Homological interpretation of extensions and biextensions of 1-motives

Published version:

DOI:10.1016/j.jnt.2012.04.005

HOMOLOGICAL INTERPRETATION OF EXTENSIONS AND BIEXTENSIONS OF 1-MOTIVES

CRISTIANA BERTOLIN

ABSTRACT. Let k be a separably closed field. Let $K_i = [A_i \stackrel{u_i}{\to} B_i]$ (for $i = 1,2,3$) be three 1-motives defined over k. We define the geometrical notions of extension of K_1 by K_3 and of biextension of (K_1, K_2) by K_3. We then compute the homological interpretation of these new geometrical notions: namely, the group $\mathrm{Biext}^0(K_1, K_2; K_3)$ of automorphisms of any biextension of (K_1, K_2) by K_3 is canonically isomorphic to the group $\mathrm{Ext}^0(K_1 \overset{\mathbb{L}}{\otimes} K_2, K_3)$, and the group $\mathrm{Biext}^1(K_1, K_2; K_3)$ of isomorphism classes of biextensions of (K_1, K_2) by K_3 is canonically isomorphic to the group $\mathrm{Ext}^1(K_1 \overset{\mathbb{L}}{\otimes} K_2, K_3)$.

CONTENTS

Introduction 1
Notation 3
1. Extensions and biextensions of 1-motives 4
2. Review on strictly commutative Picard stacks 11
3. Proof of theorem 0.1 (**b**) 15
4. The category $\Psi_{\mathrm{L}_{\cdot\cdot}}(G)$ and its homological interpretation 16
5. Geometrical description of $\Psi_{\mathrm{L}_{\cdot\cdot}}(G)$ 22
6. Proof of theorem 0.1 (**a**) 28
References 29

INTRODUCTION

Let k be a separably closed field and let $S = \mathrm{Spec}\,(k)$. A 1-motive $K = [u : A \to B]$ over S consists of an S-group scheme A which is locally for the étale topology a constant group scheme defined by a finitely generated free $\mathbb{Z}$-module, an extension B of an abelian S-scheme by an S-torus, and a morphism $u : A \to B$ of S-group schemes. Since the field k is separably closed, remark that $A = \mathbb{Z}^r$ with $r \geq 0$.

Let $\mathbf{S}$ be the big fppf site over S. A 1-motive $K = [u : A \to B]$ can be viewed also as a complex of abelian sheaves on $\mathbf{S}$ concentrated in two consecutive degrees. A morphism of 1-motives is a morphism of complexes of commutative S-group schemes (see [R], in particular Lemma 2.3.2)

Let $K_i = [u_i : A_i \to B_i]$ (for $i = 1,2,3$) be three 1-motives defined over S. In this paper we introduce the geometrical notions of extension of K_1 by K_3 and of biextension of (K_1, K_2) by K_3. We then compute the homological interpretation of

1991 *Mathematics Subject Classification.* 18G15.
Key words and phrases. extensions, biextensions, 1-motives, tensor products.

these new geometrical notions. More precisely, if $\mathrm{Biext}^0(K_1,K_2;K_3)$ is the group of automorphisms of any biextension of (K_1,K_2) by K_3, $\mathrm{Biext}^1(K_1,K_2;K_3)$ is the group of isomorphism classes of biextensions of (K_1,K_2) by K_3, $\mathcal{E}xt^0(K_1,K_3)$ is the group of automorphisms of any extension of K_1 by K_3, and $\mathcal{E}xt^1(K_1,K_3)$ is the group of isomorphism classes of extensions of K_1 by K_3, then we prove

Theorem 0.1. *We have the following canonical isomorphisms*

(a) $\mathrm{Biext}^1(K_1,K_2;K_3) \cong \mathrm{Ext}^1(K_1 \overset{\mathbb{L}}{\otimes} K_2, K_3) = \mathrm{Hom}_{\mathcal{D}(\boldsymbol{S})}\big(K_1 \overset{\mathbb{L}}{\otimes} K_2, K_3[1]\big)$,

(b) $\mathrm{Biext}^0(K_1,K_2;K_3) \cong \mathrm{Ext}^0(K_1 \overset{\mathbb{L}}{\otimes} K_2, K_3) = \mathrm{Hom}_{\mathcal{D}(\boldsymbol{S})}\big(K_1 \overset{\mathbb{L}}{\otimes} K_2, K_3\big)$,

(c) $\mathcal{E}xt^1(K_1,K_3) \cong \mathrm{Ext}^1(K_1,K_3) = \mathrm{Hom}_{\mathcal{D}(\boldsymbol{S})}(K_1,K_3[1])$,

(d) $\mathcal{E}xt^0(K_1,K_3) \cong \mathrm{Ext}^0(K_1,K_3) = \mathrm{Hom}_{\mathcal{D}(\boldsymbol{S})}(K_1,K_3)$,

where $K_1 \overset{\mathbb{L}}{\otimes} K_2$ is the derived functor of the functor $K_2 \to K_1 \otimes K_2$ in the derived category $\mathcal{D}(\boldsymbol{S})$ of complexes of abelian sheaves on $\boldsymbol{S}$.

The homological interpretation (c)-(d) of extensions of 1-motives is a special case of the homological interpretation (a)-(b) of biextensions of 1-motives: in fact, if $K_2 = [0 \to \mathbb{Z}]$

(1) the category of biextensions of $(K_1,[0 \to \mathbb{Z}])$ by K_3 is equivalent to the category of extensions of K_1 by K_3, and

(2) in the derived category $\mathrm{Ext}^i(K_1 \overset{\mathbb{L}}{\otimes} [0 \to \mathbb{Z}], K_3) \cong \mathrm{Ext}^i(K_1,K_3)$ for $i = 0,1$.

Applications of Theorem 0.1 are given by the isomorphism

$$\mathrm{Biext}^1(K_1,K_2;K_3) \cong \mathrm{Ext}^1(K_1 \overset{\mathbb{L}}{\otimes} K_2, K_3) = \mathrm{Hom}_{\mathcal{D}(\mathcal{C})}(K_1 \overset{\mathbb{L}}{\otimes} K_2, K_3[1]) \tag{0.1}$$

which makes explicit the link between biextensions and bilinear morphisms. A classical example of this link is given by the Poincaré biextension of an abelian variety which defines the Weil pairing on the Tate modules. Other examples are furnished by [B08] and [BM], where we prove that

- the group of isomorphism classes of biextensions of (K_1,K_2) by K_3 is isomorphic to the group of morphisms of the category $\mathcal{MHS}$ of mixed Hodge structures from the tensor product $\mathrm{T_H}(K_1) \otimes \mathrm{T_H}(K_2)$ of the Hodge realizations of K_1 and K_2 to the Hodge realization $\mathrm{T_H}(K_3)$ of K_3:
$$\mathrm{Biext}^1(K_1,K_2;K_3) \cong \mathrm{Hom}_{\mathcal{MHS}}(\mathrm{T_H}(K_1) \otimes \mathrm{T_H}(K_2), \mathrm{T_H}(K_3)).$$
- modulo isogenies the group of isomorphism classes of biextensions of (K_1,K_2) by K_3 is isomorphic to the group of morphisms of the category $\mathcal{MR}_{\mathbb{Z}}(k)$ of mixed realizations with integral structure from the tensor product $\mathrm{T}(K_1) \otimes \mathrm{T}(K_2)$ of the realizations of K_1 and K_2 to the realization $\mathrm{T}(K_3)$ of K_3:
$$\mathrm{Biext}^1(K_1,K_2;K_3) \otimes \mathbb{Q} \cong \mathrm{Hom}_{\mathcal{MR}_{\mathbb{Z}}(k)}\big(\mathrm{T}(K_1) \otimes \mathrm{T}(K_2), \mathrm{T}(K_3)\big).$$
Following Deligne's philosophy of motives described in [D89] 1.11, this isomorphism means that the notion of biextensions of 1-motives furnishes the geometrical origin of the morphisms of $\mathcal{MR}_{\mathbb{Z}}(k)$ from the tensor product of the realizations of two 1-motives to the realization of another 1-motive, which are therefore motivic morphisms.
- modulo isogenies the group of isomorphism classes of biextensions of (K_1,K_2) by K_3 is isomorphic to the group of morphisms of Voevodsky's triangulated category $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ of effective geometrical motives with rational coefficients

from the tensor product $\mathcal{O}(K_1) \otimes \mathcal{O}(K_2)$ of the images of K_1 and K_2 in the category $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ to the image $\mathcal{O}(K_3)$ of K_3 in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$:

$$\mathrm{Biext}^1(K_1, K_2; K_3) \otimes \mathbb{Q} \cong \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k,\mathbb{Q})}(\mathcal{O}(K_1) \otimes \mathcal{O}(K_2), \mathcal{O}(K_3)).$$

In [BM] we have used Theorem 0.1 (a) in order to show the above isomorphism.

In [Be11] and [Be] we have introduced the notions of extension and biextension for arbitrary length 2 complexes of abelian sheaves and we have computed their homological interpretation. The definitions and the results of [Be11] and [Be] are a generalization of the definitions and the results of this paper (in particular of Theorem 0.1) to arbitrary length 2 complexes of abelian sheaves.

The idea of the proof of Theorem 0.1 is the following one: Let $K = [A \xrightarrow{u} B]$ be a 1-motives and let L.. be a complex of 1-motives $R \to Q \to P \to 0$. To the complex K and to the bicomplex L.. we associate a category $\Psi_{\mathrm{L..}}(K)$ which has the following *homological description*:

$$\Psi^i_{\mathrm{L..}}(K) \cong \mathrm{Ext}^i(\mathrm{Tot}(\mathrm{L..}), K) \qquad (i = 0, 1) \tag{0.2}$$

where $\Psi^0_{\mathrm{L..}}(K)$ is the group of automorphisms of any object of $\Psi_{\mathrm{L..}}(K)$ and $\Psi^1_{\mathrm{L..}}(K)$ is the group of isomorphism classes of objects of $\Psi_{\mathrm{L..}}(K)$. Then, to any 1-motive $K = [A \xrightarrow{u} B]$ we associate a canonical flat partial resolution $\mathrm{L..}(K)$ whose components are direct sums of objects of the kind $\mathbb{Z}[A]$ and $\mathbb{Z}[B]$. Here "partial resolution" means that we have an isomorphism between the homology groups of K and of this partial resolution only in degree 1 and 0. This is enough for our goal since only the groups Ext^1 and Ext^0 are involved in the statement of Theorem 0.1. Consider now three 1-motives K_i (for $i = 1, 2, 3$). The categories $\Psi_{\mathrm{L..}(K_1)}(K_3)$ and $\Psi_{\mathrm{L..}(K_1)\otimes \mathrm{L..}(K_2)}(K_3)$ admit the following *geometrical description*:

$$\begin{aligned} \Psi_{\mathrm{L..}(K_1)}(K_3) &\simeq \mathbf{Ext}(K_1, K_3) \\ \Psi_{\mathrm{L..}(K_1)\otimes \mathrm{L..}(K_2)}(K_3) &\simeq \mathbf{Biext}(K_1, K_2; K_3) \end{aligned} \tag{0.3}$$

Putting together this geometrical description (0.3) with the homological description (0.2), we get the proof of Theorem 0.1.

Notation

In this paper, k is a separably closed field, $S = \mathrm{Spec}\,(k)$ and $\mathbf{S}$ is the big fppf site over S. If I is a sheaf on $\mathbf{S}$, we denote by $\mathbb{Z}[I]$ the free $\mathbb{Z}$-module generated by I (see [D73] Exposé IV 11).

Denote by $\mathcal{K}(\mathbf{S})$ the category of complexes of abelian sheaves on the site $\mathbf{S}$: all complexes that we consider in this paper are cochain complexes. Let $\mathcal{K}^{[-1,0]}(\mathbf{S})$ be the subcategory of $\mathcal{K}(\mathbf{S})$ consisting of complexes $K = (K^i)_i$ such that $K^i = 0$ for $i \neq -1$ or 0. The good truncation $\tau_{\leq n} K$ of a complex K of $\mathcal{K}(\mathbf{S})$ is the following complex: $(\tau_{\leq n} K)^i = K^i$ for $i < n$, $(\tau_{\leq n} K)^n = \ker(d^n)$ and $(\tau_{\leq n} K)^i = 0$ for $i > n$. For any $i \in \mathbb{Z}$, the shift functor $[i] : \mathcal{K}(\mathbf{S}) \to \mathcal{K}(\mathbf{S})$ acts on a complex $K = (K^n)_n$ as $(K[i])^n = K^{i+n}$ and $d^n_{K[i]} = (-1)^i d^{n+i}_K$.

Denote by $\mathcal{D}(\mathbf{S})$ the derived category of the category of abelian sheaves on $\mathbf{S}$, and let $\mathcal{D}^{[-1,0]}(\mathbf{S})$ be the subcategory of $\mathcal{D}(\mathbf{S})$ consisting of complexes K such that $\mathrm{H}^i(K) = 0$ for $i \neq -1$ or 0. If K and K' are complexes of $\mathcal{D}(\mathbf{S})$, the group $\mathrm{Ext}^i(K, K')$ is by definition $\mathrm{Hom}_{\mathcal{D}(\mathbf{S})}(K, K'[i])$ for any $i \in \mathbb{Z}$. Let $\mathrm{RHom}(-,-)$ be the derived functor of the bifunctor $\mathrm{Hom}(-,-)$. The cohomology groups $\mathrm{H}^i\big(\mathrm{RHom}(K,K')\big)$ of $\mathrm{RHom}(K,K')$ are isomorphic to $\mathrm{Hom}_{\mathcal{D}(\mathbf{S})}(K,K'[i])$.

1. Extensions and biextensions of 1-motives

Let G be abelian sheaf on $\mathbf{S}$. A **G-torsor** is a sheaf on $\mathbf{S}$ endowed with an action of G, which is locally isomorphic to G acting on itself by translation.

Let P,G be abelian sheaves on $\mathbf{S}$. An **extension of P by G** is an exact sequence

$$0 \longrightarrow G \longrightarrow E \longrightarrow P \longrightarrow 0.$$

Since in this paper we consider only commutative extensions, E is in fact an abelian sheaf on $\mathbf{S}$. We denote by $\mathbf{Ext}(P,G)$ the category of extensions of P by G. It is a classical result that the Baer sum of extensions defines a group law for the objects of the category $\mathbf{Ext}(P,G)$, which is therefore a strictly commutative Picard category.

Let P,G be abelian sheaves on $\mathbf{S}$. Denote by $m : P\times P \to P$ the group law of P and by $pr_i : P\times P \to P$ with $i=1,2$ the two projections of $P\times P$ in P. According to [G] Exposé VII 1.1.6 and 1.2, the category of extensions of P by G is equivalent to the category of 4-tuples (P,G,E,φ), where E is a G_P-torsor over P, and $\varphi : pr_1^*E \wedge pr_2^*E \to m^*E$ is an isomorphism of torsors over $P\times P$ satisfying some associativity and commutativity conditions (see [G] Exposé VII diagrams (1.1.4.1) and (1.2.1)):

$$\begin{aligned}\mathbf{Ext}(P,G) \quad &\simeq \quad \Big\{(P,G,E,\varphi)\;\Big|\; E = G_P-\text{torsor over } P \text{ and}\\ &\qquad \varphi : pr_1^*E \wedge pr_2^*E \cong m^*E \text{ with ass. and comm. conditions}\Big\}.\end{aligned} \tag{1.1}$$

Here pr_i^*E is the pull-back of E via the projection $pr_i : P\times P \to P$ for $i=1,2$ and $pr_1^*E\wedge pr_2^*E$ is the contracted product of pr_1^*E and pr_2^*E (see 1.3 Chapter III [G71]). It will be useful in what follows to look at the isomorphism of torsors φ as an associative and commutative group law on the fibres:

$$+ : E_p\ E_{p'} \longrightarrow E_{p+p'}$$

where p,p' are points of $P(U)$ with U an S-scheme.

Let I be a sheaf on $\mathbf{S}$ and let G be an abelian sheaf on $\mathbf{S}$. Concerning extensions of free commutative groups, by [G] Exposé VII 1.4 the category of extensions of $\mathbb{Z}[I]$ by G is equivalent to the category of G_I-torsors over I:

$$\mathbf{Ext}(\mathbb{Z}[I],G) \simeq \mathbf{Tors}(I,G_I). \tag{1.2}$$

Let P,Q and G be abelian sheaves on $\mathbf{S}$. A **biextension of (P,Q) by G** is a $G_{P\times Q}$-torsor B over $P\times Q$, endowed with a structure of commutative extension of Q_P by G_P and a structure of commutative extension of P_Q by G_Q, which are compatible one with another (for the definition of compatible extensions see [G] Exposé VII Définition 2.1). If m_P,p_1,p_2 (resp. m_Q,q_1,q_2) denote the three morphisms $P\times P\times Q \to P\times Q$ (resp. $P\times Q\times Q \to P\times Q$) deduced from the three morphisms $P\times P\to P$ (resp. $Q\times Q\to Q$) group law, first and second projection, the equivalence of categories (1.1) furnishes the following equivalent definition: a biextension of (P,Q) by G is a $G_{P\times Q}$-torsor B over $P\times Q$ endowed with two isomorphisms of torsors

$$\varphi : p_1^*E\ p_2^*E \longrightarrow m_P^*E \qquad\qquad \psi : q_1^*E\ q_2^*E \longrightarrow m_Q^*E$$

over $P \times P \times Q$ and $P \times Q \times Q$ respectively, satisfying some associativity, commutativity and compatible conditions (see [G] Exposé VII diagrams (2.0.5),(2.0.6),(2.0.8), (2.0.9), (2.1.1)). As for extensions, we will look at the isomorphisms of torsors φ and ψ as two associative and commutative group laws on the fibres which are compatible with one another:

$$+_1 : E_{p,q}\ E_{p',q} \longrightarrow E_{p+p',q} \qquad\qquad +_2 : E_{p,q}\ E_{p,q'} \longrightarrow E_{p,q+q'}$$

where p, p' (resp. q, q') are points of $P(U)$ (resp. of $Q(U)$) with U any sheaf on $\mathbf{S}$.

Let $K_i = [u_i : A_i \to B_i]$ (for $i = 1, 2$) be two 1-motives defined over S.

Definition 1.1. An **extension** (E, β, γ) **of** K_1 **by** K_2 consists of

1. an extension E of B_1 by B_2;
2. a trivialization β of the extension $u_1^* E$ of A_1 by B_2 obtained as pull-back of the extension E via $u_1 : A_1 \to B_1$;
3. a trivial extension $T = (T, \gamma)$ of A_1 by A_2 (i.e. an extension T of A_1 by A_2 endowed with a trivialization γ) and an isomorphism of extensions $\Theta : u_{2*}T \to u_1^* E$ between the push-down via $u_2 : A_2 \to B_2$ of T and $u_1^* E$. Through this isomorphism the trivialization $u_2 \circ \gamma$ of $u_{2*}T$ is compatible with the trivialization β of $u_1^* E$.

Condition (3) can be rewritten as

(3') an homomorphism $\gamma : A_1 \to A_2$ such that $u_2 \circ \gamma$ is compatible with β.

Note that to have a trivialization $\beta : A_1 \to u_1^* E$ of $u_1^* E$ is the same thing as to have a lifting $\widetilde{\beta} : A_1 \to E$ of $u_1 : A_1 \to B_1$. In fact, if we denote $p : E \to B_1$ the canonical surjection of the extension E, a morphism $\widetilde{\beta} : A_1 \to E$ such that $p \circ \widetilde{\beta} = u_1$ induces a splitting $\beta : A_1 \to u_1^* E$ that composes with $u_1^* E \to E \stackrel{p}{\to} B_1$ to $u_1 : A_1 \to B_1$, and vice versa.

Remark 1.2. We can summarize the above definition with the following diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & B_2 \arrow[r] & E \arrow[r] & B_1 \arrow[r] & 0 \\
0 \arrow[r] & B_2 \arrow[r] \arrow[u, equal] & u_1^* E \cong u_{2*}T \arrow[r] \arrow[u] & A_1 \arrow[r] \arrow[u, "u_1"'] \arrow[l, bend left, "\beta"] & 0 \\
0 \arrow[r] & A_2 \arrow[r] \arrow[u, "u_2"] & T \arrow[r] \arrow[u] & A_1 \arrow[r] \arrow[u, equal] \arrow[l, bend left, "\gamma"] & 0
\end{tikzcd}$$

In particular, we observe that the short sequence of complexes in $\mathcal{K}(\mathbf{S})$

$$0 \longrightarrow K_2 \longrightarrow [T \to E] \longrightarrow K_1 \longrightarrow 0$$

is exact. On the other hand if $0 \to K_2 \to G \to K_1 \to 0$ is a short exact sequence of $\mathcal{K}(\mathbf{S})$, then the complex G is an extension of 1-motives of K_1 by K_2 as defined in Definition 1.1, i.e. G is a complex of the kind $[T \to E]$, with T a trivial extension of A_1 by A_2 and E an extension of B_1 by B_2. In fact, over a separably closed field the groups $\mathrm{Ext}^1(A_1, A_2)$ and $\mathrm{Ext}^1(A_1, B_2)$ are trivial.

Let $K_i = [A_i \stackrel{u_i}{\to} B_i]$ and $K'_i = [A'_i \stackrel{u'_i}{\to} B'_i]$ (for $i = 1, 2$) be 1-motives defined over S. Let (E, β, γ) be an extension of K_1 by K_2 and let (E', β', γ') be an extension of K'_1 by K'_2.

Definition 1.3. A **morphism of extensions**

$$(\underline{F}, \underline{\Upsilon}, \underline{\Phi}) : (E, \beta, \gamma) \longrightarrow (E', \beta', \gamma')$$

consists of

(1) a morphism $\underline{F} = (F, f_1, f_2) : E \to E'$ from the extension E to the extension E'. In particular, $F : E \to E'$ is a morphism of the sheaves underlying E and E', and

$$f_1 : B_1 \longrightarrow B'_1 \qquad f_2 : B_2 \longrightarrow B'_2$$

are morphisms of abelian sheaves on $\mathbf{S}$;

(2) a morphism of extensions $\underline{\Upsilon} = (\Upsilon, g_1, f_2) : u_1^*E \to {u'_1}^*E'$ compatible with the morphism $\underline{F} = (F, f_1, f_2)$ and with the trivializations β and β'. In particular, $\Upsilon : u_1^*E \to {u'_1}^*E'$ is a morphism of the sheaves underlying u_1^*E and ${u'_1}^*E'$, and

$$g_1 : A_1 \longrightarrow A'_1$$

is an morphism of abelian sheaves on $\mathbf{S}$;

(3) a morphism of extensions $\underline{\Phi} = (\Phi, g_1, g_2) : T \to T'$ compatible with the morphism $\underline{\Upsilon} = (\Upsilon, g_1, f_2)$ and with the trivializations γ and γ'. In particular, $\Phi : T \to T'$ is a morphism of the sheaves underlying T and T', and

$$g_2 : A_2 \longrightarrow A'_2$$

is an morphism of abelian sheaves on $\mathbf{S}$.

Condition (3) can be rewritten as

(3') an morphism $g_2 : A_2 \to A'_2$ of abelian sheaves on $\mathbf{S}$ compatible with u_2 and u'_2 (i.e. $u'_2 \circ g_2 = f_2 \circ u_2$) and such that

$$\gamma' \circ g_1 = g_2 \circ \gamma.$$

Explicitly, the compatibility of $\underline{\Upsilon}$ with $\underline{F}$, β and β' means that the following diagram is commutative:

$$\begin{array}{ccccc} A_1 & \xrightarrow{\beta} & u_1^*E & \longrightarrow & E \\ {\scriptstyle g_1}\downarrow & & {\scriptstyle \Upsilon}\downarrow & & \downarrow{\scriptstyle F} \\ A'_1 & \xrightarrow{\beta'} & {u'_1}^*E' & \longrightarrow & E'. \end{array}$$

The compatibility of $\underline{\Phi}$ with $\underline{\Upsilon}$, γ and γ' means that the following diagram is commutative:

$$\begin{array}{ccccccc} A_1 & \xrightarrow{\gamma} & T & \longrightarrow & u_{2*}T & \overset{\Theta}{\cong} & u_1^*E \\ {\scriptstyle g_1}\downarrow & & {\scriptstyle \Phi}\downarrow & & & & \downarrow{\scriptstyle \Upsilon} \\ A'_1 & \xrightarrow{\gamma'} & T' & \longrightarrow & u'_{2*}T' & \overset{\Theta'}{\cong} & {u'_1}^*E'. \end{array}$$

We denote by $\mathbf{Ext}(K_1, K_2)$ the category of extensions of K_1 by K_2. As for extensions of abelian sheaves, it is possible to define the Baer sum of extensions of 1-motives. This notion of sum furnishes a group law for the objects of the category $\mathbf{Ext}(K_1, K_2)$ which is therefore a strictly commutative Picard category (see [G] Exposé VII 2.5). The zero object (E_0, β_0, γ_0) of $\mathbf{Ext}(K_1, K_2)$ with respect to this group law consists of

- the trivial extension $E_0 = B_1 \times B_2$ of B_1 by B_2, i.e. the zero object of $\mathbf{Ext}(B_1, B_2)$, and

- the trivialization $\beta_0 = (id_{A_1}, 0)$ of the extension $u_1^* E_0 = A_1 \times B_2$ of A_1 by B_2. We can consider β_0 as the lifting $(u_1, 0) : A_1 \to B_1 \times B_2$ of $u_1 : A_1 \to B_1$.
- the trivial extension T_0 of A_1 by A_2 (i.e. $T_0 = (T_0, \gamma_0)$ with $T_0 = A_1 \times A_2$ and $\gamma_0 = (id_{A_1}, 0)$) and the isomorphism of extension $\Theta_0 = (id_{A_1}, id_{B_2}) : {u_2}_* T_0 \to u_1^* E_0$.

Denote by $\mathcal{E}xt^0(K_1, K_2)$ the group of automorphisms of any object (E, β, γ) of $\mathbf{Ext}(K_1, K_2)$. It is canonically isomorphic to the group of automorphisms $\mathrm{Aut}(E_0, \beta_0, \gamma_0)$ of the zero object (E_0, β_0, γ_0) of $\mathbf{Ext}(K_1, K_2)$: to an automorphism $(\underline{F}, \underline{\Upsilon}, \underline{\Phi})$ of (E_0, β_0, γ_0) the canonical isomorphism associates the automorphism $(\underline{F}, \underline{\Upsilon}, \underline{\Phi}) + id_{(E,\beta,\gamma)}$ of $(E_0, \beta_0, \gamma_0) + (E, \beta, \gamma) \cong (E, \beta, \gamma)$. Explicitly, $\mathcal{E}xt^0(K_1, K_2)$ consists of the couple (f_0, f_1) where

- $f_0 : B_1 \to B_2$ is an automorphism of the trivial extension E_0 (i.e. $f_0 \in \mathrm{Aut}(E_0) = \mathrm{Ext}^0(B_1, B_2)$), and
- $f_1 : A_1 \to A_2$ is an automorphism of the trivial extension T_0 (i.e. $f_1 \in \mathrm{Aut}(T_0) = \mathrm{Ext}^0(A_1, A_2)$) such that, via the isomorphism of extensions $\Theta_0 : {u_2}_* T_0 \to u_1^* E_0$, the push-down ${u_2}_* f_1$ of the automorphism f_1 of T_0 is compatible with the pull-back $u_1^* f_0$ of the automorphism f_0 of E_0, i.e. $u_2 \circ f_1 = f_0 \circ u_1$.

We have therefore the canonical isomorphism

$$\mathcal{E}xt^0(K_1, K_2) \cong \mathrm{Hom}_{\mathcal{K}(\mathbf{S})}(K_1, K_2).$$

The group law of the category $\mathbf{Ext}(K_1, K_2)$ induces a group law on the set of isomorphism classes of objects of $\mathbf{Ext}(K_1, K_2)$ which we denote by $\mathcal{E}xt^1(K_1, K_2)$.

Let $K_i = [u_i : A_i \to B_i]$ (for $i = 1, 2, 3$) be three 1-motives defined over S.

Definition 1.4. A **biextension** $(\mathcal{B}, \Psi_1, \Psi_2, \lambda)$ **of** (K_1, K_2) **by** K_3 consists of

1. a biextension $\mathcal{B}$ of (B_1, B_2) by B_3;
2. a trivialization
$$\Psi_1 : A_1 \times B_2 \longrightarrow (u_1, id_{B_2})^* \mathcal{B}$$
of the biextension $(u_1, id_{B_2})^* \mathcal{B}$ of (A_1, B_2) by B_3 obtained as pull-back of $\mathcal{B}$ via $(u_1, id_{B_2}) : A_1 \times B_2 \to B_1 \times B_2$, and a trivialization
$$\Psi_2 : B_1 \times A_2 \longrightarrow (id_{B_1}, u_2)^* \mathcal{B}$$
of the biextension $(id_{B_1}, u_2)^* \mathcal{B}$ of (B_1, A_2) by B_3 obtained as pull-back of $\mathcal{B}$ via $(id_{B_1}, u_2) : B_1 \times A_2 \to B_1 \times B_2$. These two trivializations Ψ_1 and Ψ_2 have to coincide over $A_1 \times A_2$;
3. a trivial biextension $\mathcal{T}_1 = (\mathcal{T}_1, \lambda_1)$ of (A_1, B_2) by A_3, an isomorphism of biextensions
$$\Theta_1 : {u_3}_* \mathcal{T}_1 \longrightarrow (u_1, id_{B_2})^* \mathcal{B}$$
between the push-down via $u_3 : A_3 \to B_3$ of $\mathcal{T}_1$ and $(u_1, id_{B_2})^* \mathcal{B}$, a trivial biextension $\mathcal{T}_2 = (\mathcal{T}_2, \lambda_2)$ of (B_1, A_2) by A_3 and an isomorphism of biextensions
$$\Theta_2 : {u_3}_* \mathcal{T}_2 \longrightarrow (id_{B_1}, u_2)^* \mathcal{B}$$
between the push-down via $u_3 : A_3 \to B_3$ of $\mathcal{T}_2$ and $(id_{B_1}, u_2)^* \mathcal{B}$. Through the isomorphism Θ_1 the trivialization $u_3 \circ \lambda_1$ of ${u_3}_* \mathcal{T}_1$ is compatible with the trivialization Ψ_1 of $(u_1, id_{B_2})^* \mathcal{B}$, and through the isomorphism Θ_2 the trivialization $u_3 \circ \lambda_2$ of ${u_3}_* \mathcal{T}_2$ is compatible with the trivialization Ψ_2 of

$(id_{B_1}, u_2)^*\mathcal{B}$. The two trivializations λ_1 and λ_2 have to coincide over $A_1 \times A_2$, i.e. $(id_{A_1}, u_2)^*\mathcal{T}_1 = (u_1, id_{A_2})^*\mathcal{T}_2$ (we will denote this biextension by $\mathcal{T} = (\mathcal{T}, \lambda)$ with λ the restriction of the trivializations λ_1 and λ_2 over $A_1 \times A_2$). Moreover, we require an isomorphism of biextensions

$$\Theta : u_{3\,*}\mathcal{T} \longrightarrow (u_1, u_2)^*\mathcal{B}$$

which is compatible with the isomorphisms Θ_1 and Θ_2 and through which the trivialization $u_3 \circ \lambda$ of $u_{3\,*}\mathcal{T}$ is compatible with the restriction Ψ of the trivializations Ψ_1 and Ψ_2 over $A_1 \times A_2$.

Condition (3) can be rewritten as
(3') an morphism $\lambda : A_1 \otimes A_2 \to A_3$ such that $u_3 \circ \lambda$ is compatible with Ψ.

Let $K_i = [u_i : A_i \to B_i]$ and $K'_i = [u'_i : A'_i \to B'_i]$ (for $i = 1, 2, 3$) be 1-motives defined over S. Let $(\mathcal{B}, \Psi_1, \Psi_2, \lambda)$ be a biextension of (K_1, K_2) by K_3 and let $(\mathcal{B}', \Psi'_1, \Psi'_2, \lambda')$ be a biextension of (K'_1, K'_2) by K'_3.

Definition 1.5. A **morphism of biextensions**

$$(\underline{F}, \underline{\Upsilon}_1, \underline{\Upsilon}_2, \underline{\Phi}) : (\mathcal{B}, \Psi_1, \Psi_2, \lambda) \longrightarrow (\mathcal{B}', \Psi'_1, \Psi'_2, \lambda')$$

consists of

(1) a morphism $\underline{F} = (F, f_1, f_2, f_3) : \mathcal{B} \to \mathcal{B}'$ from the biextension $\mathcal{B}$ to the biextension $\mathcal{B}'$. In particular, $F : \mathcal{B} \to \mathcal{B}'$ is a morphism of the sheaves underlying $\mathcal{B}$ and $\mathcal{B}'$, and

$$f_1 : B_1 \longrightarrow B'_1 \qquad f_2 : B_2 \longrightarrow B'_2 \qquad f_3 : B_3 \longrightarrow B'_3$$

are morphisms abelian sheaves on $\mathbf{S}$.

(2) a morphism of biextensions

$$\underline{\Upsilon}_1 = (\Upsilon_1, g_1, f_2, f_3) : (u_1, id_{B_2})^*\mathcal{B} \longrightarrow (u'_1, id_{B'_2})^*\mathcal{B}'$$

compatible with the morphism $\underline{F} = (F, f_1, f_2, f_3)$ and with the trivializations Ψ_1 and Ψ'_1, and a morphism of biextensions

$$\underline{\Upsilon}_2 = (\Upsilon_2, f_1, g_2, f_3) : (id_{B_1}, u_2)^*\mathcal{B} \longrightarrow (id_{B'_1}, u'_2)^*\mathcal{B}'$$

compatible with the morphism $\underline{F} = (F, f_1, f_2, f_3)$ and with the trivializations Ψ_2 and Ψ'_2. In particular, $\Upsilon_1 : (u_1, id_{B_2})^*\mathcal{B} \to (u'_1, id_{B'_2})^*\mathcal{B}'$ is a morphism of the sheaves underlying $(u_1, id_{B_2})^*\mathcal{B}$ and $(u'_1, id_{B'_2})^*\mathcal{B}'$, $\Upsilon_2 : (id_{B_1}, u_2)^*\mathcal{B} \to (id_{B'_1}, u'_2)^*\mathcal{B}'$ is a morphism of the sheaves underlying $(id_{B_1}, u_2)^*\mathcal{B}$ and $(id_{B'_1}, u'_2)^*\mathcal{B}'$, and

$$g_1 : A_1 \longrightarrow A'_1 \qquad g_2 : A_2 \longrightarrow A'_2$$

are morphisms abelian sheaves on $\mathbf{S}$. By pull-back, the two morphisms $\underline{\Upsilon}_1 = (\Upsilon_1, g_1, f_2, f_3)$ and $\underline{\Upsilon}_2 = (\Upsilon_2, f_1, g_2, f_3)$ define a morphism of biextensions

$$\underline{\Upsilon} = (\Upsilon, g_1, g_2, f_3) : (u_1, u_2)^*\mathcal{B} \longrightarrow (u'_1, u'_2)^*\mathcal{B}'$$

compatible with the morphism $\underline{F} = (F, f_1, f_2, f_3)$ and with the trivializations Ψ (restriction of Ψ_1 and Ψ_2 over $A_1 \times A_2$) and Ψ' (restriction of Ψ'_1 and Ψ'_2 over $A'_1 \times A'_2$).

(3) a morphism of biextensions

$$\underline{\Phi}_1 = (\Phi_1, g_1, f_2, g_3) : \mathcal{T}_1 \longrightarrow \mathcal{T}'_1$$

compatible with the morphism $\underline{\Upsilon}_1 = (\Upsilon, g_1, f_2, f_3)$ and with the trivializations λ_1 and λ'_1, and a morphism of biextensions

$$\underline{\Phi}_2 = (\Phi_2, f_1, g_2, g_3) : \mathcal{T}_2 \longrightarrow \mathcal{T}'_2$$

compatible with the morphism $\underline{\Upsilon}_2 = (\Upsilon_2, f_1, g_2, f_3)$ and with the trivializations λ_2 and λ'_2. In particular, $\Phi_1 : \mathcal{T}_1 \to \mathcal{T}'_1$ is a morphism of the sheaves underlying $\mathcal{T}_1$ and $\mathcal{T}'_1$, $\Phi_2 : \mathcal{T}_2 \to \mathcal{T}'_2$ is a morphism of the sheaves underlying $\mathcal{T}_2$ and $\mathcal{T}'_2$, and

$$g_3 : A_3 \longrightarrow A'_3$$

is an morphism abelian sheaves on $\mathbf{S}$. By pull-back, the two morphisms $\underline{\Phi}_1 = (\Phi_1, g_1, f_2, g_3)$ and $\underline{\Phi}_2 = (\Phi_2, f_1, g_2, g_3)$ define a morphism of biextensions

$$\underline{\Phi} = (\Phi, g_1, g_2, g_3) : \mathcal{T} \longrightarrow \mathcal{T}'$$

compatible with the morphism $\underline{\Upsilon} = (\Upsilon, g_1, g_2, f_3)$ and with the trivializations λ (restriction of λ_1 and λ_2 over $A_1 \times A_2$) and λ' (restriction of λ'_1 and λ'_2 over $A_1 \times A_2$).

Condition (3) can be rewritten as

(3') an morphism $g_3 : A_3 \to A'_3$ abelian sheaves on $\mathbf{S}$ compatible with u_3 and u'_3 (i.e. $u'_3 \circ g_3 = f_3 \circ u_3$) and such that

$$\lambda' \circ (g_1 \times g_2) = g_3 \circ \lambda.$$

Explicitly, the compatibility of $\underline{\Upsilon}_1$ with $\underline{F}$, Ψ_1 and Ψ'_1 means that the following diagram is commutative:

$$\begin{array}{ccccc} A_1 \times B_2 & \xrightarrow{\Psi_1} & (u_1, id_{B_2})^*\mathcal{B} & \longrightarrow & \mathcal{B} \\ {\scriptstyle g_1 \times f_2}\downarrow & & {\scriptstyle \Upsilon_1}\downarrow & & \downarrow{\scriptstyle F} \\ A'_1 \times B'_2 & \xrightarrow{\Psi'_1} & (u'_1, id_{B'_2})^*\mathcal{B}' & \longrightarrow & \mathcal{B}'. \end{array}$$

The compatibility of $\underline{\Upsilon}_2$ with $\underline{F}$, Ψ_2 and Ψ'_2 means that the following diagram is commutative:

$$\begin{array}{ccccc} B_1 \times A_2 & \xrightarrow{\Psi_2} & (id_{B_1}, u_2)^*\mathcal{B} & \longrightarrow & \mathcal{B} \\ {\scriptstyle f_1 \times g_2}\downarrow & & {\scriptstyle \Upsilon_2}\downarrow & & \downarrow{\scriptstyle F} \\ B'_1 \times A'_2 & \xrightarrow{\Psi'_2} & (id_{B'_1}, u'_2)^*\mathcal{B}' & \longrightarrow & \mathcal{B}'. \end{array}$$

The compatibility of $\underline{\Upsilon}$ with $\underline{F}$, Ψ and Ψ' means that the following diagram is commutative:

$$\begin{array}{ccccc} A_1 \times A_2 & \xrightarrow{\Psi} & (u_1, u_2)^*\mathcal{B} & \longrightarrow & \mathcal{B} \\ {\scriptstyle g_1 \times g_2}\downarrow & & {\scriptstyle \Upsilon}\downarrow & & \downarrow{\scriptstyle F} \\ A'_1 \times A'_2 & \xrightarrow{\Psi'} & (u'_1, u'_2)^*\mathcal{B}' & \longrightarrow & \mathcal{B}'. \end{array}$$

The compatibility of $\underline{\Phi}_1$ with $\underline{\Upsilon}_1$, λ_1 and λ'_1 means that the following diagram is commutative:

$$\begin{array}{ccccccc} A_1 \times B_2 & \xrightarrow{\lambda_1} & \mathcal{T}_1 & \longrightarrow & {u_3}_*\mathcal{T}_1 & \overset{\Theta_1}{\cong} & (u_1, id_{B_2})^*\mathcal{B} \\ {\scriptstyle g_1 \times f_2}\downarrow & & {\scriptstyle \Phi_1}\downarrow & & & & \downarrow{\scriptstyle \Upsilon_1} \\ A'_1 \times B'_2 & \xrightarrow{\lambda'_1} & \mathcal{T}'_1 & \longrightarrow & {u'_3}_*\mathcal{T}'_1 & \overset{\Theta'_1}{\cong} & (u'_1, id_{B'_2})^*\mathcal{B}'. \end{array}$$

The compatibility of $\underline{\Phi}_2$ with $\underline{\Upsilon}_2$, λ_2 and λ'_2 means that the following diagram is commutative:

$$\begin{array}{ccccccc} B_1 \times A_2 & \stackrel{\lambda_2}{\longrightarrow} & \mathcal{T}_2 & \longrightarrow & u_{3\,*}\mathcal{T}_2 & \stackrel{\Theta_2}{\cong} & (id_{B_1}, u_2)^*\mathcal{B} \\ {\scriptstyle f_1 \times g_2}\downarrow & & {\scriptstyle \Phi_2}\downarrow & & & & \downarrow{\scriptstyle \Upsilon_2} \\ B'_1 \times A'_2 & \stackrel{\lambda'_2}{\longrightarrow} & \mathcal{T}'_2 & \longrightarrow & u'_{3\,*}\mathcal{T}'_2 & \stackrel{\Theta'_2}{\cong} & (id_{B'_1}, u'_2)^*\mathcal{B}'. \end{array}$$

Finally, the compatibility of $\underline{\Phi}$ with $\underline{\Upsilon}$, λ and λ' means that the following diagram is commutative:

$$\begin{array}{ccccccc} A_1 \times A_2 & \stackrel{\lambda}{\longrightarrow} & \mathcal{T} & \longrightarrow & u_{3\,*}\mathcal{T} & \stackrel{\Theta}{\cong} & (u_1, u_2)^*\mathcal{B} \\ {\scriptstyle g_1 \times g_2}\downarrow & & {\scriptstyle \Phi}\downarrow & & & & \downarrow{\scriptstyle \Upsilon} \\ A'_1 \times A'_2 & \stackrel{\lambda'}{\longrightarrow} & \mathcal{T}' & \longrightarrow & u'_{3\,*}\mathcal{T}' & \stackrel{\Theta'}{\cong} & (u'_1, u'_2)^*\mathcal{B}'. \end{array}$$

We denote by $\mathbf{Biext}(K_1, K_2; K_3)$ the category of biextensions of (K_1, K_2) by K_3. The Baer sum of extensions defines a group law for the objects of this category which is therefore a strictly commutative Picard category (see [G] Exposé VII 2.5). The zero object $(\mathcal{B}_0, \Psi_{01}, \Psi_{02}, \lambda_0)$ of $\mathbf{Biext}(K_1, K_2; K_3)$ with respect to this group law consists of

- the trivial biextension $\mathcal{B}_0 = B_1 \times B_2 \times B_3$ of (B_1, B_2) by B_3, i.e. the zero object of $\mathbf{Biext}(B_1, B_2; B_3)$, and
- the trivialization $\Psi_{01} = (id_{A_1}, id_{B_2}, 0)$ (resp. $\Psi_{02} = (id_{B_1}, id_{A_2}, 0)$) of the biextension $(u_1, id_{B_2})^*\mathcal{B}_0 = A_1 \times B_2 \times B_3$ of (A_1, B_2) by B_3 (resp. of the biextension $(id_{B_1}, u_2)^*\mathcal{B}_0 = B_1 \times A_2 \times B_3$ of $(B_1 \times A_2)$ by B_3),
- the trivial biextension $\mathcal{T}_{10}$ of (A_1, B_2) by A_3 (i.e. $\mathcal{T}_{10} = (\mathcal{T}_{10}, \lambda_{10})$ with $\mathcal{T}_{10} = A_1 \times B_2 \times A_3$ and $\lambda_{10} = (id_{A_1}, id_{B_2}, 0)$), the isomorphism of biextensions $\Theta_{10} = (id_{A_1}, id_{B_2}, id_{B_3}) : u_{3\,*}\mathcal{T}_{10} \to (u_1, id_{B_2})^*\mathcal{B}_0$, the trivial biextension $\mathcal{T}_{01}$ of (B_1, A_2) by A_3 (i.e. $\mathcal{T}_{01} = (\mathcal{T}_{01}, \lambda_{01})$ with $\mathcal{T}_{10} = B_1 \times A_2 \times A_3$ and $\lambda_{01} = (id_{B_1}, id_{A_2}, 0)$) and the isomorphism of biextensions $\Theta_{01} = (id_{B_1}, id_{A_2}, id_{B_3}) : u_{3\,*}\mathcal{T}_{10} \to (id_{B_1}, u_2)^*\mathcal{B}_0$. In particular the restriction of λ_{10} and λ_{01} over $A_1 \times A_2$ is $\lambda_0 = (id_{A_1}, id_{A_2}, 0)$

We denote by $\mathrm{Biext}^0(K_1, K_2; K_3)$ the group of automorphisms of any object of $\mathbf{Biext}(K_1, K_2; K_3)$. It is canonically isomorphic to the group of automorphisms of the zero object $(\mathcal{B}_0, \Psi_{01}, \Psi_{02}, \lambda_0)$. Explicitly, $\mathrm{Biext}^0(K_1, K_2; K_3)$ consists of the couple $(f_0, (f_{10}, f_{01}))$ where

- $f_0 : B_1 \otimes B_2 \to B_3$ is an automorphism of the trivial biextension $\mathcal{B}_0$ (i.e. $f_0 \in \mathrm{Biext}^0(B_1, B_2; B_3)$), and
- $f_{10} : A_1 \otimes B_2 \to A_3$ is an automorphism of the trivial biextension $\mathcal{T}_{10}$ (i.e. $f_{10} \in \mathrm{Biext}^0(A_1, B_2; A_3)$) and $f_{01} : B_1 \otimes A_2 \to A_3$ is an automorphism of the trivial biextension $\mathcal{T}_{01}$ (i.e. $f_{01} \in \mathrm{Biext}^0(B_1, A_2; A_3)$) such that, via the isomorphisms of biextensions $\Theta_{10} : u_{3\,*}\mathcal{T}_{10} \to (u_1, id_{B_2})^*\mathcal{B}_0$ and $\Theta_{01} : u_{3\,*}\mathcal{T}_{01} \to (id_{B_1}, u_2)^*\mathcal{B}_0$, the push-down $u_{3\,*}f_{10}$ of f_{10} is compatible with the pull-back $(u_1, id_{B_2})^*f_0$ of f_0, and the push-down $u_{3\,*}f_{01}$ of f_{01} is compatible with the pull-back $(id_{B_1}, u_2)^*f_0$ of f_0, i.e. such that the

following diagram commute

$$\begin{tikzcd} A_1 \otimes B_2 + B_1 \otimes A_2 \arrow[r, "(u_1,id)+(id,u_2)"] \arrow[d, "f_{10}+f_{01}"'] & B_1 \otimes B_2 \arrow[d, "f_0"] \\ A_3 \arrow[r, "u_3"'] & B_3. \end{tikzcd}$$

We have therefore the canonical isomorphism

$$\mathrm{Biext}^0(K_1, K_2; K_3) \cong \mathrm{Hom}_{\mathcal{K}(\mathbf{S})}(K_1 \overset{\mathbb{L}}{\otimes} K_2, K_3).$$

The group law of the category $\mathbf{Biext}(K_1, K_2; K_3)$ induces a group law on the set of isomorphism classes of objects of $\mathbf{Biext}(K_1, K_2; K_3)$, that we denote by $\mathrm{Biext}^1(K_1, K_2; K_3)$.

Remark 1.6. According to the above geometrical definitions of extensions and biextensions of 1-motives, we have the following equivalence of categories

$$\mathbf{Biext}(K_1, [0 \to \mathbb{Z}]; K_3) \simeq \mathbf{Ext}(K_1, K_3).$$

Moreover we have also the following isomorphisms

$$\mathrm{Biext}^i(K_1, [\mathbb{Z} \to 0]; K_3) = \begin{cases} \mathrm{Hom}(B_1, A_3), & i = 0; \\ \mathrm{Hom}(K_1, K_3), & i = 1. \end{cases}$$

Note that we get the same results applying the homological interpretation of biextensions furnished by our main Theorem 0.1.

2. Review on strictly commutative Picard stacks

Let $\mathbf{S}$ be a site. For the notions of $\mathbf{S}$-pre-stack, $\mathbf{S}$-stack and morphisms of $\mathbf{S}$-stacks we refer to [G71] Chapter II 1.2.

A **strictly commutative Picard S-stack** is an $\mathbf{S}$-stack of groupoids $\mathcal{P}$ endowed with a functor $+ : \mathcal{P} \times_{\mathbf{S}} \mathcal{P} \to \mathcal{P}$, $(a, b) \mapsto a + b$, and two natural isomorphisms of associativity σ and of commutativity τ, which are described by the functorial isomorphisms

$$\sigma_{a,b,c} \quad : \quad (a + b) + c \xrightarrow{\cong} a + (b + c) \qquad \forall\ a, b, c \in \mathcal{P}, \tag{2.1}$$

$$\tau_{a,b} \quad : \quad a + b \xrightarrow{\cong} b + a \qquad \forall\ a, b \in \mathcal{P}; \tag{2.2}$$

such that for any object U of $\mathbf{S}$, $(\mathcal{P}(U), +, \sigma, \tau)$ is a strictly commutative Picard category (i.e. it is possible to make the sum of two objects of $\mathcal{P}(U)$ and this sum is associative and commutative, see [D73] 1.4.2 for more details). Here "strictly" means that $\tau_{a,a}$ is the identity for all $a \in \mathcal{P}$. Any strictly commutative Picard $\mathbf{S}$-stack admits a global neutral object e and the sheaf of automorphisms of the neutral object $\underline{\mathrm{Aut}}(e)$ is abelian.

Let $\mathcal{P}$ and $\mathcal{Q}$ be two strictly commutative Picard $\mathbf{S}$-stacks. An **additive functor** $(F, \sum) : \mathcal{P} \to \mathcal{Q}$ between strictly commutative Picard $\mathbf{S}$-stacks is a morphism of $\mathbf{S}$-stacks (i.e. a cartesian $\mathbf{S}$-functor, see [G71] Chapter I 1.1) endowed with a natural isomorphism $\sum$ which is described by the functorial isomorphisms

$$\sum_{a,b} : F(a + b) \xrightarrow{\cong} F(a) + F(b) \qquad \forall\ a, b \in \mathcal{P}$$

and which is compatible with the natural isomorphisms σ and τ of $\mathcal{P}$ and $\mathcal{Q}$. A **morphism of additive functors** $u : (F, \sum) \to (F', \sum')$ is an $\mathbf{S}$-morphism of cartesian **S**-functors (see [G71] Chapter I 1.1) which is compatible with the natural isomorphisms $\sum$ and $\sum'$ of F and F' respectively.

An **equivalence of strictly commutative Picard S-stacks** between $\mathcal{P}$ and $\mathcal{Q}$ is an additive functor $(F,\sum): \mathcal{P} \to \mathcal{Q}$ with F an equivalence of **S**-stacks. Two strictly commutative Picard **S**-stacks are **equivalent as strictly commutative Picard S-stacks** if there exists an equivalence of strictly commutative Picard **S**-stacks between them.

To any strictly commutative Picard **S**-stack $\mathcal{P}$, we associate the sheaffification $\pi_0(\mathcal{P})$ of the pre-sheaf which associates to each object U of **S** the group of isomorphism classes of objects of $\mathcal{P}(U)$, and the sheaf $\pi_1(\mathcal{P})$ of automorphisms $\underline{\mathrm{Aut}}(e)$ of the neutral object of $\mathcal{P}$.

In [D73] §1.4 Deligne associates to each complex K of $\mathcal{K}^{[-1,0]}(\mathbf{S})$ a strictly commutative Picard **S**-stack $st(K)$ and to each morphism of complexes $g: K \to L$ an additive functor $st(g): st(K) \to st(L)$. Moreover, if Picard(**S**) denotes the category whose objects are small strictly commutative Picard **S**-stacks and whose arrows are isomorphism classes of additive functors, Deligne proves the following equivalence of category

$$\begin{array}{rcl} st: \mathcal{D}^{[-1,0]}(\mathbf{S}) & \longrightarrow & \mathrm{Picard}(\mathbf{S}) \\ K & \mapsto & st(K) \\ K \stackrel{f}{\to} L & \mapsto & st(K) \stackrel{st(f)}{\to} st(L). \end{array} \tag{2.3}$$

constructing explicitly the inverse equivalence of st, that we denote by $[\ ]$.

Example 2.1. Let $\mathcal{P}, \mathcal{Q}$ and $\mathcal{G}$ be three strictly commutative Picard **S**-stacks.
I) Let

$$\mathrm{HOM}(\mathcal{P}, \mathcal{Q})$$

be the strictly commutative Picard **S**-stack defined as followed: for any object U of **S**, the objects of the category $\mathrm{HOM}(\mathcal{P}, \mathcal{Q})(U)$ are additive functors from $\mathcal{P}_{|U}$ to $\mathcal{Q}_{|U}$ and its arrows are morphisms of additive functors. According [D73] 1.4.18 we have the equality $[\mathrm{HOM}(\mathcal{P}, \mathcal{Q})] = \tau_{\leq 0}\mathrm{RHom}\big([\mathcal{P}], [\mathcal{Q}]\big)$ in the derived category $\mathcal{D}^{[-1,0]}(\mathbf{S})$.
II) A **biadditive functor** $(F, l, r): \mathcal{P} \times \mathcal{Q} \to \mathcal{G}$ is a morphism of **S**-stacks $F: \mathcal{P} \times \mathcal{Q} \to \mathcal{G}$ endowed with two natural isomorphisms, which are described by the functorial isomorphisms

$$\begin{array}{rclr} l_{a,b,c}: F(a+b,c) & \stackrel{\cong}{\longrightarrow} & F(a,c)+F(b,c) & \forall\ a,b \in \mathcal{P},\ \forall\ c \in \mathcal{Q} \\ r_{a,c,d}: F(a,c+d) & \stackrel{\cong}{\longrightarrow} & F(a,c)+F(a,d) & \forall\ a \in \mathcal{P},\ \forall\ c,d \in \mathcal{Q}, \end{array}$$

such that

- for any fixed $a \in \mathcal{P}$, $F(a,-)$ is compatible with the natural isomorphisms σ and τ of $\mathcal{P}$ and $\mathcal{G}$,
- for any fixed $c \in \mathcal{Q}$, $F(-,c)$ is compatible with the natural isomorphisms σ and τ of $\mathcal{Q}$ and $\mathcal{G}$,

- for any fixed $a, b \in \mathcal{P}$ and $c, d \in \mathcal{Q}$ is the following diagram commute

$$\begin{tikzcd}
F(a+b,c+d) \arrow[r, "r"] \arrow[d, "l"'] & F(a+b,c)+F(a+b,d) \arrow[r, "l+l"] & F(a,c)+F(b,c)+F(a,d)+F(b,d) \\
F(a,c+d)+F(b,c+d) \arrow[rr, "r+r"'] & & F(a,c)+F(a,d)+F(b,c)+F(b,d). \arrow[u, "id_{\mathcal{G}}+\tau+id_{\mathcal{G}}"']
\end{tikzcd}$$

A **morphism of biadditive functors** $\alpha : (F, l, r) \Rightarrow (F', l', r')$ is a morphism of morphisms of $\mathbf{S}$-stacks $\alpha : F \Rightarrow F'$ which is compatible with the natural isomorphisms l, r and l', r of F and F' respectively. Let

$$\mathrm{HOM}(\mathcal{P}, \mathcal{Q}; \mathcal{G})$$

be the strictly commutative Picard $\mathbf{S}$-stack defined as followed: for any object U of $\mathbf{S}$, the objects of the category $\mathrm{HOM}(\mathcal{P}, \mathcal{Q}; \mathcal{G})(U)$ are biadditive functors from $\mathcal{P}_{|U} \times \mathcal{Q}_{|U}$ to $\mathcal{G}_{|U}$ and its arrows are morphisms of biadditive functors.
III) Let

$$\mathcal{P} \otimes \mathcal{Q}$$

be the strictly commutative Picard $\mathbf{S}$-stack endowed with a biadditive functor $\otimes : \mathcal{P} \times \mathcal{Q} \to \mathcal{P} \otimes \mathcal{Q}$ such that for any strictly commutative Picard $\mathbf{S}$-stack $\mathcal{G}$, the biadditive functor $\otimes$ defines the following equivalence of strictly commutative Picard $\mathbf{S}$-stacks:

$$\mathrm{HOM}(\mathcal{P} \otimes \mathcal{Q}, \mathcal{G}) \cong \mathrm{HOM}(\mathcal{P}, \mathcal{Q}; \mathcal{G}). \tag{2.4}$$

According to [D73] 1.4.20, in the derived category $\mathcal{D}^{[-1,0]}(\mathbf{S})$ we have the equality $[\mathcal{P} \otimes \mathcal{Q}] = \tau_{\geq -1}([\mathcal{P}] \otimes^{\mathbb{L}} [\mathcal{Q}])$.

By §2 [Be11] we have the following operations on strictly commutative Picard $\mathbf{S}$-stacks:
(1) The **product** of two strictly commutative Picard $\mathbf{S}$-stacks $\mathcal{P}$ and $\mathcal{Q}$ is the strictly commutative Picard $\mathbf{S}$-stack $\mathcal{P} \times \mathcal{Q}$ defined as followed:

- for any object U of $\mathbf{S}$, an object of the category $\mathcal{P} \times \mathcal{Q}(U)$ is a pair (p, q) of objects with p an object of $\mathcal{P}(U)$ and q an object of $\mathcal{Q}(U)$;
- for any object U of $\mathbf{S}$, if (p, q) and (p', q') are two objects of $\mathcal{P} \times \mathcal{Q}(U)$, an arrow of $\mathcal{P} \times \mathcal{Q}(U)$ from (p, q) to (p', q') is a pair (f, g) of arrows with $f : p \to p'$ an arrow of $\mathcal{P}(U)$ and $g : q \to q'$ an arrow of $\mathcal{Q}(U)$.

(2) Let $G : \mathcal{P} \to \mathcal{Q}$ and $F : \mathcal{P}' \to \mathcal{Q}$ be additive functors between strictly commutative Picard $\mathbf{S}$-stacks. The **fibered product** of $\mathcal{P}$ and $\mathcal{P}'$ over $\mathcal{Q}$ via F and G is the strictly commutative Picard $\mathbf{S}$-stack $\mathcal{P} \times_{\mathcal{Q}} \mathcal{P}'$ defined as followed:

- for any object U of $\mathbf{S}$, the objects of the category $(\mathcal{P} \times_{\mathcal{Q}} \mathcal{P}')(U)$ are triplets (p, p', f) where p is an object of $\mathcal{P}(U)$, p' is an object of $\mathcal{P}'(U)$ and $f : G(p) \stackrel{\cong}{\to} F(p')$ is an isomorphism of $\mathcal{Q}(U)$ between $G(p)$ and $F(p')$;
- for any object U of $\mathbf{S}$, if (p_1, p'_1, f) and (p_2, p'_2, g) are two objects of $(\mathcal{P} \times_{\mathcal{Q}} \mathcal{P}')(U)$, an arrow of $(\mathcal{P} \times_{\mathcal{Q}} \mathcal{P}')(U)$ from (p_1, p'_1, f) to (p_2, p'_2, g) is a pair (f, g) of arrows with $\alpha : p_1 \to p_2$ of arrow of $\mathcal{P}(U)$ and $\beta : p'_1 \to p'_2$ an arrow of $\mathcal{P}'(U)$ such that $g \circ G(\alpha) = F(\beta) \circ f$.

The fibered product $\mathcal{P} \times_{\mathcal{Q}} \mathcal{P}'$ is also called the **pull-back** $F^*\mathcal{P}$ of $\mathcal{P}$ via $F : \mathcal{P}' \to \mathcal{Q}$ or the **pull-back** $G^*\mathcal{P}'$ of $\mathcal{P}'$ via $G : \mathcal{P} \to \mathcal{Q}$.
(3) Let $G : \mathcal{Q} \to \mathcal{P}$ and $F : \mathcal{Q} \to \mathcal{P}'$ be additive functors between strictly commutative Picard $\mathbf{S}$-stacks. The **fibered sum** of $\mathcal{P}$ and $\mathcal{P}'$ under $\mathcal{Q}$ via F and G is the

strictly commutative Picard $\mathbf{S}$-stack $\mathcal{P} +^{\mathcal{Q}} \mathcal{P}'$ generated by the following strictly commutative Picard $\mathbf{S}$-pre-stack $\mathcal{D}$:

- for any object U of $\mathbf{S}$, the objects of the category $\mathcal{D}(U)$ are pairs (p, p') with p an object of $\mathcal{P}(U)$ and p' an object of $\mathcal{P}'(U)$;
- for any object U of $\mathbf{S}$, if (p_1, p'_1) and (p_2, p'_2) are two objects of $\mathcal{D}(U)$, an arrow of $\mathcal{D}(U)$ from (p_1, p'_1) to (p_2, p'_2) is an equivalence class of triplets (q, α, β) with q an object of $\mathcal{Q}(U)$, $\alpha : p_1 + G(q) \to p_2$ an arrow of $\mathcal{P}(U)$ and $\beta : p'_1 + F(q) \to p'_2$ an arrow of $\mathcal{P}'(U)$. Two triplets (q_1, α_1, β_1) and (q_2, α_2, β_2) are equivalent it there is an arrow $\gamma : q_1 \to q_2$ in $\mathcal{Q}(U)$ such that $\alpha_2 \circ (id + G(\gamma)) = \alpha_1$ and $(F(\gamma) + id) \circ \beta_1 = \beta_2$.

The fibered sum $\mathcal{P} +^{\mathcal{Q}} \mathcal{P}'$ is also called the **push-down** $F_*\mathcal{P}$ of $\mathcal{P}$ via $F : \mathcal{Q} \to \mathcal{P}'$ or the **push-down** $G_*\mathcal{P}'$ of $\mathcal{P}'$ via $G : \mathcal{Q} \to \mathcal{P}$.

We have analogous operations on complexes of $\mathcal{K}^{[-1,0]}(\mathbf{S})$:
(1) The **product** of two complexes $P = [d^P : P^{-1} \to P^0]$ and $Q = [d^Q : Q^{-1} \to Q^0]$ of $\mathcal{K}^{[-1,0]}(\mathbf{S})$ is the complex $P + Q = [(d^P, d^Q) : P^{-1} + Q^{-1} \to P^0 + Q^0]$. Via the equivalence of category (2.3) we have that $st(P + Q) = st(P) \times st(Q)$.
(2) Let $P = [d^P : P^{-1} \to P^0], Q = [d^Q : Q^{-1} \to Q^0]$ and $G = [d^G : G^{-1} \to G^0]$ be complexes of $\mathcal{K}^{[-1,0]}(\mathbf{S})$ and let $f : P \to G$ and $g : Q \to G$ be morphisms of complexes. The **fibered product** $P \times_G Q$ of P and Q over G is the complex $[d_P \times_{d_G} d_Q : P^{-1} \times_{G^{-1}} Q^{-1} \to P^0 \times_{G^0} Q^0]$, where for $i = -1, 0$ the abelian sheaf $P^i \times_{G^i} Q^i$ is the fibered product of P^i and of Q^i over G^i and the morphism of abelian sheaves $d_P \times_{d_G} d_Q$ is given by the universal property of the fibered product $P^0 \times_{G^0} Q^0$. The fibered product $P \times_G Q$ is also called the **pull-back** g^*P of P via $g : Q \to G$ or the **pull-back** f^*Q of Q via $f : P \to G$. Remark that $st(P \times_G Q) = st(P) \times_{st(G)} st(Q)$ via the equivalence of category (2.3).
(3) Let $P = [d^P : P^{-1} \to P^0], Q = [d^Q : Q^{-1} \to Q^0]$ and $G = [d^G : G^{-1} \to G^0]$ be complexes of $\mathcal{K}^{[-1,0]}(\mathbf{S})$ and let $f : G \to P$ and $g : G \to Q$ be morphisms of complexes. The **fibered sum** $P +^G Q$ of P and Q under G is the complex $[d_P +^{d_G} d_Q : P^{-1} +^{G^{-1}} Q^{-1} \to P^0 +^{G^0} Q^0]$, where for $i = -1, 0$ the abelian sheaf $P^i +^{G^i} Q^i$ is the fibered sum of P^i and of Q^i under G^i and the morphism of abelian sheaves $d_P +^{d_G} d_Q$ is given by the universal property of the fibered sum $P^{-1} +^{G^{-1}} Q^{-1}$. The fibered sum $P +^G Q$ is also called the **push-down** g_*P of P via $g : G \to Q$ or the **push-down** f_*Q of Q via $f : G \to P$. We have $st(P +^G Q) = st(P) +^{st(G)} st(Q)$ via the equivalence of category (2.3).

If $\mathcal{P}$ and $\mathcal{G}$ are strictly commutative Picard $\mathbf{S}$-stacks, by §3 [Be11] an **extension** $\mathcal{E} = (\mathcal{E}, I, J)$ of $\mathcal{P}$ by $\mathcal{G}$ consists of a strictly commutative Picard $\mathbf{S}$-stack $\mathcal{E}$, two additive functors $I : \mathcal{G} \to \mathcal{E}$ and $J : \mathcal{E} \to \mathcal{P}$, and an isomorphism of additive functors $J \circ I \cong 0$, such that the following equivalent conditions are satisfied:

(a) $\pi_0(J) : \pi_0(\mathcal{E}) \to \pi_0(\mathcal{P})$ is surjective and I induces an equivalence of strictly commutative Picard $\mathbf{S}$-stacks between $\mathcal{G}$ and $\ker(J)$;

(b) $\pi_1(I) : \pi_1(\mathcal{G}) \to \pi_1(\mathcal{E})$ is injective and J induces an equivalence of strictly commutative Picard $\mathbf{S}$-stacks between $\mathrm{coker}(I)$ and $\mathcal{P}$.

In terms of complexes of $\mathcal{K}^{[-1,0]}(\mathbf{S})$, an **extension** $E = (E, i, j)$ of P by G consists of a complex E of $\mathcal{K}^{[-1,0]}(\mathbf{S})$, two morphisms of complexes $i : G \to E$ and $j : E \to P$ of $\mathcal{K}^{[-1,0]}(\mathbf{S})$, and an homotopy between $j \circ i$ and 0, such that the following equivalent conditions are satisfied:

(a) $\mathrm{H}^0(j) : \mathrm{H}^0(E) \to \mathrm{H}^0(P)$ is surjective and i induces a quasi-isomorphism between G and $\tau_{\leq 0}(MC(j)[-1])$;

(b) $\mathrm{H}^{-1}(i) : \mathrm{H}^{-1}(G) \to \mathrm{H}^{-1}(E)$ is injective and j induces a quasi-isomorphism between $\tau_{\geq -1}MC(i)$ and P.

As recalled in the introduction we can see 1-motives as complexes of abelian sheaves on $\mathbf{S}$ concentrated in two consecutive degrees. Hence via (2.3) to each 1-motives is associated a strictly commutative Picard $\mathbf{S}$-stack and in particular, we can apply all what we have recalled in this section to 1-motives. Moreover, since a short exact sequence in $\mathcal{K}^{[-1,0]}(\mathbf{S})$ is an extension of complexes in the above sense (see [Be11] Remark 3.6.), extensions of 1-motives are also extensions of complexes in the above sense, i.e. they furnishes extensions of strictly commutative Picard $\mathbf{S}$-stacks (see Remark 1.2).

3. Proof of theorem 0.1 (b)

Proof of Theorem 0.1 **b**. Via the equivalence of category (2.3), to the trivial biextension of (K_1, K_2) by K_3 corresponds the trivial biextension $\mathcal{B}_0 = st(K_3) \times st(K_1) \times st(K_2)$ of $(st(K_1), st(K_2))$ by $st(K_3)$ (see [Be] Definition 5.1). In particular $\mathcal{B}_0$ is a Picard stack and so the group of isomorphism classes of arrows from $\mathcal{B}_0$ to itself is the cohomology group $\mathrm{H}^0([\mathrm{HOM}(\mathcal{B}_0, \mathcal{B}_0)])$, where $\mathrm{HOM}(\mathcal{B}_0, \mathcal{B}_0)$ is the strictly commutative Picard stack of additive functors from $\mathcal{B}_0$ to itself. Therefore, in order to compute $\mathrm{Biext}^0(K_1, K_2; K_3)$ it is enough to compute the complex $[\mathrm{HOM}(\mathcal{B}_0, \mathcal{B}_0)]$.
Let $F : \mathcal{B}_0 \to \mathcal{B}_0$ be an additive functor. Since F is first of all an arrow from the $st(K_3)$-torsor over $st(K_1) \times st(K_2)$ underlying $\mathcal{B}_0$ to itself, F is given by the formula

$$F(b) = b + IF'J(b) \qquad \forall\, b \in \mathcal{B}_0$$

where $F' : st(K_1) \times st(K_2) \to st(K_3)$ is an additive functor and $J : \mathcal{B}_0 \to st(K_1) \times st(K_2)$ and $I : st(K_3) \to \mathcal{B}_0$ are the additive functors underlying the structure of $st(K_3)$-torsor over $st(K_1) \times st(K_2)$ of $\mathcal{B}_0$. Now $F : \mathcal{B}_0 \to \mathcal{B}_0$ must be compatible with the structures of extension of $st(K_2)_{st(K_1)}$ by $st(K_3)_{st(K_1)}$ and of extension of $st(K_1)_{st(K_2)}$ by $st(K_3)_{st(K_2)}$ underlying $\mathcal{B}_0$, and so $F' : st(K_1) \times st(K_2) \to st(K_3)$ must be a biadditive functor, i.e. an object of $\mathrm{HOM}(st(K_1), st(K_2); st(K_3))$. Hence $\mathrm{HOM}(\mathcal{B}_0, \mathcal{B}_0)$ is equivalent as Picard stack to $\mathrm{HOM}(st(K_1), st(K_2); st(K_3))$ via the following additive functor

$$\begin{array}{rcl} \mathrm{HOM}(st(K_1), st(K_2); st(K_3)) & \longrightarrow & \mathrm{HOM}(\mathcal{B}_0, \mathcal{B}_0) \\ F' & \mapsto & \big(b \mapsto b + IF'J(b)\big). \end{array}$$

By (2.4), $\mathrm{HOM}(st(K_1), st(K_2); st(K_3)) \cong \mathrm{HOM}(st(K_1) \otimes st(K_2), st(K_3))$ and so

$$[\mathrm{HOM}(\mathcal{B}_0, \mathcal{B}_0)] = \tau_{\leq 0}\mathrm{RHom}\Big(\tau_{\geq -1}(K_1 \otimes^{\mathbb{L}} K_2), K_3\Big), \tag{3.1}$$

and in particular the group of isomorphism classes of additive functors from $\mathcal{B}_0$ to itself is isomorphic to the group

$$\mathrm{Hom}_{\mathcal{D}(\mathbf{S})}(K_1 \otimes^{\mathbb{L}} K_2, K_3).$$

This implies that $\mathrm{Biext}^0(K_1, K_2; K_3) \cong \mathrm{Hom}_{\mathcal{D}(\mathbf{S})}(K_1 \otimes^{\mathbb{L}} K_2, K_3)$.

In Section 6 we gives another proof of Theorem 0.1 **b**. Remark that by (3.1) $\mathrm{H}^{-1}([\mathrm{HOM}(\mathcal{B}_0,\mathcal{B}_0)]) \cong \mathrm{Hom}_{\mathcal{D}(\mathbf{S})}(K_1 \otimes^{\mathbb{L}} K_2, K_3[-1])$. Since $K_i = [A_i \to B_i]$ are 1-motives, $\mathrm{Hom}(B_j, A_i) = 0$ for $i,j = 1,2,3$ (see [B09] Lemma 1.1.1), and hence the group $\mathrm{H}^{-1}([\mathrm{HOM}(\mathcal{B}_0,\mathcal{B}_0)])$ is trivial.

4. The category $\Psi_{\mathrm{L}^{\cdot\cdot}}(G)$ and its homological interpretation

Consider the following complex of 1-motives defined over S

$$R \xrightarrow{D^R} Q \xrightarrow{D^Q} P \longrightarrow 0 \tag{4.1}$$

Explicitly, $R = [d^R : R^{-1} \to R^0], Q = [d^Q : Q^{-1} \to Q^0], P = [d^P : P^{-1} \to P^0]$ and $D^R = (d^{R,-1}, d^{R,0}), D^Q = (d^{Q,-1}, d^{Q,0})$. This complex is a bicomplex $\mathrm{L}^{\cdot\cdot}$ of abelian sheaves on $\mathbf{S}$,

$$\begin{tikzcd}
R^{-1} \arrow[r, "d^{R,-1}"] \arrow[d, "d^R"'] & Q^{-1} \arrow[r, "d^{Q,-1}"] \arrow[d, "d^Q"] & P^{-1} \arrow[r] \arrow[d, "d^P"] & 0 \\
R^0 \arrow[r, "d^{R,0}"] & Q^0 \arrow[r, "d^{Q,0}"] & P^0 \arrow[r] & 0
\end{tikzcd}$$

where $P^0, P^{-1}, Q^0, Q^{-1}, R^0, R^{-1}$ are respectively in degrees $(0,0), (0,-1), (-1,0)$, $(-1,-1), (-2,0), (-2,-1)$. Denote by $\mathrm{Tot}(\mathrm{L}^{\cdot\cdot})$ its total complex. Let $G = [d^G : G^{-1} \to G^0]$ be a 1-motive defined over S.

Definition 4.1. Denote by $\Psi_{\mathrm{L}^{\cdot\cdot}}(G)$ the category

(1) whose objects are pairs (E, I) with E an extension of 1-motives of P by G and I a trivialization of the extension $(D^Q)^*E$ of Q by G obtained as pull-back of E by D^Q. Moreover we require that the corresponding trivialization $(D^R)^*I$ of $(D^R)^*(D^Q)^*E$ is the trivialization arising from the isomorphism of transitivity $(D^R)^*(D^Q)^*E \cong (D^Q \circ D^R)^*E$ and the relation $D^Q \circ D^R = 0$. Note that to have such a trivialization I is the same thing as to have a lifting $I : Q \to E$ of $D^Q : Q \to P$ such that $I \circ D^R = 0$;

(2) whose arrows $F : (E, I) \to (E', I')$ are morphisms of extensions $F : E \to E'$ of 1-motives compatible with the trivializations I, I', i.e. we have an isomorphism of additive functors $F \circ I \cong I'$.

In order to compute the homological interpretation of the category $\Psi_{\mathrm{L}^{\cdot\cdot}}(G)$, the language of Picard stacks will be very useful. Hence now we translate the construction of the category $\Psi_{\mathrm{L}^{\cdot\cdot}}(G)$ in terms of Picard stacks : Let $\mathcal{R} = st(R), \mathcal{Q} = st(Q), \mathcal{P} = st(P), \mathcal{G} = st(G), D^{\mathcal{R}} = st(D^R)$ and $D^{\mathcal{Q}} = st(D^Q)$. The complex of 1-motives (4.1) furnishes the following complex of strictly commutative Picard $\mathbf{S}$-stacks

$$\mathcal{L}^\cdot : \qquad \mathcal{R} \xrightarrow{D^{\mathcal{R}}} \mathcal{Q} \xrightarrow{D^{\mathcal{Q}}} \mathcal{P} \xrightarrow{D^{\mathcal{P}}} 0$$

with $\mathcal{P}, \mathcal{Q}$ and $\mathcal{R}$ in degrees 0,-1 and -2 respectively. Via the equivalence of categories (2.3), to the category $\Psi_{\mathrm{L}^{\cdot\cdot}}(G)$ is associated the category $\Psi_{\mathrm{L}^{\cdot\cdot}}(G)$ $\Psi_{\mathcal{L}^\cdot}(\mathcal{G})$

(1) whose objects are pairs $(\mathcal{E}, I)$ with $\mathcal{E}$ an extension of $\mathcal{P}$ by $\mathcal{G}$ and I a trivialization of the extension $(D^{\mathcal{Q}})^*\mathcal{E}$ of $\mathcal{Q}$ by $\mathcal{G}$ obtained as pull-back of $\mathcal{E}$ by $D^{\mathcal{Q}}$. Moreover we require that the corresponding trivialization $(D^{\mathcal{R}})^*I$ of $(D^{\mathcal{R}})^*(D^{\mathcal{Q}})^*\mathcal{E}$ is the trivialization arising from the isomorphism of transitivity $(D^{\mathcal{R}})^*(D^{\mathcal{Q}})^*\mathcal{E} \cong (D^{\mathcal{Q}} \circ D^{\mathcal{R}})^*\mathcal{E}$ and the relation $D^{\mathcal{Q}} \circ D^{\mathcal{R}} \cong 0$.

Note that to have such a trivialization I is the same thing as to have a lifting $I : \mathcal{Q} \to \mathcal{E}$ of $D^{\mathcal{Q}} : \mathcal{Q} \to \mathcal{P}$ such that $I \circ D^{\mathcal{R}} \cong 0$;

(2) whose arrows $F : (\mathcal{E}, I) \to (\mathcal{E}', I')$ are morphisms of extensions $F : \mathcal{E} \to \mathcal{E}'$ compatible with the trivializations I, I', i.e. we have an isomorphism of additive functors $F \circ I \cong I'$.

As observed at the end of section 2, extensions of 1-motives furnishes extensions of strictly commutative Picard stacks and so the sum of extensions of strictly commutative Picard stacks introduced in [Be11] 4.6 defines a group law on the set of isomorphism classes of objects of $\Psi_{\mathrm{L}^{\cdot\cdot}}(G)$. We denote this group by $\Psi^1_{\mathrm{L}^{\cdot\cdot}}(G)$. The neutral object of $\Psi_{\mathrm{L}^{\cdot\cdot}}(G)$ is the object (E_0, I_0) where E_0 is the trivial extension $G \times P$ of P by G and I_0 is the trivialization $(Id_Q, 0)$ of the extension $(D^Q)^* E_0 = G \times Q$ of Q by G. We can consider I_0 as the lifting $(D^Q, 0)$ of $D^Q : Q \to P$.

The monoid of automorphisms of an object (E, I) of $\Psi_{\mathrm{L}^{\cdot\cdot}}(G)$ is canonically isomorphic to the monoid of automorphisms of (E_0, I_0): to an automorphism $F : (E_0, I_0) \to (E_0, I_0)$ the canonical isomorphism associates the automorphism $F + Id_{(E,I)}$ of $(E_0, I_0) + (E, I) \cong (E, I)$. The monoid of automorphisms of (E_0, I_0) is a commutative group via the composition law $(F, G) \mapsto F + G$ (here $F + G$ is the automorphism of $(E_0, I_0) + (E_0, I_0) \cong (E_0, I_0)$). Hence we can conclude that the set of automorphisms of an object of $\Psi_{\mathrm{L}^{\cdot\cdot}}(G)$ is a commutative group that we denote by $\Psi^0_{\mathrm{L}^{\cdot\cdot}}(G)$.

We can now state the homological interpretation of the groups $\Psi^i_{\mathrm{L}^{\cdot\cdot}}(G)$.

Theorem 4.2.

$$\Psi^i_{\mathrm{L}^{\cdot\cdot}}(G) \cong \mathrm{Ext}^i\big(\mathrm{Tot}(\mathrm{L}^{\cdot\cdot}), G\big) = \mathrm{Hom}_{\mathcal{D}(\boldsymbol{S})}\big(\mathrm{Tot}(\mathrm{L}^{\cdot\cdot}), G[i]\big) \qquad i = 0, 1.$$

Proof of the case i=0. For this proof we will work with the category $\Psi_{\mathcal{L}^{\cdot}}(\mathcal{G})$. As observed above, $\Psi^0_{\mathcal{L}^{\cdot}}(\mathcal{G})$ is canonically isomorphic to the group of isomorphism classes of arrows from the neutral object $(\mathcal{E}_0, I_0)$ of $\Psi_{\mathcal{L}^{\cdot}}(\mathcal{G})$ to itself. By definition of arrows in the category $\Psi_{\mathcal{L}^{\cdot}}(\mathcal{G})$, the additive functor $F : \mathcal{E}_0 \to \mathcal{E}_0$ is an arrow from $(\mathcal{E}_0, I_0)$ to itself if we have an isomorphism of additive functors $F \circ D^{\mathcal{Q}} \cong 0$, i.e. if F is an object of the strictly commutative Picard $\mathbf{S}$-stack

$$\mathcal{K} = \ker\big(\mathrm{HOM}(\mathcal{P}, \mathcal{G}) \stackrel{D^{\mathcal{Q}}}{\to} \mathrm{HOM}(\mathcal{Q}, \mathcal{G})\big).$$

Therefore we have the equality

$$\Psi^0_{\mathcal{L}^{\cdot}}(\mathcal{G}) = \mathrm{H}^0\big([\mathcal{K}]\big) \tag{4.2}$$

and in order to conclude, it is enough to compute the complex $[\mathcal{K}]$ of $\mathcal{K}^{[-1,0]}(\mathbf{S})$. By [Be11] Lemma 3.4 we have

$$[\mathcal{K}] = \tau_{\leq 0}\Big(MC\big(\tau_{\leq 0}\mathrm{RHom}([\mathcal{P}], [\mathcal{G}]) \stackrel{(d^{R,-1}, d^{R,0})}{\longrightarrow} \tau_{\leq 0}\mathrm{RHom}([\mathcal{Q}], [\mathcal{G}])\big)[-1]\Big).$$

Explicitly, we get

$$[\mathcal{K}] = \big[\mathrm{Hom}(P^0, G^{-1}) \stackrel{((d^G, d^P), d^{Q,0})}{\longrightarrow} K_1 + K_2\big] \tag{4.3}$$

where

$$K_1 = \ker\big(\mathrm{Hom}(P^0, G^0) + \mathrm{Hom}(P^{-1}, G^{-1}) \stackrel{(d^{Q,0}, d^{Q,-1})}{\to} \mathrm{Hom}(Q^0, G^0) + \mathrm{Hom}(Q^{-1}, G^{-1})\big)$$

$$K_2 = \ker\big(\mathrm{Hom}(Q^0, G^{-1}) \stackrel{(d^G, d^Q)}{\to} \mathrm{Hom}(Q^0, G^0) + \mathrm{Hom}(Q^{-1}, G^{-1})\big).$$

In order to simplify notation let $C^{\cdot} : C^{-3} \to C^{-2} \to C^{-1} \to C^{0}$ be the total complex $\mathrm{Tot}([\mathcal{L}^{\cdot}])$. In particular $C^0 = P^0, C^{-1} = P^{-1} + Q^0$ and $C^{-2} = Q^{-1} + R^0$. The stupid filtration of the complexes $C^{\cdot}$ and G furnishes the spectral sequence

$$\mathrm{E}_1^{pq} = \bigoplus_{p_2 - p_1 = p} \mathrm{Ext}^q(C^{p_1}, G^{p_2}) \Longrightarrow \mathrm{Ext}^*(C^{\cdot}, G). \tag{4.4}$$

This spectral sequence is concentrated in the region of the plane defined by $-1 \leq p \leq 3$ and $q \geq 0$. We are interested on the total degrees -1 and 0. The rows $q = 1$ and $q = 0$ are

$$\mathrm{Ext}^1(C^0, G^{-1}) \to \mathrm{Ext}^1(C^0, G^0) \oplus \mathrm{Ext}^1(C^{-1}, G^{-1}) \to \mathrm{Ext}^1(C^{-1}, G^0) \oplus \mathrm{Ext}^1(C^{-2}, G^{-1}) \to ...$$

$$\mathrm{Hom}(C^0, G^{-1}) \stackrel{d_1^{-10}}{\to} \mathrm{Hom}(C^0, G^0) \oplus \mathrm{Hom}(C^{-1}, G^{-1}) \stackrel{d_1^{00}}{\to} \mathrm{Hom}(C^{-1}, G^0) \oplus \mathrm{Hom}(C^{-2}, G^{-1}) \to ...$$

Since $\mathrm{Ext}^1(C^0, G^{-1}) = 0$, i.e. the only extension of $[G^{-1} \to 0]$ by $[0 \to C^0]$ is the trivial one, we obtain

$$\mathrm{Hom}_{\mathcal{D}(\mathbf{S})}(C^{\cdot}, G[-1]) = \mathrm{Ext}^{-1}(C^{\cdot}, G) = \mathrm{E}_2^{-10} = \ker(d_1^{-10}), \tag{4.5}$$

$$\mathrm{Hom}_{\mathcal{D}(\mathbf{S})}(C^{\cdot}, G) = \mathrm{Ext}^{0}(C^{\cdot}, G) = \mathrm{E}_2^{00} = \ker(d_1^{00})/\mathrm{im}(d_1^{-10}). \tag{4.6}$$

Comparing the above equalities with the explicit computation (4.3) of the complex $[\mathcal{K}]$, we get

$$\mathrm{Ext}^i(C^{\cdot}, G) = \mathrm{H}^i\big([\mathcal{K}]\big) \qquad i = -1, 0. \tag{4.7}$$

These equalities together with equality (4.2) give the expected statement.

Remark 4.3. In the computation (4.3) the term $\mathrm{Hom}(P^{-1}, G^0)$ does not appear because we work with the good truncation $\tau_{\leq 0}\mathrm{RHom}([\mathcal{P}], [\mathcal{G}])$. In the spectral sequence (4.4) this term appear but we are interested in elements which become zero in $\mathrm{Hom}(P^{-1}, G^0)$.

Remark 4.4. If $\mathcal{H}(\mathbf{S})$ denotes the category of complexes of abelian sheaves on $\mathbf{S}$ modulo homotopy, by equality (4.6) we have $\mathrm{Hom}_{\mathcal{D}(\mathbf{S})}(C^{\cdot}, G) = \mathrm{Hom}_{\mathcal{H}(\mathbf{S})}(C^{\cdot}, G)$. Moreover, since P and G are 1-motives we have that $\mathrm{Hom}(C^0, G^{-1}) = 0$ ([B09] Lemma 1.1.1) and so $\mathrm{Hom}_{\mathcal{D}(\mathbf{S})}(C^{\cdot}, G) = \mathrm{Hom}_{\mathcal{K}(\mathbf{S})}(C^{\cdot}, G)$.

Remark 4.5. The category $\Psi_{\mathcal{L}^{\cdot}}(\mathcal{G})$ should be a 2-category, but it is just a category because we are working with strictly commutative Picard stacks defined by 1-motives. In fact, if A is a group scheme which is locally for the étale topology a constant group scheme defined by a finitely generated free $\mathbb{Z}$-module and B is an extension of an abelian scheme by a torus, then the group $\mathrm{Hom}(B, A)$ is trivial ([B09] Lemma 1.1.1). Because of (4.2), (4.5), (4.7), this implies that the group $\Psi_{\mathcal{L}^{\cdot}}^{-1}(\mathcal{G})$ of automorphisms of arrows from an object of $\Psi_{\mathcal{L}^{\cdot}}(\mathcal{G})$ to itself is trivial:

$$\Psi_{\mathcal{L}^{\cdot}}^{-1}(\mathcal{G}) \cong \mathrm{H}^{-1}\big([\mathcal{K}]\big) \cong \mathrm{Ext}^{-1}(C^{\cdot}, G) = \ker(d_1^{-10}) = 0.$$

Proof of the case i=1. First we show how an object (E, I) of $\Psi_{\mathrm{L}^{\cdot\cdot}}(G)$ defines a morphism $\mathrm{Tot}(\mathrm{L}^{\cdot\cdot}) \to G[1]$ in the derived category $\mathcal{D}(\mathbf{S})$. Recall that E is an extension of 1-motives of P by G. Denote $j : E \to P$ the surjective morphism underlying the extension E. Since the trivialization I can be seen as a lifting $Q \to E$ of $D^Q : Q \to P$ such that $I \circ D^R = 0$, we have the following diagram in the category $\mathcal{K}(\mathbf{S})$ of complexes of abelian sheaves on $\mathbf{S}$

$$\begin{tikzcd} \mathrm{L}^{\cdot\cdot}: & R \arrow[r, "D^R"] \arrow[d] & Q \arrow[r, "D^Q"] \arrow[d, "i"'] & P \arrow[r] \arrow[d, "id_P"] & 0 \\ MC(j): & 0 \arrow[r] & E \arrow[r, "j"] & P \arrow[r] & 0 \end{tikzcd} \tag{4.8}$$

where $i \circ D^R = 0$ and $j \circ i = id_P \circ D^Q$. Putting the complex P in degree 0, the above diagram gives an arrow

$$c(E,I) : \mathrm{Tot}(\mathrm{L}^{\cdot\cdot}) \longrightarrow MC(j)$$

in the derived category $\mathcal{D}(\mathbf{S})$. The complex E is an extension of 1-motives of P by G and so as observed at the end of section 2, G is quasi-isomorphic to $\tau_{\leq 0}(MC(j)[-1])$. Hence we have constructed a canonical arrow

$$\begin{array}{rcl} c : \Psi^1_{\mathrm{L}^{\cdot\cdot}}(G) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}(\mathbf{S})}\big(\mathrm{Tot}(\mathrm{L}^{\cdot\cdot}), G[1]\big) \\ (E,I) & \mapsto & c(E,I). \end{array} \tag{4.9}$$

Now we will show that this arrow is bijective. The proof that this bijection is additive, i.e. that c is an isomorphism of groups, is left to the reader.

Injectivity: Let (E,I) be an object of $\Psi_{\mathrm{L}^{\cdot\cdot}}(G)$ such that the morphism $c(E,I)$ that it defines in $\mathcal{D}(\mathbf{S})$ is the zero morphism. The hypothesis that $c(E,I)$ is zero in $\mathcal{D}(\mathbf{S})$ implies that there exists a resolution of G

$$V^0 \longrightarrow V^1 \longrightarrow V^2 \longrightarrow ...$$

and a quasi isomorphism

$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r, "j"] \arrow[d, "v^0"'] & P \arrow[r] \arrow[d, "v^1"] & 0 & \\ 0 \arrow[r] & V^0 \arrow[r, "k"] & V^1 \arrow[r] & V^2 \arrow[r] & ... \end{tikzcd} \tag{4.10}$$

such that the composite

$$\begin{tikzcd} R \arrow[r, "D^R"] & Q \arrow[r, "D^Q"] \arrow[d, "i"'] & P \arrow[r] \arrow[d, "id_P"] & 0 & \\ 0 \arrow[r] & E \arrow[r, "j"] \arrow[d, "v^0"'] & P \arrow[r] \arrow[d, "v^1"] & 0 & \\ 0 \arrow[r] & V^0 \arrow[r, "k"] & V^1 \arrow[r] & V^2 \arrow[r] & ... \end{tikzcd}$$

is homotopic to zero. We can assume $V^i \in \mathcal{K}^{[-1,0]}(\mathbf{S})$ for all i and $V^i = 0$ for $i \geq 2$ (instead of the complex of complexes $(V^i)_i$ consider its good truncation in degree 1). The complex of complexes $(V^i)_i$ is a resolution of G, and so the short sequence

$$0 \longrightarrow G \longrightarrow V^0 \longrightarrow V^1 \longrightarrow 0$$

is exact, i.e. V^0 is an extension of W by G. Since the quasi-isomorphism (4.10) induces the identity on G, the extension E is in fact the fibred product $P \times_{V^1} V^0$

of P and V^0 over V^1. Therefore, the morphism $s : P \to V^0$ inducing the homotopy $(v^0, v^1) \circ c(\mathcal{E}, I) \sim 0$, i.e. satisfying $k \circ s = v^1 \circ id_P$, factorizes through a morphism

$$h : P \longrightarrow E = P \times_{V^1} V^0$$

satisfying

$$j \circ h = id_P \qquad\qquad h \circ D^Q = i.$$

These two equalities mean that h splits the extension E, which is therefore the trivial extension of P by G, and that h is compatible with the trivializations I. Hence we can conclude that the object (E, I) lies in the isomorphism class of the zero object of $\Psi_{\mathrm{L}^{..}}(G)$.

Surjectivity: Now we show that for any morphism f of $\mathrm{Hom}_{\mathcal{D}(\mathbf{S})}(\mathrm{Tot}(\mathrm{L}^{..}), G[1])$, there is an element of $\Psi^1_{\mathrm{L}^{..}}(G)$ whose image via c is f. The hypothesis that f is an element of $\mathcal{D}(\mathbf{S})$ implies that there exists a resolution of G

$$V^0 \longrightarrow V^1 \longrightarrow V^2 \longrightarrow ...$$

such that the morphism f can be described in the category $\mathcal{H}(\mathbf{S})$ of complexes modulo homotopy via the following diagram

$$\begin{tikzcd} & R \arrow[r, "D^R"] & Q \arrow[r, "D^Q"] \arrow[d, "v^0"'] & P \arrow[r] \arrow[d, "v^1"] & 0 & \\ & 0 \arrow[r] & V^0 \arrow[r, "k"] & V^1 \arrow[r] & V^2 \arrow[r] & ... \end{tikzcd} \tag{4.11}$$

We can assume $V^i \in \mathcal{K}^{[-1,0]}(\mathbf{S})$ for all i and $V^i = 0$ for $i \geq 2$ (instead of the complex of complexes $(V^i)_i$ consider its good truncation in degree 1). Since the complex of complexes $(V^i)_i$ is a resolution of G, the short sequence of complexes

$$0 \longrightarrow G \longrightarrow V^0 \longrightarrow V^1 \longrightarrow 0$$

is exact, i.e. V^0 is an extension of V^1 by G. Consider the extension of P by G

$$Z = (v^1)^* V^0 = V^0 \times_{V^1} P$$

obtained as pull-back of V^0 via $v^1 : P \to V^1$. The pull-back of a short exact sequence is again a short exact sequence, and so $0 \to G \to Z \to P \to 0$ is exact. Moreover, as observed in Remark 1.2, since P and G are 1-motives the complex Z is an extension of 1-motives. The condition $v^1 \circ D^Q = k \circ v^0$ implies that $v^0 : Q \to V^0$ factories through a morphism

$$z : Q \to Z$$

satisfying $l \circ z = D^Q$, with $l : Z \to P$ the canonical surjection of the extension Z. Moreover the equalities $v^0 \circ D^R = D^Q \circ D^R = 0$ furnish $z \circ D^R = 0$. Therefore the datum (Z, z) is an object of the category $\Psi_{\mathrm{L}^{..}}(G)$. Consider now the morphism $c(Z, z) : \mathrm{Tot}(\mathrm{L}^{..}) \to G[1]$ associated to (Z, z). By construction, the morphism f (4.11) is the composite of the morphism $c(Z, z)$

$$\begin{tikzcd} R \arrow[r, "D^R"] \arrow[d] & Q \arrow[r, "D^Q"] \arrow[d, "z"'] & P \arrow[r] \arrow[d, "id_P"] & 0 \\ 0 \arrow[r] & Z \arrow[r, "l"] & P \arrow[r] & 0 \end{tikzcd}$$

with the morphism

$$\begin{tikzcd} 0 \arrow[r] & Z \arrow[r, "l"] \arrow[d, "h"'] & P \arrow[r] \arrow[d, "v^1"] & 0 \\ 0 \arrow[r] & V^0 \arrow[r, "k"] & V^1 \arrow[r] & 0, \end{tikzcd}$$

where $h : Z = (v^1)^* V^0 \to V^0$ is the canonical projection underlying the pull-back Z. Since this last morphism is a morphism of resolutions of G (inducing the identity on G), we can conclude that in the derived category $\mathcal{D}(\mathbf{S})$ the morphism $f : \mathrm{Tot}(\mathrm{L}^{\cdot\cdot}) \to G[1]$ (4.11) is the morphism $c(Z, z)$.

Using the above homological description of the groups $\Psi^i_{\mathrm{L}^{\cdot\cdot}}(G)$ for $i = 0, 1$ we can study how the category $\Psi_{\mathrm{L}^{\cdot\cdot}}(G)$ varies with respect to the bicomplex $\mathrm{L}^{\cdot\cdot}$. Let $R' \to Q' \to P' \to 0$ be another complex of 1-motives defined over S. Denote by $\mathrm{L}'^{\cdot\cdot}$ its total bicomplex. Consider a morphism of bicomplexes

$$F : \mathrm{L}'^{\cdot\cdot} \longrightarrow \mathrm{L}^{\cdot\cdot}$$

given by the following commutative diagram

$$\begin{tikzcd} R' \arrow[r, "D^{R'}"] \arrow[d, "F^{-2}"'] & Q' \arrow[r, "D^{Q'}"] \arrow[d, "F^{-1}"'] & P' \arrow[r] \arrow[d, "F^0"] & 0 \\ R \arrow[r, "D^R"'] & Q \arrow[r, "D^Q"'] & P \arrow[r] & 0. \end{tikzcd} \tag{4.12}$$

The morphism F defines a canonical functor

$$F^* : \Psi_{\mathrm{L}^{\cdot\cdot}}(G) \longrightarrow \Psi_{\mathrm{L}'^{\cdot\cdot}}(G)$$

as follows: if (E, I) is an object of $\Psi_{\mathrm{L}^{\cdot\cdot}}(G)$, $F^*(E, I)$ is the object (E', I') where

- E' is the extension $(F^0)^*E$ of P' by G obtained as pull-back of E via $F^0 : P' \to P$;
- I' is the trivialization $(F^{-1})^*I$ of $(D^{Q'})^*E'$ induced by the trivialization I of $(D^Q)^*E$ via the commutativity of the first square of (4.12).

The commutativity of the diagram (4.12) implies that (E', I') is in fact an object of $\Psi_{\mathrm{L}'^{\cdot\cdot}}(G)$ (the condition $I' \circ D^{Q'} = 0$ is easily deducible from the corresponding conditions on I and from the commutativity of the diagram (4.12)).

Proposition 4.6. *Let $F : \mathrm{L}'^{\cdot\cdot} \to \mathrm{L}^{\cdot\cdot}$ be morphism of bicomplexes. The corresponding functor $F^* : \Psi_{\mathrm{L}^{\cdot\cdot}}(G) \to \Psi_{\mathrm{L}'^{\cdot\cdot}}(G)$ is an equivalence of categories if and only if the homomorphisms*

$$\mathrm{H}^i\big(\mathrm{Tot}(F)\big) : \mathrm{H}^i\big(\mathrm{Tot}(\mathrm{L}'^{\cdot\cdot})\big) \longrightarrow \mathrm{H}^i\big(\mathrm{Tot}(\mathrm{L}^{\cdot\cdot})\big) \qquad i = 0, 1$$

are isomorphisms.

Proof. The functor $F^* : \Psi_{\mathrm{L}^{\cdot\cdot}}(G) \to \Psi_{\mathrm{L}'^{\cdot\cdot}}(G)$ induces the homomorphisms

$$\Psi^i_{\mathrm{L}^{\cdot\cdot}}(G) \longrightarrow \Psi^i_{\mathrm{L}'^{\cdot\cdot}}(G) \qquad i = 0, 1. \tag{4.13}$$

On the other hand the morphism of bicomplexes $F : \mathrm{L}'^{\cdot\cdot} \to \mathrm{L}^{\cdot\cdot}$ defines the homomorphisms

$$\mathrm{Ext}^i\big(\mathrm{Tot}(\mathrm{L}^{\cdot\cdot}), -\big) \longrightarrow \mathrm{Ext}^i\big(\mathrm{Tot}(\mathrm{L}'^{\cdot\cdot}), -\big) \qquad i \in \mathbb{Z}. \tag{4.14}$$

Since the homomorphisms (4.13) and (4.14) are compatible with the canonical isomorphisms obtained in Theorem 4.2, the following diagrams (with $i = 0, 1$) are commutative:

$$\begin{array}{ccc} \Psi^i_{\mathrm{L}^{\cdot\cdot}}(G) & \rightarrow & \mathrm{Ext}^i\big(\mathrm{Tot}(\mathrm{L}^{\cdot\cdot}), G\big) \\ \downarrow & & \downarrow \\ \Psi^i_{\mathrm{L}'^{\cdot\cdot}}(G) & \rightarrow & \mathrm{Ext}^i\big(\mathrm{Tot}(\mathrm{L}'^{\cdot\cdot}), G\big). \end{array}$$

The functor $F^* : \Psi_{\mathrm{L}^{\cdot\cdot}}(G) \rightarrow \Psi_{\mathrm{L}'^{\cdot\cdot}}(G)$ is an equivalence of categories if and only if the homomorphisms (4.13) are isomorphisms, and so using the above commutative diagrams we are reduced to prove that the homomorphisms (4.14) are isomorphisms if and only if the homomorphisms $\mathrm{H}^i\big(\mathrm{Tot}(F)\big) : \mathrm{H}^i\big(\mathrm{Tot}(\mathrm{L}'^{\cdot\cdot})\big) \rightarrow \mathrm{H}^i\big(\mathrm{Tot}(\mathrm{L}^{\cdot\cdot})\big)$ are isomorphisms. This last assertion is clearly true. $\square$

5. Geometrical description of $\Psi_{\mathrm{L}..}(G)$

In this section we switch from cohomological notation to homological.

Let $K = [u : A \rightarrow B]$ be a 1-motive defined over S with A in degree 1 and B in degree 0. We start constructing **a canonical flat partial resolution** $\mathrm{L}..(K)$ of the complex K. But before, we introduce the following notations: if P is an abelian sheaf on $\mathbf{S}$, we denote by $[p]$ the point of $\mathbb{Z}[P](U)$ defined by the point p of $P(U)$ with U an S-scheme. In an analogous way, if p, q and r are points of $P(U)$ we denote by $[p, q]$, $[p, q, r]$ the elements of $\mathbb{Z}[P \times P](U)$ and $\mathbb{Z}[P \times P \times P](U)$ respectively.

Consider the following complexes of $\mathcal{D}^{[1,0]}(\mathbf{S})$

$$\begin{array}{rcl} P & = & [\mathbb{Z}[A] \stackrel{D_{00}}{\longrightarrow} \mathbb{Z}[B]] \\ Q & = & [0 \longrightarrow \mathbb{Z}[B \times B]] \\ R & = & [0 \longrightarrow \mathbb{Z}[B \times B] + \mathbb{Z}[B \times B \times B]] \end{array} \tag{5.1}$$

and the following morphisms of complexes

$$\begin{array}{rcl} (\epsilon_1, \epsilon_0) & : & P \longrightarrow K \\ (0, d_{00}) & : & Q \longrightarrow P \\ (0, d_{01}) & : & R \longrightarrow Q \end{array}$$

where for any U and for any $a \in A(U), b_1, b_2, b_3 \in B(U)$, we set

$$\begin{array}{rcl} \epsilon_0[b] & = & b \\ \epsilon_1[a] & = & a \\ d_{00}[b_1, b_2] & = & [b_1 + b_2] - [b_1] - [b_2] \\ d_{01}[b_1, b_2] & = & [b_1, b_2] - [b_2, b_1] \\ d_{01}[b_1, b_2, b_3] & = & [b_1 + b_2, b_3] - [b_1, b_2 + b_3] + [b_1, b_2] - [b_2, b_3] \\ D_{00}[a] & = & [u(a)]. \end{array} \tag{5.2}$$

These data define the bicomplex L..(K)

$$
\begin{array}{rcccccccc}
 & & \overbrace{}^{\mathrm{L}_{2*}(K)} & & \overbrace{}^{\mathrm{L}_{1*}(K)} & & \overbrace{}^{\mathrm{L}_{0*}(K)} & & \\
\mathrm{L}_{*3}(K) & \{ & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
R{=}\mathrm{L}_{*2}(K) & \{ & 0 & \rightarrow & 0 & \rightarrow & \mathbb{Z}[B\times B]+\mathbb{Z}[B\times B\times B] & \rightarrow & 0 \\
 & & & & \downarrow & & \downarrow d_{01} & & \\
Q{=}\mathrm{L}_{*1}(K) & \{ & 0 & \rightarrow & 0 & \rightarrow & \mathbb{Z}[B\times B] & \rightarrow & 0 \\
 & & & & \downarrow & & \downarrow d_{00} & & \\
P{=}\mathrm{L}_{*0}(K) & \{ & 0 & \rightarrow & \mathbb{Z}[A] & \stackrel{D_{00}}{\rightarrow} & \mathbb{Z}[B] & \rightarrow & 0 \\
 & & & & \downarrow \epsilon_1 & & \downarrow \epsilon_0 & & \\
K & \{ & 0 & \rightarrow & A & \stackrel{u}{\rightarrow} & B & \rightarrow & 0
\end{array}
$$

which satisfies $\mathrm{L}_{ij}(K)=0$ for $(ij)\neq(00),(01),(02),(10)$ and which is endowed with an augmentation map $\epsilon.=(\epsilon_1,\epsilon_0): P\rightarrow K$. Note that the relation $\epsilon_0\circ d_{00}=0$ is just the group law on B, and the relation $d_{00}\circ d_{01}=0$ decomposes in two relations which express the commutativity and the associativity of the group law on B. This augmented bicomplex L..(K) depends functorially on K: in fact, any morphism $f: K\rightarrow K'$ of 1-motives furnishes a commutative diagram

$$
\begin{array}{ccc}
\mathrm{L..}(K) & \stackrel{\mathrm{L..}(f)}{\longrightarrow} & \mathrm{L..}(K') \\
\epsilon.\downarrow & & \downarrow\epsilon. \\
K & \stackrel{f}{\longrightarrow} & K'.
\end{array}
$$

Moreover the components of the bicomplex L..(K) are flat since they are free $\mathbb{Z}$-modules. In order to conclude that L..(K) is a canonical flat partial resolution of K we need the following Lemma. Let $K'=[u':A'\rightarrow B']$ be a 1-motive defined over S.

Lemma 5.1. *The category* $\mathbf{Ext}(K,K')$ *of extensions of* K *by* K' *is equivalent to the category* $\Psi_{\mathrm{L..}(K)}(K')$:

$$\mathbf{Ext}(K;K')\simeq\Psi_{\mathrm{L..}(K)}(K'). \tag{5.3}$$

Proof. In order to describe explicitly the objects of the category $\Psi_{\mathrm{L..}(K)}(K')$ we use the description (1.2) of extensions of free commutative groups in terms of torsors:

- an extension of $\mathbb{Z}[B]$ by B' is a $(B')_B$-torsor,
- an extension of $\mathbb{Z}[A]$ by B' is a $(B')_A$-torsor,
- an extension of $\mathbb{Z}[B\times B]$ by B' is a $(B')_{B\times B}$-torsor, and finally
- an extension of $\mathbb{Z}[B\times B]+\mathbb{Z}[B\times B\times B]$ by B' consists of a couple of a $(B')_{B\times B}$-torsor and a $(B')_{B\times B\times B}$-torsor.

According to these considerations an object (E,I) of $\Psi_{\mathrm{L..}(K)}(K')$ consists of

(1) an extension E of $P=[D_{00}:\mathbb{Z}[A]\rightarrow\mathbb{Z}[B]]$ by $K'=[u':A'\rightarrow B']$, i.e.
   (a) a B'-torsor E over B,
   (b) a trivializations β of the B'-torsor D_{00}^*E over A obtained as pull-back of E via $D_{00}:\mathbb{Z}[A]\rightarrow\mathbb{Z}[B]$,
   (c) an homomorphism $\gamma: A\rightarrow A'$ such that the composite $u'\circ\gamma$ is compatible with β;

(2) a trivialization I of the extension $(0,d_{00})^*E$ of Q by K' obtained as pull-back of E by $(0,d_{00}):Q\rightarrow P$, i.e. a trivialization I of the B'-torsor d_{00}^*E

over $B \times B$ obtained as pull-back of E via $d_{00} : \mathbb{Z}[B \times B] \to \mathbb{Z}[B]$. This trivialization can be interpreted as a group law on the fibres of the B'-torsor E:

$$+ : E_{b_1}\ E_{b_2} \longrightarrow E_{b_1+b_2}$$

where b_1, b_2 are points of $B(U)$ with U an S-scheme. The compatibility of I with the relation $(0, d_{00}) \circ (0, d_{01}) = 0$ imposes on the datum $(E, +)$ two relations through the two torsors over $B \times B$ and $B \times B \times B$. These two relations are the relations of commutativity and of associativity of the group law $+$, which mean that $+$ defines over E a structure of commutative extension of B by B'.

Hence the object $(E, +, \beta, \gamma)$ of $\Psi_{\mathrm{L}..(K)}(K')$ is an extension of K by K' and we can conclude that the category $\Psi_{\mathrm{L}..(K)}(K')$ is equivalent to the category $\mathbf{Ext}(K, K')$. $\square$

Proposition 5.2. *The augmentation map* $\epsilon. : \mathrm{L}.{}_0(K) \to K$ *induces the isomorphisms* $\mathrm{H}_1(\mathrm{Tot}(\mathrm{L}..(K))) \cong \mathrm{H}_1(K)$ *and* $\mathrm{H}_0(\mathrm{Tot}(\mathrm{L}..(K))) \cong \mathrm{H}_0(K)$.

Proof. Applying Proposition 4.6 to the augmentation map $\epsilon. : \mathrm{L}.{}_0(K) \to K$, we just have to prove that for any 1-motive $K' = [u' : A' \to B']$ the functor

$$\epsilon.^* : \Psi_K(K') \to \Psi_{\mathrm{L}..(K)}(K')$$

is an equivalence of categories (in the symbol $\Psi_K(K')$, K is seen as a bicomplex whose only non trivial entries are A in degree (10) and B in degree (00)). According to definition 4.1, it is clear that the category $\Psi_K(K')$ is equivalent to the category $\mathbf{Ext}(K, K')$ of extensions of K by K'. On the other hand, by Lemma 5.1 also the category $\Psi_{\mathrm{L}..(K)}(K')$ is equivalent to the category $\mathbf{Ext}(K, K')$. Hence we can conclude. $\square$

Let $K_i = [u_i : A_i \to B_i]$ (for $i = 1, 2, 3$) be 1-motives defined over S and let $\mathrm{L}..(K_i)$ be its canonical flat partial resolution. Denote by $\mathrm{L}..(K_1, K_2)$ the bicomplex $\mathrm{L}..(K_1) \otimes \mathrm{L}..(K_2)$.

Theorem 5.3. *The category* $\mathbf{Biext}(K_1, K_2; K_3)$ *of biextensions of* (K_1, K_2) *by* K_3 *is equivalent to the category* $\Psi_{\tau_{\leq (1*)}\mathrm{L}..(K_1,K_2)}(K_3)$:

$$\mathbf{Biext}(K_1, K_2; K_3) \simeq \Psi_{\tau_{\leq (1*)}\mathrm{L}..(K_1,K_2)}(K_3)$$

Proof. Explicitly, the non trivial components of $\mathrm{L}_{ij}(K_1,K_2)$ are

$$\begin{array}{rcl}
\mathrm{L}_{00}(K_1,K_2) & = & \mathrm{L}_{00}(K_1)\otimes \mathrm{L}_{00}(K_2) \\
 & = & \mathbb{Z}[B_1\times B_2] \\
\mathrm{L}_{01}(K_1,K_2) & = & \mathrm{L}_{00}(K_1)\otimes \mathrm{L}_{01}(K_2)+\mathrm{L}_{01}(K_1)\otimes \mathrm{L}_{00}(K_2) \\
 & = & \mathbb{Z}[B_1\times B_2\times B_2]+\mathbb{Z}[B_1\times B_1\times B_2] \\
\mathrm{L}_{02}(K_1,K_2) & = & \mathrm{L}_{00}(K_1)\otimes \mathrm{L}_{02}(K_2)+\mathrm{L}_{02}(K_1)\otimes \mathrm{L}_{00}(K_2)+\mathrm{L}_{01}(K_1)\otimes \mathrm{L}_{01}(K_2) \\
 & = & \mathbb{Z}[B_1\times B_2\times B_2]+\mathbb{Z}[B_1\times B_2\times B_2\times B_2]+ \\
 & = & \mathbb{Z}[B_1\times B_1\times B_2]+\mathbb{Z}[B_1\times B_1\times B_1\times B_2]+ \\
 & = & \mathbb{Z}[B_1\times B_1\times B_2\times B_2] \\
\mathrm{L}_{03}(K_1,K_2) & = & \mathrm{L}_{01}(K_1)\otimes \mathrm{L}_{02}(K_2)+\mathrm{L}_{02}(K_1)\otimes \mathrm{L}_{01}(K_2) \\
\mathrm{L}_{04}(K_1,K_2) & = & \mathrm{L}_{02}(K_1)\otimes \mathrm{L}_{02}(K_2) \\
\mathrm{L}_{10}(K_1,K_2) & = & \mathrm{L}_{10}(K_1)\otimes \mathrm{L}_{00}(K_2)+\mathrm{L}_{00}(K_1)\otimes \mathrm{L}_{10}(K_2) \\
 & = & \mathbb{Z}[A_1\times B_2]+\mathbb{Z}[B_1\times A_2] \\
\mathrm{L}_{11}(K_1,K_2) & = & \mathrm{L}_{10}(K_1)\otimes \mathrm{L}_{01}(K_2)+\mathrm{L}_{01}(K_1)\otimes \mathrm{L}_{10}(K_2) \\
 & = & \mathbb{Z}[A_1\times B_2\times B_2]+\mathbb{Z}[B_1\times B_1\times A_2] \\
\mathrm{L}_{12}(K_1,K_2) & = & \mathrm{L}_{10}(K_1)\otimes \mathrm{L}_{02}(K_2)+\mathrm{L}_{02}(K_1)\otimes \mathrm{L}_{10}(K_2) \\
\mathrm{L}_{20}(K_1,K_2) & = & \mathrm{L}_{10}(K_1)\otimes \mathrm{L}_{10}(K_2) \\
 & = & \mathbb{Z}[A_1\times A_2]
\end{array}$$

The differential operators of $\mathrm{L}_{..}(K_1,K_2)$ can be computed from the below diagram, where we don't have written the identity homomorphisms in order to avoid too heavy notation (for example instead of $(id\times D_{00}^{K_2}, D_{00}^{K_1}\times id)$ we have written just $(D_{00}^{K_2}, D_{00}^{K_1})$):

(5.4)

$$\begin{tikzcd}
 & \mathrm{L}_{2*}(K) & \mathrm{L}_{1*}(K) & \mathrm{L}_{0*}(K) \\
\mathrm{L}_{*2}(K) & & 0 \arrow[r] \arrow[d] & \mathrm{L}_{02}(K_1,K_2) \arrow[d, "d_{01}^{K_2}+d_{01}^{K_1}+(d_{00}^{K_1},d_{00}^{K_2})"] \\
\mathrm{L}_{*1}(K) & & \mathrm{L}_{11}(K_1,K_2) \arrow[r, "D_{00}^{K_1}+D_{00}^{K_2}"] \arrow[d, "d_{00}^{K_2}+d_{00}^{K_1}"'] & \mathrm{L}_{01}(K_1,K_2) \arrow[d, "d_{00}^{K_2}+d_{00}^{K_1}"] \\
\mathrm{L}_{*0}(K) & \mathrm{L}_{20}(K_1,K_2) \arrow[r, "(D_{00}^{K_2},D_{00}^{K_1})"] & \mathrm{L}_{10}(K_1,K_2) \arrow[r, "D_{00}^{K_1}+D_{00}^{K_2}"] & \mathrm{L}_{00}(K_1,K_2).
\end{tikzcd}$$

These operators have to satisfy the well-known conditions on differential operators of bicomplexes that we recall explicitly here:

- the following sequences are exact:

(5.5) $$\mathbb{Z}[B_1\times B_2\times B_2]+\mathbb{Z}[B_1\times B_2\times B_2\times B_2]\xrightarrow{d_{01}^{K_2}}\mathbb{Z}[B_1\times B_2\times B_2]\xrightarrow{d_{00}^{K_2}}\mathbb{Z}[B_1\times B_2]$$

(5.6) $$\mathbb{Z}[B_1\times B_1\times B_2]+\mathbb{Z}[B_1\times B_1\times B_1\times B_2]\xrightarrow{d_{01}^{K_1}}\mathbb{Z}[B_1\times B_1\times B_2]\xrightarrow{d_{00}^{K_1}}\mathbb{Z}[B_1\times B_2]$$

- the following diagrams are anticommutative:

$$\begin{array}{ccc} \mathbb{Z}[B_1 \times B_1 \times B_2 \times B_2] & \stackrel{d_{00}^{K_2}}{\longrightarrow} & \mathbb{Z}[B_1 \times B_1 \times B_2] \\ {\scriptstyle d_{00}^{K_1}}\downarrow & & \downarrow{\scriptstyle d_{00}^{K_1}} \\ \mathbb{Z}[B_1 \times B_2 \times B_2] & \stackrel{d_{00}^{K_2}}{\longrightarrow} & \mathbb{Z}[B_1 \times B_2] \end{array} \tag{5.7}$$

$$\begin{array}{ccc} \mathbb{Z}[A_1 \times B_2 \times B_2] & \stackrel{D_{00}^{K_1}}{\longrightarrow} & \mathbb{Z}[B_1 \times B_2 \times B_2] \\ {\scriptstyle d_{00}^{K_2}}\downarrow & & \downarrow{\scriptstyle d_{00}^{K_2}} \\ \mathbb{Z}[A_1 \times B_2] & \stackrel{D_{00}^{K_1}}{\longrightarrow} & \mathbb{Z}[B_1 \times B_2] \end{array} \tag{5.8}$$

$$\begin{array}{ccc} \mathbb{Z}[B_1 \times B_1 \times A_2] & \stackrel{D_{00}^{K_2}}{\longrightarrow} & \mathbb{Z}[B_1 \times B_1 \times B_2] \\ {\scriptstyle d_{00}^{K_1}}\downarrow & & \downarrow{\scriptstyle d_{00}^{K_1}} \\ \mathbb{Z}[B_1 \times A_2] & \stackrel{D_{00}^{K_2}}{\longrightarrow} & \mathbb{Z}[B_1 \times B_2] \end{array} \tag{5.9}$$

$$\begin{array}{ccc} \mathbb{Z}[A_1 \times A_2] & \stackrel{D_{00}^{K_2}}{\longrightarrow} & \mathbb{Z}[A_1 \times B_2] \\ {\scriptstyle D_{00}^{K_1}}\downarrow & & \downarrow{\scriptstyle D_{00}^{K_1}} \\ \mathbb{Z}[B_1 \times A_2] & \stackrel{D_{00}^{K_2}}{\longrightarrow} & \mathbb{Z}[B_1 \times B_2] \end{array} \tag{5.10}$$

The bicomplex $\tau_{\leq (1*)}\mathrm{L}..(K_1, K_2)$ is furnished by the bicomplex (5.4) where instead of $\mathrm{L}_{10}(K_1)$ we have

$$\begin{aligned} \mathrm{L}'_{10}(K_1, K_2) &= \mathrm{L}_{10}(K_1, K_2) \Big/ (D_{00}^{K_2}, D_{00}^{K_1})\, \mathrm{L}_{20}(K_1, K_2) \\ &= \mathbb{Z}[A_1 \times B_2] + \mathbb{Z}[B_1 \times A_2] \Big/ (id \times u_2) + (u_1 \times id)\, \mathbb{Z}[A_1 \times A_2] \end{aligned} \tag{5.11}$$

In order to describe explicitly the objects of $\Psi_{\tau_{\leq (1*)}\mathrm{L}..(K_1,K_2)}(K_3)$ we use the description (1.2) of extensions of free commutative groups in terms of torsors:

- an extension of $\mathrm{L}_{00}(K_1, K_2)$ by B_3 is a $(B_3)_{B_1 \times B_2}$-torsor,
- an extension of $\mathrm{L}'_{10}(K_1, K_2)$ by B_3 consists of a $(B_3)_{A_1 \times B_2}$-torsor and a $(B_3)_{B_1 \times A_2}$-torsor,
- an extension of $\mathrm{L}_{02}(K_1, K_2)$ by B_3 consists of a system of 5 torsors under the groups deduced from B_3 by base change over the bases $B_1 \times B_2 \times B_2,\ B_1 \times B_2 \times B_2 \times B_2,\ B_1 \times B_1 \times B_2,\ B_1 \times B_1 \times B_1 \times B_2,\ B_1 \times B_1 \times B_2 \times B_2$.

By these considerations an object (E, I) of $\Psi_{\tau_{\leq (1*)}\mathrm{L}..(K_1,K_2)}(K_3)$ consists of

(1) an extension E of $[D_{00}^{K_1} + D_{00}^{K_2} : \mathrm{L}'_{10}(K_1, K_2) \to \mathrm{L}_{00}(K_1, K_2)]$ by K_3, i.e.
  (a) a B_3-torsor E over $B_1 \times B_2$,
  (b) a couple of trivializations (Ψ_1, Ψ_2) of the couple of B_3-torsors $((D_{00}^{K_1} \times id)^* E, (id \times D_{00}^{K_2})^* E)$ over $A_1 \times B_2$ and $B_1 \times A_2$ respectively, which are the pull-back of E via

$$(D_{00}^{K_1} \times id) + (id \times D_{00}^{K_2}) : \mathbb{Z}[A_1 \times B_2] + \mathbb{Z}[B_1 \times A_2] \to \mathbb{Z}[B_1 \times B_2];$$

  We consider the factor $\mathrm{L}'_{10}(K_1, K_2)$ (5.11) instead of $\mathrm{L}_{10}(K_1, K_2)$ and this means that the restriction of the trivializations (Ψ_1, Ψ_2) have to coincide over $A_1 \times A_2$,

(c) a homomorphism $\gamma : \mathbb{Z}[A_1] \otimes \mathbb{Z}[A_2] \to A_3$ such that the composite $\mathbb{Z}[A_1]\otimes\mathbb{Z}[A_2] \stackrel{\gamma}{\to} \mathbb{Z}[A_1]\otimes\mathbb{Z}[A_2] \stackrel{u_3}{\to} B_3$ is compatible with the restriction of the trivializations Ψ_1, Ψ_2 over $\mathbb{Z}[A_1] \otimes \mathbb{Z}[A_2]$.

(2) a trivialization I of the extension $(d_{00}^{K_2}+d_{00}^{K_1}, d_{00}^{K_2}+d_{00}^{K_1})^*E$ of $[D_{00}^{K_1}+D_{00}^{K_2} : \mathrm{L}_{11}(K_1,K_2) \to \mathrm{L}_{01}(K_1,K_2)]$ by K_3 obtained as pull-back of E via

$$(d_{00}^{K_2}+d_{00}^{K_1}, d_{00}^{K_2}+d_{00}^{K_1}) : [\mathrm{L}_{11}(K_1,K_2) \to \mathrm{L}_{01}(K_1,K_2)] \longrightarrow [\mathrm{L}'_{10}(K_1,K_2) \to \mathrm{L}_{00}(K_1,K_2)],$$

i.e. a couple of trivializations $\alpha = (\alpha_1, \alpha_2)$ of the couple of B_3-torsors over $B_1 \times B_2 \times B_2$ and $B_1 \times B_1 \times B_2$ which are the pull-back of E via $(id \times d_{00}^{K_2}) + (d_{00}^{K_1} \times id) : \mathbb{Z}[B_1 \times B_2 \times B_2] + \mathbb{Z}[B_1 \times B_1 \times B_2] \to \mathbb{Z}[B_1 \times B_2]$. The trivializations (α_1, α_2) can be viewed as two group laws on the fibres of the B_3-torsor E over $B_1 \times B_2$:

$$+_2 : E_{b_1,b_2}\ E_{b_1,b'_2} \longrightarrow E_{b_1,b_2+b'_2} \qquad\qquad +_1 : E_{b_1,b_2}\ E_{b'_1,b_2} \longrightarrow E_{b_1+b'_1,b_2}$$

where b_2, b'_2 (resp. b_1, b'_1) are points of $B_2(U)$ (resp. of $B_1(U)$) with U any S-scheme.

The trivialization I, i.e. the two group laws, must be compatible with the trivializations (Ψ_1, Ψ_2) underlying the trivialization E. This compatibility is expressed through the 2 torsors arising from the factors $\mathrm{L}_{11}(K_1,K_2)$:

- the anticommutative diagram (5.8) furnishes a relation of compatibility between the group law $+_2$ of E and the trivialization Ψ_1 of the pull-back $(D_{00}^{K_1} \times id)^*E$ of E over $A_1 \times B_2$, which means that Ψ_1 is an additive section;
- the anticommutative diagram (5.9) furnishes a relation of compatibility between the group law $+_1$ of E and the trivialization Ψ_2 of the pull-back $(id \times D_{00}^{K_2})^*E$ of E over $B_1 \times A_2$, which means that also Ψ_2 is an additive section.

Finally, the compatibility of I with the relation

$$\big(d_{00}^{K_2} + d_{00}^{K_1}, d_{00}^{K_2} + d_{00}^{K_1}\big) \circ \big(d_{01}^{K_2} + d_{01}^{K_1} + (d_{00}^{K_1}, d_{00}^{K_2})\big) = 0$$

imposes on the datum $(E, +_1, +_2)$ 5 relations of compatibility through the system of 5 torsors over $B_1 \times B_2 \times B_2,\ B_1 \times B_2 \times B_2 \times B_2,\ B_1 \times B_1 \times B_2,\ B_1 \times B_1 \times B_1 \times B_2,\ B_1 \times B_1 \times B_2 \times B_2$ arising from $\mathrm{L}_{02}(K_1,K_2)$:

- the exact sequence (5.5) furnishes the two relations of commutativity and of associativity of the group law $+_2$, which mean that $+_2$ defines over E a structure of commutative extension of $(B_2)_{B_1}$ by $(B_3)_{B_1}$;
- the exact sequence (5.6) expresses the two relations of commutativity and of associativity of the group law $+_1$, which mean that $+_1$ defines over E a structure of commutative extension of $(B_1)_{B_2}$ by $(B_3)_{B_2}$;
- the anticommutative diagram (5.7) means that these two group laws are compatible.

Therefore these 5 conditions implies that the B_3-torsor E is endowed with a structure of biextension of (B_1, B_2) by B_3.

The object $(E, \Psi_1, \Psi_2, \gamma)$ of $\Psi_{\tau_{\leq(1*)}\mathrm{L}..(K_1,K_2)}(K_3)$ is therefore a biextension of (K_1, K_2) by K_3. □

In the above proof we have not used diagram (5.10) because we work with the truncated bicomplex $\tau_{\leq(1*)}\mathrm{L}..(K_1,K_2)$ (see (5.11)).

6. Proof of theorem 0.1 (a)

Let $K_i = [A_i \stackrel{u_i}{\to} B_i]$ (for $i = 1, 2, 3$) be three 1-motives defined over S. Denote by $\mathrm{L}..(K_i)$ (for $i = 1, 2$) the canonical flat partial resolution of K_i introduced in §5. According to Proposition 5.2, there exists an arbitrary flat resolution $\mathrm{L}'..(K_i)$ (for $i = 1, 2$) of K_i such that the groups $\mathrm{Tot}(\mathrm{L}..(K_i))_j$ and $\mathrm{Tot}(\mathrm{L}'..(K_i))_j$ are isomorphic for $j = 0, 1$. We have therefore two canonical homomorphisms of bicomplexes

$$\mathrm{L}..(K_1) \longrightarrow \mathrm{L}'..(K_1) \qquad \mathrm{L}..(K_2) \longrightarrow \mathrm{L}'..(K_2)$$

inducing a canonical homomorphism between the corresponding total complexes

$$\mathrm{Tot}(\mathrm{L}..(K_1) \otimes \mathrm{L}..(K_2)) \longrightarrow \mathrm{Tot}(\mathrm{L}'..(K_1) \otimes \mathrm{L}'..(K_2))$$

which is an isomorphism in degrees 0 and 1. Denote by $\mathrm{L}..(K_1, K_2)$ (resp. $\mathrm{L}'..(K_1, K_2)$) the bicomplex $\mathrm{L}..(K_1)\otimes\mathrm{L}..(K_2)$ (resp. $\mathrm{L}'..(K_1)\otimes\mathrm{L}'..(K_2)$). Remark that $\mathrm{Tot}(\mathrm{L}'..(K_1, K_2))$ represents $K_1 \stackrel{\mathbb{L}}{\otimes} K_2$ in the derived category $\mathcal{D}(\mathbf{S})$:

$$\mathrm{Tot}(\mathrm{L}'..(K_1, K_2)) = K_1 \stackrel{\mathbb{L}}{\otimes} K_2.$$

By Proposition 4.6 we have the equivalence of categories

$$\Psi_{\tau_{\leq(1*)}\mathrm{L}..(K_1,K_2)}(K_3) \simeq \Psi_{\tau_{\leq(1*)}\mathrm{L}'..(K_1,K_2)}(K_3).$$

Hence applying Theorem 5.3, which furnishes the following geometrical description of the category $\Psi_{\tau_{\leq(1*)}\mathrm{L}..(K_1,K_2)}(K_3)$:

$$\mathbf{Biext}(K_1, K_2; K_3) \simeq \Psi_{\tau_{\leq(1*)}\mathrm{L}..(K_1,K_2)}(K_3),$$

and applying Theorem 4.2, which furnishes the following homological description of the groups $\Psi^i_{\tau_{\leq(1*)}\mathrm{L}'..(K_1,K_2)}(K_3)$ for $i = 0, 1$:

$$\Psi^i_{\tau_{\leq(1*)}\mathrm{L}'..(K_1,K_2)}(K_3) \cong \mathrm{Ext}^i\big(\mathrm{Tot}\big(\tau_{\leq(1*)}\mathrm{L}'..(K_1, K_2)\big), K_3\big) \cong \mathrm{Ext}^i(K_1 \stackrel{\mathbb{L}}{\otimes} K_2, K_3),$$

we get Theorem 0.1, i.e.

$$\mathrm{Biext}^i(K_1, K_2; K_3) \cong \mathrm{Ext}^i(K_1 \stackrel{\mathbb{L}}{\otimes} K_2, K_3) \qquad (i = 0, 1).$$

Remark 6.1. From the exact sequence $0 \to B_3 \to K_3 \to A_3[1] \to 0$ we get the long exact sequence

$$0 \to \Psi^0_{\tau_{\leq(1*)}\mathrm{L}'..(K_1,K_2)}(B_3) \to \Psi^0_{\tau_{\leq(1*)}\mathrm{L}'..(K_1,K_2)}(K_3) \to \Psi^0_{\tau_{\leq(1*)}\mathrm{L}'..(K_1,K_2)}(A_3[1])$$

$$\to \Psi^1_{\tau_{\leq(1*)}\mathrm{L}'..(K_1,K_2)}(B_3) \to \Psi^1_{\tau_{\leq(1*)}\mathrm{L}'..(K_1,K_2)}(K_3) \to \Psi^1_{\tau_{\leq(1*)}\mathrm{L}'..(K_1,K_2)}(A_3[1]).$$

The homological interpretation of this long exact sequence is

$$0 \to \mathrm{Hom}(T, B_3) \to \mathrm{Hom}(T, K_3) \to \mathrm{Hom}(T, A_3[1])$$

$$\to \mathrm{Ext}^1(T, B_3) \to \mathrm{Ext}^1(T, K_3) \to \mathrm{Ext}^1(T, A_3[1]),$$

where we set $T = \mathrm{Tot}\big(\tau_{\leq(1*)}\mathrm{L}'..(K_1, K_2)\big)$, and its geometrical interpretation is

$$0 \to \mathrm{Hom}(B_1 \otimes B_2, B_3) \to \mathrm{Hom}(K_1 \stackrel{\mathbb{L}}{\otimes} K_2, K_3) \to \mathrm{Hom}(A_1 \otimes B_2 + B_1 \otimes A_2, A_3)$$

$$\to \mathrm{Biext}^1(K_1, K_2; B_3) \to \mathrm{Biext}^1(K_1, K_2; K_3) \to \mathrm{Hom}(A_1 \otimes A_2, A_3).$$

References

[B08] C. Bertolin, *Multilinear morphisms between 1-motives*, J. Reine Angew. Math. 637 (2009), pp. 141–174.

[B09] C. Bertolin, *Extensions and biextensions of locally constant group schemes, tori and abelian schemes*, Math. Z. (2009) 261, pp. 845–868.

[Be11] C. Bertolin, *Extensions of Picard stacks and their homological interpretation*, J. of Algebra 331 (2011), no. 1, pp. 28–45.

[Be] C. Bertolin, *Biextensions of Picard stacks and their homological interpretation*, arXiv:1103.0723v1 [math.AG], 2011.

[BM] C. Bertolin, C. Mazza, *Biextensions of 1-motives in Voevodsky's category of motives*, Int. Math. Res. Not. IMRN (2009), no. 19, pp. 3747–3757.

[D73] P. Deligne, *La formule de dualité globale*, Théorie des topos et cohomologie étale des schémas, Tome 3. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4). Lecture Notes in Mathematics, Vol. 305. Springer-Verlag, Berlin-New York, 1973.

[D89] P. Deligne, *Le groupe fondamental de la droite projective moins trois points*, Galois groups over Q (Berkeley, CA, 1987), pp. 79–297, Math. Sci. Res. Inst. Publ., 16, Springer, New York, 1989.

[G71] J. Giraud, *Cohomologie non abélienne*, Die Grundlehren der mathematischen Wissenschaften, Band 179. Springer-Verlag, Berlin-New York, 1971.

[G] A. Grothendieck and others, *Groupes de Monodromie en Géométrie Algébrique*, SGA 7 I, Lecture Notes in Mathematics, Vol. 288. Springer-Verlag, Berlin-New York, 1972.

[R] M. Raynauld, *1-motifs et monodromie géométrique*, Périodes p-adiques, Astérisque 223 (1994) pp. 295–319.

Dip. di Matematica, Università di Torino, Via Carlo Alberto 10, I-10123 Torino
E-mail address: cristiana.bertolin@googlemail.com