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MOTIVIC GALOIS GROUPS OF 1-MOTIVES: A SURVEY

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Abstract. We investigate the structure of the motivic Galois groups of 1-motives defined over a field of characteristic 0.

In this note we review the main results of [B03] and [B04].

Let $k$ be a field of characteristic 0 and let $\bar{k}$ be its algebraic closure. Let $T$ be a Tannakian category over $k$. The tensor product of $T$ allows us to define the notion of Hopf algebras in the category $\text{Ind} T$ of Ind-objects of $T$. The category of affine $T$-schemes is the opposite of the category of Hopf algebras in $\text{Ind} T$.

The fundamental group $\pi(T)$ of $T$ is the affine group $\text{Sp}(\Lambda)$, whose Hopf algebra $\Lambda$ is endowed for each object $X$ of $T$ with a morphism $X \longrightarrow \Lambda \otimes X$ functorial in $X$, and is universal for these properties. Those morphisms $\{X \longrightarrow \Lambda \otimes X\}_{X \in T}$ define an action of the fundamental group $\pi(T)$ on each object of $T$.

For each fibre functor $\omega$ of $T$ over a $k$-scheme $S$, $\omega\pi(T)$ is the affine group $S$-scheme $\text{Aut}^\otimes_S(\omega)$ which represents the functor which associates to each $S$-scheme $T$, $u : T \longrightarrow S$, the group of automorphisms of $\otimes$-functors of the functor $u^*\omega$.

If $T(k)$ is a Tannakian category generated by motives defined over $k$ (in an appropriate category of mixed realizations), the fundamental group $\pi(T(k))$ is called the motivic Galois group $G_{\text{mot}}(T(k))$ of $T(k)$ and for each embedding $\sigma : k \longrightarrow \mathbb{C}$, the fibre functor $\omega_{\sigma}$ “Hodge realization” furnishes the $\mathbb{Q}$-algebraic group

$$\omega_{\sigma}G_{\text{mot}}(T) = \text{Spec} (\omega_{\sigma}(\Lambda)) = \text{Aut}^\otimes_S(\omega_{\sigma})$$

which is the Hodge realization of the motivic Galois group of $T(k)$.

EXAMPLES:

(1) From the main theorem on neutral Tannakian categories, we know that the Tannakian category $\text{Vec}(k)$ of finite dimensional $k$-vector spaces is equivalent to the category of finite-dimensional $k$-representations of $\text{Spec}(k)$. In this case, affine group $T$-schemes are affine group $k$-schemes and $\pi(\text{Vec}(k))$ is $\text{Spec}(k)$.

(2) Let $T = \text{Rep}_k(G)$ be the Tannakian category of $k$-representations of an affine group $k$-scheme $G$. The affine group $T$-schemes are affine $k$-schemes endowed with an action of $G$ and the fundamental group $\pi(T)$ of $T$ is $G$ endowed with its action on itself by inner automorphisms (see [D89] 6.3).

(3) Let $\mathcal{T}_0(k)$ be the Tannakian category of Artin motives over $k$, i.e. the Tannakian sub-category of the Tannakian category of mixed realizations for absolute Hodge cycles (see [J90] I 2.1) generated by pure realizations of 0-dimensional varieties over $k$. The motivic Galois group $G_{\text{mot}}(\mathcal{T}_0(k))$ of $\mathcal{T}_0(k)$ is the affine group $\mathbb{Q}$-scheme $\text{Gal}(\bar{k}/k)$ endowed with its action on itself by inner automorphisms. We denote it by $\mathcal{GAC}(\bar{k}/k)$. In particular,
for any fibre functor \( \omega \) over Spec(\( k \)) of \( \mathcal{T}_0(k) \), the affine group scheme 
\( \omega(\mathcal{GAL}(\overline{k}/k)) = \text{Aut}_\text{Spec}(\mathbb{Q})(\omega) \) is canonically isomorphic to Gal(\( k/\overline{k} \)).

(4) The motivic Galois group \( \mathcal{G}_{\text{mot}}(\mathbb{Z}(0)) \) of the unit object \( \mathbb{Z}(0) \) of \( \mathcal{T}_0(k) \) is the affine group \( (\mathbb{Z}(0))^{\otimes} \)-scheme \( \text{Sp}(\mathbb{Z}(0)) \). For each fibre functor “Hodge realization” \( \omega_\sigma \), we have that \( \omega_\sigma(\mathcal{G}_{\text{mot}}(\mathbb{Z}(0))) := \text{Spec}(\omega_\sigma(\mathbb{Z}(0))) = \text{Spec}(\mathbb{Q}) \), which is the Mumford-Tate group of \( \mathcal{T}_\sigma(\mathbb{Z}(0)) \).

(5) Let \( (\mathbb{Z}(1))^{\otimes} \) be the Tannakian category over \( k \) defined by the \( k \)-torus \( \mathbb{Z}(1) \). The motivic Galois group \( \mathcal{G}_{\text{mot}}(\mathbb{Z}(1)) \) of the torus \( \mathbb{Z}(1) \) is the affine group \( (\mathbb{Z}(1))^{\otimes} \)-scheme \( \mathbb{G}_m \) defined by the \( Q \)-scheme \( \mathbb{G}_m/Q \). For each fibre functor “Hodge realization” \( \omega_\sigma \), we have that \( \omega_\sigma(\mathcal{G}_m) = \mathbb{G}_m/Q \), which is the Mumford-Tate group of \( \mathcal{T}_\sigma(\mathbb{Z}(1)) \).

(6) If \( k \) is algebraically closed, the motivic Galois group of motives of CM-type over \( k \) is the Serre group (cf. [M94] 4.8).

(7) The Tannakian category \( \mathcal{T}_1(k) \) of 1-motives over \( k \) is the Tannakian subcategory of the Tannakian category of mixed realizations (for absolute Hodge cycles) generated by mixed realizations of 1-motives over \( k \). Recall that a 1-motive \( M = [X \xrightarrow{u} G] \) over \( k \) consists of

- a group scheme \( X \) over \( k \), which is locally for the étale topology, a constant group scheme defined by a finitely generated free \( Z \)-module,
- a semi-abelian variety \( G \) defined over \( k \), i.e. an extension of an abelian variety \( A \) by a torus \( Y(1) \), which cocharacter group \( Y \),
- a morphism \( u : X \longrightarrow G \) of group schemes over \( k \).

1-motives are mixed motives of level \( \leq 1 \): the weight filtration \( \mathcal{W}_* \) on \( M \) is \( \mathcal{W}_i(M) = M \) for each \( i \geq 0 \), \( \mathcal{W}_{-1}(M) = G, \mathcal{W}_{-2}(M) = Y(1), \mathcal{W}_{j}(M) = 0 \) for each \( j \leq -3 \). If \( \mathcal{G}_{\text{mot}}(\mathcal{W}_i) = \mathcal{W}_i/\mathcal{W}_{i-1} \), we have the quotients \( \mathcal{G}_{\text{mot}}(\mathcal{W}_i) = X, \mathcal{G}_{\text{mot}}(\mathcal{W}_2(M)) = A \) and \( \mathcal{G}_{\text{mot}}(\mathcal{W}_3(M)) = Y(1) \). We will denote by \( \mathcal{W}_{-1}\mathcal{T}_1(k) \) (resp. \( \mathcal{G}_{\text{mot}}(\mathcal{W}_i) \)) the Tannakian sub-category of \( \mathcal{T}_1(k) \) generated by all \( \mathcal{W}_{i}M \) (resp. \( \mathcal{G}_{\text{mot}}(\mathcal{W}_i) \)) with \( M \) a 1-motive. With this notation we can easily compute the following motivic Galois groups

- \( \mathcal{G}_{\text{mot}}(\mathcal{G}_{\text{mot}}(\mathcal{T}_1(k))) = \mathcal{G}_{\text{mot}}(\mathcal{T}_0(\overline{k})) \times \mathbb{G}_m \),
- \( \mathcal{G}_{\text{mot}}(\mathcal{G}_{\text{mot}}(\mathcal{T}_1(k))) = \mathcal{G}_{\text{mot}}(\mathcal{T}_0(\overline{k})) = \text{Sp}(\mathbb{Z}(0)) \),
- \( \mathcal{G}_{\text{mot}}(\mathcal{G}_{\text{mot}}(\mathcal{T}_1(k))) = \mathbb{G}_m \).

1. Motivic Galois theory

For each Tannakian sub-category \( \mathcal{T}' \) of \( \mathcal{T} \), let \( H_{\mathcal{T}}(\mathcal{T}') \) be the kernel of the faithfully flat morphism of group \( T \)-schemes \( \pi : \mathcal{T} \longrightarrow i\pi(\mathcal{T}') \) corresponding to the inclusion functor \( i : \mathcal{T}' \longrightarrow \mathcal{T} \). In particular we have the short exact sequence of group \( \pi(\mathcal{T}) \)-schemes

\[
0 \longrightarrow H_{\mathcal{T}}(\mathcal{T}') \longrightarrow \pi(\mathcal{T}) \longrightarrow i\pi(\mathcal{T}') \longrightarrow 0.
\]

In [D89] 6.6, Deligne proves that the Tannakian category \( \mathcal{T}' \) is equivalent, as tensor category, to the sub-category of \( T \) generated by those objects on which the action of \( \pi(\mathcal{T}) \) induces a trivial action of \( H_{\mathcal{T}}(\mathcal{T}') \). In particular, this implies that the fundamental group \( \pi(\mathcal{T}') \) of \( \mathcal{T}' \) is isomorphic to the group \( T \)-scheme \( \pi(\mathcal{T})/H_{\mathcal{T}}(\mathcal{T}') \). The group \( T \)-scheme \( H_{\mathcal{T}}(\mathcal{T}') \) characterizes the Tannakian sub-category \( \mathcal{T}' \). In fact
we have a clear dictionary between Tannakian sub-categories of $T$ and normal affine group sub-$\mathcal{T}$-schemes of the fundamental group $\pi(T)$ of $T$.

**Theorem 1.0.1.** There is bijection between the Tannakian sub-categories of $\mathcal{T}$ and the normal affine group sub-$\mathcal{T}$-schemes of $\pi(T)$, which associates
- to each Tannakian sub-category $\mathcal{T}'$ of $T$, the kernel $H_T(\mathcal{T}')$ of the morphism of $\mathcal{T}$-schemes $1 : \pi(T) \to \pi(\mathcal{T}')$ corresponding to the inclusion $i : \mathcal{T}' \to \mathcal{T}$;
- to each normal affine group sub-$\mathcal{T}$-scheme $H$ of $\pi(T)$, the Tannakian sub-category $T(H)$ of objects of $\mathcal{T}$ on which the action of $\pi(T)$ induces a trivial action of $H$.

2. The case of motives of level $\leq 1$

In order to study the category $\mathcal{T}_1(k)$ of motives of level $\leq 1$, in [B04] we have applied the above theorem to some sub-categories of $\mathcal{T}_1(k)$. The weight filtration $W_*$ of 1-motives induces an increasing filtration $W_*$ of 3 steps on the motivic Galois group $G_{\text{mot}}(\mathcal{T}_1(k))$ which we describe through the action of $G_{\text{mot}}(\mathcal{T}_1(k))$ on the generators of $\mathcal{T}_1(k)$: For each 1-motive $M$ over $k$, we have that

\begin{itemize}
  \item $W_0(G_{\text{mot}}(\mathcal{T}_1(k))) = G_{\text{mot}}(\mathcal{T}_1(k))$
  \item $W_{-1}(G_{\text{mot}}(\mathcal{T}_1(k))) = \{g \in G_{\text{mot}}(\mathcal{T}_1(k)) \mid (g - id)M \subseteq W_{-1}(M), (g - id)W_{-1}(M) = 0\}$,
  \item $W_{-2}(G_{\text{mot}}(\mathcal{T}_1(k))) = \{g \in G_{\text{mot}}(\mathcal{T}_1(k)) \mid (g - id)M \subseteq W_{-2}(M), (g - id)W_{-2}(M) = 0\}$,
  \item $W_{-3}(G_{\text{mot}}(\mathcal{T}_1(k))) = 0$.
\end{itemize}

According to the motivic analogue of [Br83] §2.2, $Gr^W_{\mathcal{T}_1(k)}$ is a reductive group sub-$\mathcal{T}_1(k)$-scheme of $G_{\text{mot}}(\mathcal{T}_1(k))$ and $W_{-1}(G_{\text{mot}}(\mathcal{T}_1(k)))$ is the unipotent radical of $G_{\text{mot}}(\mathcal{T}_1(k))$. Each of these 3 steps $W_{-i}(G_{\text{mot}}(\mathcal{T}_1(k)))$ $(i = 0, 1, 2)$ can be reconstructed as intersection of group sub-$\mathcal{T}_1(k)$-schemes of $G_{\text{mot}}(\mathcal{T}_1(k))$ associated to Tannakian sub-categories of $\mathcal{T}_1(k)$ through the bijection 1.0.1:

**Lemma 2.0.2.**
1. $W_{-i}(G_{\text{mot}}(\mathcal{T}_1(k))) = \cap_{i=1}^{2} H_{\mathcal{T}_1(k)}(Gr^W_{\mathcal{T}_i(k)})(G_{\text{mot}}(\mathcal{T}_1(k)))$.
2. $W_{-2}(G_{\text{mot}}(\mathcal{T}_1(k))) = H_{\mathcal{T}_1(k)}(W_{-1}(\mathcal{T}_1(k))) = H_{\mathcal{T}_1(k)}(W_{0}/W_{-2}(\mathcal{T}_1(k)))$.

The explicit computation of these group sub-$\mathcal{T}_1(k)$-schemes involved in the above lemma will provide four exact short sequences of group sub-$\mathcal{T}_1(k)$-schemes of $G_{\text{mot}}(\mathcal{T}_1(k))$:

**Theorem 2.0.3.** We have the following diagram of affine group $\mathcal{T}_1(k)$-schemes

\[
\begin{array}{ccccccccc}
0 & \to & \text{Res}_{\mathbb{F}_k/K} G_{\text{mot}}(\mathcal{T}_1(\mathbb{F})) & \to & G_{\text{mot}}(\mathcal{T}_1(k)) & \to & G_{\text{AL}}(\mathbb{F}/k) & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \text{Res}_{\mathbb{F}/k} H_{\mathcal{T}_1}(\mathbb{F})((\mathbb{Z}/l)^\times) & \to & G_{\text{mot}}(\mathcal{T}_1(k)) & \to & G_{\text{AL}}(\mathbb{F}/k) \times \mathbb{G}_m & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & W_{-1}G_{\text{mot}}(\mathcal{T}_1(k)) & \to & G_{\text{mot}}(\mathcal{T}_1(k)) & \to & G_{\text{mot}}(Gr^W_{\mathcal{T}_1}(k)) & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & W_{-2}G_{\text{mot}}(\mathcal{T}_1(k)) & \to & G_{\text{mot}}(\mathcal{T}_1(k)) & \to & G_{\text{mot}}(W_{-1}(\mathcal{T}_1(k))) & \to & 0 \\
\end{array}
\]

where all horizontal short sequences are exact and where the vertical arrows on the left are inclusions and those on the right are surjections.
3. The case of a 1-motive

Let $M = [X \to G]$ be a 1-motive defined over $k$. The motivic Galois group $G_{\text{mot}}(M)$ of $M$ is the fundamental group of the Tannakian sub-category $(M)^{\otimes}$ of $T(k)$ generated by $M$ i.e. the affine group $(M)^{\otimes}$-scheme $\text{Sp}(\Lambda)$, where $\Lambda$ is the Hopf algebra of $(M)^{\otimes}$ universal for the following property: for each object $X$ of $(M)^{\otimes}$, there is a morphism $\lambda_X : X^\vee \otimes X \to \Lambda$ functorial in $X$. The morphisms $\{\lambda_X\}$, which can be rewritten in the form $X \to X \otimes \Lambda$, define an action of the group $G_{\text{mot}}(M)$ on each object $X$ of $(M)^{\otimes}$, and in particular on itself. The main result of [B03] is that

**Theorem 3.0.4.** The unipotent radical $W_{-1}(\text{Lie} G_{\text{mot}}(M))$ of the Lie algebra of $G_{\text{mot}}(M)$ is the semi-abelian variety defined by the adjoint action of the graded $\text{Gr}^W_1(\text{Lie} G_{\text{mot}}(M))$ on itself.

The idea of the proof is as follows: Before recall that according to [D75] (10.2.14), to have $M$ is equivalent to have the 7-uptuple $(X, Y^\vee, A, A^*, v, v^*, \psi)$ where

- $X$ and $Y^\vee$ are two group $k$-schemes, which are locally for the étale topology, constant group schemes defined by a finitely generated free $\mathbb{Z}$-module;
- $A$ and $A^*$ are two abelian varieties defined over $k$, dual to each other;
- $v : X \to A$ and $v^* : Y^\vee \to A^*$ are two morphisms of group $k$-schemes;
- $\psi$ is a trivialization of the pull-back $(v, v^*)^* \mathcal{P}_A$ by $(v, v^*)$ of the Poincaré biextension $\mathcal{P}_A$ of $(A, A^*)$.

Observe that the 4-uptuple $(X, Y^\vee, A, A^*)$ corresponds to the pure part of the 1-motive, i.e. it defines the pure motives underlying $M$, and the 3-uptuple $(v, v^*, \psi)$ represents the “mixity” of $M$.

Consider the motive $E = W_{-1}(\text{End}(\text{Gr}^W_1M))$ : it is a split 1-motive of weight -1 and -2 obtained from the endomorphisms of the graded $\text{Gr}^W_1M$. The composition of endomorphisms endowed $E$ with a Lie algebra structure $(E, [\cdot, \cdot])$, whose crochet $[\cdot, \cdot]$ corresponds to a $\Sigma - X^\vee \otimes Y(1)$-torsor $B$ living over $A \otimes X^\vee + A^* \otimes Y$. The action of $E$ on the motive $\text{Gr}^W_1(M)$ is described by a morphism

$$E \otimes \text{Gr}^W_1(M) \to \text{Gr}^W_1(M)$$

which endowed the motive $\text{Gr}^W_1(M)$ with a structure of $(E, [\cdot, \cdot])$-module.

Denote by $b = (b_1, b_2)$ the $k$-rational point $b = (b_1, b_2)$ of the abelian variety $A \otimes X^\vee + A^* \otimes Y$ defining the morphisms $v : X \to A$ and $v^* : Y^\vee \to A^*$. Let $B$ be the smallest abelian sub-variety of $X^\vee \otimes A + A^* \otimes Y$ containing this point $b = (b_1, b_2)$. The restriction $i^*B$ of the $\Sigma - X^\vee \otimes Y(1)$-torsor $B$ by the inclusion $i : B \to X^\vee \otimes A \times A^* \otimes Y$ is a $\Sigma - X^\vee \otimes Y(1)$-torsor over $B$. Denote by $Z_1$ the smallest $\text{Gal}(\overline{k}/k)$-module of $X^\vee \otimes Y$ such that the torus $Z_1(1)$, that it defines, contains the image of the restriction $[\cdot, \cdot] : B \otimes B \to X^\vee \otimes Y(1)$ of the Lie crochet to $B \otimes B$. The direct image $p_*i^*B$ of the $\Sigma - X^\vee \otimes Y(1)$-torsor $i^*B$ by the projection $p : X^\vee \otimes Y(1) \to (X^\vee \otimes Y/Z_1)(1)$ is a trivial $\Sigma - (X^\vee \otimes Y/Z_1)(1)$-torsor over $B$. We denote by $\pi : p_*i^*B \to (X^\vee \otimes Y/Z_1)(1)$ the canonical projection. The morphism $u : X \to G$ defines a point $b$ in the fibre of $B$ over $b$. We denote again by $\tilde{b}$ the points of $i^*B$ and of $p_*i^*B$ over the point $b$ of $B$. Let $Z$ be the smallest sub-$\text{Gal}(\overline{k}/k)$-module of $X^\vee \otimes Y$, containing $Z_1$ and such that the sub-torus $(Z/Z_1)(1)$ of $(X^\vee \otimes Y/Z_1)(1)$ contains $\pi(\tilde{b})$. If we put $Z_2 = Z/Z_1$, we have that $Z(1) = Z_1(1) \times Z_2(1)$. 
With these notations, the unipotent radical $W_{-1}(\text{Lie} \mathcal{G}_{\text{mot}}(M))$ of the Lie algebra of $\mathcal{G}_{\text{mot}}(M)$ is the extension of the abelian variety $B$ by the torus $Z(1)$ defined by the adjoint action of $(B + Z(1), [, ])$ on itself. Since in the construction of $B$ and $Z(1)$ are involved only the parameters $v$, $v^*$ and $u$, the computation of the unipotent radical $W_{-1}(\text{Lie} \mathcal{G}_{\text{mot}}(M))$ of the Lie algebra of $\mathcal{G}_{\text{mot}}(M)$ depends only on the 3-uplet $(v, v^*, \psi)$, i.e. on the “mixity” of the 1-motive $M$.

**EXAMPLES:**

1. Let $M$ be the split 1-motive $\mathbb{Z} \oplus A \oplus \mathbb{G}_m$. In this case all is trivial: $W_{-1}(\text{Lie} \mathcal{G}_{\text{mot}}(M)) = B = Z(1) = 0$.

2. Let $M = [\mathbb{Z} \rightarrow E]$ be a 1-motive over $k$ defined by $u(1) = P$ with $P$ a non-torsion $k$-rational point of the elliptic curve $E$. We have that the torus $Z(1)$ is trivial and the unipotent radical $W_{-1}(\text{Lie} \mathcal{G}_{\text{mot}}(M))$ is the elliptic curve $B = E$.

3. Let $M = [\mathbb{Z} \rightarrow \mathbb{G}_m^3 \times A]$ be a 1-motive over $k$ defined by $u(1) = (q_1, q_2, 1, 0)$ with $q_1, q_2$ two elements of $\mathbb{G}_m(k) - \mu_{\infty}$ multiplicatively independents ($\mu_{\infty}$ is the group of roots of the unity in $\overline{k}$). In this example the abelian variety $B$ is trivial and the unipotent radical $W_{-1}(\text{Lie} \mathcal{G}_{\text{mot}}(M))$ is the torus $Z(1) = \mathbb{G}_m^2$.

With the above notations we have also that

**Proposition 3.0.5.** The derived group of the unipotent radical $W_{-1}(\text{Lie} \mathcal{G}_{\text{mot}}(M))$ of the Lie algebra of $\mathcal{G}_{\text{mot}}(M)$ is the torus $Z_1(1)$.

**Proposition 3.0.6.**

$$\dim \text{Lie} \mathcal{G}_{\text{mot}}(M) = \dim B + \dim Z(1) + \dim \text{Lie} \mathcal{G}_{\text{mot}}(\text{Gr}^W M).$$

**REFERENCES**


