm{op}} \to \mathsf{NCoalg}(\mathsf{Alg}_\Bbbk)$ is right adjoint to $(-)^\circ : \mathsf{NCoalg}(\mathsf{Alg}_\Bbbk) \to \mathsf{NAlg}(\mathsf{Coalg}_\Bbbk)^{\mathrm{op}}$.

Proof. Pick $(C, \Delta_C, \varepsilon_C, m_C, u_C)$ in $\mathsf{NAlg}(\mathsf{Coalg}_\Bbbk)$. First of all, we need to show that $C^\bullet$ is an object in $\mathsf{Alg}_\Bbbk$. Since we already know that C^* is an algebra and that j_C is injective, it is enough for us to endow $C^\bullet$ with a multiplication and a unit in such a way that j_C is multiplicative and unital. As a consequence, $C^\bullet$ will become a subalgebra of C^* and hence an algebra itself.

Since $\mathsf{Id}_\Bbbk \in \Bbbk^* = \Bbbk^\bullet$, we can compute $\varepsilon_C{}^\bullet(\mathsf{Id}_\Bbbk)$ obtaining that $\varepsilon_C = \varepsilon_C{}^\bullet(\mathsf{Id}_\Bbbk) \in C^\bullet$. Thus we can set $1_{C^\bullet} := \varepsilon_C$. The multiplication is $m_{C^\bullet} := \Delta_C{}^\bullet \circ \varphi'_{C,C}$. If we compute

$$j_C \circ m_{C^\bullet} = j_C \circ \Delta_C{}^\bullet \circ \varphi'_{C,C} \overset{(2.42)}{=} \Delta_C{}^* \circ j_{C\otimes C} \circ \varphi'_{C,C} \overset{(2.55)}{=} \Delta_C{}^* \circ \varphi_{C,C} \circ (j_C \otimes j_C) = m_{C^*} \circ (j_C \otimes j_C).$$

and $j_C(1_{C^\bullet}) = 1_{C^*}$, it turns out that j_C is multiplicative and unital with respect to these choices and so $C^\bullet$ is an algebra as claimed. Explicitly, for all $f, g \in C^\bullet$ we have that $m_{C^\bullet}(f \otimes g) = f * g$. Moreover, $C^\bullet$ inherits a comultiplication $\Delta_{C^\bullet} := \left(\varphi'_{C,C}\right)^{-1} \circ m_C{}^\bullet$ and a counit $\varepsilon_{C^\bullet} : f \mapsto f(1)$ as in Lemma 2.4.2. We have

$$\begin{aligned} \sum (f * g)_1 (x) (f * g)_2 (y) &\overset{(2.43)}{=} (f * g)(xy) = \sum f((xy)_1)\, g((xy)_2) = \sum f(x_1 y_1)\, g(x_2 y_2) \\ &\overset{(2.43)}{=} \sum f_1(x_1) f_2(y_1) g_1(x_2) g_2(y_2) = \sum (f_1 * g_1)(x)(f_2 * g_2)(y), \end{aligned}$$

and $\sum (\varepsilon_C)_1 (x) (\varepsilon_C)_2 (y) \overset{(2.43)}{=} \varepsilon_C(xy) = \varepsilon_C(x) \varepsilon_C(y)$ for all $x, y \in C$ and $f, g \in C^\bullet$. This implies that

$$\sum (f * g)_1 \otimes (f * g)_2 = \sum (f_1 * g_1) \otimes (f_2 * g_2) \qquad \text{and} \qquad \sum (\varepsilon_C)_1 \otimes (\varepsilon_C)_2 = \varepsilon_C \otimes \varepsilon_C.$$

Thus $\Delta_{C^\bullet}$ is multiplicative and unital. Moreover

$$\varepsilon_{C^\bullet}(f * g) = (f * g)(1) = f(1)\, g(1) = \varepsilon_{C^\bullet}(f)\, \varepsilon_{C^\bullet}(g)$$

and $\varepsilon_{C^\bullet}(\varepsilon_C) = \varepsilon_C(1) = 1$ so that $\varepsilon_{C^\bullet}$ is multiplicative and unital as well and hence $C^\bullet$ is an object in $\mathsf{NCoalg}(\mathsf{Alg}_\Bbbk)$.

Take now a map $f : C \to D$ in $\mathsf{NAlg}\left(\mathsf{Coalg}_{\Bbbk}\right)$. By Lemma 2.4.6, we know that $f^{\bullet} : D^{\bullet} \to C^{\bullet}$ is a coalgebra morphism. It remains to check that it is multiplicative and unital. For every $x \in C$,

$$f^{\bullet}\left(1_{D^{\bullet}}\right)(x) = 1_{D^{\bullet}}\big(f(x)\big) = \varepsilon_D\big(f(x)\big) = \varepsilon_C(x) = 1_{C^{\bullet}}(x),$$

where in the third equality we used the fact that f is a counital map. Furthermore, for every α and β in $D^{\bullet}$

$$\begin{aligned} f^{\bullet}(\alpha * \beta)(x) &= (\alpha * \beta)\big(f(x)\big) = \sum \alpha\big(f(x)_1\big)\beta\big(f(x)_2\big) = \sum \alpha\big(f(x_1)\big)\beta\big(f(x_2)\big) \\ &= \big(f^{\bullet}(\alpha) * f^{\bullet}(\beta)\big)(x), \end{aligned}$$

where we have used the fact that f is comultiplicative. This establishes the stated functor which is clearly a contravariant one.

In order to prove the isomorphism in (2.57), let us consider the unit $\eta_A : A \to A^{\circ *}$ and the counit $\epsilon_C : C \to C^{*\circ}$ of the classical adjunction $\left((-)^{\circ}, (-)^{*}\right) : \mathsf{Alg}_{\Bbbk} \to {\mathsf{Coalg}_{\Bbbk}}^{\mathrm{op}}$. In light of what we observed right above Lemma 2.4.14, these η_A and ϵ_C satisfies relations (2.53) and (2.52) respectively, where now $(A, m_A, u_A, \Delta_A, \varepsilon_A)$ is in $\mathsf{NCoalg}\left(\mathsf{Alg}_{\Bbbk}\right)$ and $(C, \Delta_C, \varepsilon_C, m_C, u_C)$ is in $\mathsf{NAlg}\left(\mathsf{Coalg}_{\Bbbk}\right)$ (notice that the roles played by η and ϵ here are reversed with respect to the η and ϵ in Lemma 2.4.6, because there $(-)^{\bullet} : {\mathsf{NAlg}_{\Bbbk}}^{\mathrm{op}} \to \mathsf{NCoalg}_{\Bbbk}$ was right adjoint to $(-)^{*}$, while now $(-)^{\circ} : \mathsf{Alg}_{\Bbbk} \to {\mathsf{Coalg}_{\Bbbk}}^{\mathrm{op}}$ is left adjoint to $(-)^{*}$). We already know that η_A is multiplicative and unital and we claim that it lands into $A^{\circ\bullet}$. Indeed, let us show that $\mathsf{im}(\eta_A)$ is a good subspace of $A^{\circ *}$. For all $a \in A$ and $f, g \in A^{\circ}$ we have that

$$\begin{aligned} {m_{A^{\circ}}}^{*}\left(\eta_A(a)\right)(f \otimes g) &= \eta_A(a)(f * g) = (f * g)(a) = \sum f(a_1) g(a_2) \\ &= \sum \eta_A(a_1)(f)\eta_A(a_2)(g) = \Big(\sum \eta_A(a_1) \otimes \eta_A(a_2)\Big)(f \otimes g), \end{aligned}$$

that is, for all $a \in A$, ${m_{A^{\circ}}}^{*}(\eta_A(a)) \in \varphi_{A^{\circ}, A^{\circ}}\left(\mathrm{Im}(\eta_A) \otimes \mathrm{Im}(\eta_A)\right)$ where $\varphi_{-,-}$ is the canonical inclusion of equation (2.39). We denote by $\xi_A : A \to A^{\circ\bullet}$ the corestriction of η_A. The above computation entails that

$$\varphi_{A^{\circ}, A^{\circ}} \circ (j_{A^{\circ}} \otimes j_{A^{\circ}}) \circ (\xi_A \otimes \xi_A) \circ \Delta_A = {m_{A^{\circ}}}^{*} \circ j_{A^{\circ}} \circ \xi_A \overset{(2.43)}{=} \varphi_{A^{\circ}, A^{\circ}} \circ (j_{A^{\circ}} \otimes j_{A^{\circ}}) \circ \Delta_{A^{\circ\bullet}} \circ \xi_A$$

hence, by injectivity of $\varphi_{A^{\circ}, A^{\circ}}$ and of $j_{A^{\circ}}$, ξ_A is comultiplicative. Moreover, ξ_A is also counital since

$$\varepsilon_{A^{\circ\bullet}}\left(\xi_A(a)\right) = \xi_A(a)\left(1_{A^{\circ}}\right) = \xi_A(a)\left(\varepsilon_A\right) = \varepsilon_A(a)$$

for all $a \in A$. By the foregoing, ξ_A is a morphism in the category $\mathsf{NCoalg}\left(\mathsf{Alg}_{\Bbbk}\right)$. Now we can check the naturality in A of ξ_A. Pick a morphism $f \colon A \to B$ in $\mathsf{NCoalg}\left(\mathsf{Alg}_{\Bbbk}\right)$ and consider the diagram

$$\begin{tikzcd} A \arrow[r, "\xi_A"] \arrow[d, "f"'] & A^{\circ\bullet} \arrow[r, "j_{A^{\circ}}"] \arrow[d, "f^{\circ\bullet}"] & A^{\circ *} \arrow[d, "f^{\circ *}"] \\ B \arrow[r, "\xi_B"'] & B^{\circ\bullet} \arrow[r, "j_{B^{\circ}}"'] & B^{\circ *} \end{tikzcd}$$

The commutativity of the outer diagram encodes the naturality of η, while the right-hand side diagram follows by (2.42). Hence the left-hand side diagram commutes too whence the naturality of ξ is settled. To construct the counit one proceeds in a very similar way. Explicitly, for an object $(C, \Delta_C, \varepsilon_C, m_C, u_C)$ in $\mathsf{NAlg}\left(\mathsf{Coalg}_{\Bbbk}\right)$, the map ϵ_C induces the counit ϑ_C which is given by

$$\vartheta_C := \left(C \xrightarrow{\epsilon_C} C^{*\circ} \xrightarrow{{j_C}^{*}} C^{\bullet\circ} \right), \qquad \left(x \mapsto \Big[C^{\bullet} \to \Bbbk;\ \big[g \mapsto g(x)\big] \Big] \right). \tag{2.58}$$

It remains to check the commutativity of the following two diagrams

$$\begin{tikzcd} C^{\bullet\circ\bullet} \arrow[r, "{\vartheta_C}^{\bullet}"] & C^{\bullet} \\ C^{\bullet} \arrow[u, "\xi_{C^{\bullet}}"] \arrow[ur, "="'] & \end{tikzcd} \qquad \begin{tikzcd} A^{\circ\bullet\circ} \arrow[d, "{\xi_A}^{\circ}"'] & A^{\circ} \arrow[l, "\vartheta_{A^{\circ}}"'] \arrow[dl, "="] \\ A^{\circ} & \end{tikzcd} \tag{2.59}$$

As for the first one, for every $g \in C^{\bullet}$ and for every $c \in C$ a direct calculation shows that

$$\vartheta_C{}^{\bullet}\left(\xi_{C^{\bullet}}(g)\right)(c) = \xi_{C^{\bullet}}(g)\left(\vartheta_C(c)\right) = \vartheta_C(c)(g) = g(c)$$

while for the second one, for every $f \in A^{\circ}$ and for every $a \in A$,

$$\xi_A{}^{\circ}\left(\vartheta_{A^{\circ}}(f)\right)(a) = \vartheta_{A^{\circ}}(f)\left(\xi_A(a)\right) = \xi_A(a)(f) = f(a). \qquad \square$$

As a consequence, we constructed a duality

$$\mathsf{NAlg}\left(\mathsf{Coalg}_{\Bbbk}\right) \underset{(-)^{\circ}}{\overset{(-)^{\bullet}}{\rightleftarrows}} \mathsf{NCoalg}\left(\mathsf{Alg}_{\Bbbk}\right).$$

The process we followed may be summarized in the following (non-commutative) diagram

$$\begin{tikzcd}
\mathsf{Vect}_{\Bbbk} \arrow[d, shift right, "(-)^*"'] & \mathsf{NCoalg}_{\Bbbk} \arrow[l] \arrow[d, "(-)^*"'] & \mathsf{NCoalg}\left(\mathsf{Alg}_{\Bbbk}\right) \arrow[l] \arrow[d, dotted, shift right, "(-)^{\circ}"'] \\
\mathsf{Vect}_{\Bbbk}{}^{\mathrm{op}} \arrow[u, shift right, "(-)^*"'] & \mathsf{NAlg}_{\Bbbk}{}^{\mathrm{op}} \arrow[l] \arrow[u, dotted, shift right, "(-)^{\bullet}"'] & \mathsf{NAlg}\left(\mathsf{Coalg}_{\Bbbk}\right)^{\mathrm{op}} \arrow[l] \arrow[u, shift right, "(-)^{\bullet}"']
\end{tikzcd}$$

where the horizontal functor are the forgetful ones and the dotted arrows are the functors constructed to lift the adjunctions on their left. If we denote by $\mathsf{NCoalg}\left(\mathsf{Vect}_{\Bbbk}\right)$ the category of non-coassociative coalgebras in $\mathsf{Vect}_{\Bbbk}$ then $\mathsf{NCoalg}_{\Bbbk} = \mathsf{NCoalg}\left(\mathsf{Vect}_{\Bbbk}\right)$ and $\mathsf{NCoalg}\left(\mathsf{Alg}_{\Bbbk}\right) = \mathsf{Alg}\left(\mathsf{NCoalg}_{\Bbbk}\right)$.

Now, notice that a quasi-bialgebra is an object $(A, m, u, \Delta, \varepsilon)$ in the category $\mathsf{NCoalg}\left(\mathsf{Alg}_{\Bbbk}\right)$ which is endowed with a reassociator Φ that takes care of the quasi-coassociativity of the comultiplication. On the other hand, a coquasi-bialgebra is an object $(H, \Delta, \varepsilon, m, u)$ in the category $\mathsf{NAlg}\left(\mathsf{Coalg}_{\Bbbk}\right)$ endowed with a reassociator ω which takes care of the quasi-associativity of the multiplication. As a consequence, we may try to extend the duality we developed for (co)algebras with (co)multiplication and (co)unit to a duality between quasi and coquasi-bialgebras.

We first recall that the functor $(-)^{\circ}$ of Lemma 2.4.14 can be lifted further to a functor from the category $\mathsf{QBialg}_{\Bbbk}$ to $\mathsf{CQBialg}_{\Bbbk}$, as we already saw in Proposition 2.3.25 and the subsequent Remark 2.3.27. Namely, if $(A, m, u, \Delta, \varepsilon, \Phi)$ is a quasi-bialgebra, then we may consider the image A° by the functor $(-)^{\circ}$ of its underlying object $(A, m, u, \Delta, \varepsilon) \in \mathsf{NCoalg}\left(\mathsf{Alg}_{\Bbbk}\right)$ and we know that it is an object in $\mathsf{NAlg}\left(\mathsf{Coalg}_{\Bbbk}\right)$. Set $H = A^{\circ}$ and consider the unit of the adjunction of Theorem 2.4.15 at $A \otimes A \otimes A$

$$A \otimes A \otimes A \xrightarrow{\eta_{A\otimes A\otimes A}} \Big(\left(A \otimes A \otimes A\right)^{\circ}\Big)^{*} \overset{(2.54)}{\cong} \Big(A^{\circ} \otimes A^{\circ} \otimes A^{\circ}\Big)^{*} = \left(H \otimes H \otimes H\right)^{*},$$

which by construction is an algebra map that satisfies (2.53). Therefore,

$$\omega := \eta_{A\otimes A\otimes A}(\Phi) : H \otimes H \otimes H \to \Bbbk, \quad \Big(f \otimes g \otimes h \mapsto \sum f(\Phi^1) g(\Phi^2) h(\Phi^3)\Big), \tag{2.60}$$

is an invertible element in the convolution algebra $(H \otimes H \otimes H)^{*}$, since Φ is so in the algebra $A \otimes A \otimes A$ and it is the same reassociator we provided in Remark 2.3.27. Nevertheless, one may check directly that $(H, m^{\circ}, u^{\circ}, \Delta^{\circ}, \varepsilon^{\circ}, \omega)$ is now a coquasi-bialgebra.

Furthermore, by definition any morphism $f : (A, m, u, \Delta, \varepsilon, \Phi) \to (A', m', u', \Delta', \varepsilon', \Phi')$ of quasi-bialgebras is a morphism in the category $\mathsf{NCoalg}\left(\mathsf{Alg}_{\Bbbk}\right)$. Then, by applying the functor $(-)^{\circ}$ of Lemma 2.4.14, we get that $f^{\circ} : A'^{\circ} \to A^{\circ}$ is a morphism in the category $\mathsf{NAlg}\left(\mathsf{Coalg}_{\Bbbk}\right)$. Therefore, we only need to check the compatibility condition with reassociators constructed in equation (2.60), which is derived as follows

$$\begin{aligned}
\omega\Big(\left(f^{\circ} \otimes f^{\circ} \otimes f^{\circ}\right)(\alpha \otimes \beta \otimes \gamma)\Big) &= \omega\Big((\alpha \circ f) \otimes (\beta \circ f) \otimes (\gamma \circ f)\Big) \\
&= (\alpha \otimes \beta \otimes \gamma)\Big((f \otimes f \otimes f)(\Phi)\Big) \overset{(1.19)}{=} (\alpha \otimes \beta \otimes h)(\Phi') = \omega'(\alpha \otimes \beta \otimes \gamma).
\end{aligned}$$

Hence f° satisfies (1.26) and it is a morphism of coquasi-bialgebras. Then, we have a contravariant functor

$$(-)^\circ : \mathsf{QBialg}_\Bbbk \to \mathsf{CQBialg}_\Bbbk, \tag{2.61}$$

which obviously makes the following diagram commute

$$\begin{tikzcd} \mathsf{QBialg}_\Bbbk \arrow[r, dashed, "(-)^\circ"] \arrow[d] & \mathsf{CQBialg}_\Bbbk \arrow[d] \\ \mathsf{NCoalg}\left(\mathsf{Alg}_\Bbbk\right) \arrow[r, "(-)^\circ"] & \mathsf{NAlg}\left(\mathsf{Coalg}_\Bbbk\right) \end{tikzcd}$$

where the vertical functors are the canonical forgetful functors.

The other way around, take an object $(H, \Delta, \varepsilon, m, u, \omega)$ in the category $\mathsf{CQBialg}_\Bbbk$. We may consider the image of its underlying object $(H, \Delta, \varepsilon, m, u)$ via the functor $(-)^\bullet$ of Theorem 2.4.15, that is, the object $(H^\bullet, \Delta^\bullet, \varepsilon^\bullet, m^\bullet, u^\bullet)$ in $\mathsf{NCoalg}\left(\mathsf{Alg}_\Bbbk\right)$ (up to the isomorphism $\varphi'_{H,H}$ of (2.54)). The problem now is to construct a reassociator Φ for $H^\bullet$. It seems that, *a priori*, there is no obvious way to deduce this cocycle directly from the starting datum $(H, \Delta, \varepsilon, m, u, \omega)$. To this aim, an additional assumption is needed. First, consider the following natural transformation

$$\zeta := \left(H^\bullet \otimes H^\bullet \otimes H^\bullet \subseteq H^* \otimes H^* \otimes H^* \xrightarrow{\varphi_{H,H} \otimes H^*} (H \otimes H)^* \otimes H^* \xrightarrow{\varphi_{H \otimes H, H}} (H \otimes H \otimes H)^* \right) \tag{2.62}$$

which, up to the isomorphism $(H \otimes H \otimes H)^\bullet \cong H^\bullet \otimes H^\bullet \otimes H^\bullet$ of equation (2.54), coincides with the canonical injection of the total good subspace of $(H \otimes H \otimes H)^*$. Notice that ζ is an algebra map, as it is a composition of algebra maps.

Proposition 2.4.16. *Let $(H, \Delta, \varepsilon, m, u, \omega)$ be a coquasi-bialgebra. Assume there exists an invertible element $\Phi \in H^\bullet \otimes H^\bullet \otimes H^\bullet$ such that $\zeta(\Phi) = \omega$, then $(H^\bullet, \Delta^\bullet, \varepsilon^\bullet, m^\bullet, u^\bullet, \Phi)$ is a quasi-bialgebra.*

Proof. Write $\Phi = \sum \Phi^1 \otimes \Phi^2 \otimes \Phi^3$. Then $\omega(x \otimes y \otimes z) = \sum \Phi^1(x)\Phi^2(y)\Phi^3(z)$, for every $x, y, z \in H$. Using this equality, equations (1.25a), (1.25b) and (1.25c) are easily transferred to equations (1.16a), (1.23a) and (1.17a), respectively. This concludes the proof. □

Corollary 2.4.17. *Let $(A, m, u, \Delta, \varepsilon, \Phi)$ be a quasi-bialgebra. Then $(A^{\circ\bullet}, m^{\circ\bullet}, u^{\circ\bullet}, \Delta^{\circ\bullet}, \varepsilon^{\circ\bullet})$ is still a quasi-bialgebra with reassociator $\Psi := (\xi_A \otimes \xi_A \otimes \xi_A)(\Phi)$, where ξ is the unit of the adjunction of Theorem 2.4.15.*

Proof. We already know from (2.61) that $(A^\circ, m^\circ, u^\circ, \Delta^\circ, \varepsilon^\circ, \omega)$ is a coquasi-bialgebra with reassociator given by $\omega = \zeta\big((\xi_A \otimes \xi_A \otimes \xi_A)(\Phi)\big)$. Now apply Proposition 2.4.16 to conclude. □

Let us denote by $\mathsf{SCQBialg}_\Bbbk$ the full subcategory of the category $\mathsf{CQBialg}_\Bbbk$ whose objects are *split coquasi-bialgebras*, that is, coquasi-bialgebras $(H, \Delta, \varepsilon, m, u, \omega)$ such that there exists an invertible element $\Phi \in H^\bullet \otimes H^\bullet \otimes H^\bullet$ with $\zeta(\Phi) = \omega$. In this way the assignment described in Proposition 2.4.16 yields the functor

$$(-)^\bullet : \mathsf{SCQBialg}_\Bbbk \to \mathsf{QBialg}_\Bbbk, \tag{2.63}$$

acting on morphisms as in Lemma 2.4.6 (see also Theorem 2.4.15). The compatibility with reassociators follows by using the natural transformation of (2.62).

Theorem 2.4.18. *The contravariant adjunction of Theorem 2.4.15 induces a contravariant adjunction*

$$\begin{tikzcd} \mathsf{SCQBialg}_\Bbbk \arrow[r, shift left, "(-)^\bullet"] & \mathsf{QBialg}_\Bbbk \arrow[l, shift left, "(-)^\circ"] \end{tikzcd}$$

where the functor $(-)^\circ$ is the one of (2.61) and $(-)^\bullet$ is the one of (2.63).

Proof. The only thing we need to check is that the unit and the counit of the adjunction of Theorem 2.4.15 preserve the reassociator of a quasi-bialgebra and that of a coquasi-bialgebra respectively. For the unit $\xi : \mathsf{Id}_{\mathsf{NCoalg}(\mathsf{Alg}_{\Bbbk})} \to (-)^{\bullet} \circ (-)^{\circ}$, which is given as in the proof of Theorem 2.4.15, this follows directly from Corollary 2.4.17. As for the counit $\vartheta : \mathsf{Id}_{\mathsf{NAlg}(\mathsf{Coalg}_{\Bbbk})} \to (-)^{\circ} \circ (-)^{\bullet}$, which is given by (2.58), consider a coquasi-bialgebra $(H, \Delta, \varepsilon, m, u, \omega)$ in $\mathsf{SCQBialg}_{\Bbbk}$. This means that there exists an element $\Phi \in H^{\bullet} \otimes H^{\bullet} \otimes H^{\bullet}$ such that $\zeta(\Phi) = \omega$ and that $(H^{\bullet}, \Delta^{\bullet}, \varepsilon^{\bullet}, m^{\bullet}, u^{\bullet}, \Phi)$ is a quasi-bialgebra, where ζ is the natural transformation of (2.62). From the definition of the functor $(-)^{\circ}$ in (2.61), we have that the reassociator for the coquasi-bialgebra $(H^{\bullet\circ}, \Delta^{\bullet\circ}, \varepsilon^{\bullet\circ}, m^{\bullet\circ}, u^{\bullet\circ})$ is given by $\zeta\left((\xi_{H^{\bullet}} \otimes \xi_{H^{\bullet}} \otimes \xi_{H^{\bullet}})(\Phi)\right)$. Now, the following computation

$$\begin{aligned}\zeta\Big(\big(\xi_{H^{\bullet}} \otimes \xi_{H^{\bullet}} \otimes \xi_{H^{\bullet}}\big)(\Phi)\Big) \circ (\vartheta_H \otimes \vartheta_H \otimes \vartheta_H) &= (\vartheta_H \otimes \vartheta_H \otimes \vartheta_H)^{*}\Big(\zeta\Big(\big(\xi_{H^{\bullet}} \otimes \xi_{H^{\bullet}} \otimes \xi_{H^{\bullet}}\big)(\Phi)\Big)\Big) \\ &\overset{(\text{nat. of } \zeta)}{=} \zeta\Big(\big({\vartheta_H}^{\bullet} \otimes {\vartheta_H}^{\bullet} \otimes {\vartheta_H}^{\bullet}\big)\Big(\big(\xi_{H^{\bullet}} \otimes \xi_{H^{\bullet}} \otimes \xi_{H^{\bullet}}\big)(\Phi)\Big)\Big) \overset{(2.59)}{=} \zeta(\Phi) = \omega\end{aligned}$$

shows that ϑ preserves reassociators as desired. Hence, the unit comes out to be a quasi-bialgebra map and the counit a coquasi-bialgebra map, settling the adjunction. □

This result fulfils only partially our aims, as it does not establish a contravariant adjunction between quasi and coquasi-bialgebras, but just between quasi-bialgebras and a full subcategory of $\mathsf{CQBialg}_{\Bbbk}$. Nevertheless, in Example 2.4.20 we will exhibit a coquasi-bialgebra H such that $H^{\bullet}$ cannot be a quasi-bialgebra. This is why we consider Theorem 2.4.18 the best result which can be extracted from this approach.

Remark 2.4.19. Recall that a subcategory $\mathcal{B}$ of a category $\mathcal{A}$ is *closed under sources* whenever for any morphism $f : a \to b$ in $\mathcal{A}$, if b is in $\mathcal{B}$ then a is in $\mathcal{B}$. Let us check that $\mathsf{SCQBialg}_{\Bbbk}$ is closed under sources when regarded as a subcategory of $\mathsf{CQBialg}_{\Bbbk}$. Let $g : (H', \omega') \to (H, \omega)$ be a morphism in $\mathsf{CQBialg}_{\Bbbk}$ such that (H, ω) is an object in $\mathsf{SCQBialg}_{\Bbbk}$. By assumption, there exists $\Phi = \sum \Phi^1 \otimes \Phi^2 \otimes \Phi^3 \in H^{\bullet} \otimes H^{\bullet} \otimes H^{\bullet}$ such that $\omega = \zeta(\Phi)$. Since g preserves the reassociator, we have that

$$\omega' = \omega \circ (g \otimes g \otimes g) = (g \otimes g \otimes g)^{*}(\zeta(\Phi)) \overset{(\text{nat. of } \zeta)}{=} \zeta\Big(\big(g^{\bullet} \otimes g^{\bullet} \otimes g^{\bullet}\big)(\Phi)\Big).$$

This means that ω' itself comes out to be the image via ζ of $\big(g^{\bullet} \otimes g^{\bullet} \otimes g^{\bullet}\big)(\Phi)$, that lies in $\big(H'\big)^{\bullet} \otimes \big(H'\big)^{\bullet} \otimes \big(H'\big)^{\bullet}$. Therefore, (H', ω') belongs to $\mathsf{SCQBialg}_{\Bbbk}$.

Example 2.4.20 (A non-split coquasi-bialgebra)**.** Let $\Bbbk$ be a field and consider $\Bbbk[X]$, the ring of polynomials in the indeterminate X endowed with its monoid bialgebra structure, that is, $\Delta(X) = X \otimes X$ and $\varepsilon(X) = 1$. Let us consider a map $\varphi : \Bbbk[X] \to \Bbbk$ not in $\Bbbk[X]^{\circ}$, the ordinary finite dual of $\Bbbk[X]$ (which, in this case, coincides with $\Bbbk[X]^{\bullet}$), and such that $\varphi(1) = 1$, $\varphi(X^n) \neq 0$ for all $n \geq 1$. Let us build a 3-cocycle ω that does not split by mean of φ. Since a basis for $\Bbbk[X] \otimes \Bbbk[X] \otimes \Bbbk[X]$ is given by the elements $X^n \otimes X^k \otimes X^m$ for $m, k, n \geq 0$, let us define ω on this basis as follows. For all $m, n, k \geq 0$ let us set

$$\omega(1 \otimes X^n \otimes X^m) := 1 =: \omega(X^n \otimes 1 \otimes X^m) := 1 =: \omega(X^n \otimes X^m \otimes 1) \quad \text{and}$$

$$\omega\left(X^n \otimes X^{k+1} \otimes X^m\right) := \frac{\varphi(X^{n+k})\,\varphi(X^{m+k})}{\varphi(X^k)^2} \quad \text{for } n, m \geq 1.$$

Observe that the given comultiplication ensures that we have

$$\omega^{-1}\left(X^n \otimes X^k \otimes X^m\right) = \omega\left(X^n \otimes X^k \otimes X^m\right)^{-1} = \frac{1}{\omega(X^n \otimes X^k \otimes X^m)}$$

for all $m, k, n \geq 0$. Now, let us show that ω is actually a unital 3-cocycle. It is unital by definition. If $0 \in \{m, n, r, s\}$ then we trivially have

$$\omega\left(X^m\otimes X^r\otimes X^s\right)\omega\left(X^n\otimes X^{m+r}\otimes X^s\right)\omega\left(X^n\otimes X^m\otimes X^r\right)$$
$$=\omega\left(X^n\otimes X^m\otimes X^{r+s}\right)\omega\left(X^{n+m}\otimes X^r\otimes X^s\right).$$

For all $m,n,r,s\geq 1$ we have

$$\begin{aligned}
&\omega(X^m\otimes X^r\otimes X^s)\omega(X^n\otimes X^{m+r}\otimes X^s)\omega(X^n\otimes X^m\otimes X^r)\\
&=\frac{\varphi\left(X^{m+r-1}\right)\varphi\left(X^{s+r-1}\right)}{\varphi\left(X^{r-1}\right)^2}\frac{\varphi\left(X^{n+m+r-1}\right)\varphi\left(X^{s+m+r-1}\right)}{\varphi\left(X^{m+r-1}\right)^2}\frac{\varphi\left(X^{n+m-1}\right)\varphi\left(X^{r+m-1}\right)}{\varphi\left(X^{m-1}\right)^2}\\
&=\frac{\varphi\left(X^{n+m-1}\right)\varphi\left(X^{s+m+r-1}\right)}{\varphi\left(X^{m-1}\right)^2}\frac{\varphi\left(X^{s+r-1}\right)\varphi\left(X^{n+m+r-1}\right)}{\varphi\left(X^{r-1}\right)^2}\\
&=\omega\left(X^n\otimes X^m\otimes X^{r+s}\right)\omega\left(X^{n+m}\otimes X^r\otimes X^s\right).
\end{aligned}$$

This proves that ω is a 3-cocycle. If $\omega\in\Bbbk[X]^\bullet\otimes\Bbbk[X]^\bullet\otimes\Bbbk[X]^\bullet$, then

$$\varphi=\frac{1}{\varphi(X)}\omega(-\otimes X\otimes X)=\Big(\Bbbk[X]^\bullet\otimes\eta(X)\otimes\eta(X)\Big)\left(\frac{1}{\varphi(X)}\omega\right)\in\Bbbk[X]^\bullet$$

where $\eta=\eta_{\Bbbk[X]}$ is the unit of the classical adjunction $((-)^\circ,(-)^*):\mathsf{Alg}_\Bbbk\to\mathsf{Coalg}_\Bbbk$ and it coincides with the map $\epsilon_{\Bbbk[X]}$ of equation (2.53). This contradicts our choice of φ. Since the comultiplication Δ is cocommutative, the datum $(\Bbbk[X],\Delta,\varepsilon,m,u,\omega)$ defines a coquasi-bialgebra whose reassociator does not split. An example of a map φ as above is exhibited in the subsequent Lemma 2.4.21. For the moment, assume by contradiction that the algebra with comultiplication and counit $\Bbbk[X]^\bullet$ can be endowed with a reassociator Φ in such a way that it becomes a quasi-bialgebra. Then $\Bbbk[X]^{\bullet\circ}$ would turn out to be a split coquasi-bialgebra by construction. However in such a case we may consider the counit $\vartheta_{\Bbbk[X]}:\Bbbk[X]\to\Bbbk[X]^{\bullet\circ}$ of the adjunction in Theorem 2.4.15 and Remark 2.4.19 would imply that $\Bbbk[X]$ is an object in $\mathsf{SCQBialg}_\Bbbk$. Since this contradicts the construction of the coquasi-bialgebra $\Bbbk[X]$ performed above, it follows that $\Bbbk[X]^\bullet$ cannot be a quasi-bialgebra.

Lemma 2.4.21. *Let $\Bbbk$ be a field of characteristic 0 and consider $\Bbbk[X]$ the (bi)algebra of polynomials in the indeterminate X. The map $\varphi:\Bbbk[X]\to\Bbbk$ given by $\varphi\left(X^n\right):=n!$ and extended by linearity does not belong to $\Bbbk[X]^\circ$, the ordinary finite dual of $\Bbbk[X]$.*

Proof. Assume, by contradiction, that $\varphi\in\Bbbk[X]^\circ$. Then there exists a (finite-codimensional) ideal $I:=\langle p\left(X\right)\rangle$ in $\Bbbk[X]$ with $\varphi\left(I\right)=0$. Consider the system of the equations $\varphi\left(X^ip\left(X\right)\right)=0$ for $i=0,\ldots,n$. If we write $p\left(X\right)=\sum_{j=0}^n p_jX^j$, these equations become $\sum_{j=0}^n p_j\varphi\left(X^{i+j}\right)=0$ for $i=0,\ldots,n$. The matrix associated to this system is

$$T:=\begin{pmatrix}0! & 1! & \cdots & (n-1)! & n!\\ 1! & 2! & \cdots & n! & (n+1)!\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ (n-1)! & n! & \cdots & (2n-2)! & (2n-1)!\\ n! & (n+1)! & \cdots & (2n-1)! & (2n)!\end{pmatrix}$$

Thus $T=\Big((i+j)!\Big)$ for i,j that run from 0 to n. We claim that $\det\left(T\right)\neq 0$, or equivalently that T is invertible, which is impossible since $p(X)\neq 0$ as I is finite-codimensional. To show this, let us consider the n-th Pascal matrix $Q_n=(q_{ij})$, that is the matrix whose entries are given by the relation $q_{ij}:=\binom{i+j}{i}$. Then

$$\det\left(T\right)=\det\Big((i+j)!\Big)=\det\left(i!j!q_{ij}\right)=\prod_{i=0}^n i!\prod_{j=0}^n j!\det\left(Q_n\right)=\mathsf{sf}(n)^2\det\left(Q_n\right),$$

where $\mathsf{sf}(n)$ stands for the *super factorial* of n and it is defined as the product of the first n factorials (i.e. $\mathsf{sf}(n)=1!\cdot 2!\cdot\cdots\cdot(n-1)!\cdot n!$ for all $n\geq 0$. The terminology is borrowed from [PS]). In view of [BP, Discussion preceding Theorem 4], we know that $\det\left(Q_n\right)=1$, whence $\det(T)\neq 0$ and the claim is proved. □

Remark 2.4.22. The fact that the map $\varphi(X^n) = n!$ is not in $\Bbbk[X]^\circ$ seems to be well-known, see [FMT, Section 2]. This depends on the correspondence between elements in $\Bbbk[X]^\circ$ and linearly recursive sequences, see e.g. [LT]. Since we could not find an explicit proof that $n!$ defines a non-linearly recursive sequence, we included the previous lemma.

We conclude this subsection with some considerations on the relation between duality on the one side and preantipodes and quasi-antipodes on the other.

Remark 2.4.23. A further natural step in the study of the duality between quasi and coquasi-bialgebras would be to see how the adjunction of Theorem 2.4.18 behaves with respect to preantipodes or quasi-antipodes, that is to say, if it is true or not that the finite dual of a quasi-bialgebra with preantipode (quasi-Hopf algebra) is a (split) coquasi-bialgebra with preantipode (coquasi-Hopf algebra) and conversely. We already saw at the end of §2.3.3 that the functor $(-)^\circ$ preserves preantipodes, in the sense that for a quasi-bialgebra A the (co)restriction S° of the dual map S^* of a preantipode S to the finite dual A° is a preantipode for A° itself. Given instead a quasi-antipode (s, α, β) on A, a direct computation shows that the triple composed by the (co)restriction of s^* to A° and the linear forms $\alpha_\circ : A^\circ \to \Bbbk, f \mapsto f(\alpha)$ and $\beta_\circ : A^\circ \to \Bbbk, f \mapsto f(\beta)$, is a quasi-antipode for A°. In particular, the finite dual of a quasi-bialgebra with preantipode (quasi-Hopf algebra) is in fact a coquasi-bialgebra with preantipode (coquasi-Hopf algebra). However, dealing with the reverse implication seems to be definitely less straightforward. Indeed, assume we are given a coquasi-bialgebra H with preantipode S. A natural candidate for a preantipode on $H^\bullet$ would be S^*, but the difficulties arise in proving that $S^*(f) \in H^\bullet$ for every $f \in H^\bullet$. Whether we resort to the definition of $(-)^\bullet$ or we try to apply one of the criteria we will study in the forthcoming §2.4.3, sooner or later we are forced to deal with some kind of anti-multiplicativity of the preantipode (which is not known yet for coquasi-bialgebras). A similar problem arises with coquasi-Hopf algebras, where instead a formula for the anti-multiplicativity of the quasi-antipode is supposed to exist, but only up to conjugation by a suitable invertible element.

A second natural step could be to try to extend the duality of Theorem 2.4.18 in such a way that it involves the whole category of coquasi-bialgebras. Due to the content of Example 2.4.20, this would probably require to introduce a variation on the quasi-bialgebra notion in order to encompass also the duals of coquasi-bialgebras which are not split.

Presently, however, we don't have any result in none of the foregoing directions and we leave these questions for future investigation.

2.4.3 An alternative description of the finite dual

In this section we give an alternative description of the finite dual in the non-associative case. Given a linear map, several useful criteria are shown in order to guarantee that this map belongs to the finite dual. Further characterizations can be found in [ACM].

Given a vector space V and $S \subseteq V^*$, we set

$$S^\perp := \left\{ v \in V \mid s(v) = 0, \text{ for all } s \in S \right\}.$$

Let A be a non-associative algebra. Mimicking the constructions performed in Example 1.3.2, for every $a \in A$ and $f \in A^*$, we define in A^* the elements $a \rightharpoonup f$ and $f \leftharpoonup a$ by setting

$$(a \rightharpoonup f)(b) := f(ba) \qquad \text{and} \qquad (f \leftharpoonup a)(b) := f(ab), \tag{2.64}$$

for every $b \in A$. Furthermore, the vector subspace of A^* generated by the set $\{a \rightharpoonup f \mid a \in A\}$ will be simply denoted by $A \rightharpoonup f$. A similar notation will be adopted for the right action $\leftharpoonup$. The subsequent lemma is an analogue of [Sw, Proposition 6.0.3] or [Mo, Lemma 9.1.1] and can be proved by the same argument.

Lemma 2.4.24. *Let $f \in A^*$. Then the following assertions are equivalent*

(1) $m^*(f) \in \mathsf{im}(\varphi_{A,A})$.

(2) $\dim_{\Bbbk}\left(A \rightharpoonup f\right) < \infty$.

(3) $\dim_{\Bbbk}\left(f \leftharpoonup A\right) < \infty$.

However one cannot expect, as in the associative case [Mo, Lemma 9.1.1], that the equivalent conditions *(1)*-*(3)* in Lemma 2.4.24, imply either that $\dim_{\Bbbk}\left(A \rightharpoonup (f \leftharpoonup A)\right) < \infty$ or that $\dim_{\Bbbk}\left((A \rightharpoonup f) \leftharpoonup A\right) < \infty$ (even if the converse remains true) nor that they characterize the membership of f to $A^{\bullet}$, as the subsequent Example 2.4.25 shows.

Example 2.4.25. Let $\mathsf{char}\,(\Bbbk) = 0$ and $A := \bigoplus_{n\geq 0}\Bbbk X^n$ be the vector space with basis $\{X^n \mid n \geq 0\}$ (i.e. the underlying vector space of the polynomial algebra $\Bbbk[X]$). Set by definition $X^0 = E$. Define a multiplication $A \otimes A \to A \colon v \otimes w \mapsto v \cdot w$ on A given for all $m, n \geq 1$ by

$$E \cdot E = E, \qquad X^n \cdot E = X^n = E \cdot X^n, \qquad X^m \cdot X^n := \Big((m+n)!\Big)X^{m+n}$$

and extended by $\Bbbk$-linearity. Since

$$(X \cdot X^2) \cdot X^3 = \big(3!\big)X^3 \cdot X^3 = \big(3!\big)\big(6!\big)X^6 \quad \neq \quad \big(5!\big)\big(6!\big)X^6 = \big(5!\big)X \cdot X^5 = X \cdot (X^2 \cdot X^3),$$

the algebra $(A, \cdot, E)$ is strictly non-associative. Consider the maps $\psi, \varepsilon, \gamma, \tau$ in A^* given by

$$\psi(X^n) = \frac{1}{n!}, \qquad \varepsilon(X^n) = \delta_{n,0}, \qquad \gamma(X^n) = 1 - \delta_{n,0}, \qquad \tau(X^n) = (2 - \delta_{n,0})\psi(X^n)$$

for all $n \geq 0$, where $\delta_{i,j}$ is the Kronecker delta. Let us show that

$$(\cdot)^*(\psi) = \varphi_{A,A}\left(\frac{1}{2}(\tau \otimes \varepsilon + \varepsilon \otimes \tau) + \gamma \otimes \gamma\right).$$

Indeed, for all $m, n \geq 1$

$$\begin{aligned}
\psi(E \cdot E) = \psi(E) = 1 &= \frac{1}{2}\tau(E)\varepsilon(E) + \frac{1}{2}\varepsilon(E)\tau(E) + \gamma(E)\gamma(E),\\
\psi(X^n \cdot E) = \psi(X^n) = \frac{1}{n!} &= \frac{1}{2}\tau(X^n)\varepsilon(E) + \frac{1}{2}\varepsilon(X^n)\tau(E) + \gamma(X^n)\gamma(E),\\
\psi(X^m \cdot X^n) = \Big((m+n)!\Big)\psi(X^{m+n}) = 1 &= \frac{1}{2}\tau(X^m)\varepsilon(X^n) + \frac{1}{2}\varepsilon(X^m)\tau(X^n) + \gamma(X^m)\gamma(X^n),
\end{aligned}$$

so that it satisfies condition *(1)* of Lemma 2.4.24. We want to show now that ψ does not belong to any good subspace of A^*. To this aim, assume by contradiction that $\psi \in V$ for a good subspace $V \subseteq A^*$. Then $(\cdot)^*(\psi) = \varphi_{A,A}\left(\sum_{i=1}^t \alpha_i \otimes \beta_i\right) \in \varphi_{A,A}(V \otimes V)$ so that $\sum_{i=1}^t \alpha_i \otimes \beta_i \in V \otimes V$. If the elements β_i for $i = 1, \ldots, t$ are linearly independent over $\Bbbk$ then it follows that $\alpha_i \in V$ for all i. In particular, $(\cdot)^*(\alpha_i) \in \mathsf{im}(\varphi_{A,A})$ as well. The next steps are to show that the elements $\varepsilon, \tau, \gamma \in A^*$ are linearly independent over $\Bbbk$ and that $(\cdot)^*(\gamma)$ cannot belong to $\mathsf{im}(\varphi_{A,A})$, reaching the desired contradiction. For the first step, assume that $x\varepsilon + y\tau + z\gamma \equiv 0$ with $x, y, z \in \Bbbk$. Then

$$\begin{cases} x + y = (x\varepsilon + y\tau + z\gamma)(E) = 0 \\ 2y + z = (x\varepsilon + y\tau + z\gamma)(X) = 0 \\ y + z = (x\varepsilon + y\tau + z\gamma)(X^2) = 0 \end{cases}$$

from which one easily deduces that $x = 0 = y = 0 = z$ and so $\varepsilon, \tau, \gamma \in A^*$ are linearly independent in A^* (alternatively, one may observe directly that the elements $E - X + X^2$, $X - X^2$ and $3X^3 + X^2 - X$ in A allows one to 'distinguish' the three maps $\varepsilon, \tau, \gamma$). For the second step, we are going to prove that $\dim_{\Bbbk}\left(A \rightharpoonup \gamma\right) = \infty$, whence $(\cdot)^*(\gamma) \notin \mathsf{im}(\varphi_{A,A})$ in light of Lemma 2.4.24. Namely, we want to show that the elements $X^n \rightharpoonup \gamma$ are linearly independent for all $n \geq 1$. Therefore, assume that $\sum_{n=1}^t x_n\left(X^n \rightharpoonup \gamma\right) \equiv 0$ for some $t \in \mathbb{N}$ and $x_1, \ldots, x_t \in \Bbbk$. For all $m = 1, \ldots, t$ we consider

$$0 = \sum_{n=1}^t x_n\left(X^n \rightharpoonup \gamma\right)(X^m) = \sum_{n=1}^t x_n \gamma(X^m \cdot X^n) = \sum_{n=1}^t x_n \Big((m+n)!\Big).$$

These relations give rise to a system of t equations in the t indeterminates x_n whose associated matrix is

$$T := \begin{pmatrix} 2! & 3! & \cdots & t! & (t+1)! \\ 3! & 4! & \cdots & (t+1)! & (t+2)! \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t! & (t+1)! & \cdots & (2t-2)! & (2t-1)! \\ (t+1)! & (t+2)! & \cdots & (2t-1)! & (2t)! \end{pmatrix}$$

Thus $T = \Big((i+j)!\Big)$ for i, j that run from 1 to t. Let $P_t := (p_{i,j})$ where $p_{i,j} = \binom{i+j}{i}$ and i, j run from 1 to t.[3] As we did in the proof of Lemma 2.4.21,

$$\det(T) = \det\left(\Big((i+j)!\Big)_{i,j=1}^{t}\right) = \det\Big((i!j!p_{i,j})_{i,j=1}^{t}\Big) = \prod_{i=1}^{t} i! \prod_{j=1}^{t} j! \det(P_t) = \mathsf{sf}(t)^2 \det(P_t).$$

Hence, if we are able to prove that P_t is non-singular, then T turns out to be non-singular as well and the unique solution to the system above is $x_n = 0$ for all $n = 1, \ldots, t$, that is, the elements $X^n \rightharpoonup \gamma$ are linearly independent over $\Bbbk$. This will be done in Lemma 2.4.26. Summing up, we showed that $\psi : A \to \Bbbk, X^n \mapsto 1/n!$, satisfies the equivalent conditions of Lemma 2.4.24 but it does not belong to $A^{\bullet}$, since it cannot belong to any good subspace of A^*.

Lemma 2.4.26. *Let n be an integer, $n \geq 1$. Consider the n-th Pascal matrix $Q_n := (q_{i,j})$ where $q_{i,j} = \binom{i+j}{i}$ and i, j run from 0 to n. Then every square sub-matrix of Q_n made up of adjacent rows of Q_n is non-singular.*

Proof. Let $P = (p_{i,j})$ be the square sub-matrix of Q_n with $i, j = s, \ldots, t$, for some $0 \leq s < t \leq n$. Consider the following polynomials with coefficients in $\mathbb{Q}$

$$p_i(X) := \frac{1}{i!}(X+i)(X+i-1)\cdots(X+1), \qquad \deg(p_i) = i, \tag{2.65}$$

that one may also represent in a very evocative and compact way by writing $p_i(X) = \binom{i+X}{i}$. It is clear that $p_{i,j} = p_i(j)$ for all $i, j = s, \ldots, t$. The plan of the proof is to show first that these polynomials are linearly independent over $\mathbb{Q}$ and then to deduce from this that the rows of the matrix P are linearly independent. For the first claim, it's enough to observe that they are polynomials of different (in fact, increasing) degree, so that they are obviously linearly independent. For the second claim, assume that there exist elements $a_i \in \mathbb{Q}$, $i = s, \ldots, t$, such that $\sum_{i=s}^{t} a_i p_i(j) = 0$ for all $j = s, \ldots, t$. From the expanded expression (2.65), one may notice that for all $i = s, \ldots, t-1$ we have

$$p_{i+1}(X) = \frac{(X+i+1)}{(i+1)} p_i(X),$$

whence the polynomial $\sum_{i=s}^{t} a_i p_i(X)$ already admits s different roots, namely those in the set $\{-1, -2, \ldots, -s+1, -s\}$. The additional conditions $\sum_{i=s}^{t} a_i p_i(j) = 0$ for all $j = s, \ldots, t$ entail that it admits other $t-s+1$ different roots, namely $\{s, s+1, \ldots, t-1, t\}$, for a total amount of $t-s+1+s = t+1$ roots. However, the polynomial $\sum_{i=s}^{t} a_i p_i(X)$ has degree at most t, so that it has to be the 0 polynomial. In light of the linear independence of the p_i's, we conclude that $a_i = 0$ for all $i = s, \ldots, t$ and hence that the matrix P is non-singular. $\square$

Example 2.4.27. Let us work out a bit the procedure of the previous proof on a small concrete example. Assume we have the 4-th Pascal matrix

$$Q_4 := \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix}$$

[3] Notice that P_t is the sub-matrix that one obtains from the t-th Pascal matrix Q_t by removing the first row and the first column.

and we want to show that the sub-matrix

$$P_4 := \begin{pmatrix} 2 & 3 & 4 \\ 3 & 6 & 10 \\ 4 & 10 & 20 \end{pmatrix}$$

is non-singular. Then $s = 1$ and $t = 3$. We have the three polynomials

$$p_1(X) = (X+1), \quad p_2(X) = \frac{1}{2}(X+2)(X+1), \quad \text{and} \quad p_3(X) = \frac{1}{6}(X+3)(X+2)(X+1).$$

which are clearly linearly independent over $\mathbb{Q}$ and share the common root -1. Therefore, if P_4 was singular, that is to say if the rows of P_4 were linearly dependent, then the degree 3 polynomial

$$a_1(X+1) + \frac{a_2}{2}(X+2)(X+1) + \frac{a_3}{6}(X+3)(X+2)(X+1)$$

would have had $\{-1, 1, 2, 3\}$ as roots, which means exactly that it has to be zero as a polynomial. We conclude by observing that the proof of Lemma 2.4.26 serves also as a proof of the fact that the Pascal matrices are non-singular.

Remark 2.4.28. The result of Lemma 2.4.26 seems to be folklore. However, since we were not able to find an explicit reference, we added the revised proof above.

It is natural then to look for some conditions that characterizes when a certain $f \in A^*$ actually belongs to $A^\bullet$.

Remark 2.4.29. Let V and W be vector spaces endowed with a linear map $\phi^1_{V,W} : V \to \mathsf{End}_{\Bbbk}(W)$. Denote by $T(-)$ the tensor algebra functor and by $\iota_V : V \to T(V)$ the canonical inclusion, for all V in $\mathsf{Vect}_{\Bbbk}$. The map $\phi^1_{V,W}$ induces a unique algebra map $\phi_{V,W} : T(V) \to \mathsf{End}_{\Bbbk}(W)^{\mathrm{op}}$ such that $\phi_{V,W} \circ \iota_V = \phi^1_{V,W}$ and that restricted to $\Bbbk$ gives the unit $\Bbbk \to \mathsf{End}_{\Bbbk}(W)^{\mathrm{op}} : k \mapsto k\mathsf{Id}_W$. Then W becomes a right $T(V)$-module via $\blacktriangleleft$ defined, for every $z \in T(V), w \in W$, by setting

$$w \blacktriangleleft z := \phi_{V,W}(z)(w).$$

Hence we can consider the left $T(V)$-module structure on W^* uniquely defined by setting

$$(z \blacktriangleright f)(w) := f(w \blacktriangleleft z),$$

for every $z \in T(V), w \in W$ and $f \in W^*$.

Example 2.4.30. Consider the so-called enveloping algebra $A^{\mathrm{e}} := A \otimes A^{\mathrm{op}}$ as V and A as W. Then one can consider the map

$$\phi^1_{V,W} : A^{\mathrm{e}} \to \mathsf{End}_{\Bbbk}(A) : l \otimes r \mapsto [a \mapsto r(al)].$$

For shortness, we set

$$\phi^1_A := \phi^1_{V,W} \qquad \text{and} \qquad \phi_A := \phi_{V,W}.$$

In particular, for every $l, r \in A$, we get

$$x \blacktriangleleft (l \otimes r) = \phi_A(l \otimes r)(x) = \phi^1_A(l \otimes r)(x) = r(xl) \tag{2.66}$$

and

$$((l \otimes r) \blacktriangleright f)(a) = f(a \blacktriangleleft (l \otimes r)) \overset{(2.66)}{=} f(r(al)) = (l \rightharpoonup (f \leftharpoonup r))(a)$$

so that

$$(l \otimes r) \blacktriangleright f = (l \rightharpoonup (f \leftharpoonup r)). \tag{2.67}$$

For a subset $S \subseteq T(A^{\mathrm{e}})$ and an element $f \in A^*$, we denote by $S \blacktriangleright f$ the vector subspace of A^* spanned by the set of elements $\{s \blacktriangleright f \mid s \in S\}$.

Proposition 2.4.31. *Let (A, m, u) be in $\mathsf{NAlg}_{\Bbbk}$. Then*

$$A^{\bullet} = \Big\{ f \in A^{*} \mid \dim_{\Bbbk}\left((A^{\mathrm{e}})^{\otimes n} \blacktriangleright f \right) < \infty, \ \textit{for every } n \in \mathbb{N} \Big\}. \tag{2.68}$$

Proof. Set $T := T(A^{\mathrm{e}})$. We write a generator of $(A^{\mathrm{e}})^{\otimes i}$ in the form $(l_1 \otimes r_1) \otimes \cdots \otimes (l_i \otimes r_i)$ where $l_1, \ldots, l_i \in A$ and $r_1, \ldots, r_i \in A^{\mathrm{op}}$. Note that

$$\begin{aligned}
[\phi_A (1 \otimes r) \circ \phi_A (l \otimes 1)] (a) &= [\phi_A^1 (1 \otimes r) \circ \phi_A^1 (l \otimes 1)] (a) = \phi_A^1 (1 \otimes r) (al) = r (al) \\
&= \phi_A^1 (l \otimes r) (a) = \phi_A (l \otimes r) (a)
\end{aligned}$$

and hence

$$\phi_A (l \otimes r) = \phi_A (1 \otimes r) \circ \phi_A (l \otimes 1) = \phi_A (l \otimes 1) \circ^{\mathrm{op}} \phi_A (1 \otimes r),$$

where the notation $\circ^{\mathrm{op}}$ stands for the multiplication of $\mathsf{End}_{\Bbbk}(A)^{\mathrm{op}}$. Thus

$$\begin{aligned}
\phi_A [(l_1 \otimes r_1) \otimes \cdots \otimes (l_i \otimes r_i)] &= \phi_A [(l_1 \otimes r_1) \cdot_T \cdots \cdot_T (l_i \otimes r_i)] = \phi_A (l_1 \otimes r_1) \circ^{\mathrm{op}} \cdots \circ^{\mathrm{op}} \phi_A (l_i \otimes r_i) \\
&= \phi_A (l_1 \otimes 1) \circ^{\mathrm{op}} \phi_A (1 \otimes r_1) \circ^{\mathrm{op}} \cdots \circ^{\mathrm{op}} \phi_A (l_i \otimes 1) \circ^{\mathrm{op}} \phi_A (1 \otimes r_i) \\
&= \phi_A [(l_1 \otimes 1) \cdot_T (1 \otimes r_1) \cdot_T \cdots \cdot_T (l_i \otimes 1) \cdot_T (1 \otimes r_i)] \\
&= \phi_A [(l_1 \otimes 1) \otimes (1 \otimes r_1) \cdots \otimes (l_i \otimes 1) \otimes (1 \otimes r_i)]
\end{aligned}$$

where the notation $\cdot_T$ stands for the multiplication of T. Therefore

$$\begin{aligned}
a \blacktriangleleft [(l_1 \otimes r_1) \otimes \cdots \otimes (l_i \otimes r_i)] &= \phi_A [(l_1 \otimes r_1) \otimes \cdots \otimes (l_i \otimes r_i)] (a) \\
&= \phi_A [(l_1 \otimes 1) \otimes (1 \otimes r_1) \cdots \otimes (l_i \otimes 1) \otimes (1 \otimes r_i)] (a) \\
&= a \blacktriangleleft [(l_1 \otimes 1) \cdot_T (1 \otimes r_1) \cdot_T \cdots \cdot_T (l_i \otimes 1) \cdot_T (1 \otimes r_i)]
\end{aligned}$$

Set $L := A \otimes 1$ and $R := 1 \otimes A^{\mathrm{op}}$. For shortness we write $l \in L$ for $l \otimes 1$ and $r \in R$ for $1 \otimes r$. We also omit the product over T. Using this notation, we obtain

$$a \blacktriangleleft [(l_1 \otimes r_1) \otimes \cdots \otimes (l_i \otimes r_i)] = a \blacktriangleleft (l_1 r_1 \cdots l_i r_i).$$

For every $n \geq 1$ and $f \in A^{*}$, we set

$$W_n (f) := \mathrm{Span}_{\Bbbk} \big\{ (a_1 a_2 \cdots a_{n-1} a_n) \blacktriangleright f \mid a_1, \ldots, a_n \in L \cup R \big\}.$$

Set also $W_0 (f) := \Bbbk f$. Since both A and A^{op} contain 1, it is clear that $W_i (f) \subseteq W_j (f)$ for $i \leq j$. Note further that $W_i (f) \subseteq \left((A^{\mathrm{e}})^{\otimes i} \blacktriangleright f \right) \subseteq W_{2i} (f)$ for every $i \in \mathbb{N}$ so that $\dim_{\Bbbk} \left((A^{\mathrm{e}})^{\otimes n} \blacktriangleright f \right) < \infty$ if and only if $\dim_{\Bbbk} (W_n (f)) < \infty$ and this for every $n \in \mathbb{N}$. Set

$$B := \big\{ f \in A^{*} \mid \dim_{\Bbbk} (W_n (f)) < \infty \text{ for every } n \in \mathbb{N} \big\}.$$

It remains to prove that $A^{\bullet} = B$.

($A^{\bullet} \subseteq B$). It suffices to prove that $V \subseteq B$ for every $V \in \mathcal{G}$, the family of good subspaces of A^{*}. Let us prove that $W_n (f)$ is finite-dimensional for every $f \in V$ by induction on $n \in \mathbb{N}$. For $n = 0$ there is nothing to prove.

Let $n > 0$ be such that $W_{n-1} (v)$ is finite-dimensional for every $v \in V$. Let $f \in V$. Write $\Delta_V (f) = \sum_{i=1}^{t} g_i \otimes h_i \in V \otimes V$. Let $a_1, \ldots, a_n \in L \cup R$ and $w := a_1 a_2 \cdots a_{n-1}$. Thus

$$((w a_n) \blacktriangleright f) (x) = f (x \blacktriangleleft (w a_n)) = f ((x \blacktriangleleft w) \blacktriangleleft a_n).$$

If $a_n = l \in L$, then

$$\begin{aligned}
((w a_n) \blacktriangleright f) (x) = f ((x \blacktriangleleft w) \blacktriangleleft l) &\overset{(2.66)}{=} f ((x \blacktriangleleft w) l) \\
&= \sum_{i=1}^{t} g_i (x \blacktriangleleft w) h_i (l) = \sum_{i=1}^{t} (w \blacktriangleright g_i) (x) h_i (l)
\end{aligned}$$

so that $(wa_n) \blacktriangleright f = \sum_{i=1}^n h_i(l) \cdot (w \blacktriangleright g_i) \in \sum_{i=1}^n W_{n-1}(g_i)$. If $a_n = r \in R$, then

$$\begin{aligned}((wa_n) \blacktriangleright f)(x) &= f((x \blacktriangleleft w) \blacktriangleleft r) \overset{(2.66)}{=} f(r(x \blacktriangleleft w)) \\ &= \sum_{i=1}^t g_i(r) h_i(x \blacktriangleleft w) = \sum_{i=1}^t g_i(r)(w \blacktriangleright h_i)(x)\end{aligned}$$

so that $(wa_n) \blacktriangleright f = \sum_{i=1}^t g_i(r) \cdot (w \blacktriangleright h_i) \in \sum_{i=1}^t W_{n-1}(h_i)$. In both cases,

$$(a_1a_2a_3a_4 \cdots a_{n-1}a_n) \blacktriangleright f \in \sum_{i=1}^t W_{n-1}(g_i) + \sum_{i=1}^t W_{n-1}(h_i)$$

for every $a_1, \ldots, a_n \in L \cup R$, which means that

$$W_n(f) \subseteq \sum_{i=1}^t W_{n-1}(g_i) + \sum_{i=1}^t W_{n-1}(h_i).$$

Since, by inductive hypothesis, the latter is finite-dimensional so is $W_n(f)$.

$(A^\bullet \supseteq B)$. Let $f \in B$ and let us prove that $V := (T \blacktriangleright f)$ is a good subspace. This will imply that $f = (1 \blacktriangleright f) \in V \subseteq A^\bullet$. Consider an element $v \in V$. Then there is $z \in T$ such that $v = z \blacktriangleright f$. Write $z := \sum_{i=0}^n z_i$ with $z_i \in (A^e)^{\otimes i}$ so that

$$v = z \blacktriangleright f = \sum_{i=0}^n z_i \blacktriangleright f \in \sum_{i=0}^n \left((A^e)^{\otimes i} \blacktriangleright f\right) \subseteq \left((A^e)^{\otimes n} \blacktriangleright f\right).$$

Henceforth it is not restrictive to assume $z \in (A^e)^{\otimes n}$. We have then that

$$(A \rightharpoonup v) \overset{(2.67)}{\subseteq} (A^e) \blacktriangleright v \subseteq (A^e) \blacktriangleright (z \blacktriangleright f) \subseteq (A^e) \blacktriangleright \left((A^e)^{\otimes n} \blacktriangleright f\right) \subseteq (A^e)^{\otimes(n+1)} \blacktriangleright f$$

and the latter is finite-dimensional. Hence $(A \rightharpoonup v)$ is finite-dimensional and, by Lemma 2.4.24, we have that $m^*(v) \in \mathsf{im}(\varphi_{A,A})$. Since $0 \neq \dim_{\Bbbk}(A \rightharpoonup v) < \infty$, there exist a basis $\{g_1, \ldots, g_n\}$ of $(A \rightharpoonup v)$ and elements $\{a_1, \ldots, a_n\}$ in A such that $g_i(a_j) = \delta_{i,j}$ (see e.g. [Ho, Lemma 1.1]). This implies that the assignments

$$h_i : A \to \Bbbk, \quad \left(b \mapsto (b \rightharpoonup v)(a_i)\right)$$

satisfies $(b \rightharpoonup v) = \sum_{i=1}^n h_i(b) g_i$. Therefore, $v(ab) = (b \rightharpoonup v)(a) = \sum_{i=1}^n h_i(b) g_i(a)$ for all $a, b \in A$ implies that $m^*(v) = \varphi_{A,A}\left(\sum_{i=1}^n g_i \otimes h_i\right)$ where $\{g_i \mid i = 1, \ldots, n\}$ forms a basis of $(A \rightharpoonup v)$ and there exist $a_1, \ldots, a_n \in A$ such that $g_i(a_j) = \delta_{i,j}$. We compute

$$((1 \otimes a_j) \blacktriangleright v)(x) \overset{(2.67)}{=} (v \leftharpoonup a_j)(x) = v(a_j x) = \sum_{i=1}^n g_i(a_j) h_i(x) = h_j(x)$$

so that $h_j = (1 \otimes a_j) \blacktriangleright v \in (A^e \blacktriangleright v) \subseteq V$ by definition of V. We have so proved that $m^*(v) = \sum_{i=1}^n g_i \otimes h_i \in A^* \otimes V$. A similar argument shows that $m^*(v) \in V \otimes A^*$ and hence

$$m^*(v) \in (A^* \otimes V) \cap (V \otimes A^*) = V \otimes V. \qquad \square$$

Remark 2.4.32. Let $f \in A^*$ be such that $f(I) = 0$ for some finite-codimensional ideal in A, as in Example 2.4.1. Let $l \otimes r \in A^e$ and let $x \in I$. We have that

$$((l \otimes r) \blacktriangleright f)(x) = f(r(xl)) \subseteq f(I) = 0.$$

Inductively, if $z \in (A^e)^{\otimes n}$, $z = (l_1 \otimes r_1) \otimes \cdots \otimes (l_{n-1} \otimes r_{n-1}) \otimes (l_n \otimes r_n) = w \otimes (l_n \otimes r_n)$, then

$$(z \blacktriangleright f)(x) = ((l_n \otimes r_n) \blacktriangleright f)(x \blacktriangleleft w) = f(r_n((x \blacktriangleleft w) l_n)) \subseteq f(r_n(I l_n)) \subseteq f(I) = 0.$$

Therefore for all $n \in \mathbb{N}$, $(A^e)^{\otimes n} \blacktriangleright f$ is contained in $I^\perp$, that injects into $\left(\frac{A}{I}\right)^*$, which has finite dimension. Hence, if f vanishes on a finite-codimensional ideal of A, then $f \in A^\bullet$. This is an alternative way to show that A° is contained in $A^\bullet$, see Remark 2.4.3.

Remark 2.4.33. Another description of $A^{\bullet}$ by using the so-called *standard filtration* $\left(T_{(n)}\right)_{n\in\mathbb{N}}$ of $T := T\left(A^{\mathrm{e}}\right)$ is also possible. Precisely, this filtration is defined by setting $T_{(n)} := \bigoplus_{i=0}^{n}\left(A^{\mathrm{e}}\right)^{\otimes i}$, where $\left(A^{\mathrm{e}}\right)^{\otimes 0} := \Bbbk$. Then

$$A^{\bullet} = \left\{f \in A^{*} \mid \dim_{\Bbbk}\left(T_{(n)} \blacktriangleright f\right) < \infty \text{ for every } n \in \mathbb{N}\right\}.$$

In fact

$$\left(T_{(n)} \blacktriangleright f\right) \subseteq \left(\left(\bigoplus_{i=0}^{n}\left(A^{\mathrm{e}}\right)^{\otimes i}\right) \blacktriangleright f\right) \subseteq \sum_{i=0}^{n}\left(\left(A^{\mathrm{e}}\right)^{\otimes i} \blacktriangleright f\right) \subseteq \left(\left(A^{\mathrm{e}}\right)^{\otimes n} \blacktriangleright f\right)$$

so that $\left(T_{(n)} \blacktriangleright f\right) = \left(\left(A^{\mathrm{e}}\right)^{\otimes n} \blacktriangleright f\right)$.

We now give a characterization of $A^{\bullet}$ in the spirit of [Mo, Definition 1.2.3].

Proposition 2.4.34. *Let (A, m, u) be in $\mathsf{NAlg}_{\Bbbk}$ and let $f \in A^{*}$. Then the following are equivalent*

(i) $f \in A^{\bullet}$;

(ii) There is a family $\left(I_n\right)_{n\in\mathbb{N}}$ of subspaces of A of finite codimension such that, for each $n \geq 1$,

$$\left(I_n \blacktriangleleft A^{\mathrm{e}}\right) \subseteq I_{n-1}, \qquad \textit{and} \qquad f\left(I_0\right) = 0.$$

Moreover, if one the these conditions hold true, then we can choose for every $n > 0$

$$I_0 = \mathsf{ker}(f) \qquad \textit{and} \qquad I_n = \left\{a \in A \mid a \blacktriangleleft A^{\mathrm{e}} \subseteq I_{n-1}\right\}.$$

Proof. ***(i)*** $\Rightarrow$ ***(ii)***. Assume $f \in A^{\bullet}$ and set $I_n := \left(\left(A^{\mathrm{e}}\right)^{\otimes n} \blacktriangleright f\right)^{\perp}$. For every $n \geq 1$, $u \in I_n, z \in A^{\mathrm{e}}, w \in \left(A^{\mathrm{e}}\right)^{\otimes(n-1)}$,

$$(w \blacktriangleright f)(u \blacktriangleleft z) = (z \blacktriangleright (w \blacktriangleright f))(u) = ((zw) \blacktriangleright f)(u) \in \left(\left(A^{\mathrm{e}}\right)^{\otimes n} \blacktriangleright f\right)(u) = 0$$

so that $u \blacktriangleleft z \in \left(\left(A^{\mathrm{e}}\right)^{\otimes(n-1)} \blacktriangleright f\right)^{\perp} = I_{n-1}$ and hence $\left(I_n \blacktriangleleft A^{\mathrm{e}}\right) \subseteq I_{n-1}$. Since $\left(\left(A^{\mathrm{e}}\right)^{\otimes 0} \blacktriangleright f\right) = (\Bbbk \blacktriangleright f) = \Bbbk f$ we get that $f\left(I_0\right) = 0$.

(i) $\Leftarrow$ ***(ii)***. Inductively one proves that $\left(I_n \blacktriangleleft \left(A^{\mathrm{e}}\right)^{\otimes n}\right) \subseteq I_0$ so that we have

$$\left(\left(A^{\mathrm{e}}\right)^{\otimes n} \blacktriangleright f\right)\left(I_n\right) \subseteq f\left(I_n \blacktriangleleft \left(A^{\mathrm{e}}\right)^{\otimes n}\right) \subseteq f\left(I_0\right) = 0.$$

Therefore $\left(\left(A^{\mathrm{e}}\right)^{\otimes n} \blacktriangleright f\right) \subseteq I_n^{\perp}$ which is finite-dimensional as I_n has finite codimension, which by Proposition 2.4.31 implies that $f \in A^{\bullet}$.

Let us conclude by proving the last claim of the statement. For $n = 0$ we have that

$$I_0 = \left(\left(A^{\mathrm{e}}\right)^{\otimes 0} \blacktriangleright f\right)^{\perp} = (\Bbbk \blacktriangleright f)^{\perp} = (\Bbbk f)^{\perp} = \mathsf{ker}(f),$$

and for $n > 0$ we have:

$$\begin{aligned} I_n &:= \left(\left(A^{\mathrm{e}}\right)^{\otimes n} \blacktriangleright f\right)^{\perp} = \left\{a \in A \mid \left(\left(A^{\mathrm{e}}\right)^{\otimes n} \blacktriangleright f\right)(a) = 0\right\} = \left\{a \in A \,\middle|\, \left(\left(A^{\mathrm{e}}\right)^{\otimes(n-1)} \blacktriangleright f\right)\left(a \blacktriangleleft A^{\mathrm{e}}\right) = 0\right\} \\ &= \left\{a \in A \,\middle|\, \left(a \blacktriangleleft A^{\mathrm{e}}\right) \subseteq \left(\left(A^{\mathrm{e}}\right)^{\otimes(n-1)} \blacktriangleright f\right)^{\perp}\right\} = \left\{a \in A \mid \left(a \blacktriangleleft A^{\mathrm{e}}\right) \subseteq I_{n-1}\right\}, \end{aligned}$$

and this finishes the proof. $\square$

Let C be a coalgebra. The coalgebra structure of C, through the universal property of the tensor algebra, induces a bialgebra structure on $T(C)$ so that it makes sense to use the notation $\Delta_{T(C)}(z) := \sum z_1 \otimes z_2$ for any $z \in T(C)$ to denote the comultiplication of $T(C)$ (see e.g. [Rad, Theorem 5.3.1]). Recall from [Mo] that if B is any bialgebra, then a *left* (resp. *right*) *B-module coalgebra* is a coalgebra in the monoidal category $\left({}_{B}\mathfrak{M}, \otimes, \Bbbk\right)$ of left B-modules (resp. in the monoidal category $\left(\mathfrak{M}_{B}, \otimes, \Bbbk\right)$ of right B-modules). Analogously one defines left and right B-module algebras.

Lemma 2.4.35. *Let C and D be two coalgebras with a $\Bbbk$-linear map $\phi^1_{C,D} : C \to \mathrm{End}_{\Bbbk}(D)$ as in Remark 2.4.29. Assume that $D \otimes C \to D : d \otimes c \mapsto d \blacktriangleleft c$ is a coalgebra map. Then D is a right $T(C)$-module coalgebra through $\blacktriangleleft$ and (D^*, m_{D^*}, u_{D^*}) is a left $T(C)$-module algebra through $\blacktriangleright$ where*

$$m_{D^*}(f \otimes g) = f * g \text{ (convolution product)} \qquad \text{and} \qquad u_{D^*}(k) = k\varepsilon_D$$

for every $f, g \in D^*, k \in \Bbbk$.

Proof. By hypothesis for every $c \in C, d \in D$, we have that

$$\sum (d_1 \blacktriangleleft c_1) \otimes (d_2 \blacktriangleleft c_2) = \sum (d \blacktriangleleft c)_1 \otimes (d \blacktriangleleft c)_2,$$
$$\varepsilon_D(d)\varepsilon_C(c) = \varepsilon_D(d \blacktriangleleft c).$$

We need to prove that for every $z \in T(C), d \in D$, we have

$$\sum (d_1 \blacktriangleleft z_1) \otimes (d_2 \blacktriangleleft z_2) = \sum (d \blacktriangleleft z)_1 \otimes (d \blacktriangleleft z)_2,$$
$$\varepsilon_D(d)\varepsilon_{T(C)}(z) = \varepsilon_D(d \blacktriangleleft z).$$

For $k \in \Bbbk$ we have

$$d \blacktriangleleft k = \phi_{C,D}(k)(d) = (k\mathsf{Id}_D)(d) = kd.$$

Then, for every $z \in \Bbbk = T^0(C)$ we have

$$\begin{aligned}\sum (d_1 \blacktriangleleft z_1) \otimes (d_2 \blacktriangleleft z_2) &= \sum (d_1 \blacktriangleleft z1) \otimes (d_2 \blacktriangleleft 1) = \sum zd_1 \otimes d_2\\ &= \sum (zd)_1 \otimes (zd)_2 = \sum (d \blacktriangleleft z)_1 \otimes (d \blacktriangleleft z)_2,\end{aligned}$$

and

$$\varepsilon_D(d)\varepsilon_{T(C)}(z) = \varepsilon_D(d)z = \varepsilon_D(dz) = \varepsilon_D(d \blacktriangleleft z).$$

Let $c_1, \ldots, c_n \in C$. Let us prove, by induction on $n \geq 1$, that

$$\sum (d_1 \blacktriangleleft z_1) \otimes (d_2 \blacktriangleleft z_2) = \sum (d \blacktriangleleft z)_1 \otimes (d \blacktriangleleft z)_2 \qquad \text{and} \qquad \varepsilon_D(d)\varepsilon_{T(C)}(z) = \varepsilon_D(d \blacktriangleleft z),$$

where $z := c_1 \cdots c_n$ is the multiplication of the c_i's, each one viewed as an element in $T(C)$.

For $n = 1$ there is nothing to prove, as it is the hypothesis. Let $n > 1$ and assume the statement true for $n - 1$. If we set $z' := c_1 \cdots c_{n-1}$, then we get on the one hand that

$$\begin{aligned}(d_1 \blacktriangleleft z_1) \otimes (d_2 \blacktriangleleft z_2) &= (d_1 \blacktriangleleft (z'c_n)_1) \otimes (d_2 \blacktriangleleft (z'c_n)_2) = (d_1 \blacktriangleleft z'_1 (c_n)_1) \otimes (d_2 \blacktriangleleft z'_2 (c_n)_2)\\ &= ((d_1 \blacktriangleleft z'_1) \blacktriangleleft (c_n)_1) \otimes ((d_2 \blacktriangleleft z'_2) \blacktriangleleft (c_n)_2)\\ &= ((d \blacktriangleleft z')_1 \blacktriangleleft (c_n)_1) \otimes ((d \blacktriangleleft z')_2 \blacktriangleleft (c_n)_2)\\ &= ((d \blacktriangleleft z') \blacktriangleleft c_n)_1 \otimes ((d \blacktriangleleft z') \blacktriangleleft c_n)_2\\ &= (d \blacktriangleleft (z'c_n))_1 \otimes (d \blacktriangleleft (z'c_n))_2 = (d \blacktriangleleft z)_1 \otimes (d \blacktriangleleft z)_2\end{aligned}$$

and on the other hand that

$$\begin{aligned}\varepsilon_D(d)\varepsilon_{T(C)}(z) &= \varepsilon_D(d)\varepsilon_{T(C)}(z'c_n) = \varepsilon_D(d)\varepsilon_{T(C)}(z')\varepsilon_{T(C)}(c_n)\\ &= \varepsilon_D(d \blacktriangleleft z')\varepsilon_{T(C)}(c_n) = \varepsilon_D((d \blacktriangleleft z') \blacktriangleleft c_n)\\ &= \varepsilon_D(d \blacktriangleleft (z'c_n)) = \varepsilon_D(d \blacktriangleleft z).\end{aligned}$$

This shows the claimed formulae for every $z \in T(C)$. Therefore D is a right $T(C)$-module coalgebra through $\blacktriangleleft$. Since $(D, \Delta_D, \varepsilon_D)$ is a coassociative coalgebra, we know that (D^*, m_{D^*}, u_{D^*}) is an associative algebra. Let us check that it is a left $T(C)$-module algebra through $\blacktriangleright$. For all $f, g \in D^*, z \in T(C), d \in D$ we have

$$\sum \Big[(z_1 \blacktriangleright f) * (z_2 \blacktriangleright g) \Big](d) = \sum (z_1 \blacktriangleright f)(d_1)(z_2 \blacktriangleright g)(d_2) = \sum f(d_1 \blacktriangleleft z_1)\, g(d_2 \blacktriangleleft z_2)$$

$$= \sum f\Big((d \blacktriangleleft z)_1\Big) g\Big((d \blacktriangleleft z)_2\Big) = (f * g)(d \blacktriangleleft z) = (z \blacktriangleright (f * g))(d)$$

so that $\sum (z_1 \blacktriangleright f) * (z_2 \blacktriangleright g) = (z \blacktriangleright (f * g))$. Moreover

$$(z \blacktriangleright \varepsilon_D)(d) = \varepsilon_D(d \blacktriangleleft z) = \varepsilon_D(d)\,\varepsilon_{T(C)}(z)$$

so that $z \blacktriangleright \varepsilon_D = \varepsilon_{T(C)}(z)\,\varepsilon_D$. This proves that (D^*, m_{D^*}, u_{D^*}) is a left $T(C)$-module algebra through $\blacktriangleright$. $\square$

Remark 2.4.36. More generally, given a bialgebra B the contravariant functor $(-)^* : \mathfrak{M}_B \to {}_B\mathfrak{M}$ from the category of right B-modules to the category of left ones is lax monoidal so that it induces a covariant functor $(-)^* : \mathfrak{M}_B \to ({}_B\mathfrak{M})^{\mathrm{op}}$ which is colax monoidal. Thus the latter functor induces a functor $\mathsf{Coalg}((-)^*) : \mathsf{Coalg}(\mathfrak{M}_B) \to \mathsf{Coalg}({}_B\mathfrak{M}^{\mathrm{op}}) \equiv \mathsf{Alg}({}_B\mathfrak{M})^{\mathrm{op}}$ which means that $(-)^*$ maps right B-module coalgebras to left B-module algebras as in the particular case of Lemma 2.4.35.

References

This chapter contains the most part of the results in the framework of quasi and coquasi-bialgebras, with or without preantipode, on which the author worked. Up to our knowledge, the first section, §2.1, appears here for the first time and can be considered an original contribution to the subject. Section 2.2 is a revised version of the content of [Sa2] in light of Section 2.1. Instead, the material in Sections 2.3 and 2.4 reflects respectively the content of [Sa1] and of the joint paper [AES] with A. Ardizzoni and L. El Kaoutit.

Chapter 3

Completion bifunctor and complete commutative Hopf algebroids

In this last chapter we will deal with the completion procedure for filtered bimodules, but applied to the theory of Hopf algebroids and their linear and finite duals. The first section is dedicated then to filtered rings and modules and to recall how this procedure works. It does not contain any particularly new result, but the presentation is different from the classical ones that can be found in textbooks: we decided to follow a bicategorical approach, taking advantage of the fact that filtered (complete) bimodules over filtered (complete) algebras can be seen as objects of the internal bicategory of bimodules and algebras (in the sense of §1.6) in the monoidal category of filtered (complete) modules over $\Bbbk$ (see Proposition 3.1.9, Proposition 3.1.34 and Theorem 3.1.35).

After this, we apply these tools to introduce complete commutative Hopf algebroids (§3.2.1) and to show that the full linear dual U^* of a cocommutative Hopf algebroid (A, U) with an admissible filtration (notion borrowed from Kapranov [Kp]) is a complete commutative Hopf algebroid (§3.3.1). Parallel to the full linear dual, we may also consider the so-called finite dual Hopf algebroid (A, U°) of (A, U) (§3.3.2.1), which comes equipped with a structure of filtered commutative Hopf algebroid together with a canonical filtered structure map $\zeta : U^\circ \to U^*$. The very last part of the chapter (§3.3.2) focuses on the study of the main morphism of complete commutative Hopf algebroids $\widehat{\zeta} : \widehat{U^\circ} \to U^*$ obtained by completion of the canonical map (see Theorem 3.3.22) and the study of conditions under which this is a filtered isomorphism (see Proposition 3.3.23).

3.1 Complete bimodules and the completion bifunctor

In this section we revise some notions on linear topology of rings and modules which maybe are well-known or folklore, perhaps apart from the adjunction between the complete tensor product and the Hom functor retrieved in Theorem 3.1.27. However, we will give a new (up to our knowledge) presentation of these results that relies on a bicategorical approach which makes use of the results in Section 1.6. One of the reasons why we decided to add this section is to present in a comprehensive way all the material needed for dealing with complete Hopf algebroids in Subsection 3.2.1. For further reading on the subject, we refer the reader to [Bk3, Gt, LvO, NvO2].

3.1.1 Filtered bimodules over filtered algebras

Let again $\Bbbk$ be a fixed commutative ring. As usual all $\Bbbk$-modules are considered, implicitly, as central $\Bbbk$-bimodules when needed. As far as we will be concerned with this, a *linear topology* on an algebraic structure is a topology on the underlying set with respect to which all structure maps are continuous. A $\Bbbk$-module V is said to be *filtered over* $\mathbb{Z}$ or $\mathbb{Z}$*-filtered* if there exists a chain of $\Bbbk$-submodules

$$\cdots \subseteq F_{-n}V \subseteq \cdots \subseteq F_{-1}V \subseteq F_0V \subseteq F_1V \subseteq \cdots \subseteq F_nV \subseteq \cdots \subseteq V$$

such that $n \geq 0$. The filtration is said to be *exhaustive* if $V = \bigcup_{n\in\mathbb{Z}} F_nV$. In this section we will deal mainly with *decreasingly* filtered $\Bbbk$-modules, which means that we have a chain of $\Bbbk$-subbimodules

$$\cdots \subseteq F_nV \subseteq \cdots \subseteq F_1V \subseteq F_0V = V$$

for $n \geq 0$. These are just $\mathbb{Z}$-filtered modules where $F_0V = V$ and where we re-labelled the terms of the filtration for the sake of simplicity (in particular, the filtration is exhaustive). We will denote it as a pair (V, F_nV) or we will just say that V is filtered. If (V, F_nV) is a filtered $\Bbbk$-module, then the k*-shifted (filtered) module* $V[k]$, $k \in \mathbb{N}$, is the same $\Bbbk$-module as V, but with the filtration shifted by k[(1)]. Namely, $F_n(V[k]) = F_{n+k}V$.

If V is a filtered $\Bbbk$-module then it can be endowed with a linear topology such that the given filtration forms a fundamental system of neighbourhoods of 0. A basis for this topology is given by the open sets $\{v + F_nV \mid v \in V, n \in \mathbb{N}\}$. A filtration $\{F_nV \mid n \in \mathbb{N}\}$ on an $\Bbbk$-module V is said to be *finer* than another filtration $\{G_nV \mid n \in \mathbb{N}\}$ on V if and only if for every $n \in \mathbb{N}$ there exists an $m \in \mathbb{N}$, which we may always assume to be greater than n, such that $F_mV \subseteq G_nV$ (see e.g. [Bk3, I.38, § 6.3, Proposition 4]). As a consequence, the linear topology induced by the filtration $\{F_nV \mid n \in \mathbb{N}\}$ is finer than the one induced by the filtration $\{G_nV \mid n \in \mathbb{N}\}$ (notice that also the converse is true: if the topology induced by $\{F_nV \mid n \in \mathbb{N}\}$ is finer than the one induced by $\{G_nV \mid n \in \mathbb{N}\}$, then the filtration $\{F_nV \mid n \in \mathbb{N}\}$ is finer than $\{G_nV \mid n \in \mathbb{N}\}$). If $\{F_nV \mid n \in \mathbb{N}\}$ is finer than $\{G_nV \mid n \in \mathbb{N}\}$, then we may also say that $\{G_nV \mid n \in \mathbb{N}\}$ is *coarser* than $\{F_nV \mid n \in \mathbb{N}\}$. Two filtrations $\{F_nV \mid n \in \mathbb{N}\}$ and $\{G_nV \mid n \in \mathbb{N}\}$ on an $\Bbbk$-module V are said to be *equivalent* if and only if each one is finer than the other one. In particular, two filtrations are equivalent if and only if the induced topologies are so.

Remark 3.1.1. We will always endow the base ring $\Bbbk$ with the *discrete filtration* $D_0\Bbbk = \Bbbk$ and $D_n\Bbbk = 0$ for every $n \geq 1$. This filtration induces on $\Bbbk$ the discrete topology because $\{0\}$ turns out to be open by definition, whence every point is open. This topology is always compatible with all algebraic structures on $\Bbbk$, even

$$(-)^{-1} \colon \Bbbk^\times \to \Bbbk^\times, \ k \mapsto k^{-1}$$

in case $\Bbbk$ is a field (see for example [Bk3, III.55, § 6.7, Example 1]).

Given two filtered $\Bbbk$-modules (V, F_nV) and (W, F_nW), a $\Bbbk$-linear map $f : V \to W$ is said to be a *filtered morphism* is it satisfies $f(F_nV) \subseteq F_nW$ for every $n \in \mathbb{N}$. We may often denote by $f_n : F_nV \to F_nW$ the restriction of the filtered morphism $f : V \to W$ to the n-th component of the filtration. A *filtered isomorphism* is a filtered morphism which is an isomorphism of $\Bbbk$-modules and whose inverse is filtered as well.

Remark 3.1.2. Independently from being filtered or not, a $\Bbbk$-module homomorphism $f \colon V \to W$ is continuous with respect to the linear topologies induced by the filtrations if and only if for every $n \in \mathbb{N}$ there exists $m(n) \in \mathbb{N}$ such that $f(F_{m(n)}V) \subseteq F_nW$. In particular, any morphism of filtered $\Bbbk$-modules is continuous. The converse is not true in general, but given V, W two filtered $\Bbbk$-modules, one can prove that a $\Bbbk$-module homomorphism $f \colon V \to W$ is continuous with respect to the linear topologies induced by the given filtrations if and only if there exists a sub-filtration on V equivalent to the former one and with respect to which f is filtered.

In light of Remark 3.1.2, we will distinguish *homeomorphism* as topological spaces (which may be called *topological isomorphism*) from *filtered isomorphism* as filtered modules. Note that every filtered isomorphism is in fact an homeomorphism.

Filtered $\Bbbk$-modules and filtered morphisms form a category $\mathfrak{M}^{\mathsf{flt}}$. It turns out that this is an additive (but not abelian) category which is complete and cocomplete in such a way that the

(1) In [NvO1] the operation of shifting by k is called the *k-th suspension functor*.

functor $\mathcal{U} : \mathfrak{M}^{\mathsf{flt}} \to \mathfrak{M}$ that forgets the filtration lifts limits and colimits uniquely. Indeed, for example, if $\left\{ (V_i, F_n V_i), f_{i,j} \right\}_{\mathcal{J}}$ is a (small) diagram of filtered $\Bbbk$-modules ($\mathcal{J}$ a small category), we may compute the limit $\left(\varprojlim (V_i) \xrightarrow{\sigma_j} V_j \right)_{\mathcal{J}}$ in $\mathfrak{M}$ and then endow $\varprojlim (V_i)$ with the *initial*[2] or *projective limit filtration*

$$F_n\big(\varprojlim (V_i)\big) := \varprojlim (F_n (V_i)), \tag{3.1}$$

which is the coarsest filtration such that each structure map $\sigma_j : \varprojlim (V_i) \to V_j$ is filtered. Analogously, one may compute the colimit $\left(V_j \xrightarrow{\tau_j} Q := \varinjlim (V_i) \right)_{\mathcal{J}}$ in $\mathfrak{M}$ and then endow Q with the *final* or *inductive limit filtration*

$$F_n\big(\varinjlim (V_i)\big) := \sum_{i \in \mathcal{J}} \tau_i\big(F_n V_i\big), \tag{3.2}$$

which is the finest filtration such that all morphisms $\tau_i : V_i \to Q$ are filtered.

Example 3.1.3. As a particular case, let $(V, F_n V)$ be a filtered $\Bbbk$-module and let $W \subseteq V$ be a $\Bbbk$-submodule. Endow W with the *induced filtration*, the coarsest filtration such that the inclusion map $W \hookrightarrow V$ is filtered. Namely, $F_n W := W \cap F_n V$ for all $n \in \mathbb{N}$. The quotient $\Bbbk$-module V/W endowed with the colimit filtration $F_n(V/W) = (F_n V + W)/W$ is then the cokernel of $W \hookrightarrow V$ in $\mathfrak{M}^{\mathsf{flt}}$. Notice that this filtration is the finest such that the canonical projection $V \twoheadrightarrow V/W$ is filtered, whence we may call it the *quotient filtration*[3]. Unless claimed otherwise, these will be the filtrations that we will consider on submodules and quotient modules.

Remark 3.1.4. Analogously to what we did with the limit of a diagram in $\mathfrak{M}^{\mathsf{flt}}$, for every $n \in \mathbb{N}$ we may consider the colimit $\varinjlim (F_n V_i)$ in $\mathfrak{M}$ and this comes endowed with a unique $\Bbbk$-linear map $\iota_n : \varinjlim (F_n V_i) \to \varinjlim (V_i)$ such that $\iota_n \circ \tau_{i,n} = \tau_i \circ \iota_{i,n}$, given by the universal property of $\varinjlim (F_n V_i)$. Thus one may consider $F_n\big(\varinjlim (V_i)\big) := \mathsf{im}(\iota_n)$ for $n \in \mathbb{N}$ as the filtration on $\varinjlim (V_i)$, as it is done for example in [NvO2, §D.I page 281]. However, in view of [Sr, Corollary 8.5] we have that $\mathsf{im}(\iota_n) = \sum_{i \in \mathcal{J}} \mathsf{im}(\tau_i \circ \iota_{i,n}) = \sum_{i \in \mathcal{J}} \tau_i (F_n V_i)$, because $\mathfrak{M}$ is a cocomplete category. We opted for this last description of the filtration on the colimit because it is easier to handle.

Since $\mathcal{U}$ lifts limits and colimits uniquely and $\mathfrak{M}$ is complete and cocomplete, it preserves limits and colimits as well. However, notice that $\mathcal{U} : \mathfrak{M}^{\mathsf{flt}} \to \mathfrak{M}$ does not create limits, as any filtration on $L := \varprojlim (V_i)$ which is finer than the initial one makes of it a source for the diagram $\left\{ (V_i, F_n V_i), f_{i,j} \right\}_{\mathcal{J}}$ in $\mathfrak{M}^{\mathsf{flt}}$. The same happens for colimits and coarser filtrations.

Remark 3.1.5. Every $\Bbbk$-module V can be considered a filtered $\Bbbk$-module via the *discrete filtration* $D_0 V = V$, $D_n V = 0$ for all $n \geq 1$ or via the *trivial filtration* $Z_n V = V$ for all $n \geq 0$. If we denote by $\mathcal{D} : \mathfrak{M} \to \mathfrak{M}^{\mathsf{flt}}$ the functor that endows every $\Bbbk$-module with the discrete filtration then

$$\eta_W = \mathsf{Id}_W : W \to \mathcal{U}\mathcal{D}(W) \qquad \text{and} \qquad \epsilon_V = \mathsf{Id}_V : \mathcal{D}\mathcal{U}((V, F_n V)) \to (V, F_n V)$$

are the unit and the counit of an adjunction $(\mathcal{D}, \mathcal{U}, \eta, \epsilon)$ and moreover η is always a natural isomorphism (equivalently, $\mathcal{D}$ is full and faithful). On the other hand, ϵ is bijective (in particular, $\mathcal{U}$ is faithful) but it is not an isomorphism, so that $(\mathcal{D}, \mathcal{U})$ is not an equivalence of categories. To see why ϵ is not an isomorphism, consider the $\Bbbk$-module $V = \Bbbk \oplus \Bbbk$ with filtrations $\{D_n V \mid n \in \mathbb{N}\}$ given by $D_0 V = V$, $D_n V = 0$ for all $n \geq 1$, and $\{F_n V \mid n \in \mathbb{N}\}$ given by $F_0 V = V$, $F_1 V = \Bbbk \oplus 0$ and $F_n V = 0$ for all $n \geq 2$. Then $\mathsf{Id}_V : (V, D_n V) \to (V, F_n V)$ is filtered but $\mathsf{Id}_V : (V, F_n V) \to (V, D_n V)$ is not. In such a case $\epsilon_V = \mathsf{Id}_V$ is not an isomorphism in $\mathfrak{M}^{\mathsf{flt}}$ since its inverse is not filtered. This is also the reason why $\mathfrak{M}^{\mathsf{flt}}$ is not an abelian category.

---

(2) The terminology is inspired from the topological framework, where the *initial topology* on a set X with respect to a family of maps $f_i : X \to Y_i$, $i \in I$, is the coarsest topology on X such that each f_i is continuous.

(3) More generally, whenever we have a $\Bbbk$-linear surjective morphism $p : V \twoheadrightarrow W$ between a filtered $\Bbbk$-module $(V, F_n V)$ and another $\Bbbk$-module W, we will call *quotient filtration* the final filtration on W with respect to p, that is, the finest one such that p is filtered.

On the other hand, one may consider the functor $\mathcal{Z} : \mathfrak{M} \to \mathfrak{M}^{\mathsf{flt}}$ which endows every $\Bbbk$-module with the trivial filtration. Thus the maps

$$\gamma_V = \mathsf{Id}_V : (V, F_n V) \to \mathcal{Z}\mathcal{U}\left((V, F_n V)\right) \quad \text{and} \quad \theta_W = \mathsf{Id}_W : \mathcal{U}\mathcal{Z}(W) \to W$$

are the unit and the counit of an adjunction $(\mathcal{U}, \mathcal{Z}, \gamma, \theta)$. Moreover, θ is a natural isomorphism so that $\mathcal{Z}$ is fully faithful, while γ_V is bijective for all V but not an isomorphism, as above.

Notice that Remark 3.1.5 provides a different way to see that $\mathcal{U}$ preserves both limits and colimits, as it is a left and a right adjoint at the same time.

Remark 3.1.6. As every function from a discrete topological space to any topological space is continuous, every morphism from a discretely filtered module to any filtered module is automatically filtered. Analogously, any morphism to a trivially filtered module is automatically filtered.

Now, if V and W are filtered $\Bbbk$-modules, then there is a natural filtration on their tensor product $V \otimes W$ given by

$$F_n\left(V \otimes W\right) := \sum_{p+q=n} \mathsf{im}\left(F_p V \otimes F_q W\right) \tag{3.3}$$

for all $n \in \mathbb{N}$, where the notation $\mathsf{im}(-)$ on the right-hand side stands for the image of $F_p V \otimes F_q W$ in $V \otimes W$[(4)]. We will consider this one as the standard filtration on the tensor product of filtered $\Bbbk$-modules. Moreover, if we denote by $\mathsf{Hom}^{\mathsf{flt}}_{\Bbbk}\left(V, W\right)$ the abelian group of filtered morphisms $f \colon V \to W$, then it is a filtered $\Bbbk$-module as well with filtration given by

$$F_n\left(\mathsf{Hom}^{\mathsf{flt}}_{\Bbbk}\left(V, W\right)\right) = \mathsf{Hom}^{\mathsf{flt}}_{\Bbbk}\left(V, W[n]\right) = \Big\{ f \in \mathsf{Hom}_{\Bbbk}\left(V, W\right) \mid f\left(F_k V\right) \subseteq F_{n+k} W \text{ for all } k \geq 0 \Big\}.$$

Proposition 3.1.7. *If $f \colon V \to V'$ and $g \colon W \to W'$ are morphisms of filtered $\Bbbk$-modules, then $f \otimes g : V \otimes W \to V' \otimes W'$ is a morphism of filtered $\Bbbk$-modules. In particular, the category $\mathfrak{M}^{\mathsf{flt}}$ of filtered modules is a monoidal category with tensor product $\otimes$ and unit $\Bbbk$. Furthermore, for every U, V, W in $\mathfrak{M}^{\mathsf{flt}}$ we have a bijections*

$$\mathsf{Hom}^{\mathsf{flt}}_{\Bbbk}\left(V, \mathsf{Hom}^{\mathsf{flt}}_{\Bbbk}\left(U, W\right)\right) \cong \mathsf{Hom}^{\mathsf{flt}}_{\Bbbk}\left(U \otimes V, W\right) \cong \mathsf{Hom}^{\mathsf{flt}}_{\Bbbk}\left(U, \mathsf{Hom}^{\mathsf{flt}}_{\Bbbk}\left(V, W\right)\right) \tag{3.4}$$

natural in V, W and U, W respectively. Therefore, for every V in $\mathfrak{M}^{\mathsf{flt}}$ the endofunctors $V \otimes -$ and $- \otimes V$ are left adjoints of the functor $\mathsf{Hom}^{\mathsf{flt}}_{\Bbbk}\left(V, -\right)$.

Proof. Since the first claim is just an easy check, let us start by proving the first bijection in (3.4). The classical hom-tensor adjunction provides us with a bijection

$$\begin{array}{ccc}
\mathsf{Hom}_{\Bbbk}\left(V, \mathsf{Hom}_{\Bbbk}\left(U, W\right)\right) & \underset{\psi}{\overset{\varphi}{\rightleftarrows}} & \mathsf{Hom}_{\Bbbk}\left(U \otimes V, W\right) \\
\left[f : V \to \mathsf{Hom}_{\Bbbk}\left(U, W\right)\right] & \longmapsto & \left[\varphi(f) : U \otimes V \to W;\ u \otimes v \mapsto \big(f(v)\big)(u)\right] \\
\Big[\psi(g) : V \to \mathsf{Hom}_{\Bbbk}\left(U, W\right);\ v \mapsto \left[u \mapsto g(u \otimes v)\right]\Big] & \longleftarrow & \left[g : U \otimes V \to W\right]
\end{array}$$

Denote by $i_n : F_n U \to U$ and $j_m : F_m V \to V$ the canonical inclusions, so that $\mathsf{im}\left(F_p U \otimes F_q V\right) = i_p(F_p U) \otimes j_q(F_q V) \subseteq U \otimes V$. If $f \in \mathsf{Hom}^{\mathsf{flt}}_{\Bbbk}\left(V, \mathsf{Hom}^{\mathsf{flt}}_{\Bbbk}\left(U, W\right)\right)$ then for all $p + q = n$, $u \in F_p U$ and $v \in F_q V$, we have

$$\varphi(f)(i_p(u) \otimes j_q(v)) = \Big(f\left(j_q(v)\right)\Big)\left(i_p(u)\right) \in F_p\left(W[q]\right) = F_{p+q} W = F_n W,$$

because $f\left(j_q(v)\right) \in F_q\left(\mathsf{Hom}^{\mathsf{flt}}_{\Bbbk}\left(U, W\right)\right) = \mathsf{Hom}^{\mathsf{flt}}_{\Bbbk}\left(U, W[q]\right)$, thus $\varphi(f) \in \mathsf{Hom}^{\mathsf{flt}}_{\Bbbk}\left(U \otimes V, W\right)$. Conversely, if $g \in \mathsf{Hom}^{\mathsf{flt}}_{\Bbbk}\left(U \otimes V, W\right)$ then for all $m, n \in \mathbb{N}$

$$\Big(\big(\psi(g)\big)(j_m(v))\Big)\left(i_n(u)\right) = g\left(i_n(u) \otimes j_m(v)\right) \in F_{n+m} W.$$

[(4)] More precisely, one should say that $\mathsf{im}\left(F_p V \otimes F_q W\right)$ is the $\Bbbk$-submodule of $V \otimes W$ generated by all the elements of the form $v \otimes w$ where $v \in F_p V$ and $w \in F_q W$, whence $F_n\left(V \otimes W\right)$ is the $\Bbbk$-submodule of $V \otimes W$ generated by all the elements of the form $v \otimes w$ where $v \in F_p V$ and $w \in F_q W$ and $p + q = n$.

In particular, for $m = 0$ we get that $\psi(g)(v) \in \mathsf{Hom}_{\Bbbk}^{\mathsf{flt}}(U, W)$ for all $v \in V$. For m varying in $\mathbb{N}$ we get also that $\psi(g)(j_m(v)) \in \mathsf{Hom}_{\Bbbk}^{\mathsf{flt}}(U, W[m]) = F_m\left(\mathsf{Hom}_{\Bbbk}^{\mathsf{flt}}(U, W)\right)$ so that $\psi(g) \in \mathsf{Hom}_{\Bbbk}^{\mathsf{flt}}\left(V, \mathsf{Hom}_{\Bbbk}^{\mathsf{flt}}(U, W)\right)$. Therefore, φ and ψ induce the desired bijection

$$
\begin{tikzcd}
\mathsf{Hom}_{\Bbbk}^{\mathsf{flt}}\left(V, \mathsf{Hom}_{\Bbbk}^{\mathsf{flt}}(U, W)\right) \arrow[r, shift left, "\varphi"] & \mathsf{Hom}_{\Bbbk}^{\mathsf{flt}}(U \otimes V, W) \arrow[l, shift left, "\psi"] \\
\left[f : V \to \mathsf{Hom}_{\Bbbk}^{\mathsf{flt}}(U, W)\right] \arrow[r, mapsto] & \left[\varphi(f) : U \otimes V \to W;\ u \otimes v \mapsto \big(f(v)\big)(u)\right] \\
\left[\psi(g) : V \to \mathsf{Hom}_{\Bbbk}^{\mathsf{flt}}(U, W);\ v \mapsto \left[u \mapsto g(u \otimes v)\right]\right] & \left[g : U \otimes V \to W\right] \arrow[l, mapsto]
\end{tikzcd}
$$

The proof of the other adjunction follows the same steps and hence it is omitted. □

Remark 3.1.8. A first observation which is in order is that the adjunctions of Proposition 3.1.7 may also be deduced from [NvO2, Lemma D.VIII.1] by restricting to the 0-th component. We point out however that the foregoing result is slightly different from the cited one, since $\mathsf{Hom}_{\Bbbk}^{\mathsf{flt}}(U, V) \subsetneq \mathsf{HOM}(U, V)$. To see why the inclusion is strict, consider again $V = \Bbbk \oplus \Bbbk$ with the two filtrations of Remark 3.1.5, i.e. the discrete filtration and $F_0(V) = V$, $F_1(V) = \Bbbk \oplus 0$, $F_n(V) = 0$ for all $n \geq 2$. The identity $\mathsf{Id}_V : (V, F_nV) \to (V, D_nV)$ is a morphism of degree -1 (in the sense that $\mathsf{Id}_V(F_nV) \subseteq D_{n-1}V$ for every $n \geq 0$) and so it belongs to $\mathsf{HOM}(V, V)$, but we already notice that it does not belong to $\mathsf{Hom}_{\Bbbk}^{\mathsf{flt}}(V, V)$.

Secondly, observe that both morphisms φ and ψ are in fact $\Bbbk$-linear morphisms. Moreover, fix $m \in \mathbb{N}$. If $f \in F_n\left(\mathsf{Hom}_{\Bbbk}^{\mathsf{flt}}\left(V, \mathsf{Hom}_{\Bbbk}^{\mathsf{flt}}(U, W)\right)\right) = \mathsf{Hom}_{\Bbbk}^{\mathsf{flt}}\left(V, \mathsf{Hom}_{\Bbbk}^{\mathsf{flt}}(U, W)[n]\right)$, if $p + q = m$ and if $u \in F_pU$ and $v \in F_qV$, then we have

$$\varphi(f)(i_p(u) \otimes j_q(v)) = \Big(f\left(j_q(v)\right)\Big)(i_p(u)) \in F_{q+n+p}W$$

because $f\left(j_q(v)\right) \in F_q\left(\mathsf{Hom}_{\Bbbk}^{\mathsf{flt}}(U, W)[n]\right) = F_{q+n}\left(\mathsf{Hom}_{\Bbbk}^{\mathsf{flt}}(U, W)\right) = \mathsf{Hom}_{\Bbbk}^{\mathsf{flt}}(U, W[q+n])$, so that $\varphi(f)\left(F_m(U \otimes V)\right) \subseteq F_{m+n}W$ and hence $\varphi(f) \in \mathsf{Hom}_{\Bbbk}^{\mathsf{flt}}(U \otimes V, W[n]) = F_n\left(\mathsf{Hom}_{\Bbbk}^{\mathsf{flt}}(U \otimes V, W)\right)$. On the other hand, if $g \in F_n\left(\mathsf{Hom}_{\Bbbk}^{\mathsf{flt}}(U \otimes V, W)\right) = \mathsf{Hom}_{\Bbbk}^{\mathsf{flt}}(U \otimes V, W[n])$ then

$$\Big(\psi(g)\left(j_q(v)\right)\Big)(i_p(u)) = g(i_p(u) \otimes j_q(v)) \in g\left(F_{p+q}(U \otimes V)\right) \subseteq F_{p+q+n}W$$

for all $p, q \in \mathbb{N}$, $u \in F_pU$, $v \in F_qV$, so that

$$\psi(g)\left(j_q(v)\right) \in \mathsf{Hom}_{\Bbbk}^{\mathsf{flt}}(U, W[n+q]) = F_{n+q}\left(\mathsf{Hom}_{\Bbbk}^{\mathsf{flt}}(U, W)\right) = F_q\left(\mathsf{Hom}_{\Bbbk}^{\mathsf{flt}}(U, W)[n]\right)$$

for all $q \in \mathbb{N}$ and so $\psi(g) \in \mathsf{Hom}_{\Bbbk}^{\mathsf{flt}}\left(V, \mathsf{Hom}_{\Bbbk}^{\mathsf{flt}}(U, W)[n]\right) = F_n\left(\mathsf{Hom}_{\Bbbk}^{\mathsf{flt}}\left(V, \mathsf{Hom}_{\Bbbk}^{\mathsf{flt}}(U, W)\right)\right)$. Therefore, the bijection on (3.4) is in fact an isomorphism of filtered $\Bbbk$-modules.

As a consequence, $(\mathfrak{M}^{\mathsf{flt}}, \otimes, \Bbbk, a, l, r)$ is a monoidal category, $\mathfrak{M}^{\mathsf{flt}}$ is cocomplete and for any object V in $\mathfrak{M}^{\mathsf{flt}}$, the endofunctors $V \otimes -$ and $- \otimes V$ are cocontinuous (because they are left adjoints). In light of Theorem 1.6.5, we have a bicategory $\mathscr{B}im_{\Bbbk}^{\mathsf{flt}}$ whose 0-cells are algebras in $\mathfrak{M}^{\mathsf{flt}}$ and whose categories of $\{1, 2\}$-cells are the categories of bimodules over these algebras.

Explicitly, an algebra R in $\mathfrak{M}^{\mathsf{flt}}$ is a $\Bbbk$-module endowed with a decreasing filtration of $\Bbbk$-submodules

$$\cdots \subseteq F_1R \subseteq F_0R = R \tag{3.5}$$

and with two filtered morphisms $u : \Bbbk \to R$ and $m : R \otimes R \to R$ such that m is associative and unital with unit u. Claiming that m has to be filtered is equivalent to say that the filtration (3.5) satisfies $F_nR \cdot F_mR \subseteq F_{n+m}R$ for every $m, n \in \mathbb{N}$, where we denoted by $\cdot$ the multiplication in R. Thus R is what is usually called a *filtered* $\Bbbk$-*algebra* (compare for example with [NvO2, §D.I.1]).

If R, S are filtered $\Bbbk$-algebras, then an (S, R)-bimodule is a filtered $\Bbbk$-module (M, F_nM) endowed with two filtered morphisms

$$\mu_{S,M} : S \otimes M \to M \qquad \text{and} \qquad \mu_{M,R} : M \otimes R \to M$$

that are actions of S and R over M from the left and the right respectively and that are compatible in a suitable way as expressed by (1.10). As before, this is equivalent to claim that for all $m, n \in \mathbb{N}$ we have

$$F_n S \cdot F_m M \subseteq F_{n+m} M \qquad \text{and} \qquad F_n M \cdot F_m R \subseteq F_{n+m} M$$

where now we denoted by $\cdot$ the action of S or R on M indifferently. This means that M is a *filtered* (S,R)*-bimodule*. A morphism of filtered (S,R)-bimodules $f : M \to N$ is a $\Bbbk$-linear filtered morphism which is also (S,R)-bilinear. Thus, the categories of $\{1,2\}$-cells are the categories of filtered bimodules over the filtered $\Bbbk$-algebras. The vertical composition is the composition of (S,R)-bilinear morphisms, which turns out to be filtered since the composing morphisms are so.

Given ${}_S M_R$ and ${}_R N_T$ two filtered bimodules, their horizontal composition is given by their tensor product over R, which is the coequalizer

$$M \otimes R \otimes N \underset{M \otimes \mu_{R,N}}{\overset{\mu_{M,R} \otimes N}{\rightrightarrows}} M \otimes N \xrightarrow{\omega_{M,N}} M \otimes_R N$$

in the category of (S,T)-bimodules. Explicitly, $M \otimes_R N$ is the classical tensor product of R-modules

$$M \otimes_R N = \frac{M \otimes N}{\langle m \cdot r \otimes n - m \otimes r \cdot n \mid m \in M, n \in N, r \in R \rangle},$$
$$\omega_{M,N} : M \otimes N \twoheadrightarrow M \otimes_R N, \quad \big(m \otimes n \mapsto m \otimes_R n\big),$$

filtered with the filtration given in (3.2), that is,

$$\sum_{p+q+r=n} \mathsf{im}\left(F_p M \otimes F_q R \otimes F_r N\right) \underset{M \otimes \mu_{R,N}}{\overset{\mu_{M,R} \otimes N}{\rightrightarrows}} \sum_{h+k=n} \mathsf{im}\left(F_h M \otimes F_k N\right) \xrightarrow{\omega_{M,N}} F_n\left(M \otimes_R N\right),$$

which amounts to say that, for all $n \in \mathbb{N}$,

$$F_n\left(M \otimes_R N\right) = \sum_{p+q=n} \mathsf{im}\left(F_p M \otimes_R F_q N\right). \tag{3.6}$$

Summing up, let us collect our achievements in the following result for future reference.

Proposition 3.1.9. *There is a bicategory $\mathscr{B}im_{\Bbbk}^{\mathsf{flt}}$ which has filtered $\Bbbk$-algebras as 0-cells and whose categories of $\{1,2\}$-cells are the categories of filtered bimodules over these filtered algebras with vertical and horizontal compositions given by the composition of bilinear morphisms and the usual tensor product, filtered as in (3.6), respectively.*

Remark 3.1.10. Similar to the case of usual bimodules, we have an isomorphism between the category ${}_S\mathfrak{M}_R^{\mathsf{flt}}$ of filtered (S,R)-bimodules and the category ${}_{S \otimes R^{\mathrm{op}}}\mathfrak{M}^{\mathsf{flt}}$ of filtered $S \otimes R^{\mathrm{op}}$-modules, where R^{op} denotes the opposite algebra of R and $S \otimes R^{\mathrm{op}}$ is a filtered algebra with filtration as in (3.6).

Proposition 3.1.11. *For S, R filtered $\Bbbk$-algebras the forgetful functor $\mathcal{U} : {}_S\mathfrak{M}_R^{\mathsf{flt}} \to \mathfrak{M}^{\mathsf{flt}}$ creates limits and colimits. In particular, the category ${}_S\mathfrak{M}_R^{\mathsf{flt}}$ is complete and cocomplete.*

Proof. In view of Remark 3.1.10, ${}_S\mathfrak{M}_R^{\mathsf{flt}}$ is complete and cocomplete if and only if ${}_{S \otimes R^{\mathrm{op}}}\mathfrak{M}^{\mathsf{flt}}$ is. Notice that ${}_{S \otimes R^{\mathrm{op}}}\mathfrak{M}^{\mathsf{flt}} = {}_{S \otimes R^{\mathrm{op}}}\left(\mathfrak{M}^{\mathsf{flt}}\right)$ is the Eilenberg-Moore category of algebras over the monad $(S \otimes R^{\mathrm{op}}) \otimes - : \mathfrak{M}^{\mathsf{flt}} \to \mathfrak{M}^{\mathsf{flt}}$. Therefore, since $\mathfrak{M}^{\mathsf{flt}}$ is a cocomplete category, it follows from Proposition 1.4.2 that the forgetful functor $\mathcal{U} : {}_{S \otimes R^{\mathrm{op}}}\mathfrak{M}^{\mathsf{flt}} \to \mathfrak{M}^{\mathsf{flt}}$ creates all limits and it creates all those colimits which are preserves by $(S \otimes R^{\mathrm{op}}) \otimes -$. However, we already noticed that $(S \otimes R^{\mathrm{op}}) \otimes -$ is cocontinuous, so that $\mathcal{U}$ creates all colimits as well. $\square$

Explicitly, if S, R are filtered $\Bbbk$-algebras and $\big\{ (M_i, F_n M_i), f_{i,j} \big\}_{\mathcal{J}}$ is a diagram of filtered (S,R)-bimodules over a scheme $\mathcal{J}$, then its limit $\varprojlim (M_i)$ and its colimit $\varinjlim (M_i)$ are filtered with the initial and the final filtrations

$$F_n\big(\varprojlim (M_i)\big) = \varprojlim \left(F_n\left(M_i\right)\right) \qquad \text{and} \qquad F_n\big(\varinjlim (M_i)\big) = \sum_{i \in \mathcal{J}} \tau_i\big(F_n M_i\big) \tag{3.7}$$

as in (3.1) and (3.2), respectively.

Remark 3.1.12. Unless both R and S are discretely filtered $\Bbbk$-algebras, the discrete filtration $\{D_0M = M, D_nM = 0 \mid n \geq 1\}$ on an (S,R)-bimodule M does not endow it with a structure of filtered (S,R)-bimodule as $F_hR \cdot D_0M \cdot F_kS \neq 0 = D_{h+k}M$ in general. Thus the functor $\mathcal{D}$ of Remark 3.1.5 needs to be substituted by a functor $\mathcal{D}\colon {}_S\mathfrak{M}_R \to {}_S\mathfrak{M}_R^{\mathsf{flt}}$ which associates to M in ${}_S\mathfrak{M}_R$ the (S,R)-bimodule M with filtration

$$F_nM := \sum_{h+k=n} F_kS \cdot M \cdot F_hR$$

for all $n \in \mathbb{N}$. This filtration is called the *induced filtration* and it coincides with the discrete one in case both S and R are discretely filtered. The new functor $\mathcal{D}$ is still left adjoint of $\mathcal{U}$ with the same unit and counit as in Remark 3.1.5, which justifies also why we use the same notation for both.

Apart from the discrete filtration (which has been taken care in Remark 3.1.12), all the constructions and notions that we introduced for filtered $\Bbbk$-modules hold as well for filtered bimodules over filtered $\Bbbk$-algebras. For example, if ${}_SM_R$, ${}_SP_T$ and ${}_RN_T$ are bimodules as denoted, then the abelian group $\mathsf{Hom}_T^{\mathsf{flt}}(N,P)$ of filtered morphisms $f\colon N \to P$ which are T-linear is an object in ${}_S\mathfrak{M}_R^{\mathsf{flt}}$ with filtration given by

$$F_n\Big(\mathsf{Hom}_T^{\mathsf{flt}}(N,P)\Big) = \mathsf{Hom}_T^{\mathsf{flt}}(N,P[n]) = \Big\{f \in \mathsf{Hom}_T^{\mathsf{flt}}(N,P) \mid f(F_kN) \subseteq F_{n+k}P, k \geq 0\Big\}. \tag{3.8}$$

and we have a bijection

$${}_S\mathsf{Hom}_T^{\mathsf{flt}}(M \otimes_R N, P) \cong {}_S\mathsf{Hom}_R^{\mathsf{flt}}\big(M, \mathsf{Hom}_T^{\mathsf{flt}}(N,P)\big) \tag{3.9}$$

natural in M and P. Therefore from now on we will use them freely both for $\Bbbk$-modules and (S,R)-bimodules.

Remark 3.1.13. Notice that the filtration in (3.8) is the one induced by the filtered bimodule of all morphisms of finite degree $\mathsf{HOM}_T(N,P)$ onto its subgroup $F_0\big(\mathsf{HOM}_T(N,P)\big) = \mathsf{Hom}_T^{\mathsf{flt}}(N,P)$ (see e.g. [LvO, I.2.5]).

3.1.2 The completion bifunctor

In this subsection we will recall the construction of the completion functor from the category of filtered bimodules to the one of complete bimodules. As a classical reference for the material presented here, we suggest [NvO2, Chap. D, §§ I-II] and [LvO, Chap. I, §3]. Nevertheless, consistently with Subsection 3.1.1, we will prefer to follow a bicategorical approach.

Let (V, F_nV) be a filtered $\Bbbk$-module. We recall that V is *Hausdorff* (or *separable*) if and only if for every pair of elements $x, y \in V$ there exist two open sets $X, Y \subseteq V$ such that $x \in X$, $y \in Y$ and $X \cap Y = \emptyset$. However, by definition of the linear topology on V, this is equivalent to say that $\bigcap_{n\in\mathbb{N}} F_nV = 0$. Moreover, a sequence $\{v_k \mid k \geq 0\}$ in a Hausdorff filtered $\Bbbk$-module V is a *Cauchy sequence* if and only if for every $p \in \mathbb{N}$, there exists $q \in \mathbb{N}$ such that for all $k, h \geq q$ we have that $v_k - v_h \in F_pV$. It is *convergent* to an element $v \in V$ if and only if for every $p \in \mathbb{N}$, there exists $q \in \mathbb{N}$ such that for all $k \geq q$ we have that $v - v_k \in F_pV$. A Hausdorff filtered $\Bbbk$-module (V, F_nV) is said to be *complete* with respect to the linear topology induced by the filtration if and only if every Cauchy sequence is convergent.

Inside the category $\mathfrak{M}^{\mathsf{flt}}$ of filtered $\Bbbk$-modules we may isolate the full subcategory $\mathfrak{M}^{\mathsf{c}}$ of complete $\Bbbk$-modules, whose objects are complete Hausdorff filtered $\Bbbk$-modules and whose morphisms are all the filtered morphisms between them.

Now, given a filtered $\Bbbk$-module (V, F_nV) the filtration on V gives rise to a projective (or inverse) system of $\Bbbk$-modules given by

$$\pi_{m,n}\colon \frac{V}{F_mV} \to \frac{V}{F_nV}; \quad \big(x + F_mV \mapsto x + F_nV\big) \tag{3.10}$$

for all $m \geq n$ and this allows us to give an effective characterization of when a filtered $\Bbbk$-module is Hausdorff and complete, as well as a universal construction of its Hausdorff completion. To this aim, set

$$\widehat{V} := \varprojlim \left(\frac{V}{F_n V} \right)$$

and consider the canonical morphism $\gamma_V \colon V \to \widehat{V}$ rendering commutative the diagram

$$\begin{tikzcd} V \arrow[rr, dashed, "\gamma_V"] \arrow[dr, "\pi_n"'] & & \widehat{V} \arrow[dl, "p_n"] \\ & \frac{V}{F_n V} & \end{tikzcd} \tag{3.11}$$

for all $n \in \mathbb{N}$, where $p_n \colon \widehat{V} \to V/F_n V$ are the structure maps of the limit. The subsequent result can be proven directly (see also [NvO2, Proposition D.II.3]).

Proposition 3.1.14. *An object $(V, F_n V)$ in $\mathfrak{M}^{\mathsf{flt}}$ is complete and Hausdorff as a topological space if and only if the map γ_V of diagram (3.11) is an isomorphism.*

Remark 3.1.15. Recall that a realization of the inverse limit $\varprojlim (V/F_n V)$ in $\mathfrak{M}^{\mathsf{flt}}$ is given by

$$\left\{ \left(v_k + F_k V\right)_{k \geq 0} \in \prod_{k \geq 0} \frac{V}{F_k V} \;\middle|\; \text{for all } k \geq 0 \text{ and } h \geq k,\ v_k + F_k V = v_h + F_k V \right\}. \tag{3.12}$$

If $\left(v_k + F_k V\right)_{k \geq 0} \in \varprojlim (V/F_n V)$ then for all $N \in \mathbb{N}$ and for every $k, h \geq N$ one has

$$v_k + F_N V = v_N + F_N V = v_h + F_N V,$$

that is, $v_k - v_h \in F_N V$. Considering $p = q = N$ in the definition of a Cauchy sequence one can see then that $\{v_k \mid k \geq 0\}$ is Cauchy. Conversely, if $\{v_k \mid k \geq 0\}$ is a Cauchy sequence in V, then we can define a subsequence $\{v'_n \mid n \geq 0\}$ in the following way. For every $n \in \mathbb{N}$, there exists $N_n \in \mathbb{N}$ such that for all $k, h \geq N_n$, $v_k - v_h \in F_n V$. Set then $v'_n := v_{N_n}$. By construction, we have that $\left(v'_k + F_k V\right)_{k \geq 0} \in \varprojlim (V/F_n V)$. As a consequence, one may work with elements of $\widehat{V}$ as if they were Cauchy sequences in V.

Proposition 3.1.14 and Remark 3.1.15 justify in some sense the following definition.

Definition 3.1.16. For a filtered $\Bbbk$-module V, we define its *Hausdorff completion* to be the inverse limit $\widehat{V}$ over the projective system (3.10).

Example 3.1.17. Let V be a filtered $\Bbbk$-module with a filtration $\{F_n V \mid n \in \mathbb{N}\}$ such that $F_m V = 0$ for some $m \in \mathbb{N}$. Clearly $\bigcap_{n \in \mathbb{N}} F_n V \subseteq F_m V = 0$, so that it is Hausdorff and γ_V is injective. Furthermore, pick $\left(v_k + F_k V\right)_{k \geq 0} \in \varprojlim (V/F_n V)$. In view of the realization (3.12), it follows that $v_m - v_k \in F_k V$ for all $k < m$ and $v_k - v_m \in F_m V = 0$ for all $k \geq m$, so that $\left(v_k + F_k V\right)_{k \geq 0} = \left(v_m + F_k V\right)_{k \geq 0} = \gamma_V(v_m)$ and γ_V is surjective as well. Thus V is a complete Hausdorff filtered $\Bbbk$-module.

As a matter of terminology, from now on we will understand that a complete $\Bbbk$-module is Hausdorff as well, whence we will just refer to *complete* $\Bbbk$-modules and *completions* of filtered $\Bbbk$-modules. Recall that the completion $\widehat{V}$ may be filtered with the filtration given in (3.7), which in this case satisfies

$$F_m \widehat{V} = \mathsf{ker}\left(p_m : \widehat{V} \to \frac{V}{F_m V} \right). \tag{3.13}$$

In particular, the canonical $\Bbbk$-linear morphism $\gamma_V \colon V \to \widehat{V}$ is always filtered. Moreover, every $V/F_n V$ with the quotient filtration is a complete $\Bbbk$-module, because it satisfies the condition of Example 3.1.17.

The fact that Definition 3.1.16 is consistent (i.e. that the completion $\widehat{V}$ of a filtered $\Bbbk$-module V is a complete $\Bbbk$-module) follows from the subsequent Lemma 3.1.18 (see also [NvO2, Proposition D.II.3]), whose proof is omitted.

Lemma 3.1.18. *Let (V, F_nV) be a filtered $\Bbbk$-module. For all $n \geq 0$ we have an isomorphism $\widehat{V}/F_n\widehat{V} \cong V/F_nV$ in $\mathfrak{M}^{\mathsf{flt}}$ which is compatible with the morphisms of the projective system (3.10). In particular, $\widehat{V} = \varprojlim (V/F_nV)$ is a complete $\Bbbk$-module.*

Remark 3.1.19. Assume that V is a complete $\Bbbk$-module. The inverse morphism of $\gamma_V : V \to \widehat{V}$ is given by the assignment

$$\sigma_V : \widehat{V} \to V; \quad \left(x_k + F_kV\right)_{k\geq 0} \mapsto \lim_{k\to\infty} (x_k), \tag{3.14}$$

which is well-defined because the limit $\lim_{k\to\infty} (x_k)$ is independent of the representatives chosen for the equivalence classes $x_k + F_kV \in V/F_kV$, $k \geq 0$. Notice also that $\lim_{k\to\infty} (x_k) + F_nV = x_n + F_nV$ for all $n \geq 0$. If we perform the completion twice, that is, if we consider $\widehat{\widehat{V}}$, then we claim that

$$\sigma_{\widehat{V}} : \widehat{\widehat{V}} \to \widehat{V}; \qquad \left(\left(\left(v_n^k + F_nV\right)_{n\geq 0} + F_k\widehat{V}\right)_{k\geq 0} \mapsto \left(v_n^n + F_nV\right)_{n\geq 0}\right). \tag{3.15}$$

To see why, notice that for every $k \geq 0$ we have $\left(v_n^k + F_nV\right)_{n\geq 0} - \left(v_n^n + F_nV\right)_{n\geq 0} \in F_k\widehat{V} = \mathsf{ker}(p_k)$. Hence for every $M \geq 0$, there exists N, indeed $N = M$, such that for all $k \geq N$

$$\left(v_n^k + F_nV\right)_{n\geq 0} - \left(v_n^n + F_nV\right)_{n\geq 0} \in F_k\widehat{V} \subseteq F_M\widehat{V},$$

which means exactly that $\lim_{k\to\infty} \left(\left(v_n^k + F_nV\right)_{n\geq 0}\right) = \left(v_n^n + F_nV\right)_{n\geq 0}$.

The following conventions turn out to be very useful in dealing with completions.

Notation 3.1.20. Given a filtered $\Bbbk$-module V, by a slight abuse of notation we are going to denote the elements of its completion $\widehat{V}$ by $\widehat{x}_\infty$, meaning by that an $\mathbb{N}$-tuple $\left(x_n + F_nV\right)_{n\geq 0} \in \prod_{n\geq 0} V/F_nV$ satisfying $x_{n+1} - x_n \in F_nV$ for all $n \geq 0$.[(5)] If $x \in V$, then its image via γ_V in $\widehat{V}$ will be denoted by $\widehat{x}$, which is the $\mathbb{N}$-tuple $\left(x + F_nV\right)_{n\geq 0}$. When V is complete, and so γ_V and σ_V are mutually inverse functions, the element $x_\infty := \sigma_V\left(\widehat{x}_\infty\right) = \lim_{n\to\infty} (x_n)$ belongs to V.

For example, with these conventions it becomes easier to check that the morphism σ_V is filtered as well, so that γ_V is a filtered isomorphism.

Lemma 3.1.21. *If (V, F_nV) is a complete $\Bbbk$-module, then $\gamma_V : V \to \widehat{V}$ is a filtered isomorphism.*

Proof. If we have $\widehat{x}_\infty = (x_k + F_kV)_{k\geq 0} \in \mathsf{ker}\left(p_n : \widehat{V} \to V/F_nV\right)$ and if $x_\infty = \sigma_V\left(\widehat{x}_\infty\right)$ is the limit of the sequence $\{x_k \mid k \geq 0\}$ as in Remark 3.1.19, then $0 = p_n\left(\widehat{x_\infty}\right) = x_\infty + F_nV$. In particular $x_\infty \in F_nV$, so that σ_V is filtered. $\square$

The construction of the completion of a filtered $\Bbbk$-module provides us with a functor

$$\widehat{(-)} : \mathfrak{M}^{\mathsf{flt}} \to \mathfrak{M}^{\mathsf{c}}$$

that associates every filtered $\Bbbk$-module with its completion and every morphism of filtered $\Bbbk$-modules $f : V \to W$ with the morphism $\widehat{f} := \varprojlim \left(\widetilde{f_n}\right)$, where $\widetilde{f_n} : V/F_nV \to W/F_nW$ is the map induced by f on the quotients (see e.g. [LvO, Chapter I, §3]). The other way around, we have a functor $\mathcal{U} : \mathfrak{M}^{\mathsf{c}} \to \mathfrak{M}^{\mathsf{flt}}$ which "forgets" the completeness, that is, it associates every complete $\Bbbk$-module with itself, but seen just as a filtered $\Bbbk$-module. For all intents and purposes, it is the inclusion functor of the full subcategory $\mathfrak{M}^{\mathsf{c}}$ in $\mathfrak{M}^{\mathsf{flt}}$. By using Proposition 3.1.14 and Lemma 3.1.21 one may check that the following result holds.

[(5)] This condition is in fact equivalent to claim that $(x_n + F_nV)_{n\geq 0} \in \varprojlim (V/F_nV)$.

Lemma 3.1.22. *The functor* $\widehat{(-)} : \mathfrak{M}^{\mathsf{flt}} \to \mathfrak{M}^{\mathsf{c}}$ *is left adjoint to the functor* $\mathcal{U} : \mathfrak{M}^{\mathsf{c}} \to \mathfrak{M}^{\mathsf{flt}}$*, that is, we have a bijection*

$$\mathsf{Hom}^{\mathsf{flt}}_{\Bbbk}\left(V, \mathcal{U}(W)\right) \cong \mathsf{Hom}^{\mathsf{c}}_{\Bbbk}\left(\widehat{V}, W\right) \tag{3.16}$$

natural in both components. The unit is the canonical map $\gamma_V \colon V \to \mathcal{U}\big(\widehat{V}\big)$ *for* $V \in \mathfrak{M}^{\mathsf{flt}}$*. The counit is "its inverse"* $\sigma_W \colon \widehat{\mathcal{U}(W)} \to W$ *for all* $W \in \mathfrak{M}^{\mathsf{c}}$*. In particular, the counit is always a filtered isomorphism. Furthermore, the bijection (3.16) is in fact a filtered isomorphism.*

Proof. Let $f \in \mathsf{Hom}^{\mathsf{flt}}_{\Bbbk}(V, W[t])$ and consider its completion $\widehat{f} : \widehat{V} \to \widehat{W}$. This is the unique $\Bbbk$-linear morphism such that $p^W_k \circ \widehat{f} = \widetilde{f_k} \circ p^V_k$ for every $k \geq 0$, where $p^V_k : \widehat{V} \to V/F_kV$ are the structure morphisms of the projective limit as usual and $\widetilde{f_k} \colon V/F_kV \to W/F_kW$ are the maps induced by f on the quotients. Notice that the hypothesis on f implies that for every $k \geq 0$, $F_kV/F_{k+t}V \subseteq \ker\left(\widetilde{f_{k+t}}\right)$, because for every $v + F_{k+t}V \in F_kV/F_{k+t}V$ we have $\widetilde{f_{k+t}}\left(v + F_{k+t}V\right) = f(v) + F_{k+t}W = F_{k+t}W$. Consider an element $\widehat{v}_\infty \in F_n\widehat{V}$, so that $0 = p^V_n\left(\widehat{v}_\infty\right) = \pi^V_{n+t,n}\left(p^V_{n+t}\left(\widehat{v}_\infty\right)\right)$ and $p^V_{n+t}\left(\widehat{v}_\infty\right) \in F_nV/F_{n+t}V \subseteq \ker\left(\widetilde{f_{n+t}}\right)$. Thus, $p^W_{n+t}\left(\widehat{f}\left(\widehat{v}_\infty\right)\right) = \widetilde{f_{n+t}}\left(p^V_{n+t}\left(\widehat{v}_\infty\right)\right) = 0$ and hence $\widehat{f}\left(F_n\widehat{V}\right) \subseteq F_{n+t}\widehat{W}$. This implies that the assignments

$$\begin{tikzcd}
\mathsf{Hom}^{\mathsf{flt}}_{\Bbbk}\left(V, \mathcal{U}(W)\right) \arrow[r, shift left] & \mathsf{Hom}^{\mathsf{c}}_{\Bbbk}\left(\widehat{V}, W\right) \arrow[l, shift left] \\
f \arrow[r, mapsto] & \sigma_W \circ \widehat{f} \\
\mathcal{U}(g) \circ \gamma_V & g \arrow[l, mapsto]
\end{tikzcd}$$

are filtered. One may also check easily that they are inverses each other. $\square$

Before proceeding, let us discuss some consequences that can be drawn from Lemma 3.1.22. First of all, it is worthy to point out that the bijection in equation (3.16) encodes the universal property of the completion: every filtered morphism $g \colon V \to W$ from a filtered $\Bbbk$-module V to a complete $\Bbbk$-module W factors through the completion of V, that is, we have a commutative diagram of filtered morphisms

$$\begin{tikzcd}
V \arrow[rr, "g"] \arrow[dr, "\gamma_V"'] & & W \\
& \widehat{V} \arrow[ur, dashed, "\widehat{g}"'] &
\end{tikzcd} \tag{3.17}$$

Secondly, since we already noticed that $\mathfrak{M}^{\mathsf{c}}$ is a full subcategory of $\mathfrak{M}^{\mathsf{flt}}$ with inclusion $\mathcal{U}$, Lemma 3.1.22 is equivalent to claim that $\mathfrak{M}^{\mathsf{c}}$ is in fact a *reflective subcategory* (i.e., full subcategory whose inclusion functor admits a left adjoint) of $\mathfrak{M}^{\mathsf{flt}}$ with *reflection* (i.e., the left adjoint of the inclusion functor) $\widehat{(-)}$[(6)]. By the general properties of reflective subcategories, one may deduce that $\mathcal{U} : \mathfrak{M}^{\mathsf{c}} \to \mathfrak{M}^{\mathsf{flt}}$ detects colimits, whence the cocompleteness of $\mathfrak{M}^{\mathsf{c}}$ (see e.g. [Brx1, Proposition 3.5.4]). We collect these facts in the following lemma for future reference.

Lemma 3.1.23. *The category* $\mathfrak{M}^{\mathsf{c}}$ *of complete* $\Bbbk$*-modules is a reflective subcategory of* $\mathfrak{M}^{\mathsf{flt}}$*. Thus the functor* $\mathcal{U} : \mathfrak{M}^{\mathsf{c}} \to \mathfrak{M}^{\mathsf{flt}}$ *detects colimits and* $\mathfrak{M}^{\mathsf{c}}$ *is cocomplete.*

Notation 3.1.24. Observe that a sequence $\{x_n \mid n \in \mathbb{N}\}$ in V is Cauchy if and only if the sequence $\{\widehat{x_n} \mid n \in \mathbb{N}\}$ in $\widehat{V}$ is Cauchy. It turns out that $\widehat{x}_\infty = \lim\limits_{n \to \infty}\left(\widehat{x_n}\right)$ in $\widehat{V}$, where $\{x_n \mid n \in \mathbb{N}\}$ is the Cauchy sequence associated with $\widehat{x}_\infty$ as in Remark 3.1.15. Hence, by a slightly but consistent abuse

(6) This should not be surprising, one key example of reflection being the completion of metric spaces (see [AHS, Examples 4.17(C)]. Even if it is unessential, notice that $\mathfrak{M}^{\mathsf{c}}$ is also replete as a subcategory of $\mathfrak{M}^{\mathsf{flt}}$, that is, every object V in $\mathfrak{M}^{\mathsf{flt}}$ which is isomorphic to $\mathcal{U}(W)$ for some W in $\mathfrak{M}^{\mathsf{c}}$ already belongs to $\mathfrak{M}^{\mathsf{c}}$ (see [Brx1, Definition 3.5.2]).

of notation, we are going to write $\widehat{x}_\infty = \lim\limits_{n\to\infty}(x_n)$ whenever $\widehat{x}_\infty = (x_n + F_nV)_{n\geq 0}$. This proves to be very useful when one will have to compute, for example, $\widehat{f}(\widehat{x}_\infty)$ for a given $f : V \to W$. Indeed

$$\widehat{f}\left(\lim_{n\to\infty}(x_n)\right) = \widehat{f}(\widehat{x}_\infty) = (f(x_n) + F_nV)_{n\in\mathbb{N}} = \lim_{n\to\infty}(f(x_n)) =: \widehat{f(x)}_\infty. \tag{3.18}$$

Notice finally that for two given Cauchy sequences $\{x_n \mid n \in \mathbb{N}\}$ and $\{y_n \mid n \in \mathbb{N}\}$ in V, we have that $\lim\limits_{n\to\infty}(x_n) = \lim\limits_{n\to\infty}(y_n)$ in $\widehat{V}$ if and only if $\widehat{x}_\infty = \widehat{y}_\infty$, if and only if $x_n - y_n \in F_nV$ for all $n \in \mathbb{N}$.

We conclude this subsection by recalling the following fact.

Lemma 3.1.25. *For every complete $\Bbbk$-module V the assignment*

$$\mathsf{Hom}^{\mathsf{c}}_{\Bbbk}(V, -) : \mathfrak{M}^{\mathsf{c}} \to \mathfrak{M}^{\mathsf{c}} \tag{3.19}$$

gives a well-defined functor.

Proof. The proof of the fact that the functor is well-defined can be adapted from [LvO, Proposition 6.7], in light of the fact that $\mathsf{Hom}^{\mathsf{c}}_{\Bbbk}(V, W) = F_0(\mathsf{HOM}^{\mathsf{c}}_{\Bbbk}(V, W))$. The main idea is to show that for every $\widehat{f}_\infty = \left(f_k + F_k\left(\mathsf{Hom}^{\mathsf{c}}_{\Bbbk}(V, W)\right)\right)_{k\geq 0}$ in $\widehat{\mathsf{Hom}^{\mathsf{c}}_{\Bbbk}(V, W)}$, the morphism $f : V \to W$ defined by $f(v) = \lim\limits_{n\to\infty}(f_n(v))$ satisfies $\gamma_{\mathsf{Hom}^{\mathsf{c}}_{\Bbbk}(V,W)}(f) = \widehat{f}_\infty$. □

3.1.3 The complete tensor product of filtered bimodules

Henceforth, we will often omit to write explicitly the functor $\mathcal{U} : \mathfrak{M}^{\mathsf{c}} \to \mathfrak{M}^{\mathsf{flt}}$ in the computations to increase the readability, unless we need to stress the category in which we are working or the domain or codomain of particular morphisms.

Let V, W be complete $\Bbbk$-modules. In general, $V \otimes W$ is not a complete $\Bbbk$-module itself. However, we may consider the completion $\widehat{\mathcal{U}(V) \otimes \mathcal{U}(W)}$, which we will denote simply by $\widehat{V \otimes W}$.

Definition 3.1.26. Given V, W two complete $\Bbbk$-modules, we define their *complete tensor product* to be the completion $V \mathbin{\widehat{\otimes}} W := \widehat{V \otimes W}$.

This complete tensor product turns out to enjoy many of the nice properties that the canonical tensor product enjoys.

Theorem 3.1.27. *For every complete $\Bbbk$-modules U, V, W we have filtered isomorphisms*

$$\mathsf{Hom}^{\mathsf{c}}_{\Bbbk}(U, \mathsf{Hom}^{\mathsf{c}}_{\Bbbk}(V, W)) \cong \mathsf{Hom}^{\mathsf{c}}_{\Bbbk}\left(U \mathbin{\widehat{\otimes}} V, W\right) \cong \mathsf{Hom}^{\mathsf{c}}_{\Bbbk}(V, \mathsf{Hom}^{\mathsf{c}}_{\Bbbk}(U, W))$$

which are natural in U, W and V, W respectively. In particular, the functors $- \mathbin{\widehat{\otimes}} V$ and $V \mathbin{\widehat{\otimes}} -$ are left adjoints to the functor $\mathsf{Hom}^{\mathsf{c}}_{\Bbbk}(V, -) : \mathfrak{M}^{\mathsf{c}} \to \mathfrak{M}^{\mathsf{c}}$.

Proof. Recall that for every complete $\Bbbk$-modules V, W we have that $\mathsf{Hom}^{\mathsf{c}}_{\Bbbk}(V, W) = \mathsf{Hom}^{\mathsf{flt}}_{\Bbbk}(V, W)$ by definition. Up to the omission of the functor $\mathcal{U} : \mathfrak{M}^{\mathsf{c}} \to \mathfrak{M}^{\mathsf{flt}}$, from Proposition 3.1.7, Remark 3.1.8 and Lemma 3.1.22 we deduce that

$$\mathsf{Hom}^{\mathsf{c}}_{\Bbbk}(U, \mathsf{Hom}^{\mathsf{c}}_{\Bbbk}(V, W)) \cong \mathsf{Hom}^{\mathsf{flt}}_{\Bbbk}(U \otimes V, W) \cong \mathsf{Hom}^{\mathsf{c}}_{\Bbbk}\left(U \mathbin{\widehat{\otimes}} V, W\right).$$

The other isomorphism is proved in the same way. □

Let us consider the category $\mathfrak{M}^{\mathsf{c}}$ endowed with the tensor product $\widehat{\otimes}$ and the distinguished object $\Bbbk$. It turns out that the functor $\widehat{(-)} : \mathfrak{M}^{\mathsf{flt}} \to \mathfrak{M}^{\mathsf{c}}$ preserves the tensor products in the sense of the following proposition.

Proposition 3.1.28. *Assume that V and W are two filtered $\Bbbk$-modules. Then we have a filtered isomorphism of $\Bbbk$-modules $\widehat{V \otimes W} \cong \widehat{V} \mathbin{\widehat{\otimes}} \widehat{W}$, natural in both variables, explicitly given by*

$$\varphi_{V,W} : \widehat{V} \mathbin{\widehat{\otimes}} \widehat{W} \to \widehat{V \otimes W}, \quad \left(\lim_{n\to\infty} \left(\lim_{k\to\infty} (v_k^n) \otimes \lim_{h\to\infty} (w_h^n) \right) \mapsto \lim_{n\to\infty} (v_n^n \otimes w_n^n) \right), \tag{3.20}$$

$$\psi_{V,W} : \widehat{V \otimes W} \to \widehat{V} \mathbin{\widehat{\otimes}} \widehat{W}, \quad \left(\lim_{n\to\infty} (v_n \otimes w_n) \mapsto \lim_{n\to\infty} (\widehat{v_n} \otimes \widehat{w_n}) \right) \tag{3.21}$$

(summations over the tensor products are understood to increase readability).

Proof. The morphism $\psi_{V,W}$ is simply $\widehat{\gamma_V \otimes \gamma_W}$, whence it is filtered. On the other hand, consider the $\Bbbk$-linear filtered morphism

$$\iota_n : \frac{V}{F_nV} \otimes \frac{W}{F_nW} \to \frac{V \otimes W}{F_n(V \otimes W)}; \quad (v + F_nV) \otimes (w + F_nW) \mapsto v \otimes w + F_n(V \otimes W)$$

for every $n \geq 0$ and compose it with $p_n^V \otimes p_n^W$. These make of $\widehat{V} \otimes \widehat{W}$ a source for the projective system $\{(V \otimes W)/F_n(V \otimes W)\}_{n \in \mathbb{N}}$ in $\mathfrak{M}^{\mathsf{flt}}$ and hence there exists a unique filtered morphism $\phi_{V,W} : \widehat{V} \otimes \widehat{W} \to \widehat{V \otimes W}$ which, in light of (3.17), may be lifted uniquely to the morphism $\varphi_{V,W} : \widehat{V} \mathbin{\widehat{\otimes}} \widehat{W} \to \widehat{V \otimes W}$ of Equation (3.20). Obviously, $\varphi_{V,W} \circ \psi_{V,W} = \mathsf{Id}_{\widehat{V \otimes W}}$. The other way around, recall that $F_n\widehat{V} = \ker(p_n)$ and observe that $(v_n^n + F_kV)_{k\in\mathbb{N}} + F_n\widehat{V} = (v_k^n + F_kV)_{k\in\mathbb{N}} + F_n\widehat{V}$, because $p_n\left((v_n^n + F_kV)_{k\in\mathbb{N}}\right) = v_n^n + F_nV = p_n\left((v_k^n + F_kV)_{k\in\mathbb{N}}\right)$, whence

$$\begin{aligned} &(v_n^n + F_kV)_{k\in\mathbb{N}} \otimes (w_n^n + F_hW)_{h\in\mathbb{N}} - (v_k^n + F_kV)_{k\in\mathbb{N}} \otimes (w_h^n + F_hW)_{h\in\mathbb{N}} + F_n(\widehat{V} \otimes \widehat{W}) \\ &= \left[\begin{array}{l} \left((v_n^n + F_kV)_{k\in\mathbb{N}} - (v_k^n + F_kV)_{k\in\mathbb{N}}\right) \otimes (w_n^n + F_hW)_{h\in\mathbb{N}} + \\ + (v_k^n + F_kV)_{k\in\mathbb{N}} \otimes \left((w_n^n + F_hW)_{h\in\mathbb{N}} - (w_h^n + F_hW)_{h\in\mathbb{N}}\right) + F_n(\widehat{V} \otimes \widehat{W}) \end{array} \right] = F_n(\widehat{V} \otimes \widehat{W}) \end{aligned}$$

holds in $(\widehat{V} \otimes \widehat{W})/F_n(\widehat{V} \otimes \widehat{W})$ for every $n \geq 0$, so that $\psi_{V,W} \circ \varphi_{V,W} = \mathsf{Id}_{\widehat{V} \mathbin{\widehat{\otimes}} \widehat{W}}$ as well. □

In light of Lemma 3.1.23 and Proposition 3.1.28, we may apply the following technical result.

Proposition 3.1.29. *Let $(\mathcal{M}, \otimes, \mathbb{I}, \mathsf{\alpha}^{\mathcal{M}}, \mathsf{\lambda}^{\mathcal{M}}, \mathsf{\rho}^{\mathcal{M}})$ be a monoidal category and let $\mathcal{C}$ be a reflective subcategory of $\mathcal{M}$. There exists a monoidal structure $(\mathcal{C}, \boxtimes, \mathbb{J}, \mathsf{\alpha}^{\mathcal{C}}, \mathsf{\lambda}^{\mathcal{C}}, \mathsf{\rho}^{\mathcal{C}})$ on $\mathcal{C}$ such that the left adjoint $\mathcal{L}$ of the inclusion functor $\mathcal{R} : \mathcal{C} \to \mathcal{M}$ is a monoidal functor if and only if $\mathcal{L}(\eta_M \otimes \eta_N)$ is an isomorphism for all M, N in $\mathcal{M}$, where η is the unit of the adjunction $(\mathcal{L}, \mathcal{R}, \eta, \epsilon)$. Moreover, the monoidal structure above is unique up to an isomorphism of monoidal categories.*

Even if the conclusions of Proposition 3.1.29 may be expected and they are already (implicitly) contained in Day's Reflection Theorem (see [Da, §1] and the proof in [Da, §4]), we are going to take the proof out explicitly for the sake of completeness and for future reference.

Proof. Assume firstly that $\mathcal{L}(\eta_M \otimes \eta_N)$ is an isomorphism for all M, N in $\mathcal{M}$ and let us construct a monoidal structure on $\mathcal{C}$ in such a way that $\mathcal{L}$ becomes a monoidal functor. For all X, Y in $\mathcal{C}$ set

$$X \boxtimes Y := \mathcal{L}(\mathcal{R}(X) \otimes \mathcal{R}(Y)) \quad \text{and} \quad \mathbb{J} := \mathcal{L}(\mathbb{I}),$$

which are going to be the tensor product and the unit object of $\mathcal{C}$. Set also $\psi_0 = \mathsf{Id}_{\mathcal{L}(\mathbb{I})}$ and

$$\psi_{M,N} := \left(\mathcal{L}(M \otimes N) \xrightarrow{\mathcal{L}(\eta_M \otimes \eta_N)} \mathcal{L}(\mathcal{RL}(M) \otimes \mathcal{RL}(N)) = \mathcal{L}(M) \boxtimes \mathcal{L}(N) \right),$$

which are going to be the structure isomorphisms of $\mathcal{L}$. Since $\mathsf{\alpha}^{\mathcal{C}}$ has to be compatible with $\mathcal{L}$ and $\mathsf{\alpha}^{\mathcal{M}}$ it has to make the following diagram commute

$$\begin{tikzcd} \mathcal{L}((M \otimes N) \otimes P) \arrow[r, "\psi_{M\otimes N,P}"] \arrow[d, "\mathcal{L}(\alpha^{\mathcal{M}}_{M,N,P})"'] & \mathcal{L}(M \otimes N) \boxtimes \mathcal{L}(P) \arrow[r, "\psi_{M,N} \boxtimes \mathcal{L}(P)"] & (\mathcal{L}(M) \boxtimes \mathcal{L}(N)) \boxtimes \mathcal{L}(P) \arrow[d, dotted, "\alpha^{\mathcal{C}}_{\mathcal{L}(M),\mathcal{L}(N),\mathcal{L}(P)}"] \\ \mathcal{L}(M \otimes (N \otimes P)) \arrow[r, "\psi_{M,N\otimes P}"'] & \mathcal{L}(M) \boxtimes \mathcal{L}(N \otimes P) \arrow[r, "\mathcal{L}(M)\boxtimes\psi_{N,P}"'] & \mathcal{L}(M) \boxtimes (\mathcal{L}(N) \boxtimes \mathcal{L}(P)) \end{tikzcd} \tag{3.22}$$

and in this way we have defined $\alpha^{\mathcal{C}}$ on elements of the form $\mathcal{L}(M)$. To define it on every triple of objects X, Y and Z in $\mathcal{C}$ recall that $\mathcal{R}$ fully faithful implies that the counit ϵ is a natural isomorphism. Hence $\alpha^{\mathcal{C}}$ is uniquely determined by the commutativity of

$$
\begin{tikzcd}
(\mathcal{L}\mathcal{R}(X) \boxtimes \mathcal{L}\mathcal{R}(Y)) \boxtimes \mathcal{L}\mathcal{R}(Z) \arrow[r, "(\epsilon_X \boxtimes \epsilon_Y) \boxtimes \epsilon_Z"] \arrow[d, "\alpha^{\mathcal{C}}_{\mathcal{L}\mathcal{R}(X),\mathcal{L}\mathcal{R}(Y),\mathcal{L}\mathcal{R}(Z)}"'] & (X \boxtimes Y) \boxtimes Z \arrow[d, dotted, "\alpha^{\mathcal{C}}_{X,Y,Z}"] \\
\mathcal{L}\mathcal{R}(X) \boxtimes (\mathcal{L}\mathcal{R}(Y) \boxtimes \mathcal{L}\mathcal{R}(Z)) \arrow[r, "\epsilon_X \boxtimes (\epsilon_Y \boxtimes \epsilon_Z)"'] & X \boxtimes (Y \boxtimes Z)
\end{tikzcd}
\tag{3.23}
$$

Now we have to show that the $\alpha^{\mathcal{C}}$ we constructed is associative. We prove it firstly on quadruples of the form $\mathcal{L}(M)$, $\mathcal{L}(N)$, $\mathcal{L}(P)$, $\mathcal{L}(Q)$ and then extend it to all X, Y, Z, W objects in $\mathcal{C}$. Let us perform the following computation

$$
\begin{aligned}
&\left[\begin{array}{l}
\alpha^{\mathcal{C}}_{\mathcal{L}(M),\mathcal{L}(N),\mathcal{L}(P)\boxtimes\mathcal{L}(Q)} \circ \alpha^{\mathcal{C}}_{\mathcal{L}(M)\boxtimes\mathcal{L}(N),\mathcal{L}(P),\mathcal{L}(Q)} \circ \Big((\psi_{M,N} \boxtimes \mathcal{L}(P)) \boxtimes \mathcal{L}(Q) \Big) \circ \\
\circ \left(\psi_{M\otimes N,P} \boxtimes \mathcal{L}(Q)\right) \circ \psi_{(M\otimes N)\otimes P,Q}
\end{array}\right] \\
&= \left[\begin{array}{l}
\alpha^{\mathcal{C}}_{\mathcal{L}(M),\mathcal{L}(N),\mathcal{L}(P)\boxtimes\mathcal{L}(Q)} \circ \Big(\psi_{M,N} \boxtimes \left(\mathcal{L}(P) \boxtimes \mathcal{L}(Q)\right) \Big) \circ \alpha^{\mathcal{C}}_{\mathcal{L}(M\otimes N),\mathcal{L}(P),\mathcal{L}(Q)} \circ \\
\circ \left(\psi_{M\otimes N,P} \boxtimes \mathcal{L}(Q)\right) \circ \psi_{(M\otimes N)\otimes P,Q}
\end{array}\right] \\
&\overset{(3.22)}{=} \left[\begin{array}{l}
\alpha^{\mathcal{C}}_{\mathcal{L}(M),\mathcal{L}(N),\mathcal{L}(P)\boxtimes\mathcal{L}(Q)} \circ \Big(\psi_{M,N} \boxtimes \left(\mathcal{L}(P) \boxtimes \mathcal{L}(Q)\right) \Big) \circ \left(\mathcal{L}(M \otimes N) \boxtimes \psi_{P,Q}\right) \circ \\
\circ\, \psi_{M\otimes N,P\otimes Q} \circ \mathcal{L}\left(\alpha^{\mathcal{M}}_{M\otimes N,P,Q}\right)
\end{array}\right] \\
&= \left[\begin{array}{l}
\alpha^{\mathcal{C}}_{\mathcal{L}(M),\mathcal{L}(N),\mathcal{L}(P)\boxtimes\mathcal{L}(Q)} \circ \Big(\left(\mathcal{L}(M) \boxtimes \mathcal{L}(N)\right) \boxtimes \psi_{P,Q} \Big) \circ \left(\psi_{M,N} \boxtimes \mathcal{L}(P \otimes Q)\right) \circ \\
\circ\, \psi_{M\otimes N,P\otimes Q} \circ \mathcal{L}\left(\alpha^{\mathcal{M}}_{M\otimes N,P,Q}\right)
\end{array}\right] \\
&= \left[\begin{array}{l}
\Big(\mathcal{L}(M) \boxtimes \left(\mathcal{L}(N) \boxtimes \psi_{P,Q}\right) \Big) \circ \alpha^{\mathcal{C}}_{\mathcal{L}(M),\mathcal{L}(N),\mathcal{L}(P\otimes Q)} \circ \left(\psi_{M,N} \boxtimes \mathcal{L}(P \otimes Q)\right) \circ \\
\circ\, \psi_{M\otimes N,P\otimes Q} \circ \mathcal{L}\left(\alpha^{\mathcal{M}}_{M\otimes N,P,Q}\right)
\end{array}\right] \\
&\overset{(3.22)}{=} \left[\begin{array}{l}
\Big(\mathcal{L}(M) \boxtimes \left(\mathcal{L}(N) \boxtimes \psi_{P,Q}\right) \Big) \circ \left(\mathcal{L}(M) \boxtimes \psi_{N,P\otimes Q}\right) \circ \psi_{M,N\otimes(P\otimes Q)} \circ \\
\circ\, \mathcal{L}\left(\alpha^{\mathcal{M}}_{M,N,P\otimes Q}\right) \circ \mathcal{L}\left(\alpha^{\mathcal{M}}_{M\otimes N,P,Q}\right)
\end{array}\right] \\
&= \left[\begin{array}{l}
\Big(\mathcal{L}(M) \boxtimes \left(\mathcal{L}(N) \boxtimes \psi_{P,Q}\right) \Big) \circ \left(\mathcal{L}(M) \boxtimes \psi_{N,P\otimes Q}\right) \circ \psi_{M,N\otimes(P\otimes Q)} \circ \\
\circ\, \mathcal{L}\left(M \otimes \alpha^{\mathcal{M}}_{N,P,Q}\right) \circ \mathcal{L}\left(\alpha^{\mathcal{M}}_{M,N\otimes P,Q}\right) \circ \mathcal{L}\left(\alpha^{\mathcal{M}}_{M,N,P} \otimes Q\right)
\end{array}\right] \\
&= \left[\begin{array}{l}
\Big(\mathcal{L}(M) \boxtimes \left(\mathcal{L}(N) \boxtimes \psi_{P,Q}\right) \Big) \circ \left(\mathcal{L}(M) \boxtimes \psi_{N,P\otimes Q}\right) \circ \left(\mathcal{L}(M) \boxtimes \mathcal{L}\left(\alpha^{\mathcal{M}}_{N,P,Q}\right)\right) \circ \\
\circ\, \psi_{M,(N\otimes P)\otimes Q} \circ \mathcal{L}\left(\alpha^{\mathcal{M}}_{M,N\otimes P,Q}\right) \circ \mathcal{L}\left(\alpha^{\mathcal{M}}_{M,N,P} \otimes Q\right)
\end{array}\right] \\
&\overset{(3.22)}{=} \left[\begin{array}{l}
\Big(\mathcal{L}(M) \boxtimes \mathcal{L}\left(\alpha^{\mathcal{C}}_{\mathcal{L}(N),\mathcal{L}(P),\mathcal{L}(Q)}\right) \Big) \circ \Big(\mathcal{L}(M) \boxtimes \left(\psi_{N,P} \boxtimes \mathcal{L}(Q)\right) \Big) \circ \left(\mathcal{L}(M) \boxtimes \psi_{N\otimes P,Q}\right) \circ \\
\circ\, \psi_{M,(N\otimes P)\otimes Q} \circ \mathcal{L}\left(\alpha^{\mathcal{M}}_{M,N\otimes P,Q}\right) \circ \mathcal{L}\left(\alpha^{\mathcal{M}}_{M,N,P} \otimes Q\right)
\end{array}\right] \\
&\overset{(3.22)}{=} \left[\begin{array}{l}
\Big(\mathcal{L}(M) \boxtimes \mathcal{L}\left(\alpha^{\mathcal{C}}_{\mathcal{L}(N),\mathcal{L}(P),\mathcal{L}(Q)}\right) \Big) \circ \Big(\mathcal{L}(M) \boxtimes \left(\psi_{N,P} \boxtimes \mathcal{L}(Q)\right) \Big) \circ \alpha^{\mathcal{C}}_{\mathcal{L}(M),\mathcal{L}(N\otimes P),\mathcal{L}(Q)} \circ \\
\circ \left(\psi_{M,N\otimes P} \boxtimes \mathcal{L}(Q)\right) \circ \psi_{M\otimes(N\otimes P),Q} \circ \mathcal{L}\left(\alpha^{\mathcal{M}}_{M,N,P} \otimes Q\right)
\end{array}\right] \\
&= \left[\begin{array}{l}
\Big(\mathcal{L}(M) \boxtimes \mathcal{L}\left(\alpha^{\mathcal{C}}_{\mathcal{L}(N),\mathcal{L}(P),\mathcal{L}(Q)}\right) \Big) \circ \alpha^{\mathcal{C}}_{\mathcal{L}(M),\mathcal{L}(N)\boxtimes\mathcal{L}(P),\mathcal{L}(Q)} \circ \Big(\left(\mathcal{L}(M) \boxtimes \psi_{N,P}\right) \boxtimes \mathcal{L}(Q) \Big) \circ \\
\circ \left(\psi_{M,N\otimes P} \boxtimes \mathcal{L}(Q)\right) \circ \left(\mathcal{L}\left(\alpha^{\mathcal{M}}_{M,N,P}\right) \boxtimes \mathcal{L}(Q)\right) \circ \psi_{(M\otimes N)\otimes P,Q}
\end{array}\right] \\
&\overset{(3.22)}{=} \left[\begin{array}{l}
\Big(\mathcal{L}(M) \boxtimes \mathcal{L}\left(\alpha^{\mathcal{C}}_{\mathcal{L}(N),\mathcal{L}(P),\mathcal{L}(Q)}\right) \Big) \circ \alpha^{\mathcal{C}}_{\mathcal{L}(M),\mathcal{L}(N)\boxtimes\mathcal{L}(P),\mathcal{L}(Q)} \circ \left(\alpha^{\mathcal{C}}_{\mathcal{L}(M),\mathcal{L}(N),\mathcal{L}(P)} \boxtimes \mathcal{L}(Q)\right) \circ \\
\circ \Big(\left(\psi_{M,N} \boxtimes \mathcal{L}(P)\right) \boxtimes \mathcal{L}(Q) \Big) \circ \left(\psi_{M\otimes N,P} \boxtimes \mathcal{L}(Q)\right) \circ \psi_{(M\otimes N)\otimes P,Q}
\end{array}\right]
\end{aligned}
$$

from which we deduce

$$\begin{aligned}\alpha^{\mathcal{C}}_{\mathcal{L}(M),\mathcal{L}(N),\mathcal{L}(P)\boxtimes\mathcal{L}(Q)}\circ\alpha^{\mathcal{C}}_{\mathcal{L}(M)\boxtimes\mathcal{L}(N),\mathcal{L}(P),\mathcal{L}(Q)} &= \\ &= \Big(\mathcal{L}(M)\boxtimes\mathcal{L}\big(\alpha^{\mathcal{C}}_{\mathcal{L}(N),\mathcal{L}(P),\mathcal{L}(Q)}\big)\Big)\circ\alpha^{\mathcal{C}}_{\mathcal{L}(M),\mathcal{L}(N)\boxtimes\mathcal{L}(P),\mathcal{L}(Q)}\circ\Big(\alpha^{\mathcal{C}}_{\mathcal{L}(M),\mathcal{L}(N),\mathcal{L}(P)}\boxtimes\mathcal{L}(Q)\Big).\end{aligned}$$

By applying repeatedly (3.23) we conclude that $\alpha^{\mathcal{C}}$ satisfies the Pentagon Axiom and hence it is associative. Furthermore, we can define the natural transformations $\lambda^{\mathcal{C}}$ and $\rho^{\mathcal{C}}$ by requiring that they make the following diagrams commute

$$\begin{tikzcd} \mathbb{J}\boxtimes\mathcal{L}(M) \arrow[r, dotted, "\lambda^{\mathcal{C}}_{\mathcal{L}(M)}"] & \mathcal{L}(M) \\ \mathcal{L}(\mathbb{I})\boxtimes\mathcal{L}(M) \arrow[u, "\psi_0\boxtimes\mathcal{L}(M)"] & \mathcal{L}(\mathbb{I}\otimes M) \arrow[l, "\psi_{\mathbb{I},M}"] \arrow[u, "\mathcal{L}(\lambda^{\mathcal{M}}_M)"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{L}(M)\boxtimes\mathbb{J} \arrow[r, dotted, "\rho^{\mathcal{C}}_{\mathcal{L}(M)}"] & \mathcal{L}(M) \\ \mathcal{L}(M)\boxtimes\mathcal{L}(\mathbb{I}) \arrow[u, "\mathcal{L}(M)\boxtimes\psi_0"] & \mathcal{L}(M\otimes\mathbb{I}) \arrow[l, "\psi_{M,\mathbb{I}}"] \arrow[u, "\mathcal{L}(\rho^{\mathcal{M}}_M)"'] \end{tikzcd} \tag{3.24}$$

and extending them to all objects X in $\mathcal{C}$ via naturality as above, by asking the commutativity of

$$\begin{tikzcd} \mathbb{J}\boxtimes\mathcal{L}\mathcal{R}(X) \arrow[r, "\lambda^{\mathcal{C}}_{\mathcal{L}\mathcal{R}(X)}"] \arrow[d, "\mathbb{J}\boxtimes\epsilon_X"'] & \mathcal{L}\mathcal{R}(X) \arrow[d, "\epsilon_X"] \\ \mathbb{J}\boxtimes X \arrow[r, dotted, "\lambda^{\mathcal{C}}_X"'] & X \end{tikzcd} \qquad \begin{tikzcd} \mathcal{L}\mathcal{R}(X)\boxtimes\mathbb{J} \arrow[r, "\rho^{\mathcal{C}}_{\mathcal{L}\mathcal{R}(X)}"] \arrow[d, "\epsilon_X\boxtimes\mathbb{J}"'] & \mathcal{L}\mathcal{R}(X) \arrow[d, "\epsilon_X"] \\ X\boxtimes\mathbb{J} \arrow[r, dotted, "\rho^{\mathcal{C}}_X"'] & X \end{tikzcd} \tag{3.25}$$

Let us show that these satisfies the Triangle Axiom. By the following computation

$$\begin{aligned} &\big(\mathcal{L}(M)\boxtimes\lambda^{\mathcal{C}}_{\mathcal{L}(N)}\big)\circ\alpha^{\mathcal{C}}_{\mathcal{L}(M),\mathbb{J},\mathcal{L}(N)}\circ\Big(\big(\mathcal{L}(M)\boxtimes\psi_0\big)\boxtimes\mathcal{L}(N)\Big)\circ\big(\psi_{M,\mathbb{I}}\boxtimes\mathcal{L}(N)\big)\circ\psi_{M\otimes\mathbb{I},N} \\ &= \ \big(\mathcal{L}(M)\boxtimes\lambda^{\mathcal{C}}_{\mathcal{L}(N)}\big)\circ\Big(\mathcal{L}(M)\boxtimes\big(\psi_0\boxtimes\mathcal{L}(N)\big)\Big)\circ\alpha^{\mathcal{C}}_{\mathcal{L}(M),\mathcal{L}(\mathbb{I}),\mathcal{L}(N)}\circ\big(\psi_{M,\mathbb{I}}\boxtimes\mathcal{L}(N)\big)\circ\psi_{M\otimes\mathbb{I},N} \\ &\overset{(3.22)}{=} \big(\mathcal{L}(M)\boxtimes\lambda^{\mathcal{C}}_{\mathcal{L}(N)}\big)\circ\Big(\mathcal{L}(M)\boxtimes\big(\psi_0\boxtimes\mathcal{L}(N)\big)\Big)\circ\big(\mathcal{L}(M)\boxtimes\psi_{\mathbb{I},N}\big)\circ\psi_{M,\mathbb{I}\otimes N}\circ\mathcal{L}\big(\alpha^{\mathcal{M}}_{M,\mathbb{I},N}\big) \\ &\overset{(3.24)}{=} \big(\mathcal{L}(M)\boxtimes\mathcal{L}(\lambda^{\mathcal{M}}_N)\big)\circ\psi_{M,\mathbb{I}\otimes N}\circ\mathcal{L}\big(\alpha^{\mathcal{M}}_{M,\mathbb{I},N}\big) = \psi_{M,N}\circ\mathcal{L}\big(M\otimes\lambda^{\mathcal{M}}_N\big)\circ\mathcal{L}\big(\alpha^{\mathcal{M}}_{M,\mathbb{I},N}\big) \\ &= \ \psi_{M,N}\circ\mathcal{L}\big(\rho^{\mathcal{M}}_M\otimes N\big) = \big(\mathcal{L}(r_M)\boxtimes\mathcal{L}(N)\big)\circ\psi_{M\otimes\mathbb{I},N} \\ &\overset{(3.24)}{=} \big(\rho^{\mathcal{C}}_{\mathcal{L}(M)}\boxtimes\mathcal{L}(N)\big)\circ\Big(\big(\mathcal{L}(M)\boxtimes\psi_0\big)\boxtimes\mathcal{L}(N)\Big)\circ\big(\psi_{M,\mathbb{I}}\boxtimes\mathcal{L}(N)\big)\circ\psi_{M\otimes\mathbb{I},N} \end{aligned}$$

we deduce that

$$\big(\mathcal{L}(M)\boxtimes\lambda^{\mathcal{C}}_{\mathcal{L}(N)}\big)\circ\alpha^{\mathcal{C}}_{\mathcal{L}(M),\mathbb{J},\mathcal{L}(N)} = \rho^{\mathcal{C}}_{\mathcal{L}(M)}\boxtimes\mathcal{L}(N),$$

whence, by applying repeatedly (3.25), that the Triangle Axiom is satisfied for all X in $\mathcal{C}$. Summing up, $(\mathcal{C},\boxtimes,\mathbb{J},\alpha^{\mathcal{C}},\lambda^{\mathcal{C}},\rho^{\mathcal{C}})$ is now a monoidal category and $\mathcal{L}:\mathcal{M}\to\mathcal{C}$ is a monoidal functor with structure isomorphisms ψ_0 and $\psi_{M,N}$, for M,N in $\mathcal{M}$.

Conversely, if $(\mathcal{C},\boxtimes,\mathbb{J},\alpha^{\mathcal{C}},\lambda^{\mathcal{C}},\rho^{\mathcal{C}})$ is a monoidal category and $\mathcal{L}:\mathcal{M}\to\mathcal{C}$ is a monoidal functor with structure isomorphisms $\psi_0:\mathcal{L}(\mathbb{I})\to\mathbb{J}$ and $\psi_{M,N}:\mathcal{L}(M\otimes N)\to\mathcal{L}(M)\boxtimes\mathcal{L}(N)$ for M,N in $\mathcal{M}$, then the commutativity (given by the naturality of ψ) of the diagram

$$\begin{tikzcd} \mathcal{L}(M\otimes N) \arrow[r, "\psi_{M,N}"] \arrow[d, "\mathcal{L}(\eta_M\otimes\eta_N)"'] & \mathcal{L}(M)\boxtimes\mathcal{L}(N) \arrow[d, "\mathcal{L}(\eta_M)\boxtimes\mathcal{L}(\eta_N)"] \\ \mathcal{L}(\mathcal{R}\mathcal{L}(M)\otimes\mathcal{R}\mathcal{L}(N)) \arrow[r, "\psi_{\mathcal{R}\mathcal{L}(M),\mathcal{R}\mathcal{L}(N)}"'] & \mathcal{L}\mathcal{R}\mathcal{L}(M)\boxtimes\mathcal{L}\mathcal{R}\mathcal{L}(N) \end{tikzcd}$$

implies that $\mathcal{L}(\eta_M\otimes\eta_N)$ is an isomorphism for all M and N in $\mathcal{M}$.

Finally, assume that $(\mathcal{C},\boxtimes',\mathbb{J}',\alpha',\lambda',\rho')$ is another monoidal structure on $\mathcal{C}$ such that $(\mathcal{L},\varphi_0,\varphi)$ is a monoidal functor. The assignments

$$\gamma_{X,Y} := \left(X\boxtimes Y = \mathcal{L}(\mathcal{R}(X)\otimes\mathcal{R}(Y)) \xrightarrow{\varphi_{\mathcal{R}(X),\mathcal{R}(Y)}} \mathcal{L}\mathcal{R}(X)\boxtimes'\mathcal{L}\mathcal{R}(Y) \xrightarrow{\epsilon_X\boxtimes'\epsilon_Y} X\boxtimes' Y \right)$$

and $\gamma_0 := \left(\mathbb{J} = \mathcal{L}(\mathbb{I}) \xrightarrow{\varphi_0} \mathbb{J}' \right)$ convert $(\mathsf{Id}_{\mathcal{C}}, \gamma_0, \gamma)$ into an isomorphism of monoidal categories. $\square$

Therefore, we may conclude that the category of complete $\Bbbk$-modules $\mathfrak{M}^{\mathsf{c}}$ is monoidal with tensor product $\widehat{\otimes}$ and unit object $\Bbbk$ and moreover that the completion functor $\widehat{(-)} \colon \mathfrak{M}^{\mathsf{flt}} \to \mathfrak{M}^{\mathsf{c}}$ is a monoidal functor. By adapting the constructions of Proposition 3.1.29 to this framework (and by omitting the functor $\mathcal{U} : \mathfrak{M}^{\mathsf{c}} \to \mathfrak{M}^{\mathsf{flt}}$) we find out that the constraints are given as follows. For every complete $\Bbbk$-module V, the left unit constraint is given by $\lambda_V := \sigma_V \circ \widehat{l_V} \colon \Bbbk \,\widehat{\otimes}\, V \to V$ and the right unit constraint $\rho_V \colon V \,\widehat{\otimes}\, \Bbbk \to V$ is given by $\rho_V := \sigma_V \circ \widehat{r_V}$. The associativity constraint $\alpha_{U,V,W} \colon \left(U \,\widehat{\otimes}\, V\right) \widehat{\otimes}\, W \to U \,\widehat{\otimes}\, \left(V \,\widehat{\otimes}\, W\right)$ is the unique $\Bbbk$-linear map that makes all the following diagrams commute for every U, V and W filtered $\Bbbk$-modules

$$\begin{tikzcd}
\left(\widehat{U} \,\widehat{\otimes}\, \widehat{V}\right) \widehat{\otimes}\, \widehat{W} \arrow[d, dotted, "\alpha_{\widehat{U},\widehat{V},\widehat{W}}"'] & \widehat{U \otimes V} \,\widehat{\otimes}\, \widehat{W} \arrow[l, "\psi_{U,V} \widehat{\otimes} \widehat{W}"'] & \widehat{(U \otimes V) \otimes W} \arrow[l, "\psi_{U \otimes V, W}"'] \arrow[d, "\widehat{a_{U,V,W}}"] \\
\widehat{U} \,\widehat{\otimes}\, \left(\widehat{V} \,\widehat{\otimes}\, \widehat{W}\right) & \widehat{U} \,\widehat{\otimes}\, \widehat{V \otimes W} \arrow[l, "\widehat{U} \widehat{\otimes} \psi_{V,W}"] & \widehat{U \otimes (V \otimes W)} \arrow[l, "\psi_{U, V \otimes W}"]
\end{tikzcd}$$

or, more explicitly, for U, V, W complete $\Bbbk$-modules

$$\alpha_{U,V,W} = \left(\sigma_U \,\widehat{\otimes}\, \left(V \,\widehat{\otimes}\, W\right)\right) \circ \psi_{U,V\otimes W} \circ \widehat{a_{U,V,W}} \circ \varphi_{U \otimes V, W} \circ \left(\left(U \,\widehat{\otimes}\, V\right) \widehat{\otimes}\, \gamma_W\right). \tag{3.26}$$

Remark 3.1.30. For the sake of clearness, let us make explicit an obvious fact that has been used and will be used more or less implicitly in what follows. Let V, W be complete $\Bbbk$-modules. The morphism $\gamma_V : V \to \widehat{V}$ of (3.11) is at the same time the component $\gamma_{\mathcal{U}(V)} : \mathcal{U}(V) \to \mathcal{U}\left(\widehat{\mathcal{U}(V)}\right)$ in $\mathfrak{M}^{\mathsf{flt}}$ of the unit γ of the adjunction (3.16) and the inverse of the counit $\sigma_V : \widehat{\mathcal{U}(V)} \to V$ in $\mathfrak{M}^{\mathsf{c}}$. By definition, $V \,\widehat{\otimes}\, W = \widehat{\mathcal{U}(V) \otimes \mathcal{U}}(W)$. Hence

$$\varphi_{\mathcal{U}(V),\mathcal{U}(W)} \circ \left(\gamma_V \,\widehat{\otimes}\, \gamma_W\right) = \varphi_{\mathcal{U}(V),\mathcal{U}(W)} \circ (\widehat{\mathcal{U}(\gamma_V) \otimes \mathcal{U}}(\gamma_W)) = \varphi_{\mathcal{U}(V),\mathcal{U}(W)} \circ \psi_{\mathcal{U}(V),\mathcal{U}(W)} = \mathsf{Id}_{V \widehat{\otimes} W},$$
$$\left(\sigma_V \,\widehat{\otimes}\, \sigma_W\right) \circ \psi_{\mathcal{U}(V),\mathcal{U}(W)} = (\widehat{\mathcal{U}(\sigma_V) \otimes \mathcal{U}}(\sigma_W)) \circ (\widehat{\gamma_{\mathcal{U}(V)} \otimes \gamma_{\mathcal{U}(W)}}) = \mathsf{Id}_{V \widehat{\otimes} W},$$

so that for V, W complete $\Bbbk$-modules

$$\psi_{\mathcal{U}(V),\mathcal{U}(W)} = \gamma_V \,\widehat{\otimes}\, \gamma_W \quad \text{and} \quad \varphi_{\mathcal{U}(V),\mathcal{U}(W)} = \sigma_V \,\widehat{\otimes}\, \sigma_W. \tag{3.27}$$

Recall from Lemma 3.1.22 that the functor $\mathcal{U} : \mathfrak{M}^{\mathsf{c}} \to \mathfrak{M}^{\mathsf{flt}}$ is right adjoint to the completion functor. Thus, in light of [AMa, Proposition 3.84], $\mathcal{U}$ inherits a lax monoidal structure

$$\gamma_{\mathcal{U}(V) \otimes \mathcal{U}(W)} \colon \mathcal{U}(V) \otimes \mathcal{U}(W) \to \mathcal{U}\left(\widehat{\mathcal{U}(V) \otimes \mathcal{U}}(W)\right) = \mathcal{U}(V \,\widehat{\otimes}\, W), \qquad \gamma_{\Bbbk} \colon \Bbbk \to \mathcal{U}\left(\widehat{\Bbbk}\right) = \Bbbk. \tag{3.28}$$

Summing up, we have again a monoidal category $\left(\mathfrak{M}^{\mathsf{c}}, \widehat{\otimes}, \Bbbk, \alpha, \lambda, \rho\right)$ such that $\mathfrak{M}^{\mathsf{c}}$ is cocomplete and the functors $V \,\widehat{\otimes} -$ and $- \,\widehat{\otimes}\, V$ are cocontinuous. In light of Theorem 1.6.5, we have a bicategory $\mathscr{B}im^{\mathsf{c}}_{\Bbbk}$ whose 0-cells are algebras in $\mathfrak{M}^{\mathsf{c}}$ and whose categories of $\{1, 2\}$-cells are the categories of bimodules over these algebras.

Remark 3.1.31. Explicitly, an algebra R in $\mathfrak{M}^{\mathsf{c}}$ is a complete $\Bbbk$-module endowed with two filtered morphisms $\boldsymbol{m} : R \,\widehat{\otimes}\, R \to R$ and $u : \Bbbk \to R$ such that the following diagrams commute

$$\begin{tikzcd}
(R \,\widehat{\otimes}\, R) \,\widehat{\otimes}\, R \arrow[rr, "\alpha_{R,R,R}"] \arrow[d, "\boldsymbol{m} \widehat{\otimes} R"'] & & R \,\widehat{\otimes}\, (R \,\widehat{\otimes}\, R) \arrow[d, "R \widehat{\otimes} \boldsymbol{m}"] \\
R \,\widehat{\otimes}\, R \arrow[r, "\boldsymbol{m}"'] & R & R \,\widehat{\otimes}\, R \arrow[l, "\boldsymbol{m}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk \,\widehat{\otimes}\, R \arrow[r, "u \widehat{\otimes} R"] \arrow[dr, "\lambda_R"'] & R \,\widehat{\otimes}\, R \arrow[d, "\boldsymbol{m}"] & R \,\widehat{\otimes}\, \Bbbk \arrow[l, "R \widehat{\otimes} u"'] \arrow[dl, "\rho_R"] \\
& R &
\end{tikzcd}$$

This is equivalent to claim that R is a complete $\Bbbk$-module endowed with two filtered morphisms $m : R \otimes R \to R$ and $u : \Bbbk \to R$ such that (R, m, u) is an algebra in $\mathfrak{M}^{\mathsf{flt}}$. Namely given $\boldsymbol{m}$, the map m is defined as the unique filtered morphism such that

$$\begin{tikzcd}
\widehat{\mathcal{U}(R)} \widehat{\otimes} \widehat{\mathcal{U}(R)} \arrow[rr, "\sigma_R \widehat{\otimes} \sigma_R"] & & R \widehat{\otimes} R \arrow[d, "\boldsymbol{m}"] \\
\widehat{\mathcal{U}(R) \otimes \mathcal{U}(R)} \arrow[u, "\psi_{\mathcal{U}(R),\mathcal{U}(R)}"] \arrow[r, dotted, "\widehat{m}"'] & \widehat{\mathcal{U}(R)} \arrow[r, "\sigma_R"'] & R
\end{tikzcd} \tag{3.29}$$

commutes. Conversely, $\boldsymbol{m}$ is defined as the composition of the other four sides in diagram (3.29) above. That is, R is a *complete* $\Bbbk$*-algebra* (compare for example with [Dv, §A.1]).

If R, S are complete $\Bbbk$-algebras, then an (S, R)-bimodule is a complete $\Bbbk$-module $(M, F_n M)$ endowed with two filtered $\Bbbk$-linear morphisms

$$\mu_{S,M} : S \widehat{\otimes} M \to M \qquad \text{and} \qquad \mu_{M,R} : M \widehat{\otimes} R \to M$$

that are actions of S and R over M from the left and the right respectively and that are compatible in a suitable way as expressed in (1.10).

Remark 3.1.32. As before, this is equivalent to request that $((M, F_n M), \mu_{S,M}, \mu_{M,R})$ is a filtered (S, R)-bimodule with the additional property that the canonical (S, R)-bilinear filtered morphism $\gamma_M : M \to \varprojlim (M/F_n M)$ is an isomorphism. That is, $(M, F_n M)$ is a *complete* (S, R)*-bimodule* (compare for example with [NvO2, Definition D.II.1]).

A morphism of complete (S, R)-bimodules is simply an (S, R)-bilinear filtered morphism between the complete bimodules and the vertical composition is the usual composition. The category of complete (S, R)-bimodules will be denoted by ${}_S\mathfrak{M}^{\mathsf{c}}_T$. Given $({}_S M_R, F_n M)$ and $({}_R N_T, F_n N)$ two complete bimodules as denoted, their horizontal composition is given by the completion $\widehat{M \otimes_R N}$ of the tensor product over R of the two underlying filtered bimodules. Indeed, by recalling that $\widehat{(-)}$ preserves colimits, we have that the first row of the following commutative diagram is a coequalizer in $\mathfrak{M}^{\mathsf{c}}$, while the vertical arrows are isomorphisms

$$\begin{tikzcd}
(M \widehat{\otimes R}) \otimes N \arrow[rr, "\widehat{\mu_{M,R} \otimes N}"] \arrow[dr, "\widehat{a_{M,R,N}}"] \arrow[dd, "\psi_{M \otimes R, N}"] & & \widehat{M \otimes N} \arrow[r, "\widehat{\omega_{M,N}}"] \arrow[dd, "\psi_{M,N}"] & \widehat{M \otimes_R N} \arrow[r] & 0 \\
& M \otimes \widehat{(R \otimes N)} \arrow[ur, "\widehat{M \otimes \mu_{R,N}}"'] \arrow[dd, dotted, "\psi_{M, R \otimes N}"] & & & \\
\widehat{M \otimes R} \widehat{\otimes} \widehat{N} \arrow[rr, dotted, "\widehat{\mu_{M,R}} \widehat{\otimes} \widehat{N}"] \arrow[d, "\psi_{M,R} \widehat{\otimes} \widehat{N}"] & & \widehat{M} \widehat{\otimes} \widehat{N} \arrow[ur, dashed, "\omega_{\widehat{M},\widehat{N}}"] \arrow[ddd, "\sigma_M \widehat{\otimes} \sigma_N"] & & \\
(\widehat{M} \widehat{\otimes} \widehat{R}) \widehat{\otimes} \widehat{N} \arrow[urr, dashed, "\mu_{\widehat{M},\widehat{R}} \widehat{\otimes} \widehat{N}"] \arrow[dr, dashed, "\alpha_{\widehat{M},\widehat{R},\widehat{N}}"] \arrow[dd, "\sigma_M \widehat{\otimes} \sigma_R \widehat{\otimes} \sigma_N"] & \widehat{M} \widehat{\otimes} \widehat{R \otimes N} \arrow[ur, dotted, "\widehat{M} \widehat{\otimes} \widehat{\mu_{R,N}}"] \arrow[d, dotted, "\widehat{M} \widehat{\otimes} \psi_{R,N}"] & & & \\
& \widehat{M} \widehat{\otimes} (\widehat{R} \widehat{\otimes} \widehat{N}) \arrow[uur, dashed, "\widehat{M} \widehat{\otimes} \mu_{\widehat{R},\widehat{N}}"'] \arrow[dd, dotted] & & & \\
(M \widehat{\otimes} R) \widehat{\otimes} N \arrow[rr, "\mu_{M,R} \widehat{\otimes} N"] \arrow[dr, "\alpha_{M,R,N}"'] & & M \widehat{\otimes} N \arrow[uuuuur, squiggly, "\omega_{M,N}"'] & & \\
& M \widehat{\otimes} (R \widehat{\otimes} N) \arrow[ur, "M \widehat{\otimes} \mu_{R,N}"'] & & &
\end{tikzcd} \tag{3.30}$$

The dotted lines have been added to clarify why the diagram commutes, while the role played by the dashed ones will be clarified soon. Notice also that the vertical composition are in fact the identity maps, in light of Remark 3.1.30. Thus, $\widehat{M \otimes_R N}$ with $\omega_{M,N} := \widehat{\omega_{M,N}} \circ \varphi_{M,N} \circ \left(\sigma_M^{-1} \mathbin{\widehat{\otimes}} \sigma_N^{-1}\right)$ is also the coequalizer of the two bottom parallel arrows. As a matter of notation, we are going to set $M \mathbin{\widehat{\otimes}}_R N := \widehat{M \otimes_R N}$ and we will refer to it as the *complete tensor product* of the complete bimodules M and N over the complete $\Bbbk$-algebra R.

Remark 3.1.33. Diagram (3.30) encodes another important property. Assume that neither M and N nor R are complete. Nevertheless, since $\widehat{(-)}$ is monoidal, both $\widehat{M}$ and $\widehat{N}$ are complete $\widehat{R}$-modules with actions

$$\mu_{\widehat{M},\widehat{R}} = \widehat{\mu_{M,R}} \circ \varphi_{M,R} \qquad \text{and} \qquad \mu_{\widehat{R},\widehat{N}} = \widehat{\mu_{R,N}} \circ \varphi_{R,N}.$$

The dashed path express the fact that $\left(\widehat{M \otimes_R N}, \omega_{\widehat{M},\widehat{N}}\right)$ is still the coequalizer of these two actions. Thus $\widehat{M \otimes_R N} \cong \widehat{M} \mathbin{\widehat{\otimes}}_{\widehat{R}} \widehat{N}$ via the unique map $\varphi_{M,N} : \widehat{M} \mathbin{\widehat{\otimes}}_{\widehat{R}} \widehat{N} \to \widehat{M \otimes_R N}$ of equation (1.14). We will come back on these facts in Lemma 3.1.38.

Summing up, we have the following analogue of Proposition 3.1.9.

Proposition 3.1.34. *There is a bicategory $\mathscr{B}im_{\Bbbk}^{\mathsf{c}}$ which has complete $\Bbbk$-algebras as 0-cells and whose categories of $\{1,2\}$-cells are the categories of complete bimodules over these complete algebras. The vertical compositions are given by the ordinary compositions of morphisms. The horizontal compositions are given by the composition functors $- \mathbin{\widehat{\otimes}}_R - := \widehat{(-)} \circ (- \otimes_R -)$*

$$- \mathbin{\widehat{\otimes}}_R - : {}_S\mathfrak{M}_R^{\mathsf{c}} \times {}_R\mathfrak{M}_T^{\mathsf{c}} \to {}_S\mathfrak{M}_T^{\mathsf{c}}$$

for all complete algebras R, S, T.

Since the notations and conventions we introduced in Notation 3.1.20 and 3.1.24 can be easily adapted to filtered and complete bimodules, we will use them freely from now on.

Recall that we already know from the application of Proposition 3.1.29 that the completion functor $\widehat{(-)} : \mathfrak{M}^{\mathsf{flt}} \to \mathfrak{M}^{\mathsf{c}}$ is a monoidal functor and consequently that $\mathcal{U} : \mathfrak{M}^{\mathsf{c}} \to \mathfrak{M}^{\mathsf{flt}}$ is a lax monoidal functor, hence we may apply Theorem 1.6.6 to conclude that they can be lifted to two bifunctor between the bicategories of filtered and complete bimodules.

Theorem 3.1.35. *Let $\Bbbk$ be a commutative ring which we consider discretely filtered. Then the completion construction developed in this section induces a bifunctor*

$$\begin{array}{lccc}
 & \widehat{(-)} : \mathscr{B}im_{\Bbbk}^{\mathsf{flt}} & \longrightarrow & \mathscr{B}im_{\Bbbk}^{\mathsf{c}} \\
\text{0-cells} & R & \longmapsto & \widehat{R} \\
\text{1-cells} & {}_SM_R & \longmapsto & {}_{\widehat{S}}\widehat{M}_{\widehat{R}} \\
\text{2-cells} & \left[f : M \to N\right] & \longmapsto & \left[\widehat{f} : \widehat{M} \to \widehat{N}\right]
\end{array}$$

from the bicategory $\mathscr{B}im_{\Bbbk}^{\mathsf{flt}}$ of filtered $\Bbbk$-algebras and filtered bimodules to the bicategory $\mathscr{B}im_{\Bbbk}^{\mathsf{c}}$ of complete $\Bbbk$-algebras and complete bimodules. The other way around, $\mathcal{U} : \mathfrak{M}^{\mathsf{c}} \to \mathfrak{M}^{\mathsf{flt}}$ as well can be lifted to a bifunctor

$$\mathscr{U} : \mathscr{B}im_{\Bbbk}^{\mathsf{c}} \longrightarrow \mathscr{B}im_{\Bbbk}^{\mathsf{flt}}.$$

In particular, the completion $\widehat{R}$ of a filtered $\Bbbk$-algebra R is a complete $\Bbbk$-algebra and given a filtered (S,R)-bimodule M, its completion $\widehat{M}$ as a $\Bbbk$-module is a complete $\left(\widehat{S}, \widehat{R}\right)$-bimodule.

Example 3.1.36. Let $\Bbbk[X]$ be the $\Bbbk$-algebra of polynomials in the indeterminate X and consider it endowed with the adic filtration induced by $\langle X \rangle$, that is $F_n\Bbbk[X] = \langle X^n \rangle$ for all $n \geq 0$. The family of $\Bbbk$-algebra morphisms

$$\phi_n : \Bbbk[[X]] \to \frac{\Bbbk[X]}{\langle X^n \rangle}, \qquad \left(\sum_{k \geq 0} a_k X^k \mapsto \sum_{k=0}^{n-1} a_k X^k + \langle X^n \rangle\right)$$

induces a unique $\Bbbk$-linear map $\phi\colon \Bbbk[[X]] \to \widehat{\Bbbk[X]}$ acting as $\sum_{k\geq 0} a_k X^k \mapsto \lim_{n\to\infty}\left(\sum_{k=0}^{n-1} a_k X^k\right)$ which is easily seen to be an injective algebra map. It is also surjective, since every element $\lim_{n\to\infty}\left(\sum_{k=0}^{n-1} a_{n,k} X^k\right) \in \widehat{\Bbbk[X]}$ satisfies

$$a_{n+1,n+1}X^{n+1} + \sum_{k=0}^{n}\left(a_{n+1,k} - a_{n,k}\right)X^k = \sum_{k=0}^{n+1} a_{n+1,k}X^k - \sum_{k=0}^{n} a_{n,k}X^k \in \langle X^{n+1}\rangle,$$

so that $a_{n+1,k} = a_{n,k}$ for all $0 \leq k \leq n$ and hence $\lim_{n\to\infty}\left(\sum_{k=0}^{n-1} a_{n,k} X^k\right)$ coincides with the image of $\sum_{k\geq 0} a_{k,k}X^k$ via ϕ. Thus $\Bbbk[[X]] \cong \widehat{\Bbbk[X]}$ is a complete $\Bbbk$-algebra.

Remark 3.1.37. Let R be a filtered $\Bbbk$-algebra. Its completion $\widehat{R}$ is a complete $\Bbbk$-algebra with multiplication given by

$$\widehat{R}\,\widehat{\otimes}\,\widehat{R} \xrightarrow{\varphi_{R,R}} \widehat{R\otimes R} \xrightarrow{\widehat{m}} \widehat{R}.$$

Since $\mathcal{U}$ is lax monoidal, with structure morphism $\vartheta_{V,W} : \mathcal{U}(V)\otimes\mathcal{U}(W) \to \mathcal{U}\left(V\,\widehat{\otimes}\,W\right)$ given in (3.28), $\mathcal{U}(\widehat{R})$ is again a filtered $\Bbbk$-algebra with multiplication

$$\mathcal{U}(\widehat{R})\otimes\mathcal{U}(\widehat{R}) \xrightarrow{\vartheta_{\widehat{R},\widehat{R}}} \mathcal{U}\left(\widehat{R}\,\widehat{\otimes}\,\widehat{R}\right) \xrightarrow{\mathcal{U}(\varphi_{R,R})} \mathcal{U}\left(\widehat{R\otimes R}\right) \xrightarrow{\mathcal{U}(\widehat{m})} \mathcal{U}(\widehat{R})$$

and $\gamma_R : R \to \mathcal{U}(\widehat{R})$ is a filtered $\Bbbk$-algebra morphism. Analogously, one proves that for a complete $\Bbbk$-algebra R, the canonical morphism $\sigma_R : \widehat{\mathcal{U}(R)} \to R$ is an isomorphism of complete $\Bbbk$-algebras.

It is also worthy to consider explicitly the structure morphisms uniquely determined by (1.14).

Lemma 3.1.38. *For every filtered $\Bbbk$-algebras S, R, T and filtered bimodules ${}_SM_R$ and ${}_RN_T$ we have a unique natural isomorphism*

$$\varphi_{M,N} : \widehat{M}\,\widehat{\otimes}_{\widehat{R}}\,\widehat{N} \to \widehat{M\otimes_R N}$$

such that the following diagram commutes

$$\begin{tikzcd}
\widehat{M}\,\widehat{\otimes}_{\widehat{R}}\,\widehat{N} \arrow[r, "\varphi_{M,N}"] & \widehat{M\otimes_R N} \\
\widehat{M}\,\widehat{\otimes}\,\widehat{N} \arrow[u, "\omega_{\widehat{M},\widehat{N}}"] \arrow[r, "\varphi_{M,N}"'] & \widehat{M\otimes N} \arrow[u, "\widehat{\omega_{M,N}}"']
\end{tikzcd}$$

and whose inverse is given by

$$\psi_{M,N} := \left(\widehat{M\otimes_R N} \xrightarrow{(\gamma_M\widehat{\otimes}\gamma_N)} \widehat{\widehat{M}\otimes_R\widehat{N}} \xrightarrow{\widehat{\pi}} \widehat{M}\,\widehat{\otimes}_{\widehat{R}}\,\widehat{N},\right)$$

where π is the obvious projection. They are explicitly given by

$$\varphi_{M,N} : \widehat{M}\,\widehat{\otimes}_{\widehat{R}}\,\widehat{N} \to \widehat{M\otimes_R N},\ \left(\lim_{n\to\infty}\left(\lim_{k\to\infty}(x_{n,k})\otimes_{\widehat{R}}\lim_{l\to\infty}(y_{n,l})\right) \mapsto \lim_{n\to\infty}(x_{n,n}\otimes_R y_{n,n})\right), \tag{3.31}$$

$$\psi_{M,N} : \widehat{M\otimes_R N} \to \widehat{M}\,\widehat{\otimes}_{\widehat{R}}\,\widehat{N},\ \left(\lim_{n\to\infty}(x_n\otimes_R y_n) \mapsto \lim_{n\to\infty}\left(\widehat{x_n}\otimes_{\widehat{R}}\widehat{y_n}\right)\right). \tag{3.32}$$

Proof. Let us omit the functor $\mathcal{U}$ to increase the readability and keep the notations of diagram (3.30). Denote by $\omega_{\widehat{M},\widehat{N}}\colon \widehat{M}\otimes\widehat{N} \to \widehat{M}\otimes_R\widehat{N}$ and by $\omega'_{\widehat{M},\widehat{N}}\colon \widehat{M}\otimes\widehat{N} \to \widehat{M}\otimes_{\widehat{R}}\widehat{N}$ the obvious coequalizer maps. Consider the projection $\pi : \widehat{M} \otimes_R \widehat{N} \twoheadrightarrow \widehat{M} \otimes_{\widehat{R}} \widehat{N}$ which makes the following diagram commute

$$\begin{tikzcd}
 & \widehat{M} \otimes_R \widehat{N} \arrow[dd, "\pi"] \\
\widehat{M} \otimes \widehat{N} \arrow[ur, "\omega_{\widehat{M},\widehat{N}}"] \arrow[dr, "\omega'_{\widehat{M},\widehat{N}}"'] & \\
 & \widehat{M} \otimes_{\widehat{R}} \widehat{N}
\end{tikzcd}$$

and compose it with $\gamma_M \otimes_R \gamma_N$. Since as usual $\widehat{\gamma_M} = \sigma^{-1}_{\widehat{M}}$, the following diagram commutes

$$\begin{tikzcd}
\widehat{M} \widehat{\otimes}_{\widehat{R}} \widehat{N} & \widehat{\widehat{M} \otimes_R \widehat{N}} \arrow[l, "\widehat{\pi}"'] & \widehat{M \otimes_R N} \arrow[l, "\widehat{\gamma_M \otimes_R \gamma_N}"'] \\
\widehat{M} \widehat{\otimes} \widehat{N} \arrow[u, "\omega_{\widehat{M},\widehat{N}}"] & & \widehat{M \otimes N} \arrow[ll, "\psi_{M,N}"] \arrow[u, "\widehat{\omega_{M,N}}"']
\end{tikzcd}$$

where $\omega_{\widehat{M},\widehat{N}} = \widehat{\omega'_{\widehat{M},\widehat{N}}} \circ \varphi_{\widehat{M},\widehat{N}} \circ \left(\widehat{\widehat{\gamma_M} \otimes \widehat{\gamma_N}}\right) = \widehat{\omega'_{\widehat{M},\widehat{N}}}$. In view of the commutativity of Diagram (3.30), the universal property of $\widehat{M \otimes_R N}$ (as a coequalizer) entails that there is one and only one morphism $\psi_{M,N} : \widehat{M \otimes_R N} \to \widehat{M} \widehat{\otimes}_{\widehat{R}} \widehat{N}$ such that the above diagram commutes, whence $\psi_{M,N} = \widehat{\pi} \circ \left(\widehat{\gamma_M \otimes \gamma_N}\right)$. On the other hand, we already know from Theorem 1.6.6 that the universal property of $\widehat{M} \widehat{\otimes}_{\widehat{R}} \widehat{N}$ implies that there exists a unique morphism $\varphi_{M,N} : \widehat{M} \widehat{\otimes}_{\widehat{R}} \widehat{N} \to \widehat{M \otimes_R N}$ such that the diagram in the statement commutes. Since $\varphi_{M,N}$ and $\psi_{M,N}$ are inverses each other, it turns out that the uniqueness of the morphisms $\psi_{M,N}$ and $\varphi_{M,N}$ implies that they are inverses each other as well. $\square$

Remark 3.1.39. The morphism $\gamma_M \widehat{\otimes_R} \gamma_N$ of Lemma 3.1.38 is the unique morphism that makes the following diagram commute (recall that $M \otimes_R N$ is a colimit and that $\widehat{(-)}$ preserves colimits)

$$\begin{tikzcd}
\widehat{(M \otimes R) \otimes N} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "(\gamma_M \widehat{\otimes R}) \otimes \gamma_N"'] & \widehat{M \otimes N} \arrow[r, "\widehat{\omega_{M,N}}"] \arrow[d, "\gamma_M \widehat{\otimes} \gamma_N"'] & \widehat{M \otimes_R N} \arrow[r] \arrow[d, dotted, "\gamma_M \widehat{\otimes_R} \gamma_N"] & 0 \\
\widehat{(\widehat{M} \otimes R) \otimes \widehat{N}} \arrow[r, shift left] \arrow[r, shift right] & \widehat{\widehat{M} \otimes \widehat{N}} \arrow[r, "\widehat{\omega_{\widehat{M},\widehat{N}}}"] & \widehat{\widehat{M} \otimes_R \widehat{N}} \arrow[r] & 0
\end{tikzcd}$$

where the unlabelled arrows are the obvious ones. By omitting the functor $\mathcal{U}$, the first two vertical maps make the following diagrams commute,

$$\begin{tikzcd}
\widehat{(M \otimes R) \otimes N} \arrow[r, "\psi_{M \otimes R, N}"] \arrow[d, "(\gamma_M \widehat{\otimes R}) \otimes \gamma_N"'] & \widehat{M \otimes R} \widehat{\otimes} \widehat{N} \arrow[r, "\psi_{M,R} \widehat{\otimes} \widehat{N}"] & \left(\widehat{M} \widehat{\otimes} \widehat{R}\right) \widehat{\otimes} \widehat{N} \arrow[d, shift left, "\left(\widehat{\gamma_M} \widehat{\otimes} \widehat{R}\right) \widehat{\otimes} \widehat{\gamma_N}"] \\
\widehat{\left(\widehat{M} \otimes R\right) \otimes \widehat{N}} \arrow[r, "\psi_{\widehat{M} \otimes R, \widehat{N}}"'] & \left(\widehat{\widehat{M} \otimes R}\right) \widehat{\otimes} \widehat{\widehat{N}} \arrow[r, "\psi_{\widehat{M},R} \widehat{\otimes} \widehat{\widehat{N}}"'] & \left(\widehat{\widehat{M}} \widehat{\otimes} \widehat{R}\right) \widehat{\otimes} \widehat{\widehat{N}} \arrow[u, dotted, shift left, "\left(\sigma_{\widehat{M}} \widehat{\otimes} \widehat{R}\right) \widehat{\otimes} \sigma_{\widehat{N}}"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
\widehat{M \otimes N} \arrow[r, "\psi_{M,N}"] \arrow[d, "\gamma_M \widehat{\otimes} \gamma_N"'] & \widehat{M} \widehat{\otimes} \widehat{N} \arrow[d, shift left, "\widehat{\gamma_M} \widehat{\otimes} \widehat{\gamma_N}"] \\
\widehat{\widehat{M} \otimes \widehat{N}} \arrow[r, "\psi_{\widehat{M},\widehat{N}}"'] & \widehat{\widehat{M}} \widehat{\otimes} \widehat{\widehat{N}} \arrow[u, dotted, shift left, "\sigma_{\widehat{M}} \widehat{\otimes} \sigma_{\widehat{N}}"]
\end{tikzcd}$$

thus they are isomorphisms. As a consequence, $\widehat{\gamma_M \otimes_R \gamma_N}$ is an isomorphism as well and hence we get that

$$\widehat{M \otimes_R N} \cong \widehat{\widehat{M} \otimes_R \widehat{N}} \cong \widehat{M} \,\widehat{\otimes}_{\widehat{R}}\, \widehat{N}$$

In light of this, we may set $\widehat{M} \,\widehat{\otimes}_R\, \widehat{N} := \widehat{\widehat{M} \otimes_R \widehat{N}}$ even for a non complete $\Bbbk$-algebra R.

The last Remark 3.1.39 introduces implicitly a fact about which it is worthy to spend a few more words. Notice in fact that we may also speak about complete bimodules over non-necessarily complete $\Bbbk$-algebras. If S and R are filtered $\Bbbk$-algebras, nothing prevent us from having a filtered (S, R)-bimodule (M, F_nM) such that $\gamma_M : M \to \varprojlim (M/F_nM)$ is an isomorphism. Therefore, in principle, we may also consider the full subcategory ${}_S\mathfrak{M}^{\mathsf{c}}_R$ of ${}_S\mathfrak{M}^{\mathsf{flt}}_R$ given by complete (S, R)-bimodules for every filtered $\Bbbk$-algebras S and R. Objects are filtered (S, R)-bimodules (M, F_nM) such that $M \cong \varprojlim (M/F_nM)$ as bimodules and arrows are filtered morphisms of complete bimodules

$${}_S\mathsf{Hom}^{\mathsf{c}}_R (M, N) = {}_S\mathsf{Hom}^{\mathsf{flt}}_R (M, N) .$$

However, the subsequent Proposition 3.1.40 shows that these do not add anything essentially new to the picture.

Proposition 3.1.40. *We have an equivalence of categories between* ${}_S\mathfrak{M}^{\mathsf{c}}_R$ *and* ${}_{\widehat{S}}\mathfrak{M}^{\mathsf{c}}_{\widehat{R}}$.

Proof. Since $\gamma_S : S \to \mathcal{U}(\widehat{S})$ and $\gamma_R : R \to \mathcal{U}(\widehat{R})$ are filtered $\Bbbk$-algebra morphisms, the restriction of scalars along γ_S and γ_R associates every complete $(\widehat{S}, \widehat{R})$-bimodule $\left((M, F_nM), \mu_{\widehat{S},M}, \mu_{M,\widehat{R}}\right)$ with a complete (S, R)-bimodule $\left((M, F_nM), \mu_{\widehat{S},M} \circ (\gamma_S \otimes M), \mu_{M,\widehat{R}} \circ (M \otimes \gamma_R)\right)$.

Conversely, for every complete (S, R)-bimodule (M, F_nM) we may consider the underlying filtered $\Bbbk$-module $\mathcal{U}(M)$ which is also a filtered (S, R)-bimodule. We know then that $\widehat{\mathcal{U}(M)}$ is a complete $(\widehat{S}, \widehat{R})$-bimodule and via $\sigma_M : \widehat{\mathcal{U}(M)} \to M$ we can endow M with an $(\widehat{S}, \widehat{R})$-bimodule structure which makes of it a complete $(\widehat{S}, \widehat{R})$-bimodule.

It is easy to see that these assignments define in fact two functors that together form an equivalence of categories between ${}_S\mathfrak{M}^{\mathsf{c}}_R$ and ${}_{\widehat{S}}\mathfrak{M}^{\mathsf{c}}_{\widehat{R}}$. □

As a consequence, Proposition 3.1.40 allows us to work in the category ${}_S\mathfrak{M}^{\mathsf{c}}_R$ as if we were working in ${}_{\widehat{S}}\mathfrak{M}^{\mathsf{c}}_{\widehat{R}}$, depending on our needs or on what we would like to stress, even when the algebras S and R are not themselves complete. For example, we may consider the lifted completion and forgetful functors $\widehat{(-)} : {}_S\mathfrak{M}^{\mathsf{flt}}_R \to {}_S\mathfrak{M}^{\mathsf{c}}_R$ and $\mathcal{U} : {}_S\mathfrak{M}^{\mathsf{c}}_R \to {}_S\mathfrak{M}^{\mathsf{flt}}_R$ without caring about completeness of S and R and they are adjoint functors. Namely, we have a natural isomorphism

$${}_S\mathsf{Hom}^{\mathsf{flt}}_R (N, \mathcal{U}(M)) \cong {}_S\mathsf{Hom}^{\mathsf{c}}_R \left(\widehat{N}, M\right) . \tag{3.33}$$

Unit and counit of this adjunction are the same of Lemma 3.1.22, i.e. $\gamma_N : N \to \mathcal{U}(\widehat{N})$ and $\sigma_M : \widehat{\mathcal{U}(M)} \to M$ for all $N \in {}_S\mathfrak{M}^{\mathsf{flt}}_R$ and $M \in {}_S\mathfrak{M}^{\mathsf{c}}_R$. Again, the bijection in equation (3.33) encodes the universal property of the completion: every filtered morphism $g : N \to M$ from a filtered (S, R)-bimodule N to a complete (S, R)-bimodule M factors through the completion of N, i.e. , we have a commutative diagram of filtered morphisms

$$\begin{tikzcd} N \arrow[rr, "g"] \arrow[dr, "\gamma_N"'] & & M \\ & \widehat{N} \arrow[ur, dotted, "\widehat{g}"'] & \end{tikzcd} \tag{3.34}$$

For the reason above, we will often assume that algebras we are working with are just filtered, even if we intend to formulate results for complete bimodules. For example, the following Corollary 3.1.41 collects some few facts as a future reference.

Corollary 3.1.41. *Let R be a filtered $\Bbbk$-algebra. Then the category of complete R-bimodules ${}_R\mathfrak{M}^{\mathsf{c}}_R$ is monoidal with tensor product the complete tensor product $-\widehat{\otimes}_R-$ and with unit the algebra R itself. Moreover, the completion functor $\widehat{(-)}\colon {}_R\mathfrak{M}^{\mathsf{flt}}_R \to {}_{\widehat{R}}\mathfrak{M}^{\mathsf{c}}_{\widehat{R}}$ is a monoidal functor.*

As it happens for $\Bbbk$-modules, given filtered algebras R, S, T and a complete (R,T)-bimodule N the assignment

$$\mathsf{Hom}^{\mathsf{c}}_T(N,-)\colon {}_S\mathfrak{M}^{\mathsf{c}}_T \to {}_S\mathfrak{M}^{\mathsf{c}}_R \tag{3.35}$$

gives a well-defined functor which is right adjoint to the complete tensor product functor. Namely, for ${}_SM_R$, ${}_RN_T$, ${}_SP_T$ complete bimodules we have a filtered isomorphism

$${}_S\mathsf{Hom}^{\mathsf{c}}_R(M, \mathsf{Hom}^{\mathsf{c}}_T(N,P)) \cong {}_S\mathsf{Hom}^{\mathsf{c}}_T\left(M \widehat{\otimes}_R N, P\right) \tag{3.36}$$

which is natural in M and P. The proof is the same of Theorem 3.1.27, simply using (3.9) instead of (3.4).

Remark 3.1.42. In light of the adjunction (3.36), it is reasonable to call this complete tensor product a *topological* tensor product as it is the left adjoint to the *continuous* Hom functor between complete bimodules (in fact, we did it in [ES2])[(7)]. We point out however that our definition of a topological tensor product satisfies a different universal property with respect to e.g. [Se, Definition 2.1] or [Sm, Theorem 20.1.2]. Namely, assume that ${}_SM_R, {}_RN_T, {}_SP_T$ are complete bimodules over filtered algebras as indicated. Endow $M \times N$ with the filtration $F_k(M\times N) = F_kM \times F_kN$. The induced linear topology coincides with the product linear topology, i.e. the coarsest linear topology for which the canonical projections are continuous, and $M\times N$ is a complete (S,T)-bimodule with respect to this filtration. The canonical morphism $M \times N \to M \otimes_R N$ maps $F_k(M\times N)$ into $\mathsf{im}\,(F_kM \otimes_R F_kN) \subseteq \mathcal{F}_{2k}(M\otimes_R N) \subseteq \mathcal{F}_k(M \otimes_R N)$, whence it is filtered (and continuous) and the same hold for the composition $\tau := \left(M\times N \to M\otimes_R N \to M \widehat{\otimes}_R N\right)$. Endow $M \times N$ with the bi-filtration $F_{h,k}(M\times N) = F_hM \times F_kN$ (for the definition of a bi-filtration seee.g. [Brl, §X.2]). We observe that τ is bi-filtered[(8)] as well. The bijective correspondence between R-balanced (S,T)-bilinear morphisms ${}_SM_R \times {}_RN_T \to {}_SP_T$ and morphisms in ${}_S\mathsf{Hom}_R(M, \mathsf{Hom}_T(N,P))$ restricts to a bijective correspondence between R-balanced (S,T)-bilinear bi-filtered morphisms $M\times N \to P$ and elements in ${}_S\mathsf{Hom}^{\mathsf{c}}_R(M, \mathsf{Hom}^{\mathsf{c}}_T(N,P))$. From this it follows that the complete tensor product could be considered as a topological tensor product in the sense that it satisfies the following universal property: there exists a complete (S,T)-bimodule and a bi-filtered R-balanced (S,T)-bilinear morphism $\tau : M\times N \to M\widehat{\otimes}_R N$ such that for every other complete (S,T)-bimodule P and every bi-filtered R-balanced (S,T)-bilinear morphism $f : M\times N \to P$ there exists a unique filtered (S,T)-bilinear morphism $\widetilde{f} : M \widehat{\otimes}_R N \to P$ such that $f = \widetilde{f}\circ\tau$.

3.2 Hopf algebroids

This section is devoted to recall the definitions of commutative and cocommutative Hopf algebroid, as well as to retrieve explicitly the structure maps involved in the definition of complete commutative Hopf algebroids. We will make use of the notations and notions expounded in §3.1. Following the standard literature on the subject, we will assume $\Bbbk$ to be a field, even if many of the results presented here can be stated for a commutative ring. All algebras are assumed to be $\Bbbk$-algebras if not differently specified.

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(7) In some particular cases, for example when $R = \mathbb{C}[[h]]$, this terminology has already been used. See e.g. [Ks, §XVI.3]. However we decided not to do so here in order to avoid confusion with other notions of topological tensor product and in accordance with the literature we are referring to.

(8) By a bi-filtered morphism we mean a morphism $f : M \times N \to P$ such that $f(F_hM\times F_kN) \subseteq F_{h+k}P$. In particular, if f is bi-filtered then $f(-,n) : M \to P$ and $f(m,-) : N \to P$ are filtered for all $m\in M$ and $n \in N$. Bi-filtered morphisms can be seen as a counterpart of *separately continuous* functions (for an account on the subject we refer the reader to [Pi]).

3.2.1 Commutative and complete Hopf algebroids

Recall from [Rav, Appendix A1] that a *commutative Hopf algebroid* is a cogroupoid object in the category $\mathsf{CAlg}_{\Bbbk}$ of commutative algebras (equivalently a groupoid in the category of affine schemes, see for example [DG, III, §2, n°1] and [Ga, p.253]).

Namely, given a category $\mathcal{C}$ with enough pullbacks, a *groupoid* in $\mathcal{C}$ (also called *groupoid object* or *internal groupoid*) consist of an object "of objects" C_0 and an object "of arrows" C_1 together with source and target morphisms $s, t : C_1 \to C_0$, identity morphism $e : C_0 \to C_1$, composition morphism $c : C_1 \times_{C_0} C_1 \to C_1$ and "inversion" $i : C_1 \to C_1$ such that

- the pullback $C_{1t} \times_{C_0} {}_sC_1$ is defined via the square

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "p_2"] \arrow[d, "p_1"'] & C_1 \arrow[d, "s"] \\ C_1 \arrow[r, "t"'] & C_0 \end{tikzcd}$$

- source and target of the identity morphism are the starting object itself, that is

$$\begin{tikzcd} C_0 & C_1 \arrow[l, "t"'] \arrow[r, "s"] & C_0 \\ & C_0 \arrow[ul, "\mathsf{Id}_{C_0}"] \arrow[u, "e"'] \arrow[ur, "\mathsf{Id}_{C_0}"'] & \end{tikzcd}$$

- the source and the target of a composition are the source of the first and the target of the second composed maps, that is

$$\begin{tikzcd} C_1 \arrow[d, "s"'] & C_1 \times_{C_0} C_1 \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "c"'] & C_1 \arrow[d, "t"] \\ C_0 & C_1 \arrow[l, "s"] \arrow[r, "t"'] & C_0 \end{tikzcd}$$

- the associative and unital laws for composition of morphisms hold, that is

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \times_{C_0} C_1 \arrow[r, "c \times_{C_0} C_1"] \arrow[d, "C_1 \times_{C_0} c"'] & C_1 \times_{C_0} C_1 \arrow[d, "c"] \\ C_1 \times_{C_0} C_1 \arrow[r, "c"'] & C_1 \end{tikzcd} \qquad \begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[dr, "c"] & C_1 \times_{C_0} C_0 \arrow[l, "C_1 \times_{C_0} e"'] \arrow[d, "p_1"] \\ C_0 \times_{C_0} C_1 \arrow[u, "e \times_{C_0} C_1"] \arrow[r, "p_2"'] & C_1 \end{tikzcd}$$

- the inversion satisfies $i^2 = \mathsf{Id}_{C_1}$, it switches the source and the target of an arrow and the composition of an arrow with its inverse on both sides gives the identity, that is[9]

$$\begin{tikzcd} C_1 \arrow[d, "i"'] \arrow[dr, "s"] & C_1 \arrow[l, "i"'] \arrow[d, "t"] \\ C_1 \arrow[r, "t"'] & C_0 \end{tikzcd} \qquad \begin{tikzcd} C_1 & C_0 \arrow[l, "e"'] & C_1 \arrow[l, "s"'] \arrow[d, "\mathrm{diag}"] \\ C_{1t} \times_{C_0} {}_sC_1 \arrow[u, "c"] & & C_{1t} \times_{C_0} {}_tC_1 \arrow[ll, "C_1 \times_{C_0} i"] \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[r, "t"] \arrow[d, "\mathrm{diag}"'] & C_0 \arrow[r, "e"] & C_1 \\ C_{1s} \times_{C_0} {}_sC_1 \arrow[rr, "i \times_{C_0} C_1"'] & & C_{1t} \times_{C_0} {}_sC_1 \arrow[u, "c"'] \end{tikzcd}$$

see e.g. [J, Definition 2.3.1 (a) and Example 2.3.12 (c)]. A *cogroupoid* (or *cogroupoid object* or *internal cogroupoid*) in $\mathcal{C}$ is a groupoid in $\mathcal{C}^{\mathrm{op}}$.

Since the opposite category $\mathsf{CAlg}_{\Bbbk}{}^{\mathrm{op}}$ of the category of commutative algebras is equivalent to the category AS of *affine schemes* (i.e. representable functors $\mathcal{F} : \mathsf{CAlg}_{\Bbbk} \to \mathsf{Set}$), a commutative Hopf

[9] The unusual notation $C_{1t} \times_{C_0} {}_tC_1$ means that the pullback is taken with respect to t in both entrances.

algebroid consists of a pair of commutative algebras (A, H) together with a diagram of algebra maps

$$\begin{tikzcd} A \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & H \arrow[l, "\varepsilon"'] \arrow[loop above, "\mathcal{S}"] \arrow[r, "\Delta"] & H \otimes_A H, \end{tikzcd} \tag{3.37}$$

where to perform the tensor product, H is considered as an A-bimodule of the form ${}_sH_t$, that is, A acts on the left through s while it acts on the right through t. The maps $s, t : A \to H$ are called the *source* and the *target* respectively, $\varepsilon : H \to A$ the *counit*, $\Delta : H \to H \otimes_A H$ the *comultiplication* and $\mathcal{S} : H \to H$ the *antipode*. These have to satisfy the following compatibility conditions.

(HA1) The datum (H, Δ, ε) has to be a coassociative and counital comonoid[10] in ${}_A\mathfrak{M}_A$, that is, an A-*coring*. At the level of groupoids, this reflects the unital and associative composition law between arrows.

(HA2) The antipode has to satisfy $\mathcal{S} \circ s = t$, $\mathcal{S} \circ t = s$ and $\mathcal{S}^2 = \mathsf{Id}_H$, which reflect the fact that the inverse of an arrow interchanges source and target and that the inverse of the inverse is the original arrow.

(HA3) The antipode has to satisfy also $\sum \mathcal{S}(h_1)h_2 = (t \circ \varepsilon)(h)$ and $\sum h_1\mathcal{S}(h_2) = (s \circ \varepsilon)(h)$, which reflect the fact that the composition of a morphism with its inverse on either side gives an identity morphism.

A *morphism of commutative Hopf algebroids* is a pair of algebra maps $(\phi_0, \phi_1) : (A, H) \to (B, L)$ such that

$$\begin{aligned} \phi_1 \circ s_H &= s_L \circ \phi_0, & \phi_1 \circ t_H &= t_L \circ \phi_0, \\ \Delta_L \circ \phi_1 &= \chi \circ (\phi_1 \otimes_A \phi_1) \circ \Delta_H, & \varepsilon_L \circ \phi_1 &= \phi_0 \circ \varepsilon_H, \\ \mathcal{S}_L \circ \phi_1 &= \phi_1 \circ \mathcal{S}_H \end{aligned}$$

where $\chi : L \otimes_A L \to L \otimes_B L$ is the obvious map induced by ϕ_0, that is, $\chi(h \otimes_A k) = h \otimes_B k$. The category so obtained will be denoted by $\mathsf{CHAlgd}_{\Bbbk}$.

Example 3.2.1. Here are some examples of commutative Hopf algebroids (see [EK] for more)

(1) Every commutative Hopf algebra H gives rise to a commutative Hopf algebroid $(\Bbbk, H)$.

(2) Let A be a commutative algebra. Then the pair $(A, A \otimes A)$ admits a commutative Hopf algebroid structure given by $s(a) = a \otimes 1$, $t(a) = 1 \otimes a$, $\mathcal{S}(a \otimes a') = a' \otimes a$, $\varepsilon(a \otimes a') = aa'$ and $\Delta(a \otimes a') = (a \otimes 1) \otimes_A (1 \otimes a')$, for any $a, a' \in A$.

(3) Let $(B, \Delta, \varepsilon, S)$ be a commutative Hopf algebra and A a right B-comodule commutative algebra with coaction $A \to A \otimes B$, $a \mapsto \sum a_0 \otimes a_1$. This means that A is right B-comodule and the coaction is an algebra map (see e.g. [Mo, §4]). Consider the algebra $H = A \otimes B$ with algebra extension $\eta : A \otimes A \to H$, $a' \otimes a \mapsto \sum a'a_0 \otimes a_1$. Then (A, H) has a structure of commutative Hopf algebroid, known as *split Hopf algebroid*

$$\Delta(a \otimes b) = \sum (a \otimes b_1) \otimes_A (1_A \otimes b_2), \qquad \varepsilon(a \otimes b) = a\varepsilon(b), \qquad \mathcal{S}(a \otimes b) = \sum a_0 \otimes a_1 S(b).$$

(4) Let B be as in (3) and A any commutative algebra. Then $(A, A \otimes B \otimes A)$ admits in a canonical way a structure of commutative Hopf algebroid. For $a, a' \in A$ and $b \in B$, its structure maps are given as follows

$$s(a) = a \otimes 1_B \otimes 1_A, \quad t(a) = 1_A \otimes 1_B \otimes a, \quad \varepsilon(a \otimes b \otimes a') = aa'\varepsilon(b),$$
$$\Delta(a \otimes b \otimes a') = \sum \big(a \otimes b_1 \otimes 1_A\big) \otimes_A \big(1_A \otimes b_2 \otimes a'\big), \quad \mathcal{S}(a \otimes b \otimes a') = a' \otimes S(b) \otimes a.$$

[10] In this chapter we prefer to use the terms monoids and comonoids instead of algebras and coalgebras in monoidal category, in order to avoid confusion between A-algebras (which are algebras in the symmetric monoidal category $\mathfrak{M}_A$ of A-modules) and algebras in the monoidal category ${}_A\mathfrak{M}_A$ of A-bimodules, which are A-rings.

Recall from §3.1.1 that the tensor product of two filtered A-bimodules M, N is again a filtered A-bimodule with filtration

$$F_n(M \otimes_A N) = \sum_{p+q=n} \mathsf{im}\left(F_p M \otimes_A F_q N\right)$$

as in equation (3.6). This allows us to adapt the definition of commutative Hopf algebroid to the filtered context. Namely, a pair (A, H) of filtered commutative algebras is said to be a *filtered commutative Hopf algebroid* provided that there is a diagram of filtered $\Bbbk$-algebra maps as in Equation (3.37) that satisfy the compatibility conditions (HA2) - (HA3) plus

(HA1′) The datum (H, Δ, ε) has to be a coassociative and counital comonoid in ${}_A\mathfrak{M}_A^{\mathsf{flt}}$.

Equivalently, it is a cogroupoid object in the category of filtered commutative algebras. Morphisms between filtered commutative Hopf algebroids are easily understood and the category so obtained will be denoted by $\mathsf{CHAlgd}_{\Bbbk}^{\mathsf{flt}}$. The following example shows that there is a canonical functor $\mathsf{CHAlgd}_{\Bbbk} \to \mathsf{CHAlgd}_{\Bbbk}^{\mathsf{flt}}$.

Example 3.2.2. Consider a commutative Hopf algebroid (A, H). Assume A to be discretely filtered and set $K := \mathsf{ker}(\varepsilon : H \to A)$. Endow H with the *augmentation filtration* $F_0 H = H$ and $F_n H := K^n$ for every $n \geq 1$ and endow $H \otimes_A H$ with the usual tensor product filtration

$$F_0\left(H \otimes_A H\right) = H \otimes_A H, \qquad F_n\left(H \otimes_A H\right) = \sum_{p+q=n} \mathsf{im}\left(K^p \otimes_A K^q\right)$$

for every $n \geq 1$. Then (A, H) is a filtered commutative Hopf algebroid.

As it is already known, the usual tensor product of complete bimodules is not necessarily a complete bimodule. In order to introduce complete Hopf algebroid, one needs to use the complete tensor product of §3.1.3. Thus, a *complete commutative Hopf algebroid* is a pair (A, H) of complete commutative algebras together with a diagram of filtered algebra maps

$$\begin{tikzcd} A \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & H \arrow[l, "\varepsilon"' description] \arrow[loop above, "\mathcal{S}"] \arrow[r, "\Delta"] & H \,\widehat{\otimes}_A\, H, \end{tikzcd} \tag{3.38}$$

that satisfy the compatibility conditions (HA2) - (HA3) plus

(HA1″) The datum (H, Δ, ε) has to be a coassociative and counital comonoid in ${}_A\mathfrak{M}_A^{\mathsf{c}}$.

In different but equivalent words, a complete commutative Hopf algebroid is a cogroupoid object in the category of complete commutative algebras (see e.g. [Dv, §1]). Since we will be interested only in complete Hopf algebroids which are commutative, henceforth by a complete Hopf algebroid we will always mean a complete commutative Hopf algebroid. The category of complete Hopf algebroids and their morphisms will be denoted by $\mathsf{CHAlgd}_{\Bbbk}^{\mathsf{c}}$.

The following result can be seen as a consequence of Theorem 3.1.35. Nevertheless, for the convenience of reader, we outline here a proof.

Proposition 3.2.3. *The completion functor induces a functor*

$$\mathsf{CHAlgd}_{\Bbbk}^{\mathsf{flt}} \xrightarrow{\widehat{(-)}} \mathsf{CHAlgd}_{\Bbbk}^{c}$$

to the category of complete Hopf algebroids with filtered morphisms of Hopf algebroids. In particular, we have the following composition of functors

$$\begin{tikzcd} \mathsf{CHAlgd}_{\Bbbk}^{\mathsf{flt}} \arrow[rr, "\widehat{(-)}"] & & \mathsf{CHAlgd}_{\Bbbk}^{c} \\ & \mathsf{CHAlgd}_{\Bbbk} \arrow[ul] \arrow[ur, dashed] & \end{tikzcd}$$

Proof. Let (A,H) be a filtered Hopf algebroid with filtered algebra maps $s,t,\varepsilon,\Delta,\mathcal{S}$. In particular, H is an object in ${}_A\mathfrak{M}_A^{\mathsf{flt}}$. Consider $\widehat{A}$ and $\widehat{H}$, which are complete modules as well as complete algebras (see Theorem 3.1.35). We have that $\widehat{H}$ is an object in ${}_{\widehat{A}}\mathfrak{M}^{\mathsf{c}}_{\widehat{A}}$ and we have complete algebra maps

$$\widehat{s}\ ,\ \widehat{t}\colon \widehat{A}\to\widehat{H},\qquad \widehat{\varepsilon}\colon \widehat{H}\to\widehat{A},$$
$$\widehat{\Delta}\colon \widehat{H}\to\left(\widehat{H\otimes_A H}\right)\overset{(3.32)}{\cong}\widehat{H}\ \widehat{\otimes}_{\widehat{A}}\ \widehat{H},\quad \text{and}\quad \widehat{\mathcal{S}}\colon\widehat{H}\to\widehat{H}.$$

These maps satisfy the same axioms as the original ones, because $\widehat{(-)}:\mathscr{B}im_{\Bbbk}^{\mathsf{flt}}\to\mathscr{B}im_{\Bbbk}^{\mathsf{c}}$ is a bifunctor. The unique detail that needs perhaps a few words more is the antipode condition. Consider the maps $c_l:H\otimes_A H\to H$ and $c_r:H\otimes_A H\to H$ such that $c_l(x\otimes_A y)=\mathcal{S}(x)y$ and $c_r(x\otimes_A y)=x\mathcal{S}(y)$ respectively, for all $x,y\in H$. These allow us to write the antipode conditions as the commutativity of the diagram

$$\begin{tikzcd}
H & & H\otimes H \arrow[ll, "m\circ(\mathcal{S}\otimes H)"'] \arrow[rr, "m\circ(H\otimes\mathcal{S})"] \arrow[d, "p"] & & H \\
& & H\otimes_A H \arrow[ull, "c_l"] \arrow[urr, "c_r"'] & & \\
A \arrow[uu, "t"] & & H \arrow[ll, "\varepsilon"'] \arrow[rr, "\varepsilon"] \arrow[u, "\Delta"] & & A \arrow[uu, "s"']
\end{tikzcd}$$

where p, c_l and c_r are all filtered. Indeed, $p:=\omega_{H,H}$ is the canonical projection of the coequalizer in $\mathfrak{M}^{\mathsf{flt}}$ and c_l and c_r are the factorizations through the coequalizer of the filtered maps $m\circ(\mathcal{S}\otimes H)$ and $m\circ(H\otimes\mathcal{S})$. We can now apply the functor $\widehat{(-)}$ to get a commutative diagram

$$\begin{tikzcd}
\widehat{H} & & \widehat{H}\,\widehat{\otimes}\,\widehat{H} \arrow[ll, "\widehat{m}\circ\left(\widehat{\mathcal{S}}\widehat{\otimes}\widehat{H}\right)"'] \arrow[rr, "\widehat{m}\circ\left(\widehat{H}\widehat{\otimes}\widehat{\mathcal{S}}\right)"] \arrow[d, "\widehat{p}"] & & \widehat{H} \\
& & \widehat{H}\,\widehat{\otimes}_{\widehat{A}}\widehat{H} \arrow[ull, "\widehat{c_l}"] \arrow[urr, "\widehat{c_r}"'] & & \\
\widehat{A} \arrow[uu, "\widehat{t}"] & & \widehat{H} \arrow[ll, "\widehat{\varepsilon}"'] \arrow[rr, "\widehat{\varepsilon}"] \arrow[u, "\widehat{\Delta}"] & & \widehat{A} \arrow[uu, "\widehat{s}"']
\end{tikzcd} \tag{3.39}$$

which shows that $\widehat{\mathcal{S}}$ is the antipode of $\widehat{H}$. Therefore, $\left(\widehat{A},\widehat{H}\right)$ is a complete Hopf algebroid.

Let $(\phi_0,\phi_1):(A,H)\to(B,L)$ be a morphism of filtered Hopf algebroids. Hence we can consider $\widehat{\phi_0}\colon\widehat{A}\to\widehat{B}$ and $\widehat{\phi_1}\colon\widehat{H}\to\widehat{L}$ and these are morphisms of complete algebras. Since $\chi:L\otimes_A L\to L\otimes_B L$ is filtered, $\left(\widehat{\phi_0},\widehat{\phi_1}\right)$ becomes a morphism of complete Hopf algebroids by the functoriality of $\widehat{(-)}$. In light of Example 3.2.2, $\widehat{(-)}$ restricts to a functor

$$\widehat{(-)}\colon \mathsf{CHAlgd}_{\Bbbk}\to\mathsf{CHAlgd}_{\Bbbk}^{\mathsf{c}},$$

and this finishes the proof. $\square$

Example 3.2.4 (The complete Hopf algebroid of infinite jets)**.** Let A be an algebra and consider the Hopf algebroid $(A,A\otimes A)$ as in (2) of Example 3.2.1. Then the pair $(A,\widehat{A\otimes A})$ is a complete Hopf algebroid by Proposition 3.2.3, where A is discretely filtered and $A\otimes A$ is given the K-adic filtration where $K:=\mathsf{ker}\,(m_A:A\otimes A\to A)$ is the kernel of the multiplication. For every quotient A-bimodule $\mathcal{J}^n(A):=(A\otimes A)/K^n$, set $j_n:A\to\mathcal{J}^n(A)$, $a\mapsto(1\otimes a)+K^n$. The A-bimodules $\mathcal{J}^n(A)$ turn out to be commutative A-algebras with $A\to\mathcal{J}^n(A):a\mapsto aj^n(1)$ and product $(aj^n(b))\cdot(a'j^n(b'))=aa'j^n(bb')$. They are called the *algebras of n-jets* of the algebra A. The complete algebra $\widehat{A\otimes A}$ is known under the name *algebra of infinite jets* of the algebra A (see e.g. [Kr, §6]). This terminology may be justified by considering the case $A = \mathcal{C}^\infty(\mathcal{M})$ for $\mathcal{M}$ a (finite-dimensional) smooth manifold. Namely, in this case, the geometrization

$$\mathscr{G}\left(\mathcal{J}^k(A)\right) = \frac{\mathcal{J}^k(A)}{\bigcap_{z\in\mathcal{M}} \mu_z \mathcal{J}^k(A)},$$

$\mu_z = \{f \in \mathcal{C}^\infty(\mathcal{M}) \mid f(z) = 0\}$, of the module of k-jets of $\mathcal{C}^\infty(A)$ is isomorphic to the module $\mathscr{J}^k(\mathcal{M}) = \Gamma(\pi_{J^k} : J^k\mathcal{M} \to \mathcal{M})$ of k-jets of functions on $\mathcal{M}$ (see e.g. [Kr, Proposition 9.4 (iv)]). For further details on these constructions we refer to [N, Kr, KLV].

3.2.2 Co-commutative Hopf algebroids and Lie-Rinehart algebras

Next, we recall the definition of a (right) cocommutative Hopf algebroid. It can be considered as a revised (right-handed and cocommutative) version of the notion of a $\times_A$-Hopf algebra as it appears in [Sc3, Theorem and Definition 3.5]. However, to define the underlying right bialgebroid structure we preferred to mimic [L] as presented in [BM, Definition 2.2] (in light of [BM, Theorem 3.1], this is something we may do). See also [Kp, A.3.6] and compare with [Ko, Definition 2.5.1] and [Sz1, §4.1] as well.

Definition 3.2.5. A *(right) cocommutative bialgebroid* over a commutative algebra A is the datum of a possibly non-commutative algebra U together with an algebra map $\tau : A \to U$ landing not necessarily in the center of U, such that the symmetric A-bimodule U_A (i.e. $a \cdot u = u \cdot a = u\tau(a)$ for all $a \in A, u \in U$) comes endowed with a comultiplication $\Delta : U \to U \otimes_A U$ and a counit $\varepsilon : U \to A$ which make of $(U_A, \Delta, \varepsilon)$ a comonoid in the monoidal category $({}_A\mathfrak{M}_A, \otimes_A, A)$ of A-bimodules. Moreover, the counit ε is required to be a *right character* in the sense of [BS2, Lemma 2.5] (i.e. it satisfies $\varepsilon(1_U) = 1_A$, $\varepsilon(u\tau(a)) = \varepsilon(u)a$ and

$$\varepsilon(uv) = \varepsilon(\tau\varepsilon(u)v) \tag{3.40}$$

for all $u, v \in U$, $a \in A$) and Δ is required to satisfy

$$\mathsf{im}\,(\Delta) \subseteq U \times_A U := \left\{ \sum_i u_i \otimes_A v_i \in U_A \otimes_A U_A \;\middle|\; \sum_i \tau(a)u_i \otimes_A v_i = \sum_i u_i \otimes_A \tau(a)v_i \right\} \tag{3.41}$$

so that it factors through an A-ring map $\Delta : U \to U \times_A U$, where the module $U \times_A U$ is endowed with the algebra structure

$$\left(\sum_i u_i \times_A v_i\right) \cdot \left(\sum_j u'_j \times_A v'_j\right) = \sum_{i,j} u_iu'_j \times_A v_iv'_j, \qquad 1_{U\times_A U} = 1_U \otimes_A 1_U$$

and the A-ring structure given by the algebra map $A \to U \times_A U : a \mapsto (\tau(a) \times_A 1_U = 1_U \times_A \tau(a))$.

If furthermore the canonical $\Bbbk$-linear map (also called *Hopf-Galois map*)

$$\mathfrak{can}\colon U_A \otimes_A {}_AU \to U_A \otimes_A U_A, \quad u \otimes_A v \mapsto \sum uv_1 \otimes_A v_2 \tag{3.42}$$

is bijective, then we say that (A, U) is a cocommutative right *Hopf algebroid*. As a matter of terminology, the map $\mathfrak{can}^{-1}(1 \otimes_A -) : U \to U_A \otimes_A {}_AU$ is called the *translation map*. The following is a standard notation

$$\delta(u) := \mathfrak{can}^{-1}(1 \otimes_A u) = \sum u_- \otimes_A u_+. \tag{3.43}$$

The module $U_A \times_A U_A$ is (one of) the so-called *Takeuchi product*. There are many others. In this thesis we will make use of this one, which we may denote also by ${}^AU_A \times_A {}^AU_A$ to highlight with respect to which actions we are considering the tensor product (lower indices) or the invariants (upper indices), and the following one

$${}^AU_A \times_A {}_AU^A := \left\{ \sum_i u_i \otimes_A v_i \in U_A \otimes_A {}_AU \;\middle|\; \sum_i \tau(a)u_i \otimes_A v_i = \sum_i u_i \otimes_A v_i\tau(a) \right\}.$$

It turns out that ${}^AU_A^{\mathrm{op}} \times_A {}_AU^A$ is an A-ring as well, with componentwise multiplication.

Remark 3.2.6. Before proceeding, let us collect here some immediate consequences of the previous definition for future reference.

(a) Since ε has to be a right character, it turns out that $a = \varepsilon(1_U)a = \varepsilon(\tau(a))$ and hence τ is injective. For this reason we may often omit to write it and we will denote the A-actions on U by simple juxtaposition: $\tau(a)u = au$ and $u\tau(a) = ua$. With these convention, the counital property of the comultiplication rewrites

$$\sum u_1\varepsilon(u_2) = u = \sum u_2\varepsilon(u_1). \tag{3.44}$$

(b) If can is invertible, then for all $v \in U$

$$1 \otimes_A v = \mathsf{can}\left(\mathsf{can}^{-1}\left(1 \otimes_A v\right)\right) = \sum \mathsf{can}(v_- \otimes_A v_+) = \sum v_- v_{+,1} \otimes_A v_{+,2}. \tag{3.45}$$

Consider the U action on $U_A \otimes_A U_A$ given by $u \centerdot (v \otimes_A w) := uv \otimes_A w$ for all $u, v, w \in U$. Thus

$$\sum \mathsf{can}\left(uv_- \otimes_A v_+\right) = \sum uv_- v_{+,1} \otimes_A v_{+,2} = u \centerdot \sum \left(v_- v_{+,1} \otimes_A v_{+,2}\right) = u \otimes_A v$$

and so

$$\sum uv_- \otimes_A v_+ = \mathsf{can}^{-1}\left(u \otimes_A v\right). \tag{3.46}$$

(c) For every $v \in U$, $\mathsf{can}(1 \otimes_A v) \in {}^AU_A \times_A {}^AU_A$. Moreover, since

$$\sum av_- \otimes_A v_+ = \sum a_- v_- \otimes_A v_+ a_+ = \sum v_- \otimes_A v_+ a$$

for all $a \in A, v \in U$ (see [Sc3, Proposition 3.7, (3.5)]), we have that δ factors through the Takeuchi product ${}^AU_A \times_A {}_AU^A$ and $\delta\colon U \to {}^AU_A^{\mathrm{op}} \times_A {}_AU^A$ turns out to be a $\Bbbk$-algebra map (see [Sc3, Proposition 3.7, (3.4)]). Hence $\mathsf{can}^{-1}(1 \otimes_A v) \in {}^AU_A \times_A {}_AU^A$.

(d) Both ${}_\tau U_A \otimes_A {}_AU_\tau$ and ${}_\tau U_A \otimes_A U_{A,\tau}$ are A-bimodules as highlighted with the τ-actions and can is A-bilinear with respect to these actions.

The main example we will consider of a cocommutative right Hopf algebroid is the so-called universal enveloping algebra of a Lie-Rinehart algebra.

Let A be a commutative algebra over $\Bbbk$ and denote by $\mathsf{Der}_\Bbbk(A)$ the Lie algebra of all linear derivations of A. Following [Rin], a *Lie-Rinehart algebra* over A is a Lie algebra L endowed with an A-module structure $a \otimes X \mapsto a \cdot X$ and a Lie algebra map $\omega : L \to \mathsf{Der}_\Bbbk(A)$ such that

$$\omega(a \cdot X) = a \cdot \omega(X) \qquad \text{and} \qquad [X, a \cdot Y] = a \cdot [X, Y] + \omega(X)(a) \cdot Y, \tag{3.47}$$

for all $X, Y \in L$ and $a \in A$. The map ω is called the *anchor map*. The shortened notation $X(a) = \omega(X)(a)$ is often used in the literature.

Remark 3.2.7. Recall from Example 1.3.2 that $\mathsf{End}_\Bbbk(A)$ is an A-bimodule via

$$(a \cdot f \leftharpoonup b)(c) = af(bc)$$

for all $a, b, c \in A$ and $f \in \mathsf{End}_\Bbbk(A)$. The A-module structure on $\mathsf{Der}_\Bbbk(A)$ is the one that makes of it a left A-subbimodule of $\mathsf{End}_\Bbbk(A)$, that is,

$$(a \cdot \delta)(b) = a\delta(b).$$

The first relation of equation (3.47) says that ω is A-linear.

Apart from $(A, \mathsf{Der}_\Bbbk(A))$ (with anchor the identity map), a basic source of examples of Lie-Rinehart algebras is provided by the smooth global sections of Lie algebroids over smooth manifolds.

Example 3.2.8. A *Lie algebroid* over a smooth manifold $\mathcal{M}$ is a vector bundle $\mathcal{L} \to \mathcal{M}$ over $\mathcal{M}$, together with a map $\omega : \mathcal{L} \to T\mathcal{M}$ of vector bundles and a Lie structure $[-,-]$ on the vector space $\Gamma(\mathcal{L})$ of smooth global sections of $\mathcal{L}$ such that the induced map $\Gamma(\omega) : \Gamma(\mathcal{L}) \to \Gamma(T\mathcal{M})$ is a Lie algebra homomorphism and for all $X, Y \in \Gamma(\mathcal{L})$ and for any $f \in \mathcal{C}^\infty(\mathcal{M})$ one has that $[X, f \cdot Y] = f \cdot [X, Y] + \Gamma(\omega)(X)(f) \cdot Y$. Then $\Gamma(\mathcal{L})$ is a Lie-Rinehart algebra over $\mathcal{C}^\infty(\mathcal{M})$.

The *(right) universal enveloping algebra* of a Lie-Rinehart algebra is by definition an algebra $\mathcal{V}_A(L)$ endowed with an algebra morphism $\iota_A : A \to \mathcal{V}_A(L)$ and a Lie algebra morphism $\iota_L : L \to \mathcal{V}_A(L)$ such that

$$\iota_L(a \cdot X) = \iota_L(X)\iota_A(a) \quad \text{and} \quad \iota_L(X)\iota_A(a) - \iota_A(a)\iota_L(X) = \iota_A(\omega(X)(a)) \tag{3.48}$$

for all $a \in A$ and $X \in L$, which is universal with respect to this property. That is, if (W, ϕ_A, ϕ_L) is another algebra with an algebra morphism $\phi_A : A \to W$ and a Lie algebra morphism $\phi_L : L \to W$ such that

$$\phi_L(a \cdot X) = \phi_L(X)\phi_A(a) \quad \text{and} \quad \phi_L(X)\phi_A(a) - \phi_A(a)\phi_L(X) = \phi_A(\omega(X)(a)),$$

then there exists a unique algebra morphism $\Phi : \mathcal{V}_A(L) \to W$ such that $\Phi \circ \iota_A = \phi_A$ and $\Phi \circ \iota_L = \phi_L$.

Notice that the first relation in (3.48) says that ι_L has to be *right* A-linear and the compatibility condition $\Phi \circ \iota_A = \phi_A$ says that in fact Φ is an A-ring map. There exists also a notion of *left* universal enveloping algebra in which we require ι_L to be *left* A-linear (see e.g. [CG]).

Remark 3.2.9. A *left representation* of a Lie-Rinehart (A, L, ω) is an A-module V together with a Lie algebra map $\rho : L \to \mathsf{End}_\Bbbk(V)$ such that

$$\rho\,(a \cdot X)\,(v) = a \cdot \rho\,(X)\,(v), \qquad \rho\,(X)\,(a \cdot v) = \omega\,(X)\,(a) \cdot v + a \cdot \rho\,(X)\,(v)$$

for all $a \in A$, $X \in L$ and $v \in V$ (see e.g. [Hu2, Hu3]). A *right representation* of (A, L, ω) is a left representation of the *opposite* Lie-Rinehart algebra $(A, L^{\mathrm{op}}, -\omega)$. That is, an A-module V with a $\Bbbk$-linear map $\rho\colon L \to \mathsf{End}_\Bbbk(V)$ such that

$$\rho\,([X,Y])\,(v) = \rho(Y)\big(\rho(X)(v)\big) - \rho(X)\big(\rho(Y)(v)\big), \qquad \rho\,(a \cdot X)\,(v) = a \cdot \rho\,(X)\,(v),$$
$$\rho\,(X)\,(a \cdot v) = -\omega\,(X)\,(a) \cdot v + a \cdot \rho\,(X)\,(v)\,.$$

As one may expect, right (resp. left) representations of (A, L, ω) are in one-to-one correspondence with right (resp. left) modules over the right (resp. left) universal enveloping algebra.

We point out that our definition of a right representation differs slightly from the one given in [Hu2, page 430]. The reason is threefold: first of all this is more symmetric, secondly this one ensures that A is a right representation as much naturally as it is a left one, that is to say, via the anchor map ω, and thirdly because with this definition right representations correspond to right modules over the right universal enveloping algebra in a natural way.

The existence of the universal enveloping algebra has already been settled, in light of the well-known constructions of [Rin] and [MM2]. However, it admits other concrete realizations. In this thesis we opted for the following, which we consider easier to handle. Set $\eta : L \to A \otimes L$, $X \mapsto 1 \otimes X$ and consider the tensor A-ring $T_A\,(A \otimes L)$ of the A-bimodule $A \otimes L$. It can be shown that

$$\mathcal{V}_A\,(L) \cong \frac{T_A\,(A \otimes L)}{\mathcal{J}}$$

where $\mathcal{J}$ is the two sided ideal

$$\mathcal{J} := \left\langle \begin{array}{c} \eta\,(X) \otimes_A \eta\,(Y) - \eta\,(Y) \otimes_A \eta\,(X) - \eta\,([X,Y])\,, \\ \eta\,(X) \cdot a - a \cdot \eta\,(X) - \omega\,(X)\,(a) \end{array} \;\middle|\; X, Y \in L,\ a \in A \right\rangle$$

with algebra map $\iota_A : A \to \mathcal{V}_A\,(L)\,, a \mapsto a + \mathcal{J}$ and Lie map $\iota_L : L \to \mathcal{V}_A\,(L)\,, X \mapsto \eta\,(X) + \mathcal{J}$. They satisfy the compatibility conditions (3.48) and $T_A\,(A \otimes L)\,/\mathcal{J}$ satisfies the universal property.

Remark 3.2.10. By definition, $-\omega : L \to \mathsf{End}(A)^{\mathrm{op}}$ makes of A a right representation of (A, L, ω). For M, N two right representations, the assignments

$$m \otimes_A n \mapsto \big(\rho_M(X)(m) \otimes_A n + m \otimes_A \rho_N(X)(n)\big)$$

for X running in L make of $M \otimes_A N$ a right representation of (A, L, ω). In light of Remark 3.2.9, these observations entail that the category of right $\mathcal{V}_A(L)$-modules is monoidal in such a way that the forgetful functor $\mathcal{U} : \mathfrak{M}_{\mathcal{V}_A(L)} \to \mathfrak{M}_{A^e}$ is monoidal as well.

Now, in light of Remark 3.2.10 and of a right-handed version of e.g. [Sc2, Theorem 5.1] (see also [BM, Theorem 3.1] and [EKG, page 4]) it turns out that $\mathcal{V}_A(L)$ has to be a right bialgebroid. By a direct computation using its universal property, one sees that $\mathcal{V}_A(L)$ is in fact a cocommutative right A-Hopf algebroid with structure maps induced by the assignments

$$\begin{gathered}
\varepsilon\left(\iota_A(a)\right) = a, \qquad \varepsilon\left(\iota_L(X)\right) = 0, \\
\Delta\left(\iota_A(a)\right) = \iota_A(a) \otimes_A 1 = 1 \otimes_A \iota_A(a), \\
\Delta\left(\iota_L(X)\right) = \iota_L(X) \otimes_A 1 + 1 \otimes_A \iota_L(X), \\
\mathfrak{can}^{-1}\left(1 \otimes_A \iota_A(a)\right) = \iota_A(a) \otimes_A 1 = 1 \otimes_A \iota_A(a), \\
\mathfrak{can}^{-1}\left(1 \otimes_A \iota_L(X)\right) = 1 \otimes_A \iota_L(X) - \iota_L(X) \otimes_A 1,
\end{gathered}$$

for all $a \in A, X \in L$. The unique detail that perhaps is worth to highlight is the following. The assignments

$$a \mapsto \iota_A(a) \otimes_A 1 = 1 \otimes_A \iota_A(a) \quad \text{and} \quad X \mapsto 1 \otimes_A \iota_L(X) - \iota_L(X) \otimes_A 1$$

actually land into ${}^A\mathcal{V}_A(L)_A \times_A {}_A\mathcal{V}_A(L)^A$ (see Remark 3.2.6). This is a $\Bbbk$-algebra with componentwise product but opposite algebra structure on the left-hand factor, i.e. ${}^A\mathcal{V}_A(L)_A^{\mathrm{op}} \times_A {}_A\mathcal{V}_A(L)^A$. Thus it is fair to apply the universal property of $\mathcal{V}_A(L)$ to define the inverse of the canonical map.

As a consequence, we will sometimes refer to $\mathcal{V}_A(L)$ as the universal enveloping Hopf algebroid of (A, L, ω).

Example 3.2.11. Let $A = \Bbbk[X]$ be the algebra of polynomials in one indeterminate X over a field $\Bbbk$ of characteristic 0. Consider its associated first Weyl algebra $U := A[Y, \partial/\partial X]$, that is, its algebra of differential operators (see e.g. [Co, Theorem 2.3]). Then the pair (A, U) is a (left) cocommutative Hopf algebroid with structure maps

$$\Delta(Y) = 1 \otimes_A Y + Y \otimes_A 1, \quad \varepsilon(Y) = 0, \quad \sum Y_- \otimes_A Y_+ = 1 \otimes_A Y - Y \otimes_A 1.$$

In fact, it is the (left) universal enveloping algebra of the Lie-Rinehart algebra $(A, \mathsf{Der}_\Bbbk(A))$.

More generally, let $\mathcal{M}$ be a smooth real manifold and let $A := \mathcal{C}^\infty(\mathcal{M})$ be the algebra of smooth functions on $\mathcal{M}$. Up to isomorphism, $\mathsf{Der}_\Bbbk(A)$ is the Lie algebra of vector fields on $\mathcal{M}$, that is, $\mathsf{Der}_\Bbbk(A) \cong \Gamma(T\mathcal{M} \to \mathcal{M})$, the smooth global sections of the tangent bundle. Thus, up to isomorphism, $\mathcal{V}_A(\mathsf{Der}_\Bbbk(A))$ is the algebra (in fact, cocommutative Hopf algebroid) of (globally defined) differential operators on $\mathcal{M}$ (see e.g. [Hu3, page 64] or [Kp, Example (1.2.5)]).

3.3 The linear dual and the finite dual of a cocommutative Hopf algebroid

Let $\mathcal{L} \to \mathcal{M}$ be a Lie algebroid over the smooth manifold $\mathcal{M}$. Set $A := \mathcal{C}^\infty(\mathcal{M})$ and $L := \Gamma(\mathcal{L})$ and denote by $\mathcal{U}_A(L)$ the left universal enveloping algebra of the Lie-Rinehart algebra (A, L). Mimicking [MM2, Definition 1.3] we say that $\mathcal{U}_A(L)$ is the (left) universal enveloping algebra of $\mathcal{L}$. In [Kp, A.5.10] it is shown that ${}^*\mathcal{U}_A(L) = {}_A\mathsf{Hom}(\mathcal{U}_A(L), A)$ represents a formal groupoid which formally integrates $\mathcal{L}$ (even if the author explicitly decided to do not care about the antipode). However, as we mentioned in the Introduction, we are in the conditions to associate to $\mathcal{L}$ a *finite dual Hopf* *algebroid* ${}^{\circ}\mathcal{U}_A(L)$ as well. This section is dedicated to study the connection that exists between these two constructions. In fact, we will do this in a right-handed context, by considering the (right) universal enveloping Hopf algebroid $\mathcal{V}_A(L)$ of $\mathcal{L}$, its full right linear dual $\mathcal{V}_A(L)^* = \mathsf{Hom}_A(\mathcal{V}_A(L), A)$ and the right finite dual commutative Hopf algebroid $\mathcal{V}_A(L)^{\circ}$ in the sense of [EKG], for a matter of consistency with the works we are referring to. Nevertheless, we would like to point out that the results we are going to present admit a left-right symmetric version which allows one to apply them to the aforementioned geometric situation.

Up to this section, by a filtered (bi)module we meant a $\Bbbk$-module V endowed with a decreasing filtration

$$\cdots \subseteq F_n V \subseteq \cdots \subseteq F_1 V \subseteq F_0 V = V$$

plus other structures suitably compatible with this filtration. From this section on, we will also need the notion of an *increasingly* filtered (bi)module, which for us is a $\Bbbk$-module W endowed with an increasing filtration

$$0 \subseteq F^0 W \subseteq F^1 W \subseteq \cdots \subseteq F^n W \subseteq \cdots \subseteq W.$$

Referring to Subsection 3.1.1, this is nothing more than a $\mathbb{Z}$-filtered $\Bbbk$-module W where $F_{-1}W = 0$ and where we raised the subscripts to superscripts in order to distinguish this filtration from a decreasing one.

3.3.1 The complete commutative Hopf algebroid structure of the convolution algebra

Our task in this subsection is to show that the convolution algebra of a given (right) cocommutative Hopf algebroid endowed with an admissible filtration (see §3.3.1.1) and whose translation map is filtered is a complete Hopf algebroid in the sense of §3.2.1. At the level of suitably filtered (left) bialgebroids, this was already mentioned in [Kp, A.5]. However, it seems that the literature is lacking in a construction for an antipode. Here we will compensate for this by providing the explicit description of all the involved maps in the complete Hopf algebroid structure on the convolution algebra. Such a description will be also needed to prove the results of the forthcoming subsection. The prototype example which we have in mind and fulfils the above assumptions is the convolution algebra of the universal enveloping algebra of a Lie algebroid.

In order to make the exposition more flowing, we consigned some proofs to Appendix B.

3.3.1.1 Admissible filtrations on general rings.

Let U be an A-*ring* or, equivalently, let $A \to U$ be a ring extension[(11)]. Throughout this subsection we assume A to be discretely filtered (i.e. $F^n A = A$ for all $n \geq 0$)[(12)]. Mimicking [Kp, Definition A.5.1], we say that U has a *right admissible filtration* if $A \subset U$ (as a ring) and there is an increasing exhaustive filtration $U = \bigcup_{n\in\mathbb{N}} F^n U$ of A-subbimodules such that $F^0 U = A$, $F^n U \cdot F^m U \subseteq F^{n+m} U$ and each one of the quotient modules in $\left\{F^n U / F^{n-1} U\right\}_{n\geq 0}$ is a finitely generated and projective right A-module. In particular, any A-ring with a right admissible filtration is a locally finitely generated and projective A-module in the sense of Appendix B. We will denote by $\tau_n : F^n U \to U$ and by $\tau_{n,m} : F^n U \to F^m U$ the canonical inclusions for $m \geq n \geq 0$. Notice that U can be identified with the direct limit of the system $\{F^n U, \tau_{n,m}\}$, that is, $U = \varinjlim (F^n U)$. The subsequent Proposition 3.3.1 summarizes the properties of rings with an admissible filtration and we refer to Appendix B for a more detailed treatment in the framework of (increasingly) filtered bimodules.

Proposition 3.3.1. *Let U be an A-ring endowed with a right admissible filtration $\{F^n U \mid n \geq 0\}$. The following properties hold true*

(11) Even if we plan to apply the results of this subsection to a cocommutative Hopf algebroid (A, U), we think that these are interesting on their own and that this justifies the choice of presenting them in the present form.

(12) By referring to Subsection 3.1.1, this is the $\mathbb{Z}$-filtration on A given by $F_n A = A$ for all $n \geq 0$ and 0 otherwise, which is exactly the same $\mathbb{Z}$-filtration introduced in Remark 3.1.1 and that induces the discrete topology on A.

1. *Each of the structural maps $\tau_{n,n+1} : F^nU \to F^{n+1}U$ is a split monomorphism of right A-modules. In particular, each of the submodules F^nU is a finitely generated and projective right A-module and each one of the monomorphisms $\tau_n : F^nU \to U$ splits as map of right A-modules.*

2. *As a filtered right A-module, U satisfies*

$$U \cong gr(U) = A \oplus \frac{F^1U}{A} \oplus \frac{F^2U}{F^1U} \oplus \cdots \oplus \frac{F^nU}{F^{n-1}U} \oplus \cdots$$

In particular U is a faithfully flat right A-module.

Proof. Since U is locally finitely generated and projective as right A-module, the proposition follows from Lemma B.1, Corollary B.2 and Remark B.3 of the Appendices. The faithfully flatness is a consequence of the fact that U is the direct sum of the faithfully flat right A-module A and the flat right A-module $\bigoplus_{n\in\mathbb{N}} F^{n+1}U/F^nU$ (see e.g. [Bk2, Proposition 9, I §3]). □

For every $n \geq 0$, we will denote by θ_n the right A-linear retraction of τ_n.

Remark 3.3.2. Given a right admissible filtration $\{F^nU \mid n \geq 0\}$ on an A-ring U, it follows from Lemma B.1 that we have right A-linear isomorphisms $\psi_n : F^nU \cong \bigoplus_{k=0}^n F^kU/F^{k-1}U$ for all $n \geq 0$. Let us fix a dual basis $\{(u_i^n, \gamma_i^n) \mid i = 1, \ldots, d'_n\}$ for every $F^nU/F^{n-1}U$, $n \geq 0$. These induce a distinguished dual basis $\left\{(e_i^n, \lambda_i^n) \mid i = 1, \ldots, d_n := \sum_{j=0}^n d'_j\right\}$ on F^nU for all $n \geq 0$, which is given as follows. The generating set $\{e_i^n \mid i = 1, \ldots, d_n\}$ is given by $\{\psi_n^{-1}(u_i^k) \mid k = 0, \ldots, n, i = 1, \ldots, d'_k\}$, that is, $\psi_n^{-1}(u_i^k) = e_{i+d_{k-1}}^n$ for all $0 \leq k \leq n$ and all $1 \leq i \leq d'_k$ ($d_{-1} = 0$ by convention). The dual elements are given by first extending $\gamma_i^k : F^kU/F^{k-1}U \to A$ to $(\gamma')_i^k : \bigoplus_{k=0}^n F^kU/F^{k-1}U \to A$ letting $(\gamma')_i^k \mid_{F^hU/F^{h-1}U} = 0$ for $h \neq k$ and then composing with ψ_n, i.e. $(\gamma')_i^k \circ \psi_n = \lambda_{i+d_{k-1}}^n$ for all k, i as above.

This distinguished dual basis enjoys the following property which turns out to be quite useful. Let us denote by $j_{m,n} \colon \bigoplus_{h=0}^m F^hU/F^{h-1}U \to \bigoplus_{k=0}^n F^kU/F^{k-1}U$ the inclusion morphisms for $m \leq n$. Then $\psi_n \circ \tau_{m,n} = j_{m,n} \circ \psi_m$, whence $\tau_{m,n}(e_i^m) = e_i^n$ for all $i = 1, \ldots, d_m$ and $\lambda_i^n \circ \tau_{m,n} = \lambda_i^m$ if $i = 1, \ldots, d_m$, $\lambda_i^n \circ \tau_{m,n} = 0$ otherwise.

3.3.1.2 The convolution algebra of a Hopf algebroid with an admissible filtration

Let (A, U) be a cocommutative (right) Hopf algebroid. By mimicking [Kp, A.5.8], we say that (A, U) is endowed with a *(right) admissible filtration* if the A-ring U has a right admissible filtration $U = \bigcup_{n\in\mathbb{N}} F^nU$ as in §3.3.1.1 which is also compatible with the comultiplication, in the sense that

$$\Delta(F^nU) \subseteq \sum_{p+q=n} \mathsf{im}\,(F^pU_A \otimes_A F^qU_A) = \sum_{p+q=n} F^pU_A \otimes_A F^qU_A = F^n\,(U_A \otimes_A U_A)$$

(the counit is automatically filtered and the identification of $F^pU_A \otimes_A F^qU_A$ with its image $\mathsf{im}\,(F^pU_A \otimes_A F^qU_A)$ is fair since the $\tau_n \colon F^nU_A \to U_A$'s are split monomorphisms of right A-modules). The morphisms τ from Definition 3.2.5 plays the role of the inclusion $\tau_0 \colon A \to U$.

Example 3.3.3. Consider the universal enveloping Hopf algebroid $U := \mathcal{V}_A(L)$ of a given Lie-Rinehart algebra (A, L, ω). Since the tensor A-ring $T_A\,(A \otimes L)$ is endowed with a natural increasing filtration

$$F^n\,(T_A\,(A \otimes_{\Bbbk} L)) := \bigoplus_{k=0}^{n} T_A\,(A \otimes_{\Bbbk} L)^k\,,$$

where $T_A\,(A \otimes_{\Bbbk} L)^k := (A \otimes_{\Bbbk} L) \otimes_A (A \otimes_{\Bbbk} L) \otimes_A \cdots \otimes_A (A \otimes_{\Bbbk} L)$ for k times, this induces a filtration on U via the canonical projection. Thus, the n-th term of the filtration F^nU is the A-subbimodule generated by the products of the images of elements of L in U of length at most n, that is to say,

$F^nU = A + \iota_L(L) + \cdots + \iota_L(L)^n$. If we assume as usual that A is discretely filtered then both $\iota_A : A \to U$ and $\varepsilon : U \to A$ are filtered. Moreover, from

$$\Delta\left(\iota_L(X)\right) = \iota_L(X) \otimes_A 1_U + 1_U \otimes_A \iota_L(X) \in \mathsf{im}\left(F^1U \otimes_A F^0U\right) + \mathsf{im}\left(F^0U \otimes_A F^1U\right) = F^1\left(U \otimes_A U\right)$$

it follows that

$$\Delta\left(F^nU\right) \subseteq \sum_{k=0}^{n} \Delta\left(\iota_L(L)^k\right) \subseteq \sum_{k=0}^{n} \Delta\left(\iota_L(L)\right)^k \subseteq \sum_{k=0}^{n} F^k\left(U \otimes_A U\right) \subseteq F^n\left(U \otimes_A U\right)$$

(notice that this makes sense since $\mathsf{im}(\Delta) \subseteq U \times_A U$, which is a filtered algebra with filtration induced by the one of $U \otimes_A U$). Summing up, U is what we may call a *filtered cocommutative bialgebroid*. Furthermore, if L is a projective A-module, then we have a graded isomorphism of A-algebras $\mathsf{gr}(U) \cong S_A(L)$, the symmetric algebra of L (see e.g. [Rin, Theorem 3.1]). From this, one deduces that if L is also finitely generated, then the quotient modules $F^nU/F^{n-1}U$ are finitely generated and projective as right A-modules. Therefore, under this additional hypothesis (which is always satisfied when $L = \Gamma(\mathcal{L})$ for a Lie algebroid $\mathcal{L}$ over a connected smooth manifold $\mathcal{M}$[13]), U turns out to be a cocommutative right Hopf algebroid endowed with an admissible filtration.

Lemma 3.3.4. *If (A, U) is a cocommutative Hopf algebroid endowed with an admissible filtration, then the canonical morphism can is a morphism of filtered $\Bbbk$-modules. Its inverse can^{-1} is filtered if and only if the translation map δ is.*

Proof. Endow $U_A \otimes_A {}_AU$ and $U_A \otimes_A U_A$ with the increasing filtrations

$$F^n\left(U_A \otimes_A {}_AU\right) = \sum_{p+q=n} \mathsf{im}\left(F^pU_A \otimes_A {}_AF^qU\right) \quad \text{and} \quad F^n\left(U_A \otimes_A U_A\right) = \sum_{p+q=n} F^pU_A \otimes_A F^qU_A.$$

Since the comultiplication is compatible with the filtrations, by applying can to each term of the canonical filtration of $U_A \otimes_A {}_AU$ we find out that

$$\begin{aligned}\mathsf{can}\Big(F^n\left(U_A \otimes_A {}_AU\right)\Big) &\subseteq \sum_{p+l+k=n} F^pU \cdot F^kU_A \otimes_A F^lU_A \subseteq \sum_{p+l+k=n} F^{p+k}U_A \otimes_A F^lU_A \\ &\subseteq \sum_{i+j=n} F^iU_A \otimes_A F^jU_A = F^n(U_A \otimes_A U_A).\end{aligned}$$

This means that can is a filtered morphism of $\Bbbk$-modules as claimed.

For the second claim, if can^{-1} is filtered then obviously δ is so. Conversely, assume that δ is filtered. We already know from Equation (3.46) that $\mathsf{can}^{-1}(u \otimes_A v) = \sum uv_- \otimes_A v_+$ for all $u, v \in U$, whence for every $n \geq 0$

$$\sum_{p+q=n} \mathsf{can}^{-1}\left(F^pU_A \otimes_A F^qU_A\right) \subseteq \sum_{p+h+k=n} \mathsf{im}\left(F^pU \cdot F^hU_A \otimes_A {}_AF^qU\right) \subseteq F^n\left(U_A \otimes_A {}_AU\right)$$

and so can^{-1} is filtered as well. □

In general, right A-linear maps $U_A \to A$ form the $\Bbbk$-module U^* which comes endowed with a structure of algebra given by the *convolution product*

$$f * g : U \to A, \quad \left(u \mapsto \sum f(u_1)g(u_2)\right). \tag{3.49}$$

In addition, this comes endowed with an algebra map

$$\vartheta : A \otimes A \to U^*, \quad \left(a \otimes a' \mapsto [u \mapsto \varepsilon(a'u)a]\right). \tag{3.50}$$

As a matter of notation, we will set $s_*(a) = \vartheta(a \otimes 1)$ and $t_*(a) = \vartheta(1 \otimes a)$ for all $a \in A$ and we will refer to them as the source and the target of U^* respectively.

[13] If $\mathcal{M}$ is a compact smooth manifold, this is the Swan part of the Serre-Swan Theorem. However, its proof can be generalized to arbitrary smooth manifolds (see e.g. [N, Theorem 11.32] and the subsequent Remark).

Remark 3.3.5. It may be useful to notice that the A-bimodule structure induced on U^* by ϑ coincides with the expected one, that is $a \triangleright f \triangleleft b = a \cdot f \leftharpoonup b$. Indeed, for all $f \in U^*$, $u \in U$ and all $a, b \in A$ we have that

$$\begin{aligned}(a \triangleright f \triangleleft b)(u) &= (\vartheta(a \otimes 1) * f * \vartheta(1 \otimes b))(u) = \sum \vartheta(a \otimes 1)(u_1) f(u_2) \vartheta(1 \otimes b)(u_3) \\ &= \sum a\varepsilon(u_1) f(u_2) \varepsilon(bu_3) = \sum af(u_2\varepsilon(u_1)\varepsilon(bu_3)) \overset{(3.44)}{=} \sum af(u_1\varepsilon(bu_2)) \\ &\overset{(3.41)}{=} \sum af(bu_1\varepsilon(u_2)) \overset{(3.44)}{=} af(bu).\end{aligned}$$

We are going to show now that the convolution algebra U^* of certain cocommutative right Hopf algebroids (A, U) with an admissible filtration is a complete Hopf algebroid, where the comultiplication $\Delta_* : U^* \to U^*_A \widehat{\otimes}_A {}_AU^*$ is induced by the multiplication of U and the counit is the map $\varepsilon_* \colon U^* \to A$ such that $f \mapsto f(1)$. The base algebra A is always assumed to be discretely filtered and U^* is considered as an A-bimodule via the source and the target maps induced by the algebra morphism $\vartheta : A \otimes A \to U^*$ of equation (3.50). The construction of the antipode for U^* will require an additional hypothesis and it will be treated separately in §3.3.1.4.

First of all, notice that $U^* \cong \varprojlim((F^nU)^*)$ as A-bimodules via the isomorphism

$$\begin{array}{ccc} U^* & \overset{\Phi}{\rightleftarrows} & \varprojlim((F^nU)^*) \\ f & \longmapsto & (\tau_n^*(f))_{n\geq 0} \\ g := \varinjlim(g_n) & \longleftarrow & (g_n)_{n\geq 0} \end{array} \tag{3.51}$$

and it can be endowed with a natural decreasing filtration

$$F_0(U^*) := U^* \quad \text{and} \quad F_{n+1}(U^*) := \mathsf{ker}(\tau_n^*) = \mathsf{Ann}(F^nU) \tag{3.52}$$

where $\mathsf{Ann}(F^nU) = \{f \in U^* \mid f(F^nU) = 0\}$ (see §B.2 for the general case). Moreover, this filtration enjoys the following central property: for every $n \geq 0$ we have an isomorphism of A-bimodules $U^*/F_{n+1}(U^*) \cong (F^nU)^*$ (see Remark B.3). Hence U^* is a complete A-bimodule and so a complete $\Bbbk$-module as well. Besides, U^* is a projective limit of $(A \otimes A)$-algebras as well, where the projective system $\{(F^nU)^* \mid n \in \mathbb{N}\}$ is endowed with the algebra maps $\vartheta_n = (\tau_n)^* \circ \vartheta : A \otimes A \to (F^nU)^*$ and where $(F^nU)^*$ is the convolution algebra of the A-coring $(F^nU)_A$.

Lemma 3.3.6. *The pair $(U^*, F_n(U^*))$ gives a complete A-bimodule as well as a complete algebra.*

Proof. We already observed that U^* is a complete A-bimodule as well as a complete $\Bbbk$-module. Thus, in light of Remark 3.1.31, it is enough for us to prove that the filtration $F_n(U^*)$ is compatible with the convolution product to conclude the proof. Notice that the $\mathsf{Ann}(F^nU)$'s are ideals, whence we have that $F_n(U^*) * F_m(U^*) \subseteq F_{n+m}(U^*)$ whenever n or m is 0. If $mn > 0$ then, given $f \in F_n(U^*)$ and $g \in F_m(U^*)$, we have that

$$(f * g)(F^{m+n-1}U) \subseteq \sum_{p+q=n+m-1} f(F^pU)\, g(F^qU) = 0,$$

because whenever $p \geq n$ it happens that $q = m + n - 1 - p \leq m - 1$ and hence g vanishes on F^qU. Therefore, $f * g \in \mathsf{Ann}(F^{n+m-1}U) = F_{n+m}(U^*)$ and hence $F_n(U^*) * F_m(U^*) \subseteq F_{n+m}(U^*)$ for all $m, n \geq 0$. Once recalled that we consider A discretely filtered, it is clear that $F_n(U^*)$ induces on U^* a filtration as a algebra and as an A-bimodule at the same time. $\square$

Remark 3.3.7. We already know that the convolution algebra U^* is an augmented one and the augmentation is given by the algebra map (which is going to be the counit) $\varepsilon_* : U^* \to A$, $f \mapsto f(1)$. Therefore, one can consider the $\mathcal{I}$-adic topology on U^* with respect to the two-sided ideal $\mathcal{I} := \mathsf{ker}(\varepsilon_*)$. If we compare this with the filtration (3.52), we see that $\mathcal{I} = F_1(U^*)$ and so $\mathcal{I}^n \subseteq F_n(U^*)$, for every $n \geq 0$. Thus the $\mathcal{I}$-adic topology is finer than the linear topology obtained from the filtration $\{F_n(U^*) \mid n \in \mathbb{N}\}$ of Equation (3.52).

3.3.1.3 The comultiplication and the counit of U^*

Next we want to show that the multiplication $m\colon U\otimes U\to U$ induces a comultiplication $\Delta_*\colon U^*\to U^*\widehat{\otimes}_A U^*$ which endows U^* with a structure of comonoid in the monoidal category of complete bimodules $\left({}_A\mathfrak{M}^{\mathfrak{c}}_A,\widehat{\otimes}_A, A\right)$. We keep the conventions of §3.3.1.1 and we will often make use of the notations introduced in Notation 3.1.20 and 3.1.24.

Let us perform the tensor product $U_A\otimes_A {}_AU$. The multiplication $m: U\otimes U\to U$ which gives the algebra structure to U factors through the tensor product over A

$$m\left(x\tau_0(a)\otimes y\right)=x\tau_0(a)y=m(x\otimes\tau_0(a)y),$$

and it is A-linear with respect to both regular A-actions on $U_A\otimes_A {}_AU$, namely

$$a(x\otimes_A y)=(\tau_0(a)x)\otimes_A y\quad\text{and}\quad (x\otimes_A y)a=x\otimes_A(y\tau_0(a)).$$

Therefore, it induces a filtered A-bilinear morphism $m^*: U^*\to(U\otimes_A U)^*$ and A-bilinear maps $m_{q,p}: F^qU_A\otimes_A {}_AF^pU\to F^{q+p}U$, which dually give rise to a family of morphisms of A-bimodules

$$\Delta_{q,p}: U^*\xrightarrow{\tau_{p+q}^*}\left(F^{p+q}U\right)^*\xrightarrow{m_{q,p}^*}\left(F^qU_A\otimes_A {}_AF^pU\right)^*,$$

such that $\Delta_{q,p}(f)\left(x\otimes_A y\right)=f(xy)$ for every $q,p\in\mathbb{N}$ and for all $f\in U^*$, $x\in F^qU$ and $y\in F^pU$. Given $f\in U^*$, for each element $u\in U$ we define $f\leftharpoonup u: U_A\to A_A$ to be the linear map which acts as $v\mapsto f(uv)$.

Lemma 3.3.8. *For any $f\in U^*$ and for all $q,p\in\mathbb{N}$, we have*

$$\left(\phi_{p,q}^{-1}\circ\Delta_{q,p}\right)(f)=\sum_{k=1}^{d_q}\tau_p^*\left(f\leftharpoonup\tau_q\left(e_k^q\right)\right)\otimes_A\lambda_k^q\in\left(F^pU\right)^*{}_A\otimes_A {}_A\left(F^qU\right)^* \tag{3.53}$$

where $\{e_k^q,\lambda_k^q\mid k=1,\cdots,d_q\}$ is the dual basis of $(F^qU)_A$ given in Remark 3.3.2 and the morphisms $\phi_{p,q}:\left(F^pM^*\right)_A\otimes_A {}_A\left(F^qM^*\right)\cong\left(F^qM_A\otimes_A {}_AF^pM\right)^*$ are canonical isomorphisms.

Proof. The existence of the canonical isomorphisms $\phi_{p,q}$ descends from Corollary B.2. The proof then follows by applying $\phi_{p,q}$ to both sides of (3.53). □

Recall that since U is endowed with an admissible filtration, it is an A-bimodule which is locally finitely generated and projective on the right and so we may apply, in particular, the results from §B.3. It follows then, from Lemma 3.3.8 and from the fact that $(U\otimes_A U)^*\cong U^*\widehat{\otimes}_A U^*$ as filtered bimodules via the completion of the canonical map

$$\begin{aligned}(U^*)_A\otimes_A {}_A(U^*)&\xrightarrow{\phi_{U,U}}\left(U_A\otimes_A {}_AU\right)^*\\ f\otimes_A g&\longmapsto[x\otimes_A y\mapsto f(g(x)y)]\end{aligned}$$

(see Proposition B.7), that we have an A-bilinear comultiplication

$$\Delta_*:=\psi_{U,U}\circ m^*\colon U^*\to(U^*)_A\widehat{\otimes}_A {}_A(U^*)$$

which makes the following diagram commute

$$\begin{tikzcd}
U^* \arrow[rrrr, dashed, "\Delta_*"] \arrow[ddd, "\tau_{p+q}^*"'] \arrow[ddrr, "m^*"] & & & & U^*\widehat{\otimes}_A U^* \arrow[ddd, "\Pi_{p,q}"] \arrow[ddll, shift left, "\psi_{U,U}"] \\
& & & & \\
& & (U\otimes_A U)^* \arrow[uurr, shift left, "\widehat{\phi_{U,U}}"] \arrow[d, "(\tau_q\otimes_A\tau_p)^*"] & & \\
(F^{q+p}U)^* \arrow[rr, "m_{q,p}^*"] & & (F^qU\otimes_A F^pU)^* & & (F^pU)^*\otimes_A(F^qU)^* \arrow[ll, "\phi_{p,q}"']
\end{tikzcd} \tag{3.54}$$

for all $p, q \geq 0$. The projections $\Pi_{p,q}$ (see Lemma B.14) are defined in such a way that $U^* \widehat{\otimes}_A U^*$ becomes the limit of the projective system $\left\{(F^m U)^* \otimes_A (F^n U)^*, \tau^*_{p,m} \otimes_A \tau^*_{q,n}\right\}_{\mathbb{N}^2}$ where the maps $\tau_{q,n} : F^q U \to F^n U$ are the canonical inclusions for $n \geq q$. Furthermore, the comultiplication Δ_* is uniquely determined by the following rule: for every $f \in U^*$,

$$\Delta_*(f) = \lim_{n\to\infty} \left(\sum f_{(1),n} \otimes_A f_{(2),n}\right) \Leftrightarrow \left[f(uv) = \lim_{n\to\infty} \left(\sum f_{(1),n}\big(f_{(2),n}(u)v\big)\right), \forall\, u, v \in U\right] \tag{3.55}$$

(it is enough to apply $\widehat{\phi_{U,U}}$ to both sides of the left-hand equality).

Remark 3.3.9. Thanks to relation (B.25) of the Appendices and (3.53) and by resorting to the notations introduced in Remark 3.3.2, one may write explicitly

$$\Delta_*(f) = \lim_{n\to\infty} \left(\sum_{i=1}^{d_n} \big(f \leftharpoonup \tau_n(e_i^n)\big) \otimes_A E_{\lambda_i^n}\right), \tag{3.56}$$

where we set $E_{\lambda_i^n} := \theta_n^*(\lambda_i^n)$. Indeed,

$$\begin{aligned}
\Delta_*(f) &\overset{(B.25)}{=} \lim_{n\to\infty} \left(\xi_{n,n}\left(\Pi_{n,n}\left(\Delta_*(f)\right)\right)\right) \overset{(3.53)}{=} \lim_{n\to\infty} \left(\xi_{n,n}\left(\sum_{k=1}^{d_n} \tau_n^*\big(f \leftharpoonup \tau_n(e_k^n)\big) \otimes_A \lambda_k^n\right)\right) \\
&= \lim_{n\to\infty} \left(\sum_{i=1}^{d_n} (\theta_n^* \circ \tau_n^*)\big(f \leftharpoonup \tau_n(e_i^n)\big) \otimes_A \theta_n^*(\lambda_i^n)\right) = \lim_{n\to\infty} \left(\sum_{i=1}^{d_n} \big(f \leftharpoonup \tau_n(e_i^n)\big) \otimes_A E_{\lambda_i^n}\right)
\end{aligned}$$

(recall that for $g \in U^*$ we have that $g - (\theta_n^* \circ \tau_n^*)(g) \in \mathsf{ker}(\tau_n^*) = F_{n+1}(U^*)$).

Now we can state the subsequent lemma.

Lemma 3.3.10. *Endow $U^* \widehat{\otimes}_A U^*$ with the projective limit (decreasing) filtration (see e.g. (3.7) or adapt (3.13)) and A with the discrete one. Then the comultiplication $\Delta_* \colon U^* \to U^* \widehat{\otimes}_A U^*$ and the counit $\varepsilon_* \colon U^* \to A$, $f \mapsto f(1_U)$, are morphisms of filtered A-bimodules. Moreover, they are morphisms of complete algebras as well.*

Proof. Both properties for ε_* are easy checks, thus we will focus on the comultiplication only. By definition of Δ_*, the first claim follows from the fact that $(U \otimes_A U)^* \cong U^* \widehat{\otimes}_A U^*$ is a filtered isomorphism and that the transpose of a filtered morphism of increasingly filtered modules is filtered with respect to the induced decreasing filtrations (3.52) on the duals. To show that Δ_* is unital, recall first that the unit of U^* is the counit $\varepsilon = 1_{U^*}$ of U and the unit of $U^* \otimes_A U^*$ is $\varepsilon \otimes_A \varepsilon = 1_{U^* \otimes_A U^*}$, so $1_{U^* \widehat{\otimes_A} U^*} = \widehat{\varepsilon \otimes_A \varepsilon}$ (the notation is that of Notation 3.1.24). Since

$$\begin{aligned}
\widehat{\phi_{U,U}}\left(\widehat{\varepsilon \otimes_A \varepsilon}\right)(u \otimes_A v) &= \widehat{\phi_{U,U}}\left(\gamma_{U^* \otimes_A U^*}(\varepsilon \otimes_A \varepsilon)\right)(u \otimes_A v) = \phi_{U,U}(\varepsilon \otimes_A \varepsilon)(u \otimes_A v) \\
&= \varepsilon(\varepsilon(u)v) \overset{(3.40)}{=} \varepsilon(uv)
\end{aligned}$$

it follows from (3.55) that $\Delta_*(\varepsilon) = \widehat{\varepsilon \otimes_A \varepsilon}$.

Now consider $\left(U^* \widehat{\otimes}_A U^*\right) \widehat{\otimes} \left(U^* \widehat{\otimes}_A U^*\right) \to \left(U^* \widehat{\otimes}_A U^*\right)$, the multiplication of the complete algebra $U^* \widehat{\otimes}_A U^*$, which is, up to the isomorphisms of Lemma 3.1.38, the completion of the factor-wise multiplication $(U^* \otimes_A U^*) \otimes (U^* \otimes_A U^*) \to (U^* \otimes_A U^*)$, $(x \otimes_A y) \otimes (x' \otimes_A y') \mapsto x * x' \otimes_A y * y'$. Denote by $\boldsymbol{m} : \left(U^* \widehat{\otimes}_A U^*\right) \otimes \left(U^* \widehat{\otimes}_A U^*\right) \to \left(U^* \widehat{\otimes}_A U^*\right)$ the associated filtered multiplication. In view of the adjunction (3.33) and of the commutativity of the following diagram

$$\begin{tikzcd}
 & U^* \otimes U^* \arrow[rr, "\Delta_* \otimes \Delta_*"] \arrow[dl, "\gamma"'] & & (U^* \widehat{\otimes}_A U^*) \otimes (U^* \widehat{\otimes}_A U^*) \arrow[dl, "\gamma"'] \arrow[ddl, "\boldsymbol{m}"] \\
U^* \widehat{\otimes} U^* \arrow[rr, "\Delta_* \widehat{\otimes} \Delta_*"] \arrow[d, "\widehat{*}"'] & & (U^* \widehat{\otimes}_A U^*) \widehat{\otimes} (U^* \widehat{\otimes}_A U^*) \arrow[d, "\widehat{\boldsymbol{m}}"] & \\
U^* \arrow[rr, "\Delta_*"'] & & U^* \widehat{\otimes}_A U^* &
\end{tikzcd}$$

(where we simplified the notation as much as we could), to prove that Δ_* is multiplicative it is enough to show that $\Delta_*(f*g) = \boldsymbol{m}\left(\Delta_*(f)\otimes\Delta_*(g)\right)$, for every $f,g\in U^*$. In view of (3.54), to show the last equality amounts to check that $\Pi_{p,q}\left(\Delta_*(f*g)\right) = \Pi_{p,q}\left(\boldsymbol{m}\left(\Delta_*(f)\otimes\Delta_*(g)\right)\right)$ for all $p,q\in\mathbb{N}$. By employing the notation of (3.56), we know that

$$\Delta_*(f*g) = \lim_{n\to\infty}\left(\sum_{k=1}^{d_n}\left((f*g)\leftharpoonup \tau_n\left(e_k^n\right)\right)\otimes_A E_{\lambda_k^n}\right), \tag{3.57}$$

$$\boldsymbol{m}\left(\Delta_*(f)\otimes\Delta_*(g)\right) = \lim_{n\to\infty}\left(\sum_{i,j,=1}^{d_n,d_n}\left(f\leftharpoonup \tau_n\left(e_i^n\right)\right)*\left(g\leftharpoonup\tau_n\left(e_j^n\right)\right)\otimes_A\left(E_{\lambda_i^n}*E_{\lambda_j^n}\right)\right). \tag{3.58}$$

Let $x\in F^qU$, $y\in F^pU$ for some $p+q=n$ and set $k=n+1=p+q+1$. We compute

$$\begin{aligned}
&\phi_{p,q}\left(\Pi_{p,q}\Big(\boldsymbol{m}\left(\Delta_*(f)\,\widehat{\otimes}_A\,\Delta_*(g)\right)\Big)\right)(x\otimes_A y)\\
&\overset{(B.24)}{=} \sum_{i,j}^{d_k,d_k}\left(\left(f\leftharpoonup\tau_k\left(e_j^k\right)\right)*\left(g\leftharpoonup\tau_k\left(e_j^k\right)\right)\right)\left(\left(E_{\lambda_i^k}*E_{\lambda_j^k}\right)\left(\tau_q(x)\right)\tau_p(y)\right)\\
&\overset{(**)}{=} \sum\sum_{i,j}^{d_k,d_k}\left(\left(f\leftharpoonup e_i^k\right)*\left(g\leftharpoonup e_j^k\right)\right)\left(E_{\lambda_i^k}(x_1)E_{\lambda_j^k}(x_2)y\right)\\
&\overset{(*)}{=} \sum\sum_{i,j}^{d_k,d_k} f\left(\left(e_i^kE_{\lambda_i^k}(x_1)\right)E_{\lambda_j^k}(x_2)y_1\right)g(e_j^ky_2)\\
&\overset{(\triangle)}{=} \sum\sum_{j}^{d_k} f\left(x_1E_{\lambda_j^k}(x_2)y_1\right)g(e_j^ky_2) \overset{(3.41)}{=} \sum f(x_1y_1)g\left(\sum_j^{d_k}(e_j^kE_{\lambda_j^k}(x_2))y_2\right)\\
&\overset{(\triangle)}{=} \sum f(x_1y_1)g(x_2y_2) = \sum f((xy)_1)g((xy)_2)\\
&= (f*g)(xy) \overset{(3.54)}{=} \phi_{p,q}\left(\Pi_{p,q}\left(\Delta_*(f*g)\right)\right)(x\otimes_A y)
\end{aligned}$$

where in $(*)$ we used the left A-linearity of Δ and from $(**)$ up to the end of the computation, we omitted the inclusions τ_h's. The equalities $(\triangle)$ follow from the fact that Δ is compatible with the filtration and from the following computation. Let $x\in F^pU$ and pick $k\geq p$, then in light of Remark 3.3.2

$$\begin{aligned}
\sum_{i=1}^{d_k}\tau_k\left(e_i^k\right)E_{\lambda_i^k}\left(\tau_p(x)\right) &= \sum_{i=1}^{d_k}\tau_k\left(e_i^k\right)\lambda_i^k\left(\theta_k\left(\tau_p(x)\right)\right) \overset{(B.8)}{=} \sum_{i=1}^{d_k}\tau_k\left(e_i^k\right)\lambda_i^k\left(\tau_{p,k}(x)\right)\\
&= \tau_k\left(\sum_{i=1}^{d_p}\tau_{p,k}\left(e_i^p\right)\lambda_i^p(x)\right) = \tau_p(x).
\end{aligned}$$

In conclusion, we have

$$\Pi_{p,q}\Big(\boldsymbol{m}\left(\Delta_*(f)\otimes\Delta_*(g)\right)\Big) = \Pi_{p,q}\left(\Delta_*(f*g)\right),$$

for every $p,q\geq 0$, whence Δ_* is multiplicative as well. □

Proposition 3.3.11. *Let (A,U) be a cocommutative Hopf algebroid with U endowed with an admissible filtration $\{F^nU \mid n\geq 0\}$. Then $(U^*,\Delta_*,\varepsilon_*)$ is a coalgebra in the monoidal category $\left({}_A\mathfrak{M}^{\mathsf{c}}_A,\ \widehat{\otimes}_A, A\right)$ of complete A-bimodules.*

Proof. We already know from Lemma 3.3.6 that $U^*\cong \varprojlim\left(U^*/\mathsf{Ann}\left(F^nU\right)\right)$ is a complete A-bimodule and from Lemma 3.3.10 that the maps Δ_* and ε_* are A-bilinear and filtered.

Let us prove then that Δ_* is coassociative and counital, with counit ε_*. Let us begin with counitality. Since ε_* is filtered, $\varepsilon_* \otimes_A U^*$ is filtered and hence we have $\varepsilon_* \widehat{\otimes}_A U^* \colon U^* \widehat{\otimes}_A U^* \to U^*$ which acts as

$$\left(\varepsilon_* \widehat{\otimes}_A U^*\right) \left(\lim_{n\to\infty} \left(\sum_{i=1}^{r_n} f_i^{(n)} \otimes_A g_i^{(n)} \right) \right) = \lim_{n\to\infty} \left(\sum_{i=1}^{r_n} f_i^{(n)} (1_U) \cdot g_i^{(n)} \right).$$

Applying this formula to $\Delta_*(f)$ for any $f \in U^*$, we get

$$\begin{aligned}\left(\varepsilon_* \widehat{\otimes}_A U^*\right) (\Delta_*(f)) &\overset{(3.56)}{=} \lim_{n\to\infty} \left(\sum_{i=1}^{d_n} f(\tau_n(e_i^n)) \cdot E_{\lambda_i^n} \right) = \lim_{n\to\infty} \left(\sum_{i=1}^{d_n} f(\tau_n(e_i^n)) \cdot \lambda_i^n \theta_n \right) \\ &= \lim_{n\to\infty} (f\tau_n\theta_n) = f,\end{aligned}$$

because the dual basis $\{e_i^n, \lambda_i^n \mid i = 1 \cdots d_n\}$ of F^nU is the one introduced in Remark 3.3.2 and $f\tau_n\theta_n - f \in \mathsf{Ann}(F^{n-1}U) = F_nU^*$. This shows that $\left(\varepsilon_* \widehat{\otimes}_A U^*\right) \circ \Delta_* = \mathsf{Id}_{U^*}$. Analogously, we obtain $\left(U^* \widehat{\otimes}_A \varepsilon_*\right) \circ \Delta_* = \mathsf{Id}_{U^*}$.

Finally, we have to check the coassociativity of the comultiplication. For a given $f \in U^*$,

$$\begin{aligned}\left(\Delta_* \widehat{\otimes}_A U^*\right) (\Delta_*(f)) &= \lim_{n\to\infty} \left(\sum \left(\left(\lim_{k\to\infty} \left(\sum (f_{(11),n,k} \otimes_A f_{(12),n,k}) \right) \right) \otimes_A f_{(2),n} \right) \right), \\ \left(U^* \widehat{\otimes}_A \Delta_*\right) (\Delta_*(f)) &= \lim_{n\to\infty} \left(\sum \left(f_{(1),n} \otimes_A \left(\lim_{k\to\infty} \left(\sum (f_{(21),n,k} \otimes_A f_{(22),n,k}) \right) \right) \right) \right)\end{aligned}$$

and we need to prove that

$$\left(U^* \widehat{\otimes}_A \Delta_*\right) (\Delta_*(f)) = \boldsymbol{\alpha}_{U^*,U^*,U^*} \left(\left(\Delta_* \widehat{\otimes}_A U^*\right) (\Delta_*(f)) \right)$$

where $\boldsymbol{\alpha}_{U^*,U^*,U^*}$ is the associativity constraint induced by the one in (3.26). It descends from Theorem 3.1.35 that $\widehat{(-)} : {}_A\mathfrak{M}_A^{\mathsf{flt}} \to {}_A\mathfrak{M}_A^{\mathsf{c}}$ is monoidal with the structure isomorphisms given in Lemma 3.1.38. In particular, the following diagram commutes

$$\begin{tikzcd}
\left(U^* \widehat{\otimes}_A U^*\right) \widehat{\otimes}_A U^* \arrow[r, "\varphi_{U^*,U^*} \widehat{\otimes}_A U^*"] \arrow[d, "\boldsymbol{\alpha}_{U^*,U^*,U^*}"'] & (\widehat{U^* \otimes_A U^*}) \widehat{\otimes}_A U^* \arrow[r, "\varphi_{U^* \otimes_A U^*, U^*}"] & (U^* \otimes_A \widehat{U^*) \otimes_A} U^* \arrow[d, "\boldsymbol{\alpha}_{\widehat{U^*,U^*},U^*}"] \\
U^* \widehat{\otimes}_A \left(U^* \widehat{\otimes}_A U^*\right) \arrow[r, "U^* \widehat{\otimes}_A \varphi_{U^*,U^*}"] & U^* \widehat{\otimes}_A (\widehat{U^* \otimes_A U^*}) \arrow[r, "\varphi_{U^*,U^* \otimes_A U^*}"] & U^* \otimes_A \widehat{(U^* \otimes_A} U^*)
\end{tikzcd}$$

hence the coassociativity of Δ_* will follow once it will be shown that

$$\lim_{n\to\infty} \left(\sum f_{(11),n,n} \otimes_A f_{(12),n,n} \otimes_A f_{(2),n} \right) = \lim_{n\to\infty} \left(\sum f_{(1),n} \otimes_A f_{(21),n,n} \otimes_A f_{(22),n,n} \right) \tag{3.59}$$

Observe that, in light of (3.55), for all $u, v, w \in U$ we have

$$\begin{aligned}&\phi_{\widehat{U \otimes_A U},U} \left(\phi_{\widehat{U,U} \otimes_A U^*} \right) \left(\lim_{n\to\infty} \left(\sum f_{(11),n,n} \otimes_A f_{(12),n,n} \otimes_A f_{(2),n} \right) \right) (u \otimes_A v \otimes_A w) \\ &\qquad = \lim_{n\to\infty} \left(\sum f_{(11),n,n} \Big(f_{(12),n,n}(f_{(2),n}(u)v)w \Big) \right) \overset{(3.55)}{=} f(u(vw)), \\ &\phi_{\widehat{U,U \otimes_A U}} \left(\widehat{U^* \otimes_A \phi_{U,U}} \right) \left(\lim_{n\to\infty} \left(\sum f_{(1),n} \otimes_A f_{(21),n,n} \otimes_A f_{(22),n,n} \right) \right) (u \otimes_A v \otimes_A w) \\ &\qquad = \lim_{n\to\infty} \left(\sum f_{(1),n} \Big(f_{(21),n,n}(f_{(22),n,n}(u)v)w \Big) \right) \overset{(3.55)}{=} f((uv)w).\end{aligned}$$

Comparing this last equations leads to equality (3.59) and then to the coassociativity of Δ_*. $\square$

3.3.1.4 An antipode for U^*

Now we proceed to construct an antipode for U^*, under the additional hypothesis that the translation map of U is a filtered morphism of algebras. Notice that we already know from Lemma 3.3.4 that can is filtered as a morphism of $\Bbbk$-modules, but nothing ensures *a priori* that can^{-1} is filtered as well. Nevertheless, such an assumption is always fulfilled in the case of the universal enveloping Hopf algebroid of a Lie-Rinehart algebra with finitely generated and projective module L_A, as the next example shows.

Example 3.3.12. Take (A, L, ω) a Lie-Rinehart algebra with L finitely generated and projective and $U = \mathcal{V}_A(L)$ its universal enveloping algebra as in Example 3.3.3. For every $k \geq 1$, the translation map δ satisfies

$$\delta\left(\iota_L(L)\right)^k \subseteq \Big(F^1\left(U \otimes_A U\right)\Big)^k \subseteq F^k\left(U \otimes_A U\right). \tag{3.60}$$

Then, in light of (c) of Remark 3.2.6 and in particular of (3.66) below, the following computation

$$\delta\left(F^n U\right) \subseteq \sum_{k=0}^{n} \delta\Big(\iota_L\left(L\right)^k\Big) \overset{(3.66)}{\subseteq} \sum_{k=0}^{n} \delta\left(\iota_L\left(L\right)\right)^k \overset{(3.60)}{\subseteq} \sum_{k=0}^{n} F^k\left(U \otimes_A U\right) \subseteq F^n\left(U \otimes_A U\right)$$

shows that δ is a filtered algebra map, which implies that can^{-1} is also filtered.

As a matter of terminology, if we have a cocommutative Hopf algebroid composed by filtered $\Bbbk$-algebras and filtered $\Bbbk$-algebra maps (translation map included), we may call it a *filtered cocommutative Hopf algebroid*. Now, getting back to the point, at the level of the algebra structure the antipode is provided by the following map

$$\mathcal{S}_*: U^* \to U^*, \quad \Big(f \longmapsto \Big[u \mapsto \sum \varepsilon\big(f(u_-)u_+\big)\Big]\Big), \tag{3.61}$$

where $\delta: U \to U_A \otimes_A {}_AU$, $u \mapsto \mathsf{can}^{-1}(1 \otimes_A u) = \sum u_- \otimes_A u_+$ is the translation map (compare with [Ko, §4.3] and [CGK, Theorem 5.1.1] for the case when U is finitely generated and projective right A-module). As it was shown in [Sc3, Proposition 3.7], the map δ enjoys a series of properties. Here we recall few of them, suitably adapted to our framework, which will be needed in the sequel. First recall from (b) of Remark 3.2.6 that $\mathsf{can}^{-1}(u \otimes_A v) = \sum uv_- \otimes_A v_+$. Then for all $u, v \in U$ and $a \in A$ we have

$$1 \otimes_A u = \sum u_- u_{+,1} \otimes_A u_{+,2} \qquad \text{in} \quad U_A \otimes_A U_A \tag{3.62}$$

$$\sum u_{1,-} \otimes_A u_{1,+} \otimes_A u_2 = \sum u_- \otimes_A u_{+,1} \otimes_A u_{+,2} \qquad \text{in} \quad (U_A \otimes_A {}_AU) \otimes_A U_A \tag{3.63}$$

$$\sum u_{+,-} \otimes_A u_- \otimes_A u_{+,+} = \sum u_{-,1} \otimes_A u_{-,2} \otimes_A u_+ \qquad \text{in} \quad U_A \otimes_A U_A \otimes_A U_A \tag{3.64}$$

$$\sum u_- u_+ = \tau_0(\varepsilon(u)) \qquad \text{in} \quad F^0 U = A \tag{3.65}$$

$$\sum (uv)_- \otimes_A (uv)_+ = \sum v_- u_- \otimes_A u_+ v_+ \qquad \text{in} \quad U_A \otimes_A {}_AU \tag{3.66}$$

$$a \otimes_A 1 \; = \; 1 \otimes_A a = \sum a_- \otimes_A a_+ \qquad \text{in} \quad U_A \otimes_A {}_AU. \tag{3.67}$$

In particular, by equation (3.66) we conclude that δ is an algebra map, viewed as a morphism

$$\delta\colon U \to {}^AU_A^{\mathrm{op}} \times_A {}_AU^A. \tag{3.68}$$

Remark 3.3.13. If $(A, \mathcal{V}_A(L))$ is the universal enveloping Hopf algebroid of a Lie-Rinehart algebra as in Example 3.3.12, then the translation map is filtered and hence the map $\mathcal{S}_*$ of equation (3.61) is filtered as well. Indeed, since

$$\sum \tau_n(u)_- \otimes_A \tau_n(u)_+ \in \sum_{p+q=n} \mathsf{im}\left(\tau_p \otimes_A \tau_q\right),$$

for every $u \in F^n U$ and $n \geq 0$, we have that

$$\big(\mathcal{S}_*(f)\big)(\tau_n(u)) \overset{(3.61)}{=} \sum \varepsilon\Big(f\big(\tau_n(u)_-\big)\tau_n(u)_+\Big) = 0$$

for all $f \in F_{n+1}(U^*) = \mathsf{Ann}\,(F^n U)$. Therefore, $\mathcal{S}_*(F_n(U^*)) \subseteq F_n(U^*)$, for every $n \geq 0$.

The subsequent lemma will be needed to show that $\mathcal{S}_*$ is multiplicative.

Lemma 3.3.14. *Let $f, g, h \in U^*$ and $u \in U$. Then we have*

$$\begin{aligned} \mathcal{S}_*(f * g)(u) &= \sum \mathcal{S}_*(f)\Big(g(u_-)u_+\Big) \\ \big(\mathcal{S}_*(f) * h\big)(u) = \big(h \leftharpoonup f(u_-)\big)(u_+) &= \Big(\big(\varepsilon \leftharpoonup f(u_-)\big) * h\Big)(u_+) \end{aligned} \tag{3.69}$$

Proof. We will implicitly use the co-commutativity of the comultiplication of U as well as the A-linearity of δ. Computing the left hand side of the first equality gives

$$\begin{aligned} \mathcal{S}_*(f * g)(u) &\overset{(3.61)}{=} \sum \varepsilon\Big((f * g)(u_-)u_+\Big) = \sum \varepsilon\Big(f(u_{-,1})g(u_{-,2})u_+\Big) \\ &\overset{(3.64)}{=} \sum \varepsilon\Big(f(u_{+,-})(g(u_-)u_{+,+})\Big) \overset{(3.61)}{=} \sum \mathcal{S}_*(f)\Big(g(u_-)u_+\Big), \end{aligned}$$

where in the last equality we used (3.61) and (3.67). This leads to the stated first equality.

As for the second one, we have

$$(\mathcal{S}_*(f) * h)(u) = \mathcal{S}_*(f)(u_1)h(u_2) \overset{(3.61)}{=} \varepsilon\Big(f(u_{1,-})u_{1,+}\Big)h(u_2) \overset{(3.63)}{=} \varepsilon\Big(f(u_-)u_{+,1}\Big)h(u_{+,2})$$

from which one deduces on the one hand that

$$(\mathcal{S}_*(f) * h)(u) = \varepsilon\Big(f(u_-)u_{+,1}\Big)h(u_{+,2}) = \Big((\varepsilon \leftharpoonup f(u_-)) * h\Big)(u_+)$$

and on the other hand that

$$(\mathcal{S}_*(f) * h)(u) = \varepsilon\Big(f(u_-)u_{+,1}\Big)h(u_{+,2}) = \varepsilon(u_{+,1})h(f(u_-)u_{+,2}) = h(f(u_-)u_+). \qquad \square$$

Proposition 3.3.15. *Let (A, U) be a cocommutative (right) Hopf algebroid endowed with an admissible filtration and assume δ is a filtered algebra map. Then the map $\mathcal{S}_*$ of equation (3.61) is a morphism of complete algebras such that $\mathcal{S}_* \circ s_* = t_*$ and $\mathcal{S}_* \circ t_* = s_*$.*

Proof. We need to check that $\mathcal{S}_*$ is multiplicative and that it exchanges the source with the target, as we already know that it preserves the filtration in view of Remark 3.3.13. Recall that the unit of U^* is given by $\vartheta : A \otimes A \to U^*$ sending $a \otimes a' \mapsto [u \mapsto a\varepsilon(a'u)]$. Given $a, a' \in A$ we have

$$\begin{aligned} \mathcal{S}_*(\vartheta(a \otimes a'))(u) &\overset{(3.61)}{=} \varepsilon\Big(\vartheta(a \otimes a')(u_-)u_+\Big) = \varepsilon\Big(\varepsilon(a'u_-)au_+\Big) = \varepsilon\Big(a'u_-au_+\Big) \overset{(3.68)}{=} \varepsilon\Big(u_-au_+a'\Big) \\ &= \varepsilon\Big(u_-au_+\Big)a' = \varepsilon\Big((au)_-(au)_+\Big)a' \overset{(3.65)}{=} \varepsilon(au)a', \end{aligned}$$

whence $\mathcal{S}_*(\vartheta(a \otimes a')) = \vartheta(a' \otimes a)$. Therefore, $\mathcal{S}_* \circ s_* = t_*$ and $\mathcal{S}_* \circ t_* = s_*$, where s_*, t_* are as in equation (3.50). Let us check that $\mathcal{S}_*$ is multiplicative. If we consider $f, g \in U^*$ and $u \in U$, we have

$$\Big(\mathcal{S}_*(f) * \mathcal{S}_*(g)\Big)(u) \overset{(3.69)}{=} \Big(\mathcal{S}_*(g) \leftharpoonup f(u_-)\Big)(u_+) = \mathcal{S}_*(g)\Big(f(u_-)u_+\Big) \overset{(3.69)}{=} \mathcal{S}_*(g * f)(u) = \mathcal{S}_*(f * g)(u).$$

This shows that $\mathcal{S}_*(f * g) = \mathcal{S}_*(f) * \mathcal{S}_*(g)$, concluding the proof. $\square$

The following is the main result of this subsection.

Proposition 3.3.16. *Let (A,U) and δ be as in Proposition 3.3.15. Then (A,U^*) is a complete Hopf algebroid with structure maps s_*, t_*, Δ_*, ε_* and $\mathcal{S}_*$. In particular, this is the case for the universal enveloping Hopf algebroid $U=\mathcal{V}_A(L)$ of a Lie-Rinehart algebra (A,L,ω), where L is a finitely generated and projective A-module.*

Proof. We only need to check that the algebra map $\mathcal{S}_*$ enjoys the properties for being an algebraic antipode since it is already filtered. Let $f\in U^*$ and take an arbitrary element $u\in U$. Then

$$\lim_{n\to\infty}\Big(\sum\big(f_{1,n}*\mathcal{S}_*(f_{2,n})\big)(u)\Big)=\lim_{n\to\infty}\Big(\sum\big(\mathcal{S}_*(f_{2,n})*f_{1,n}\big)(u)\Big)\overset{(3.69)}{=}\lim_{n\to\infty}\Big(\sum f_{1,n}\big(f_{2,n}(u_-)u_+\big)\Big)$$
$$\overset{(3.55)}{=}\sum f(u_-u_+)\overset{(3.65)}{=}f(\varepsilon(u))=f(1)\varepsilon(u)=s_*(\varepsilon_*(f))(u).$$

Therefore, for every $f\in U^*$, we have

$$\lim_{n\to\infty}\Big(\sum\big(f_{1,n}*\mathcal{S}_*(f_{2,n})\big)\Big)=s_*(\varepsilon_*(f)).$$

Now let us check that $\mathcal{S}_*^2=\mathsf{Id}_{U^*}$ which will be sufficient to claim that $\mathcal{S}_*$ is an antipode for U^*. Recall that $\delta: U_\tau\to U_A\otimes_A {}_AU_\tau$ is right A-linear with respect to the highlighted actions (in particular, $U_A\otimes_A {}_AU$ has the regular right action), so that we can consider the map

$$(\delta\otimes_A {}_AU)\circ\delta\colon U\to\left(U_A\otimes_A {}_AU\right)_A\otimes_A {}_AU,\quad\Big(u\mapsto\sum u_{-,-}\otimes_A u_{-,+}\otimes_A u_+\Big)$$

Let us compute the image of the element $\sum u_{-,-}\otimes_A u_{-,+}u_+\in U_A\otimes_A {}_AU$ by the map $\mathfrak{can}$

$$\begin{aligned}\sum\mathfrak{can}\big(u_{-,-}\otimes_A u_{-,+}u_+\big)&=\sum u_{-,-}\left(u_{-,+}u_+\right)_1\otimes_A\left(u_{-,+}u_+\right)_1\\&=\sum u_{-,-}u_{-,+,1}u_{+,1}\otimes_A u_{-,+,2}u_{+,2}\\&\overset{(3.62)}{=}\sum u_{+,1}\otimes_A u_-u_{+,2}=\sum u_{+,2}\otimes_A u_-u_{+,1}\\&\underset{(*)}{\overset{(3.62)}{=}}u\otimes_A 1=\mathfrak{can}(u\otimes_A 1),\end{aligned}$$

where the step $(*)$ is fair as we are considering U as an A-bimodule via the central action induced by the right one (as we do when we consider the A-coring (U_A,Δ,ε)), so that the switch morphism $U_A\otimes_A U_A\to U_A\otimes_A U_A$, $u\otimes_A v\mapsto v\otimes_A u$ is well-defined. Therefore, for every $u\in U$, we have

$$\sum u_{-,-}\otimes_A u_{-,+}u_+=u\otimes_A 1\in U_A\otimes_A {}_AU.\tag{3.70}$$

In this way, if we take a function $f\in U^*$ and an element $u\in U$, we get

$$\mathcal{S}_*^2(f)(u)\overset{(3.61)}{=}\sum\varepsilon\Big(\mathcal{S}_*(f)(u_-)u_+\Big)\overset{(3.61)}{=}\sum\varepsilon\Big(f(u_{-,-})u_{-,+}u_+\Big)\overset{(3.70)}{=}\varepsilon\big(f(u)1\big)=f(u),$$

whence $\mathcal{S}_*^2=\mathsf{Id}_{U^*}$ and this finishes the proof of the fact that $(U^*,*,\varepsilon,\Delta_*,\varepsilon_*,\mathcal{S}_*)$ is a complete Hopf algebroid. The particular case follows immediately from Remark 3.3.13. □

3.3.2 The main morphism of complete commutative Hopf algebroids

In this section we prove the existence of a morphism of complete Hopf algebroids connecting the completion of the so-called finite dual Hopf algebroid (to be recalled in §3.3.2.1) of a cocommutative Hopf algebroid (A,U) with admissible filtration and its full linear dual U^*. As before, a possible application is for $(A,U)=\big(\mathcal{C}^\infty(\mathcal{M}),\mathcal{V}_A(\Gamma(\mathcal{L}))\big)$, where $\mathcal{L}\to\mathcal{M}$ is a Lie algebroid over a smooth manifold $\mathcal{M}$. We hope that this construction, together with the approach presented in [Kp], may shed new light on the integration problem for Lie algebroids.

3.3.2.1 The finite dual of a cocommutative Hopf algebroid.

We recall from [EKG] the construction of what is known as *the finite dual Hopf algebroid*. Following [EKG], given a (right) cocommutative Hopf algebroid (A, U) we consider the category $\mathcal{A}_U$ of those right U-modules whose underlying right A-module structure is finitely generated and projective (sometimes called A-*profinite* right U-modules). This category is a symmetric rigid monoidal $\Bbbk$-linear category with identity object A, whose right U-action is given by $a \centerdot u = \varepsilon(au)$. The tensor product of two right U-modules M and N is the A-module $M \otimes_A N$ endowed with the following right U-action

$$(m \otimes_A n) \centerdot u = \sum (m \centerdot u_1) \otimes_A (n \centerdot u_2).$$

The dual object of a right U-module M belonging to $\mathcal{A}_U$ is the A-module $M^* = \mathsf{Hom}_A(M, A)$ with the right U-action

$$\varphi \centerdot u : M \to A, \quad \left(m \mapsto \sum \varphi(m \centerdot u_-) \centerdot u_+\right), \tag{3.71}$$

where as usual $\sum u_- \otimes_A u_+ = \mathsf{can}^{-1}(1 \otimes_A u)$.

Furthermore, the forgetful functor $\boldsymbol{\omega} : \mathcal{A}_U \to \mathsf{proj}(A)$ to the category of finitely generated and projective A-modules, which is a (non trivial) symmetric strict monoidal faithful functor, plays the role of a *fibre functor* as in the Tannaka reconstruction process performed in [Brg].

The commutative Hopf algebroid constructed from the data $(\mathcal{A}_U, \boldsymbol{\omega})$, will be denoted by (A, U°) and referred to as *the finite dual Hopf algebroid* of (A, U). If we set $T_{M,N} := \mathsf{Hom}_{\mathcal{A}_U}(M, N)$, $T_M := T_{M,M}$ and we denote by M and M^* the objects $\boldsymbol{\omega}(M)$ and $\boldsymbol{\omega}(M)^*$ in $\mathsf{proj}(A)$, then U° may be realized as the quotient algebra

$$U^\circ = \frac{\bigoplus_{M \in \mathrm{Ob}(\mathcal{A}_U)} M^* \otimes_{T_M} M}{\mathfrak{J}_{\mathcal{A}_U}} \tag{3.72}$$

by the two sided ideal $\mathfrak{J}_{\mathcal{A}_U}$ generated by the set

$$\Big\{ \left(\varphi \otimes_{T_N} f(m)\right) - \left((\varphi \circ f) \otimes_{T_M} m\right) \;\Big|\; \varphi \in N^*, m \in M, f \in T_{M,N}, M, N \in \mathrm{Ob}(\mathcal{A}_U) \Big\},$$

where we identified each element of the form $\varphi \otimes_{T_M} m \in M^* \otimes_{T_M} M$ with its image in the direct sum $\bigoplus_{M \in \mathrm{Ob}(\mathcal{A}_U)} M^* \otimes_{T_M} M$.

The structure maps of the finite dual Hopf algebroid (A, U°) are given as follows. Write $\overline{\varphi \otimes_{T_M} m}$ for the equivalence class of the image of an element of the form $\varphi \otimes_{T_M} m \in M^* \otimes_{T_M} M$, for some object $M \in \mathcal{A}_U$. Since all involved maps are linear, we will be dealing most of all just with elements of the form $\overline{\varphi \otimes_{T_M} m}$, by-passing the more general summation notation. Thus the structure maps on U° are given by

$$u_\circ : \Bbbk \to U^\circ, \; \left(1_\Bbbk \mapsto \overline{\mathsf{Id}_A \otimes_\Bbbk 1_A}\right), \qquad \eta_\circ : A \otimes A \to U^\circ, \; \left(a \otimes b \mapsto \overline{l_a \otimes_\Bbbk b}\right),$$

$$m_\circ : U^\circ \otimes U^\circ \to U^\circ, \; \left(\overline{\psi \otimes_{T_N} n} \otimes \overline{\varphi \otimes_{T_M} m} \mapsto \overline{(\psi \star \varphi) \otimes_{T_{M \otimes_A N}} (m \otimes_A n)}\right),$$

$$\Delta_\circ : U^\circ \to U^\circ \otimes_A U^\circ, \; \left(\overline{\varphi \otimes_{T_M} m} \mapsto \sum_{i=1}^{r} \overline{\varphi \otimes_{T_M} e_i} \otimes_A \overline{e_i^* \otimes_{T_M} m}\right),$$

$$\varepsilon_\circ : U^\circ \to A, \; \left(\overline{\varphi \otimes_{T_M} m} \mapsto \varphi(m)\right), \qquad \mathcal{S}_\circ : U^\circ \to U^\circ, \; \left(\overline{\varphi \otimes_{T_M} m} \mapsto \overline{\mathrm{ev}_m \otimes_{T_{M^*}} \varphi}\right).$$

where $l_a : A \to A$, $b \mapsto ab$, is the left multiplication by a, $\{e_i, e_i^* \mid i = 1, \ldots, d_M\}$ is a dual basis of M_A, $\mathrm{ev}_m : M^* \to A$ is the "evaluation at m" map and for every $\psi \in N^*$ and $\varphi \in M^*$, the map $\psi \star \varphi \colon M \otimes_A N \to A$ acts as $m \otimes_A n \mapsto \varphi(m)\, \psi(n)$.

Remark 3.3.17. In fact, every element in U° is of the form $\overline{\varphi \otimes_{T_M} m}$ for some M in $\mathcal{A}_U$, $m \in M$ and $\varphi \in M^*$. Indeed, assume that we have an element of the form $\overline{\varphi \otimes_{T_M} m} + \overline{\psi \otimes_{T_N} n}$. Consider $P := M \oplus N$ and $i_M : M \to P$, $i_N : N \to P$ the obvious inclusions. Set $\varphi \oplus \psi : P \to A$ for the codiagonal map of φ and ψ such that $(m,n) \mapsto \varphi(m) + \psi(n)$. Then

$$\begin{aligned}\overline{\varphi \otimes_{T_M} m} + \overline{\psi \otimes_{T_N} n} &= \overline{((\varphi \oplus \psi) \circ i_M) \otimes_{T_M} m} + \overline{((\varphi \oplus \psi) \circ i_N) \otimes_{T_N} n} \\ &= \overline{(\varphi \oplus \psi) \otimes_{T_P} i_M(m)} + \overline{(\varphi \oplus \psi) \otimes_{T_P} i_N(n)} \\ &= \overline{(\varphi \oplus \psi) \otimes_{T_P} (m,n)}.\end{aligned}$$

Notice that there is a linear map

$$\zeta : U^{\circ} \longrightarrow U^*, \quad \left(\overline{\varphi \otimes_{T_M} m} \longmapsto [u \mapsto \varphi(m \centerdot u)]\right). \tag{3.73}$$

The following lemma is a straightforward computation, see [EKG].

Lemma 3.3.18. *The linear map ζ is an homomorphism of $(A \otimes A)$-algebras.*

Remark 3.3.19. It is noteworthy to mention that the algebra map ζ, in contrast with the case of algebras over a field, is not known to be injective. However, if the base algebra A is a Dedekind domain for example, then ζ is injective for every U (see [EKG] for more details).

Example 3.3.20. Let $A = \mathbb{C}[X]$ and $U = A[Y, \partial/\partial X]$ as in the Example 3.2.11. The category $\mathcal{A}_U$ of right U-modules with finitely generated and projective (in fact, free of finite rank) underlying right A-module structure coincides with the category of differential modules over A. If we consider the forgetful functor $\mathcal{A}_U \to \mathsf{proj}(A)$ and we perform the Tannaka reconstruction process as in this section then the outcome is the Hopf algebroid U° and the canonical morphism ζ of Lemma 3.3.18 is injective by Remark 3.3.19.

3.3.2.2 The completion of the finite dual and the convolution algebra

Let (A, U) be a filtered cocommutative (right) Hopf algebroid and consider its finite dual (A, U°) as a commutative Hopf algebroid with structure maps given in §3.3.2.1. Here we assume that U is endowed with an admissible (increasing) filtration $\{F^n U\}_{n \in \mathbb{N}}$ such that the translation map is filtered, as in §3.3.1.4. The admissible filtration on the Hopf algebroid U induces a filtration on the convolution algebra U^* as given in (3.52) of §3.3.1.2. As we already showed, (A, U^*) with this filtration is a complete Hopf algebroid with structure maps explicitly given along §3.3.1.

Proposition 3.3.21. *Let (A, U) be a cocommutative (right) Hopf algebroid with an admissible filtration and consider its finite dual (A, U°). Set $\mathcal{I} := \mathsf{ker}\,(\varepsilon_{\circ} : U^{\circ} \to A)$, that is the kernel of the counit of U°. Then the canonical map $\zeta : U^{\circ} \to U^*$ of equation (3.73) is filtered with respect to the filtrations $F_n\,(U^{\circ}) = \mathcal{I}^n$ and $F_{n+1}\,(U^*) = \mathsf{Ann}\,(F^n U)$ for all $n \geq 0$, as in (3.52).*

Proof. It can be easily checked that $\varepsilon_* \circ \zeta = \varepsilon_{\circ}$, where $\varepsilon_* : U^* \to A$ and $\varepsilon_{\circ} : U^{\circ} \to A$ are the counits. In particular this implies that $\zeta\,(\mathcal{I}) \subseteq \mathsf{ker}\,(\varepsilon_*)$. Hence the claim will be proved if we will be able to show that $\mathsf{ker}\,(\varepsilon_*) \subseteq F_1\,(U^*) = \mathsf{Ann}\,(F^0 U) = \mathsf{Ann}\,(A)$, because in this case the multiplicativity of ζ will imply that

$$\zeta\,(F_n U^{\circ}) = \zeta(\mathcal{I}^n) \subseteq \zeta(\mathcal{I})^n \subseteq (F_1\,(U^*))^n \subseteq F_n\,(U^*)\,.$$

However, if $f \in \mathsf{ker}\,(\varepsilon_*)$ then $f(1_U) = 0$, whence $f(\tau_0(a)) = f(1_U \cdot a) = f(1_U)a = 0$. Consequently, $F^0 U = A \subseteq \mathsf{ker}\,(f)$, from which it follows that $\mathsf{ker}\,(\varepsilon_*) \subseteq \mathsf{Ann}\,(F^0 U)$ as desired. □

In light of Proposition 3.2.3, $(A, \widehat{U^{\circ}})$ is a complete Hopf algebroid. On the other hand, we know from Proposition 3.3.16 that (A, U^*) admits a structure of complete Hopf algebroid whenever the translation map of U is filtered algebra map. Combining all these facts allows us to improve the content of Lemma 3.3.18 as follows.

Theorem 3.3.22. *Let (A,U) be a filtered cocommutative (right) Hopf algebroid with an admissible filtration. Then the $(A\otimes A)$-algebra map $\zeta : U^\circ \to U^*$ of equation (3.73) factors through a filtered morphism $\widehat{\zeta} : \widehat{U^\circ} \to U^*$ of complete Hopf algebroids. Thus we have a commutative diagram*

$$\begin{tikzcd} U^\circ \arrow[rr, "\zeta"] \arrow[dr, "\gamma"'] & & U^* \\ & \widehat{U^\circ} \arrow[ur, "\widehat{\zeta}"'] & \end{tikzcd}$$

In particular, this applies to $U = \mathcal{V}_A(L)$, the universal enveloping Hopf algebroid of any Lie-Rinehart algebra (A,L,ω) with L finitely generated and projective.

Proof. In Proposition 3.3.21 we showed that ζ is a filtered algebra map. Thus, by applying the completion bifunctor of Theorem 3.1.35 to ζ (A is discretely filtered), we obtain that $\widehat{\zeta}$ is a filtered morphism of complete algebras. Now, since we already know that $\varepsilon_* \circ \zeta = \varepsilon_\circ$ and in view of Lemma 3.3.18, we are left to show that $\widehat{\zeta}$ is compatible with the comultiplications and the antipodes. That is, that the following relations hold

$$\left(\widehat{\zeta}\,\widehat{\otimes}_A\,\widehat{\zeta}\right)\circ\widehat{\Delta_\circ} = \Delta_*\circ\widehat{\zeta} \quad \text{and} \quad \widehat{\zeta}\circ\widehat{\mathcal{S}_\circ} = \mathcal{S}_*\circ\widehat{\zeta}.$$

However, in light of the adjunction (3.33), it will be enough to show the following ones

$$\gamma_{U^*\otimes_A U^*}\circ(\zeta\otimes_A\zeta)\circ\Delta_\circ = \Delta_*\circ\zeta \quad \text{and} \quad \zeta\circ\mathcal{S}_\circ = \mathcal{S}_*\circ\zeta.$$

Hence, let us consider an element of the form $\overline{\varphi\otimes_{T_M} m} \in U^\circ$. So we obtain an element in $\widehat{U^*\otimes_A U^*} = U^*\,\widehat{\otimes}_A\,U^*$ given by

$$\begin{aligned}(\gamma_{U^*\otimes_A U^*}\circ(\zeta\otimes_A\zeta)\circ\Delta_\circ)\left(\overline{\varphi\otimes_{T_M} m}\right) &= \gamma_{U^*\otimes_A U^*}\left(\sum_i \zeta\left(\overline{\varphi\otimes_{T_M} e_i}\right)\otimes_A\zeta\left(\overline{e_i^*\otimes_{T_M} m}\right)\right)\\ &= \lim_{n\to\infty}\left(\sum_i \zeta\left(\overline{\varphi\otimes_{T_M} e_i}\right)\otimes_A\zeta\left(\overline{e_i^*\otimes_{T_M} m}\right)\right).\end{aligned}$$

Notice that for every $u,v\in U$ it satisfies

$$\begin{aligned}\lim_{n\to\infty}\left(\sum_i \zeta\left(\overline{\varphi\otimes_{T_M} e_i}\right)\left(\zeta\left(\overline{e_i^*\otimes_{T_M} m}\right)(u)v\right)\right) &= \lim_{n\to\infty}\left(\sum_i \varphi\left(e_ie_i^*(m\centerdot u)\centerdot v\right)\right) = \varphi(m\centerdot(uv))\\ &= \zeta\left(\overline{\varphi\otimes_{T_M} m}\right)(uv)\end{aligned}$$

whence, by the criterion of equation (3.55), we have that $\gamma_{U^*\otimes_A U^*}\circ(\zeta\otimes_A\zeta)\circ\Delta_\circ = \Delta_*\circ\zeta$. Moreover,

$$\begin{aligned}((\zeta\circ\mathcal{S}_\circ)\left(\overline{\varphi\otimes_{T_M} m}\right))(u) &= \zeta\left(\overline{\mathrm{ev}_m\otimes_{T_{M^*}}\varphi}\right)(u) = (\varphi\centerdot u)(m)\\ &\overset{(3.71)}{=} \varepsilon\left(\varphi\left(m\centerdot u_-\right)u_+\right) \overset{(3.61)}{=} \mathcal{S}_*\left(\zeta\left(\overline{\varphi\otimes_{T_M} m}\right)\right)(u)\end{aligned}$$

for every $u\in U$, M in $\mathcal{A}_U$, $m\in M$ and $\varphi\in M^*$, so that the proof is complete. $\square$

As in Example 3.2.4, we are going to consider the A-bimodule $A\otimes A$ to be endowed with the K-adic filtration given by $K := \ker(m_A : A\otimes A\to A)$, even if A itself is discretely filtered.

Proposition 3.3.23. *Let (A,U) and (A,U°) be as in Proposition 3.3.21 and assume further that $\zeta : U^\circ\to U^*$ is injective. Then the following assertions are equivalent*

(a) the morphism $\widehat{\zeta} : \widehat{U^\circ}\to U^*$ is a filtered isomorphism,

(b) the morphism $\mathsf{gr}\left(\widehat{\zeta}\right) : \mathsf{gr}\left(\widehat{U^\circ}\right)\to\mathsf{gr}(U^*)$ is a graded isomorphism,

(c) the morphism $\widehat{\zeta}$ is surjective and the $\mathcal{I}$-adic filtration on U° coincides with the one induced from U^*,

(d) the graded morphism $\mathsf{gr}\left(\widehat{\zeta}\right) : \mathsf{gr}\left(\widehat{U^\circ}\right) \to \mathsf{gr}(U^*)$ is surjective and the $\mathcal{I}$-adic filtration on U° coincides with the one induced from U^*,

(e) the graded morphism $\mathsf{gr}(\zeta) : \mathsf{gr}(U^\circ) \to \mathsf{gr}(U^*)$ is surjective and the $\mathcal{I}$-adic filtration on U° coincides with the one induced from U^*,

Moreover, the following assertions are equivalent as well

(f) the morphism $\widehat{\zeta} : \widehat{U^\circ} \to U^*$ is an homeomorphism,

(g) the morphism $\widehat{\zeta} : \widehat{U^\circ} \to U^*$ is open and injective and U° is dense in U^*,

(h) the $\mathcal{I}$-adic topology on U° is equivalent to the one induced from U^* and U° is dense in U^*.

(i) the $\mathcal{I}$-adic filtration on U° is equivalent to the one induced from U^* and U° is dense in U^*.

If in addition the morphism $\widehat{\vartheta}$ induced by the algebra map $\vartheta : A \otimes A \to U^*$ of equation (3.50) is a filtered isomorphism, then all the assertions from (a) to (i) are equivalent.

Proof. Since ζ is injective, we may omit it from the computations by identifying U° with its image in U^*. Before proceeding with the proof, there are some facts that have to be highlighted or recalled. First of all, notice that injectivity of ζ implies that the filtration on U° is separated. Secondly, recall that a morphism of filtered bimodules $f : M \to N$ is said to be *strict* if $f(F_kM) = f(M) \cap F_kN$ for all $k \geq 0$. In particular, ζ is strict if and only if the $\mathcal{I}$-adic filtration on U° coincides with the one induced from U^* via ζ itself. Thirdly, a filtered morphism (as $\widehat{\zeta}$ for example) is a filtered isomorphism if and only if it is bijective and strict. Finally, we have that $\mathsf{gr}(\gamma_{U^\circ}) : \mathsf{gr}(U^\circ) \to \mathsf{gr}\left(\widehat{U^\circ}\right)$ is always an isomorphism (see e.g. [NvO2, Proposition D.3.1]), so that $\mathsf{gr}\left(\widehat{\zeta}\right)$ is injective (resp. surjective, bijective) if and only if $\mathsf{gr}(\zeta)$ is. Now, by applying [NvO2, Cor. D.III.5, D.III.6 and D.III.7] one proves that *(c)* $\Leftrightarrow$ *(a)* $\Leftrightarrow$ *(b)* $\Leftrightarrow$ *(d)* $\Leftrightarrow$ *(e)*.

The equivalence *(h)* $\Leftrightarrow$ *(i)* follows because two filtrations are equivalent if and only if the induced topologies are so (we already mentioned this in the introduction to §3.1.1). For the remaining equivalent facts, notice that $\widehat{\zeta}$ is surjective if and only if for every $x \in U^*$ and for all $k \geq 0$, there exists $m_k \in U^\circ$ such that $x - \zeta(m_k) \in F_k(U^*)$ or, equivalently, if and only if (the image of) U° is dense in U^*. This proves the equivalence between *(f)* and *(g)*, so that we may focus on *(g)* $\Leftrightarrow$ *(h)*. Assume initially that $\widehat{\zeta}$ is an open and injective map. We may then omit it as well from the computations by identifying $\widehat{U^\circ}$ with its image in U^*. From this it follows that for all $h \geq 0$, $F_h\left(\widehat{U^\circ}\right)$ is open in U^*. In particular, there exists $k \geq 0$ such that $F_k(U^*) \subseteq F_h\left(\widehat{U^\circ}\right)$. Thus, $U^\circ \cap F_k(U^*) \subseteq U^\circ \cap F_h\left(\widehat{U^\circ}\right) = F_h(U^\circ)$, which expresses the fact that the $\mathcal{I}$-adic topology is equivalent to the induced one. Conversely, assume that these two topologies are equivalent and that U° is dense in U^* (that is, that $\widehat{\zeta}$ is surjective). We plan to prove first that every $\widehat{\zeta}\left(F_t\left(\widehat{U^\circ}\right)\right)$ is open in U^* (which implies that $\widehat{\zeta}$ is open) and then that $\widehat{\zeta}$ is injective. To this aim, pick $t \geq 0$ and consider k (which we may assume greater or equal than t) such that $U^\circ \cap F_k(U^*) \subseteq F_t(U^\circ)$. Then every $y \in F_k(U^*)$ is of the form $y = \widehat{\zeta}\left((m_i + F_i(U^\circ))_{i\geq 0}\right) = (\zeta(m_i) + F_i(U^*))_{i\geq 0}$ for some $(m_i + F_i(U^\circ))_{i\geq 0} \in \widehat{U^\circ}$ such that $\zeta(m_k) \in F_k(U^*) \cap U^\circ \subseteq F_t(U^\circ)$, because $0 = \pi_k(y) = p_k(\widehat{y}) = \zeta(m_k) + F_k(U^*)$. Hence

$$m_t + F_t(U^\circ) = m_k + F_t(U^\circ) = 0$$

in the quotient $U^\circ/F_t(U^\circ)$ and so $(m_i + F_i(U^\circ))_{i\geq 0} \in F_t\left(\widehat{U^\circ}\right)$. Summing up, we showed that for every $t \geq 0$, there exists a $k \geq t$ such that $F_k(U^*) \subseteq \widehat{\zeta}\left(F_t\left(\widehat{U^\circ}\right)\right)$ and hence that $\widehat{\zeta}$ is an open map. Let us show now that it is injective as well. To this aim, let $(m_i + F_i(U^\circ))_{i\geq 0}$ be an element in $\ker\left(\widehat{\zeta}\right)$. This implies that $\zeta(m_k) \in F_k(U^*) \cap U^\circ$ for all $k \geq 0$ and that, since the two topologies are equivalent, for every $k \geq 0$ there exists j_k (which we may assume greater or equal than k) such that $F_{j_k}(U^*) \cap U^\circ \subseteq F_k(U^\circ)$, whence

$$m_k + F_k(U^\circ) = m_{j_k} + F_k(U^\circ) \in (F_{j_k}(U^*) \cap U^\circ) + F_k(U^\circ) = F_k(U^\circ),$$

so that $(m_i + F_i(U^\circ))_{i \geq 0} = 0$. With this we conclude the proof that *(f)* $\Leftrightarrow$ *(g)* $\Leftrightarrow$ *(h)*.

Finally, *(a)* clearly implies *(f)*. Conversely, assume *(f)*. Since $\zeta \circ \eta_\circ = \vartheta$, if $\widehat{\vartheta}$ is a filtered isomorphism then $\widehat{\zeta}$ admits the filtered section $\widehat{\eta_\circ} \circ \widehat{\vartheta}^{-1}$, which is forced to be its inverse. Thus, it is a filtered isomorphism. □

The subsequent Corollary is the point of contact with the theory of integration for Lie algebroids retrieved in [Kp]. Its proof is a direct application of [EKG, Theorem 4.2.2] and so it is omitted.

Corollary 3.3.24. *Let (A, L) be a Lie-Rinehart algebra and consider $U = \mathcal{V}_A(L)$ its universal enveloping Hopf algebroid. Assume that U° is an Hausdorff topological space with respect to the $\mathcal{I}$-adic topology and that $\widehat{\zeta}$ is an homeomorphism. Then ζ is injective and hence there is an equivalence of symmetric rigid monoidal categories between the category of right L-modules and the category of right U°-comodules with finitely generated and projective underlying A-module structure.*

Remark 3.3.25. As a final remark, we want to point out that the completion of ζ might fail to be an homeomorphism, even if ζ is injective and A is the base field, as we are going to show in the next subsection for an apparently trivial example: namely the enveloping Hopf algebra $U = U(L)$ of the one-dimensional Lie algebra $L = \Bbbk X$ (see [ES1]). Nevertheless, we believe that in the Hopf algebroid framework some unexpected result may show up. For instance, we just mention the fact that the classical Sweedler dual coalgebra U^o of the first Weyl algebra U as in Example 3.2.11 is zero, while the finite dual Hopf algebroid U° is not. This, in our opinion, suggests that the problem of $\widehat{\zeta}$ being an homeomorphism or not for universal enveloping Hopf algebroids is still worthy to be studied. In fact the presence of the algebra of infinite jets could make some difference.

3.3.2.3 Not an example: the one-dimensional Lie algebra

We keep working with $\Bbbk$ a field, now algebraically closed of characteristic zero. Let $L := \Bbbk X$ be the one-dimensional (abelian) Lie algebra and take $A = \Bbbk$ itself. Obviously, $\mathsf{Der}_\Bbbk(\Bbbk) = 0$ and hence $(A, L, 0)$ is trivially a Lie-Rinehart algebra. Its universal enveloping Hopf algebroid $\mathcal{V}_A(L)$ coincides with $U(L)$, the ordinary universal enveloping algebra, which in turn coincides with $\Bbbk[X]$, the Hopf algebra of polynomials in one indeterminate. Comultiplication, counit and antipode are the algebra morphisms induced by the assignments

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \qquad \varepsilon(X) = 0, \qquad S(X) = -X.$$

Furthermore, the obvious filtration $F^n\Bbbk[X] = \bigoplus_{k=0}^n \Bbbk X^k$ is an admissible filtration in the sense of §3.3.1.2 for the cocommutative Hopf algebra $\Bbbk[X]$.

In light of [EKG, §3.4], the finite dual construction in this case coincides with the ordinary Sweedler dual (we already saw this in Proposition 2.3.25), whence

$$\mathcal{V}_A(L)^\circ = \Bbbk[X]^\circ = \{f \in \Bbbk[X]^* \mid f(\langle p(X) \rangle) = 0 \text{ for } p(X) \in \Bbbk[X]\} \subseteq \Bbbk[X]^*$$

and ζ is simply the right-most inclusion above.

For what concerns the filtrations, set $I := \mathsf{ker}(\varepsilon_* : \Bbbk[X]^* \to \Bbbk)$ and $J := \mathsf{ker}(\varepsilon_\circ : \Bbbk[X]^\circ \to \Bbbk)$, where both ε_* and $\varepsilon_\circ$ are given by the evaluation at $1_\Bbbk$. Recall that $F_{n+1}(\Bbbk[X]^*) = \mathsf{Ann}(F^n\Bbbk[X])$ and $F_n(\Bbbk[X]^\circ) = J^n$ for all $n \geq 0$. For the sake of simplicity, let us show that in this case

$$F_n(\Bbbk[X]^*) = I^n. \tag{3.74}$$

Remark 3.3.26. For every $k \geq 0$, set $e_k := X^k/k! \in \Bbbk[X]$. It is well-known that the assignment

$$\Theta \colon \Bbbk[X]^* \to \Bbbk[[Z]], \qquad \left(f \mapsto s_f := \sum_{k \geq 0} f(e_k) Z^k \right) \tag{3.75}$$

gives an algebra isomorphism between $\Bbbk[X]^*$ and the algebra of formal power series, where $\Bbbk[X]^*$ is endowed with the convolution product (1.24) and the multiplication in $\Bbbk[[Z]]$ is

$$\left(\sum_{i\geq 0} a_i Z^i\right)\left(\sum_{j\geq 0} b_j Z^j\right) = \sum_{k\geq 0}\left(\sum_{i+j=k} a_i b_j\right) Z^k.$$

Now, on the one hand $f \in F_n(\Bbbk[X]^*)$ if and only if $f(X^k) = 0$ for all $0 \leq k \leq n-1$, if and only if $s_f \in \langle Z^n\rangle$. On the other hand, $g \in \mathsf{ker}(\varepsilon_*)$ if and only if $s_g \in \langle Z\rangle$. Therefore, if we pick $f \in F_n(\Bbbk[X]^*)$ then $s_f \in \langle Z^n\rangle = \langle Z\rangle^n$ and so we can write $s_f = \sum s_{g_1}\cdots s_{g_n}$ for some $s_{g_i} \in \langle Z\rangle$. Coming back to $\Bbbk[X]^*$, this implies that $f = \sum g_1 * \cdots * g_n \in I^n$, so that $F_n(\Bbbk[X]^*) \subseteq I^n$. Since the other inclusion is evident, we have the equality (3.74).

Summing up, both $\Bbbk[X]^*$ and $\Bbbk[X]^\circ$ are filtered with the adic filtrations $F_n(\Bbbk[X]^*) = I^n$ and $F_n(\Bbbk[X]^\circ) = J^n$, $n \geq 0$. Moreover, $\Bbbk[X]^\circ$ inherits the induced filtration $F'_n(\Bbbk[X]^\circ) = I^n \cap \Bbbk[X]^\circ$ from the inclusion $\zeta\colon \Bbbk[X]^\circ \subseteq \Bbbk[X]^*$ and it is clear that $F_n(\Bbbk[X]^\circ) \subseteq F'_n(\Bbbk[X]^\circ)$. Hence, the J-adic filtration on $\Bbbk[X]^\circ$ is finer than the induced one. As we will show, it is in fact strictly finer.

For every $\lambda \in \Bbbk$, we set $\phi_\lambda : \Bbbk[X] \to \Bbbk$ to be the algebra map such that $\phi_\lambda(X) = \lambda$. The set $G_a := \mathsf{Alg}_\Bbbk(\Bbbk[X], \Bbbk) = \{\phi_\lambda \mid \lambda \in \Bbbk\}$ is a group with group structure given by

$$\phi_\lambda \cdot \phi_{\lambda'} := \phi_\lambda * \phi_{\lambda'} = \phi_{\lambda+\lambda'}, \qquad e_{G_a} := \varepsilon = \phi_0, \qquad (\phi_\lambda)^{-1} := \phi_\lambda \circ S = \phi_{-\lambda}.$$

Lemma 3.3.27 ([Mo, Example 9.1.7])**.** *Denote by ξ the distinguished element in $\Bbbk[X]^*$ which satisfies $\xi(X^n) = \delta_{n,1}$ for all $n \geq 0$ (Kronecker's delta). Then the convolution product induces an isomorphism of commutative Hopf algebras*

$$\Psi : \Bbbk[\xi] \otimes \Bbbk G_a \to \Bbbk[X]^\circ, \quad \Big(\xi^n \otimes \phi_\lambda \mapsto \xi^n * \phi_\lambda\Big), \tag{3.76}$$

where $\Bbbk G_a$ is the group algebra on G_a and $\Bbbk[\xi]$ is the Hopf algebra of polynomials in ξ.

We denote by

$$\iota := \left(\Bbbk[\xi] \xhookrightarrow{\Psi} \Bbbk[X]^\circ \xhookrightarrow{\zeta} \Bbbk[X]^* \right) \tag{3.77}$$

the algebra monomorphism induced by Ψ.

Remark 3.3.28. It is worthy to point out that Lemma 3.3.27 is a particular instance of the renowned *Cartier-Gabriel-Kostant-Milnor-Moore Theorem*, which states that for a cocommutative Hopf $\Bbbk$-algebra H over an algebraically closed field $\Bbbk$ of characteristic zero, the multiplication in H induces an isomorphism of Hopf algebras $U(P(H))\#\Bbbk G(H) \cong H$, where the left-hand side is endowed with the smash product algebra structure (see [Mo, Corollary 5.6.4 and Theorem 5.6.5], [Sw, Theorems 8.1.5, 13.0.1 and §13.1] and [Rad, Theorem 15.3.4]).

Denote by $\varepsilon_a : \Bbbk G_a \to \Bbbk$ the counit of the group algebra, which acts via $\varepsilon_a(\phi_\lambda) = \phi_\lambda(1) = 1$ for all $\lambda \in \Bbbk$, and by $\varepsilon_\xi : \Bbbk[\xi] \to \Bbbk$ the counit of the polynomial algebra in ξ defined by $\varepsilon_\xi(\xi) = 0$. These maps are in fact the restrictions of the counit $\varepsilon_\circ : \Bbbk[X]^\circ \to \Bbbk$ to the vector subspaces of $\Bbbk[X]^\circ$ generated by G_a and $\{\xi^n \mid n \geq 0\}$, respectively. Thus, up to the isomorphism Ψ of equation (3.76), we have $\varepsilon_\circ = \varepsilon_\xi \otimes \varepsilon_a$.

Lemma 3.3.29. *The isomorphism Ψ of (3.76) induces an isomorphism of vector spaces*

$$\Bbbk\bar{\xi} \oplus \frac{\mathsf{ker}(\varepsilon_a)}{\mathsf{ker}(\varepsilon_a)^2} \cong \frac{J}{J^2},$$

where $\bar{\xi} = \xi + \langle \xi^2\rangle$ in the quotient $\langle\xi\rangle/\langle\xi^2\rangle$.

Proof. First of all, as Ψ is an isomorphism of Hopf algebras, it induces an isomorphism of vector spaces between J/J^2 and $\mathsf{ker}(\varepsilon_\xi \otimes \varepsilon_a)/\mathsf{ker}(\varepsilon_\xi \otimes \varepsilon_a)^2$. Set $K := \mathsf{ker}(\varepsilon_\xi \otimes \varepsilon_a)$. The family of assignments

$$\frac{\langle \xi^k\rangle}{\langle \xi^{k+1}\rangle} \otimes \frac{\mathsf{ker}(\varepsilon_a)^h}{\mathsf{ker}(\varepsilon_a)^{h+1}} \to \frac{K^n}{K^{n+1}}, \qquad \left(\left(\xi^k + \langle\xi^{k+1}\rangle\right) \otimes \left(x + \mathsf{ker}(\varepsilon_a)^{h+1}\right) \mapsto \left(\xi^k \otimes x\right) + K^{n+1}\right)$$

for $h, k \geq 0$ and $n = h + k$ induces a graded isomorphism of graded vector spaces

$$\mathsf{gr}(\Bbbk[\xi]) \otimes \mathsf{gr}(\Bbbk G_a) \cong \mathsf{gr}\big(\Bbbk[\xi] \otimes \Bbbk G_a\big),$$

see e.g. [NvO2, Lemma D.VIII.2]. In particular, the degree 1 component of this together with Ψ induces the stated isomorphism

$$\Bbbk\bar{\xi} \oplus \frac{\mathsf{ker}\,(\varepsilon_a)}{\mathsf{ker}\,(\varepsilon_a)^2} \cong \left(\frac{\langle\xi\rangle}{\langle\xi^2\rangle} \otimes \frac{\Bbbk G_a}{\mathsf{ker}\,(\varepsilon_a)}\right) \oplus \left(\frac{\Bbbk[\xi]}{\langle\xi\rangle} \otimes \frac{\mathsf{ker}\,(\varepsilon_a)}{\mathsf{ker}\,(\varepsilon_a)^2}\right) \cong \frac{K}{K^2} \cong \frac{J}{J^2}. \qquad \square$$

The key fact to prove that the J-adic filtration on $\Bbbk[X]^\circ$ is finer than the induced one is that the quotient $\mathsf{ker}\,(\varepsilon_a)/\mathsf{ker}\,(\varepsilon_a)^2$ does not vanish, as we will show in the subsequent lemma. To this aim, recall the isomorphism Θ from Equation (3.75) and notice that for all $\lambda \in \Bbbk$ we have

$$\Theta\left(\phi_\lambda\right) = \sum_{k\geq 0} \frac{(\lambda Z)^k}{k!} = \mathsf{exp}\left(\lambda Z\right), \qquad \Theta(\varepsilon) = 1 \qquad \text{and} \qquad \Theta(\xi) = Z. \tag{3.78}$$

Lemma 3.3.30. *The element $\phi_1 - \varepsilon + \mathsf{ker}\,(\varepsilon_a)^2$ is non-zero in the quotient $\mathsf{ker}\,(\varepsilon_a)/\mathsf{ker}\,(\varepsilon_a)^2$.*

Proof. Assume by contradiction that $\phi_1 - \varepsilon \in \mathsf{ker}\,(\varepsilon_a)^2$. By applying Ψ, this implies that $\phi_1 - \varepsilon \in J^2$, whence $\phi_1 - \varepsilon \in I^2$ in $\Bbbk[X]^*$. Since Θ induces a bijection between I^n and $\langle Z^n\rangle \subseteq \Bbbk[[Z]]$ for all $n \geq 1$, claiming that $\phi_1 - \varepsilon \in I^2$ in $\Bbbk[X]^*$ would imply that $\sum_{k\geq 1} Z^k/k! \in \langle Z^2\rangle$, which is a contradiction. Thus, $\phi_1 - \varepsilon \notin \mathsf{ker}\,(\varepsilon_a)^2$. $\square$

It follows from Lemma 3.3.29 and Lemma 3.3.30 that the elements $\xi + J^2$ and $\phi_1 - \varepsilon + J^2$ are linearly independent in J/J^2. In particular, $\phi_1 - \varepsilon - \xi \notin J^2$. However, since

$$\Theta\left(\phi_1 - \varepsilon - \xi\right) \overset{(3.78)}{=} \mathsf{exp}(Z) - 1 - Z \in \langle Z^2\rangle,$$

we have that $\phi_1 - \varepsilon - \xi \in I^2$ as an element of $\Bbbk[X]^*$. This shows that $\phi_1 - \varepsilon - \xi$ belongs to $\Bbbk[X]^\circ \cap I^2$ but not to J^2, so that $J^2 \subsetneq \Bbbk[X]^\circ \cap I^2$. In general, for all $n \geq 2$ the computation

$$\Theta\left(\phi_1 - \left(\sum_{k=0}^{n-1} \frac{1}{k!}\xi^k\right)\right) \overset{(3.78)}{=} \mathsf{exp}(Z) - \left(\sum_{k=0}^{n-1} \frac{Z^k}{k!}\right) = Z^n \cdot \left(\sum_{k\geq 0} \frac{Z^k}{(n+k)!}\right) \in \langle Z^n\rangle$$

implies that the element

$$\phi_1 - \left(\sum_{k=0}^{n-1} \frac{1}{k!}\xi^k\right) \tag{3.79}$$

belongs to $I^n \cap \Bbbk[X]^\circ$. By induction on $n \geq 2$, however, one may check that it does not belong to J^n, so that the two filtrations do not coincide.

Summing up, we have shown that the J-adic filtration on $\Bbbk[X]^\circ$ is strictly finer than the filtration induced from the inclusion $\zeta : \Bbbk[X]^\circ \to \Bbbk[X]^*$. This is already enough to claim that $\widehat{\zeta}$ is not a filtered isomorphism, in light of Proposition 3.3.23. However, for the sake of completeness, we want to see that it is not even an homeomorphism (notice that in this case the completion of $\vartheta : \Bbbk \otimes \Bbbk \to \Bbbk[X]^*$ is not a filtered isomorphism, so that we cannot apply directly the last claim of the above mentioned proposition).

Let us consider again the algebra monomorphism $\iota : \Bbbk[\xi] \to \Bbbk[X]^*$ of equation (3.77). Assume $\Bbbk[\xi]$ to be endowed with the adic filtration induced by $\langle\xi\rangle$. Since $\iota(\xi) \in \mathsf{ker}\,(\varepsilon_*) = I$, we have that ι is a morphism of filtered algebras and so we may consider its completion $\widehat{\iota} : \widehat{\Bbbk[\xi]} \to \widehat{\Bbbk[X]^*} \cong \Bbbk[X]^*$. Therefore, up to the canonical isomorphism $\widehat{\Bbbk[\xi]} \cong \Bbbk[[\xi]] \cong \Bbbk[[Z]]$ (see Example 3.1.36), the map $\widehat{\iota}$ turns out to be the inverse of Θ. A useful consequence of this is that every element $g \in \Bbbk[X]^*$ can be written as

$$g = \sum_{k\geq 0} g(e_k)\xi^k, \tag{3.80}$$

where as before $e_k = X^k/k!$ for all $k \geq 0$. By the right-hand side of equation (3.80), we mean the image in $\Bbbk[X]^*$ of the element

$$\left(\sum_{k=0}^{n} g(e_k)\xi^k + I^{n+1}\right)_{n\geq 0} = \lim_{n\to\infty}\left(\sum_{k=0}^{n} g(e_k)\xi^k\right)$$

via the isomorphism $\widehat{\iota}$. Since $\xi^i(e_j) = \delta_{i,j}$ for all $i, j \geq 0$, given any $p = \sum_{i=0}^{t} p_i e_i \in \Bbbk[X]$ the sequence $\left(\sum_{k=0}^{n} g(e_k)\xi^k\right)(p)$, $n \geq 0$, eventually becomes constant and it equals the element $\sum_{i=0}^{t} p_i g(e_i) = g(p)$. In light of this interpretation, $I^n = \langle \xi^n \rangle$ for all $n \geq 0$, in the algebra $\Bbbk[X]^*$.

Remark 3.3.31. Looking at the conditions in Proposition 3.3.23, it is worthy to mention that $\Bbbk[X]^\circ$ is dense in $\Bbbk[X]^*$ with respect to the finite topology on $\Bbbk[X]^*$ (the one induced by the product topology on $\Bbbk^{\Bbbk[X]}$), see for instance [DNR, Exercise 1.5.21]. On the other hand, since for every $f \in \Bbbk[X]^*$ and for all $n \geq 0$, we have that $f + \langle \xi^n \rangle = \mathcal{O}(f; e_0, e_1, \ldots, e_{n-1})$, the space of linear maps which coincide with f on $e_0, e_1, \ldots, e_{n-1}$, it turns out that the I-adic topology on $\Bbbk[X]^*$ is coarser then the linear one. It follows then that $\Bbbk[X]^\circ \subseteq \Bbbk[X]^*$ is dense with respect to the I-adic topology as well and hence one may check that

$$\varprojlim\left(\frac{\Bbbk[X]^\circ}{\Bbbk[X]^\circ \cap I^n}\right) \cong \varprojlim\left(\frac{\Bbbk[X]^*}{I^n}\right) \cong \Bbbk[X]^*.$$

Now, consider the completion $\widehat{\psi} : \Bbbk[[\xi]] \to \widehat{\Bbbk[X]^\circ}$, where ψ is the filtered monomorphism of algebras given in (3.77). By the definition of ι (see (3.77)), one shows that $\widehat{\zeta} \circ \widehat{\psi} = \widehat{\iota}$. Therefore, $\widehat{\zeta}$ is a split epimorphism, as $\widehat{\iota}$ is an homeomorphism whose inverse is Θ.

The subsequent proposition gives conditions under which $\widehat{\zeta}$ becomes an homeomorphism.

Proposition 3.3.32. *The following assertions are equivalent*

(1) the canonical map $\widehat{\zeta} : \widehat{\Bbbk[X]^\circ} \to \Bbbk[X]^*$ is a filtered isomorphism,

(2) the J-adic and the induced filtrations on $\Bbbk[X]^\circ$ coincide,

(3) the canonical map $\widehat{\zeta} : \widehat{\Bbbk[X]^\circ} \to \Bbbk[X]^*$ is an homeomorphism,

(4) the canonical map $\widehat{\zeta} : \widehat{\Bbbk[X]^\circ} \to \Bbbk[X]^*$ is injective,

(5) the J-adic and the induced topologies on $\Bbbk[X]^\circ$ are equivalent,

(6) the J-adic and the induced filtrations on $\Bbbk[X]^\circ$ are equivalent.

Proof. We already know from Remark 3.3.31 that $\Bbbk[X]^\circ$ is dense in $\Bbbk[X]^*$. Moreover, $\widehat{\psi} \circ \Theta$ is a filtered section of $\widehat{\zeta}$, which is surjective. Thus, if $\widehat{\zeta}$ is injective then it is bijective with inverse $\widehat{\psi} \circ \Theta$ and so a filtered isomorphism (in particular, an open map). This proves the implication *(4)* $\Rightarrow$ *(1)*. Furthermore, in light of these observations the statements *(1)*, *(2)*, *(3)*, *(4)*, *(5)* and *(6)* correspond to the statements *(a)*, *(c)*, *(f)*, *(g)*, *(h)* and *(i)* of Proposition 3.3.23 respectively. Whence we have the remaining chain of implications

$$(1) \Leftrightarrow (2) \Rightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5) \Leftrightarrow (6).$$ ☐

In conclusion, it follows that none of the equivalent conditions in Proposition 3.3.32 holds, as the two filtrations do not coincide. An explicit non-zero element which lies in the kernel of $\widehat{\zeta}$ is exactly the one coming from equation (3.79). Indeed, on the one hand

$$\left(\phi_1 - \sum_{k=0}^{n} \frac{1}{k!}\xi^k + J^{n+1}\right)_{n\geq 0} \in \widehat{\Bbbk[X]^\circ}$$

is non-zero, but on the other hand a direct check shows that in $\Bbbk[X]^*$ we have

$$\widehat{\zeta}\left(\left(\phi_1 - \sum_{k=0}^{n} \frac{1}{k!}\xi^k + J^{n+1}\right)_{n\geq 0}\right) = \left(\phi_1 - \sum_{k=0}^{n} \frac{1}{k!}\xi^k + I^{n+1}\right)_{n\geq 0} = 0.$$

Remark 3.3.33. Recall that an element $(f_n + J^{n+1})_{n\geq 0}$ in $\widehat{\Bbbk[X]^\circ}$ can be considered as the formal limit $\lim_{n\to\infty}(f_n)$ of the Cauchy sequence $\{f_n \mid n \geq 0\}$ in $\Bbbk[X]^\circ$ with the J-adic topology. The element $(\phi_1 + J^{n+1})_{n\geq 0}$ can be identified with ϕ_1 itself, as limit of a constant sequence. On the other hand, the element $(\sum_{k=0}^{n} \xi^k/k! + J^{n+1})_{n\geq 0}$ can be considered as the limit $\lim_{n\to\infty}(\sum_{k=0}^{n}\xi^k/k!)$. As we already noticed, ϕ_1 is associated with the exponential function, in the sense that its power series expansion in $\Bbbk[X]^*$ is $\sum_{k\geq 0}\xi^k/k! = \mathsf{exp}(\xi)$. However, it follows from what we showed that in $\widehat{\Bbbk[X]^\circ}$ the Cauchy sequence $\{\sum_{k=0}^{n}\xi^k/k! \mid n \geq 0\}$ does not converge to ϕ_1.

References

The material for this chapter comes almost entirely from the joint papers with Laiachi El Kaoutit [ES1] and [ES2]. Section 3.1 is a revised version of Appendix A in [ES2] and many of the results contained there can be found (maybe up to some reinterpretation) in any textbook on filtered and graded modules (we found inspiration in [NvO2] and [Bk3], for example), but we preferred to follow a personal approach in the presentation, to set them in the wider framework of bicategories. All the other sections, apart from §3.3.2.3 which collects the content of [ES1], come from the main body of [ES2].

Appendix A

Units of the tensor algebra

In Example 2.4.11 we claimed that the tensor $\Bbbk$-algebra $T(V)$ over a vector space V cannot admit non-trivial reassociators. This appendix contains few technical results aimed at proving this. The following general result is probably well-known but we were not able to find a reference.

Lemma A.1. *Let $A = \bigoplus_{n\in\mathbb{N}} A_n$ be an $\mathbb{N}$-graded ring. Suppose that the product of two non-zero homogeneous elements is non-zero. Then the units of A are concentrated in A_0 and A is a domain.*

Proof. Let $x, y \in A$ be non-zero elements. Write $x = x_0+x_1+\cdots+x_s$ and $y = y_0+y_1+\cdots+y_t$, where $x_i, y_i \in A_i$, with $x_s \neq 0$ and $y_t \neq 0$. By assumption, $x_s y_t \neq 0$ and it is clearly the homogeneous element with highest degree of xy. If $xy = 1$, then the unique option is $s + t = 0$, whence $s = 0$ which means $x \in A_0$. If $xy = 0$, then we must have $x_s y_t = 0$, which is a contradiction. □

Corollary A.2. *Given a vector space V, the group of units of the tensor algebra $T(V)$ is $\Bbbk \setminus \{0\}$ and $T(V)$ is a domain.*

Proof. We have that $T = T(V)$ is graded with respect to $T_n := V^{\otimes n}$. Given $x \in T_s$ and $y \in T_t$ non-zero elements, we have that $x \cdot y = x \otimes y$ which is non-zero. □

Lemma A.3. *Let R be a $\Bbbk$-algebra that is also a domain. Then $T(V) \otimes R$ is a domain.*

Proof. By the Axiom of Choice we can choose a totally ordered basis $\mathcal{B} := \{v_i \mid i \in I\}$ for V. Mimicking [Gr, Example 2 and 3] we can construct an admissible graded lexicographic order on the basis $\mathcal{B}_T := \{v_{i_1}v_{i_2}\cdots v_{i_n} \mid n \geq 1 \text{ and } i_1,\ldots,i_n \in I\} \cup \{1_\Bbbk\}$ of $T = T(V)$ as follows

$$v_{i_1}v_{i_2}\cdots v_{i_n} < v_{j_1}v_{j_2}\cdots v_{j_m}$$

if $n < m$ or $n = m$, $v_{i_s} = v_{j_s}$ for $0 \leq s \leq (t-1) < n$ and $v_{i_t} < v_{j_t}$ with respect to the total order on $\mathcal{B}$. Let $x, y \in T \otimes R$ with $x \neq 0$ and $y \neq 0$. We can write $x = b_{i_1} \otimes x_1 + \cdots + b_{i_s} \otimes x_s$ where $x_1,\ldots,x_s \in R$ with $x_s \neq 0$, $b_{i_1},\cdots,b_{i_s} \in \mathcal{B}_T$ and $b_{i_1} < \cdots < b_{i_s}$. Analogously write $y = b_{j_1} \otimes y_1 + \cdots + b_{j_t} \otimes y_t$ where $y_1,\ldots,y_t \in R$ with $y_t \neq 0$, $b_{j_1},\cdots,b_{j_t} \in \mathcal{B}_T$ and $b_{j_1} < \cdots < b_{j_t}$. Since R is a domain, $x_s y_t \neq 0$. Moreover, $b_{i_s}b_{j_t} \in \mathcal{B}_T$ whence $(b_{i_s} \otimes x_s)(b_{j_t} \otimes y_t) = b_{i_s}b_{j_t} \otimes x_s y_t \neq 0$. Note that $xy = b_{i_1}b_{j_1} \otimes x_1 y_1 + \cdots + b_{i_s}b_{j_t} \otimes x_s y_t$ where $b_{i_s}b_{j_t}$ is the greatest of all the first entries of the summands involved. Thus $xy \neq 0$. □

Corollary A.4. *Given a vector space V and $n \in \mathbb{N}$, the group of units of the tensor algebra $T(V)^{\otimes n}$ is $\Bbbk \setminus \{0\}$ and $T(V)^{\otimes n}$ is a domain.*

Proof. In view of Corollary A.2 and of Lemma A.3, $T(V)^{\otimes n}$ is a domain by induction on n. Moreover, since $T(V)$ is a graded algebra, $T(V)^{\otimes n}$ is graded too. By Lemma A.1, the group of units of $T(V)^{\otimes n}$ is concentrated in degree zero. □

Appendix B

Locally finitely generated and projective filtered bimodules

In this appendix we plan to study the linear dual of the tensor product of two locally finitely generated and projective filtered modules (for instance, rings with an admissible filtration as in §3.3.1.1). In particular, we will show that this bimodule is isomorphic as a filtered bimodule to the complete tensor product of the duals.

B.1 Locally fgp filtered modules

Let $\Bbbk$ be a commutative ring as usual. Let R be a $\Bbbk$-algebra and M be a right R-module endowed with an exhaustive ascending filtration $\{F^nM \mid n \in \mathbb{N}\}$ (see the introductions to §3.1.1 and §3.3). In view of our aims, we assume R to be discretely filtered (i.e. $F^nR = R$ for $n \geq 0$ and 0 otherwise). We denote by $\mathsf{gr}^n(M)$ the quotient module $F^nM/F^{n-1}M$ for all $n \geq 0$ ($F^{-1}M = 0$ by convention), and by $\mathsf{gr}(M)$ the associated graded module $\mathsf{gr}(M) = \bigoplus_{n\geq 0}\mathsf{gr}^n(M)$. Henceforth and in line with §3.3, we denote increasing filtrations with upper indices and decreasing ones with lower indices. Moreover, $\tau_{m,n} : F^nM \to F^mM$ and $\tau_n : F^nM \to M$ for all $m \geq n \geq 0$ will denote the canonical inclusions.

Lemma B.1. *Let R be any $\Bbbk$-algebra, M a right R-module endowed with an ascending filtration $\{F^kM \mid k \in \mathbb{N}\}$ and let $n \in \mathbb{N}$. If the quotient modules $F^kM/F^{k-1}M$ are projective right R-modules for all $0 \leq k \leq n$, then $F^nM \cong \mathsf{gr}(F^nM)$ as filtered modules. In particular, F^nM is projective. If moreover the quotient modules $F^kM/F^{k-1}M$ are finitely generated for $0 \leq k \leq n$, then F^nM is finitely generated as well. Finally, if the filtration is exhaustive and the quotient modules $F^nM/F^{n-1}M$ are projective for all $n \in \mathbb{N}$, then there exists an isomorphism of filtered modules $M \cong \mathsf{gr}(M)$ and M_R itself is projective.*

Proof. Since every quotient module $F^kM/F^{k-1}M$ is projective as right R-module for all $0 \leq k \leq n$, we have a split exact sequence of right R-modules

$$\begin{tikzcd} 0 \arrow[r] & F^{n-1}M \arrow[r, shift left, "\tau_{n-1,n}"] & F^nM \arrow[l, dotted, shift left] \arrow[r, shift left] & \dfrac{F^nM}{F^{n-1}M} \arrow[l, dotted, shift left] \arrow[r] & 0 \end{tikzcd}$$

from which it follows that, as right R-modules,

$$F^nM \cong F^{n-1}M \oplus \frac{F^nM}{F^{n-1}M}.$$

Proceeding inductively, we have that

$$F^nM \cong \bigoplus_{k=0}^{n} \frac{F^kM}{F^{k-1}M} = \mathsf{gr}(F^nM). \tag{B.1}$$

Observing that $F^m(\mathrm{gr}(F^nM)) = \bigoplus_{k=0}^m F^kM/F^{k-1}M = \mathrm{gr}(F^mM)$ and $F^m(F^nM) = F^mM$ for all $m \leq n$, it is clear that the isomorphism preserves the filtrations as claimed. Moreover, as direct sum of projective right R-modules, F^nM is projective as well. The second claim is clear, as the direct sum is finite. About the last claim in the statement, saying that the filtration is exhaustive means that $M \cong \varinjlim(F^nM)$ as filtered modules. Since $F^nM \cong \mathrm{gr}(F^nM) \cong F^n(\mathrm{gr}(M))$ as filtered modules, we have that $M \cong \varinjlim(F^nM) \cong \varinjlim(F^n(\mathrm{gr}(M))) \cong \mathrm{gr}(M)$ as claimed. As direct sum of projective right R-modules, M is itself projective. $\square$

Assumption. *Henceforth, all ascending filtrations will be exhaustive.*

In analogy with [Crn, §4], we will say that an increasingly filtered right R-module M such that the quotient modules $F^nM/F^{n-1}M$ are finitely generated and projective is a *locally finitely generated and projective* (filtered) module (*locally fgp*, for short).

B.2 The filtration on the linear dual of a locally fgp filtered bimodule

Assume that we are given an increasingly filtered R-bimodule M which is locally fgp as a filtered right R-module. In particular, this means that each member of the increasing filtration $\{F^nM \mid n \in \mathbb{N}\}$ is actually an R-subbimodule with a monomorphism $\tau_n : F^nM \to M$ and that the factors $F^nM/F^{n-1}M$ are finitely generated and projective right R-modules. Since the filtration $\{F^nM \mid n \in \mathbb{N}\}$ is exhaustive, we may identify the right R-module M_R with the inductive limit $M = \varinjlim(F^nM)$ of the system $\{F^nM, \tau_{n,n+1}\}_{n\in\mathbb{N}}$. Therefore, $M^* = \mathrm{Hom}_A(M, A) \cong \varprojlim((F^nM)^*)$ as a left R-module via the left R-linear isomorphism

$$\begin{array}{rcl} M^* & \xleftrightarrow{\cong} & \varprojlim((F^nM)^*) \\ f & \longmapsto & (\tau_n^*(f))_{n\geq 0} \\ g := \varinjlim(g_n) & \longleftarrow & (g_n)_{n\geq 0} \end{array} \tag{B.2}$$

where $(r \cdot f)(x) = rf(x)$ for all $f \in M^*$, $r \in R$ and $x \in M$. However, M^* is also a right R-module with $(f \leftharpoonup x)(m) = f(x \cdot m)$ for all $f \in M^*$, $m \in M$ and $x \in R$, and it turns out that the isomorphism (B.2) is right R-linear as well (in fact, τ_n^* is R-bilinear for all $n \geq 0$). Therefore, $M^* \cong \varprojlim((F^nM)^*)$ as R-bimodules. Notice that $g : M \to R$ is the unique right R-linear map that extends all the g_n's at the same time, that is, $g \circ \tau_n = g_n$ for all $n \geq 0$.

Corollary B.2. *Let M be an increasingly filtered R-bimodule which is locally fgp as right R-module. The following properties hold true.*

(i) Each of the subbimodules F^nM is a finitely generated and projective right R-module and each of the structural maps $\tau_{n,n+1} : F^nM \to F^{n+1}M$ is a split monomorphism of right R-modules.

(ii) Each canonical inclusion $\tau_n : F^nM \to M$ is a split monomorphism of right R-modules, too.

(iii) For every $m, n \geq 0$, we have an isomorphism of R-bimodules

$$\phi_{m,n} : (F^mM)^*{}_R \otimes_R {}_R(F^nM)^* \cong (F^nM_R \otimes_R {}_RF^mM)^*$$

such that $\phi_{m,n}(f \otimes_R g)(x \otimes_R y) = f(g(x)y)$ for all $x \in F^nM$, $y \in F^mM$, $f \in (F^mM)^*$ and $g \in (F^nM)^*$.

Proof. Claim *(i)* follows directly from Lemma B.1 and to prove *(ii)*, we may proceed as follows. For every $n \geq 0$ consider a right R-linear retraction $\rho_{n,n+1} : F^{n+1}M \to F^nM$ of the structure map $\tau_{n,n+1}$. If $m \geq n$ then the composition $\rho_{n,m} := \rho_{n,n+1} \circ \rho_{n+1,n+2} \circ \cdots \circ \rho_{m-1,m}$ is a right R-linear retraction for the canonical inclusion $\tau_{n,m} : F^nM \to F^mM$. Now, the following family of right R-linear maps

$$\begin{cases} \tau_{k,n} : F^kN \to F^nN & k \leq n \\ \rho_{n,k} : F^kN \to F^nN & k > n \end{cases}$$

makes of F^nM a sink for the diagram $\{F^kM, \tau_{k,k+1}\}_{k\in\mathbb{N}}$ in $\mathfrak{M}_R$. By the universal property of $M = \varinjlim (F^nM)$ we have that there exists a unique right R-linear morphism $\theta_n : M \to F^nM$ such that $\theta_n \circ \tau_k = \rho_{n,k}$ if $k > n$ and $\theta_n \circ \tau_k = \tau_{k,n}$ if $k \leq n$. In particular, $\theta_n \circ \tau_n = \mathsf{Id}_{F^nM}$, whence it is a retraction of τ_n. Finally, in view of [BSZ, Lemma 11.3] and the hom-tensor adjunction respectively, we have the chain of isomorphisms of R-bimodules

$$(F^mM)^*{}_R \otimes_R {}_R(F^nM)^* \cong \mathsf{Hom}_R\left(F^nM_R, (F^mM)^*{}_R\right) \cong \mathsf{Hom}_R\left(F^nM_R \otimes_R {}_RF^mM, R\right)$$

which proves *(iii)*. $\square$

Remark B.3. From the previous proof (of Corollary B.2) we get that there is a right R-linear retraction $\theta_n : M \to F^nM$ of τ_n. In particular, each of the maps $\tau_n^* : M^* \to (F^nM)^*$ is a split epimorphism of left R-modules with section $\theta_n^* : (F^nM)^* \to M^*$. Denote temporarily by $\pi_n : M^* \to M^*/\mathsf{ker}(\tau_n^*)$ the canonical projection. Even if θ_n^* is just left R-linear, the composition $\pi_n \circ \theta_n^* : (F^nM)^* \to M^*/\mathsf{ker}(\tau_n^*)$ is R-bilinear as it is the inverse of the R-bilinear isomorphism $\widetilde{\tau_n^*} : M^*/\mathsf{ker}(\tau_n^*) \to (F^nM)^*$ induced by the factorization of τ_n^* through the quotient.

Now, the right linear dual M^* inherits naturally a decreasing filtration which converts it into a complete R-bimodule. Namely, mimicking [MSS, Appendix A.2], let us consider the filtration

$$F_0(M^*) = M^* \quad \text{and} \quad F_{n+1}(M^*) = \mathsf{ker}(\tau_n^*), \quad \text{for } n \geq 0.^{(1)} \tag{B.3}$$

In view of *(i)* of Corollary B.2, we have an isomorphism of R-bimodules $(F^nM)^* \cong M^*/F_{n+1}(M^*)$. From this together with the isomorphism (B.2), Proposition 3.1.14 and Remark 3.1.32, we deduce that the filtration $\{F_n(M^*) \mid n \in \mathbb{N}\}$ induces a linear topology over M^* with respect to which it is a complete R-bimodule.

Remark B.4. In order to be able to evaluate limits of Cauchy sequences in M^* on an element of M it is useful to notice the following. Let $\{f_n\}_{n\geq 0}$ be a Cauchy sequence of right R-linear maps in M^* and let $f = \lim_{n\to\infty}(f_n)$ denote its limit in M^*. Therefore we have that $f - f_n \in F_n(M^*) = \mathsf{ker}\left(\tau_{n-1}^*\right)$ for every $n \geq 1$. For all $x \in M$, there exists an $l \geq 0$ such that $x \in F^lM$ and hence for every $k \geq l+1$ we have that

$$f_k(x) = f_k(\tau_l(x)) = f(\tau_l(x)) = f(x).$$

This means that the sequence of elements $\{f_n(x)\}_{n\geq 0}$ eventually becomes constant in A and equal to the value of f on x. Thus, it is meaningful to set $f(x) = \left(\lim_{n\to\infty}(f_n)\right)(x) := \lim_{n\to\infty}(f_n(x))$.

Notice also that if we consider the inductive limit function of the inductive cone $\{\tau_n^*(f_{n+1})\}_{n\in\mathbb{N}}$, we find out that $\varinjlim(\tau_n^*(f_{n+1})) = \varinjlim(\tau_n^*(f)) = f = \lim_{n\to\infty}(f_n)$.

B.3 The complete tensor product of fgp modules

It is useful to point out that the full subcategory of R-bimodules which are locally fgp on the right is closed under taking tensor products (compare with [Ma, Theorem C.24] in light of Lemma B.1). Indeed, let M, N be filtered R-bimodules which are locally fgp on the right and consider the filtration

$$\left\{ F^n(M \otimes_R N) = \sum_{p+q=n} F^pM \otimes_R F^qN \;\middle|\; n \in \mathbb{N} \right\}$$

on $M \otimes_R N$ with inclusion maps $\boldsymbol{\tau}_n : F^n(M \otimes_R N) \to M \otimes_R N$. Then we have an R-bilinear isomorphism

$$\bigoplus_{p+q=n} \left(\frac{F^pM}{F^{p-1}M} \otimes_R \frac{F^qN}{F^{q-1}N} \right) \to \frac{F^n(M \otimes_R N)}{F^{n-1}(M \otimes_R N)},$$

(1) Since $\mathsf{ker}(\tau_n^*) = \{f \in M^* \mid F^nM \subseteq \mathsf{ker}(f)\}$, we will often use the notation $\mathsf{Ann}(F^nM)$ to refer to it.

induced by the assignments

$$(x_p + F^{p-1}M) \otimes_R (y_q + F^{q-1}N) \mapsto (x_p \otimes_R y_q) + F^{n-1}(M \otimes_R N)$$

for $p+q=n$. Thus the factors $F^n(M \otimes_R N)/F^{n-1}(M \otimes_R N)$ are finitely generated and projective as right R-modules and $M \otimes_R N$ is locally fgp as claimed.

Next, for M, N two R-bimodules which are locally fgp on the right we want to compare the linear dual $(N \otimes_R M)^*$ with the complete R-bimodule $\widehat{M^* \otimes_R N^*} = M^* \widehat{\otimes}_R N^*$. At the algebraic level, we have a canonical R-bilinear map

$$\begin{aligned} (M^*)_R \otimes_{R\ R}(N^*) &\xrightarrow{\phi_{M,N}} (N_R \otimes_{R\ R}M)^* \\ f \otimes_R g &\longmapsto [y \otimes_R x \mapsto f(g(y)x)] \end{aligned} \tag{B.4}$$

which makes the following diagram commute

$$\begin{tikzcd} M^* \otimes_R N^* \arrow[r, "\phi_{M,N}"] \arrow[d, "(\tau_m^M)^* \otimes_R (\tau_n^N)^*"'] & (N \otimes_R M)^* \arrow[d, "(\tau_n^N \otimes_R \tau_m^M)^*"] \\ (F^m M)^* \otimes_R (F^n N)^* \arrow[r, "\phi_{m,n}"] & (F^n N \otimes_R F^m M)^* . \end{tikzcd} \tag{B.5}$$

Our plan for what is left of the section will be to show that the natural transformation $\phi_{M,N}$ is filtered and that it induces an isomorphism of filtered R-bimodules between $(N \otimes_R M)^*$ and the completion of $M^* \otimes_R N^*$ with respect to the filtration

$$F_n (M^* \otimes_R N^*) = \sum_{p+q=n} \mathsf{im}\left(F_p (M^*) \otimes_R F_q (N^*)\right). \tag{B.6}$$

To this aim, we start with a couple of intermediate results. The first technical lemma, Lemma B.5, is about the relation between intersections and tensor products. Maybe it is well-known, but we were not able to find an explicit reference. The second one, Lemma B.6, allows us to rewrite the n-th term of the filtration (B.6) as intersection of suitable kernels.

Lemma B.5 (see also [BW, §40.16])**.** *Let R be any $\Bbbk$-algebra. Let W be a left R-module and $p\colon W \to W_2$ be a surjective left R-linear morphism with kernel $f\colon W_1 \to W$, where W_2 is projective over R. Let also $g\colon V_1 \to V$ be an injective morphism of right R-modules. Then*

$$\mathsf{im}\,(V \otimes_R W_1) \cap \mathsf{im}\,(V_1 \otimes_R W) = \mathsf{im}\,(V_1 \otimes_R W_1)$$

in $V \otimes_R W$, where $\mathsf{im}\,(\cdot)$ denotes the canonical image in the tensor product.

Proof. First of all, notice that the hypothesis on p implies that we have a split short exact sequence

$$0 \longrightarrow W_1 \xrightarrow{f} W \xrightarrow{p} W_2 \longrightarrow 0.$$

We want to apply [W, §10.3(2)]. To this aim, consider the following diagram of abelian groups with commutative squares and lower exact row

$$\begin{tikzcd} 0 \arrow[r] & V_1 \otimes_R W_1 \arrow[r, "V_1 \otimes_R f"] \arrow[d, "g \otimes_R W_1"'] & V_1 \otimes_R W \arrow[r, "g \otimes_R p"] \arrow[d, "g \otimes_R W"] & V \otimes_R W_2 \arrow[d, equal] & \\ 0 \arrow[r] & V \otimes_R W_1 \arrow[r, "V \otimes_R f"'] & V \otimes_R W \arrow[r, "V \otimes_R p"'] & V \otimes_R W_2 \arrow[r] & 0 \end{tikzcd}$$

Since f splits, $V_1 \otimes_R f$ is injective. Clearly $(g \otimes_R p) \circ (V_1 \otimes_R f) = g \otimes_R (p \circ f) = 0$, so that to have that the upper row is exact as well we are left to prove that $\mathsf{ker}\,(g \otimes_R p) \subseteq \mathsf{im}\,(V_1 \otimes_R f)$. Let us pick $z \in \operatorname{ker}(g \otimes_R p)$. Since W_2 is projective as left R-module, $g \otimes_R W_2$ is still injective, whence $(V_1 \otimes_R p)(z) = 0$ and $z \in \operatorname{ker}(V_1 \otimes_R p) = \operatorname{im}(V_1 \otimes_R f)$. Thus $\operatorname{ker}(g \otimes_R p) \subseteq \operatorname{im}(V_1 \otimes_R f)$ and the first row is exact as well.

Passing to the images of the vertical arrows, we have that

$$\begin{tikzcd} 0 \arrow[r] & g(V_1)\otimes_R W_1 \arrow[r, "V\otimes_R f"] \arrow[d, "\subseteq"'] & g(V_1)\otimes_R W \arrow[r, "V\otimes_R p"] \arrow[d, "\subseteq"] & V\otimes_R W_2 \arrow[d, equal] & \\ 0 \arrow[r] & V\otimes_R W_1 \arrow[r, "V\otimes_R f"'] & V\otimes_R W \arrow[r, "V\otimes_R p"'] & V\otimes_R W_2 \arrow[r] & 0 \end{tikzcd}$$

where by $g(V_1) \otimes_R W$ we meant the $\mathbb{Z}$-submodule of $V \otimes_R W$ generated by elements of the form $g(v) \otimes_R w$ for $v \in V_1$ and $w \in W$. It is still a diagram of abelian groups with exact rows and commuting squares. Being $V \otimes_R f$ injective, we may identify its domain with its image, so that

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{im}(V_1\otimes_R W_1) \arrow[r, "\subseteq"] \arrow[d, "\subseteq"'] & \operatorname{im}(V_1\otimes_R W) \arrow[r, "V\otimes_R p"] \arrow[d, "\subseteq"] & V\otimes_R W_2 \arrow[d, equal] & \\ 0 \arrow[r] & \operatorname{im}(V\otimes_R W_1) \arrow[r, "\subseteq"'] & V\otimes_R W \arrow[r, "V\otimes_R p"'] & V\otimes_R W_2 \arrow[r] & 0 \end{tikzcd}$$

is still a diagram of abelian groups with exact rows and commuting squares. In view of [W, §10.3(2)], the left-most square is a pull-back diagram, which means exactly that in $V \otimes_R W$ we have $\operatorname{im}(V_1 \otimes_R W_1) = \operatorname{im}(V_1 \otimes_R W) \cap \operatorname{im}(V \otimes_R W_1)$. $\square$

Lemma B.6 (compare with [AMe])**.** *Let R be a $\Bbbk$-algebra and V, W be decreasingly filtered R-bimodules such that W/F_nW is projective as left R-module for all $n \in \mathbb{N}$. Then*

$$\sum_{p+q=n} \operatorname{im}(F_pV \otimes_R F_qW) = \bigcap_{p+q=n+1} \operatorname{ker}\left(\pi_p^V \otimes_R \pi_q^W\right)$$

where $\pi_p^V : V \to V/F_pV$ and $\pi_q^W : W \to W/F_qW$ are the canonical projections. In particular, for M and N R-bimodules such that N is locally fgp on the right, we have

$$F_n(M^* \otimes_R N^*) = \bigcap_{p+q=n-1} \operatorname{ker}\Big(\left(\tau_p^M\right)^* \otimes_R \left(\tau_q^N\right)^*\Big). \tag{B.7}$$

Proof. The inclusion from left to right is trivial, so let us prove the other one. The hypotheses on the W/F_nW's imply that every injection $\tau_n^W : F_nW \to W$ admits a retraction ρ_n^W and every projection $\pi_n^W : W \to W/F_nW$ admits a section σ_n^W which are left R-linear. Since every τ_n^W is filtered, for all $m \geq n$ it induces $\tilde{\tau}_{n,m}^W : F_nW/F_mW \to W/F_mW$ such that $\tilde{\tau}_{n,m}^W \circ \pi_{n,m}^W = \pi_m^W \circ \tau_n^W$, where $\pi_{n,m}^W : F_nW \to F_nW/F_mW$ is the canonical projection. Since $\rho_n^W \circ \tau_m^W = \rho_n^W \circ \tau_n^W \circ \tau_{m,n}^W = \tau_{m,n}^W$, it turns out that ρ_n^W induces a retraction $\tilde{\rho}_n^W : W/F_mW \to F_nW/F_mW$ of $\tilde{\tau}_{n,m}^W$ such that $\tilde{\rho}_n^W \circ \pi_m^W = \pi_{n,m}^W \circ \rho_n^W$. Analogously, σ_m^W induces a section $\tilde{\sigma}_m^W : F_nW/F_mW \to F_nW$ of $\pi_{n,m}^W$ such that $\tau_n^W \circ \tilde{\sigma}_m^W = \sigma_m^W \circ \tilde{\tau}_{n,m}^W$. Since

$$\pi_{n-i}^V \otimes_R \left(\pi_{i+1}^W \circ \tau_i^W \circ \rho_i^W \circ \cdots \circ \tau_1^W \circ \rho_1^W\right) = \pi_{n-i}^V \otimes_R \left(\tilde{\tau}_{i,i+1}^W \circ \pi_{i,i+1}^W \circ \rho_i^W \circ \cdots \circ \tau_1^W \circ \rho_1^W\right),$$

it follows that $\left(\pi_{n-i}^V \otimes_R \left(\pi_{i+1}^W \circ \tau_i^W \circ \rho_i^W \circ \cdots \circ \tau_1^W \circ \rho_1^W\right)\right)(x) = 0$ if and only if

$$\left(V \otimes_R \left(\pi_{i,i+1}^W \circ \rho_i^W \circ \cdots \circ \tau_1^W \circ \rho_1^W\right)\right)(x) \in \operatorname{ker}\left(\pi_{n-i}^V \otimes_R \frac{F_iW}{F_{i+1}W}\right) = \operatorname{im}\left(\tau_{n-i}^V \otimes_R \frac{F_iW}{F_{i+1}W}\right).$$

Pick $x \in \bigcap_{p+q=n+1} \operatorname{ker}\left(\pi_p^V \otimes_R \pi_q^W\right)$ and write it as

$$\begin{aligned} x = {} & \left(V \otimes_R \left(\sigma_1^W \circ \pi_1^W\right)\right)(x) + \sum_{i=1}^{n-1} \left(V \otimes_R \left(\sigma_{i+1}^W \circ \pi_{i+1}^W \circ \tau_i^W \circ \rho_i^W \circ \cdots \circ \tau_1^W \circ \rho_1^W\right)\right)(x) + \\ & + \left(V \otimes_R \left(\tau_n^W \circ \rho_n^W \circ \tau_{n-1}^W \circ \rho_{n-1}^W \circ \cdots \circ \tau_1^W \circ \rho_1^W\right)\right)(x). \end{aligned}$$

Notice that, by the hypothesis on x,

$$\left(V \otimes_R \pi_1^W\right)(x) \in \ker\left(\pi_n^V \otimes_R \frac{W}{F_1 W}\right) = \operatorname{im}\left(\tau_n^V \otimes_R \frac{W}{F_1 W}\right)$$

so that $\left(V \otimes_R \left(\sigma_1^W \circ \pi_1^W\right)\right)(x) \in \operatorname{im}\left(F_n V \otimes_R W\right)$ and $\left(\pi_{n-1}^V \otimes_R \left(\pi_2^W \circ \sigma_1^W \circ \pi_1^W\right)\right)(x) = 0$. By proceeding inductively one shows that we have

$$\left(V \otimes_R \left(\pi_{i+1}^W \circ \tau_i^W \circ \rho_i^W \circ \cdots \circ \tau_1^W \circ \rho_1^W\right)\right)(x) \in \ker\left(\pi_{n-i}^V \otimes_R \frac{W}{F_{i+1} W}\right) = \operatorname{im}\left(\tau_{n-i}^V \otimes_R \frac{W}{F_{i+1} W}\right),$$

whence $\left(V \otimes_R \left(\sigma_{i+1}^W \circ \pi_{i+1}^W \circ \tau_i^W \circ \rho_i^W \circ \cdots \circ \tau_1^W \circ \rho_1^W\right)\right)(x) \in \operatorname{im}\left(F_{n-i} V \otimes_R W\right) \subseteq V \otimes_R W$ for all $0 \leq i \leq n-1$. On the other hand, since

$$\left(V \otimes_R \left(\pi_{i+1}^W \circ \tau_i^W \circ \rho_i^W \circ \cdots \circ \tau_1^W \circ \rho_1^W\right)\right)(x) \in \operatorname{im}\left(V \otimes_R \frac{F_i W}{F_{i+1} W}\right) \subseteq V \otimes_R \frac{W}{F_{i+1} W},$$

we have that $\left(V \otimes_R \left(\sigma_{i+1}^W \circ \pi_{i+1}^W \circ \tau_i^W \circ \rho_i^W \circ \cdots \circ \tau_1^W \circ \rho_1^W\right)\right)(x) \in \operatorname{im}\left(V \otimes_R F_i W\right) \subseteq V \otimes_R W$ for all $0 \leq i \leq n-1$. Thus, in view of Lemma B.5,

$$\begin{aligned}\left(V \otimes_R \left(\sigma_{i+1}^W \circ \pi_{i+1}^W \circ \tau_i^W \circ \rho_i^W \circ \cdots \circ \tau_1^W \circ \rho_1^W\right)\right)(x) \in \operatorname{im}\left(V \otimes_R F_i W\right) \cap \operatorname{im}\left(F_{n-i} V \otimes_R W\right) = \\ = \operatorname{im}\left(F_{n-i} V \otimes_R F_i W\right) \subseteq V \otimes_R W\end{aligned}$$

for all $0 \leq i \leq n-1$. Clearly, the summand $\left(V \otimes_R \left(\tau_n^W \circ \rho_n^W \circ \tau_{n-1}^W \circ \rho_{n-1}^W \circ \cdots \circ \tau_1^W \circ \rho_1^W\right)\right)(x)$ lives in $\operatorname{im}\left(V \otimes_R F_n W\right)$, so that the first assertion is proved. For what concerns the second one, if N is locally fgp on the right then, by definition of $F_{k+1}\left(N^*\right)$ and by *(i)* of Corollary B.2, $N^* / F_{k+1}\left(N^*\right) \cong \left(F^k N\right)^*$ for all $k \geq 0$, which is a finitely generated and projective left R-module. Thus we may apply the first assertion to claim that

$$\sum_{p+q=n} \operatorname{im}\left(F_p\left(M^*\right) \otimes_R F_q\left(N^*\right)\right) = \bigcap_{p+q=n+1} \ker\left(\pi_p^{M^*} \otimes_R \pi_q^{N^*}\right) = \bigcap_{p+q=n+1} \ker\left(\left(\tau_{p-1}^M\right)^* \otimes_R \left(\tau_{q-1}^N\right)^*\right).$$

Notice however that since both morphisms τ_{-1}^M and τ_{-1}^N are the 0 morphism, we have that $\ker\left(\left(\tau_{p-1}^M\right)^* \otimes_R \left(\tau_{q-1}^N\right)^*\right) = M^* \otimes_R N^*$ for the pairs $(p,q) \in \{(0,n+1),(n+1,0)\}$ and so they do not contribute to the intersection. Therefore,

$$F_n(M^* \otimes_R N^*) = \bigcap_{\substack{p+q=n+1 \\ p,q \geq 1}} \ker\left(\left(\tau_{p-1}^M\right)^* \otimes_R \left(\tau_{q-1}^N\right)^*\right) = \bigcap_{p+q=n-1} \ker\left(\left(\tau_p^M\right)^* \otimes_R \left(\tau_q^N\right)^*\right). \qquad \square$$

Now we are ready to state and prove Proposition B.7. It gives the relation between the complete bimodules $\widehat{M^* \otimes_R N^*} = M^* \widehat{\otimes}_R N^*$ and $(N \otimes_R M)^*$, where $M^* \otimes_R N^*$ is endowed with the filtration (B.6) and the decreasing filtration on $(N \otimes_R M)^*$ is given as in (B.3), that is, $F_0\left((N \otimes_R M)^*\right) = (N \otimes_R M)^*$ and $F_n\left((N \otimes_R M)^*\right) = \ker\left(\boldsymbol{\tau}_{n-1}^*\right)$ for $n \geq 1$. The proof is long and a bit technical, whence we split it into smaller results.

Proposition B.7. *Let M and N be two R-bimodules, locally fgp as right R-modules. The natural transformation $\phi_{M,N}$ of equation* (B.4) *induces a filtered isomorphism $M^* \widehat{\otimes}_R N^* \cong (N \otimes_R M)^*$ such that the following diagram is commutative*

$$\begin{tikzcd}
\left(M^*\right)_R \otimes_R {}_R\left(N^*\right) \arrow[rr, "\phi_{M,N}"] \arrow[dr, "\gamma_{M^* \otimes_R N^*}"'] & & \left(N_R \otimes_R {}_R M\right)^* \\
& \left(M^*\right)_R \widehat{\otimes}_R {}_R\left(N^*\right) \arrow[ur, dashed, "\cong", "\widehat{\phi_{M,N}}"'] &
\end{tikzcd}$$

Let us devote the remaining part of this section to the proof of this proposition. Henceforth we will assume that M and N are R-bimodules which are locally fgp on the right and, for the sake of clearness, we will denote by

$$\gamma : M^* \otimes_R N^* \to \widehat{M^* \otimes_R} N^*, \qquad \pi_k : M^* \otimes_R N^* \to \frac{M^* \otimes_R N^*}{F_k\left(M^* \otimes_R N^*\right)},$$
$$\pi_{l,k} : \frac{M^* \otimes_R N^*}{F_l\left(M^* \otimes_R N^*\right)} \to \frac{M^* \otimes_R N^*}{F_k\left(M^* \otimes_R N^*\right)}, \qquad p_k : M^* \widehat{\otimes}_R N^* \to \frac{M^* \otimes_R N^*}{F_k\left(M^* \otimes_R N^*\right)},$$

the obvious maps, for all $k \geq 0$ and all $l \geq k$. Notice that since θ_m^M is a retraction of τ_m^M (see Remark B.3), we know that for all $p \leq m$ we have

$$\theta_m^M \circ \tau_p^M = \theta_m^M \circ \tau_m^M \circ \tau_{p,m}^M = \tau_{p,m}^M \tag{B.8}$$

and hence $\left(\tau_p^M\right)^* \circ (\theta_m^M)^* = \left(\tau_{p,m}^M\right)^*$ for all $m \geq 0$ and all $p \leq m$.

Our first aim is to show that the completion $\widehat{\phi_{M,N}}$ actually exists.

Lemma B.8. *The dual bimodule $(N \otimes_R M)^*$ is a complete R-bimodule with respect to the filtration*

$$F_{k+1}\left((N \otimes_R M)^*\right) = \mathsf{ker}\left(\boldsymbol{\tau}_k^*\right) = \mathsf{Ann}\left(F^k(N \otimes_R M)\right) \tag{B.9}$$

(see §B.2). Moreover, the canonical morphism $\phi_{M,N}$ is filtered and hence it induces a morphism of complete bimodules

$$\widehat{\phi_{M,N}} : M^* \widehat{\otimes}_R N^* \to (N \otimes_R M)^*.$$

Proof. As we observed at the beginning of the section, the category of bimodules which are locally fgp on the right is closed under taking tensor products. Therefore, $N \otimes_R M$ is a locally fgp right R-module and hence $(N \otimes_R M)^*$ is a complete R-bimodule with respect to the filtration (B.9). Thus, it is enough for us to prove that $\phi_{M,N}$ is filtered.

In view of (B.5), for all $m + n = k$ we have that

$$\left(\tau_n^N \otimes_R \tau_m^M\right)^* \left(\phi_{M,N}\left(F_{k+1}\left(M^* \otimes_R N^*\right)\right)\right) = \phi_{m,n}\left(\left(\left(\tau_m^M\right)^* \otimes_R \left(\tau_n^N\right)^*\right)\left(F_{k+1}\left(M^* \otimes_R N^*\right)\right)\right) \overset{\text{(B.7)}}{=} 0.$$

In particular, this implies that $\phi_{M,N}$ is filtered as claimed and $\widehat{\phi_{M,N}}$ is well-defined. □

Furthermore, we deduce from the argument of the above proof that for all $k \geq 0$ and for all $m, n \geq 0$ such that $m + n = k$, there exists a unique R-bilinear morphism

$$\sigma_{m,n} : \frac{\left(M^* \otimes_R N^*\right)}{F_{k+1}\left(M^* \otimes_R N^*\right)} \longrightarrow \left(F^n N \otimes_R F^m M\right)^*$$

such that

$$\sigma_{m,n} \circ \pi_{k+1} = \left(\tau_n^N \otimes_R \tau_m^M\right)^* \circ \phi_{M,N}. \tag{B.10}$$

Lemma B.9. *For every $m, n \geq 0$, $q \leq n$, $p \leq m$, $k = m + n$ and $h = p + q$, the morphism $\sigma_{m,n}$ satisfies also*

$$\left(\tau_{q,n}^N \otimes_R \tau_{p,m}^M\right)^* \circ \sigma_{m,n} = \sigma_{p,q} \circ \pi_{k+1,h+1}. \tag{B.11}$$

Proof. We may compute directly

$$\begin{aligned}\left(\tau_{q,n}^N \otimes_R \tau_{p,m}^M\right)^* \circ \sigma_{m,n} \circ \pi_{m+n+1} &\overset{\text{(B.10)}}{=} \left(\tau_{q,n}^N \otimes_R \tau_{p,m}^M\right)^* \circ \left(\tau_n^N \otimes_R \tau_m^M\right)^* \circ \phi_{M,N} \\ &= \left(\tau_q^N \otimes_R \tau_p^M\right)^* \circ \phi_{M,N} \overset{\text{(B.10)}}{=} \sigma_{p,q} \circ \pi_{h+1} = \sigma_{p,q} \circ \pi_{k+1,h+1} \circ \pi_{k+1}.\end{aligned}$$ □

These facts will be needed soon in the next part of the proof. Notice moreover that the completion $\widehat{\phi_{M,N}}$ of the filtered morphism $\phi_{M,N}$ fits into the commutative diagram

$$\begin{tikzcd} M^* \widehat{\otimes}_R N^* \arrow[r, "\widehat{\phi_{M,N}}"] \arrow[d, "p_{k+1}"'] & \left(N \otimes_R M\right)^* \arrow[d, "(\tau_n^N \otimes_R \tau_m^M)^*"] \\ \frac{M^* \otimes_R N^*}{F_{k+1}(M^* \otimes_R N^*)} \arrow[r, "\sigma_{m,n}"] & \left(F^n N \otimes_R F^m M\right)^* \end{tikzcd} \tag{B.12}$$

for every $k \geq 0$ and for all $m, n \geq 0$ such that $m + n = k$.

Our next aim is to construct explicitly a filtered inverse for $\widehat{\phi_{M,N}}$. To do this, we are going to show that $M^* \widehat{\otimes}_R N^*$ is the projective limit of the projective system $\{(F^m M)^* \otimes_R (F^n N)^* \mid m, n \in \mathbb{N}\}$ with structure maps $\left(\tau_{p,m}^M\right)^* \otimes_R \left(\tau_{q,n}^N\right)^*$ for all $p \leq m$ and $q \leq n$ and then use this fact to construct a suitable map $(N \otimes_R M)^* \to M^* \widehat{\otimes}_R N^*$ which will prove to be the inverse of $\widehat{\phi_{M,N}}$.

For all $m + n = k$ consider the composition

$$\Pi_{m,n} := \phi_{m,n}^{-1} \circ \sigma_{m,n} \circ p_{k+1} \tag{B.13}$$

which gives an R-bilinear morphism $\Pi_{m,n} : M^* \widehat{\otimes}_R N^* \to (F^m M)^* \otimes_R (F^n N)^*$ (the candidate structure maps). Basically by definition it satisfies $\Pi_{m,n} \circ \gamma = (\tau_m^M)^* \otimes_R (\tau_n^N)^*$ and moreover the subsequent lemma holds.

Lemma B.10. *For all $k \geq 0$ we have*

$$F_{k+1}\left(M^* \widehat{\otimes}_R N^*\right) = \bigcap_{m+n=k} \mathsf{ker}\left(\Pi_{m,n}\right), \tag{B.14}$$

where as usual $F_{k+1}\left(M^* \widehat{\otimes}_R N^*\right) = \mathsf{ker}\left(p_{k+1}\right)$.

Proof. If we consider the commutative diagram

$$\begin{tikzcd} M^* \widehat{\otimes}_R N^* \arrow[r, "\Pi_{m,n}"] \arrow[d, "p_{k+1}"'] & (F^m M)^* \otimes_R (F^n N)^* \arrow[d, "\phi_{m,n}"] \\ \frac{M^* \otimes_R N^*}{F_{k+1}(M^* \otimes_R N^*)} \arrow[r, "\sigma_{m,n}"'] & (F^n N \otimes_R F^m M)^* \end{tikzcd}$$

for every $k \geq 0$ and every pair $m, n \geq 0$ such that $m + n = k$, then $\mathsf{ker}\left(p_{k+1}\right) \subseteq \mathsf{ker}\left(\Pi_{m,n}\right)$ and so $\mathsf{ker}\left(p_{k+1}\right) \subseteq \bigcap_{m+n=k} \mathsf{ker}\left(\Pi_{m,n}\right)$. Conversely, assume that $z \in \bigcap_{m+n=k} \mathsf{ker}\left(\Pi_{m,n}\right)$ and consider $p_{k+1}(z)$. There exists $x \in M^* \otimes_R N^*$ such that $p_{k+1}(z) = \pi_{k+1}(x)$ and for all $m, n \geq 0$ satisfying $m + n = k$ it turns out that

$$\begin{aligned} 0 &= \Pi_{m,n}(z) \overset{\text{(B.13)}}{=} \left(\phi_{m,n}^{-1} \circ \sigma_{m,n} \circ p_{k+1}\right)(z) = \left(\phi_{m,n}^{-1} \circ \sigma_{m,n} \circ \pi_{k+1}\right)(x) \\ &\overset{\text{(B.10)}}{=} \left(\phi_{m,n}^{-1} \circ \left(\tau_n^N \otimes_R \tau_m^M\right)^* \circ \phi_{M,N}\right)(x) \overset{\text{(B.5)}}{=} \left(\left(\tau_m^M\right)^* \otimes_R \left(\tau_n^N\right)^*\right)(x). \end{aligned}$$

Therefore, $x \in \mathsf{ker}\Big(\left(\tau_m^M\right)^* \otimes_R \left(\tau_n^N\right)^*\Big)$ for all $m + n = k$, whence

$$x \in \bigcap_{m+n=k} \mathsf{ker}\Big(\left(\tau_m^M\right)^* \otimes_R \left(\tau_n^N\right)^*\Big) \overset{\text{(B.7)}}{=} F_{k+1}\left(M^* \otimes_R N^*\right).$$

In particular, $0 = \pi_{k+1}(x) = p_{k+1}(z)$ and so $z \in \mathsf{ker}\left(p_{k+1}\right)$ as claimed. ☐

Now, for every $k \geq 0$ and all $m, n \geq 0$ such that $k = m + n$ consider the R-bilinear maps

$$\phi_{m,n}^{-1} \circ \sigma_{m,n} \colon \frac{M^* \otimes_R N^*}{F_{k+1}\left(M^* \otimes_R N^*\right)} \longrightarrow (F^m M)^* \otimes_R (F^n N)^* \tag{B.15}$$

and

$$\xi_{m,n} \colon (F^m M)^* \otimes_R (F^n N)^* \longrightarrow \frac{M^* \otimes_R N^*}{F_{h+1}(M^* \otimes_R N^*)}$$

where $h = \min(m,n)$ and for all $f \in (F^m M)^*, g \in (F^n N)^*$

$$\xi_{m,n}(f \otimes_R g) = \left(\left(\theta_m^M\right)^* (f) \otimes_R \left(\theta_n^N\right)^* (g) \right) + F_{h+1}(M^* \otimes_R N^*). \tag{B.16}$$

Lemma B.11. *For all $m,n \geq 0$, the morphisms $\xi_{m,n}$ are well-defined*[(2)].

Proof. The following computation

$$\begin{aligned} \left(\tau_m^M\right)^* \left(\left(\theta_m^M\right)^* (f \leftharpoondown r) \right) \otimes_R \left(\tau_n^N\right)^* \left(\left(\theta_n^N\right)^* (g) \right) &= (f \leftharpoondown r) \otimes_R g = f \otimes_R rg \\ &= \left(\tau_m^M\right)^* \left(\left(\theta_m^M\right)^* (f) \right) \otimes_R \left(\tau_n^N\right)^* \left(r \left(\theta_n^N\right)^* (g) \right) \end{aligned}$$

shows that $((\theta_m^M)^* (f \leftharpoondown r) \otimes_R (\theta_n^N)^* (g)) - ((\theta_m^M)^* (f) \otimes_R (\theta_n^N)^* (rg)) \in \mathsf{ker}\left((\tau_m^M)^* \otimes_R (\tau_n^N)^*\right)$ and since for every i,j such that $i+j = \min(m,n) = h$

$$\left(\tau_i^M\right)^* \otimes_R \left(\tau_j^N\right)^* = \left(\left(\tau_{i,m}^M\right)^* \otimes_R \left(\tau_{j,n}^N\right)^* \right) \circ \left(\left(\tau_m^M\right)^* \otimes_R \left(\tau_n^N\right)^* \right),$$

it is clear that

$$\mathsf{ker}\left(\left(\tau_m^M\right)^* \otimes_R \left(\tau_n^N\right)^* \right) \subseteq \bigcap_{i+j=h} \mathsf{ker}\left(\left(\tau_i^M\right)^* \otimes_R \left(\tau_j^N\right)^* \right) \stackrel{(B.7)}{=} F_{h+1}(M^* \otimes_R N^*) \tag{B.17}$$

and hence

$$\left(\theta_m^M\right)^* (f \leftharpoondown r) \otimes_R \left(\theta_n^N\right)^* (g) + F_{h+1}(M^* \otimes_R N^*) = \left(\theta_m^M\right)^* (f) \otimes_R \left(\theta_n^N\right)^* (rg) + F_{h+1}(M^* \otimes_R N^*).$$

A similar argument may be used to show that $\xi_{m,n}$ is also right R-linear, which is not immediate from the definition. □

These will be used to connect the projective system $\{M^* \otimes_R N^* / F_{h+1}(M^* \otimes_R N^*), \pi_{h,k}\}_{\mathbb{N}}$ with the diagram $\left\{ (F^m M)^* \otimes_R (F^n N)^*, \left(\tau_{p,m}^M\right)^* \otimes_R \left(\tau_{q,n}^N\right)^* \right\}_{\mathbb{N}^2}$.

Lemma B.12. *For all $m,n,p,q \geq 0$ such that $p+q = h = \min(m,n)$ and $l = \min(p,q)$, the following relations hold*

$$\sigma_{p,q} \circ \xi_{m,n} = \phi_{p,q} \circ \left(\left(\tau_{p,m}^M\right)^* \otimes_R \left(\tau_{q,n}^N\right)^* \right), \tag{B.18}$$

$$\xi_{p,q} \circ \phi_{p,q}^{-1} \circ \sigma_{p,q} = \pi_{h+1,l+1}. \tag{B.19}$$

Proof. To prove (B.18) observe that for all $f \in (F^m M)^*$ and all $g \in (F^n N)^*$ we have

$$\begin{aligned} \sigma_{p,q}\left(\xi_{m,n}(f \otimes_R g)\right) &= \sigma_{p,q}\Big(\left(\theta_m^M\right)^* (f) \otimes_R \left(\theta_n^N\right)^* (g) + F_{h+1}(M^* \otimes_R N^*) \Big) \\ &= \sigma_{p,q}\Big(\pi_{h+1}\Big(\left(\theta_m^M\right)^* (f) \otimes_R \left(\theta_n^N\right)^* (g) \Big) \Big) \\ &\stackrel{(B.10)}{=} \left(\tau_q^N \otimes_R \tau_p^M\right)^* \Big(\phi_{M,N}\Big(\left(\theta_m^M\right)^* (f) \otimes_R \left(\theta_n^N\right)^* (g) \Big) \Big) \\ &\stackrel{(B.5)}{=} \phi_{p,q}\Big(\Big(\left(\tau_p^M\right)^* \circ \left(\theta_m^M\right)^* \Big)(f) \otimes_R \Big(\left(\tau_q^N\right)^* \circ \left(\theta_n^N\right)^* \Big)(g) \Big) \\ &\stackrel{(B.8)}{=} \phi_{p,q}\Big(\left(\tau_{p,m}^M\right)^* (f) \otimes_R \left(\tau_{q,n}^N\right)^* (g) \Big) \end{aligned}$$

[(2)]In general, we cannot perform the tensor product $\left(\theta_m^M\right)^* \otimes_R \left(\theta_n^N\right)^*$ as the maps $\left(\theta_m^M\right)^*$ are just left R-linear and the claimed relation (B.18) follows by R-bilinearity of $\sigma_{p,q}\circ\xi_{m,n}$ and $\phi_{p,q}\circ\left(\left(\tau_{p,m}^M\right)^*\otimes_R\left(\tau_{p,m}^M\right)^*\right)$. Instead, to prove (B.19) notice firstly that since $\left(\theta_p^M\right)^*$ is a section of $\left(\tau_p^M\right)^*$ it follows that for all $f\in M^*$ one has $\left(\left(\theta_p^M\right)^*\circ\left(\tau_p^M\right)^*\right)(f)\in f+\ker\left(\left(\tau_p^M\right)^*\right)=f+F_{p+1}\left(M^*\right)$ and hence, since $l=\min(p,q)\leq p,q$, one has also that for all $g\in N^*$

$$\left(\left(\left(\theta_p^M\right)^*\circ\left(\tau_p^M\right)^*\right)(f)\otimes_R g\right)-(f\otimes_R g)\in F_{p+1}(M^*\otimes_R N^*)\subseteq F_{l+1}(M^*\otimes_R N^*). \tag{B.20}$$

An analogous result holds for all $g\in N^*$. Therefore, for all $f\in(F^mM)^*$ and all $g\in(F^nN)^*$ we have

$$\begin{aligned}\left(\xi_{p,q}\circ\phi_{p,q}^{-1}\circ\sigma_{p,q}\circ\pi_{h+1}\right)(f\otimes_R g)&\overset{(B.10)}{=}\left(\xi_{p,q}\circ\phi_{p,q}^{-1}\circ\left(\tau_q^N\otimes_R\tau_p^M\right)^*\circ\phi_{M,N}\right)(f\otimes_R g)\\&\overset{(B.5)}{=}\left(\xi_{p,q}\circ\left(\left(\tau_p^M\right)^*\otimes_R\left(\tau_q^N\right)^*\right)\right)(f\otimes_R g)\\&=\left(\left(\theta_p^M\right)^*\circ\left(\tau_p^M\right)^*\right)(f)\otimes_R\left(\left(\theta_q^N\right)^*\circ\left(\tau_q^N\right)^*\right)(g)+F_{l+1}\left(M^*\otimes_R N^*\right)\\&\overset{(B.20)}{=}f\otimes_R g+F_{l+1}\left(M^*\otimes_R N^*\right)=\left(\pi_{h+1,l+1}\circ\pi_{h+1}\right)(f\otimes_R g).\end{aligned}$$ □

Getting back to the point, in light of (B.11) the first family (B.15) of maps makes all the following diagrams commute

$$\begin{tikzcd}\frac{M^*\otimes_R N^*}{F_{m+n+1}(M^*\otimes_R N^*)} \arrow[r, "\sigma_{m,n}"] \arrow[d, "\pi_{m+n+1,p+q+1}"'] & (F^nN\otimes_R F^mM)^* \arrow[r, "\phi_{m,n}^{-1}"] \arrow[d, "(\tau_{q,n}^N\otimes_R\tau_{p,m}^M)^*"] & (F^mM)^*\otimes_R(F^nN)^* \arrow[d, "(\tau_{p,m}^M)^*\otimes_R(\tau_{q,n}^N)^*"] \\ \frac{M^*\otimes_R N^*}{F_{p+q+1}(M^*\otimes_R N^*)} \arrow[r, "\sigma_{p,q}"] & (F^nN\otimes_R F^mM)^* \arrow[r, "\phi_{p,q}^{-1}"] & (F^pM)^*\otimes_R(F^qN)^*\end{tikzcd} \tag{B.21}$$

Let us show that the second family (B.16) makes almost the same the other way around.

Lemma B.13. *For all $m,n\geq 0$, $p\leq m$, $q\leq n$, all the following diagrams commute*

$$\begin{tikzcd}(F^mM)^*\otimes_R(F^nN)^* \arrow[r, "\xi_{m,n}"] \arrow[d, "(\tau_{p,m}^M)^*\otimes_R(\tau_{q,n}^N)^*"'] & \frac{M^*\otimes_R N^*}{F_{l+1}(M^*\otimes_R N^*)} \arrow[d, "\pi_{l+1,h+1}"] \\ (F^pM)^*\otimes_R(F^qN)^* \arrow[r, "\xi_{p,q}"'] & \frac{M^*\otimes_R N^*}{F_{h+1}(M^*\otimes_R N^*)}\end{tikzcd} \tag{B.22}$$

where $h=\min(p,q)$ and $l=\min(m,n)$.

Proof. By a direct computation,

$$\begin{aligned}\left(\tau_p^M\right)^*\left(\left(\theta_p^M\right)^*\left(\left(\tau_{p,m}^M\right)^*(f)\right)\right)\otimes_R\left(\tau_q^N\right)^*\left(\left(\theta_q^N\right)^*\left(\left(\tau_{q,n}^N\right)^*(g)\right)\right)&=\left(\tau_{p,m}^M\right)^*(f)\otimes_R\left(\tau_{q,n}^N\right)^*(g)\\&\overset{(B.8)}{=}\left(\tau_p^M\right)^*\left(\left(\theta_m^M\right)^*(f)\right)\otimes_R\left(\tau_q^N\right)^*\left(\left(\theta_n^N\right)^*(g)\right).\end{aligned}$$

Thus, by using the same argument that we used in the proof of Lemma B.11 and, in particular, Relation (B.17), one may check that for all $f\in(F^mM)^*$ and all $g\in(F^nN)^*$

$$\left(\theta_p^M\right)^*\left(\left(\tau_{p,m}^M\right)^*(f)\right)\otimes_R\left(\theta_q^N\right)^*\left(\left(\tau_{q,n}^N\right)^*(g)\right)-\left(\theta_m^M\right)^*(f)\otimes_R\left(\theta_n^N\right)^*(g)\in F_{h+1}\left(M^*\otimes_R N^*\right).$$ □

With this last lemma, we collected results enough to prove that $M^*\widehat{\otimes}_R N^*$ together with the family of maps $\{\Pi_{m,n}\mid m,n\geq 0\}$ is (isomorphic to) the projective limit in the category of R-bimodules of the projective system $(F^mM)^*\otimes_R(F^nN)^*$ with structure maps $\left(\tau_{p,m}^M\right)^*\otimes_R\left(\tau_{q,n}^N\right)^*$ for all $p\leq m$ and $q\leq n$. We state it here the more explicitly that we can for future reference.

Lemma B.14. *Let M and N be two R-bimodules which are locally fgp on the right and consider*

$$\Pi_{m,n} : M^* \widehat{\otimes}_R N^* \to (F^m M)^* \otimes_R (F^n N)^*, \qquad \left(\widehat{z}_\infty \overset{(B.13)}{\longmapsto} \left(\left(\tau_m^M\right)^* \otimes_R \left(\tau_n^N\right)^*\right)(z_{m+n+1})\right).$$

for all $m, n \in \mathbb{N}$. Then $M^* \widehat{\otimes}_R N^*$ together with the family of morphisms $\{\Pi_{m,n} \mid m, n \in \mathbb{N}\}$ is the projective limit of the projective system $\{(F^m M)^* \otimes_R (F^n N)^* \mid m, n \in \mathbb{N}\}$ with structure maps $\left(\tau_{p,m}^M\right)^* \otimes_R \left(\tau_{q,n}^N\right)^*$ for all $p \leq m$ and $q \leq n$.

Proof. In light of (B.21) and the fact that $\left(M^* \widehat{\otimes}_R N^* \xrightarrow{p_k} (M^* \otimes_R N^*) / F_k (M^* \otimes_R N^*) \right)_{\mathbb{N}}$ is a source, we have that $\left(M^* \widehat{\otimes}_R N^* \xrightarrow{\Pi_{m,n}} (F^m M)^* \otimes_R (F^n N)^* \right)_{\mathbb{N}^2}$ is a source as well. Let us assume that there is another source $\left(X \xrightarrow{\beta_{m,n}} (F^m M)^* \otimes_R (F^n N)^* \right)_{\mathbb{N}^2}$. By considering the compositions $\xi_{m,n} \circ \beta_{m,n}$, one makes of X a source for $\{(M^* \otimes_R N^*) / F_k (M^* \otimes_R N^*), \pi_{k,h}\}_{\mathbb{N}}$ and hence there exists a unique morphism of bimodules $\Phi : X \to M^* \widehat{\otimes}_R N^*$ such that for all $h \geq 0$ one has

$$p_{h+1} \circ \Phi = \xi_{m,n} \circ \beta_{m,n} \tag{B.23}$$

for all m, n such that $h = \min(m, n)$[3]. For all $m, n \geq 0$ and $k = m + n$, this Φ satisfies

$$\Pi_{m,n} \circ \Phi \overset{(B.13)}{=} \phi_{m,n}^{-1} \circ \sigma_{m,n} \circ p_{k+1} \circ \Phi \overset{(B.23)}{=} \phi_{m,n}^{-1} \circ \sigma_{m,n} \circ \xi_{k,k} \circ \beta_{k,k} \overset{(B.18)}{=} ((\tau_{m,k}^M)^* \otimes_R (\tau_{n,k}^N)^*) \circ \beta_{k,k} = \beta_{m,n}.$$

Assume that there exists a morphism $\Psi : X \to M^* \widehat{\otimes}_R N^*$ such that $\Pi_{m,n} \circ \Psi = \beta_{m,n}$ for all $m, n \in \mathbb{N}$. For any $k \geq 0$ we have

$$p_{k+1} \circ \Psi = \pi_{2k+1,k+1} \circ p_{2k+1} \circ \Psi \overset{(B.19)}{=} \xi_{k,k} \circ \Pi_{k,k} \circ \Psi = \xi_{k,k} \circ \beta_{k,k}$$

which means, by uniqueness of the morphism Φ satisfying (B.23), that $\Psi = \Phi$ and $M^* \widehat{\otimes}_R N^*$ satisfies the universal property of the projective limit as claimed. □

Lemma B.14 allows us, finally, to prove Proposition B.7.

Proof of Proposition B.7. Since the tensor product commutes with colimits, since the filtrations on M and N are exhaustive (i.e. $M = \varinjlim (F^m M)$ and $N = \varinjlim (F^n N)$) and since the Hom functor converts colimits in limits, one may claim that $(N \otimes_R M)^*$ together with the morphisms $(\tau_n^N \otimes_R \tau_m^M)^*$ is the projective limit of the projective system $\{(F^n N \otimes_R F^m M)^* \mid m, n \in \mathbb{N}\}$ with structure maps $\left(\tau_{q,n}^N \otimes_R \tau_{p,m}^M\right)^*$ for $p \leq m$ and $q \leq n$. Now, by definition of $\Pi_{m,n}$ we have that

$$\phi_{m,n} \circ \Pi_{m,n} \overset{(B.13)}{=} \sigma_{m,n} \circ p_{k+1} \overset{(B.12)}{=} \left(\tau_n^N \otimes_R \tau_m^M\right)^* \circ \widehat{\phi_{M,N}}, \tag{B.24}$$

whence $\widehat{\phi_{M,N}}$ is also the unique morphism induced by the map of projective systems $\phi_{m,n}$. By considering $\phi_{m,n}^{-1}$ instead, one deduces that there exists a unique morphism $\psi_{M,N} : (N \otimes_R M)^* \to M^* \widehat{\otimes}_R N^*$ such that $\Pi_{m,n} \circ \psi_{M,N} = \phi_{m,n}^{-1} \circ (\tau_n^N \otimes_R \tau_m^M)^*$. It is not difficult to see that $\widehat{\phi_{M,N}}$ and $\psi_{M,N}$ are mutually inverses, so that we are really left to prove that ψ is filtered. Recall that $F_{n+1}((N \otimes_R M)^*) = \mathsf{Ann}(F^n(N \otimes_R M))$. Thus, for all $p + q = n$,

$$\Pi_{p,q}\Big(\psi_{M,N}\big(\mathsf{Ann}\left(F^n(N \otimes_R M)\right)\big)\Big) = \phi_{p,q}^{-1}\Big(\left(\tau_q^N \otimes_R \tau_p^M\right)^* \big(\mathsf{Ann}\left(F^n(N \otimes_R M)\right)\big)\Big) = 0$$

so that $\psi_{M,N}(\mathsf{Ann}\left(F^n(N \otimes_R M)\right)) \subseteq \bigcap_{p+q=n} \mathsf{ker}(\Pi_{p,q}) \overset{(B.14)}{=} F_{n+1}(M^* \widehat{\otimes}_R N^*)$ and $\psi_{M,N}$ is filtered. □

[3] By using (B.22), one may check that $\xi_{m,n} \circ \beta_{m,n} = \xi_{h,h} \circ \beta_{h,h}$ where $h = \min(m, n)$, so that the previous relations depend only on h and not on the particular m or n used.

Given $z \in M^* \widehat{\otimes}_R N^*$ we already know, by adapting Notation 3.1.24, that $z = \lim_{n\to\infty} (p_n(z))$ up to a choice of a representative in $M^* \otimes_R N^*$ for each element $p_n(z)$ (this is not restrictive, in light of Remark 3.1.19). Fix $h \geq 0$. For all $m, n \geq h$ such that $h = \min(m, n)$, set $k = m + n$. Then

$$\left(\xi_{m,n} \circ \Pi_{m,n}\right)(z) \overset{(B.13)}{=} \left(\xi_{m,n} \circ \phi_{m,n}^{-1} \circ \sigma_{m,n} \circ p_{k+1}\right)(z) \overset{(B.19)}{=} \left(\pi_{k+1,h+1} \circ p_{k+1}\right)(z) = p_{h+1}(z)$$

and hence $\xi_{m,n} \circ \Pi_{m,n} = p_{h+1}$. In particular,

$$z = \lim_{h\to\infty} (p_{h+1}(z)) = \lim_{h\to\infty} \left(\left(\xi_{h,h} \circ \Pi_{h,h}\right)(z)\right). \tag{B.25}$$

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