Motivic Galois groups of 1-motives: a survey

MOTIVIC GALOIS GROUPS OF 1-MOTIVES: A SURVEY

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Abstract. We investigate the structure of the motivic Galois groups of 1-motives defined over a field of characteristic 0.

In this note we review the main results of [B03] and [B04].

Let k be a field of characteristic 0 and let $\overline{k}$ be its algebraic closure. Let $\mathcal{T}$ be a Tannakian category over k. The tensor product of $\mathcal{T}$ allows us to define the notion of Hopf algebras in the category Ind$\mathcal{T}$ of Ind-objects of $\mathcal{T}$. The category of affine group $\mathcal{T}$-schemes is the opposite of the category of Hopf algebras in Ind$\mathcal{T}$.

The *fundamental group* $\pi(\mathcal{T})$ *of* $\mathcal{T}$ is the affine group $\mathcal{T}$-scheme $\mathrm{Sp}(\Lambda)$, whose Hopf algebra Λ is endowed for each object X of $\mathcal{T}$ with a morphism $X \longrightarrow \Lambda \otimes X$ functorial in X, and is universal for these properties. Those morphisms $\{X \longrightarrow \Lambda \otimes X\}_{X \in \mathcal{T}}$ define *an action of the fundamental group* $\pi(\mathcal{T})$ *on each object of* $\mathcal{T}$. For each fibre functor ω of $\mathcal{T}$ over a k-scheme S, $\omega\pi(\mathcal{T})$ is the affine group S-scheme $\underline{\mathrm{Aut}}^{\otimes}_S(\omega)$ which represents the functor which associates to each S-scheme T, $u : T \longrightarrow S$, the group of automorphisms of $\otimes$-functors of the functor $u^*\omega$.

If $\mathcal{T}(k)$ is a Tannakian category generated by motives defined over k (in an appropriate category of mixed realizations), the fundamental group $\pi(\mathcal{T}(k))$ is called the *motivic Galois group* $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}(k))$ *of* $\mathcal{T}(k)$ and for each embedding $\sigma : k \longrightarrow \mathbb{C}$, the fibre functor ω_σ "Hodge realization" furnishes the $\mathbb{Q}$-algebraic group

$$\omega_\sigma \mathcal{G}_{\mathrm{mot}}(\mathcal{T}) = \mathrm{Spec}\,\big(\omega_\sigma(\Lambda)\big) = \underline{Aut}^{\otimes}_{\mathbb{Q}}(\omega_\sigma)$$

which is the *Hodge realization of the motivic Galois group of* $\mathcal{T}(k)$.

EXAMPLES:

(1) From the main theorem on neutral Tannakian categories, we know that the Tannakian category V$ec(k)$ of finite dimensional k-vector spaces is equivalent to the category of finite-dimensional k-representations of Spec$\,(k)$. In this case, affine group $\mathcal{T}$-schemes are affine group k-schemes and $\pi(\mathrm{V}ec(k))$ is Spec$\,(k)$.

(2) Let $\mathcal{T} = \mathrm{R}ep_k(G)$ be the Tannakian category of k-representations of an affine group k-scheme G. The affine group $\mathcal{T}$-schemes are affine k-schemes endowed with an action of G and the fundamental group $\pi(\mathcal{T})$ of $\mathcal{T}$ is G endowed with its action on itself by inner automorphisms (see [D89] 6.3).

(3) Let $\mathcal{T}_0(k)$ be the *Tannakian category of Artin motives over* k, i.e. the Tannakian sub-category of the Tannakian category of mixed realizations for absolute Hodge cycles (see [J90] I 2.1) generated by pure realizations of 0-dimensional varieties over k. The motivic Galois group $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_0(k))$ of $\mathcal{T}_0(k)$ is the affine group $\mathbb{Q}$-scheme $\mathrm{Gal}(\overline{k}/k)$ endowed with its action on itself by inner automorphisms. We denote it by $\mathcal{GAL}(\overline{k}/k)$. In particular,

for any fibre functor ω over Spec $(\mathbb{Q})$ of $\mathcal{T}_0(k)$, the affine group scheme $\omega(\mathcal{GAL}(\overline{k}/k)) = \underline{Aut}^{\otimes}_{\mathrm{Spec}\,(\mathbb{Q})}(\omega)$ is canonically isomorphic to $\mathrm{Gal}(\overline{k}/k)$.

(4) The motivic Galois group $\mathcal{G}_{\mathrm{mot}}(\mathbb{Z}(0))$ of the unit object $\mathbb{Z}(0)$ of $\mathcal{T}_0(k)$ is the affine group $\langle\mathbb{Z}(0)\rangle^{\otimes}$-scheme $\mathrm{Sp}(\mathbb{Z}(0))$. For each fibre functor "Hodge realization" ω_σ, we have that $\omega_\sigma\big(\mathcal{G}_{\mathrm{mot}}(\mathbb{Z}(0))\big) := \mathrm{S}pec\big(\omega_\sigma(\mathbb{Z}(0))\big) = \mathrm{S}pec(\mathbb{Q})$, which is the Mumford-Tate group of $\mathrm{T}_\sigma(\mathbb{Z}(0))$.

(5) Let $\langle\mathbb{Z}(1)\rangle^{\otimes}$ be the Tannakian category over $\mathbb{Q}$ defined by the k-torus $\mathbb{Z}(1)$. The motivic Galois group $\mathcal{G}_{\mathrm{mot}}(\mathbb{Z}(1))$ of the torus $\mathbb{Z}(1)$ is the affine group $\langle\mathbb{Z}(1)\rangle^{\otimes}$-scheme $\mathbb{G}_m$ defined by the $\mathbb{Q}$-scheme $\mathbb{G}_{m/\mathbb{Q}}$. For each fibre functor "Hodge realization" ω_σ, we have that $\omega_\sigma\big(\mathbb{G}_m\big) = \mathbb{G}_{m/\mathbb{Q}}$, which is the Mumford-Tate group of $\mathrm{T}_\sigma(\mathbb{Z}(1))$.

(6) If k is algebraically closed, the motivic Galois group of motives of CM-type over k is the Serre group (cf. [M94] 4.8).

(7) The *Tannakian category* $\mathcal{T}_1(k)$ *of 1-motives over* k is the Tannakian sub-category of the Tannakian category of mixed realizations (for absolute Hodge cycles) generated by mixed realizations of 1-motives over k. Recall that a 1-motive $M = [X \stackrel{u}{\longrightarrow} G]$ over k consists of
   - a group scheme X over k, which is locally for the étale topology, a constant group scheme defined by a finitely generated free $\mathbb{Z}$-module,
   - a semi-abelian variety G defined over k, i.e. an extention of an abelian variety A by a torus $Y(1)$, which cocharacter group Y,
   - a morphism $u : X \longrightarrow G$ of group schemes over k.

   1-motives are mixed motives of level ≤ 1: the weight filtration W_* on M is $\mathrm{W}_i(M) = M$ for each $i \geq 0$, $\mathrm{W}_{-1}(M) = G$,$\mathrm{W}_{-2}(M) = Y(1)$, $\mathrm{W}_j(M) = 0$ for each $j \leq -3$. If $\mathrm{Gr}_n^{\mathrm{W}} = \mathrm{W}_n/\mathrm{W}_{n-1}$, we have the quotients $\mathrm{G}r_0^{\mathrm{W}}(M) = X, \mathrm{G}r_{-1}^{\mathrm{W}}(M) = A$ and $\mathrm{G}r_{-2}^{\mathrm{W}}(M) = Y(1)$. We will denote by $W_{-1}\mathcal{T}_1(k)$ (resp. $\mathrm{Gr}_0^W\mathcal{T}_1(k)$, ...) the Tannakian sub-category of $\mathcal{T}_1(k)$ generated by all $W_{-1}M$ (resp. $\mathrm{Gr}_0^W M$, ...) with M a 1-motive. With this notation we can easily compute the following motivic Galois groups
   - $\mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_0^W\mathcal{T}_1(k)) = \mathcal{GAL}(\overline{k}/k)$,
   - $\mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_{-2}^W\mathcal{T}_1(k)) = \mathcal{GAL}(\overline{k}/k) \times \mathbb{G}_m$.
   - $\mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_0^W\mathcal{T}_1(\overline{k})) = \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_0(\overline{k})) = \mathrm{Sp}(\mathbb{Z}(0))$
   - $\mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_{-2}^W\mathcal{T}_1(\overline{k})) = \mathbb{G}_m$

1. Motivic Galois theory

For each Tannakian sub-category $\mathcal{T}'$ of $\mathcal{T}$, let $H_{\mathcal{T}}(\mathcal{T}')$ be the kernel of the faithfully flat morphism of group $\mathcal{T}$-schemes $I : \pi(\mathcal{T}) \longrightarrow i\pi(\mathcal{T}')$ corresponding to the inclusion functor $i : \mathcal{T}' \longrightarrow \mathcal{T}$. In particular we have the short exact sequence of group $\pi(\mathcal{T})$-schemes

$$0 \longrightarrow H_{\mathcal{T}}(\mathcal{T}') \longrightarrow \pi(\mathcal{T}) \longrightarrow i\pi(\mathcal{T}') \longrightarrow 0.$$

In [D89] 6.6, Deligne proves that the Tannakian category $\mathcal{T}'$ is equivalent, as tensor category, to the sub-category of $\mathcal{T}$ generated by those objects on which the action of $\pi(\mathcal{T})$ induces a trivial action of $H_{\mathcal{T}}(\mathcal{T}')$. In particular, this implies that the fundamental group $\pi(\mathcal{T}')$ of $\mathcal{T}'$ is isomorphic to the group $\mathcal{T}$-scheme $\pi(\mathcal{T})/H_{\mathcal{T}}(\mathcal{T}')$. The group $\mathcal{T}$-scheme $H_{\mathcal{T}}(\mathcal{T}')$ characterizes the Tannakian sub-category $\mathcal{T}'$. In fact we have a clear dictionary between Tannakian sub-categories of $\mathcal{T}$ and normal affine group sub-$\mathcal{T}$-schemes of the fundamental group $\pi(\mathcal{T})$ of $\mathcal{T}$:

Theorem 1.0.1. *There is bijection between the Tannakian sub-categories of $\mathcal{T}$ and the normal affine group sub-$\mathcal{T}$-schemes of $\pi(\mathcal{T})$, which associates*
- to each Tannakian sub-category $\mathcal{T}'$ of $\mathcal{T}$, the kernel $H_{\mathcal{T}}(\mathcal{T}')$ of the morphism of $\mathcal{T}$-schemes $I : \pi(\mathcal{T}) \longrightarrow i\pi(\mathcal{T}')$ corresponding to the inclusion $i : \mathcal{T}' \longrightarrow \mathcal{T}$;
- to each normal affine group sub-$\mathcal{T}$-scheme H of $\pi(\mathcal{T})$, the Tannakian sub-category $\mathcal{T}(H)$ of objects of $\mathcal{T}$ on which the action of $\pi(\mathcal{T})$ induces a trivial action of H.

2. The case of motives of level ≤ 1

In order to study the category $\mathcal{T}_1(k)$ of motives of level ≤ 1, in [B04] we have applied the above theorem to some sub-categories of $\mathcal{T}_1(k)$. The weight filtration W_* of 1-motives induces an increasing filtration W_* of 3 steps on the motivic Galois group $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$ which we describe through the action of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$ on the generators of $\mathcal{T}_1(k)$: For each 1-motive M over k, we have that

- $W_0(\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))) = \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$
- $W_{-1}(\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))) = \{g \in \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) \mid (g-id)M \subseteq W_{-1}(M),$ $(g-id)W_{-1}(M) \subseteq W_{-2}(M), (g-id)W_{-2}(M) = 0\}$,
- $W_{-2}(\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))) = \{g \in \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) \mid (g-id)M \subseteq W_{-2}(M),$ $(g-id)W_{-1}(M) = 0\}$,
- $W_{-3}(\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))) = 0$.

According to the motivic analogue of [Br83] §2.2, $\mathrm{Gr}_0^W(\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)))$ is a reductive group sub-$\mathcal{T}_1(k)$-scheme of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$ and $W_{-1}(\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)))$ is the unipotent radical of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$. Each of these 3 steps $\mathrm{W}_{-i}(\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)))$ $(i = 0, 1, 2)$ can be reconstructed as intersection of group sub-$\mathcal{T}_1(k)$-schemes of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$ associated to Tannakian sub-categories of $\mathcal{T}_1(k)$ through the bijection 1.0.1:

Lemma 2.0.2. (1) $W_{-1}(\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))) = \cap_{i=-1,-2} H_{\mathcal{T}_1(k)}(\mathrm{Gr}_i^W \mathcal{T}_1(k))$,
(2) $W_{-2}(\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))) = H_{\mathcal{T}_1(k)}(W_{-1}\mathcal{T}_1(k)) = H_{\mathcal{T}_1(k)}(W_0/W_{-2}\mathcal{T}_1(k))$.

The explicit computation of these group sub-$\mathcal{T}_1(k)$-schemes involved in the above lemma will provide four exact short sequences of group sub-$\mathcal{T}_1(k)$-schemes of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$:

Theorem 2.0.3. *We have the following diagram of affine group $\mathcal{T}_1(k)$-schemes*

$$\begin{array}{ccccccccc}
0 & \rightarrow & \mathrm{Res}_{\overline{k}/k}\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(\overline{k})) & \rightarrow & \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) & \rightarrow & \mathcal{GAL}(\overline{k}/k) & \rightarrow & 0 \\
 & & \uparrow & & || & & \uparrow & & \\
0 & \rightarrow & \mathrm{Res}_{\overline{k}/k}H_{\mathcal{T}_1(\overline{k})}(\langle \mathbb{Z}(1)\rangle^{\otimes}) & \rightarrow & \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) & \rightarrow & \mathcal{GAL}(\overline{k}/k)\times\mathbb{G}_m & \rightarrow & 0 \\
 & & \uparrow & & || & & \uparrow & & \\
0 & \rightarrow & W_{-1}\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) & \rightarrow & \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) & \rightarrow & \mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_*^W\mathcal{T}_1(k)) & \rightarrow & 0 \\
 & & \uparrow & & || & & \uparrow & & \\
0 & \rightarrow & W_{-2}\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) & \rightarrow & \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) & \rightarrow & \mathcal{G}_{\mathrm{mot}}(W_{-1}\mathcal{T}_1(k)) & \rightarrow & 0
\end{array}$$

where all horizontal short sequences are exact and where the vertical arrows on the left are inclusions and those on the right are surjections.

3. The case of a 1-motive

Let $M = [X \stackrel{u}{\longrightarrow} G]$ be a 1-motive defined over k. The motivic Galois group $\mathcal{G}_{\mathrm{mot}}(M)$ of M is the fundamental group of the Tannakian sub-category $\langle M \rangle^{\otimes}$ of $\mathcal{T}_1(k)$ generated by M i.e. the affine group $\langle M \rangle^{\otimes}$-scheme $\mathrm{Sp}(\Lambda)$, where Λ is the Hopf algebra of $\langle M \rangle^{\otimes}$ universal for the following property: for each object X of $\langle M \rangle^{\otimes}$, there is a morphism $\lambda_X : X^{\vee} \otimes X \longrightarrow \Lambda$ functorial in X. The morphisms $\{\lambda_X\}$, which can be rewritten in the form $X \longrightarrow X \otimes \Lambda$, define an action of the group $\mathcal{G}_{\mathrm{mot}}(M)$ on each object X of $\langle M \rangle^{\otimes}$, and in particular on itself. The main result of [B03] is that

Theorem 3.0.4. *The unipotent radical $W_{-1}(\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M))$ of the Lie algebra of $\mathcal{G}_{\mathrm{mot}}(M)$ is the semi-abelian variety defined by the adjoint action of the graded $\mathrm{Gr}_*^W(W_{-1}\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M))$ on itself.*

The idea of the proof is as followed: Before recall that according to [D75] (10.2.14), to have M is equivalent to have the 7-uplet $(X, Y^{\vee}, A, A^*, v, v^*, \psi)$ where

- X and $Y^{\vee}$ are two group k-schemes, which are locally for the étale topology, constant group schemes defined by a finitely generated free $\mathbb{Z}$-module;
- A and A^* are two abelian varieties defined over k, dual to each other;
- $v : X \longrightarrow A$ and $v^* : Y^{\vee} \longrightarrow A^*$ are two morphisms of group k-schemes;
- ψ is a trivialization of the pull-back $(v, v^*)^* \mathcal{P}_A$ by (v, v^*) of the Poincaré biextension $\mathcal{P}_A$ of (A, A^*).

Observe that the 4-uplet $(X, Y^{\vee}, A, A^*)$ corresponds to the pure part of the 1-motive, i.e. it defines the pure motives underlying M, and the 3-uplet (v, v^*, ψ) represents the "mixity" of M.

Consider the motive $E = W_{-1}(\underline{\mathrm{End}}(\mathrm{Gr}_*^W M))$: it is a split 1-motive of weight -1 and -2 obtained from the endomorphisms of the graded $\mathrm{Gr}_*^W M$ of M. The composition of endomorphisms endowed E with a Lie algebra structure $(E, [\,,\,])$, whose crochet $[\,,\,]$ corresponds to a $\Sigma - X^{\vee} \otimes Y(1)$-torsor $\mathcal{B}$ living over $A \otimes X^{\vee} + A^* \otimes Y$. The action of E on the motive $\mathrm{Gr}_*^W(M)$ is described by a morphism

$$E \otimes \mathrm{Gr}_*^W(M) \longrightarrow \mathrm{Gr}_*^W(M)$$

which endowed the motive $\mathrm{Gr}_*^W(M)$ with a structure of $(E, [\,,\,])$-module.

Denote by $b = (b_1, b_2)$ the k-rational point $b = (b_1, b_2)$ of the abelian variety $A \otimes X^{\vee} + A^* \otimes Y$ defining the morphisms $v : X \longrightarrow A$ and $v^* : Y^{\vee} \longrightarrow A^*$. Let B be the smallest abelian sub-variety of $X^{\vee} \otimes A + A^* \otimes Y$ containing this point $b = (b_1, b_2)$. The restriction $i^*\mathcal{B}$ of the $\Sigma - X^{\vee} \otimes Y(1)$-torsor $\mathcal{B}$ by the inclusion $i : B \longrightarrow X^{\vee} \otimes A \times A^* \otimes Y$ is a $\Sigma - X^{\vee} \otimes Y(1)$-torsor over B. Denote by Z_1 the smallest $\mathrm{Gal}(\overline{k}/k)$-module of $X^{\vee} \otimes Y$ such that the torus $Z_1(1)$, that it defines, contains the image of the restriction $[\,,\,] : B \otimes B \longrightarrow X^{\vee} \otimes Y(1)$ of the Lie crochet to $B \otimes B$. The direct image $p_* i^* \mathcal{B}$ of the $\Sigma - X^{\vee} \otimes Y(1)$-torsor $i^*\mathcal{B}$ by the projection $p : X^{\vee} \otimes Y(1) \longrightarrow (X^{\vee} \otimes Y/Z_1)(1)$ is a trivial $\Sigma - (X^{\vee} \otimes Y/Z_1)(1)$-torsor over B. We denote by $\pi : p_* i^* \mathcal{B} \longrightarrow (X^{\vee} \otimes Y/Z_1)(1)$ the canonical projection. The morphism $u : X \longrightarrow G$ defines a point $\widetilde{b}$ in the fibre of $\mathcal{B}$ over b. We denote again by $\widetilde{b}$ the points of $i^*\mathcal{B}$ and of $p_* i^* \mathcal{B}$ over the point b of B. Let Z be the smallest sub-$\mathrm{Gal}(\overline{k}/k)$-module of $X^{\vee} \otimes Y$, containing Z_1 and such that the sub-torus $(Z/Z_1)(1)$ of $(X^{\vee} \otimes Y/Z_1)(1)$ contains $\pi(\widetilde{b})$. If we put $Z_2 = Z/Z_1$, we have that $Z(1) = Z_1(1) \times Z_2(1)$.

With these notations, the unipotent radical $W_{-1}(\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M))$ of the Lie algebra of $\mathcal{G}_{\mathrm{mot}}(M)$ is the extension of the abelian variety B by the torus $Z(1)$ defined by the adjoint action of $(B + Z(1), [\,,\,])$ on itself. Since in the construction of B and $Z(1)$ are involved only the parameters v, v^* and u, *the computation of the unipotent radical* $W_{-1}(\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M))$ *of the Lie algebra of* $\mathcal{G}_{\mathrm{mot}}(M)$ *depends only on the 3-uplet* (v, v^*, ψ)*, i.e. on the "mixity" of the 1-motive* M.

EXAMPLES:

(1) Let M be the split 1-motive $\mathbb{Z} \oplus A \oplus \mathbb{G}_m$. In this case all is trivial: $W_{-1}(\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M)) = B = Z(1) = 0$.
(2) Let $M = [\mathbb{Z} \xrightarrow{u} \mathcal{E}]$ be a 1-motive over k defined by $u(1) = P$ with P a non-torsion k-rational point of the elliptic curve $\mathcal{E}$. We have that the torus $Z(1)$ is trivial and the unipotent radical $W_{-1}(\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M))$ is the elliptic curve $B = \mathcal{E}$.
(3) Let $M = [\mathbb{Z} \xrightarrow{u} \mathbb{G}_m^3 \times A]$ be a 1-motive over k defined by $u(1) = (q_1, q_2, 1, 0)$ with q_1, q_2 two elements of $\mathbb{G}_m(k) - \mu_\infty$ multiplicatively independents (μ_∞ is the group of roots of the unity in $\overline{k}$). In this example the abelian variety B is trivial and the unipotent radical $W_{-1}(\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M))$ is the torus $Z(1) = \mathbb{G}_m^2$.

With the above notations we have also that

Proposition 3.0.5. *The derived group of the unipotent radical* $W_{-1}(\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M))$ *of the Lie algebra of* $\mathcal{G}_{\mathrm{mot}}(M)$ *is the torus* $Z_1(1)$.

Proposition 3.0.6.

$$\dim \mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M) = \dim B + \dim Z(1) + \dim \mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_*^W M).$$

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