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MOTIVIC GALOIS GROUPS OF 1-MOTIVES: A SURVEY

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ABSTRACT. We investigate the structure of the motivic Galois groups of 1-motives defined over a field of characteristic 0.

In this note we review the main results of [B03] and [B04].

Let k be a field of characteristic 0 and let \bar{k} be its algebraic closure. Let \mathcal{T} be a Tannakian category over k . The tensor product of \mathcal{T} allows us to define the notion of Hopf algebras in the category $\text{Ind}\mathcal{T}$ of Ind-objects of \mathcal{T} . The category of affine group \mathcal{T} -schemes is the opposite of the category of Hopf algebras in $\text{Ind}\mathcal{T}$.

The *fundamental group* $\pi(\mathcal{T})$ of \mathcal{T} is the affine group \mathcal{T} -scheme $\text{Sp}(\Lambda)$, whose Hopf algebra Λ is endowed for each object X of \mathcal{T} with a morphism $X \rightarrow \Lambda \otimes X$ functorial in X , and is universal for these properties. Those morphisms $\{X \rightarrow \Lambda \otimes X\}_{X \in \mathcal{T}}$ define an action of the fundamental group $\pi(\mathcal{T})$ on each object of \mathcal{T} . For each fibre functor ω of \mathcal{T} over a k -scheme S , $\omega\pi(\mathcal{T})$ is the affine group S -scheme $\underline{\text{Aut}}_S^\otimes(\omega)$ which represents the functor which associates to each S -scheme T , $u : T \rightarrow S$, the group of automorphisms of \otimes -functors of the functor $u^*\omega$.

If $\mathcal{T}(k)$ is a Tannakian category generated by motives defined over k (in an appropriate category of mixed realizations), the fundamental group $\pi(\mathcal{T}(k))$ is called the *motivic Galois group* $\mathcal{G}_{\text{mot}}(\mathcal{T}(k))$ of $\mathcal{T}(k)$ and for each embedding $\sigma : k \rightarrow \mathbb{C}$, the fibre functor ω_σ “Hodge realization” furnishes the \mathbb{Q} -algebraic group

$$\omega_\sigma \mathcal{G}_{\text{mot}}(\mathcal{T}) = \text{Spec}(\omega_\sigma(\Lambda)) = \underline{\text{Aut}}_{\mathbb{Q}}^\otimes(\omega_\sigma)$$

which is the *Hodge realization of the motivic Galois group of $\mathcal{T}(k)$* .

EXAMPLES:

- (1) From the main theorem on neutral Tannakian categories, we know that the Tannakian category $\text{Vec}(k)$ of finite dimensional k -vector spaces is equivalent to the category of finite-dimensional k -representations of $\text{Spec}(k)$. In this case, affine group \mathcal{T} -schemes are affine group k -schemes and $\pi(\text{Vec}(k))$ is $\text{Spec}(k)$.
- (2) Let $\mathcal{T} = \text{Rep}_k(G)$ be the Tannakian category of k -representations of an affine group k -scheme G . The affine group \mathcal{T} -schemes are affine k -schemes endowed with an action of G and the fundamental group $\pi(\mathcal{T})$ of \mathcal{T} is G endowed with its action on itself by inner automorphisms (see [D89] 6.3).
- (3) Let $\mathcal{T}_0(k)$ be the *Tannakian category of Artin motives over k* , i.e. the Tannakian sub-category of the Tannakian category of mixed realizations for absolute Hodge cycles (see [J90] I 2.1) generated by pure realizations of 0-dimensional varieties over k . The motivic Galois group $\mathcal{G}_{\text{mot}}(\mathcal{T}_0(k))$ of $\mathcal{T}_0(k)$ is the affine group \mathbb{Q} -scheme $\text{Gal}(\bar{k}/k)$ endowed with its action on itself by inner automorphisms. We denote it by $\mathcal{GAL}(\bar{k}/k)$. In particular,

for any fibre functor ω over $\mathrm{Spec}(\mathbb{Q})$ of $\mathcal{T}_0(k)$, the affine group scheme $\omega(\mathcal{GAL}(\bar{k}/k)) = \underline{\mathrm{Aut}}_{\mathrm{Spec}(\mathbb{Q})}^{\otimes}(\omega)$ is canonically isomorphic to $\mathrm{Gal}(\bar{k}/k)$.

- (4) The motivic Galois group $\mathcal{G}_{\mathrm{mot}}(\mathbb{Z}(0))$ of the unit object $\mathbb{Z}(0)$ of $\mathcal{T}_0(k)$ is the affine group $\langle \mathbb{Z}(0) \rangle^{\otimes}$ -scheme $\mathrm{Sp}(\mathbb{Z}(0))$. For each fibre functor “Hodge realization” ω_{σ} , we have that $\omega_{\sigma}(\mathcal{G}_{\mathrm{mot}}(\mathbb{Z}(0))) := \mathrm{Spec}(\omega_{\sigma}(\mathbb{Z}(0))) = \mathrm{Spec}(\mathbb{Q})$, which is the Mumford-Tate group of $T_{\sigma}(\mathbb{Z}(0))$.
- (5) Let $\langle \mathbb{Z}(1) \rangle^{\otimes}$ be the Tannakian category over \mathbb{Q} defined by the k -torus $\mathbb{Z}(1)$. The motivic Galois group $\mathcal{G}_{\mathrm{mot}}(\mathbb{Z}(1))$ of the torus $\mathbb{Z}(1)$ is the affine group $\langle \mathbb{Z}(1) \rangle^{\otimes}$ -scheme \mathbb{G}_m defined by the \mathbb{Q} -scheme $\mathbb{G}_{m/\mathbb{Q}}$. For each fibre functor “Hodge realization” ω_{σ} , we have that $\omega_{\sigma}(\mathbb{G}_m) = \mathbb{G}_{m/\mathbb{Q}}$, which is the Mumford-Tate group of $T_{\sigma}(\mathbb{Z}(1))$.
- (6) If k is algebraically closed, the motivic Galois group of motives of CM-type over k is the Serre group (cf. [M94] 4.8).
- (7) The *Tannakian category* $\mathcal{T}_1(k)$ of 1-motives over k is the Tannakian subcategory of the Tannakian category of mixed realizations (for absolute Hodge cycles) generated by mixed realizations of 1-motives over k . Recall that a 1-motive $M = [X \xrightarrow{u} G]$ over k consists of
 - a group scheme X over k , which is locally for the étale topology, a constant group scheme defined by a finitely generated free \mathbb{Z} -module,
 - a semi-abelian variety G defined over k , i.e. an extension of an abelian variety A by a torus $Y(1)$, which cocharacter group Y ,
 - a morphism $u : X \rightarrow G$ of group schemes over k .

1-motives are mixed motives of level ≤ 1 : the weight filtration W_* on M is $W_i(M) = M$ for each $i \geq 0$, $W_{-1}(M) = G$, $W_{-2}(M) = Y(1)$, $W_j(M) = 0$ for each $j \leq -3$. If $\mathrm{Gr}_n^W = W_n/W_{n-1}$, we have the quotients $\mathrm{Gr}_0^W(M) = X$, $\mathrm{Gr}_{-1}^W(M) = A$ and $\mathrm{Gr}_{-2}^W(M) = Y(1)$. We will denote by $W_{-1}\mathcal{T}_1(k)$ (resp. $\mathrm{Gr}_0^W\mathcal{T}_1(k)$, ...) the Tannakian sub-category of $\mathcal{T}_1(k)$ generated by all $W_{-1}M$ (resp. $\mathrm{Gr}_0^W M$, ...) with M a 1-motive. With this notation we can easily compute the following motivic Galois groups

- $\mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_0^W\mathcal{T}_1(k)) = \mathcal{GAL}(\bar{k}/k)$,
- $\mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_{-2}^W\mathcal{T}_1(k)) = \mathcal{GAL}(\bar{k}/k) \times \mathbb{G}_m$.
- $\mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_0^W\mathcal{T}_1(\bar{k})) = \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_0(\bar{k})) = \mathrm{Sp}(\mathbb{Z}(0))$
- $\mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_{-2}^W\mathcal{T}_1(\bar{k})) = \mathbb{G}_m$

1. MOTIVIC GALOIS THEORY

For each Tannakian sub-category \mathcal{T}' of \mathcal{T} , let $H_{\mathcal{T}}(\mathcal{T}')$ be the kernel of the faithfully flat morphism of group \mathcal{T} -schemes $I : \pi(\mathcal{T}) \rightarrow i\pi(\mathcal{T}')$ corresponding to the inclusion functor $i : \mathcal{T}' \rightarrow \mathcal{T}$. In particular we have the short exact sequence of group $\pi(\mathcal{T})$ -schemes

$$0 \longrightarrow H_{\mathcal{T}}(\mathcal{T}') \longrightarrow \pi(\mathcal{T}) \longrightarrow i\pi(\mathcal{T}') \longrightarrow 0.$$

In [D89] 6.6, Deligne proves that the Tannakian category \mathcal{T}' is equivalent, as tensor category, to the sub-category of \mathcal{T} generated by those objects on which the action of $\pi(\mathcal{T})$ induces a trivial action of $H_{\mathcal{T}}(\mathcal{T}')$. In particular, this implies that the fundamental group $\pi(\mathcal{T}')$ of \mathcal{T}' is isomorphic to the group \mathcal{T} -scheme $\pi(\mathcal{T})/H_{\mathcal{T}}(\mathcal{T}')$. The group \mathcal{T} -scheme $H_{\mathcal{T}}(\mathcal{T}')$ characterizes the Tannakian sub-category \mathcal{T}' . In fact

we have a clear dictionary between Tannakian sub-categories of \mathcal{T} and normal affine group sub- \mathcal{T} -schemes of the fundamental group $\pi(\mathcal{T})$ of \mathcal{T} :

Theorem 1.0.1. *There is bijection between the Tannakian sub-categories of \mathcal{T} and the normal affine group sub- \mathcal{T} -schemes of $\pi(\mathcal{T})$, which associates*

- *to each Tannakian sub-category \mathcal{T}' of \mathcal{T} , the kernel $H_{\mathcal{T}}(\mathcal{T}')$ of the morphism of \mathcal{T} -schemes $I : \pi(\mathcal{T}) \longrightarrow i\pi(\mathcal{T}')$ corresponding to the inclusion $i : \mathcal{T}' \longrightarrow \mathcal{T}$;*
- *to each normal affine group sub- \mathcal{T} -scheme H of $\pi(\mathcal{T})$, the Tannakian sub-category $\mathcal{T}(H)$ of objects of \mathcal{T} on which the action of $\pi(\mathcal{T})$ induces a trivial action of H .*

2. THE CASE OF MOTIVES OF LEVEL ≤ 1

In order to study the category $\mathcal{T}_1(k)$ of motives of level ≤ 1 , in [B04] we have applied the above theorem to some sub-categories of $\mathcal{T}_1(k)$. The weight filtration W_* of 1-motives induces an increasing filtration W_* of 3 steps on the motivic Galois group $\mathcal{G}_{\text{mot}}(\mathcal{T}_1(k))$ which we describe through the action of $\mathcal{G}_{\text{mot}}(\mathcal{T}_1(k))$ on the generators of $\mathcal{T}_1(k)$: For each 1-motive M over k , we have that

- $W_0(\mathcal{G}_{\text{mot}}(\mathcal{T}_1(k))) = \mathcal{G}_{\text{mot}}(\mathcal{T}_1(k))$
- $W_{-1}(\mathcal{G}_{\text{mot}}(\mathcal{T}_1(k))) = \{g \in \mathcal{G}_{\text{mot}}(\mathcal{T}_1(k)) \mid (g - id)M \subseteq W_{-1}(M), (g - id)W_{-1}(M) \subseteq W_{-2}(M), (g - id)W_{-2}(M) = 0\},$
- $W_{-2}(\mathcal{G}_{\text{mot}}(\mathcal{T}_1(k))) = \{g \in \mathcal{G}_{\text{mot}}(\mathcal{T}_1(k)) \mid (g - id)M \subseteq W_{-2}(M), (g - id)W_{-1}(M) = 0\},$
- $W_{-3}(\mathcal{G}_{\text{mot}}(\mathcal{T}_1(k))) = 0.$

According to the motivic analogue of [Br83] §2.2, $\text{Gr}_0^W(\mathcal{G}_{\text{mot}}(\mathcal{T}_1(k)))$ is a reductive group sub- $\mathcal{T}_1(k)$ -scheme of $\mathcal{G}_{\text{mot}}(\mathcal{T}_1(k))$ and $W_{-1}(\mathcal{G}_{\text{mot}}(\mathcal{T}_1(k)))$ is the unipotent radical of $\mathcal{G}_{\text{mot}}(\mathcal{T}_1(k))$. Each of these 3 steps $W_{-i}(\mathcal{G}_{\text{mot}}(\mathcal{T}_1(k)))$ ($i = 0, 1, 2$) can be reconstructed as intersection of group sub- $\mathcal{T}_1(k)$ -schemes of $\mathcal{G}_{\text{mot}}(\mathcal{T}_1(k))$ associated to Tannakian sub-categories of $\mathcal{T}_1(k)$ through the bijection 1.0.1:

- Lemma 2.0.2.** (1) $W_{-1}(\mathcal{G}_{\text{mot}}(\mathcal{T}_1(k))) = \cap_{i=-1, -2} H_{\mathcal{T}_1(k)}(\text{Gr}_i^W \mathcal{T}_1(k)),$
 (2) $W_{-2}(\mathcal{G}_{\text{mot}}(\mathcal{T}_1(k))) = H_{\mathcal{T}_1(k)}(W_{-1}\mathcal{T}_1(k)) = H_{\mathcal{T}_1(k)}(W_0/W_{-2}\mathcal{T}_1(k)).$

The explicit computation of these group sub- $\mathcal{T}_1(k)$ -schemes involved in the above lemma will provide four exact short sequences of group sub- $\mathcal{T}_1(k)$ -schemes of $\mathcal{G}_{\text{mot}}(\mathcal{T}_1(k))$:

Theorem 2.0.3. *We have the following diagram of affine group $\mathcal{T}_1(k)$ -schemes*

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & \text{Res}_{\bar{k}/k} \mathcal{G}_{\text{mot}}(\mathcal{T}_1(\bar{k})) & \longrightarrow & \mathcal{G}_{\text{mot}}(\mathcal{T}_1(k)) & \longrightarrow & \mathcal{G}_{\mathcal{AL}}(\bar{k}/k) & \longrightarrow & 0 \\
 & & \uparrow & & \parallel & & \uparrow & & \\
 0 & \longrightarrow & \text{Res}_{\bar{k}/k} H_{\mathcal{T}_1(\bar{k})}(\langle \mathbb{Z}(1) \rangle^{\otimes}) & \longrightarrow & \mathcal{G}_{\text{mot}}(\mathcal{T}_1(k)) & \longrightarrow & \mathcal{G}_{\mathcal{AL}}(\bar{k}/k) \times \mathbb{G}_m & \longrightarrow & 0 \\
 & & \uparrow & & \parallel & & \uparrow & & \\
 0 & \longrightarrow & W_{-1} \mathcal{G}_{\text{mot}}(\mathcal{T}_1(k)) & \longrightarrow & \mathcal{G}_{\text{mot}}(\mathcal{T}_1(k)) & \longrightarrow & \mathcal{G}_{\text{mot}}(\text{Gr}_*^W \mathcal{T}_1(k)) & \longrightarrow & 0 \\
 & & \uparrow & & \parallel & & \uparrow & & \\
 0 & \longrightarrow & W_{-2} \mathcal{G}_{\text{mot}}(\mathcal{T}_1(k)) & \longrightarrow & \mathcal{G}_{\text{mot}}(\mathcal{T}_1(k)) & \longrightarrow & \mathcal{G}_{\text{mot}}(W_{-1} \mathcal{T}_1(k)) & \longrightarrow & 0
 \end{array}$$

where all horizontal short sequences are exact and where the vertical arrows on the left are inclusions and those on the right are surjections.

3. THE CASE OF A 1-MOTIVE

Let $M = [X \xrightarrow{u} G]$ be a 1-motive defined over k . The motivic Galois group $\mathcal{G}_{\text{mot}}(M)$ of M is the fundamental group of the Tannakian sub-category $\langle M \rangle^{\otimes}$ of $\mathcal{T}_1(k)$ generated by M i.e. the affine group $\langle M \rangle^{\otimes}$ -scheme $\text{Sp}(\Lambda)$, where Λ is the Hopf algebra of $\langle M \rangle^{\otimes}$ universal for the following property: for each object X of $\langle M \rangle^{\otimes}$, there is a morphism $\lambda_X : X^{\vee} \otimes X \rightarrow \Lambda$ functorial in X . The morphisms $\{\lambda_X\}$, which can be rewritten in the form $X \rightarrow X \otimes \Lambda$, define an action of the group $\mathcal{G}_{\text{mot}}(M)$ on each object X of $\langle M \rangle^{\otimes}$, and in particular on itself. The main result of [B03] is that

Theorem 3.0.4. *The unipotent radical $W_{-1}(\text{Lie } \mathcal{G}_{\text{mot}}(M))$ of the Lie algebra of $\mathcal{G}_{\text{mot}}(M)$ is the semi-abelian variety defined by the adjoint action of the graded $\text{Gr}_*^W(W_{-1}\text{Lie } \mathcal{G}_{\text{mot}}(M))$ on itself.*

The idea of the proof is as followed: Before recall that according to [D75] (10.2.14), to have M is equivalent to have the 7-uplet $(X, Y^{\vee}, A, A^*, v, v^*, \psi)$ where

- X and Y^{\vee} are two group k -schemes, which are locally for the étale topology, constant group schemes defined by a finitely generated free \mathbb{Z} -module;
- A and A^* are two abelian varieties defined over k , dual to each other;
- $v : X \rightarrow A$ and $v^* : Y^{\vee} \rightarrow A^*$ are two morphisms of group k -schemes;
- ψ is a trivialization of the pull-back $(v, v^*)^* \mathcal{P}_A$ by (v, v^*) of the Poincaré biextension \mathcal{P}_A of (A, A^*) .

Observe that the 4-uplet (X, Y^{\vee}, A, A^*) corresponds to the pure part of the 1-motive, i.e. it defines the pure motives underlying M , and the 3-uplet (v, v^*, ψ) represents the “mixture” of M .

Consider the motive $E = W_{-1}(\text{End}(\text{Gr}_*^W M))$: it is a split 1-motive of weight -1 and -2 obtained from the endomorphisms of the graded $\text{Gr}_*^W M$ of M . The composition of endomorphisms endowed E with a Lie algebra structure $(E, [,])$, whose crochet $[,]$ corresponds to a $\Sigma - X^{\vee} \otimes Y(1)$ -torsor \mathcal{B} living over $A \otimes X^{\vee} + A^* \otimes Y$. The action of E on the motive $\text{Gr}_*^W(M)$ is described by a morphism

$$E \otimes \text{Gr}_*^W(M) \rightarrow \text{Gr}_*^W(M)$$

which endowed the motive $\text{Gr}_*^W(M)$ with a structure of $(E, [,])$ -module.

Denote by $b = (b_1, b_2)$ the k -rational point $b = (b_1, b_2)$ of the abelian variety $A \otimes X^{\vee} + A^* \otimes Y$ defining the morphisms $v : X \rightarrow A$ and $v^* : Y^{\vee} \rightarrow A^*$. Let B be the smallest abelian sub-variety of $X^{\vee} \otimes A + A^* \otimes Y$ containing this point $b = (b_1, b_2)$. The restriction $i^* \mathcal{B}$ of the $\Sigma - X^{\vee} \otimes Y(1)$ -torsor \mathcal{B} by the inclusion $i : B \rightarrow X^{\vee} \otimes A + A^* \otimes Y$ is a $\Sigma - X^{\vee} \otimes Y(1)$ -torsor over B . Denote by Z_1 the smallest $\text{Gal}(\bar{k}/k)$ -module of $X^{\vee} \otimes Y$ such that the torus $Z_1(1)$, that it defines, contains the image of the restriction $[,] : B \otimes B \rightarrow X^{\vee} \otimes Y(1)$ of the Lie crochet to $B \otimes B$. The direct image $p_* i^* \mathcal{B}$ of the $\Sigma - X^{\vee} \otimes Y(1)$ -torsor $i^* \mathcal{B}$ by the projection $p : X^{\vee} \otimes Y(1) \rightarrow (X^{\vee} \otimes Y/Z_1)(1)$ is a trivial $\Sigma - (X^{\vee} \otimes Y/Z_1)(1)$ -torsor over B . We denote by $\pi : p_* i^* \mathcal{B} \rightarrow (X^{\vee} \otimes Y/Z_1)(1)$ the canonical projection. The morphism $u : X \rightarrow G$ defines a point \tilde{b} in the fibre of \mathcal{B} over b . We denote again by \tilde{b} the points of $i^* \mathcal{B}$ and of $p_* i^* \mathcal{B}$ over the point b of B . Let Z be the smallest sub- $\text{Gal}(\bar{k}/k)$ -module of $X^{\vee} \otimes Y$, containing Z_1 and such that the sub-torus $(Z/Z_1)(1)$ of $(X^{\vee} \otimes Y/Z_1)(1)$ contains $\pi(\tilde{b})$. If we put $Z_2 = Z/Z_1$, we have that $Z(1) = Z_1(1) \times Z_2(1)$.

With these notations, the unipotent radical $W_{-1}(\mathrm{Lie} \mathcal{G}_{\mathrm{mot}}(M))$ of the Lie algebra of $\mathcal{G}_{\mathrm{mot}}(M)$ is the extension of the abelian variety B by the torus $Z(1)$ defined by the adjoint action of $(B + Z(1), [\cdot, \cdot])$ on itself. Since in the construction of B and $Z(1)$ are involved only the parameters v, v^* and u , *the computation of the unipotent radical $W_{-1}(\mathrm{Lie} \mathcal{G}_{\mathrm{mot}}(M))$ of the Lie algebra of $\mathcal{G}_{\mathrm{mot}}(M)$ depends only on the 3-uplet (v, v^*, ψ) , i.e. on the “mixity” of the 1-motive M .*

EXAMPLES:

- (1) Let M be the split 1-motive $\mathbb{Z} \oplus A \oplus \mathbb{G}_m$. In this case all is trivial: $W_{-1}(\mathrm{Lie} \mathcal{G}_{\mathrm{mot}}(M)) = B = Z(1) = 0$.
- (2) Let $M = [\mathbb{Z} \xrightarrow{u} \mathcal{E}]$ be a 1-motive over k defined by $u(1) = P$ with P a non-torsion k -rational point of the elliptic curve \mathcal{E} . We have that the torus $Z(1)$ is trivial and the unipotent radical $W_{-1}(\mathrm{Lie} \mathcal{G}_{\mathrm{mot}}(M))$ is the elliptic curve $B = \mathcal{E}$.
- (3) Let $M = [\mathbb{Z} \xrightarrow{u} \mathbb{G}_m^3 \times A]$ be a 1-motive over k defined by $u(1) = (q_1, q_2, 1, 0)$ with q_1, q_2 two elements of $\mathbb{G}_m(k) - \mu_\infty$ multiplicatively independents (μ_∞ is the group of roots of the unity in \bar{k}). In this example the abelian variety B is trivial and the unipotent radical $W_{-1}(\mathrm{Lie} \mathcal{G}_{\mathrm{mot}}(M))$ is the torus $Z(1) = \mathbb{G}_m^2$.

With the above notations we have also that

Proposition 3.0.5. *The derived group of the unipotent radical $W_{-1}(\mathrm{Lie} \mathcal{G}_{\mathrm{mot}}(M))$ of the Lie algebra of $\mathcal{G}_{\mathrm{mot}}(M)$ is the torus $Z_1(1)$.*

Proposition 3.0.6.

$$\dim \mathrm{Lie} \mathcal{G}_{\mathrm{mot}}(M) = \dim B + \dim Z(1) + \dim \mathrm{Lie} \mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_*^W M).$$

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