IRIS AperTO

UNIVERSITÀ DEGLI STUDI DI TORINO

AperTO - Archivio Istituzionale Open Access dell'Università di Torino

Umbral nature of the Poisson random variables

This is the author's manuscript

Original Citation:

Availability:

This version is available http://hdl.handle.net/2318/1561362 since 2018-03-22T06:17:58Z

Publisher:

Springer-Verlag

Terms of use:

Open Access

Anyone can freely access the full text of works made available as "Open Access". Works made available under a Creative Commons license can be used according to the terms and conditions of said license. Use of all other works requires consent of the right holder (author or publisher) if not exempted from copyright protection by the applicable law.

(Article begins on next page)

28 April 2024

arXiv:math/0412054v1 [math.PR] 2 Dec 2004

Umbral nature of the Poisson random variables

Elvira Di Nardo and Domenico Senato

Dipartimento di Matematica, Università degli Studi della Basilicata,
Via N. Sauro 85, 85100 Potenza, Italy
{dinardo, senato}@unibas.it

Abstract. Extending the rigorous presentation of the "classical umbral calculus" [28], the so-called partition polynomials are interpreted with the aim to point out the umbral nature of the Poisson random variables. Among the new umbrae introduced, the main tool is the partition umbra that leads also to a simple expression of the functional composition of the exponential power series. Moreover a new short proof of the Lagrange inversion formula is given.

1 Introduction

The symbolic method, nowadays known as *umbral calculus*, has been extensively used since the nineteenth century although the mathematical community was sceptic of it, maybe owing to its lack of foundation. This method was fully developed by Rev. John Blissard in a series of papers beginning from 1861 [6]÷[16], nevertheless it is impossible to attribute the credit of the originary idea just to him since the Blissard's calculus has a mathematical source in the symbolic differentiation. In [22] Lucas even claimed that the umbral calculus has its historical roots in the writing of Leibniz for the successive derivatives of a product with two or several factors. Moreover Lucas held that this symbolic method had been subsequently developed by Laplace, by Vandermonde, by Herschel and augmented by the works of Cayley and of Sylvester in the theory of forms. Lucas's papers attracted considerable attention and the predominant contribution of Blissard to this method was kept in the background. Bell reviewed the whole subject in several papers, restoring the purport of the Blissard's idea [4] and in 1940 he tried to give a rigorous foundation of the mystery at the ground of the umbral calculus [5]. It was Gian-Carlo Rota [25] who twenty-five years later disclosed the "umbral magic art" of lowering and raising exponents bringing to the light the underlying linear functional. In [23] and [26] the ideas from [25] led Rota and his collaborators to conceive a beautiful theory originating a large variety of applications. Some years later, Roman and Rota gave rigorous form to the umbral tricks in the setting of the Hopf algebra. On the other hand, as Rota himself has written [28]: "...Although the notation of Hopf algebra satisfied the most ardent advocate of spic-and-span rigor, the translation of "classical" umbral calculus into the newly found rigorous language made the

method altogether unwieldy and unmanageable. Not only was the eerie feeling of witchcraft lost in the translation, but, after such a translation, the use of calculus to simplify computation and sharpen our intuition was lost by the wayside..." Thus in 1994 Rota and Taylor [28] started a rigorous and simple presentation of the umbral calculus in the spirit of the founders. The present article refers to this last point of view.

As it sometimes happens in the practice of the mathematical investigation, the subject we deal with does not develop the originary idea from which our research started in the spring of 1997, but this paper is closely related to it. In that period, Gian-Carlo Rota was visiting professor at the University of Basilicata and, during one of our latest conversations before his leaving, he shared with us his close interest for a research project: a combinatorial random variable theory. The delicate question arising from the underlying foundation side and the left short time led us to protract the discussion via e-mail intertwining it with different activities for several months. The following year, Gian-Carlo Rota held his last course in Cortona and we did not miss the opportunity to spend some time with him. We resumed the thread of our conversations and presented him with the doubts that gradually took hold of us. As usually, his contribution disclosed new horizons that have led us to write these pages.

Our starting point is the umbral notion of the Bell numbers. Many classical identities relating to these numbers are expressed in umbral notation attaining up to a new umbra, the partition umbra, connected with the so-called "partition polynomials" generated by expanding the exponential function $\exp(f(x))$ into an exponential power series. The hereafter developed theory of the Bell umbrae is not only an example of the computational power of the umbral calculus but it offers, we would like to believe, a natural way to interpret the functional composition of exponential power series tested by a new proof of the Lagrange inversion formula. Here the point operations extended with a new one play a central role. From a probabilistic point of view, the functional composition of exponential power series is closely related to the family of Poisson random variables so that these random variables have found a natural umbral interpretation through the Bell umbrae. In particular the probabilistic counterpoint of the partition umbra is the compound Poisson random variable. Also the less familiar randomized Poisson random variable unexpectedly find an umbral corresponding and in turn the umbral composition gives the way to generalize this last Poisson random variable.

What's more, we believe that the probabilistic interpretation of the partition umbra and of the umbral composition could give a probabilistic meaning to the Joyal species theory [21], namely a combinatorial random variable theory that we hope to deal in forthcoming publication.

2 The classical umbral calculus

We take a step forward in the program of the rigorous foundation of the classical umbral calculus initiated by Rota and Taylor [27], [28], [32].

In the following we denote by R a commutative integral domain whose quotient field is of characteristic zero and by $A = \{\alpha, \beta, \ldots\}$ a set whose elements are called *umbrae*. An umbral calculus is given when is assigned a linear functional $E : R[A, x, y] \to R[x, y]$ such that:

i) $E[1] = 1$;
ii) $E[\alpha^i \beta^j \cdots \gamma^k x^n y^m] = x^n y^m E[\alpha^i] E[\beta^j] \cdots E[\gamma^k]$ for any set of distinct umbrae in A and for $i, j, \ldots, k, n, m$ nonnegative integers (uncorrelation property);
iii) it exists an element $\epsilon \in A$ such that $E[\epsilon^n] = \delta_{0,n}$, for any nonnegative integer n, where
$$\delta_{i,j} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \quad i, j \in N;$$
iv) it exists an element $u \in A$ such that $E[u^n] = 1$, for any nonnegative integer n.

The umbra ϵ is named *augmentation* as Roman and Rota first called it [24]. We will call the umbra u the *unity* umbra.

A sequence $a_0, a_1, a_2, \ldots$ in $R[x, y]$ is said to be umbrally represented by an umbra α when
$$E[\alpha^i] = a_i, \quad \text{for } i = 0, 1, 2, \ldots$$
so that the linear functional E plays the role of an evaluation map.

As Rota suggested, there is an analogy between umbrae and random variables (r.v.) (see [32]), so we will refer to the elements a_i in $R[x, y]$ as *moments* of the umbra α. The umbra ϵ can be view as the r.v. which takes the value 0 with probability 1 and the umbra u as the r.v. which takes the value 1 with probability 1.

An umbra is said to be a scalar umbra if the moments are elements of R while it is said to be a polynomial umbra if the moments are polynomials. Note that if the sequence $a_0, a_1, a_2, \ldots$ is umbrally represented by a scalar umbra α, then it is $a_0 = 1$. In the same way, for polynomial umbrae, a sequence of polynomials $p_0(x), p_1(x), p_2(x), \ldots$ will always denote a sequence of polynomials with coefficients in R such that $p_0(x) = 1$ and $p_n(x)$ is of degree n for every positive integer n.

A polynomial $p \in R[A]$ is called an *umbral polynomial*. The *support* of p is defined to be the set of all occurring umbrae of A. Two umbral polynomials are said to be *uncorrelated* when their support are disjoint.

If α and β are either scalar either polynomial umbrae, we will say that α and β are *umbrally equivalent* when
$$E[\alpha] = E[\beta],$$

in symbols $\alpha \simeq \beta$. Two scalar (or polynomial) umbrae are said to be *similar* when $E[\alpha^k] = E[\beta^k]$, $k = 0, 1, 2, \ldots$ or

$$\alpha^k \simeq \beta^k,\ k = 0, 1, 2, \ldots$$

in symbols $\alpha \equiv \beta$. The notion of equivalence and similarity for umbral polynomials is obvious.

The formal power series

$$e^{\alpha t} = u + \sum_{n \geq 1} \alpha^n \frac{t^n}{n!}$$

is said to be the *generating function* of the umbra α. Moreover, if the sequence $a_0, a_1, a_2, \ldots$ has (exponential) generating function $f(t)$ and is umbrally represented by an umbra α then $E[e^{\alpha t}] = f(t)$, in symbols $e^{\alpha t} \simeq f(t)$. When α is regarded as a r.v., $f(t)$ is the moment generating function. The notion of equivalence and similarity are extended coefficientwise to the generating functions of umbrae so that $\alpha \equiv \beta$ if and only if $e^{\alpha t} \simeq e^{\beta t}$. Note that $e^{\epsilon t} \simeq 1$ and $e^{ut} \simeq e^x$.

2.1 The point operations

The notion of similarity among umbrae comes in handy in order to express sequences such

$$\sum_{i=0}^{n} \binom{n}{i} a_i a_{n-i},\ n = 0, 1, 2, \ldots \tag{1}$$

as moments of umbrae. The sequence (1) cannot be represented by using only the umbra α with moments $a_0, a_1, a_2, \ldots$ because $a_i a_{n-i}$ could not be written as $E[\alpha^i \alpha^{n-i}]$, being α related to itself. If we will assume that the umbral calculus we deal is saturated [27], the sequence $a_0, a_1, a_2, \ldots$ in $R[x, y]$ is represented by infinitely many distinct (and thus similar) umbrae. Therefore, if we choose two similar umbrae α, α', they are uncorrelated and

$$\sum_{i=0}^{n} \binom{n}{i} a_i a_{n-i} = E\left[\sum_{i=0}^{n} \binom{n}{i} \alpha^i \alpha^{n-i}\right] = E[(\alpha + \alpha')^n].$$

Then the sequence (1) represents the moments of the umbra $(\alpha + \alpha')$. This matter was first explicitly pointed out by E.T. Bell [5] who was not able to provide an effective notation:

[1] "...If in $\alpha x \dotplus \cdots \dotplus \xi x$ there are precisely T summands $\alpha x, \ldots, \xi x$ each of which is a scalar product of a scalar and x, we replace ($\rightarrow$) the T x's by T distinct umbrae, say $a, \ldots, x$ in any order, and indicate this replacement by writing

$$\alpha x \dotplus \cdots \dotplus \xi x \rightarrow \alpha a \dotplus \cdots \dotplus \xi x.$$

[1] The quotation needs more details. It is

$$\alpha x \dotplus \cdots \dotplus \varepsilon x \equiv (\alpha x_0 + \cdots + \varepsilon x_0, \ldots, \alpha x_N + \cdots + \varepsilon x_N, \ldots),$$

Then $(\alpha a \dot{+} \cdots \dot{+} \xi x)^N$ is to be calculated by (1.22) and the exponents are degraded as in (1.20). In the result, each of $a, \ldots, x$ is replaced ($\leftarrow$) by x; the resulting polynomial is defined to be the $N-$th power $(\alpha x \dot{+} \cdots \dot{+} \xi x)^N$ of the sum $(\alpha x \dot{+} \cdots \dot{+} \xi x)$. For example

$$\begin{aligned}
(\alpha x + \beta x)^3 &\to (\alpha a + \beta x)^3; \\
(\alpha a + \beta x)^3 &= \alpha^3 a_3 x_0 + 3\alpha^2 \beta a_2 x_1 + 3\alpha\beta^2 a_1 x_2 + \beta^3 a_0 x_3, \\
&\leftarrow \alpha^3 x_3 x_0 + 3\alpha^2 \beta x_2 x_1 + 3\alpha\beta^2 x_1 x_2 + \beta^3 x_0 x_3; \\
(\alpha x + \beta x)^3 &= (\alpha^3 + \beta^3) x_0 x_3 + 3\alpha\beta(\alpha + \beta) x_1 x_2 \ldots."
\end{aligned}$$

The last identity makes sense when the left side is replaced by $(\alpha x + \beta x')^3$ with $x \equiv x'$, but Bell did not have the notion of similar umbrae. However, the need of handling sequences like (1) leads to introduce some new operations between umbrae, as showed in the next sections.

2.2 The point product

We shall denote by the symbol $n.\alpha$ an auxiliary umbra similar to the sum $\alpha' + \alpha'' + \ldots + \alpha'''$ where $\alpha', \alpha'', \ldots, \alpha'''$ are a set of n distinct umbrae each of which is similar to the umbra α. We assume that $0.\alpha$ is an umbra similar to the augmentation ϵ. A similar notion $n.p$ is introduced for any umbral polynomial. The following statements are easily to be proved:

Proposition 1. *(i) If $n.\alpha \equiv n.\beta$ for some integer $n \neq 0$ then $\alpha \equiv \beta$;*
(ii) if $c \in R$ then $n.(c\alpha) \equiv c(n.\alpha)$ for any nonnegative integer n;
(iii) $n.(m.\alpha) \equiv (nm).\alpha \equiv m.(n.\alpha)$ for any two nonnegative integers n, m;
(iv) $(n+m).\alpha \equiv n.\alpha + m.\alpha'$ for any two nonnegative integers n, m and any two distinct umbrae $\alpha \equiv \alpha'$;
(v) $(n.\alpha + n.\beta) \equiv n.(\alpha + \beta)$ for any nonnegative integer n and any two distinct umbrae α and β.

Proposition 2. *If α is an umbra with generating function $e^{\alpha t} \simeq f(t)$, then the umbra $n.\alpha$ has generating function $e^{(n.\alpha)t} \simeq [f(t)]^n$.*

Proof. It follows from the definition of the auxiliary umbra $n.\alpha$. □

where x is an umbra. The formula (1.22) is

$$(\alpha a \dot{+} \cdots \dot{+} \varepsilon x)^N = \sum M_{S_1,\ldots,S_T} \alpha^{S_1} \cdots \varepsilon^{S_T} a^{S_1} \cdots x^{S_T}$$

with $M_{S_1,\ldots,S_T}$ the coefficient of $x_1^{S_1} \cdots x_T^{S_T}$ in the expansion of $(x_1 + \cdots + x_T)^N$ through the multinomial theorem. The formula (1.20) is

$$(\alpha a \dot{+} \cdots \dot{+} \varepsilon x)^N \equiv \sum M_{S_1,\ldots,S_T} \alpha^{S_1} \cdots \varepsilon^{S_T} a_{S_1} \cdots x_{S_T}.$$

The moments of the umbra $n.\alpha$ are the following polynomials in the variable n

$$E[(n.\alpha)^k] = q_k(n) = \sum_{i=0}^{k} (n)_i B_{k,i}, \quad k = 0, 1, 2, ... \tag{2}$$

where $B_{k,i} = B_{k,i}(a_1, a_2, \ldots, a_{k-i+1})$ for $i \leq k$ are the (partial) Bell exponential polynomials [2], $(n)_i$ is the lower factorial and a_i are the moments of the umbra α. Recalling that

$$\sum_{k=i}^{\infty} B_{k,i} \frac{t^k}{k!} = \frac{1}{i!}[f(t) - 1]^i, \tag{3}$$

the identity (2) follows from

$$[f(t)]^n = \sum_{i=0}^{\infty} (n)_i \frac{[f(t) - 1]^i}{i!} = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} (n)_i B_{k,i} \right) \frac{t^k}{k!}. \tag{4}$$

If in (2) set $\alpha = u$, then $q_k(n) = n^k$. Note that $q_0(n) = 1$, $q_k(0) = 0$ and the polynomial sequence $\{q_k(n)\}$ is of binomial type as it follows by using the statement *(iv)* of Proposition 1:

$$[(n + m).\alpha]^k \simeq [n.\alpha + m.\alpha']^k \simeq \sum_{i=0}^{k} \binom{k}{i} (n.\alpha)^i (m.\alpha')^{k-i}.$$

Moreover a variety of combinatorial identities could be umbrally interpreted. As instance in point, the classical Abel identity becomes

$$(\alpha + \beta)^n \simeq \sum_{k=0}^{n} \binom{n}{k} \alpha(\alpha - k.\gamma)^{k-1} (\beta + k.\gamma)^{n-k}, \quad n = 0, 1, 2, \ldots. \tag{5}$$

The expression of the polynomial sequence $\{q_k(n)\}$ in (2) suggests a way to define the auxiliary umbra $x.\alpha$ when $x \in R$, however it is impossibile to give an intrinsic definition. Up to similarity, the umbra $x.\alpha$ is the polynomial umbra with moments

$$E[(x.\alpha)^k] = q_k(x) = \sum_{i=0}^{k} (x)_i B_{k,i} \quad k = 0, 1, 2, ... \tag{6}$$

Note that $q_k(x) = x^k$ when $\alpha = u$.

Proposition 3. *If α is an umbra with generating function $e^{\alpha t} \simeq f(t)$, then the umbra $x.\alpha$ has generating function $e^{(x.\alpha)t} \simeq [f(t)]^x$.*

Proof. It follows from (4) and (3) with n replaced by x. □

Corollary 1. *(i) If $x.\alpha \equiv x.\beta$ for $x \in R - \{0\}$ then $\alpha \equiv \beta$;*

(ii) *if* $c \in R$ *then* $x.(c\alpha) \equiv c(x.\alpha)$ *for any* $x \in R$;
(iii) $x.(y.\alpha) \equiv (xy).\alpha \equiv y.(x.\alpha)$ *for any* $x, y \in R$;
(iv) $(x+y).\alpha \equiv x.\alpha + y.\alpha'$ *for any* $x, y \in R$ *and any two distinct umbrae* $\alpha \equiv \alpha'$;
(v) $(x.\alpha + x.\beta) \equiv x.(\alpha+\beta)$ *for any* $x \in R$ *and any two distinct umbrae* α *and* β.

Theorem 1. *Up to similarity, each polynomial sequence of binomial type is umbrally represented by an auxiliary umbra* $x.\alpha$ *and viceversa.*

Proof. ¿From the statement *(iv)* of the corollary 1, it follows that the polynomial sequence $\{q_k(x)\}$ is of binomial type. Viceversa, first observe that from (6) it is

$$D_x[q_k(x)]_{x=0} = a_k + F(a_1, a_2, \ldots, a_{k-1}) \tag{7}$$

where F is a function of the moments $a_1, a_2, \ldots, a_{k-1}$. Let $\{p_k(x)\}$ be a polynomial sequence of binomial type. Through (7), the moments of the umbra α are uniquely determined by the knowledge of the first derivative respect to x of $p_k(x)$ evaluated in 0. Moreover, the sequences of first derivate respect to x of $p_k(x)$ evaluated in 0 uniquely determines a sequence of binomial type. □

Similarly with it has been done for the auxiliary umbra $x.\alpha$, we define a point product among umbrae. Up to similarity, the umbra $\beta.\alpha$ is an auxiliary umbra whose moments are umbrally expressed through the umbral polynomials $q_{\alpha,k}(\beta)$:

$$(\beta.\alpha)^k \simeq q_{\alpha,k}(\beta) = \sum_{i=0}^{k} (\beta)_i B_{k,i} \quad k = 0, 1, 2, ... \tag{8}$$

If α is an umbra with generating function $e^{\alpha t} \simeq f(t)$, then the identity (4) could be rewritten as

$$[f(t)]^\beta \simeq \sum_{i=0}^{\infty} (\beta)_i \frac{[f(t)-1]^i}{i!} \simeq \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} (\beta)_i B_{k,i} \right) \frac{t^k}{k!} \tag{9}$$

so that $e^{(\beta.\alpha)t} \simeq [f(t)]^\beta$. Moreover if β is an umbra with generating function $e^{\beta t} \simeq g(t)$, then

$$[f(t)]^\beta \simeq e^{\beta log f(t)} \simeq g\left[\log f(t)\right].$$

This proves the following proposition.

Proposition 4. *If* α *is an umbra with generating function* $e^{\alpha t} \simeq f(t)$ *and* β *is an umbra with generating function* $e^{\beta t} \simeq g(t)$, *then the umbra* $\beta.\alpha$ *has generating function*

$$e^{(\beta.\alpha)t} \simeq [f(t)]^\beta \simeq g\left[\log f(t)\right]. \tag{10}$$

Corollary 2. *If $\gamma \equiv \gamma'$ then*

$$(\alpha+\beta).\gamma \equiv \alpha.\gamma + \beta.\gamma'.$$

Proof. Let $e^{\gamma t} \simeq h(t)$ the generating function of the umbra γ. It is

$$e^{[(\alpha+\beta).\gamma]t} \simeq [h(t)]^{\alpha+\beta} \simeq [h(t)]^{\alpha}[h(t)]^{\beta} \simeq e^{(\alpha.\gamma)t}e^{(\beta.\gamma')t}$$

from which the result follows. □

Remark 1. As Taylor suggests in [32], the auxiliary umbra $\beta.\alpha$ provides an umbral interpretation of the random sum since the moment generating function $g[\log f(t)]$ corresponds to the r.v. $S_N = X_1 + X_2 + \cdots + X_N$ with X_i indipendent identically distributed (i.i.d.) r.v. having moment generating function $f(t)$ and with N a discrete r.v. having moment generating function $g(t)$. The probabilistic interpretation of the corollary 2 states that the random sum S_{N+M} is similar to $S_N + S_M$, where N and M are two indipendent discrete r.v.

The left distributive property of the point product respect to the sum does not hold since

$$e^{[\alpha.(\beta+\gamma)]t} \simeq [g(t)]^{\alpha}[h(t)]^{\alpha} \not\simeq f[\log g(t)]f[\log h(t)]$$

where $g(t) \simeq e^{\beta t}, h(t) \simeq e^{\gamma t}$ and $f(t) \simeq e^{\alpha t}$.

Again this result runs in parallel with the probability theory. In fact, let $Z = X + Y$ be a r.v. with X and Y two indipendent r.v. The random sum $S_N = Z_1 + Z_2 + \cdots + Z_N$, with Z_i i.i.d. r.v. similar to Z, is not similar to the r.v. $S_N^X + S_N^Y$ where $S_N^X = X_1 + X_2 + \cdots + X_N$ and X_i i.i.d. r.v. similar to X, and where $S_N^Y = Y_1 + Y_2 + \cdots + Y_N$ and Y_i i.i.d. r.v. similar to Y.

Corollary 3. *(i) If $\beta.\alpha \equiv \beta.\gamma$ then $\alpha \equiv \gamma$;*
(ii) if $c \in R$ then $\beta.(c\alpha) \equiv c(\beta.\alpha)$ for any two distinct umbrae α and β;
(iii) $\beta.(\gamma.\alpha) \equiv (\beta.\gamma).\alpha$.

Proof. Via generating functions. □

To end this section, we deal with the notion of the inverse of an umbra. Two umbrae α and β are said to be *inverse* to each other when $\alpha + \beta \equiv \varepsilon$. Recall that, dealing with a saturated umbral calculus, the inverse of an umbra is not unique, but any two inverse umbrae of the umbra α are similar.

Proposition 5. *If α is an umbra with generating function $e^{\alpha t} \simeq f(t)$ then its inverse β has generating function $e^{\beta t} \simeq [f(t)]^{-1}$.*

Proof. The result follows observing that $e^{(\alpha+\beta)t} \simeq 1$. □

Similarly, for every positive integer n and for every umbra $\alpha \in A$, the inverse of the auxiliary umbra $n.\alpha$, written as $-n.\alpha'$ with $\alpha \equiv \alpha'$, is similar to $\beta' + \beta'' + \cdots + \beta'''$ where $\beta', \beta'', \ldots, \beta'''$ is any set of n distinct umbrae similar to β, being β the inverse of α. The notation $-n.\alpha'$ is justified by noting that

$$n.\alpha - n.\alpha' \equiv (n-n).\alpha \equiv 0.\alpha \equiv \varepsilon.$$

Proposition 6. *If α is an umbra with generating function $e^{\alpha t} \simeq f(t)$, then the inverse of $n.\alpha$ has generating function $e^{(-n.\alpha')t} \simeq [f(t)]^{-n}$.*

Proof. The result follows observing that $e^{(n.\alpha - n.\alpha')t} \simeq 1$. □

The inverse of the umbra $x.\alpha$ is the umbra $-x.\alpha'$ with $\alpha \equiv \alpha'$ defined by

$$x.\alpha - x.\alpha' \equiv (x-x).\alpha \equiv 0.\alpha \equiv \varepsilon.$$

2.3 The point power

As it is easy to be expected, the definition of the power of moments requires the use of similar umbrae and so of a point operation. This notion comes into this picture by a natural way, providing also an useful tool for umbral manipulation of generating function.

We shall denote by the symbol $\alpha^{\cdot n}$ an auxiliary umbra similar to the product $\alpha'\alpha'' \cdots \alpha'''$ where $\alpha', \alpha'', \ldots, \alpha'''$ are a set of n distinct umbrae each of which is similar to the umbra α. We assume that $\alpha^{\cdot 0}$ is an umbra similar to the unity umbra u. A similar notion is introduced for any umbral polynomial p. The following statements are easily to be proved:

Proposition 7. *(i) If $c \in R$ then $(c\alpha)^{\cdot n} \equiv c^n \alpha^{\cdot n}$ for any nonnegative integer $n \neq 0$;*
(ii) $(\alpha^{\cdot n})^{\cdot m} \equiv \alpha^{\cdot nm} \equiv (\alpha^{\cdot m})^{\cdot n}$ for any two nonnegative integers n, m;
(iii) $\alpha^{\cdot (n+m)} \equiv \alpha^{\cdot n}(\alpha')^{\cdot m}$ for any two nonnegative integers n, m and any two distinct umbrae $\alpha \equiv \alpha'$;
iv) $(\alpha^{\cdot n})^k \equiv (\alpha^k)^{\cdot n}$ for any two nonnegative integers n, k.

By the last statement, the moments of $\alpha^{\cdot n}$ for any integer n are:

$$E[(\alpha^{\cdot n})^k] = E[(\alpha^k)^{\cdot n}] = a_k^n, \quad k = 0, 1, 2, \ldots \tag{11}$$

so that the moments of the umbra $\alpha^{\cdot n}$ are the $n-$th power of the moments of the umbra α.

Proposition 8. *The generating function of the $n-$th point power of the umbra α is the $n-$th power of the generating function of the umbra α.*

Note that, by virtue of Propositions 2 and 8 it is

$$e^{(n.\alpha)t} \simeq (e^{\alpha t})^{\cdot n}. \tag{12}$$

The relation (12) restores the natural umbral interpretation of $[f(t)]^n$. More, let us observe that if α and β are not similar, it is

$$(\alpha+\beta)^{\cdot n} \equiv \sum_{i=0}^{n} \binom{n}{i} \alpha^{\cdot i} \beta^{\cdot (n-i)}.$$

The point power operation leads us to define the point exponential of an umbra. We shall denote by the symbol $e.^{\alpha}$ the auxiliary umbra

$$e.^{\alpha} \equiv u + \sum_{n=1}^{\infty} \frac{\alpha^{\cdot n}}{n!}. \tag{13}$$

We have immediately $e.^{\epsilon} \equiv u$.

Proposition 9. *For any umbra α, it is*

$$e.^{(n.\alpha)} \simeq (e.^{\alpha})^{\cdot n}. \tag{14}$$

Proof. It results

$$E[(e.^{\alpha})^{\cdot n}] = E[e.^{\alpha}]^n = e^{nE[\alpha]} = \sum_{k\geq 0} \frac{n^k E[\alpha]^k}{k!}$$

and also

$$E[e.^{(n.\alpha)}] = \sum_{k\geq 0} \frac{E[(n.\alpha)^{\cdot k}]}{k!} = \sum_{k\geq 0} \frac{n^k E[\alpha.^k]}{k!},$$

by which (14) follows. □

Up to similarity, the expression of the moments given in (11) justifies the definition of the auxiliary umbra $\alpha^{\cdot x}$ as the umbra whose moments are

$$E[(\alpha^{\cdot x})^k] = a_k^x, \quad k = 0, 1, 2, \ldots.$$

Proposition 10. *Let α be an umbra and $e^{\alpha t} \simeq f(t)$ its generating function. It is*

$$e^{(x.\alpha)t} \simeq (e^{\alpha t})^{\cdot x} \simeq [f(t)]^x.$$

Via moments, it is possible to prove the analogue of Proposition 7 where n and m are replaced by x and y with $x, y \in R$.

Once again, we define the auxiliary umbra $\alpha^{\cdot \beta}$ as the umbra whose moments are umbrally equivalent to

$$(\alpha^{\cdot \beta})^k \simeq a_k^{\beta}, \quad k = 0, 1, 2, \ldots$$

and we set $\epsilon^{\cdot \alpha} \equiv \epsilon$.

Proposition 11. *(i)* $(\alpha^{\cdot\beta})^{\cdot\gamma} \equiv \alpha^{\cdot(\gamma\cdot\beta)}$;
(ii) $\alpha^{\cdot(\beta+\gamma)} \equiv \alpha^{\cdot\beta}(\alpha')^{\cdot\gamma}$ *for any two distinct umbrae* $\alpha \equiv \alpha'$.

Proof. It follows via moments. □

Proposition 12. *Let* $e^{\alpha t} \simeq f(t)$ *be the generating function of the umbra* α. *It is*

$$e^{(\beta.\alpha)t} \simeq (e^{\alpha t})^{\cdot\beta} \simeq [f(t)]^{\beta}.$$

In closing, we notice that the generating function of the point product between umbrae is umbrally equivalent to the following series:

$$e^{(\beta.\alpha)t} \simeq \sum_{i=0}^{\infty} (\beta)_i \frac{[e^{\alpha t} - u]^{\cdot i}}{i!} \tag{15}$$

by the relation (9) and Proposition 8.

3 Bell umbrae

The Bell numbers B_n have a long history and their origin is unknown: Bell ascribes them to Euler even without a specific reference [3]. Usually they are referred as the number of the partitions of a finite nonempty set with n elements or as the coefficients of the Taylor series expansion of the function $\exp(e^t - 1)$. It is just writing about the Bell numbers that Gian-Carlo Rota [25] gives the first glimmering of the effectiveness of the umbral calculus in manipulating number sequences, indeed his proof of the Dobinski's formula is implicitly of umbral nature.

In this section, the umbral definition of the Bell numbers allows the proofs of several classical identities (cf. [33]) through elementary arguments and smooths the way to the umbral interpretation of the Poisson random variables.

Definition 1. An umbra β is said to be a *Bell scalar umbra* if

$$(\beta)_n \simeq 1 \qquad n = 0, 1, 2, \ldots$$

where $(\beta)_0 = 1$ and $(\beta)_n = \beta(\beta - 1) \cdots (\beta - n + 1)$ is the lower factorial.

Up to similarity, the Bell number sequence is umbrally represented by the Bell scalar umbra. Indeed, being

$$\beta^n = \sum_{k=0}^{n} S(n,k)(\beta)_k$$

where $S(n,k)$ are the Stirling numbers of second kind, then

$$E(\beta^n) = \sum_{k=0}^{n} S(n,k) E[(\beta)_k] = \sum_{k=0}^{n} S(n,k) = B_n$$

where B_n are the Bell numbers.

The following theorem provides a characterization of the Bell umbra.

Theorem 2. *A scalar umbra β is a Bell umbra iff*

$$\beta^{n+1} \simeq (\beta + u)^n \quad n = 0, 1, 2, \ldots \tag{16}$$

Proof. If β is the Bell scalar umbra, being $\beta(\beta - u)_n \simeq (\beta)_{n+1}$ it is $E[\beta(\beta - u)_n] = 1 = E[(\beta)_n]$. By the linearity it follows

$$E[\beta p(\beta - u)] = E[p(\beta)]$$

for every polynomial p in β. So, the idenity (16) follows setting $p(\beta) = (\beta + u)^n$. Viceversa, the relation (16) gives

$$E[\beta^{n+1}] = E[(\beta + u)^n] = \sum_{k=0}^{n} \binom{n}{k} E[\beta^k]$$

or setting $E[\beta^n] = B_n$ one has

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$$

that is the recursion formula of the Bell numbers. □

Corollary 4. *If β is the Bell scalar umbra, then*

$$D_t[e^{\beta t}] \simeq e^{(\beta+u)t}. \tag{17}$$

Proposition 13. *If β is the Bell scalar umbra, for any integer $k > 0$ and for any polynomial $p(x)$ the following relation holds*

$$p(\beta + k.u) \simeq (\beta)_k p(\beta) \simeq p(\beta).$$

Proof. For $n \geq k$, by the definition 1 it follows

$$(\beta)_n \simeq (\beta)_{n+k} \simeq (\beta)_k (\beta - k.u)_n.$$

Thus for any polynomial q it is

$$q(\beta) \simeq (\beta)_k q(\beta - k.u)$$

by which one has

$$(\beta + k.u)^n \simeq (\beta)_k \beta^n, \quad n = 0, 1, 2, \ldots$$

setting $q(\beta) = (\beta + k.u)^n$. The result follows by linearity. □

Proposition 14. *The generating function of the Bell umbra is*

$$e^{\beta t} \simeq e.^{e^{ut}-u}. \tag{18}$$

Proof. By the definition 1 and the relation (15) it is

$$e^{\beta t} \simeq e^{(\beta.u)t} \simeq \sum_{i=0}^{\infty} \frac{[e^{ut}-u]^{\cdot i}}{i!}.$$

Thus (18) follows from the relation (13). □

Remark 2. Let us go on with our probabilistic counterpoint noting that the Bell umbra can be view as a Poisson r.v. with parameter $\lambda = 1$. Indeed, the moment generating function of the Bell umbra is $\exp(e^t - 1)$ (see (18)) so that $P(e^t) = \exp(e^t - 1)$ where $P(t)$ is the probability generating function, and therefore $P(s) = \exp(s-1)$. By that, the moments of a Poisson r.v. with parameter 1 are the Bell numbers and its factorial moments are equal to 1.

The following theorem makes clear how the proof of Dobinski's formula becomes natural through the umbral expression of generating function.

Theorem 3 (Umbral Dobinski's formula). *The Bell umbra β satisfies the following formula:*

$$\beta^n \simeq e.^{-u} \sum_{k=0}^{\infty} \frac{(k.u)^n}{k!}.$$

Proof. Being $e^{\beta t} \simeq e.^{-u} e.^{e^{ut}}$ it is

$$e^{\beta t} \simeq e.^{-u} \sum_{k=0}^{\infty} \frac{e^{(k.u)t}}{k!} \simeq e.^{-u} \sum_{k=0}^{\infty} \frac{1}{k!} \left\{ \sum_{n=0}^{\infty} \frac{(k.u)^n t^n}{n!} \right\}$$

by which the result follows. □

3.1 The Bell polynomial umbra

Definition 2. An umbra ϕ is said to be a *Bell polynomial umbra* if

$$(\phi)_n \simeq x^n \qquad n = 0, 1, 2, \ldots.$$

Note that $\phi \equiv \beta$ for $x = 1$. Moreover, being

$$\phi^n = \sum_{k=0}^{n} S(n,k)(\phi)_k,$$

by the definition 2 it follows

$$E(\phi^n) = \sum_{k=0}^{n} S(n,k) E[(\phi)_k] = \sum_{k=0}^{n} S(n,k) x^k = \Phi_n(x). \tag{19}$$

The polynomials $\Phi_n(x)$ have a statistical origin and are known in the literature as *exponential polynomials*. Indeed, they were first introduced by Steffensen [31] and studied further by Touchard [33] and others. Rota, Kahaner and Odlyzko [26] state their basic properties via umbral operators.

Proposition 15. *The generating function of the Bell polynomial umbra is*

$$e^{\Phi t} \simeq e.^{x.(e^{ut}-u)} \tag{20}$$

Proof. By the definition 2 and the relation (15) it is

$$e^{\phi t} \simeq e^{(\phi.u)t} \simeq \sum_{i=0}^{\infty} x^i \frac{[e^{ut}-u]^{\cdot i}}{i!}.$$

Thus (20) follows from the relation (13). □

The following theorem provides a characterization of the Bell polynomial umbra.

Theorem 4. *An umbra ϕ is the Bell polynomial umbra iff*

$$\phi \equiv x.\beta$$

where β is the Bell scalar umbra.

Proof. The result comes via (20). □

Remark 3. The Bell polynomial umbra can be view as a Poisson r.v. with parameter $\lambda = x$. Indeed, the moment generating function of the Bell polynomial umbra is $\exp[x(e^t-1)]$ (see (20)) so that $P(e^t) = \exp[x(e^t-1)]$, where $P(t)$ is the probability generating function, and therefore $P(s) = \exp[x(s-1)]$. By that, the moments of a Poisson r.v. with parameter x are the exponential polynomials and its factorial moments are equal to x^n.

When $x = n$, the Bell polynomial umbra $n.\beta$ is the sum of n similar uncorrelated Bell scalar umbrae, likewise in probability theory where a Poisson r.v. of parameter n can be view as the sum of n i.i.d. (eventually uncorrelated) Poisson r.v. with parameter 1. More in general, the closure under convolution of the Poisson probability distributions i.e. $F_s \star F_t = F_{s+t}$, where F_t is a Poisson probability distribution depending on the parameter t, is umbrally translated by $x.\beta + y.\beta' \equiv (x+y).\beta$ (cf. statement *(iv)* of Proposition 1).

The next theorem is the polynomial analogue of the theorem.

Theorem 5. *An auxiliary umbra $x.\beta$ is a Bell polynomial umbra iff*

$$(x.\beta)^{n+1} \simeq x(x.\beta + u)^n, \quad n = 0, 1, 2, \ldots \tag{21}$$

Proof. Observe that $D_t[e^{(x.\beta)t}] \simeq D_t[(e^{\beta t})^{.x}] \simeq xe^{[(x-1).\beta]t}D_t[e^{\beta' t}]$, where $\beta' \equiv \beta$. From (17) it is $D_t[e^{(x.\beta)t}] \simeq xe^{(x.\beta+u)t}$ from which the result follows immediately. Viceversa, the relation (21) gives (16) for $x = 1$, by which it follows that β is the Bell scalar umbra. □

The formula (21) represents the umbral equivalent of the well known recursive formula for the exponential polynomials:

$$\Phi_{n+1}(x) = x \sum_{k=0}^{n} \binom{n}{k} \Phi_k(x).$$

Similarly, the next proposition gives an umbral analogue of the Rodrigues formula for the exponential polynomials (cf. [26]).

Proposition 16. *The Bell polynomial umbra $x.\beta$ has the following property:*

$$D_x[(x.\beta)^n] \simeq (x.\beta + u)^n - (x.\beta)^n.$$

Proof. ¿From (18) it is

$$D_x[e^{(x.\beta)t}] \simeq e^{(x.\beta)t}(e^{ut} - u) \simeq e^{(x.\beta+u)t} - e^{(x.\beta)t}$$

by which the result follows immediately. □

In closing we state the polynomial version of the umbral Dobinski's formula.

Proposition 17. *The Bell polynomial umbra $x.\beta$ satisfies the following relation:*

$$(x.\beta)^n \simeq e^{-x.u} \sum_{k=0}^{\infty} \frac{(k.u)^n x^k}{k!}.$$

Proof. Being $e^{(x.\beta)t} \simeq e.^{-x.u} e.^{(x.e^{ut})}$ it is

$$e^{(x.\beta)t} \simeq e.^{-x.u} \sum_{k=0}^{\infty} \frac{(x.e^{ut})^{.k}}{k!} \simeq e.^{-x.u} \sum_{k=0}^{\infty} \frac{x^k e^{(k.u)t}}{k!}$$

by which the result follows. □

3.2 The exponential umbral polynomials

Let us introduce a new family of umbral polynomials that turns out to be an useful tool in the umbral composition, also disclosing an unexpected probabilistic interpretation.

Set

$$\Phi_n(\alpha) = \sum_{k=0}^{n} S(n,k)\alpha^k, \quad n = 0, 1, 2, \ldots, \tag{22}$$

we will call $\Phi_n(\alpha)$ *exponential umbral polynomials*. By the identity (19), being $(x.\beta)^n \simeq \Phi_n(x)$, where β is the Bell scalar umbra, it is glaring that

$$\Phi_n(\alpha) \simeq (\alpha.\beta)^n \quad n = 0, 1, 2, \ldots$$

and

$$(\alpha.\beta)_n \simeq \alpha^n \quad n = 0, 1, 2, \ldots, \tag{23}$$

a formal proof passing through similar arguments already produced for the umbra $x.\beta$.

Proposition 18. *Let β be the Bell scalar umbra. If $e^{\alpha t} \simeq f(t)$ is the generating function of the umbra α then*

$$e^{(\alpha.\beta)t} \simeq f[e^t - 1]. \tag{24}$$

Proof. The result follows by the relation (10) observing that $e^{\beta t} \simeq e^{e^t - 1}$. □

When $f(t)$ is considered as the moment generating function of a r.v. X, a probabilistic interpretation of (24) suggests that the umbra $\alpha.\beta$ represents a Poisson r.v. N_X with random parameter X. Indeed the probability generating function of N_X is

$$\begin{aligned} P(s) &= \sum_{k=0}^{\infty} \mathbf{P}(N_X = k) s^k = \sum_{k=0}^{\infty} s^k \int_0^{\infty} \mathbf{P}(N_X = k | X = x) dF_X(x) \\ &= \sum_{k=0}^{\infty} \frac{s^k}{k!} \int_0^{\infty} x^k e^{-x} dF_X(x) = \sum_{k=0}^{\infty} \frac{(s-1)^k}{k!} E[X^k] = f(s-1) \end{aligned}$$

hence the moment generating function of N_X is $f(e^t - 1)$. To the best of our knowledge, this r.v. has been introduced in [20] as the randomized Poisson r.v. Once more the closure under convolution of the Poisson probability distributions leads us to claim that the point product $\alpha.\beta$ is the umbral corresponding of the random sum of independent Poisson r.v. with parameter 1 indexed by an integer r.v. X.

4 The partition umbra

As suggested in [26], there is a connection between polynomials of binomial type and compound Poisson processes. Two different approaches can be found in [30] and in [17]. In this section, we suggest a way, that we believe to be natural, in order to make clear this connection.

Definition 3. An umbra ψ is said to be an $\alpha-$*partition umbra* if

$$\psi \equiv \beta.\alpha$$

with β the Bell scalar umbra.

Note that the $u-$partition umbra is the Bell scalar umbra.

Proposition 19. *The generating function of the $\alpha-$partition umbra ψ is*

$$e^{\psi t} \simeq e.^{(e^{\alpha t}-u)}. \tag{25}$$

Proof. ¿From (15) and by the definition 1 one has

$$e^{(\beta.\alpha)t} \simeq \sum_{n=0}^{\infty} \frac{(e^{\alpha t}-u)^{\cdot n}}{n!}.$$

Thus (25) follows from the relation (13). □

The generating function (25) leads us to interpret a partition umbra as a compound Poisson r.v. with parameter 1. As well known (cf. [19]), a compound Poisson r.v. with parameter x is introduced as a random sum $S_N = X_1 + X_2 + \cdots + X_N$ where N has a Poisson distribution with parameter x. The point product of definition 3 fits perfectly this probabilistic notion taking into consideration that the Bell scalar umbra β plays the role of a Poisson r.v. with parameter 1. What's more, since the Poisson r.v. with parameter x is umbrally represented by the Bell polynomial umbra $x.\beta$, a compound Poisson r.v. with parameter x is represented by the *polynomial $\alpha-$partition umbra $x.\psi \equiv x.\beta.\alpha$* with generating function

$$e^{(x.\psi)t} \simeq e.^{[x.(e^{\alpha t}-u)]}. \tag{26}$$

The name "partition umbra" has also a probabilistic ground. Indeed the parameter of a Poisson r.v. is usually denoted by $x = \lambda t$, with t representing a time interval, so that when this interval is partitioned into non-overlapping ones, their contributions are stochastic independent and add to S_N. The last circumstance is umbrally expressed by the relation

$$(x+y).\beta.\alpha \equiv x.\beta.\alpha + y.\beta.\alpha \tag{27}$$

that also assures the binomial property for the polynomial sequence defined by $x.\beta.\alpha$. In terms of generating functions, the formula (27) means that

$$h_{x+y}(t) = h_x(t)h_y(t) \tag{28}$$

where $h_x(t)$ is the generating function of $x.\beta.\alpha$. Viceversa every generating function $h_x(t)$ satisfying the equality (28) is the generating function of a polynomial $\alpha-$partition umbra, namely $h_x(t)$ has an umbral expression of the form (26).

Going back the moments of a partition umbra, according to the definition of the Bell scalar umbra and from (8) it is

$$E[(\beta.\alpha)^n] = \sum_{k=1}^{n} B_{n,k}(a_1, a_2, \ldots, a_{n-k+1}) = Y_n(a_1, a_2, \ldots, a_n) \tag{29}$$

where $Y_n = Y_n(a_1, a_2, \ldots, a_n)$ are the *partition polynomials* (or complete Bell exponential polynomials) and a_i are the moments of the umbra α. Although the complexity of the partition polynomial expression, their umbral interpretation (29) allows an easy proof that they are of binomial type, simply observing that $\beta.\alpha + \beta.\gamma \equiv \beta.(\alpha + \gamma)$.

Partition polynomials have been first introduced by Bell [1] who gave a pioneer umbral version of them in [2]. Because of their generality, they include a variety of other polynomials such as the cycle indicator of the symmetric group and other of interest in number theory.

As already done for the Bell scalar umbra, the next theorem characterizes the partition umbrae and also provides the following recursive formula for the partition polynomials:

$$Y_{n+1}(a_1, a_2, \cdots, a_{n+1}) = \sum_{k=0}^{n} \binom{n}{k} a_{n-k+1} Y_k(a_1, a_2, \cdots, a_k).$$

Theorem 6. *Every $\alpha-$partition umbra verifies the following relation*

$$(\beta.\alpha)^{n+1} \simeq \alpha'(\beta.\alpha + \alpha')^n \quad \alpha' \equiv \alpha,\ n = 0, 1, 2, \ldots \tag{30}$$

and viceversa.

Proof. Let ψ an $\alpha-$partition umbra. Then from (25) it is $D_t[e^{(\beta.\alpha)t}] \simeq e^{(\beta.\alpha)t} D_t[e^{\alpha' t}]$, where $\alpha' \equiv \alpha$. The identity (30) follows observing that $D_t[e^{\alpha' t}] \simeq \alpha' e^{\alpha' t}$. Going back the previous steps, from (30) one has that $\beta.\alpha$ has generating function (25) and so it is an $\alpha-$partition umbra. □

The moments of the polynomial $\alpha-$partition umbra are

$$\begin{aligned} E[(x.\beta.\alpha)^n] &= \sum_{k=1}^{n} (x.\beta)_k B_{n,k}(a_1, a_2, \ldots, a_{n-k+1}) \\ &= \sum_{k=1}^{n} x^k B_{n,k}(a_1, a_2, \ldots, a_{n-k+1}) \end{aligned} \tag{31}$$

according to the definition 2. The same arguments given in the proof of the theorem 6 lead to state that every polynomial partition umbra verifies the following formula

$$(x.\beta.\alpha)^{n+1} \simeq x\alpha'(x.\beta.\alpha + \alpha')^n \quad \alpha' \equiv \alpha,\ n = 0, 1, 2, \ldots$$

and viceversa.

4.1 Umbral expression of the functional composition

An umbral wording of the functional composition of exponential formal power series is a thorny matter. It was broached by Rota, Shen and Taylor in [28] passing through the sequence of Abel polynomials. In this last section, we give an intrinsic umbral expression of this operation via the notion of partition umbra.

Definition 4. A *composition umbra* of the umbrae α and γ is the umbra

$$\chi \equiv \gamma.\beta.\alpha$$

where β is the Bell scalar umbra.

In other words, the composition umbra χ is the point product of the umbra γ and the $\alpha-$partition umbra $\beta.\alpha$.

Remark 4. As already stressed in section 3.2, the umbra $\gamma.\beta$ represents a randomized Poisson r.v.. Hence it is natural to look at the composition umbra as a new r.v. that we will call *compound randomized Poisson r.v.* Moreover, being $(\gamma.\beta).\alpha \equiv \gamma.(\beta.\alpha)$ (cf. statement *(ii)* of corollary 3), the previous relation allows to see this new r.v. from another side: the umbra $\gamma.(\beta.\alpha)$ generalizes the concept of a random sum of i.i.d. compound Poisson r.v. with parameter 1 indexed by an integer r.v. X, i.e. a randomized compound Poisson r.v. with random parameter X.

Proposition 20. *The generating function of the composition umbra $\gamma.\beta.\alpha$ is the functional composition of the generating functions $e^{\alpha t} \simeq f(t)$ and $e^{\gamma t} \simeq g(t)$.*

Proof. Via (25) it is $e^{(\beta.\alpha)t} \simeq e^{f(t)-1}$. The result follows by (10) observing that $e^{[\gamma.(\beta.\alpha)]t} \simeq g\left\{\log[e^{f(t)-1}]\right\}$. $\square$

The moments of the composition umbra are

$$(\gamma.\beta.\alpha)^n \simeq \sum_{k=0}^{n} \gamma^k B_{n,k}(a_1, a_2, \ldots, a_{n-k+1}) \tag{32}$$

where a_i are the moments of the umbra α. Indeed, by (8) it is

$$(\gamma.\beta.\alpha)^n \simeq \sum_{k=0}^{n} (\gamma.\beta)_k B_{n,k}(a_1, a_2, \ldots, a_{n-k+1})$$

and (32) follows from (23).

Once more, we give a characterization of the composition umbra in the next theorem.

Theorem 7. *Every composition umbra verifies the following relation*

$$(\gamma.\beta.\alpha)^{n+1} \equiv \gamma\alpha'(\gamma.\beta.\alpha + \alpha')^n \quad \alpha \equiv \alpha',\ n = 0, 1, 2, \ldots \tag{33}$$

and viceversa.

Proof. Let χ a composition umbra of α and γ. Then from Proposition 20, it is $D_t[e^{\chi t}] \simeq g'[f(t) - 1]f'(t)$. Equation (33) follows being $f'(t) \simeq \alpha' e^{\alpha' t}$ with $\alpha' \equiv \alpha$ and $g'[f(t) - 1] \simeq \gamma e^{\chi t}$. Going back the previous steps, from (33) it follows that $\gamma.\beta.\alpha$ has generating function $g[f(t) - 1]$ and so it is a composition umbra of α and γ. $\square$

At this point, as custom, we put to test the definition 4 of composition umbra, giving a proof of Lagrange inversion formula. In the literature (cf. [18] for a plenty of references) different forms of the Lagrange inversion formula are derived using umbral calculus. The main tool of our proof is the umbral expression of the (partial) Bell exponential polynomials that we state in the next proposition.

Lemma 1. *It is*

$$B_{n,k}(a_1, a_2, \ldots, a_{n-k+1}) \simeq \binom{n}{k} \alpha^{\cdot k} (k.\overline{\alpha})^{n-k} \tag{34}$$

where $\overline{\alpha}$ *is the umbra with moments* $E\left[\overline{\alpha}^n\right] = \dfrac{a_{n+1}}{a_1(n+1)}, n = 1, 2, \ldots.$

Proof. By the identity (3) it results

$$B_{n,k}(a_1, a_2, \ldots, a_{n-k+1}) = \frac{1}{k!} D_t^{(n)}[(f(t) - 1)^k]_{t=0}$$

where $D_t^{(n)}[\cdot]_{t=0}$ is the $n-$th derivative respect to t evaluated in $t = 0$ and $f(t) \simeq e^{\alpha t}$, so that

$$B_{n,k}(a_1, a_2, \ldots, a_{n-k+1}) \simeq \frac{1}{k!} D_t^{(n)}[(e^{\alpha t} - u)^{\cdot k}]_{t=0}.$$

On the other hand, by the moment expression of umbra $\overline{\alpha}$ it follows $e^{\alpha t} - u \simeq \alpha\, t\, e^{\overline{\alpha} t}$. Therefore one has

$$\begin{aligned} D_t^{(n)}[(e^{\alpha t} - u)^{\cdot k}] &\simeq \alpha^{\cdot k} D_t^{(n)}[t^k\, e^{k.\overline{\alpha}}] \\ &\simeq \alpha^{\cdot k} \sum_{j=0}^{k} \binom{n}{j} D_t^{(j)}[t^k] D_t^{(n-j)}[e^{k.\overline{\alpha}}], \end{aligned}$$

using the binomial property of the derivative operator. Finally, the result follows evaluating the right hand side of the previous formula in $t = 0$ and observing that $D_t^{(n-k)}[e^{k.\overline{\alpha}}]_{t=0} \simeq (k.\overline{\alpha})^{n-k}$. $\square$

Remark 5. Let $\alpha \equiv u$. Then $\alpha^{\cdot k} \simeq 1$ for $k = 0, 1, 2, \ldots$ and $B_{n,k}(1, 1, \ldots, 1) = S(n, k)$ the Stirling number of the second kind. Moreover it is $\overline{\alpha} \equiv (-1.\delta)$ where δ is the Bernoulli umbra whose moments are the Bernoulli numbers (cf. [28]). From Lemma 1 it results

$$S(n, k) \simeq \binom{n}{k} (-k.\delta)^{n-k}$$

as already stated by Rota and Taylor through a different approach (cf. Proposition 9.1 [27])

Theorem 8 (Lagrange inversion formula). *Let $e^{\alpha t} \simeq f(t)$ and $e^{\gamma t} \simeq g(t)$. If $g[f(t) - 1] = f[g(t) - 1] = 1 + t$ then*

$$\alpha^{\cdot k}\gamma^k \simeq (-k.\overline{\alpha})^{k-1}, k = 1, 2, \ldots. \tag{35}$$

Proof. By the formulas (32) and (34), it is

$$\chi^n \simeq \sum_{k=0}^{n} \binom{n}{k} \alpha^{\cdot k}\gamma^k(-k.\overline{\alpha})^{n-k}. \tag{36}$$

On the other hand, the Abel identity (5) gives

$$\chi^n \simeq \sum_{k=0}^{n} \binom{n}{k} \chi(\chi - k.\overline{\alpha})^{k-1}(k.\overline{\alpha})^{n-k}. \tag{37}$$

Comparing (36) with (37) one has

$$\alpha^{\cdot k}\gamma^k \simeq \chi(\chi - k.\overline{\alpha})^{k-1}$$

by which the result follows expanding the right hand side of the previous formula by the binomial theorem and observing that from $g[f(t) - 1] = 1 + t$ it is $\chi \simeq 1$ and $\chi^j \simeq 0, j = 2, 3, \ldots$. $\square$

More explicitly, the formula (35) says that the $k-$th coefficient of the generating function $g(t)$ is equal to the $(k-1)-$th coefficient of the generating function $[(f(t)-1)/t]^{-k}$, when $g[f(t)-1] = 1+t$. Note that if $f(t)-1 = te^{-t}$ then $\alpha^{\cdot k} \simeq 1$, $\overline{\alpha} \equiv -1.u$ and from (35) it is $\gamma^k \simeq (k.u)^{k-1}$.

In closing, let us observe that if $a_1 = 1$ then $f(t) - 1 \simeq te^{\overline{\alpha}t}$ and the Lagrange inversion formula (35) becomes

$$\gamma^k \equiv (-k.\overline{\alpha})^{k-1}.$$

On the other hand, if the generating function $g(t)$ is written as $g(t)-1 \simeq te^{\overline{\gamma}t}$ then the Lagrange inversion formula (35) becomes

$$k\overline{\gamma}^{k-1} \simeq (-k.\overline{\alpha})^{k-1}$$

that is equivalent to the version given in [28] by using the Abel polynomial sequence and its delta operator.

References

1. Bell, E.T.(1927) Partition polynomials. Ann. Math. **24**, 38–46.
2. Bell, E.T.(1934) Exponential polynomials. Ann. Math. **35**, 258–277.
3. Bell, E.T.(1938) The iterated exponential integers. Ann. Math. **39**, 539–557.
4. Bell, E.T. (1938) The history of Blissard's symbolic method with a sketch of its inventor's life. Amer. Math. Monthly. **45**, 414–421.
5. Bell, E.T. (1940) Postulational bases for the umbral calculus. Amer. J. Math. **62**, 717–724.
6. Blissard, J. (1861) Theory of generic equations. Quart. J. Pure Appl. Math. **4**, 279–305.
7. Blissard, J. (1862) Theory of generic equations. Quart. J. Pure Appl. Math. **5**, 58–75, 185–208.
8. Blissard, J. (1862) Note on certain remarkable properties of numbers. Quart. J. Pure Appl. Math. **5**, 184.
9. Blissard, J. (1862) On the discovery and properties of a peculiar class of algebraic formulae. Quart. J. Pure Appl. Math. **5**, 325-335.
10. Blissard, J. (1864) Examples of the use and application of representative notation. Quart. J. Pure Appl. Math. **6**, 49–64.
11. Blissard, J. (1864) On the generalization of certain formulae investigated by Mr. Walton. Quart. J. Pure Appl. Math. **6**, 167–179.
12. Blissard, J. (1864) Researches in analysis. Quart. J. Pure Appl. Math. **6**, 142–257.
13. Blissard, J. (1867) On the properties of the $\Delta^m 0^n$ class of numbers and of others analogous to them, as investigated by means of representative notation. Quart. J. Pure Appl. Math. **8**, 85–110.
14. Blissard, J. (1868) On the properties of the $\Delta^m 0^n$ class of numbers. (continued from Vol. VIII - p.110). Quart. J. Pure Appl. Math. **9**, 82–94, 154–171.
15. Blissard, J. (1868) Note on a certain formula. Quart. J. Pure Appl. Math. **9**, 71–76.
16. Blissard, J. (1868) On certain properties of the gamma function. Quart. J. Pure Appl. Math. **9**, 280–296.
17. Cerasoli, M. (1979) Enumerazione binomiale e processi stocastici di Poisson composti. Bollettino U.M.I. no. 5, **16**-A, 310–315.
18. Di Bucchianico, A., Loeb, D. (1995) A selected Survey of Umbral Calculus. Electron. J. Combin. **2** Dynamic Survey 3, 28 pp. (updated 2000).
19. Feller, W. (1950) An introduction to probability theory and Its Applications. Vol. I. John Wiley & Sons, Inc., New York.
20. Feller, W. (1966) An introduction to probability theory and its applications. Vol. II. John Wiley & Sons, Inc., New York-London-Sydney.
21. Joyal, A. (1981) Une théorie combinatoire des séries formelles. Adv. in Math. **42**, 1–82.
22. Lucas, E. (1876) Théorie des Nombres. Gauthier-Villars, Paris.
23. Mullin, R., Rota, G.-C. (1970) Theory of binomial enumeration. In Graph Theory and its Applications. Academic Press. New York. 167–213.
24. Roman, S.M., Rota, G.-C. (1978) The Umbral Calculus. Advances in Mathematics **27**, 95–188.
25. Rota, G.-C. (1964) The number of partitions of a set. Amer. Math. Monthly. **71**, No. 5, 498-504.

26. Rota, G.-C., Kahaner D., Odlyzko, A. (1973) On the foundations of combinatorial theory. VIII. Finite Operator Calculus. Jour. Math. Anal. Appl. **42**, 684–760.
27. Rota, G.-C., Taylor, B.D. (1994) The classical umbral calculus. SIAM J. Math. Anal. **25**, No. 2, 694–711.
28. Rota, G.-C., Shen, J., Taylor, B. D. (1998) All polynomials of binomial type are represented by Abel polynomials. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **25**, No. 3–4, 731–738.
29. Riordan, J. (1968) Combinatorial Identities. John Wiley and Sons, Inc. New York.
30. Stam, A. J. (1988) Polynomials of binomial type and compound Poisson processes. Jour. Math. Anal. Appl. **130**, 493–508.
31. Steffensen, J. F. (1950) Interpolation, Chelsea, New York (reprinted from 1927).
32. Taylor, B.D.(1999) Umbral presentations for polynomial sequences. Preprint.
33. Touchard, J. (1956) Nombres exponentiels et nombres be Bernoulli, Canad. J. Math. **8**, 305–320.