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SOME RESULTS ON RELATIVE STABILITY OF DISCRETE DYNAMICAL SYSTEMS

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In this paper a new concept of relative stability is extended to linear non autonomous discrete dynamical systems and the behaviour of solutions is analyzed. Enlarging the validity field of some useful results for economic theory, conditions of relative stability for unidimensional and multidimensional discrete dynamical systems are supplied.

1. Introduction

Studies on multisectoral dynamical models, where properties of time paths involving initial conditions are investigated, have led to a consideration of balanced growth models, which also have a significant meaning from the point of view of economic theory. In order to develop this approach, the asymptotic behaviour of components of solution vectors, as time increases, should be considered. Hence, in economical context, we are concerned with the relative stability of a discrete dynamical system.

In previous studies [4, 12, 13, 16, 17, 5], definitions and results have referred to discrete dynamical systems, belonging to particular classes which require rather restrictive hypothesis.

In an attempt to supply a more general and systematic treatment of relative stability problems of dynamical systems, L. Peccati [14] introduced a concept of relative stability which is slightly more general than the ones employed previously, with significant analytical advantages. Some theorems were proved in a continuous setting, which cannot be immediately extended to the discrete situation.

The main aim of this paper is to extend the results on relative stability to discrete dynamical systems. We shall demonstrate that the relative stability properties of a given system are related to the properties of global stability(1) of solutions of another system, in the sense that they are logically equivalent. In order to develop the study of the problem along the aforementioned lines, we shall use both known and new results.

(1) In our context the notion of stability of a system is that of asymptotic stability.

In section 2, we give the definition of a relatively stable system, in its more general formulation, making some useful remarks, and prove the aforesaid equivalence. In n. 3, some theorems will be proved in the case of unidimensional dynamical systems, while, in n. 4, an interesting relative stability condition will be proved for n-dimensional linear systems. In our proofs, we use a theorem which, to our knowledge, has so far only been demonstrated for continuous systems. We therefore give the discrete version, together with its proof, in appendix.

2. Relative stability of discrete dynamical systems

Let the discrete dynamical system

$$\mathbf{x}(t+1) = \mathbf{G}(\mathbf{x}(t), t) \tag{2.01}$$

be defined on the time set $T = \{0, 1, 2, \ldots\}$, where $\mathbf{x}(t) \in R^n$, and $\mathbf{G} : R^n \times T \rightarrow R^n$ satisfies the conditions that guarantee the existence and uniqueness of the solution of the system(2): system (2.01) is said to be *relatively stable* if a non-null normalizing sequence $g : T \rightarrow R$ exists such that for any solution $\mathbf{x}(t)$ of system (2.01), we have

$$\lim_{t\to+\infty} \frac{1}{g(t)} \mathbf{x}(t) = \mathbf{p}$$

with $\mathbf{p} \in R^n$, non-null and independent of the solution(3).

As we have announced previously, we develop the following remarks in a way which is analogous to what has already been done in the continuous case. Calling the initial condition $\mathbf{x}^0 = \mathbf{x}(0)$ and the solution of system (2.01) $\boldsymbol{\phi}(t, \mathbf{x}^0)$, (which is fully determined by $\mathbf{x}^0$ and t), we may represent the solution in the following way

$$\boldsymbol{\phi}(t, \mathbf{x}^0) = g(t)\mathbf{p} + \mathbf{o}(g(t)),$$

when it is relatively stable; $\mathbf{o}(g(t))$ is a n-dimensional vector of functions which, multiplied by $1/g(t)$, vanishes as $t \rightarrow +\infty$. As in the continuos case, the dependence of $\boldsymbol{\phi}(t, \mathbf{x}^0)$ on initial condition $\mathbf{x}^0$ is confined to the second term, because the product $g(t)\mathbf{p}$ is independent of $\mathbf{x}^0$.

Moreover the normalizing sequence $g(t)$ is not unique. In fact, if another non-null normalizing sequence $\hat{g}(t)$ is considered, the solution of the system (2.01) becomes

$$\boldsymbol{\phi}(t, \mathbf{x}^0) = g(t)\hat{\mathbf{p}} + \mathbf{o}(\hat{g}(t))$$

and it follows easily the two vectors $\mathbf{p}$ and $\hat{\mathbf{p}}$ are proportional and the two sequences $g(t)$ and $\hat{g}(t)$ are asimptotically and antithetically proportional too (i.e.: if $\hat{\mathbf{p}} = \lambda\mathbf{p}$ then $\hat{g}(t)/g(t) \rightarrow 1/\lambda$).

If in system (2.01) we perform the transformation

$$\mathbf{y}(t) = \frac{1}{g(t)} \mathbf{x}(t) \tag{2.02}$$

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(2) P. Mazzoleni, [11]; L. Grippo - F. Lampariello, [9].

(3) L. Peccati, [14].

with $\mathbf{y} : T \to R^n$, we obtain the system

$$\mathbf{y}(t+1) = \mathbf{F}(\mathbf{y}(t), t) \tag{2.03}$$

where $\mathbf{F} : R^n \times T \to R^n$. Therefore, the relative stability of system (2.01) is equivalent to the existence of a sequence $g : T \to R$ which transforms this system into system (2.03) where all the solutions tend to one and the same non-null vector $\mathbf{p} \in R^n$ as $t \to +\infty$.

In this paper we intend to investigate the properties of linear discrete dynamical systems (not necessarily autonomous and not necessarily homogeneous)

$$\mathbf{x}(t+1) = A(t)\mathbf{x}(t) + \mathbf{b}(t) \tag{2.04}$$

where $A(t)$ is the square coefficient matrix and $\mathbf{b}(t)$ is a sequence in R^n. To establish whether system (2.04) is relatively stable, we shall use transformation (2.02) to obtain the new system

$$\mathbf{y}(t+1) = \left[\frac{g(t)}{g(t+1)} A(t)\right]\mathbf{y}(t) + \frac{1}{g(t+1)}\mathbf{b}(t) \tag{2.05}$$

and, under appropriate conditions on $A(t)$ and $\mathbf{b}(t)$, all the solutions of system (2.05) will be proved to tend to a non-null vector $\mathbf{p} \in R^n$.

Finally we observe that transformation (2.02) does not alter the structure of the original system (2.04) since the matrix $A(t)$ is substituted by matrix

$$\frac{g(t)}{g(t+1)} A(t)$$

and the vector $\mathbf{b}(t)$, by vector

$$\frac{1}{g(t+1)}\mathbf{b}(t).$$

Further, we shall characterize system (2.04) by the requirement that matrix $A(t)$ and sequence $\mathbf{b}(t)$ satisfy come conditions. In fact, our aim is to extend some results on relative stability (see for example [13]) where systems with $A(t) = A$ (constant) and $\mathbf{b}(t) = \varrho^t\, \boldsymbol{\beta}(\boldsymbol{\beta} \in R^n;\ \varrho \in R,\ \varrho > 0)$ are considered. In particular, we shall assume these following hypotheses:

- $A(t)$ is an *almost constant* non negative matrix, that is $\lim_{t\to+\infty} a_{ij}(t) = \hat{a}_{ij} \in R$(4), $\hat{a}_{ij} \geqq 0$, $\forall i, j = 1, \ldots, n$, where $a_{ij}(t)$ is the general element of the matrix $A(t)$ at time t;
- $\mathbf{b}(t)$ is *almost exponential*, that is $\mathbf{b}(t) = h(t)\boldsymbol{\beta}(t)$ where $\boldsymbol{\beta}(t)$ is non-null almost constant, $\lim_{t\to+\infty} \boldsymbol{\beta}(t) = \hat{\boldsymbol{\beta}} \in R^n$, $\hat{\boldsymbol{\beta}} \neq \mathbf{0}$, and $h(t) \in R$ with

$$\lim_{t\to+\infty} \frac{h(t+1)}{h(t)} = \hat{h} \in R,$$

being $h(t) > 0$ for any $t \geqq 0$(5).

(4) In the following we shall write: $\lim_{t\to+\infty} A(t) = \hat{A}$.

(5) In general $h(t)$ could have constant sign; the condition $h(t) > 0$ is not restrictive because $\boldsymbol{\beta}(t)$ can absorb the eventual negative sign of $h(t)$.

3. Unidimensional linear discrete dynamical systems

First of all we are going to prove a preliminary result. It will be the premise for a more important theorem which gives a necessary and sufficient condition for the relative stability of a difference equation.

THEOREM 1. *In the unidimensional system*

$$x(t+1) = a(t)x(t) + b(t) \tag{3.01}$$

let $a(t)$ *be almost constant and* $b(t)$ *almost exponential. If* $\hat{h} > \hat{a}$ *then system* (3.01) *is relatively stable.*

PROOF. Calling the initial condition $x^0 = x(0)$ and

$$P(i, j) = \prod_{k=i}^{j} a(k), \quad \text{if } j \geqq i, \; P(i, j) = 1 \text{ otherwise}$$

the solution of system (3.01) is

$$x(t) = P(0, t-1)x^0 + \sum_{j=0}^{t-1} [P(j+1, t-1)b(j)], \quad \text{for} \quad t \geqq 1.$$

As indicated previously, the normalizing sequence may be chosen as follows

$$g(t) = \sum_{j=0}^{t-1} [b(j)P(j+1, t-1)]. \tag{3.03}$$

Since the recurrent relation $g(t+1) = a(t)g(t) + b(t)$ holds between consecutive terms of the normalizing sequence, such sequence can be proved to be definitively non-null monotone (with increase or decrease depending on the value of $\hat{a} \gtreqless 1$). Indeed T exists such that for any $t > T$ sequence a(t) is positive and sequence $\beta(t)$ has the sign of $\hat{\beta}$, from which it follows that $g(t+1) > a(t)g(t)$, if $\hat{\beta} > 0$, $g(t+1) < a(t)g(t)$ otherwise.

Presently, we point out that system (3.01) is relatively stable for at least $\hat{a} = 0$. In fact, if we write the solution of system as follows

$$x(t) = P(0, t-1)x^0 + g(t), \quad \text{for} \quad t \geq 1,$$

we obtain

$$\lim_{t \to +\infty} x(t) = \lim_{t \to +\infty} g(t)$$

that is solution $x(t)$ and sequence $g(t)$ have the same asymptotic behaviour, because, as t diverges, the product $P(0, t-1)$ vanishes[6] and normalizing sequence $g(t)$ is monotone.

In particular, since

$$\lim_{t \to +\infty} g(t+1) = \lim_{t \to +\infty} h(t),$$

(6) See K. Knopp, [10], p. 93.

the behaviour of normalizing sequence $g(t)$ and, consequently, that of solution $x(t)$ will depend on the behaviour of sequence $h(t)$. Hence, the solution $x(t)$ of system (3.01), for $\hat{a} = 0$, is infinitesimal whenever $0 < \hat{h} < 1$, it is convergent for $\hat{h} = 1$ and, finally, it is divergent for $\hat{h} > 1$.

Considering T as above and the ratio $x(t)/g(t)$, $\forall t > T$, system (3.01) is relatively stable if the ratio

$$\frac{P(0, t-1)}{\sum_{j=0}^{t-1} [b(j)P(j+1, t-1)]}$$

vanishes as $t \to +\infty$, that is if either

$$\sum_{j=0}^{t-1} [\beta(j)h(j)/P(0, j)] \tag{3.04}$$

or, by the hypothesis on $\beta(j)$,

$$\sum_{j=0}^{t-1} [h(j)/P(0, j)]$$

tends to ∞ as $t \to +\infty$.

We can consider $P(0, j) \neq 0$, $j = 0, \dots, t-1$, because for $P(0, j) = 0$, i.e. at least a null term of sequence $a(k)$ exists, for $k = 0, \dots, j$, we would have $\lim_{t \to +\infty} P(0, t-1) = 0$ (7), from which $\lim_{t \to +\infty} x(t) = \lim_{t \to +\infty} g(t)$ and, consequently, the relative stability of system (3.01). It also follows it is not restrictive to assume $a(t) > 0$, $\forall t \geqq 0$, because if only one term of sequence $a(t)$ is null then system (3.01) is relatively stable.

Sum (3.04) is a partial sum of the series with j-th term

$$\beta(j)h(j)/P(0, j). \tag{3.05}$$

Since the ratio between consecutive terms of series (3.04) is

$$\frac{\beta(t+1)}{\beta(t)} \frac{h(t+1)}{h(t)} \frac{P(0, t)}{P(0, t+1)} = \frac{\beta(t+1)}{\beta(t)} \frac{h(t+1)}{h(t)} \frac{1}{a(t+1)}$$

and

$$\lim_{t \to +\infty} \frac{\beta(t+1)}{\beta(t)} \frac{h(t+1)}{h(t)} \frac{1}{a(t+1)} = \frac{\hat{h}}{\hat{a}} \tag{3.06}$$

series

$$\sum_{j=0}^{+\infty} [\beta(j)h(j)/P(0, j)]$$

has the same asymptotic behaviour as the geometric series with positive common ratio $\hat{h}/\hat{a}$. Hence, we conclude that series (3.04) diverges as $t \to +\infty$ because, by hypothesis, the common ratio of the asymptotic geometric series is greater than 1.

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(7) See K. Knopp, [10], p. 93.

The above theorem provides a sufficient condition for the relative stability of the system. Now a more general theorem, based upon the behaviour of series (3.04), may be proved:

THEOREM 2. *The system*

$$x(t+1) = a(t)x(t) + b(t)$$

with almost constant $a(t) \geqq 0$, $\forall t \geqq 0$, *almost exponential* $b(t)$ *and* $\hat{h} > \hat{a}$, *is relatively stable if and only if*

$$\sum_{j=0}^{t-1} [h(j)/P(0,j)] \tag{3.07}$$

diverges as $t \to +\infty$.

PROOF. Condition (3.07) is clearly sufficient because it is derived from theorem 1. To prove its necessity, we assume that the system is relatively stable, i.e. a non-null normalizing sequence $\gamma(t)$ exists, According to the solution $x(t)$ and the definition of the sequence $g(t)$, we find

$$\frac{x(t)}{\gamma(t)} = \frac{P(0,t-1)}{\gamma(t)} \mathbf{x}^0 + \frac{g(t)}{\gamma(t)}.$$

Relative stability of the system implies that the coefficient of $\mathbf{x}^0$ vanishes and $g(t)/\gamma(t) \to k \neq 0$ as $t \to +\infty$. Now, if $P(0, t-1)/\gamma(t)$ is multiplied and divided by $g(t)$, we obtain equivalently;

$$\lim_{t \to +\infty} \frac{P(0,t-1)}{g(t)} = 0.$$

or

$$\lim_{t \to +\infty} \frac{1}{\sum_{j=0}^{t-1} [\beta(j)h(j)/P(0,j)]} = 0.$$

Since the series

$$\sum_{j=0}^{+\infty} [h(j)/P(0,j)]$$

and

$$\sum_{j=0}^{+\infty} [\beta(j)h(j)/P(0,j)]$$

have the same divergent behaviour, the theorem is proved.

Moreover, under the hypothesis of this theorem, the normalizing sequences are characterized by this structure:

$$y(t) = k(t) \sum_{j=0}^{t-1} [h(j)P(j+1, t-1)]$$

where $k(t)$ is any non-infinitesimal convergent sequence as $t \to +\infty$ and

$$\sum_{j=0}^{t-1} [h(j)P(j+1, t-1)]$$

is a normalizing sequence, as it can be easily proved.

REMARKS. 1. From an applied point of view, as time increases, condition (3.07) implies that the increase coefficient $h(t)/h(t-1)$, in the forcing term, « dominates » the asymptotic coefficient of autonomous increase $a(t)$.

2. If $a(t) = a$ is a positive constant, the normalizing sequence

$$g(t) = \sum_{j=0}^{t-1} h(j) \prod_{i=j+1}^{t-1} a(i)$$

becomes

$$g(t) = \sum_{j=0}^{t-1} h(j)a^{(t-1)-j}.$$

The latter expression will be useful to identify the normalizing sequence for the n-dimensional case.

4. n-Dimensional linear discrete dynamical system

Let the *norm of vector* $\mathbf{x} \in R^n$, be the scalar quantity

$$|\mathbf{x}| = \sum_{i=1}^{n} |x_i|$$

where $|x_i|$ is the absolute value of the i-th component of vector $\mathbf{x}$ and let the *norm of matrix* A be the scalar quantity

$$|A| = \sum_{i,j=1}^{n} |a_{ij}|$$

where $|a_{ij}|$ is the absolute value of a_{ij}, the generic element of matrix A.
Let the dynamical system be

$$\mathbf{x}(t+1) = A(t)\mathbf{x}(t) + \mathbf{b}(t) \tag{4.01}$$

with the square matrix of coefficient $A(t)$ almost constant (that is $\lim_{t\to+\infty} A(t) = \hat{A}$), the vector $\mathbf{b}(t) = h(t)\boldsymbol{\beta}(t)$ almost exponential, where $\boldsymbol{\beta}(t) \in R^n$, $h(t) \in R$ and $\lim_{t\to+\infty} \boldsymbol{\beta}(t) = \hat{\boldsymbol{\beta}} \in R^n$, $\hat{\boldsymbol{\beta}} \neq \mathbf{0}$ and finally, $h(t) > 0$ with $\lim_{t\to+\infty} h(t+1)/h(t) = \hat{h}$. Calling $\hat{a}$ the spectral radius of $\hat{A}$, we also assume that $\hat{h} > \hat{a}$.
Choose the following non-null normalizing sequence $g(t) \in R$

$$g(t) = \sum_{j=0}^{t-1} h(j)\hat{a}^{t-1-j}.$$

If in system (4.01) the transformation $\mathbf{x}(t) = g(t)\mathbf{y}(t)$ is performed, we obtain

$$\mathbf{y}(t+1) = \frac{A(t)g(t)}{h(t) + \hat{a}g(t)}\mathbf{y}(t) + \frac{h(t)}{h(t) + \hat{a}g(t)}\boldsymbol{\beta}(t).$$

Let us call:

$$\varrho(t) = \frac{h(t)}{g(t)} + \hat{a},$$

where $\lim_{t\to+\infty} \varrho(t) = \hat{h}$[8]. We find the transformed system

$$\mathbf{y}(t+1) = \frac{A(t)}{\varrho(t)}\mathbf{y}(t) + \left[1 - \frac{\hat{a}}{\varrho(t)}\right]\boldsymbol{\beta}(t). \tag{4.02}$$

As t diverges, the coefficient in the first term and the whole known term approach, respectively, the coefficient and the known term of the autonomous system

$$\mathbf{z}(t+1) = \frac{\hat{A}}{\hat{h}}\mathbf{z}(t) + \left[1 - \frac{\hat{a}}{\hat{h}}\right]\boldsymbol{\beta} \tag{4.03}$$

which obviously admits an equilibium point $\mathbf{z}^*$; i.e. a constant solution $\mathbf{z}(t) = \mathbf{z}^*$ for all t. In order to find $\mathbf{z}^*$, in the equation (4.03) it is sufficient to replace $\mathbf{z}(t)$ and $\mathbf{z}(t+1)$ with $\mathbf{z}^*$, so we obtain

$$\left[\frac{\hat{A}}{\hat{h}} - I\right]\mathbf{z}^* + \left[1 - \frac{\hat{a}}{\hat{h}}\right]\boldsymbol{\beta} = 0. \tag{4.04}$$

Since $\hat{h} > \hat{a}$, matrix

$$\left[I - \frac{\hat{A}}{\hat{h}}\right]$$

is non-negatively invertible, so that:

$$\mathbf{z}^* = \left[1 - \frac{\hat{a}}{\hat{h}}\right]\left[I - \frac{\hat{A}}{\hat{h}}\right]^{-1}\boldsymbol{\beta}$$

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(8) If we consider ratio

$$h(t)/g(t) = h(t)\hat{a}^{-t+1}\Big/\left[\sum_{j=0}^{t-1} h(j)\hat{a}^{-j}\right],$$

under assumption $\hat{h} > \hat{a}$, it follows that denominator diverges as $t \to +\infty$. Now, since ratio

$$\frac{h(t)\hat{a}^{-t+1} - h(t-1)\hat{a}^{-t+2}}{\sum_{j=0}^{t-1} h(j)\hat{a}^{-j} - \sum_{j=0}^{t-2} h(j)\hat{a}^{-j}} = \frac{h(t)}{h(t-1)} - \hat{a}$$

tends to $\hat{h} - \hat{a}$, as $t \to +\infty$, for the analogous De L'Hospital's theorem, which is true in discrete case (See K. Knopp [10], p. 35), we can conclude that $\lim_{t\to+\infty} h(t)/g(t) = \hat{h} - \hat{a}$, from which thesis follows.

and because $\beta \neq 0$, $z^* \neq 0$. Moreover z^* appears to be globally stable for the system since the eigenvalues of matrix $\hat{A}/\hat{h}$ are less than 1 in modulus.

If we prove that all solutions of system (4.02) tend to z^*, then the original system is consequently proved to be relatively stable. As we said in the second section, we show that the solutions of system (4.02) tend to one and the same non-null vector as t diverges, that is, for any solution $y(t)$ of system (4.02),

$$\lim_{t \to +\infty} y(t) = z^*.$$

For this purpose, we define:

$$v(t) = y(t) - z^*$$

and through substitution in (4.02) we obtain:

$$v(t+1) + z^* = \frac{A(t)}{\varrho(t)} [v(t) + z^*] + \left[1 - \frac{\hat{a}}{\varrho(t)}\right] \beta(t). \tag{4.05}$$

Now, remembering the definition of z^* (in particular equation (4.04)), we get:

$$v(t+1) = \frac{A(t)}{\varrho(t)} v(t) + \left[\frac{A(t)}{\varrho(t)} - \frac{\hat{A}}{\hat{h}}\right] z^* + \left[1 - \frac{\hat{a}}{\varrho(t)}\right] \beta(t) - \left[1 - \frac{\hat{a}}{\hat{h}}\right] \beta. \tag{4.06}$$

System (4.05) has been rewritten as system (4.06) in order to show that system (4.06) admits infinitesimal solutions, at t diverges.

For this purpose we consider the homogeneous system

$$w(t+1) = \frac{A(t)}{\varrho(t)} w(t) \tag{4.07}$$

and the system

$$w(t+1) = \frac{\hat{A}}{\hat{h}} w(t) \tag{4.08}$$

with (constant) coefficient matrix equal to the limit of the corresponding coefficient matrix of system (4.07). All the solutions of system (4.08) are infinitesimal since, by hypothesis, the eigenvalues of the coefficient matrix are all less than 1 in modulus. By comparison between systems, (4.07) and (4.08), it follows at once[9] that all the solutions of system (4.07) are infinitesimal. Now, if we assume that a constant M exists such that

$$\left| \prod_{k=t-1}^{s+1} \frac{A(k)}{\varrho(k)} \right| \leq M,$$

(9) Based on an easy theorem proved for continuous dynamical systems with almost constant coefficients, reported in Bellman [2], (theorem 2, p. 36), the discrete version of which can be found in appendix.

for any $0 \leqq s \leqq t$, $s, t \in T$ it follows that the equilibrium point of system (4.06) is also uniformly stable (theorem 2.14 of P. Mazzoleni [11]). From this and under the assumption that

$$\left[\frac{A(t)}{\varrho(t)} - \frac{\hat{A}}{\hat{h}}\right]$$

and

$$\left[\left(1 - \frac{\hat{a}}{\varrho(t)}\right)\beta(t) - \left(1 - \frac{\hat{a}}{\hat{h}}\right)\beta\right]$$

are limited it follows all solutions of system (4.06) vanish, as $t \to +\infty$, because of theorem 2.30 of P. Mazzoleni [11][10]. By this way we have proved the following theorem:

THEOREM 3. *Let the system be*

$$\mathbf{x}(t+1) = A(t)\mathbf{x}(t) + \mathbf{b}(t)$$

with the matrix of coefficients $A(t)$ almost constant, the vector $b(t)$ almost exponential and $\hat{h} > \hat{a}$, where $\hat{a}$ is the spectral radius of A. Then the system is relatively stable if a positive constant M exists such that

$$\left|\prod_{k=t-1}^{s+1} \frac{A(k)}{\varrho(k)}\right| \leqq M,$$

for $0 \leqq s \leqq t$, $s, t \in T$, and the quantities

$$\left[\frac{A(t)}{\varrho(t)} - \frac{\hat{A}}{\hat{h}}\right] \quad \text{and} \quad \left[\left(1 - \frac{\hat{a}}{\varrho(t)}\right)\beta(t) - \left(1 - \frac{\hat{a}}{\hat{h}}\right)\beta\right]$$

are limited.

APPENDIX

Writing system (4.07) as follows

$$\mathbf{w}(t+1) = \left[\frac{\hat{A}}{\hat{h}} + \left(\frac{A(t)}{\varrho(t)} - \frac{\hat{A}}{\hat{h}}\right)\right]\mathbf{w}(t)$$

we prove the

THEOREM A. *Consider the system*

(1) $$y(t+1) = Ay(t)$$

(10) Let the system be $\mathbf{x}(t+1) = A(t)\mathbf{x}(t) + \mathbf{b}(t)$ then « If the equilibrium of homogeneus system is uniformly asymptotically stable and if $\mathbf{b}(t)$ is limited with $\lim_{t\to+\infty} \mathbf{b}(t) = \mathbf{0}$, then all solutions of non-homogeneous system vanish as $t \to +\infty$ (i.e. quasi-asymptotical stability) ».

with the square matrix A non-negative of order n. If all the solutions of the system (1) are infinitesimal for $t \to +\infty$, there exists a constant $c > 0$, depending upon A, such that the solutions of the system

$$\mathbf{z}(t+1) = [A + B(t)]\mathbf{z}(t) \tag{2}$$

are infinitesimal if the elements of $B(t)$ are definitively bounded in modulus by c; i.e. T exists such that for $t \geqq T$, $| b_{rs}(t) | \leqq c$, $\forall r, s = 1, \dots, n$, where $| b_{rs}(t) |$ are the moduli of the elements of $B(t)$.

PROOF. Since, by hypothesis, the solutions of system (1) are infinitesimal, the spectral radius of A is less than 1. Let E be the square matrix of order n, with all elements equal to 1. We choose a constant $c > 0$ such that the spectral radius of the matrix $(A + cE)$ is less than 1(11). Obviously, the choice of constat c depends upon matrix A.

The general solution of system (2) is

$$\mathbf{z}(t) = \prod_{s=T}^{t-1} [A + B(s)]\mathbf{z}(T)$$

where

$$\mathbf{z}(T) = \prod_{s=0}^{T-1} [A + B(s)]\mathbf{z}^0$$

and $\mathbf{z}^0$ is the initial position of the state vector.

Let $p_{rs}(t)$ be the elements of the matrix $(A + cE)^{t-T+1}$ and $\pi_{rs}(t)$ be those of the matrix

$$\prod_{s=T}^{t-1} [A + B(s)].$$

Since $| b_{rs}(t) | \leqq c$, $\forall t \geqq T$, $\forall r, s = 1, \dots, n$, we obtain:

$$| \pi_{rs}(t) | \leqq p_{rs}(t), \qquad \forall t \geqq T, r, s = 1, \dots, n$$

as it can easily proved by induction on t.

Now, owing to the fact that $\lim_{t \to +\infty} p_{rs}(t) = 0$, $\forall r, s = 1, \dots, n$, from a comparison theorem it follows that $\lim_{t \to +\infty} \pi_{rs}(t) = 0$, $\forall r, s = 1, \dots, n$ and hence that

$$\prod_{s=T}^{t} [A + B(s)]$$

is infinitesimal for $t \to +\infty$, from which our thesis follows.

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(11) This is possible because the roots of a polynomial are continuous functions of the coefficients of polynomial; see for example A. S. Householder *The numerical treatment of a single non-linear equation*, McGraw-Hill, New York, 1970.

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RIASSUNTO

Dato il sistema dinamico discreto

$$\mathbf{x}(t+1) = \mathbf{G}(\mathbf{x}(t), t) \tag{1}$$

definito sull'insieme temporale $T = \{0, 1, 2, \ldots\}$, dove $\mathbf{x}(t) \in R^n$, e $\mathbf{G} : R^n \times T \to R^n$ soddisfacente le condizioni che garantiscono l'esistenza e unicità della soluzione del sistema, diciamo che il sistema (1) è *relativamente stabile* se esiste una successione normaliz-

zante, non nulla, $g : T \to R$ tale che per ogni soluzione $\mathbf{x}(t)$ si ha:

$$\lim_{t \to +\infty} \frac{1}{g(t)} \mathbf{x}(t) = \mathbf{p}$$

con $\mathbf{p} \in R^n$, non nullo e indipendente dalla soluzione. La relativa stabilità del sistema (1) è equivalente alla stabilità globale del sistema ottenuto da (1) mediante la trasformazione $\mathbf{x}(t) = g(t)\mathbf{y}(t)$.

In questa nota studiamo inoltre la stabilità relativa dei sistemi dinamici lineari discreti del tipo:

$$\mathbf{x}(t + 1) = A(t)\mathbf{x}(t) + \mathbf{b}(t)$$

sotto ipotesi meno restrittive di quelle finora incontrate in letteratura. Precisamente si considerano:

— $A(t)$ matrice *quasi costante* non negativa, cioè $\lim_{t \to +\infty} a_{ij}(t) = \hat{a}_{ij} \in R$, $\hat{a}_{ij} \geqq 0$, $\forall i, j = 1, \dots, n$, dove $a_{ij}(t)$ è il generico elemento della matrice $A(t)$ all'istante t;

— $\mathbf{b}(t)$ vettore *quasi esponenziale*, cioè $\mathbf{b}(t) = h(t)\boldsymbol{\beta}(t)$ dove $\boldsymbol{\beta}(t)$ è non nullo quasi costante, $\lim_{t \to +\infty} \boldsymbol{\beta}(t) = \hat{\boldsymbol{\beta}} \in R$, $\hat{\boldsymbol{\beta}} \neq \mathbf{0}$, dove $h(t) \in R$ e:

$$\lim_{t \to +\infty} \frac{h(t + 1)}{h(t)} = \hat{h} \in R,$$

con $h(t) > 0$ per ogni $t \geqq 0$.

Sotto queste ed altre ipotesi sul comportamento delle successioni $a(t)$ e $h(t)$, al divergere di t, viene fornita una condizione necessaria e sufficiente di stabilità relativa per sistemi unidimensionali. Per i sistemi multidimensionali è garantita la stabilità relativa se vengono aggiunte alcune ipotesi sui comportamenti iniziali della matrice $A(t)$ e del vettore $\mathbf{b}(t)$.

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ВВЕДЕНИЕ В АНАЛИЗ И НЕКОТОРЫЕ СПЕЦИАЛЬНЫЕ ВОПРОСЫ АНАЛИЗА

К. **Курс высшей математики. Том. 2.** Lehrder höheren Mathematik. Teil 2. Smirnow W. I. aus Russ. 16 aufl. Berlin: Dtsch. Verl. Wiss., S. (нем.) ISBN 3—326—00029—4 книги (РЖМат, 1966, 9А93К).

К. **Экстремум.** Extrema. Quaisser E., gel H.-J. Berlin: Dtsch. Verl. Wiss., 1986. (нем.) ISBN 3—326—00010—3

посвящена некоторым вопросам анализа и , связанным с нахождением минимума или задашюй величины. После некоторых сведений о возникновении понятия экс, авторы рассматривают следующие вопросы. трические экстремальные задачи. 2. Экстреоции. 3. Неравенства и экстремум. 4. Экстрезадачи высшей математики. 5. Задачи оптиуправления. Книга будет полезна школьниших классов и студентам первых курсов.

С. Скороходов

Теоремы о монотонности и трансцендентчисла e. Monotoniesatz und Transzendenz von e. er H. «Math. Semesterber.», 1986, 33, № 1, (нем.)

аются различные вопросы, связанные со свой монотонности функций, их значений и дифальном и интегральном исчислении. В качестприменений этих результатов доказаны свойства дентности числа e и иррациональности чис-

С. Скороходов

Асимптотическое поведение одной комплекспоследовательности. Asymptotic behaviour of some complex sequences. Simič Slavko. «Publ. Inst. math.», 1986, 39, 119—128 (англ.)

Рассматривается последовательность функций

$$f_n(z,\alpha,\beta)=\sum_{k=0}^{n}\binom{n}{k}\frac{z^k}{(k+\beta)^\alpha},\quad n=1,2,\ldots,\qquad (1)$$

где α и β принимают положительные действительные значения, а z — комплексные значения. В работе изучается асимптотическое поведение последовательности (1) при $n\to\infty$. Для этого сначала найдено интегральное представление

$$f_n(z,\alpha,\beta)=\frac{1}{\Gamma(\alpha)}\int_0^\infty x^{\alpha-1}e^{-\beta x}(1+ze^{-x})^n dx.$$

Далее при помощи этого представления найдены асимптотические формулы для случаев $|z+1|>1$, $|z+1|<1$ и $|z+1|=0$, $z\neq 0$. Во всех этих формулах фигурирует гамма-функция $\Gamma(\alpha)$. С. Скороходов

5 Б5. **Неравенства, связанные с обобщенными симметрическими средними.** Inequalities involving generalized symmetric means. Neuman Edward. «J. Math. Anal. and Appl.», 1986, 120, № 1, 315—320 (англ.)

Пусть $t_0, t_1, \ldots, t_k$ — неотрицательные действительные числа;

$$h_n(t_0,t_1,\ldots,t_k)=\binom{n+k}{k}^{-1}\sum_{i_0+i_1+\ldots+i_k=n} t_0^{i_0}t_1^{i_1}\ldots t_k^{i_k}$$ —

n-е симметрическое среднее чисел $t_0, t_1, \ldots, t_k$.

Основной результат (теорема 2.1): если f — выпуквыбираются те пары (s_i, d_i), у которых $\mathrm{Re}\, s_i = 0$. Тогда

$$h_{max} = \min_{h>0} \{h \mid d_i = e^{-h s_i}\}.$$

А. Кирьянен

5 Б266. **Некоторые результаты об относительной устойчивости дискретных динамических систем.** Some results on relative stability of discrete dynamical systems. Uberti Mariacristina. «Riv. mat. sci. econ. e soc.», 1986, 9, № 1, 95—107 (англ.; рез. ит.)

Линейная дискретная система

$$x(t+1) = A(t)x(t) + b(t), \quad x \in R^n, \quad t \in T = \{0, 1, \ldots\}, \quad (1)$$

называется относительно устойчивой, если существуют ненулевая последовательность $g: T \to R$ и ненулевой вектор p такие, что

$$\lim x(t)/g(t) = p \text{ при } t \to \infty$$

для любого решения $x(t)$ системы (1). Предполагается, что элементы матрицы $A(t)$ неотрицательны и стремятся к константам при $t \to +\infty$, а $b(t) = h(t)\beta(t)$, где при $t \to \infty$

$$\beta(t) \to \beta \in R^n, \quad h(t+1)/h(t) \to h \in R.$$

При этих предположениях получены достаточные условия относительной устойчивости. С. Пилюгин

5 Б267. **Колеблемость в линейных системах дифференциально-разностных уравнений произвольного порядка.** Oscillations in a linear system of differential-difference equations of arbitrary order. Gopalsamy K. «J. Math. Anal. and Appl.», 1986, 120, № 1, 360—369 (англ.)

Приводятся достаточные условия колеблемости хотя бы одной компоненты ограниченного решения системы

$$x_i^{(m)}(t) = \sum_{j=1}^{n} a_{ij} x_j (t - \tau_{ij}), \quad i = 1, \ldots, n,$$

где $a_{ij}, \tau_{ij} \in R$, $\tau_{ij} > 0$, $i, j = 1, \ldots, n$. Т. Чантурия

5 Б268. **Колеблемость дифференциальных уравнений первого порядка с запаздыванием нейтрального типа.** Oscillations of first-order neutral delay differential equations. Grammatikopoulos M. K., Grove E. A., Ladas G. «J. Math. Anal. and Appl.», 1985, 120, № 2, 510—520 (англ.)

Исследуется вопрос о поведении на бесконечности решений уравнения

$$[y(t) + py(t-\tau)]' + Q(t)y(t-\sigma) = 0,$$

где $p < -1$ или $p > 0$, $\tau > \sigma > 0$, $Q \in C[t_0, \infty)$, $Q(t) \geq q > 0$ для достаточно больших t. Т. Чантурия

5 Б269. **Колебания двумерных линейных систем.** Oscillations in two-dimensional linear systems. Folyńska Izabela. «Found. Contr. Eng.», 1986, 11, № 4, 149—156 (англ.)

Рассматривается линейная 2-D система

$$\begin{pmatrix} x^h(i+1, j) \\ x^v(i, j+1) \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x^h(i, j) \\ x^v(i, j) \end{pmatrix} \begin{matrix} x^h \in R^{n_1}, \\ x^v \in R^{n_2}, \\ i, j = 0, 1, 2, \ldots, \end{matrix}$$

со стандартным краевым условием

$$x^h(0, j) = a^h(j), \quad x^v(i, 0) = a^v(i), \quad i, j \geq 0.$$

Решения системы называются колебательными, если существует такое k, $1 \leq k \leq n_1$, что

$$x_k^h(i, j) \cdot x_k^h(i+1, j) < 0,$$

или существует такое l, $1 \leq l \leq n_2$, что

$$x_l^v(i, j) \cdot x_l^v(i, j+1) < 0.$$

Получено достаточное условие, при котором ... системы колебательное. С. ...

5 Б270. **Периодические решения разностных ... нений конечного и бесконечного порядков.** ... ков Я. В., Аксиев А. З. «Изв. АН Ки... 1986, № 5, 9—13 (рус.)

Установлены условия существования перио... решений однородной и неоднородной линейных ... разностных уравнений с конечным и беск... последействием.

5 Б271. **Асимптотики периодических решен... нейных разностных уравнений.** Быков Я. В., ... сиев А. З. «Изв. АН КиргССР», 1986, № ... (рус.)

Установлены достаточные условия сущес... периодических и особых периодических реш... линейных разностных уравнений.

5 Б272. **Зависимость от параметра и струк... шений периодических уравнений с запазд...** Parameter distribution and structure of solu... periodic delay equation. Zhang Shunian... ... «Chin. Ann. Math.», 1986, A7, № 5, ... (кит.)

Речь идет об уравнении вида

$$x'(t) = q(t)x(t) - p(t)x(t-\tau), \quad t \geq 0,$$

где p и q — непрерывные ω-периодические ф... $\tau = k\omega$, k — целое.

5 Б273. **Теорема единственности типа теор... ке для одного класса функционально-дифф... ных уравнений нейтрального типа.** Uniquenes... with Kamke's form in a class of neutral func... ferential equations. Zheng Yong. «Шусюэ... Chin. Ann. Math.», 1986, A7, № 5, 519—527...

5 Б274 ДЕП. **Краевые задачи для функц... дифференциального уравнения с нелинейным... нием решения в левую часть.** Брыкалов... Ин-т мат. и мех. Урал. науч. центра АН СССР... ловск, 1987, 33 с., ил. Библиогр. 12 назв. ... копись деп. в ВИНИТИ 28.01.87, № 664—В8...

Изучается существование решений краевы... для функционально-дифференциальных (в ... обыкновенных дифференциальных) уравне... сматриваемые задачи содержат нелинейную ... мость от искомого решения в уравнении и ... условиях.

Из реферат...

5 Б275 ДЕП. **Существование решений кра... дачи с отклоняющимся аргументом в случа... венности решения вспомогательной задачи.** Б... лов С. А.; Ин-т мат. и мех. Урал. науч. це... СССР. Свердловск, 1987, 21 с. Библиогр. 8 н... (Рукопись деп. в ВИНИТИ 28.01.87, № 66...

Изучаются нелинейные краевые задачи д... ционально-дифференциальных уравнений. П... признаки существования решения.

Из реферат...

5 Б276 ДЕП. **О существовании и единст... решения одной специальной задачи для ... нально-дифференциальных уравнений n-го ...** Мамедов Э. А., Шамилов А. Х.; Ин-т ... мех. АН АзССР. Баку, 1987, 30 с. Библиогр. ... Рус. (Рукопись деп. в ВИНИТИ 28.01.87, ... В87)

Доказаны теоремы существования и единст... решения одной специальной задачи, являюще... логом классической баллистической задачи, ... ционально-дифференциальных уравнений n-го ...

Из реферата...

5 Б277. **Асимптотически постоянные реше... гулярно возмущенных дифференциально-ра... уравнений.** Иванов А. Ф. «Ин-т мат. АН ... Препр.», 1986, № 59, 16 с., ил. (рус.)

5 Б266. Некоторые результаты об относительной устойчивости дискретных динамических систем. Some results on relative stability of discrete dynamical systems. Uberti Mariacristina. «Riv. mat. sci. econ. e soc.», 1986, 9, № 1, 95—107 (англ.; рез. ит.)

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