## Extensions of Picard stacks and their homological interpretation

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# EXTENSIONS OF PICARD STACKS AND THEIR HOMOLOGICAL INTERPRETATION 

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#### Abstract

Let $\mathbf{S}$ be a site. We introduce the notion of extensions of strictly commutative Picard S-stacks. We define the pull-back, the push-down, and the sum of such extensions and we compute their homological interpretation: if $\mathcal{P}$ and $\mathcal{Q}$ are two strictly commutative Picard $\mathbf{S}$-stacks, the equivalence classes of extensions of $\mathcal{P}$ by $\mathcal{Q}$ are parametrized by the cohomology group $\operatorname{Ext}^{1}([\mathcal{P}],[\mathcal{Q}])$, where $[\mathcal{P}]$ and $[\mathcal{Q}]$ are the complex associated to $\mathcal{P}$ and $\mathcal{Q}$ respectively.


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## Introduction

Let $\mathbf{S}$ be a site. Let $\mathcal{P}$ and $\mathcal{Q}$ two strictly commutative Picard $\mathbf{S}$-stacks. We define an extension of $\mathcal{P}$ by $\mathcal{Q}$ as a strictly commutative Picard $\mathbf{S}$-stack $\mathcal{E}$, two additive functors $I: \mathcal{Q} \rightarrow \mathcal{E}$ and $J: \mathcal{E} \rightarrow \mathcal{P}$, and an isomorphism of additive functors $J \circ I \cong 0$, such that the following equivalent conditions are satisfied:

- $\pi_{0}(J): \pi_{0}(\mathcal{E}) \rightarrow \pi_{0}(\mathcal{P})$ is surjective and $I$ induces an equivalence of strictly commutative Picard $\mathbf{S}$-stacks between $\mathcal{Q}$ and $\operatorname{ker}(J)$,
- $\pi_{1}(I): \pi_{1}(\mathcal{Q}) \rightarrow \pi_{1}(\mathcal{E})$ is injective and $J$ induces an equivalence of strictly commutative Picard $\mathbf{S}$-stacks between $\operatorname{coker}(I)$ and $\mathcal{P}$.
By [D73] §1.4 there is an equivalence of categories between the category of strictly commutative Picard $\mathbf{S}$-stacks and the derived category $\mathcal{D}^{[-1,0]}(\mathbf{S})$ of complexes $K$ of abelian sheaves on $\mathbf{S}$ such that $\mathrm{H}^{i}(K)=0$ for $i \neq-1$ or 0 . Via this equivalence, the above notion of extension of strictly commutative Picard $\mathbf{S}$-stacks furnishes the notion of extension in the derived category $\mathcal{D}^{[-1,0]}(\mathbf{S})$. Let $K$ and $L$ be two

[^0]complexes of $\mathcal{D}^{[-1,0]}(\mathbf{S})$. In this paper we prove that, as for extensions of abelian sheaves on $\mathbf{S}$, extensions of $K$ by $L$ are parametrized by the cohomology group $\operatorname{Ext}^{1}(K, L)$.

More precisely, the extensions of $\mathcal{P}$ by $\mathcal{Q}$ form a 2 -category $\operatorname{Ext}(\mathcal{P}, \mathcal{Q})$ where

- the objects are extensions of $\mathcal{P}$ by $\mathcal{Q}$,
- the 1 -arrows are additive functors between extensions,
- the 2 -arrows are morphisms of additive functors.

Equivalence classes of extensions of strictly commutative Picard S-stacks are endowed with a group law. We denote by $\mathcal{E} x t^{1}(\mathcal{P}, \mathcal{Q})$ the group of equivalence classes of objects of $\mathcal{E} x t(\mathcal{P}, \mathcal{Q})$, by $\mathcal{E} x t^{0}(\mathcal{P}, \mathcal{Q})$ the group of isomorphism classes of arrows from an object of $\mathcal{E} x t(\mathcal{P}, \mathcal{Q})$ to itself, and by $\mathcal{E} x t^{-1}(\mathcal{P}, \mathcal{Q})$ the group of automorphisms of an arrow from an object of $\mathcal{E} x t(\mathcal{P}, \mathcal{Q})$ to itself. With these notation our main Theorem is

Theorem 0.1. Let $\mathcal{P}$ and $\mathcal{Q}$ be two strictly commutative Picard $\boldsymbol{S}$-stacks. Then we have the following isomorphisms of groups
a: $\mathcal{E x t} t^{1}(\mathcal{P}, \mathcal{Q}) \cong \operatorname{Ext}^{1}([\mathcal{P}],[\mathcal{Q}])=\operatorname{Hom}_{\mathcal{D}(S)}([\mathcal{P}],[\mathcal{Q}][1])$,
b: $\mathcal{E} x t^{0}(\mathcal{P}, \mathcal{Q}) \cong \operatorname{Ext}^{0}([\mathcal{P}],[\mathcal{Q}])=\operatorname{Hom}_{\mathcal{D}(S)}([\mathcal{P}],[\mathcal{Q}])$,
c: $\mathcal{E} x t^{-1}(\mathcal{P}, \mathcal{Q}) \cong \operatorname{Ext}^{-1}([\mathcal{P}],[\mathcal{Q}])=\operatorname{Hom}_{\mathcal{D}(S)}([\mathcal{P}],[\mathcal{Q}][-1])$.
where $[\mathcal{P}]$ and $[\mathcal{Q}]$ denote the complex of $\mathcal{D}^{[-1,0]}(\boldsymbol{S})$ corresponding to $\mathcal{P}$ and $\mathcal{Q}$ respectively.

This paper is organized as followed: in Section 1 we recall some basic results on strictly commutative Picard $\mathbf{S}$-stacks. In Section 2 we introduce the notions of fibered product and fibered sum of strictly commutative Picard S-stacks. In Section 3 we define extensions of strictly commutative Picard S-stacks and morphisms between such extensions. The results of Section 2 will allow us to define a group law for equivalence classes of extensions of strictly commutative Picard S-stacks (Section 4). Finally in Section 5 we prove the main Theorem 0.1

We finish recalling that the non-abelian analogue of $\S 3$ has been developed by Breen (see in particular [B90, B92]), by A. Rousseau in (R03], by Aldrovandi and Noohi in AN09 and by A. Yekutieli in Y10.

## Acknowledgment

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## Notation

Let $\mathbf{S}$ be a site. Denote by $\mathcal{K}(\mathbf{S})$ the category of complexes of abelian sheaves on the site $\mathbf{S}$ : all complexes that we consider in this paper are cochain complexes. Let $\mathcal{K}^{[-1,0]}(\mathbf{S})$ be the subcategory of $\mathcal{K}(\mathbf{S})$ consisting of complexes $K=\left(K^{i}\right)_{i}$ such that $K^{i}=0$ for $i \neq-1$ or 0 . The good truncation $\tau_{\leq n} K$ of a complex $K$ of $\mathcal{K}(\mathbf{S})$ is the following complex: $\left(\tau_{\leq n} K\right)^{i}=K^{i}$ for $i<n,\left(\tau_{\leq n} K\right)^{n}=\operatorname{ker}\left(d^{n}\right)$ and $\left(\tau_{\leq n} K\right)_{i}=0$ for $i>n$. For any $i \in \mathbb{Z}$, the shift functor $[i]: \mathcal{K}(\mathbf{S}) \rightarrow \mathcal{K}(\mathbf{S})$ acts on a complex $K=\left(K^{n}\right)_{n}$ as $(K[i])^{n}=K^{i+n}$ and $d_{K[i]}^{n}=(-1)^{i} d_{K}^{n+i}$.

Denote by $\mathcal{D}(\mathbf{S})$ the derived category of the category of abelian sheaves on $\mathbf{S}$, and let $\mathcal{D}^{[-1,0]}(\mathbf{S})$ be the subcategory of $\mathcal{D}(\mathbf{S})$ consisting of complexes $K$ such that $\mathrm{H}^{i}(K)=0$ for $i \neq-1$ or 0 . If $K$ and $K^{\prime}$ are complexes of $\mathcal{D}(\mathbf{S})$, the group
$\operatorname{Ext}^{i}\left(K, K^{\prime}\right)$ is by definition $\operatorname{Hom}_{\mathcal{D}(\mathbf{S})}\left(K, K^{\prime}[i]\right)$ for any $i \in \mathbb{Z}$. Let RHom(,--$)$ be the derived functor of the bifunctor $\operatorname{Hom}(-,-)$. The cohomology groups $\mathrm{H}^{i}\left(\mathrm{RHom}\left(K, K^{\prime}\right)\right)$ of $\mathrm{RHom}\left(K, K^{\prime}\right)$ are isomorphic to $\operatorname{Hom}_{\mathcal{D}(\mathbf{S})}\left(K, K^{\prime}[i]\right)$.

A 2-category $\mathcal{A}=\left(A, C(a, b), K_{a, b, c}, U_{a}\right)_{a, b, c \in A}$ is given by the following data:

- a set $A$ of objects $a, b, c, \ldots$;
- for each ordered pair $(a, b)$ of objects of $A$, a category $C(a, b)$;
- for each ordered triple $(a, b, c)$ of objects $A$, a functor $K_{a, b, c}: C(b, c) \times$ $C(a, b) \longrightarrow C(a, c)$, called composition functor. This composition functor have to satisfy the associativity law;
- for each object $a$, a functor $U_{a}: \mathbf{1} \rightarrow C(a, a)$ where $\mathbf{1}$ is the terminal category (i.e. the category with one object, one arrow), called unit functor. This unit functor have to provide a left and right identity for the composition functor.
This set of axioms for a 2-category is exactly like the set of axioms for a category in which the arrows-sets $\operatorname{Hom}(a, b)$ have been replaced by the categories $C(a, b)$. We call the categories $C(a, b)$ (with $a, b \in A$ ) the categories of morphisms of the 2-category $\mathcal{A}$ : the objects of $C(a, b)$ are the 1 -arrows of $\mathcal{A}$ and the arrows of $C(a, b)$ are the $\mathcal{2}$-arrows of $\mathcal{A}$.

Let $\mathcal{A}=\left(A, C(a, b), K_{a, b, c}, U_{a}\right)_{a, b, c \in A}$ and $\mathcal{A}^{\prime}=\left(A^{\prime}, C\left(a^{\prime}, b^{\prime}\right), K_{a^{\prime}, b^{\prime}, c^{\prime}}, U_{a^{\prime}}\right)_{a^{\prime}, b^{\prime}, c^{\prime} \in A^{\prime}}$ be two 2-categories. A 2-functor (called also a morphism of 2-categories)

$$
\left(F, F_{a, b}\right)_{a, b \in A}: \mathcal{A} \longrightarrow \mathcal{A}^{\prime}
$$

consists of

- an application $F: A \rightarrow A^{\prime}$ between the objects of $\mathcal{A}$ and the objects of $\mathcal{A}^{\prime}$,
- a family of functors $F_{a, b}: C(a, b) \rightarrow C(F(a), F(b))$ (with $a, b \in A$ ) which are compatible with the composition functors and with the unit functors of $\mathcal{A}$ and $\mathcal{A}^{\prime}$.


## 1. Recall on strictly commutative Picard stacks

Let $\mathbf{S}$ be a site. For the notions of $\mathbf{S}$-pre-stack, $\mathbf{S}$-stack and morphisms of $\mathbf{S}$ stacks we refer to G71 Chapter II 1.2.

A strictly commutative Picard $\boldsymbol{S}$-stack is an $\mathbf{S}$-stack of groupoids $\mathcal{P}$ endowed with a functor $+: \mathcal{P} \times_{\mathbf{S}} \mathcal{P} \rightarrow \mathcal{P},(a, b) \mapsto a+b$, and two natural isomorphisms of associativity $\sigma$ and of commutativity $\tau$, which are described by the functorial isomorphisms

$$
\begin{align*}
\sigma_{a, b, c} & :(a+b)+c \xrightarrow{\cong} a+(b+c) \quad \forall a, b, c \in \mathcal{P},  \tag{1.1}\\
\tau_{a, b} & : a+b \xrightarrow{\cong} b+a \quad \forall a, b \in \mathcal{P} \tag{1.2}
\end{align*}
$$

such that for any object $U$ of $\mathbf{S},(\mathcal{P}(U),+, \sigma, \tau)$ is a strictly commutative Picard category (i.e. it is possible to make the sum of two objects of $\mathcal{P}(U)$ and this sum is associative and commutative, see [D73] 1.4.2 for more details). Here "strictly" means that $\tau_{a, a}$ is the identity for all $a \in \mathcal{P}$. Any strictly commutative Picard S-stack admits a global neutral object $e$ and the sheaf of automorphisms of the neutral object Aut $(e)$ is abelian.

Let $\mathcal{P}$ and $\overline{\mathcal{Q}}$ be two strictly commutative Picard $\mathbf{S}$-stacks. An additive functor $\left(F, \sum\right): \mathcal{P} \rightarrow \mathcal{Q}$ between strictly commutative Picard $\mathbf{S}$-stacks is a morphism of $\mathbf{S}$ stacks (i.e. a cartesian S-functor, see [G71] Chapter I 1.1) endowed with a natural
isomorphism $\sum$ which is described by the functorial isomorphisms

$$
\sum_{a, b}: F(a+b) \stackrel{\cong}{\leftrightarrows} F(a)+F(b) \quad \forall a, b \in \mathcal{P}
$$

and which is compatible with the natural isomorphisms $\sigma$ and $\tau$ of $\mathcal{P}$ and $\mathcal{Q}$. A morphism of additive functors $u:\left(F, \sum\right) \rightarrow\left(F^{\prime}, \sum^{\prime}\right)$ is an S-morphism of cartesian S-functors (see G71 Chapter I 1.1) which is compatible with the natural isomorphisms $\sum$ and $\sum^{\prime}$ of $F$ and $F^{\prime}$ respectively. We denote by $\boldsymbol{A d d}_{\mathbf{S}}(\mathcal{P}, \mathcal{Q})$ the category whose objects are additive functors from $\mathcal{P}$ to $\mathcal{Q}$ and whose arrows are morphisms of additive functors. The category $\boldsymbol{\operatorname { A d d }}_{\mathbf{S}}(\mathcal{P}, \mathcal{Q})$ is a groupoid, i.e. any morphism of additive functors is an isomorphism of additive functors.

An equivalence of strictly commutative Picard $\boldsymbol{S}$-stacks between $\mathcal{P}$ and $\mathcal{Q}$ is an additive functor $\left(F, \sum\right): \mathcal{P} \rightarrow \mathcal{Q}$ with $F$ an equivalence of $\mathbf{S}$-stacks. Two strictly commutative Picard $\mathbf{S}$-stacks are equivalent as strictly commutative Picard $\boldsymbol{S}$-stacks if there exists an equivalence of strictly commutative Picard $\mathbf{S}$-stacks between them.

To any strictly commutative Picard $\mathbf{S}$-stack $\mathcal{P}$, we associate the sheaffification $\pi_{0}(\mathcal{P})$ of the pre-sheaf which associates to each object $U$ of $\mathbf{S}$ the group of isomorphism classes of objects of $\mathcal{P}(U)$, the sheaf $\pi_{1}(\mathcal{P})$ of automorphisms Aut $(e)$ of the neutral object of $\mathcal{P}$, and an element $\varepsilon(\mathcal{P})$ of $\operatorname{Ext}^{2}\left(\pi_{0}(\mathcal{P}), \pi_{1}(\mathcal{P})\right)$. Two strictly commutative Picard $\mathbf{S}$-stacks $\mathcal{P}$ and $\mathcal{P}^{\prime}$ are equivalent as strictly commutative Picard S-stacks if and only if $\pi_{i}(\mathcal{P})$ is isomorphic to $\pi_{i}\left(\mathcal{P}^{\prime}\right)$ for $i=0,1$ and $\varepsilon(\mathcal{P})=\varepsilon\left(\mathcal{P}^{\prime}\right)$ (see Remark 1.3).

A strictly commutative Picard $\boldsymbol{S}$-pre-stack is an $\mathbf{S}$-pre-stack of groupoids $\mathcal{P}$ endowed with a functor $+: \mathcal{P} \times_{\mathbf{S}} \mathcal{P} \rightarrow \mathcal{P}$ and two natural isomorphisms of associativity $\sigma$ (1.1) and of commutativity $\tau$ (1.2), such that for any object $U$ of $\mathbf{S},(\mathcal{P}(U),+, \sigma, \tau)$ is a strictly commutative Picard category. If $\mathcal{P}$ is a strictly commutative Picard S-pre-stack, there exists modulo a unique equivalence one and only one pair ( $a \mathcal{P}, j$ ) where $a \mathcal{P}$ is a strictly commutative Picard $\mathbf{S}$-stack and $j: \mathcal{P} \rightarrow a \mathcal{P}$ is an additive functor. $(a \mathcal{P}, j)$ is the strictly commutative Picard $\boldsymbol{S}$-stack generated by $\mathcal{P}$.

To each complex $K=\left[K^{-1} \xrightarrow{d} K^{0}\right]$ of $\mathcal{K}^{[-1,0]}(\mathbf{S})$, we associate a strictly commutative Picard $\mathbf{S}$-stack $s t(K)$ which is the $\mathbf{S}$-stack generated by the following strictly commutative Picard S-pre-stack $p s t(K)$ : for any object $U$ of $\mathbf{S}$, the objects of $p s t(K)(U)$ are the elements of $K^{0}(U)$, and if $x$ and $y$ are two objects of $p s t(K)(U)$ (i.e. $x, y \in K^{0}(U)$ ), an arrow of $\operatorname{pst}(K)(U)$ from $x$ to $y$ is an element $f$ of $K^{-1}(U)$ such that $d f=y-x$. A morphism of complexes $g: K \rightarrow L$ induces an additive functor $s t(g): s t(K) \rightarrow s t(L)$ between the strictly commutative Picard $\mathbf{S}$-stacks associated to the complexes $K$ and $L$.

In D73] §1.4 Deligne proves the following links between strictly commutative Picard $\mathbf{S}$-stacks and complexes of $\mathcal{K}^{[-1,0]}(\mathbf{S})$, between additive functors and morphisms of complexes and between morphisms of additive functors and homotopies of complexes:

- for any strictly commutative Picard $\mathbf{S}$-stack $\mathcal{P}$ there exists a complex $K$ of $\mathcal{K}{ }^{[-1,0]}(\mathbf{S})$ such that $\mathcal{P}=s t(K) ;$
- if $K, L$ are two complexes of $\mathcal{K}^{[-1,0]}(\mathbf{S})$, then for any additive functor $F: s t(K) \rightarrow s t(L)$ there exists a quasi-isomorphism $k: K^{\prime} \rightarrow K$ and a morphism of complexes $l: K^{\prime} \rightarrow L$ such that $F$ is isomorphic as additive functor to $s t(l) \circ s t(k)^{-1}$;
- if $f, g: K \rightarrow L$ are two morphisms of complexes of $\mathcal{K}^{[-1,0]}(\mathbf{S})$, then
$\operatorname{Hom}_{\text {Adds }(s t(K), s t(L))}(s t(f), s t(g)) \cong\{$ homotopies $H: K \rightarrow L \mid g-f=d H+H d\}$.
Denote by Picard(S) the category whose objects are small strictly commutative Picard S-stacks and whose arrows are isomorphism classes of additive functors. We can summarize the above dictionary between strictly commutative Picard $\mathbf{S}$-stacks and complexes of abelian sheaves on $\mathbf{S}$ with the following Theorem:

Theorem 1.1. The functor

$$
\begin{array}{rll}
s t: \mathcal{D}^{[-1,0]}(\boldsymbol{S}) & \longrightarrow & \operatorname{Picard}(\boldsymbol{S})  \tag{1.4}\\
K & \mapsto & \operatorname{st}(K) \\
K \xrightarrow{f} L & \mapsto & \operatorname{st}(K) \xrightarrow{s t(f)} \operatorname{st}(L)
\end{array}
$$

is an equivalence of categories.
We denote by [ ] the inverse equivalence of $s t$.
Let $\mathcal{P i c a r d}(\mathbf{S})$ be the 2-category of strictly commutative Picard $\mathbf{S}$-stacks whose objects are strictly commutative Picard $\mathbf{S}$-stacks and whose categories of morphisms are the categories $\boldsymbol{A d d}_{\mathbf{S}}(\mathcal{P}, \mathcal{Q})$ (i.e. the 1 -arrows are additive functors between strictly commutative Picard $\mathbf{S}$-stacks and the 2-arrows are morphisms of additive functors).
Theorem 1.2. Via the functor st, there exists a 2-functor between
(a): the 2-category whose objects and 1-arrows are the objects and the arrows of the category $\mathcal{K}^{[-1,0]}(\boldsymbol{S})$ and whose 2-arrows are the homotopies between 1-arrows (i.e. $H$ such that $g-f=d H+H d$ with $f, g: K \rightarrow L$ 1-arrows), (b): the 2-category $\operatorname{Picard}(\boldsymbol{S})$.

Remark 1.3. Let $K=\left[K^{-1} \xrightarrow{d} K^{0}\right]$ be a complex of $\mathcal{D}^{[-1,0]}(\mathbf{S})$. The long exact sequence

$$
0 \longrightarrow \mathrm{H}^{-1}(K) \longrightarrow K^{-1} \xrightarrow{d} K^{0} \longrightarrow \mathrm{H}^{0}(K) \longrightarrow 0
$$

is an element of $\operatorname{Ext}^{2}\left(\mathrm{H}^{0}(K), \mathrm{H}^{-1}(K)\right)$ that we denote by $\varepsilon(K)$. The sheaves $\mathrm{H}^{0}, \mathrm{H}^{-1}$ and the element $\varepsilon$ of $\operatorname{Ext}^{2}\left(\mathrm{H}^{0}, \mathrm{H}^{-1}\right)$ classify objects of $\mathcal{D}^{[-1,0]}(\mathbf{S})$ modulo isomorphisms. Through the equivalence of categories (1.4), the above classification of objects of $\mathcal{D}^{[-1,0]}(\mathbf{S})$ is equivalent to the classification of strictly commutative Picard $\mathbf{S}$-stacks via the sheaves $\pi_{0}, \pi_{1}$ and the invariant $\varepsilon \in \operatorname{Ext}^{2}\left(\pi_{0}, \pi_{1}\right)$. In particular $\pi_{0}(\mathcal{P})=\mathrm{H}^{0}([\mathcal{P}]), \pi_{1}(\mathcal{P})=\mathrm{H}^{-1}([\mathcal{P}]), \varepsilon(\mathcal{P})=\varepsilon([\mathcal{P}])$.

Example Let $\mathcal{P}$ and $\mathcal{Q}$ be two strictly commutative Picard $\mathbf{S}$-stacks. Let

$$
\operatorname{HOM}(\mathcal{P}, \mathcal{Q})
$$

be the following strictly commutative Picard $\mathbf{S}$-stack:

- for any object $U$ of $\mathbf{S}$, the objects of the category $\operatorname{HOM}(\mathcal{P}, \mathcal{Q})(U)$ are additive functors from $\mathcal{P}_{\mid U}$ to $\mathcal{Q}_{\mid U}$ and its arrows are morphisms of additive functors;
- the functor $+: \operatorname{HOM}(\mathcal{P}, \mathcal{Q}) \times \operatorname{HOM}(\mathcal{P}, \mathcal{Q}) \rightarrow \operatorname{HOM}(\mathcal{P}, \mathcal{Q})$ is defined by the formula

$$
\left(F_{1}+F_{2}\right)(a)=F_{1}(a)+F_{2}(a) \quad \forall a \in \mathcal{P}
$$

and the natural isomorphism

$$
\sum:\left(F_{1}+F_{2}\right)(a+b) \stackrel{\cong}{\cong}\left(F_{1}+F_{2}\right)(a)+\left(F_{1}+F_{2}\right)(b)
$$

is given by the commutative diagram


- the natural isomorphisms of associativity $\sigma$ and of commutativity $\tau$ of $\operatorname{HOM}(\mathcal{P}, \mathcal{Q})$ are defined via the analogous natural isomorphisms of $\mathcal{Q}$.
Because of equality (1.3) and of Theorem (1.4 we have the following equality in the derived category $\mathcal{D}^{[-1,0]}(\mathbf{S})$


## Lemma 1.4.

$$
[\operatorname{HOM}(\mathcal{P}, \mathcal{Q})]=\tau_{\leq 0} \operatorname{RHom}([\mathcal{P}],[\mathcal{Q}])
$$

We can define the following bifunctor on $\operatorname{Picard}(\mathbf{S}) \times \operatorname{Picard}(\mathbf{S})$

$$
\begin{aligned}
\mathrm{HOM}: \operatorname{Picard}(\mathbf{S}) \times \operatorname{Picard}(\mathbf{S}) & \longrightarrow \operatorname{Picard}(\mathbf{S}) \\
(\mathcal{P}, \mathcal{Q}) & \mapsto
\end{aligned} \operatorname{HOM}(\mathcal{P}, \mathcal{Q}) .
$$

## 2. Operations on strictly commutative Picard stacks

We start defining the product of two strictly commutative Picard S-stacks. Let $\mathcal{P}$ and $\mathcal{Q}$ be two strictly commutative Picard $\mathbf{S}$-stacks.

Definition 2.1. The product of $\mathcal{P}$ and $\mathcal{Q}$ is the strictly commutative Picard $\mathbf{S}$-stack $\mathcal{P} \times \mathcal{Q}$ defined as followed:

- for any object $U$ of $\mathbf{S}$, an object of the category $\mathcal{P} \times \mathcal{Q}(U)$ is a pair $(p, q)$ of objects with $p$ an object of $\mathcal{P}(U)$ and $q$ an object of $\mathcal{Q}(U)$;
- for any object $U$ of $\mathbf{S}$, if $(p, q)$ and $\left(p^{\prime}, q^{\prime}\right)$ are two objects of $\mathcal{P} \times \mathcal{Q}(U)$, an arrow of $\mathcal{P} \times \mathcal{Q}(U)$ from $(p, q)$ to $\left(p^{\prime}, q^{\prime}\right)$ is a pair $(f, g)$ of arrows with $f: p \rightarrow p^{\prime}$ an arrow of $\mathcal{P}(U)$ and $g: q \rightarrow q^{\prime}$ an arrow of $\mathcal{Q}(U)$;
- the functor $+:(\mathcal{P} \times \mathcal{Q}) \times(\mathcal{P} \times \mathcal{Q}) \rightarrow \mathcal{P} \times \mathcal{Q}$ is defined by the formula

$$
(p, q)+\left(p^{\prime}, q^{\prime}\right)=\left(p+p^{\prime}, q+q^{\prime}\right)
$$

for any $p, p^{\prime} \in \mathcal{P}$ and $q, q^{\prime} \in \mathcal{Q}$;

- the natural isomorphisms of associativity $\sigma$ and of commutativity $\tau$ of $\mathcal{P} \times \mathcal{Q}$ are defined via the analogous natural isomorphisms of $\mathcal{P}$ and $\mathcal{Q}$.

In the derived category $\mathcal{D}^{[-1,0]}(\mathbf{S})$ we have the following equality

$$
[\mathcal{P} \times \mathcal{Q}]=[\mathcal{P}]+[\mathcal{Q}]
$$

which implies the following equality of abelian sheaves

$$
\pi_{i}(\mathcal{P} \times \mathcal{Q})=\pi_{i}(\mathcal{P})+\pi_{i}(\mathcal{Q}) \quad \text { for } \quad i=0,1
$$

Now we define the fibered product (called also the pull-back) and the fibered sum (called also the push-down) of strictly commutative Picard S-stacks. Let $G: \mathcal{P} \rightarrow \mathcal{Q}$ and $F: \mathcal{P}^{\prime} \rightarrow \mathcal{Q}$ be additive functors between strictly commutative Picard $\mathbf{S}$-stacks.

Definition 2.2. The fibered product of $\mathcal{P}$ and $\mathcal{P}^{\prime}$ over $\mathcal{Q}$ via $F$ and $G$ is the strictly commutative Picard $\mathbf{S}$-stack $\mathcal{P} \times{ }_{\mathcal{Q}} \mathcal{P}^{\prime}$ defined as followed:

- for any object $U$ of $\mathbf{S}$, the objects of the category $\left(\mathcal{P} \times{ }_{\mathcal{Q}} \mathcal{P}^{\prime}\right)(U)$ are triplets $\left(p, p^{\prime}, f\right)$ where $p$ is an object of $\mathcal{P}(U), p^{\prime}$ is an object of $\mathcal{P}^{\prime}(U)$ and $f$ : $G(p) \xlongequal{\cong} F\left(p^{\prime}\right)$ is an isomorphism of $\mathcal{Q}(U)$ between $G(p)$ and $F\left(p^{\prime}\right)$;
- for any object $U$ of $\mathbf{S}$, if $\left(p_{1}, p_{1}^{\prime}, f\right)$ and $\left(p_{2}, p_{2}^{\prime}, g\right)$ are two objects of $\left(\mathcal{P} \times_{\mathcal{Q}}\right.$ $\left.\mathcal{P}^{\prime}\right)(U)$, an arrow of $\left(\mathcal{P} \times_{\mathcal{Q}} \mathcal{P}^{\prime}\right)(U)$ from $\left(p_{1}, p_{1}^{\prime}, f\right)$ to $\left(p_{2}, p_{2}^{\prime}, g\right)$ is a pair $(f, g)$ of arrows with $\alpha: p_{1} \rightarrow p_{2}$ of arrow of $\mathcal{P}(U)$ and $\beta: p_{1}^{\prime} \rightarrow p_{2}^{\prime}$ an arrow of $\mathcal{P}^{\prime}(U)$ such that $g \circ G(\alpha)=F(\beta) \circ f$;
- the functor $+:\left(\mathcal{P} \times{ }_{\mathcal{Q}} \mathcal{P}^{\prime}\right) \times\left(\mathcal{P} \times{ }_{\mathcal{Q}} \mathcal{P}^{\prime}\right) \rightarrow \mathcal{P} \times{ }_{\mathcal{Q}} \mathcal{P}^{\prime}$ is induced by the functors $+: \mathcal{P} \times \mathcal{P} \rightarrow \mathcal{P}$ and $+: \mathcal{P}^{\prime} \times \mathcal{P}^{\prime} \rightarrow \mathcal{P}^{\prime}$;
- the natural isomorphisms of associativity $\sigma$ and of commutativity $\tau$ are induced by the analogous natural isomorphisms of $\mathcal{P}$ and $\mathcal{P}^{\prime}$.
The fibered product $\mathcal{P} \times{ }_{\mathcal{Q}} \mathcal{P}^{\prime}$ is also called the pull-back $F^{*} \mathcal{P}$ of $\mathcal{P}$ via $F: \mathcal{P}^{\prime} \rightarrow \mathcal{Q}$ or the pull-back $G^{*} \mathcal{P}^{\prime}$ of $\mathcal{P}^{\prime}$ via $G: \mathcal{P} \rightarrow \mathcal{Q}$. It is endowed with two additive functors $\operatorname{Pr}_{1}: \mathcal{P} \times_{\mathcal{Q}} \mathcal{P}^{\prime} \rightarrow \mathcal{P}$ and $\operatorname{Pr}_{2}: \mathcal{P} \times_{\mathcal{Q}} \mathcal{P}^{\prime} \rightarrow \mathcal{P}^{\prime}$. Moreover it satisfies the following universal property: given two additive functors $H: \mathcal{O} \rightarrow \mathcal{P}$ and $K: \mathcal{O} \rightarrow \mathcal{P}^{\prime}$, and given an isomorphism of additive functors $F \circ K \cong G \circ H$, then there exists a unique additive functor $U: \mathcal{O} \rightarrow \mathcal{P} \times{ }_{\mathcal{Q}} \mathcal{P}^{\prime}$ such that we have two isomorphisms of additive functors $H \cong P r_{1} \circ U$ and $K \cong P r_{2} \circ U$. The following square formed by the fibered sum $\mathcal{P} \times{ }_{\mathcal{Q}} \mathcal{P}^{\prime}$

is called a pull-back square or a cartesian square.
If $[\mathcal{P}]=\left[K^{-1} \xrightarrow{d_{K}} K^{0}\right],\left[\mathcal{P}^{\prime}\right]=\left[L^{-1} \xrightarrow{d_{L}} L^{0}\right]$ and $[\mathcal{Q}]=\left[M^{-1} \xrightarrow{d_{M}} M^{0}\right]$, in the derived category $\mathcal{D}^{[-1,0]}(\mathbf{S})$ we have the following equality

$$
\left[\mathcal{P} \times{ }_{\mathcal{Q}} \mathcal{P}^{\prime}\right]=\left[K^{-1} \times_{M^{-1}} L^{-1} \xrightarrow{d_{K} \times_{d_{M}} d_{L}} K^{0} \times_{M^{0}} K^{0}\right]
$$

where for $i=-1,0$ the abelian sheaf $K^{i} \times_{M^{i}} L^{i}$ is the fibered product of $K^{i}$ and of $L^{i}$ over $M^{i}$ and the morphism of abelian sheaves $d_{K} \times{ }_{d_{M}} d_{L}$ is given by the universal property of the fibered product $K^{0} \times_{M^{0}} K^{0}$.

Remark that we have the exact sequences of abelian sheaves

$$
0 \longrightarrow \pi_{1}\left(\mathcal{P} \times_{\mathcal{Q}} \mathcal{P}^{\prime}\right) \longrightarrow \pi_{1}(\mathcal{P})+\pi_{1}\left(\mathcal{P}^{\prime}\right) .
$$

Now we introduce the dual notion of fibered product: the fibered sum. Let $G: \mathcal{Q} \rightarrow \mathcal{P}$ and $F: \mathcal{Q} \rightarrow \mathcal{P}^{\prime}$ be additive functors between strictly commutative Picard S-stacks.

Definition 2.3. The fibered sum of $\mathcal{P}$ and $\mathcal{P}^{\prime}$ under $\mathcal{Q}$ via $F$ and $G$ is the strictly commutative Picard $\mathbf{S}$-stack $\mathcal{P}+{ }^{\mathcal{Q}} \mathcal{P}^{\prime}$ generated by the following strictly commutative Picard S-pre-stack $\mathcal{D}$ :

- for any object $U$ of $\mathbf{S}$, the objects of the category $\mathcal{D}(U)$ are the objects of the category $\left(\mathcal{P} \times \mathcal{P}^{\prime}\right)(U)$, i.e. pairs $\left(p, p^{\prime}\right)$ with $p$ an object of $\mathcal{P}(U)$ and $p^{\prime}$ an object of $\mathcal{P}^{\prime}(U)$;
- for any object $U$ of $\mathbf{S}$, if $\left(p_{1}, p_{1}^{\prime}\right)$ and $\left(p_{2}, p_{2}^{\prime}\right)$ are two objects of $\mathcal{D}(U)$, an arrow of $\mathcal{D}(U)$ from $\left(p_{1}, p_{1}^{\prime}\right)$ to $\left(p_{2}, p_{2}^{\prime}\right)$ is an isomorphism class of triplets $(q, \alpha, \beta)$ with $q$ an object of $\mathcal{Q}(U), \alpha: p_{1}+G(q) \rightarrow p_{2}$ an arrow of $\mathcal{P}(U)$ and $\beta: p_{1}^{\prime}+F(q) \rightarrow p_{2}^{\prime}$ an arrow of $\mathcal{P}^{\prime}(U)$;
- the functor $+: \mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D}$ is induced by the functors $+: \mathcal{P} \times \mathcal{P} \rightarrow \mathcal{P}$ and $+: \mathcal{P}^{\prime} \times \mathcal{P}^{\prime} \rightarrow \mathcal{P}^{\prime}$;
- the natural isomorphisms of associativity $\sigma$ and of commutativity $\tau$ are induced by the analogous natural isomorphisms of $\mathcal{P}$ and $\mathcal{P}^{\prime}$.

The fibered sum $\mathcal{P}+{ }^{\mathcal{Q}} \mathcal{P}^{\prime}$ is also called the push-down $F_{*} \mathcal{P}$ of $\mathcal{P}$ via $F: \mathcal{Q} \rightarrow \mathcal{P}^{\prime}$ or the push-down $G_{*} \mathcal{P}^{\prime}$ of $\mathcal{P}^{\prime}$ via $G: \mathcal{Q} \rightarrow \mathcal{P}$. It is endowed with two additive functors $I n_{1}: \mathcal{P} \rightarrow \mathcal{P}+{ }^{\mathcal{Q}} \mathcal{P}^{\prime}$ and $\operatorname{In}_{2}: \mathcal{P}^{\prime} \rightarrow \mathcal{P}+{ }^{\mathcal{Q}} \mathcal{P}^{\prime}$. Moreover it satisfies the following universal property: given two additive functors $H: \mathcal{P} \rightarrow \mathcal{O}$ and $K: \mathcal{P}^{\prime} \rightarrow \mathcal{O}$, and given an isomorphism of additive functors $K \circ F \cong H \circ G$, then there exists a unique additive functor $U: \mathcal{P}+{ }^{\mathcal{Q}} \mathcal{P}^{\prime} \rightarrow \mathcal{O}$ such that we have two isomorphisms of additive functors $H \cong U \circ I n_{1}$ and $K \cong U \circ I n_{2}$. The following square formed by the fibered $\operatorname{sum} \mathcal{P}+{ }^{\mathcal{Q}} \mathcal{P}^{\prime}$

is called a push-down square or a cocartesian square.
If $[\mathcal{P}]=\left[K^{-1} \xrightarrow{d_{K}} K^{0}\right],\left[\mathcal{P}^{\prime}\right]=\left[L^{-1} \xrightarrow{d_{L}} L^{0}\right]$ and $[\mathcal{Q}]=\left[M^{-1} \xrightarrow{d_{M}} M^{0}\right]$, in the derived category $\mathcal{D}^{[-1,0]}(\mathbf{S})$ we have the following equality

$$
\left[\mathcal{P}+{ }^{\mathcal{Q}} \mathcal{P}^{\prime}\right]=\left[K^{-1}+{ }^{M^{-1}} L^{-1} \xrightarrow{d_{K}++_{M}} d_{L} K^{0}+{ }^{M^{0}} K^{0}\right]
$$

where for $i=-1,0$ the abelian sheaf $K^{i}+{ }^{M^{i}} L^{i}$ is the fibered sum of $K^{i}$ and of $L^{i}$ under $M^{i}$ and the morphism of abelian sheaves $d_{K}+{ }^{d_{M}} d_{L}$ is given by the universal property of the fibered product $K^{-1}+{ }^{M^{-1}} K^{-1}$.

Remark that we have the exact sequences of abelian sheaves

$$
\pi_{0}(\mathcal{P})+\pi_{0}\left(\mathcal{P}^{\prime}\right) \longrightarrow \pi_{0}\left(\mathcal{P}+^{\mathcal{Q}} \mathcal{P}^{\prime}\right) \longrightarrow 0
$$

## 3. Extensions of strictly commutative Picard stacks

Let $\mathcal{P}$ and $\mathcal{Q}$ be two strictly commutative Picard $\mathbf{S}$-stacks. Consider an additive functor $F: \mathcal{P} \rightarrow \mathcal{Q}$. Denote by $\mathbf{1}$ the strictly commutative Picard $\mathbf{S}$-stack such that for any object $U$ of $\mathbf{S}, \mathbf{1}(U)$ is the category with one object and one arrow.

Definition 3.1. The kernel of $F, \operatorname{ker}(F)$, is the fibered product $\mathcal{P} \times{ }_{\mathcal{Q}} \mathbf{1}$ of $\mathcal{P}$ and 1 over $\mathcal{Q}$ via $F: \mathcal{P} \rightarrow \mathcal{Q}$ and the additive functor $1: \mathbf{1} \rightarrow \mathcal{Q}$.
The cokernel of $F, \operatorname{coker}(F)$, is the fibered sum $1+{ }^{\mathcal{P}} \mathcal{Q}$ of $\mathbf{1}$ and $\mathcal{Q}$ under $\mathcal{P}$ via $F: \mathcal{P} \rightarrow \mathcal{Q}$ and the additive functor $\mathbf{1}: \mathcal{P} \rightarrow \mathbf{1}$.

We have the cartesian and cocartesian squares

and the exact sequences of abelian sheaves

$$
\begin{equation*}
0 \longrightarrow \pi_{1}(\operatorname{ker}(F)) \longrightarrow \pi_{1}(\mathcal{P}) \quad \pi_{0}(\mathcal{Q}) \longrightarrow \pi_{0}(\operatorname{coker}(F)) \longrightarrow 0 \tag{3.1}
\end{equation*}
$$

Explicitly, according to Definition 2.2 the kernel of $F$ is the strictly commutative Picard S-stack $\operatorname{ker}(F)$ where

- for any object $U$ of $\mathbf{S}$, the objects of the category $\operatorname{ker}(F)(U)$ are pairs $(p, f)$ where $p$ is an object of $\mathcal{P}(U)$ and $f: F(p) \xlongequal{\cong} e$ is an isomorphism between $F(p)$ and the neutral object $e$ of $\mathcal{Q}$;
- for any object $U$ of $\mathbf{S}$, if $(p, f)$ and $\left(p^{\prime}, f^{\prime}\right)$ are two objects of $\operatorname{ker}(F)(U)$, an arrow $\alpha:(p, f) \rightarrow\left(p^{\prime}, f^{\prime}\right)$ of $\operatorname{ker}(F)(U)$ is an arrow $\alpha: p \rightarrow p^{\prime}$ of $\mathcal{P}(U)$ such that $f^{\prime} \circ F(\alpha)=f$.
By definition 2.3 the cokernel of $F$ is the strictly commutative Picard $\mathbf{S}$-stack coker $(F)$ generated by the following strictly commutative Picard S-pre-stack $\operatorname{coker}^{\prime}(F)$ where
- for any object $U$ of $\mathbf{S}$, the objects of $\operatorname{coker}^{\prime}(F)(U)$ are the objects of $\mathcal{Q}(U)$;
- for any object $U$ of $\mathbf{S}$, if $q$ and $q^{\prime}$ are two objects of $\operatorname{coker}^{\prime}(F)(U)$ (i.e. objects of $\mathcal{Q}(U)$ ), an arrow of $\operatorname{coker}^{\prime}(F)(U)$ from $q$ to $q^{\prime}$ is an isomorphism class of pairs $(p, \alpha)$ with $p$ an object of $\mathcal{P}(U)$ and $\alpha: q+F(p) \rightarrow q^{\prime}$ an arrow of $\mathcal{Q}(U)$.

Definition 3.2. An extension $\mathcal{E}=(\mathcal{E}, I, J)$ of $\mathcal{P}$ by $\mathcal{Q}$

$$
\begin{equation*}
\mathcal{Q} \xrightarrow{I} \mathcal{E} \xrightarrow{J} \mathcal{P} \tag{3.2}
\end{equation*}
$$

consists of
(1) a strictly commutative Picard $\mathbf{S}$-stack $\mathcal{E}$,
(2) two additive functors $I: \mathcal{Q} \rightarrow \mathcal{E}$ and $J: \mathcal{E} \rightarrow \mathcal{P}$,
(3) an isomorphism of additive functors between the composite $J \circ I$ and the trivial additive functor: $J \circ I \cong 0$,
such that the following equivalent conditions are satisfied:
(a): $\pi_{0}(J): \pi_{0}(\mathcal{E}) \rightarrow \pi_{0}(\mathcal{P})$ is surjective and $I$ induces an equivalence of strictly commutative Picard $\mathbf{S}$-stacks between $\mathcal{Q}$ and $\operatorname{ker}(J)$;
(b): $\pi_{1}(I): \pi_{1}(\mathcal{Q}) \rightarrow \pi_{1}(\mathcal{E})$ is injective and $J$ induces an equivalence of strictly commutative Picard $\mathbf{S}$-stacks between $\operatorname{coker}(I)$ and $\mathcal{P}$.

The additive functors $I: \mathcal{Q} \rightarrow \mathcal{E}$ and $J: \mathcal{E} \rightarrow \mathcal{P}$ induce the sequences of abelian sheaves

$$
\begin{aligned}
& 0 \longrightarrow \pi_{1}(\mathcal{Q}) \xrightarrow{\pi_{1}(I)} \pi_{1}(\mathcal{E}) \xrightarrow{\pi_{1}(J)} \pi_{1}(\mathcal{P}) \\
& \pi_{0}(\mathcal{Q}) \xrightarrow{\pi_{0}(I)} \pi_{0}(\mathcal{E}) \xrightarrow{\pi_{0}(J)} \pi_{0}(\mathcal{P}) \longrightarrow 0
\end{aligned}
$$

which are exact in $\pi_{1}(\mathcal{Q})$ and $\pi_{0}(\mathcal{P})$ because of the equivalences of strictly commutative Picard S-stacks $\mathcal{Q} \cong \operatorname{ker}(J)$ and $\operatorname{coker}(I) \cong \mathcal{P}$. According to [AN09] Proposition 6.2.6, we can say more about these two sequences: in fact there exists a connecting morphism of abelian sheaves

$$
\delta: \pi_{1}(\mathcal{P}) \longrightarrow \pi_{0}(\mathcal{Q})
$$

leading to the long exact sequence

$$
\begin{equation*}
0 \longrightarrow \pi_{1}(\mathcal{Q}) \xrightarrow{\pi_{1}(I)} \pi_{1}(\mathcal{E}) \xrightarrow{\pi_{1}(J)} \pi_{1}(\mathcal{P}) \xrightarrow{\delta} \pi_{0}(\mathcal{Q}) \xrightarrow{\pi_{0}(I)} \pi_{0}(\mathcal{E}) \xrightarrow{\pi_{0}(J)} \pi_{0}(\mathcal{P}) \longrightarrow 0 . \tag{3.3}
\end{equation*}
$$

Explicitly the connecting morphism $\delta: \pi_{1}(\mathcal{P}) \rightarrow \pi_{0}(\mathcal{Q})$ is defined as followed: if $f: e_{\mathcal{P}} \rightarrow e_{\mathcal{P}}$ is an element of $\pi_{1}(\mathcal{P})(U)$ with $U$ an object of $\mathbf{S}$, then $\delta(f)$ represent the isomorphism class of the element

$$
\left(e_{\mathcal{E}}, f \circ 1_{J}\right)
$$

of $\operatorname{ker}(J)(U) \cong \mathcal{Q}(U)$, where $1_{J}: J\left(e_{\mathcal{E}}\right) \xlongequal{\cong} e_{\mathcal{P}}$ is the isomorphism resulting from the additivity of the functor $J: \mathcal{E} \rightarrow \mathcal{P}$ (here $e_{\mathcal{E}}$ and $e_{\mathcal{P}}$ are the neutral objects of $\mathcal{E}$ and $\mathcal{P}$ respectively).

Let $\mathcal{P}, \mathcal{Q}, \mathcal{P}^{\prime}$ and $\mathcal{Q}^{\prime}$ be strictly commutative Picard $\mathbf{S}$-stacks. Let $\mathcal{E}=(\mathcal{E}, I, J)$ be an extension of $\mathcal{P}$ by $\mathcal{Q}$ and let $\mathcal{E}^{\prime}=\left(\mathcal{E}^{\prime}, I^{\prime}, J^{\prime}\right)$ be an extension of $\mathcal{P}^{\prime}$ by $\mathcal{Q}^{\prime}$.

Definition 3.3. A morphism of extensions

$$
(F, G, H): \mathcal{E} \longrightarrow \mathcal{E}^{\prime}
$$

consists of
(1) three additive functors $F: \mathcal{E} \rightarrow \mathcal{E}^{\prime}, G: \mathcal{P} \rightarrow \mathcal{P}^{\prime}, H: \mathcal{Q} \rightarrow \mathcal{Q}^{\prime}$,
(2) two isomorphisms of additive functors $J^{\prime} \circ F \cong G \circ J$ and $F \circ I \cong I^{\prime} \circ H$, which are compatible with the isomorphisms of additive functors $J \circ I \cong 0$ and $J^{\prime} \circ I^{\prime} \cong 0$ underlying the extensions $\mathcal{E}$ and $\mathcal{E}^{\prime}$, i.e. the composite

$$
0 \stackrel{\cong}{\longleftrightarrow} G \circ 0 \stackrel{\cong}{\longleftrightarrow} G \circ J \circ I \stackrel{\cong}{\longleftrightarrow} J^{\prime} \circ F \circ I \stackrel{\cong}{\longleftrightarrow} J^{\prime} \circ I^{\prime} \circ H \stackrel{\cong}{\longleftrightarrow} 0 \circ H \stackrel{\cong}{\longleftrightarrow} 0
$$

should be the identity.
The three additive functors $F, G$ and $H$ furnish the following commutative diagram modulo isomorphisms of additive functors


Fix two strictly commutative Picard $\mathbf{S}$-stacks $\mathcal{P}$ and $\mathcal{Q}$. The extensions of $\mathcal{P}$ by $\mathcal{Q}$ form a 2-category

$$
\mathcal{E} x t(\mathcal{P}, \mathcal{Q})
$$

where

- the objects are extensions of $\mathcal{P}$ by $\mathcal{Q}$,
- the 1-arrows are morphisms of extensions, i.e. additive functors between extensions,
- the 2-arrows are morphisms of additive functors.

Now we show which objects of the derived category $\mathcal{D}^{[-1,0]}(\mathbf{S})$ correspond via the equivalence of categories (1.4) to the strictly commutative Picard S-stacks $\operatorname{ker}(F)$, $\operatorname{coker}(F)$ and $\mathcal{P}=(\mathcal{P}, I, J)$.

Lemma 3.4. Let $K=\left[K^{-1} \xrightarrow{d^{K}} K^{0}\right]$ and $L=\left[L^{-1} \xrightarrow{d^{L}} L^{0}\right]$ be complexes of $\mathcal{K}^{[-1,0]}(\boldsymbol{S})$. Let $F: s t(K) \rightarrow s t(L)$ be an additive functor induced by a morphism of complexes $f=\left(f^{-1}, f^{0}\right): K \rightarrow L$. The strictly commutative Picard $\boldsymbol{S}$-stacks
$\operatorname{ker}(F)$ and coker $(F)$ correspond via the equivalence of categories (1.4) to the following complexes of $\mathcal{K}^{[-1,0]}(\boldsymbol{S})$ :

$$
\begin{align*}
{[\operatorname{ker}(F)] } & =\tau_{\leq 0}(M C(f)[-1])=\left[\begin{array}{lll}
K^{-1} & \left(f^{-1},-d^{K}\right) \\
\operatorname{ker}\left(d^{L}, f^{0}\right)
\end{array}\right]  \tag{3.4}\\
{[\operatorname{coker}(F)] } & =\tau_{\geq-1} M C(f)=\left[\operatorname{coker}\left(f^{-1},-d^{K}\right) \xrightarrow{\left(d^{L}, f^{0}\right)} L^{0}\right] \tag{3.5}
\end{align*}
$$

where $\tau$ denotes the good truncation and $M C(f)$ is the mapping cone of the morphism $f$.
Proof. It is enough to show that the strictly commutative $\mathbf{S}$-pre-stacks associated to $\operatorname{coker}(F)$ is equivalent to the one associated to $\tau_{\geq-1} M C(f)$, since for each strictly commutative $\mathbf{S}$-pre-stack $\mathcal{P}$, the strictly commutative $\mathbf{S}$-stack generated by $\mathcal{P}$ is unique modulo a unique equivalence (idem for $\operatorname{ker}(F)$ ). Explicitelly $M C(f)$ is the complex

$$
0 \longrightarrow K^{-1} \xrightarrow{\left(f^{-1},-d^{K}\right)} L^{-1}+K^{0} \xrightarrow{\left(d^{L}, f^{0}\right)} L^{0} \longrightarrow 0
$$

concentrated in degree $-2,-1$ and 0 .
Let $U$ be an object of $\mathbf{S}$. The objects of $\operatorname{pst}\left(\tau_{\geq-1} M C(f)\right)(U)$ are the elements of $L^{0}(U)$ and so they are the same objects of $p \operatorname{st}(\bar{L})(U)$. Moreover, if $l$ and $l^{\prime}$ are two objects of $\operatorname{pst}\left(\tau_{\geq-1} M C(f)\right)(U)$, an arrow of $\operatorname{pst}\left(\tau_{\geq-1} M C(f)\right)(U)$ from $l$ to $l^{\prime}$ is an isomorphism class of pairs $(\alpha, k)$ with $k$ an object of $K^{0}(U)$ and $\alpha$ an object of $L^{-1}(U)$ such that

$$
\left(d^{L}, f^{0}\right)(\alpha, k)=l^{\prime}-l
$$

This equality can be rewritten as $d^{L}(\alpha)=l^{\prime}-\left(l+f^{0}(k)\right)$. Therefore an arrow from $l$ to $l^{\prime}$ is an isomorphism class of pairs $(\alpha, k)$ with $k$ an object of $p s t(K)(U)$ and $\alpha: l+F(k) \rightarrow l^{\prime}$ an arrow of $\operatorname{pst}(L)(U)$. According to Definition 3.1, we can conclude that $\operatorname{pst}\left(\tau_{\geq-1} M C(f)\right) \cong \operatorname{coker}^{\prime}(F)$.
The objects of $\operatorname{pst}\left(\tau_{\leq 0}(M C(f)[-1])\right)(U)$ are pairs $(f, k)$ with $k$ an object of $p s t(K)(U)$ and $f: F(k) \rightarrow e_{p s t(L)}$ an isomorphism from $F(k)$ to the neutral object $e_{p s t(L)}$ of $p s t(L)$. If $(f, k)$ and $\left(f^{\prime}, k^{\prime}\right)$ are two objects of $p s t\left(\tau_{\leq 0}(M C(f)[-1])\right)(U)$, an arrow of $\operatorname{pst}\left(\tau_{\leq 0}(M C(f)[-1])\right)(U)$ from $(f, k)$ to $\left(f^{\prime}, k^{\prime}\right)$ is an element $g$ of $K^{-1}(U)$ such that

$$
\left(f^{-1},-d^{K}\right)(g)=\left(f^{\prime}, k^{\prime}\right)-(f, k)
$$

This equality implies the equalities $f^{-1}(g)=f^{\prime}-f$ and $-d^{K}(g)=k^{\prime}-k$. Therefore $g: k^{\prime} \rightarrow k$ is an arrow of $p s t(K)(U)$ such that the following diagram is commutative:


According to Definition 3.1, we can conclude that $\operatorname{pst}\left(\tau_{\leq 0}(M C(f)[-1])\right) \cong \operatorname{ker}(F)$.

By the above Lemma, the following notion of extension in $\mathcal{K}^{[-1,0]}(\mathbf{S})$ is equivalent to Definition 3.2 through the equivalence of categories (1.4):

Definition 3.5. Let

$$
K \xrightarrow{i} L \xrightarrow{j} M
$$

be morphisms of complexes of $\mathcal{K}^{[-1,0]}(\mathbf{S})$. The complex $L$ is an extension of $M$ by $K$ if $j \circ i=0$ and the following equivalent conditions are satisfied:
(a): $\mathrm{H}^{0}(j): \mathrm{H}^{0}(L) \rightarrow \mathrm{H}^{0}(M)$ is surjective and $i$ induces a quasi-isomorphism between $K$ and $\tau_{\leq 0}(M C(j)[-1])$;
(b): $\mathrm{H}^{-1}(i): \mathrm{H}^{-1}(\bar{K}) \rightarrow \mathrm{H}^{-1}(L)$ is injective and $j$ induces a quasi-isomorphism between $\tau_{\geq-1} M C(i)$ and $M$.

Remark 3.6. Consider a short exact sequence of complexes in $\mathcal{K}^{[-1,0]}(\mathbf{S})$

$$
0 \longrightarrow K \xrightarrow{i} L \xrightarrow{j} M \longrightarrow 0 .
$$

It exists a distinguished triangle $K \xrightarrow{i} L \xrightarrow{j} M \rightarrow+\operatorname{in} \mathcal{D}(\mathbf{S})$, and $M$ is isomorphic to $M C(i)$ in $\mathcal{D}(\mathbf{S})$. Therefore a short exact sequence of complexes in $\mathcal{K}^{[-1,0]}(\mathbf{S})$ is an extension of complexes of $\mathcal{K}^{[-1,0]}(\mathbf{S})$ according to the above definition.

Remark 3.7. If $K=\left[K^{-1} \xrightarrow{0} K^{0}\right]$ and $M=\left[M^{-1} \xrightarrow{0} M^{0}\right]$, then an extension of $M$ by $K$ consists of an extension of $M^{0}$ by $K^{0}$ and an extension of $M^{-1}$ by $K^{-1}$.

Remark 3.8. Assume that $s t(L)=(s t(L), I, J)$ is an extension of $s t(M)$ by $s t(K)$. Since $\mathrm{H}^{-1}(i)$ is injective and $\mathrm{H}^{1}(K)=0$, the distinguished triangle $K \xrightarrow{i} L \rightarrow$ $M C(i) \rightarrow+$ furnishes the long exact sequence

$$
\begin{aligned}
0 & \longrightarrow \mathrm{H}^{-1}(K) \xrightarrow{\mathrm{H}^{-1}(i)} \mathrm{H}^{-1}(L) \longrightarrow \mathrm{H}^{-1}\left(\tau_{\geq-1} M C(i)\right) \longrightarrow \\
& \longrightarrow \mathrm{H}^{0}(K) \xrightarrow{\mathrm{H}^{0}(i)} \mathrm{H}^{0}(L) \longrightarrow \mathrm{H}^{0}\left(\tau_{\geq-1} M C(i)\right) \longrightarrow 0 .
\end{aligned}
$$

Because of the equality $\tau_{\geq-1} M C(i)=M$ in $\mathcal{D}(\mathbf{S})$, we see that the above long exact sequence is just the long exact sequence (3.3).

## 4. Operations on extensions of strictly commutative Picard stacks

Using the results of $\S 2$ we define the pull-back and the push-down of extensions of strictly commutative Picard $\mathbf{S}$-stacks via additive functors. Let $\mathcal{E}=(\mathcal{E}, I: \mathcal{Q} \rightarrow$ $\mathcal{E}, J: \mathcal{E} \rightarrow \mathcal{P})$ be an extension of $\mathcal{P}$ by $\mathcal{Q}$.

Definition 4.1. The pull-back $F^{*} \mathcal{E}$ of the extension $\mathcal{E}$ via an additive functor $F: \mathcal{P}^{\prime} \rightarrow \mathcal{P}$ is the fibered product $\mathcal{E} \times_{\mathcal{P}} \mathcal{P}^{\prime}$ of $\mathcal{E}$ and $\mathcal{P}^{\prime}$ over $\mathcal{P}$ via $J$ and $F$.

Lemma 4.2. The pull-pack $F^{*} \mathcal{E}$ of $\mathcal{E}$ via $F$ is an extension of $\mathcal{P}^{\prime}$ by $\mathcal{Q}$.
Proof. Denote by $\operatorname{Pr}: F^{*} \mathcal{E} \rightarrow \mathcal{P}^{\prime}$ the additive functor underlying the pull-back of $\mathcal{E}$ via $F$. Composing the equivalence of strictly commutative Picard $\mathbf{S}$-stacks $\mathcal{Q} \cong \operatorname{ker}(J)=\mathcal{E} \times_{\mathcal{P}} \mathbf{1}$ with the natural equivalence of strictly commutative Picard S-stacks $\mathcal{E} \times_{\mathcal{P}} \mathbf{1} \cong \mathcal{E} \times_{\mathcal{P}} \mathcal{P}^{\prime} \times_{\mathcal{P}} \mathbf{1}=\operatorname{ker}(P r)$, we get that $\mathcal{Q}$ is equivalent to the strictly commutative Picard S-stack $\operatorname{ker}(\operatorname{Pr})$. Moreover the surjectivity of $\pi_{0}(J)$ : $\pi_{0}(\mathcal{E}) \rightarrow \pi_{0}(\mathcal{P})$ implies the surjectivity of $\pi_{0}(\operatorname{Pr}): \pi_{0}\left(F^{*} \mathcal{E}\right) \rightarrow \pi_{0}\left(\mathcal{P}^{\prime}\right)$. Hence $\left(F^{*} \mathcal{E}, I, \operatorname{Pr}\right)$ is an extension of $\mathcal{P}^{\prime}$ by $\mathcal{Q}$.

Definition 4.3. The push-down $G_{*} \mathcal{E}$ of the extension $\mathcal{E}$ via an additive functor $G: \mathcal{Q} \rightarrow \mathcal{Q}^{\prime}$ is the fibered $\operatorname{sum} \mathcal{E}+\mathcal{Q}^{\mathcal{Q}} \mathcal{Q}^{\prime}$ of $\mathcal{E}$ and $\mathcal{Q}^{\prime}$ under $\mathcal{Q}$ via $G$ and $I$.

Lemma 4.4. The push-down $G_{*} \mathcal{E}$ of $\mathcal{E}$ via $G$ is an extension of $\mathcal{P}$ by $\mathcal{Q}^{\prime}$.

Proof. Denote by In : $\mathcal{Q}^{\prime} \rightarrow G_{*} \mathcal{E}$ the additive functor underlying the push-down of $\mathcal{E}$ via $G$. Composing the equivalence of strictly commutative Picard $\mathbf{S}$-stacks $\operatorname{coker}(I) \cong \mathcal{P}=\mathbf{1}+{ }^{\mathcal{Q}} \mathcal{E}$ with the natural equivalence of strictly commutative Picard S-stacks $\mathbf{1}+{ }^{\mathcal{Q}} \mathcal{E} \cong \mathbf{1}+\mathcal{Q}^{\prime} \mathcal{Q}^{\prime}+{ }^{\mathcal{Q}} \mathcal{E}=\operatorname{coker}($ In $)$, we get that $\mathcal{P}$ is equivalent to the strictly commutative Picard $\mathbf{S}$-stack coker $(\operatorname{In})$. Moreover the injectivity of $\pi_{1}(I): \pi_{0}(\mathcal{Q}) \rightarrow \pi_{1}(\mathcal{E})$ implies the injectivity of $\pi_{1}(\operatorname{In}): \pi_{1}\left(\mathcal{Q}^{\prime}\right) \rightarrow \pi_{1}\left(G_{*} \mathcal{E}\right)$. Hence $\left(G_{*} \mathcal{E}, \operatorname{In}, P\right)$ is an extension of $\mathcal{Q}^{\prime}$ by $\mathcal{P}$.

Before to define a group law for extensions of $\mathcal{P}$ by $\mathcal{Q}$, we need the following
Lemma 4.5. Let $\mathcal{E}$ be an extension of $\mathcal{P}$ by $\mathcal{Q}$ and let $\mathcal{E}^{\prime}$ be an extension of $\mathcal{P}^{\prime}$ by $\mathcal{Q}^{\prime}$. Then $\mathcal{E} \times \mathcal{E}^{\prime}$ is an extension of $\mathcal{P} \times \mathcal{P}^{\prime}$ by $\mathcal{Q} \times \mathcal{Q}^{\prime}$.
Proof. Via the equivalence of categories (1.4), we have that the complex $[\mathcal{E}]=$ $([\mathcal{E}], i, j)$ (resp. $\left[\mathcal{E}^{\prime}\right]=\left(\left[\mathcal{E}^{\prime}\right], i^{\prime}, j^{\prime}\right)$ ) is an extension of $[\mathcal{P}]$ by $[\mathcal{Q}]$ ( resp. an extension of $\left[\mathcal{P}^{\prime}\right]$ by $\left.\left[\mathcal{Q}^{\prime}\right]\right)$ in the derived category $\mathcal{D}(\mathbf{S})$. Therefore $\mathrm{H}^{0}\left(j+j^{\prime}\right)=\mathrm{H}^{0}(j)+\mathrm{H}^{0}\left(j^{\prime}\right)$ : $\mathrm{H}^{0}\left([\mathcal{E}]+\left[\mathcal{E}^{\prime}\right]\right) \rightarrow \mathrm{H}^{0}\left([\mathcal{P}]+\left[\mathcal{P}^{\prime}\right]\right)$ is surjective. Moreover $i+i^{\prime}$ induces an isomorphism in $\mathcal{D}(\mathbf{S})$ between $[\mathcal{Q}]+\left[\mathcal{Q}^{\prime}\right]$ and

$$
\tau_{\leq 0}(M C(j)[-1])+\tau_{\leq 0}\left(M C\left(j^{\prime}\right)[-1]\right)=\tau_{\leq 0}\left(M C\left(j+j^{\prime}\right)[-1]\right)
$$

Hence we can conclude that $\left[\mathcal{E} \times \mathcal{E}^{\prime}\right]=\left(\left[\mathcal{E} \times \mathcal{E}^{\prime}\right], i+i^{\prime}, j+j^{\prime}\right)$ is an extension of $\left[\mathcal{P} \times \mathcal{P}^{\prime}\right]$ by $\left[\mathcal{Q} \times \mathcal{Q}^{\prime}\right]$.

Let $\mathcal{E}$ and $\mathcal{E}^{\prime}$ be two extensions of $\mathcal{P}$ by $\mathcal{Q}$. According to the above lemma, the product $\mathcal{E} \times \mathcal{E}^{\prime}$ is an extension of the product $\mathcal{P} \times \mathcal{P}$ by the product $\mathcal{Q} \times \mathcal{Q}$.

Definition 4.6. The sum $\mathcal{E}+\mathcal{E}^{\prime}$ of the extensions $\mathcal{E}$ and $\mathcal{E}^{\prime}$ is the following extension of $\mathcal{P}$ by $\mathcal{Q}$

$$
\begin{equation*}
D^{*}+_{*}\left(\mathcal{E} \times \mathcal{E}^{\prime}\right) \tag{4.1}
\end{equation*}
$$

where $D: \mathcal{P} \rightarrow \mathcal{P} \times \mathcal{P}$ is the diagonal additive functor and $+: \mathcal{Q} \times_{\mathbf{S}} \mathcal{Q} \rightarrow \mathcal{Q}$ is the functor underlying the strictly commutative Picard $\mathbf{S}$-stack $\mathcal{Q}=(\mathcal{Q},+, \sigma, \tau)$.
Lemma 4.7. The above notion of sum of extensions defines on the set of equivalence classes of extensions of $\mathcal{P}$ by $\mathcal{Q}$ an associative, commutative group law with neutral object, that we denote $\mathcal{P} \times \mathcal{Q}$.

Proof. Neutral object: it is the product $\mathcal{P} \times \mathcal{Q}$ of the extension $\mathcal{P}=(\mathcal{P}, \mathbf{1}: \mathbf{1} \rightarrow$ $\mathcal{P}, I d: \mathcal{P} \rightarrow \mathcal{P})$ of $\mathcal{P}$ by $\mathbf{1}$ with the extension $\mathcal{Q}=(\mathcal{Q}, I d: \mathcal{Q} \rightarrow \mathcal{Q}, \mathbf{1}: \mathcal{Q} \rightarrow \mathbf{1})$ of 1 by $\mathcal{Q}$. Lemma 4.5 provides that such a product is an extension of $\mathcal{P} \times \mathbf{1} \cong \mathcal{P}$ by $\mathcal{Q} \times 1 \cong \mathcal{Q}$. Commutativity: it is clear from the formula (4.1). Associativity: Consider three extensions $\mathcal{E}, \mathcal{E}^{\prime}, \mathcal{E}^{\prime \prime}$ of $\mathcal{P}$ by $\mathcal{Q}$. Using the functor $+: \mathcal{Q} \times{ }_{\mathbf{S}} \mathcal{Q} \times{ }_{\mathbf{S}} \mathcal{Q} \rightarrow$ $\mathcal{Q}$ and the diagonal functor $D: \mathcal{P} \rightarrow \mathcal{P} \times \mathcal{P} \times \mathcal{P}$, it is enough to show that the extensions $\left(\mathcal{E}+\mathcal{E}^{\prime}\right)+\mathcal{E}^{\prime \prime}$ and $\mathcal{E}+\left(\mathcal{E}^{\prime}+\mathcal{E}^{\prime \prime}\right)$ are equivalent. We left this computation to the reader.

This last Lemma implies that if $\mathcal{O}, \mathcal{P}$ and $\mathcal{Q}$ are three strictly commutative Picard S-stacks, we have the following equivalence of 2-categories

$$
\begin{aligned}
& \mathcal{E} x t(\mathcal{O} \times \mathcal{P}, \mathcal{Q}) \cong \mathcal{E} x t(\mathcal{O}, \mathcal{Q}) \times \mathcal{E} x t(\mathcal{P}, \mathcal{Q}) \\
& \mathcal{E} x t(\mathcal{O}, \mathcal{P} \times \mathcal{Q}) \cong \mathcal{E} x t(\mathcal{O}, \mathcal{P}) \times \mathcal{E} x t(\mathcal{O}, \mathcal{Q})
\end{aligned}
$$

A 2 -groupoid is a 2 -category whose 1 -arrows are invertible up to a 2 -arrow and whose 2 -arrows are strictly invertible. An $\mathbf{S}$-2-stack in 2 -groupoids $\mathbb{P}$ is a fibered 2-category in 2-groupoids over $\mathbf{S}$ such that

- for every pair of objects $X, Y$ of the 2-category $\mathbb{P}(U)$, the fibered category of morphisms $\operatorname{Arr}_{\mathbb{P}(U)}(X, Y)$ of $\mathbb{P}(U)$ is a $U$-stack (called the $U$-stack of morphisms);
- 2-descent is effective for objects in $\mathbb{P}$.

See [B09] §6 for more details. A strictly commutative Picard S-2-stack is the 2analog of a strictly commutative Picard $\mathbf{S}$-stack, i.e. it is an $\mathbf{S}$-2-stack in 2-groupoids $\mathbb{P}$ endowed with a morphism of $\mathbf{S}$-2-stacks $+: \mathbb{P} \times \mathbf{s} \mathbb{P} \rightarrow \mathbb{P}$ and with associative and commutative constraints (see [T09] Definition 2.3 for more details). With these notation Lemma 4.7 implies that extensions of $\mathcal{P}$ by $\mathcal{Q}$ form a strictly commutative Picard $\mathbf{S}$-2-stack $\underline{\mathcal{E} x t}(\mathcal{P}, \mathcal{Q})$ where

- for any object $U$ of $\mathbf{S}$, the objects of the 2-category $\underline{\mathcal{E} x t}(\mathcal{P}, \mathcal{Q})(U)$ are extensions of $\mathcal{P}_{\mid U}$ by $\mathcal{Q}_{\mid U}$, its 1 -arrows are additive functors between such extensions and its 2 -arrows are morphisms of additive functors. In particular if $\mathcal{E}$ and $\mathcal{E}^{\prime}$ are two objects of $\underline{\mathcal{E} x t}(\mathcal{P}, \mathcal{Q})(U)$, the $U$-stack of morphisms from $\mathcal{E}$ to $\mathcal{E}^{\prime}$ is the $U$-stack $\operatorname{HOM}\left(\mathcal{E}, \mathcal{E}^{\prime}\right)$;
- the functor $+: \underline{\mathcal{E} x t}(\mathcal{P}, \mathcal{Q}) \times \underline{\mathcal{E} x t}(\mathcal{P}, \mathcal{Q}) \rightarrow \underline{\mathcal{E} x t}(\mathcal{P}, \mathcal{Q})$ is defined by the formula (4.1).
As for strictly commutative Picard $\mathbf{S}$-stacks and complexes of abelian sheaves concentrated in degrees -1 and 0 , in [09] Tatar proves that there is a dictionary between strictly commutative Picard $\mathbf{S}$-2-stacks and complexes of abelian sheaves concentrated in degrees $-2,-1$ and 0 . The complex of abelian sheaves associated to the strictly commutative Picard $\mathbf{S}$-2-stack $\underline{\mathcal{E x t}}(\mathcal{P}, \mathcal{Q})$ is $\tau_{\leq 0} \operatorname{RHom}([\mathcal{P}],[\mathcal{Q}][1])$.


## 5. Proof of the main theorem

In this section we use the same notation as in the introduction.
Definition 5.1. Two extensions $\mathcal{E}$ and $\mathcal{E}^{\prime}$ of $\mathcal{P}$ by $\mathcal{Q}$ are equivalent as extensions of $\mathcal{P}$ by $\mathcal{Q}$ if there is
(1) an additive functor $F: \mathcal{E} \rightarrow \mathcal{E}^{\prime}$ and
(2) two isomorphisms of additive functors $J^{\prime} \circ F \cong I d_{\mathcal{P}} \circ J$ and $F \circ I \cong I^{\prime} \circ I d_{\mathcal{Q}}$, which are compatible with the isomorphisms of additive functors $J \circ I \cong 0$ and $J^{\prime} \circ I^{\prime} \cong 0$ underlying the extensions $\mathcal{E}$ and $\mathcal{E}^{\prime}$ (see 3.3).

The additive functor $F$ furnishes the following commutative diagram modulo isomorphisms of additive functors


Definition 5.2. An extension of $\mathcal{P}$ by $\mathcal{Q}$ is split if it is equivalent as extension of $\mathcal{P}$ by $\mathcal{Q}$ to the neutral object $\mathcal{P} \times \mathcal{Q}$ of the group law defined in 4.6

Proof of Theorem 0.1 b and $\mathbf{c}$. Let $\mathcal{E}=(I: \mathcal{Q} \rightarrow \mathcal{E}, J: \mathcal{E} \rightarrow \mathcal{P})$ be an extension of $\mathcal{P}$ by $\mathcal{Q}$. The strictly commutative Picard $\mathbf{S}$-stacks $\operatorname{HOM}(\mathcal{P}, \mathcal{Q})$ and $\operatorname{HOM}(\mathcal{E}, \mathcal{E})$ are equivalent as strictly commutative Picard $\mathbf{S}$-stacks via the following additive
functor

$$
\begin{aligned}
\operatorname{HOM}(\mathcal{P}, \mathcal{Q}) & \longrightarrow \\
F & \mapsto
\end{aligned}(\mathcal{E} \rightarrow \mathcal{E}+I F J \mathcal{E}) .
$$

By Lemma 1.4 we can conclude that $[\operatorname{HOM}(\mathcal{E}, \mathcal{E})]=\tau_{\leq 0} \operatorname{RHom}([\mathcal{P}],[\mathcal{Q}])$, i.e. the group of isomorphism classes of additive functors from $\mathcal{E}$ to itself is isomorphic to the group $\operatorname{Hom}_{\mathcal{D}(\mathbf{S})}([\mathcal{P}],[\mathcal{Q}])$, and the group of automorphisms of an additive functor from $\mathcal{E}$ to itself is isomorphic to the group $\operatorname{Hom}_{\mathcal{D}(\mathbf{S})}([\mathcal{P}],[\mathcal{Q}][-1])$ (for this last isomorphism see in particular (1.3)).

Proof of Theorem $0.1 \mathbf{a}$. First we construct a morphism from the $\operatorname{group} \mathcal{E} x t^{1}(\mathcal{P}, \mathcal{Q})$ of equivalence classes of extensions of $\mathcal{P}$ by $\mathcal{Q}$ to the group $\operatorname{Ext}^{1}([\mathcal{P}],[\mathcal{Q}])$

$$
\Theta: \mathcal{E} x t^{1}(\mathcal{P}, \mathcal{Q}) \longrightarrow \operatorname{Ext}^{1}([\mathcal{P}],[\mathcal{Q}])
$$

and a morphism from the group $\operatorname{Ext}^{1}([\mathcal{P}],[\mathcal{Q}])$ to the group $\mathcal{E} x t^{1}(\mathcal{P}, \mathcal{Q})$

$$
\Psi: \operatorname{Ext}^{1}([\mathcal{P}],[\mathcal{Q}]) \longrightarrow{\mathcal{E} x t^{1}}^{1}(\mathcal{P}, \mathcal{Q})
$$

Then we check that $\Theta \circ \Psi=I d=\Psi \circ \Theta$ and that $\Theta$ is an homomorphism of groups.
(1) Construction of $\Theta$ : Consider an extension $\mathcal{E}=(I: \mathcal{Q} \rightarrow \mathcal{E}, J: \mathcal{E} \rightarrow \mathcal{P})$ of $\mathcal{P}$ by $\mathcal{Q}$ and denote by $L=(i: K \rightarrow L, j: L \rightarrow M)$ the corresponding extension of complexes in $\mathcal{D}^{[-10]}(\mathbf{S})$. By definition we have the equality $K=\tau_{\leq 0}(M C(j)[-1])$ in the category $\mathcal{D}^{[-10]}(\mathbf{S})$. Hence the distinguished triangle $M C(j)[-1] \rightarrow L \xrightarrow{j}$ $M \rightarrow+$ furnishes the long exact sequence

$$
\begin{equation*}
\cdots \rightarrow \operatorname{Hom}(M, K) \rightarrow \operatorname{Hom}(M, L) \xrightarrow{j o} \operatorname{Hom}(M, M) \xrightarrow{\partial} \operatorname{Ext}^{1}(M, K) \rightarrow \cdots \tag{5.1}
\end{equation*}
$$

We set

$$
\Theta(\mathcal{E})=\partial\left(i d_{M}\right)
$$

The naturality of the connecting map $\partial$ implies that $\Theta(\mathcal{E})$ depends only on the equivalence class of the extension $\mathcal{E}$.

Lemma 5.3. If $\operatorname{Ext}^{1}([\mathcal{P}],[\mathcal{Q}])=0$, then every extension of $\mathcal{P}$ by $\mathcal{Q}$ is split.
Proof. By the long exact sequence (5.1), if the cohomology group $\operatorname{Ext}^{1}(M, K)$ is zero, the identity morphisms $i d_{M}: M \rightarrow M$ lifts to a morphism $f: M \rightarrow L$ of $\mathcal{D}^{[-10]}(\mathbf{S})$ which corresponds via the equivalence of categories (1.4) to an isomorphism classes of additive functors $F: \mathcal{P} \rightarrow \mathcal{E}$ such that $J \circ F \cong I d_{\mathcal{P}}$. Hence $\mathcal{E}$ is a split extension of $\mathcal{P}$ by $\mathcal{Q}$.

The above Lemma means that $\Theta(\mathcal{E})$ is an obstruction for the extension $\mathcal{E}$ to be split: $\mathcal{E}$ is split if and only if $i d_{M}: M \rightarrow M$ lifts to $\operatorname{Hom}(M, L)$ if and only if $\Theta(\mathcal{E})$ vanishes in $\operatorname{Ext}^{1}([\mathcal{P}],[\mathcal{Q}])$.
(2) Construction of $\Psi$ : Choose two complexes $P=\left[P^{-1} \xrightarrow{d_{P}} P^{0}\right]$ and $N=$ $\left[N^{-1} \xrightarrow{d_{N}} N^{0}\right]$ of $\mathcal{D}^{[-10]}(\mathbf{S})$ such that $P^{-1}, P^{0}$ are projective and the three complexes $N, P, M$ build a short exact sequence in $\mathcal{D}^{[-10]}(\mathbf{S})$

$$
\begin{equation*}
0 \longrightarrow N \xrightarrow{s} P \xrightarrow{t} M \longrightarrow 0 \tag{5.2}
\end{equation*}
$$

(because of the projectivity of the $P^{i}$ for $i=-1,0$, there exists a surjective morphism of complexes $P \rightarrow M$ and then, in order to get a short exact sequence, choose
$N^{i}=\operatorname{ker}\left(P^{i} \rightarrow M^{i}\right)$ for $\left.i=-1,0\right)$. By Remark 3.6 the above exact sequence furnishes an extension of strictly commutative Picard $\mathbf{S}$-stacks

$$
s t(N) \xrightarrow{S} s t(P) \xrightarrow{T} \mathcal{P}
$$

where $S$ and $T$ are the isomorphism classes of additive functors corresponding to the morphisms $s$ and $t$. Applying $\operatorname{Hom}(-, K[1])$ to the distinguished triangle $N \rightarrow P \rightarrow M \rightarrow+$ associated to the short exacts sequence (5.2) we get the long exact sequence

$$
\begin{equation*}
\cdots \rightarrow \operatorname{Hom}(M, K) \rightarrow \operatorname{Hom}(P, K) \xrightarrow{\circ s} \operatorname{Hom}(N, K) \xrightarrow{\partial} \operatorname{Ext}^{1}(M, K) \rightarrow 0 . \tag{5.3}
\end{equation*}
$$

Given an element $x$ of $\operatorname{Ext}^{1}(M, K)$, choose an element $u$ of $\operatorname{Hom}(N, K)$ such that $\partial(u)=x$. We set

$$
\Psi(x)=U_{*} \operatorname{st}(P),
$$

i.e. $\Psi(x)$ is the push-down $U_{*} s t(P)$ of the extension $s t(P)$ via one representative of the isomorphism class $U: s t(N) \rightarrow \mathcal{Q}$ of additive functors corresponding to the morphism $u: N \rightarrow K$ of $\mathcal{D}^{[-10]}(\mathbf{S})$. By Lemma 4.4 the strictly commutative Picard S-stack $\Psi(x)$ is an extension of $\mathcal{P}$ by $\mathcal{Q}$. Now we check that the morphism $\Psi$ is well defined, i.e. $\Psi(x)$ doesn't depend on the lift of $x$. If $u^{\prime} \in \operatorname{Hom}(N, K)$ is another lift of $x$, then there exists an element $f$ of $\operatorname{Hom}(P, K)$ such that $u^{\prime}-u=f \circ s$. Consider the push-down $\left(U^{\prime}-U\right)_{*} s t(P)$ of the extension $s t(P)$ via one representative of the isomorphism class $U^{\prime}-U: \operatorname{st}(N) \rightarrow \mathcal{Q}$ of additive functors (as for $u$, we denote here by $U^{\prime}: \operatorname{st}(N) \rightarrow \mathcal{Q}$ the isomorphism class corresponding to the morphism $u^{\prime}: N \rightarrow K$ of $\left.\mathcal{D}^{[-10]}(\mathbf{S})\right)$. Since $u^{\prime}-u=f \circ s$, by the universal property of the push-down there exists a unique additive functor $H:\left(U^{\prime}-U\right)_{*} s t(P) \rightarrow \mathcal{Q}$ such that $H \circ I n \cong I d_{\mathcal{Q}}$, where $\operatorname{In}: \mathcal{Q} \rightarrow\left(U^{\prime}-U\right)_{*} s t(P)$ is the additive functor underlying the extension $\left(U^{\prime}-U\right)_{*} s t(P)$ of $\mathcal{P}$ by $\mathcal{Q}$. Hence the extension $\left(U^{\prime}-U\right)_{*} s t(P)$ of $\mathcal{P}$ by $\mathcal{Q}$ is split and so the extensions $U_{*}^{\prime} s t(P)$ and $U_{*} s t(P)$ are equivalent.
(3) $\Theta \circ \Psi=I d:$ With the notation of (2), given an element $x$ of $\operatorname{Ext}^{1}(M, K)$, choose an element $u$ of $\operatorname{Hom}(N, K)$ such that $\partial(u)=x$. By definition $\Psi(x)=$ $U_{*} s t(P)$. Because of the naturality of the connecting map $\partial$, the following diagram commutes


Therefore $\Theta\left(U_{*} s t(P)\right)=x$, i.e. $\Theta$ surjective.
(4) $\Psi \circ \Theta=I d$ : Consider an extension $\mathcal{E}=(I: \mathcal{Q} \rightarrow \mathcal{E}, J: \mathcal{E} \rightarrow \mathcal{P})$ of $\mathcal{P}$ by $\mathcal{Q}$ and Denote by $L=(i: K \rightarrow L, j: L \rightarrow M)$ the corresponding extension of complexes in $\mathcal{D}^{[-10]}(\mathbf{S})$. Choose two complexes $P=\left[P^{-1} \rightarrow P^{0}\right]$ and $N=\left[N^{-1} \rightarrow N^{0}\right]$ as in (2). The lifting property of the complex $P$ furnishes a lift $u: P \rightarrow L$ of the morphism of complexes $t: P \rightarrow M$ and hence a commutative diagram

where $U: s t(P) \rightarrow \mathcal{E}$ is the isomorphism class of additive functors corresponding to the lift $u: P \rightarrow L$ and $U_{\mid}: s t(N) \rightarrow \mathcal{Q}$ is the restriction of $U$ to $s t(N)$. Consider now the push-down $\left(U_{\mid}\right)_{*} s t(P)$ of the extension $s t(P)$ via a representative of $U_{\mathrm{l}}$. Because of the universal property of the push-down, there exists a unique additive functor $H:\left(U_{\mid}\right)_{*} s t(P) \rightarrow \mathcal{E}$ such that the following diagram commute


Hence we have that the extensions $\left(U_{\mid}\right)_{*} s t(P)$ and $\mathcal{E}$ are equivalent, which implies that $\Psi(\Theta(\mathcal{E}))=\Psi\left(\Theta\left(\left(U_{\mid}\right)_{*} s t(P)\right)\right)=\left(U_{\mid}\right)_{*} s t(P) \cong \mathcal{E}$, i.e. $\Theta$ injective.
(5) $\Theta$ is an homomorphism of groups: Consider two extensions $\mathcal{E}, \mathcal{E}^{\prime}$ of $\mathcal{P}$ by $\mathcal{Q}$. With the notations of (2) we can suppose that $\mathcal{E}=U_{*} s t(P)$ and $\mathcal{E}^{\prime}=U_{*}^{\prime} s t(P)$ with $U, U^{\prime}: \operatorname{st}(N) \rightarrow \mathcal{Q}$ two isomorphism classes of additive functors corresponding to two morphisms $u, u^{\prime}: N \rightarrow K$ of $\mathcal{D}^{[-10]}(\mathbf{S})$. Now by Definition 4.6

$$
\begin{aligned}
\mathcal{E}+\mathcal{E}^{\prime} & =D^{*}\left(+_{\mathcal{Q}}\right)_{*}\left(U_{*} s t(P) \times U_{*}^{\prime} s t(P)\right) \\
& =D^{*}(+\mathcal{Q})_{*}\left(U \times U^{\prime}\right)_{*}(\operatorname{st}(P) \times \operatorname{st}(P)) \\
& =\left(U+U^{\prime}\right)_{*} D^{*}\left(+{ }_{s t(N)}\right)_{*}(\operatorname{st}(P) \times \operatorname{st}(P)) \\
& =\left(U+U^{\prime}\right)_{*}(\operatorname{st}(P)+\operatorname{st}(P))
\end{aligned}
$$

where $D: \mathcal{P} \rightarrow \mathcal{P} \times \mathcal{P}$ is the diagonal additive functor and $+_{\mathcal{Q}}: \mathcal{Q} \times_{\mathbf{S}} \mathcal{Q} \rightarrow \mathcal{Q}$ (resp. $\left.+_{s t(N)}: s t(N) \times_{\mathbf{S}} s t(N) \rightarrow s t(N)\right)$ is the functor underlying the strictly commutative Picard S-stack $\mathcal{Q}$ (resp. st $(N)$ ). If $\partial: \operatorname{Hom}(N, K) \rightarrow \operatorname{Ext}^{1}(M, K)$ is the connecting map of the long exact sequence (15.3), we get

$$
\Theta\left(\mathcal{E}+\mathcal{E}^{\prime}\right)=\partial\left(u+u^{\prime}\right)=\partial(u)+\partial\left(u^{\prime}\right)=\Theta(\mathcal{E})+\Theta\left(\mathcal{E}^{\prime}\right)
$$

Remark 5.4. In the construction of $\Psi: \operatorname{Ext}^{1}([\mathcal{P}],[\mathcal{Q}]) \rightarrow \mathcal{E} x t^{1}(\mathcal{P}, \mathcal{Q})$, instead of two complexes $P=\left[P^{-1} \rightarrow P^{0}\right]$ and $N$ of $\mathcal{D}^{[-10]}(\mathbf{S})$ such that $P^{-1}, P^{0}$ are projective and the three complexes $N, P, M$ build a short exact sequence $0 \rightarrow N \rightarrow P \rightarrow$ $M \rightarrow 0$, we can consider two complexes $I=\left[I^{-1} \rightarrow I^{0}\right]$ and $N^{\prime}$ of $\mathcal{D}^{[-10]}(\mathbf{S})$ such that $I^{-1}, I^{0}$ are injective and the three complexes $K, I, N^{\prime}$ build a short exact sequence $0 \rightarrow K \rightarrow I \rightarrow N^{\prime} \rightarrow 0$. In this case instead of applying $\operatorname{Hom}(-, K[1])$ we apply $\operatorname{Hom}(M,-)$ and instead of considering push-downs of extensions we consider pull-backs. This two way to construct $\Psi$ with projectives or with injectives are dual.

## References

[AN09] E. Aldrovandi and B. Noohi, Butterflies I: Morphisms of 2-group stacks, Advances in Mathematics 221 (2009), pp. 687-773.
[B90] L. Breen, Bitorseurs et cohomologie non abélienne, The Grothendieck Festschrift, Vol. I, Progr. Math., 86, Birkhäuser Boston, Boston, MA, 1990, pp. 401-476.
[B92] L. Breen, Théorie de Schreier supérieure, Ann. Sci. École Norm. Sup. (4) 25 no. 5, 1992, pp. 465-514.
[B09] L. Breen, Notes on 1-and 2-gerbes, in "Towards Higher Categories", J.C. Baez et J.P. May (eds.), The IMA Volumes in Mathematics and its Applications 152, Springer 2009, pp. 193235.
[D73] P. Deligne, La formule de dualité globale, Théorie des topos et cohomologie étale des schémas, Tome 3. Séminaire de Géométrie Algébrique du Bois-Marie 1963-1964 (SGA 4). Lecture Notes in Mathematics, Vol. 305. Springer-Verlag, Berlin-New York, 1973, pp. 481-587.
[G71] J. Giraud, Cohomologie non abélienne, Die Grundlehren der mathematischen Wissenschaften, Band 179. Springer-Verlag, Berlin-New York, 1971.
[R03] A. Rousseau, Bicatégories monoüdales et extensions de gr-catégories, Homology Homotopy Appl. 5 No.1, 2003, pp. 437-547.
[T09] A. Tatar, Length 3 Complexes of Abelian Sheaves and Picard 2-Stacks, arXiv:0906.2393v1 [math.AG], 2009.
[Y10] A. Yekutieli, Central extensions of gerbes, to appear in Advances in Mathematics.
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