Marked bases over quasi-stable modules

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MARKED BASES OVER QUASI-STABLE MODULES

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ABSTRACT. Let K be a field of any characteristic, A a Noetherian K-algebra and consider the polynomial ring $A[x_0, \ldots, x_n]$. The present paper deals with the definition of marked bases for free $A[x_0, \ldots, x_n]$-modules over a quasi-stable monomial module and the investigation of their properties. The proofs of our results are constructive and we can obtain upper bounds for the main invariants of an ideal of $A[x_0, \ldots, x_n]$ generated by a marked basis, such as Betti numbers, (Castelnuovo-Mumford) regularity or projective dimension.

INTRODUCTION

Marked bases may be considered as a form of Gröbner bases which do not depend on a term order. Instead one chooses for each generator some term as head term such that the head terms generate a prescribed monomial ideal. For a long time, it was believed that it was not possible to find marked bases which are not Gröbner bases for some term order. Indeed, in [16] it was shown that the standard normal form algorithm always terminates, if and only if the head terms are chosen via a term order. However, [16] contains no results about other normal form algorithms and in [4, 9] it was proven that the involutive normal form algorithm for the Pommaret division will terminate, whenever the head terms generate a strongly stable ideal over a coefficient field of characteristic zero.

The present work is concerned with generalizing the results of [4, 9] in several directions. Let K be a field *of any characteristic* and A a Noetherian K-algebra. For variables $\mathbf{x} := \{x_0, \ldots, x_n\}$, we consider submodules of the graded $A[\mathbf{x}]$-*module* $A[\mathbf{x}]_{\mathbf{d}}^m := \oplus_{i=1}^m A[\mathbf{x}](-d_i)$ with a *weight vector* $\mathbf{d} = (d_1, \ldots, d_m) \in \mathbb{Z}^m$. We will define marked bases over a *quasi-stable* monomial module U, i.e. over monomial modules possessing a Pommaret basis, for free submodules of $A[\mathbf{x}]_{\mathbf{d}}^m$ and investigate to what extent the algebraic properties of Pommaret bases shown in [18] carry over to marked bases. It will turn out that marked bases provide us with simple upper bounds on some homological invariants of the module generated by a marked basis such as Betti numbers, (Castelnuovo-Mumford) regularity or projective dimension (Corollary 5.8).

Marked bases are of no interest for typical applications of Gröbner bases like normal form computations, as their construction is more involved. In particular, it takes some effort to determine reasonable monomial modules U. However, marked bases provide the central tool for the derivation and the study of low degree equations for the classical Hilbert Scheme and of special loci on it, as shown in [1, 6] for the characteristic zero case. In a forthcoming paper, we will use the results obtained here to generalize the ideas in [1, 6] to positive characteristic and extend them to other special loci of the Hilbert Scheme. Another possible development of our investigations is the explicit study of Quot Schemes [15].

After summarizing the main properties of quasi-stable ideals and modules (Section 2), we fix a graded quasi-stable submodule U of $A[\mathbf{x}]_{\mathbf{d}}^m$ and we define a marked set $G \subset A[\mathbf{x}]_{\mathbf{d}}^m$ over the Pommaret basis $\mathcal{P}(U)$ of U. By the properties of Pommaret bases, it is natural to define a reduction relation which uses the elements of the $A[\mathbf{x}]$-module $\langle G \rangle$. This relation is Noetherian and confluent (Proposition 3.8). Given a $\mathcal{P}(U)$-marked set G, we define the notion of a $\mathcal{P}(U)$-marked basis and provide a number of equivalent characterizations (Theorem 3.11) involving

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some $A[\mathbf{x}]$-modules which can be constructed starting from G. Nevertheless, the equivalent properties of Theorem 3.11 are not sufficient to explicitly check whether a marked set is a basis. Theorem 3.16 gives such an explicit check and allows us to prove that the family $\mathrm{Mf}(U)$ containing all modules generated by a $\mathcal{P}(U)$-marked basis is naturally endowed with an affine scheme structure (Theorem 4.2).

In Section 5 we prove that if $G \subset A[\mathbf{x}]$ is a $\mathcal{P}(J)$-marked basis, where J is a quasi-stable ideal in $A[\mathbf{x}]$, then the module of first syzygies of $\langle G \rangle \subset A[\mathbf{x}]_{\mathbf{d}}^m$ is generated by a $\mathcal{P}(U)$-marked basis, for a suitable quasi-stable module U (Theorem 5.5). Iterating these arguments to the i-th syzygy module of the ideal I generated by the $\mathcal{P}(J)$-marked basis G, we obtain a graded free resolution of I (Theorem 5.6). Although this resolution is not minimal (see Example 5.9), it gives upper bounds on the Betti numbers, regularity and projective dimension of the ideal I (Corollary 5.8).

In Section 6, we consider the truncated ideal $J_{\geqslant m}$, where J is a saturated quasi-stable ideal, and prove that the polynomials in a marked basis over $\mathcal{P}(J_{\geqslant m})$ have a special "shape". More precisely, we prove that for suitable values of m, if G is a $\mathcal{P}(J_{\geqslant m})$-marked basis, then every term in the support of $f \in G$ (excluding the head term) has a minimal variable which is not greater than the minimal variable of the head term (Theorem 6.4). As a consequence, for m big enough, if the head term of $f \in G$ is divisible by x_0, then all the other terms in the support of f are also divisible by x_0 (Corollary 6.6).

1. NOTATIONS AND GENERALITIES

For every $n > 0$, we consider the variables $x_0, \ldots, x_n$, ordered as $x_0 < \cdots < x_{n-1} < x_n$ (see [17, 18]). This is a non-standard way to sort the variables, but it is suitable for our purposes. In some of the papers we refer to, variables are ordered in the opposite way, hence the interested reader should pay attention to this when browsing a reference. A *term* is a power product $x^\alpha = x_0^{\alpha_0} \cdots x_n^{\alpha_n}$. We denote by $\mathbb{T}$ the set of terms in the variables $x_0, \ldots, x_n$. We denote by $\max(x^\alpha)$ the largest variable that appears with non-zero exponent in x^α and, analogously, $\min(x^\alpha)$ is the smallest variable that appears with non-zero exponent in x^α. The *degree* of a term is $\deg(x^\alpha) = \sum_{i=0}^n \alpha_i = |\alpha|$.

Let K be a field and A be a Noetherian K-algebra. Consider the polynomial ring $A[\mathbf{x}] := A[x_0, \ldots, x_n]$ with the standard grading: for every $a \in A$, $\deg(a) = 0$. We write $A[\mathbf{x}]_t$ for the set of homogeneous polynomials of degree t in $A[\mathbf{x}]$. Since $A[\mathbf{x}] = \oplus_{i \geqslant 0} A[\mathbf{x}]_i$, we define $A[\mathbf{x}]_{\geqslant t} := \oplus_{i \geqslant t} A[\mathbf{x}]_i$. The ideals we consider in $A[\mathbf{x}]$ are always homogeneous. If $I \subset A[\mathbf{x}]$ is a homogeneous ideal, we write I_t for $I \cap A[\mathbf{x}]_t$ and $I_{\geqslant t}$ for $I \cap A[\mathbf{x}]_{\geqslant t}$. The ideal $I_{\geqslant t}$ is the *truncation* of I in degree t. If $F \subset A[\mathbf{x}]$ is a set of polynomials, we denote by (F) the ideal generated by F.

The ideal $J \subseteq A[\mathbf{x}]$ is *monomial* if it is generated by a set of terms. The monomial ideal J has a unique minimal set of generators made of terms and we call it *the monomial basis* of J, denoted by B_J. We define $\mathcal{N}(J) \subseteq \mathbb{T}$ as the set of terms in $\mathbb{T}$ not belonging to J. For every polynomial $f \in A[\mathbf{x}]$, $\mathrm{Supp}(f)$ is the set of terms appearing in f with non-zero coefficient: $f = \sum_{x^\alpha \in \mathrm{Supp}(f)} c_\alpha x^\alpha$, where $c_\alpha \in A$ is non-zero. In the sequel, we will simply write *module* (resp. submodule) for $A[\mathbf{x}]$-modules (resp. submodules of a $A[\mathbf{x}]$-module). For modules and submodules over other rings, we will explicitly state the ring.

A module M is *graded* if M has a decomposition

$$M = \oplus_{j \in \mathbb{N}} M_j \quad \text{such that} \quad A[\mathbf{x}]_i M_j \subseteq M_{i+j}. \tag{1.1}$$

As usual, if M is a graded module, the module $M(d)$ is the graded module isomorphic to M such that $M(d)_e = M_{d+e}$. We fix $m \geqslant 1$ and $\mathbf{d} = (d_1, \ldots, d_m) \in \mathbb{Z}^m$. We consider the free graded $A[\mathbf{x}]$-module $A[\mathbf{x}]_{\mathbf{d}}^m := \oplus_{i=1}^m A[\mathbf{x}](-d_i) e_i$, where $e_1, \ldots, e_m$ are the standard free generators. Every submodule of $A[\mathbf{x}]_{\mathbf{d}}^m$ is finitely generated and from now on we will only consider graded submodules of $A[\mathbf{x}]_{\mathbf{d}}^m$.

If F is a set of homogeneous elements of $A[\mathbf{x}]_{\mathbf{d}}^m$, we write $\langle F \rangle$ for the graded $A[\mathbf{x}]$-module generated by F in $A[\mathbf{x}]_{\mathbf{d}}^m$. If the elements in F have the same degree s, we denote by $\langle F \rangle^A$ the A-module generated by F in $(A[\mathbf{x}]_{\mathbf{d}}^m)_s$. In particular, if M is a graded submodule, every graded component M_j has the structure of A-module.

Following [10, Chapter 15], a term of $A[\mathbf{x}]_{\mathbf{d}}^m$ is an element of the form $t = x^\alpha e_i$ for $i \in \{1, \ldots, m\}$ and $x^\alpha \in \mathbb{T}$. Furthermore, we denote by $\mathbb{T}^m$ the set of terms in $A[\mathbf{x}]_{\mathbf{d}}^m$. Observe that $\mathbb{T}^m = \cup_{i=1}^m \mathbb{T}e_i$. For $x^\alpha e_i, x^\beta e_j$ in $\mathbb{T}^m$ we say that $x^\alpha e_i$ divides $x^\beta e_j$ if $i = j$ and x^α divides x^β.

A submodule U of $A[\mathbf{x}]_{\mathbf{d}}^m$ is *monomial*, if it is generated by elements in $\mathbb{T}^m$. Any monomial submodule U of F can be written as

$$U = \oplus_{k=1}^m J^{(k)} e_k \subset \oplus A[\mathbf{x}](-d_k)e_k = A[\mathbf{x}]_{\mathbf{d}}^m, \tag{1.2}$$

where $J^{(k)}$ is the monomial ideal generated by the terms x^α such that $x^\alpha e_k \in U$. We define $\mathcal{N}(U) := \cup_{k=1}^m \mathcal{N}(J^{(k)})e_k$, where $\mathcal{N}(J^{(k)}) \subseteq \mathbb{T}$.

If $M \subset A[\mathbf{x}]_{\mathbf{d}}^m$ is a submodule such that for every degree s, the homogeneous component M_s is a free A-module, we define the *Hilbert function* of M as $h_M(s) = \mathrm{rk}(M_s)$, which is the number of generators contained in an A-basis of M_s. In this case, we will also say that M *admits a Hilbert function*. In this setting, this definition corresponds to the classical one (e.g. [10, Chapter 12]), considering the localization of A in any of its maximal ideals. If we consider a monomial module U, every component U_s is always a free A-module and $h_U(s) = \sum_{k=1}^m h_{J^{(k)}}(s)$, with $J^{(k)}$ as in (1.2).

If $A = K$, then Hilbert's Syzygy Theorem guarantees that every module $M \subseteq A[\mathbf{x}]_{\mathbf{d}}^m$ has a graded free resolution of length at most n. If A is an arbitrary K-algebra, there exist generally modules in $A[\mathbf{x}]_{\mathbf{d}}^m$ whose minimal free resolution has an infinite length (see [10, Chapter 6, Section 1, Exercise 11]).

Assume that the module $M \subseteq A[\mathbf{x}]_{\mathbf{d}}^m$ has the following graded minimal free resolution

$$0 \to E_p \to \cdots \to E_1 \to E_0 \to M \to 0,$$

where $E_i = \oplus_j A[\mathbf{x}](-j)^{b_{i,j}}$. The *Betti numbers* of the module M are the set of positive integers $\{b_{i,j}\}_{0 \leqslant i \leqslant p, j \in \mathbb{Z}}$. The module M is t-regular if $t \geqslant j - i$ for every i, j such that $b_{i,j} \neq 0$. The *(Castelnuovo-Mumford) regularity* of M, denoted by $\mathrm{reg}(M)$, is the smallest t for which M is t-regular (see for instance [11]). If $M \subset A[\mathbf{x}]_{\mathbf{d}}^m$ admits a Hilbert function, we recall that $h_{A[\mathbf{x}]^m/M}(s) = P(s)$ for all degrees $s \geqslant \mathrm{reg}(M)$. The *projective dimension* of M, denoted by $\mathrm{pdim}(M)$, is defined as the length of the graded minimal free resolution, i.e. $\mathrm{pdim}(M) = p$.

If I and J are homogenous ideals in $A[\mathbf{x}]$, we define $(I : J)$ as the ideal $\{f \in A[\mathbf{x}] \mid fJ \subseteq I\}$; we will briefly write $(I : x_i)$ for $(I : (x_i))$. Further, we define $(I : J^\infty) := \cup_{j \geqslant 0}(I : J^j)$; again, we will write $(I : x_i^\infty)$ for $(I : (x_i)^\infty)$. The ideal $I \subset A[\mathbf{x}]$ is *saturated* if $I = (I : (x_0, \ldots, x_n)^\infty)$. The *saturation* of I is $I^{\mathrm{sat}} = (I : (x_0, \ldots, x_n)^\infty)$ and I is m-saturated if $I_t = I_t^{\mathrm{sat}}$ for every $t \geqslant m$.

2. Pommaret basis, Quasi-Stability and Stability

We now recall the definition and some properties of the Pommaret basis of a monomial ideal. Several of the following definitions and properties hold in a more general setting, namely for arbitrary *involutive divisions*. For a deeper insight into this topic, we refer to [17, 18] and the references therein. For a set of terms $M \subset \mathbb{T}$, we denote by (M) the ideal generated by M in the polynomial ring $A[\mathbf{x}]$.

For an arbitrary term $x^\alpha \in \mathbb{T}$, we define the following sets:

- the *multiplicative variables of x^α*: $\mathcal{X}_{\mathcal{P}}(x^\alpha) := \{x_i \mid x_i \leqslant \min(x^\alpha)\}$,
- the *nonmultiplicative variables of x^α*: $\overline{\mathcal{X}}_{\mathcal{P}}(x^\alpha) := \{x_0, \ldots, x_n\} \setminus \mathcal{X}_{\mathcal{P}}(x^\alpha)$.

Definition 2.1. [17] Consider $x^\alpha \in \mathbb{T}$. The *Pommaret cone* of $x^\alpha \in \mathbb{T}$ is the set of terms $\mathcal{C}_{\mathcal{P}}(x^\alpha) = \{x^\alpha x^\delta \mid x^\delta \in A[\mathcal{X}_{\mathcal{P}}(x^\alpha)]\}$. Let $M \subset \mathbb{T}$ be a finite set. The *Pommaret span* of M is

$$\langle M \rangle_{\mathcal{P}} := \bigcup_{x^\alpha \in M} \mathcal{C}_{\mathcal{P}}(x^\alpha). \tag{2.1}$$

The finite set of terms M is a *weak Pommaret basis* if $\langle M \rangle_{\mathcal{P}} = (M) \cap \mathbb{T}$ and it is a *Pommaret basis* if the union on the right hand side of (2.1) is disjoint.

If J is a monomial ideal, we denote its Pommaret basis (if it exists) by $\mathcal{P}(J)$. The existence of the Pommaret basis of a monomial ideal in $A[\mathbf{x}]$ is equivalent to the concept of *quasi-stability*.[1] We recall here the definition of quasi-stable and stable monomial ideals. Both properties do not depend on the characteristic of the underlying field. A thorough reference on this subject is again [18].

Definition 2.2. [8, Definition 4.4] Let $J \subset A[\mathbf{x}]$ be a monomial ideal.
  (i) J is *quasi-stable* if for every term $x^\alpha \in J \cap \mathbb{T}$ and for every non-multiplicative variable $x_j \in \overline{\mathcal{X}}_{\mathcal{P}}(x^\alpha)$ of it, there is an exponent $s \geqslant 0$ such that $x_j^s x^\alpha / \min(x^\alpha) \in J$.
  (ii) J is *stable* if for every term $x^\alpha \in J \cap \mathbb{T}$ and for every non-multiplicative variable $x_j \in \overline{\mathcal{X}}_{\mathcal{P}}(x^\alpha)$ of it we have $x_j x^\alpha / \min(x^\alpha) \in J$.

Remark 2.3. In order to establish whether J is quasi-stable or stable it is sufficient to check the conditions of Definition 2.2 on the terms $x^\alpha \in B_J$ contained in the minimal basis.

Theorem 2.4. [18, Proposition 4.4][14, Remark 2.10] *Let $J \subset A[\mathbf{x}]$ be a monomial ideal. J is quasi-stable if and only if it has a (finite) Pommaret basis, denoted by $\mathcal{P}(J)$. Furthermore, J is stable if and only if $\mathcal{P}(J) = B_J$.*

Remark 2.5. Products, intersections, sums and quotients of quasi-stable ideals are again quasi-stable (see [18, Lemma 4.6]). In particular, if $J \subset A[\mathbf{x}]$ is quasi-stable, then any truncation $J_{\geqslant m}$ is quasi-stable, too.

The following lemma collects some properties of Pommaret bases and of the ideals generated by them. They show in particular that certain invariants can be directly read off from a Pommaret basis.

Lemma 2.6. *Let J be a quasi-stable ideal in $A[\mathbf{x}]$.*
  (i) *The satiety of J is the maximal degree of a term in $\mathcal{P}(J)$ which is divisible by the smallest variable in the polynomial ring.*
  (ii) *The regularity of J is the maximal degree of a term in $\mathcal{P}(J)$.*
  (iii) *The projective dimension of J is $n - D$ where $D = \min\{\min(x^\alpha) \mid x^\alpha \in \mathcal{P}(J)\}$.*
  (iv) *If $x^\alpha \in J \setminus \mathcal{P}(J)$, then $x^\alpha / \min(x^\alpha) \in J$.*
  (v) *If $x^\eta \notin J$ and $x_i x^\eta \in J$, then either $x_i x^\eta \in \mathcal{P}(J)$ or $x_i \in \overline{\mathcal{X}}_{\mathcal{P}}(x^\eta)$.*
  (vi) *If $x^\eta \notin J$ and $x^\eta \cdot x^\delta = x^{\delta'} x^\alpha \in J$ with $x^\alpha \in \mathcal{P}(J)$ and $x^{\delta'} \in A[\mathcal{X}_{\mathcal{P}}(x^\alpha)]$, then $x^{\delta'} <_{lex} x^\delta$.*

Proof. Items (i), (ii) and (iii) are proven in [18, Lemma 4.11, Theorems 9.2 and 8.11]. Items (iv) and (v) are shown in [3, Lemma 3], item (vi) is a consequence of (v). □

Definition 2.7. Let J be a quasi-stable ideal in $A[\mathbf{x}]$. We define for every index $1 \leqslant i \leqslant n$ the following two sets of terms:
$$\mathcal{P}(J)(i) := \{x^\alpha \in \mathcal{P}(J) \mid \min(x^\alpha) = x_i\},$$
$$\overline{\mathcal{P}(J)(i)} := \left\{ \frac{x^\alpha}{x_i^{\alpha_i}} \mid x^\alpha \in \mathcal{P}(J)(i) \right\}.$$

Lemma 2.8. [18, Lemma 4.11]. *Let J be a quasi-stable ideal in $A[\mathbf{x}]$ with Pommaret basis $\mathcal{P}(J)$ and consider an index $0 \leqslant j \leqslant n$.*
  (i) *The set*
$$\overline{\mathcal{P}(J)(j)} \cup \bigcup_{i=j+1}^{n} \mathcal{P}(J)(i);$$
  is a weak Pommaret basis of the ideal $J : (x_n, \ldots, x_j)^\infty$.
  (ii) *If J is a saturated ideal, then no term in $\mathcal{P}(J)$ is divisible by x_0.*

Proposition 2.9. [18, Lemma 2.2, Lemma 2.3, Theorem 9.2, Proposition 9.6]

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[1]In the literature, one can find a number of alternative names for quasi-stable ideals like weakly stable ideals[7], ideals of nested type [2] or ideals of Borel type [12].

(i) Let $M \subset \mathbb{T}$ be a finite set of terms of degree s. If for every term $x^\alpha \in M$ and for every non-multiplicative variable $x_j \in \overline{\mathcal{X}}_\mathcal{P}(x^\alpha)$ of it the term $x_j x^\alpha / \min(x^\alpha)$ also lies in M, then M is a Pommaret basis.

(ii) Let $J \subset A[\mathbf{x}]$ be a quasi-stable ideal generated in degrees less than or equal to s. The ideal J is s-regular if and only if $J_{\geqslant s}$ is stable.

(iii) Let J be a quasi-stable ideal in $A[\mathbf{x}]$ and consider a degree $s \geqslant \operatorname{reg}(J)$. Then $J_{\geqslant s}$ is stable and the set of terms $J_s \cap \mathbb{T}$ is its Pommaret basis.

Finally, we consider briefly the extension of these results to the module case.

Definition 2.10. Let U be a monomial submodule of $A[\mathbf{x}]_\mathbf{d}^m$ and let $T \subset \mathbb{T}^m$ be a finite set of monomial generators for U. For every $\tau = x^\alpha e_k$ in T, we define the *Pommaret cone in* $A[\mathbf{x}]_\mathbf{d}^m$ of τ as

$$\mathcal{C}_\mathcal{P}^m(\tau) := \{x^\gamma e_k \mid x^\gamma \in \mathcal{C}_\mathcal{P}(x^\alpha) \subset A[\mathbf{x}]\} \subset \mathbb{T}e_k.$$

We say that T is a *Pommaret basis* of U if

$$U \cap \mathbb{T}^m = \bigsqcup_{\tau \in T} \mathcal{C}_\mathcal{P}^m(\tau).$$

The monomial submodule $U \subseteq A[\mathbf{x}]^m$ is *quasi-stable*, if it is generated by a Pommaret basis.

All the notions of this section are readily extended to a monomial submodule $U \subseteq A[\mathbf{x}]^m$: indeed, recalling that the monomial module U can be written as $U = \oplus_{k=1}^m J^{(k)} e_k$, with $J^{(k)}$ suitable monomial ideals in $A[\mathbf{x}]$ (see (1.2)), it is immediate to state that the monomial module U is quasi-stable if and only if $J^{(k)}$ is a quasi-stable ideal for every $k \in \{1, \ldots, m\}$.

3. Marked Modules

In this section, we extend the notions of a marked polynomial, a marked basis and a marked family, investigated in [4, 8, 9, 13] for ideals, to finitely generated modules in $A[\mathbf{x}]_\mathbf{d}^m$. Let $U \subset A[\mathbf{x}]_\mathbf{d}^m$ be a monomial module, so that $U = \oplus_{k=1}^m J^{(k)} e_k$, with $J^{(k)}$ monomial ideal in $A[\mathbf{x}]$. If U is a quasi-stable module, we denote by $\mathcal{P}(U)$ the Pommaret basis of U.

Definition 3.1. [16] A *marked polynomial* is a polynomial $f \in A[\mathbf{x}]$ together with a fixed term x^α in $\operatorname{Supp}(f)$ whose coefficient is equal to 1_A. This term is called *head term* of f and denoted by $\operatorname{Ht}(f)$. With a marked polynomial f, we associate the following sets:

- the *multiplicative variables of* f: $\mathcal{X}_\mathcal{P}(f) := \mathcal{X}_\mathcal{P}(\operatorname{Ht}(f))$;
- the *nonmultiplicative variables of* f: $\overline{\mathcal{X}}_\mathcal{P}(f) := \overline{\mathcal{X}}_\mathcal{P}(\operatorname{Ht}(f))$.

Definition 3.2. A *marked homogeneous module element* is a homogeneous module element in $A[\mathbf{x}]_\mathbf{d}^m$ with a fixed term in its support whose coefficient is 1_A and which is called *head term*. More precisely, a marked homogeneous module element is of the form

$$f_\alpha^k = f_\alpha e_k - \sum_{l \neq k} g_l e_l \in A[\mathbf{x}]_\mathbf{d}^m$$

where f_α is a marked polynomial with $\operatorname{Ht}(f_\alpha) = x^\alpha$, and $\operatorname{Ht}(f_\alpha^k) = \operatorname{Ht}(f_\alpha)e_k = x^\alpha e_k$.

The following definition is fundamental for this work. It is modelled on a well-known characteristic property of Gröbner bases.

Definition 3.3. Let $T \subset \mathbb{T}^m$ be a finite set and U the module generated by it in $A[\mathbf{x}]_\mathbf{d}^m$. A T-*marked set* is a finite set $G \subset A[\mathbf{x}]_\mathbf{d}^m$ of marked homogeneous module elements f_α^k with $\operatorname{Ht}(f_\alpha^k) = x^\alpha e^k \in T$ and $\operatorname{Supp}(f_\alpha^k - x^\alpha e_k) \subset \langle \mathcal{N}(U) \rangle$ (obviously, $|G| = |T|$).

The T-marked set G is a T-*marked basis*, if $\mathcal{N}(U)_s$ is a basis of $(A[\mathbf{x}]_\mathbf{d}^m)_s / \langle G \rangle_s$ as A-module, i.e. if $(A[\mathbf{x}]_\mathbf{d}^m)_s = \langle G \rangle_s \oplus \langle \mathcal{N}(U)_s \rangle^A$ for all s.

Lemma 3.4. Let $T \subset \mathbb{T}^m$ be a finite set and U the module generated by it in $A[\mathbf{x}]_\mathbf{d}^m$. Let $M \subseteq A[\mathbf{x}]_\mathbf{d}^m$ be a module such that for every s the set $\mathcal{N}(U)_s$ generates the A-module $(A[\mathbf{x}]_\mathbf{d}^m)_s / M_s$. Then for every degree s there exists a $U_s \cap \mathbb{T}^m$-marked set $F = F_s$ contained in M_s such that

$$(A[\mathbf{x}]_\mathbf{d}^m)_s = \langle F \rangle^A \oplus \langle \mathcal{N}(U)_s \rangle^A.$$

Proof. Let π be the usual projection morphism of $A[\mathbf{x}]_{\mathbf{d}}^m$ onto the quotient $A[\mathbf{x}]_{\mathbf{d}}^m/M$. For every $x^\alpha e_k \in U_s \cap \mathbb{T}^m$, we consider $\pi(x^\alpha e_k)$ and choose a representation $\pi(x^\alpha e_k) = \sum_{x^\eta e_l \in \mathcal{N}(U)_s} c_{\eta l}^{\alpha k} x^\eta e_l$, $c_{\eta l}^{\alpha k} \in A$, which exists as $\mathcal{N}(U)_s$ generates $(A[\mathbf{x}]_{\mathbf{d}}^m)_s/M_s$ as an A-module. We consider the set of marked module elements $F = \{f_\alpha^k\}_{x^\alpha e_k \in U_s}$, where $f_\alpha^k := x^\alpha e_k - \pi(x^\alpha e_k)$ and $\mathrm{Ht}(f_\alpha^k) = x^\alpha e_k$.

We now prove that $A[\mathbf{x}]_s^m = \langle F \rangle^A \oplus \langle \mathcal{N}(U)_s \rangle^A$. We first prove that every term in $\mathbb{T}_s^m$ belongs to $\langle F \rangle^A + \langle \mathcal{N}(U)_s \rangle^A$. If $x^\beta e_l \in \mathcal{N}(U)_s$, there is nothing to prove. If $x^\beta e_l \in U_s$, then there is $f_\beta^l \in F$ such that $\mathrm{Ht}(f_\beta^l) = x^\beta e_l$, hence we can write $x^\beta e_l = f_\beta^l + (x^\beta e_l - f_\beta^l) = f_\beta^l + \pi(x^\beta e_l)$.

We conclude proving that $\langle F \rangle^A \cap \langle \mathcal{N}(U)_s \rangle^A = \{0_A^m\}$. Let $g \in A[\mathbf{x}]_{\mathbf{d}}^m$ be an element belonging to $\langle F \rangle^A \cap \langle \mathcal{N}(U)_s \rangle^A$: $g = \sum_{f_\alpha^k \in F} \lambda_{\alpha k} f_\alpha^k \in \langle \mathcal{N}(U)_s \rangle$. Since the head terms of f_α^k cannot cancel each other, $\lambda_{\alpha k} = 0$ for every α and k and hence $g = 0$. □

We specialize now to the case that U is a quasi-stable module and $T = \mathcal{P}(U)$ its Pommaret basis. We study a reduction relation naturally induced by any basis marked over such a set T. In particular, we show that it is confluent and Noetherian just as the familiar reduction relation induced by a Gröbner basis.

Definition 3.5. Let $U \subseteq A[\mathbf{x}]_{\mathbf{d}}^m$ be a quasi-stable module and G be a $\mathcal{P}(U)$-marked set in $A[\mathbf{x}]_{\mathbf{d}}^m$. We introduce the following sets:
- $G^{(s)} := \left\{ x^\delta f_\alpha^k \mid f_\alpha^k \in G, x^\delta \in A[\mathcal{X}_\mathcal{P}(f_\alpha^k)], \deg x^\delta f_\alpha^k = s, \right\}$;
- $\widehat{G}^{(s)} := \left\{ x^\delta f_\alpha^k \mid f_\alpha^k \in G, x^\delta \notin A[\mathcal{X}_\mathcal{P}(f_\alpha^k)], \deg x^\delta f_\alpha^k = s \right\} = \left\{ x^\delta f_\alpha^k \mid f_\alpha^k \in G, x^\delta f_\alpha^k \notin G^{(s)} \right\}$;
- $\mathcal{N}(U, \langle G \rangle) := \langle G \rangle \cap \langle \mathcal{N}(U) \rangle$.

Lemma 3.6. *Let $U \subseteq A[\mathbf{x}]_{\mathbf{d}}^m$ be a quasi-stable module and G a $\mathcal{P}(U)$-marked set. For every product $x^\delta f_\alpha^k$ with $f_\alpha^k \in G$, each term in $\mathrm{Supp}(x^\delta x^\alpha e_k - x^\delta f_\alpha^k)$ either belongs to $\mathcal{N}(U)$ or is of the form $x^\eta x^\nu e_l \in \mathcal{C}_\mathcal{P}^m(x^\nu e_l)$ with $x^\nu e_l \in \mathcal{P}(U)$ and $x^\eta <_{lex} x^\delta$.*

Proof. It is sufficient to consider $x^\delta x^\beta e_l \in \mathrm{Supp}(x^\delta x^\alpha e_k - x^\delta f_\alpha^k) \cap U$. Then $x^\delta x^\beta \in J^{(l)}$ for some quasi-stable ideal $J^{(l)} \subset A[\mathbf{x}]$ appearing in (1.2). Therefore there exists $x^\gamma \in \mathcal{P}(J^{(l)})$ such that $x^\delta x^\beta \in \mathcal{C}_\mathcal{P}(x^\gamma)$. More precisely, if $x^\eta := x^\delta x^\beta/x^\gamma$, then $x^\eta <_{lex} x^\delta$ by Lemma 2.6 (vi). □

Note in the next definition the use of the set $G^{(s)}$, which means that we use here a generalization of the involutive reduction relation associated with the Pommaret division and not of the standard reduction relation in the theory of Gröbner bases. This modification is the key for circumventing the restrictions imposed by the results of [16]. It also entails that if a term is reducible, then there is only one element in the marked basis which can be used for its reduction.

Definition 3.7. Let $U \subseteq A[\mathbf{x}]_{\mathbf{d}}^m$ be a quasi-stable module and G a $\mathcal{P}(U)$-marked set. We denote by $\xrightarrow{G^{(s)}}$ the transitive closure of the relation $h \xrightarrow{G^{(s)}} h - \lambda x^\eta f_\alpha^k$ where $x^\eta x^\alpha e_k$ is a term that appears in h with a non-zero coefficient $\lambda \in A$ and which satisfies $\deg(x^\eta x^\alpha e_k) = s$ and $x^\eta f_\alpha^k \in G^{(s)}$. We will write $h \xrightarrow{G^{(s)}}_* g$ if $h \xrightarrow{G^{(s)}} g$ and $g \in \langle \mathcal{N}(U) \rangle$. Observe that if $h \in (A[\mathbf{x}]_{\mathbf{d}}^m)_s$, then $h \xrightarrow{G^{(s)}} g \in (A[\mathbf{x}]_{\mathbf{d}}^m)_s$.

Proposition 3.8. *Let $U \subseteq A[\mathbf{x}]_{\mathbf{d}}^m$ be a quasi-stable module and G a $\mathcal{P}(U)$-marked set. The reduction relation $\xrightarrow{G^{(s)}}$ is confluent and Noetherian.*

Proof. It is sufficient to prove that for every term $x^\gamma e_k$ in U, there is a unique $g \in A[\mathbf{x}]_{\mathbf{d}}^m$ such that $x^\gamma e_k \xrightarrow{G^{(s)}}_* g$ and $g \in \langle \mathcal{N}(U) \rangle^A$.

Since $x^\gamma e_k \in U$, there exists a unique $x^\delta f_\alpha^k \in G^{(s)}$ such that $x^\delta \mathrm{Ht}(f_\alpha^k) = x^\gamma e_k$. Hence, $x^\gamma e_k \xrightarrow{G^{(s)}} x^\gamma e_k - x^\delta f_\alpha^k$. If we could proceed in the reduction without obtaining an element in $\langle \mathcal{N}(U) \rangle$, we would obtain by Lemma 3.6 an infinite lex-descending chain of terms in $\mathbb{T}$ which is impossible since lex is a well-ordering. Hence $\xrightarrow{G^{(s)}}$ is Noetherian. Confluence is immediate by the uniqueness of the element of $G^{(s)}$ that is used at each step of reduction. □

Proposition 3.9. *Let $U \subseteq A[\mathbf{x}]^m$ be a quasi-stable module and G be a $\mathcal{P}(U)$-marked set. Every term $x^\beta e_k \in \mathbb{T}_s^m$ of degree s can be uniquely expressed in the form*

$$x^\beta e_l = \sum_i \lambda_{\delta_i \alpha_i k_i} x^{\delta_i} f_{\alpha_i}^{k_i} + g,$$

where $\lambda_{\delta_i \alpha_i k_i} \in A \setminus \{0_A\}$, $x^{\delta_i} f_{\alpha_i}^{k_i} \in G^{(s)}$, $g \in \langle \mathcal{N}(U) \rangle^A$ and the terms x^{δ_i} form a sequence which is strictly descending with respect to lex.

Proof. For terms in $\mathcal{N}(U)$, there is nothing to prove. For $x^\beta e_l \in U$, it is sufficient to consider $g \in \langle \mathcal{N}(U) \rangle^A$ such that $x^\beta e_l \xrightarrow{G^{(s)}}_* g$. The polynomials $x^{\delta_i} f_{\alpha_i}^{k_i} \in G^{(s)}$ are exactly those used during the reduction $\xrightarrow{G^{(s)}}$. They fulfill the statement on the terms x^{δ_i} by Lemma 3.6. $\qquad\square$

Corollary 3.10. *Let $U \subseteq A[\mathbf{x}]_{\mathbf{d}}^m$ be a quasi-stable module and G be a $\mathcal{P}(U)$-marked set. Consider a homogeneous element $g \in A[\mathbf{x}]_{\mathbf{d}}^m$ such that $g = \sum_{i=1}^m \lambda_i x^{\delta_i} f_{\alpha_i}^{k_i}$, with $\lambda_i \in A \setminus \{0\}$ and $x^{\delta_i} f_{\alpha_i}^{k_i} \in G^{(s)}$ with $s = \deg(g)$ and $x^{\delta_i} f_{\alpha_i}^{k_i}$ pairwise different. Then $g \neq 0_A^m$ and $g \notin \langle \mathcal{N}(U) \rangle^A$.*

Proof. The statement follows from the definition of $G^{(s)}$ and the properties of $\xrightarrow{G^{(s)}}$. $\qquad\square$

The following theorem and corollary collect some basic properties of sets marked over a Pommaret basis. They generalize analogous statements in [13, Theorems 1.7, 1.10] which consider only ideals and marked bases where the head terms generate a strongly stable ideal.

Theorem 3.11. *Let $U \subset A[\mathbf{x}]_{\mathbf{d}}^m$ be a quasi-stable module and G a $\mathcal{P}(U)$-marked set. Then, we have for every degree s the following decompositions of A-modules:*

(i) $\langle G \rangle_s = \langle G^{(s)} \rangle^A + \langle \widehat{G}^{(s)} \rangle^A$;

(ii) $(A[\mathbf{x}]_{\mathbf{d}}^m)_s = \langle G^{(s)} \rangle^A \oplus \langle \mathcal{N}(U)_s \rangle^A$;

(iii) *the A-module $\langle G^{(s)} \rangle^A$ is free of rank equal to $|G^{(s)}| = \mathrm{rk}(U_s)$ and it is generated (as an A-module) by a unique $U_s \cap \mathbb{T}^m$-marked set $\widetilde{G}^{(s)}$;*

(iv) $\langle G \rangle_s = \langle G^{(s)} \rangle^A \oplus \mathcal{N}(U, \langle G \rangle)_s$.

Moreover, the following conditions are equivalent:

(v) *G is a $\mathcal{P}(U)$-marked basis;*

(vi) *for all degrees s, $\langle G \rangle_s = \langle G^{(s)} \rangle^A$;*

(vii) $\mathcal{N}(U, \langle G \rangle) = \{0_A^m\}$;

(viii) *for all s, $\bigwedge^{Q(s)+1} \langle G \rangle_s = 0_A$, where $Q(s) := \mathrm{rk}(U_s)$.*

Proof. Item (i): immediate.

Item (ii) is a consequence of Proposition 3.9 and Corollary 3.10.

Item (iii): we can repeat the arguments of [13, Theorem 1.7]: for every s, we may construct a $U_s \cap \mathbb{T}^m$-marked set $\widetilde{G}^{(s)}$ such that $(A[\mathbf{x}]_{\mathbf{d}}^m)_s = \langle \widetilde{G}^{(s)} \rangle^A \oplus \langle \mathcal{N}(U)_s \rangle^A$ using Lemma 3.4. By item (ii), the $U_s \cap \mathbb{T}^m$-marked set $\widetilde{G}^{(s)}$ is unique and furthermore $\langle \widetilde{G}^{(s)} \rangle^A = \langle G^{(s)} \rangle^A$.

Item (iv): by items (i) and (iii), we have $\langle G \rangle_s = \langle \widetilde{G}^{(s)} \rangle^A + \langle \widehat{G}^{(s)} \rangle^A$. Recalling that $\langle \widetilde{G}^{(s)} \rangle^A \cap \langle \mathcal{N}(U)_s \rangle^A = \{0_A^m\}$ by Lemma 3.4, it is sufficient to show that every $g \in \langle \widehat{G}^{(s)} \rangle^A$ can be written $g = f + h$ with $f \in \langle \widetilde{G}^{(s)} \rangle^A$ and $h \in \langle \mathcal{N}(U)_s \rangle^A$: we express every term $x^\beta e_l \in U_s$ appearing in g with non-zero coefficient in the form $x^\beta e_l = \widetilde{f}_\beta^l + (x^\beta e_l - \widetilde{f}_\beta^l)$ where $\widetilde{f}_\beta^l$ is the unique polynomial in $\widetilde{G}^{(s)}$ with $\mathrm{Ht}(\widetilde{f}_\beta^l) = x^\beta e_l$. By construction, $h \in \mathcal{N}(U, \langle G \rangle)_s$. By item (ii), we obtain the assertion.

Items (v), (vi), (vii) are equivalent by the previous items, using again the same proof as in [13, Theorem 1.7].

With respect to [13], the only new item is (viii), which is obviously equivalent to (vi) and (vii). In fact, by (iii) and (iv) we find that $\langle G \rangle_s = \langle G^{(s)} \rangle^A \oplus \mathcal{N}(U, \langle G \rangle)_s$ and $\mathrm{rk} \langle G^{(s)} \rangle^A = \mathrm{rk}(U_s) = Q(s)$. $\qquad\square$

Remark 3.12. If $G \subset A[\mathbf{x}]^m$ is a $\mathcal{P}(U)$-marked basis, then, by Theorem 3.11 (ii), (iii) and (vi), then the $A[\mathbf{x}]$-module $\langle G \rangle$ admits a Hilbert function, which is the same as the Hilbert function of the monomial module U.

Corollary 3.13. *Let* $U \subset A[\mathbf{x}]^m_{\mathbf{d}}$ *be a quasi-stable module and* G *be a* $\mathcal{P}(U)$-*marked set. The following conditions are equivalent:*

(i) *G is a $\mathcal{P}(U)$-marked basis;*
(ii) *$\langle G \rangle_s = \langle G^{(s)} \rangle^A$ for every $s \leqslant \operatorname{reg}(U) + 1$;*
(iii) *$\mathcal{N}(U, \langle G \rangle)_s = \{0^m_A\}$ for every $s \leqslant \operatorname{reg}(U) + 1$;*
(iv) *$\bigwedge^{Q(s)+1} \langle G \rangle_s = 0_A$ for every $s \leqslant \operatorname{reg}(U) + 1$.*

Proof. By the second part of Theorem 3.11, item (i) implies item (ii) and items (ii), (iii), (iv) are equivalent. For the proof that item (ii) implies (i), it is sufficient to repeat the arguments of [13, Theorem 1.10]. $\square$

Corollary 3.14. *Let* $U \subset A[\mathbf{x}]^m_{\mathbf{d}}$ *be a quasi-stable module, such that* $U = \oplus J^{(k)} e_k$ *with* $J^{(k)}$ *saturated ideal for every k, and G be a $\mathcal{P}(U)$-marked set. Then the following conditions are equivalent:*

(i) *G is a $\mathcal{P}(U)$-marked basis;*
(ii) *$\langle G \rangle_{\operatorname{reg}(U)+1} = \langle G^{(\operatorname{reg}(U)+1)} \rangle^A$;*
(iii) *$\mathcal{N}(U, \langle G \rangle)_{\operatorname{reg}(U)+1} = \{0^m_A\}$;*
(iv) *$\bigwedge^{Q+1} \langle G \rangle_{\operatorname{reg}(U)+1} = 0_A$, where $Q := \operatorname{rk}(U_{\operatorname{reg}(U)+1})$.*

Proof. The equivalence among items (ii), (iii) and (iv) is immediate by Theorem 3.11. We only prove that items (i) and (iii) are equivalent. If G is a $\mathcal{P}(U)$-marked basis, then by Theorem 3.11 we have $\mathcal{N}(U, \langle G \rangle)_{\operatorname{reg}(U)+1} = \{0^m_A\}$.

We now assume that $\mathcal{N}(U, \langle G \rangle)_{\operatorname{reg}(U)+1} = \{0^m_A\}$ and prove that $\mathcal{N}(U, \langle G \rangle) = \{0^m_A\}$. By Corollary 3.13, it is sufficient to prove that $\mathcal{N}(U, \langle G \rangle)_s = \{0^m_A\}$ for every $s \leqslant \operatorname{reg}(U)$. If $f \in \mathcal{N}(U, \langle G \rangle)_s$, with $s \leqslant \operatorname{reg}(J)$, then $x_0^{\operatorname{reg}(U)+1-s} f \in \mathcal{N}(U, \langle G \rangle)_{\operatorname{reg}(U)+1}$, by Lemma 2.8 (ii) and Lemma 2.6 (v) applied to U. Hence $f = 0^m_A$. $\square$

Corollary 3.15. *Let* $U \subset A[\mathbf{x}]^m_{\mathbf{d}}$ *be a quasi-stable module and* $W \subset A[\mathbf{x}]^m_{\mathbf{d}}$ *be a finitely generated graded submodule such that* $(A[\mathbf{x}]^m_{\mathbf{d}})_s = W_s \oplus \langle \mathcal{N}(U)_s \rangle^A$ *for every s. Then W is generated by a $\mathcal{P}(U)$-marked basis.*

Proof. The statement is an easy consequence of Theorem 3.11 as soon as we define a $\mathcal{P}(U)$-marked set generating W.

By the hypotheses, for every degree s and every monomial $x^\alpha e_k \in \mathcal{P}(U)$ there is a unique element $h^k_\alpha \in \langle \mathcal{N}(U)_s \rangle^A$ such that $x^\alpha e_k - h^k_\alpha \in W_s$.

The collection G of the elements $x^\alpha e_k - h^k_\alpha$ is obviously a $\mathcal{P}(U)$-marked set and generates a graded submodule of W. Moreover, $(A[\mathbf{x}]^m_{\mathbf{d}})_s = W_s \oplus \langle \mathcal{N}(U)_s \rangle^A = \langle G^{(s)} \rangle^A \oplus \langle \mathcal{N}(U)_s \rangle^A$. Therefore, $W_s = \langle G^{(s)} \rangle^A \subseteq G_s \subseteq W_s$, so that G generates W as a graded $A[\mathbf{x}]$-module. $\square$

Finally, we give an algorithmic method to check whether a marked set is a marked basis using the reduction process introduced in Definition 3.7.

Theorem 3.16. *Let* $U \subset A[\mathbf{x}]^m_{\mathbf{d}}$ *be a quasi-stable module and* G *be a* $\mathcal{P}(U)$-*marked set. The set G is a $\mathcal{P}(U)$-marked basis if and only if*

$$\forall f^k_\alpha \in G, \forall x_i \in \overline{\mathcal{X}}_\mathcal{P}(f^k_\alpha) : x_i f^k_\alpha \xrightarrow{\ G^{(s)}\ } 0^m_A.$$

Proof. We can repeat the arguments used in [8, Theorem 5.13] for the ideal case. $\square$

4. THE SCHEME STRUCTURE OF $\operatorname{Mf}(\mathcal{P}(U))$

We now exhibit a natural scheme structure on the set containing all modules generated by a $\mathcal{P}(U)$-marked basis with U a quasi-stable module as in the previous section. Let $\mathcal{P}(U) \subset \mathbb{T}^m$

be the Pommaret basis of the quasi-stable module $U \subseteq A[\mathbf{x}]_{\mathbf{d}}^m$. We consider the functor of the marked bases on $\mathcal{P}(U)$ from the category of Noetherian K-algebras to the category of sets

$$\underline{\mathbf{Mf}}_{\mathcal{P}(U)}^{m,\mathbf{d}} : \underline{\text{Noeth } K\text{–Alg}} \longrightarrow \underline{\text{Sets}}$$

that associates to any Noetherian K-algebra A the set

$$\underline{\mathbf{Mf}}_{\mathcal{P}(U)}^{m,\mathbf{d}}(A) := \{G \subset A[\mathbf{x}]_{\mathbf{d}}^m \mid G \text{ is a } \mathcal{P}(U)\text{-marked basis}\},$$

or, equivalently by Corollary 3.15,

$$\underline{\mathbf{Mf}}_{\mathcal{P}(U)}^{m,\mathbf{d}}(A) := \{W \subset A[\mathbf{x}]_{\mathbf{d}}^m \mid W \text{ is generated by a } \mathcal{P}(U)\text{-marked basis}\},$$

and to any morphism $\sigma : A \to B$ the map

$$\begin{array}{rccc} \underline{\mathbf{Mf}}_{\mathcal{P}(U)}^{m,\mathbf{d}}(\sigma) : & \underline{\mathbf{Mf}}_{\mathcal{P}(U)}^{m,\mathbf{d}}(A) & \longrightarrow & \underline{\mathbf{Mf}}_{\mathcal{P}(U)}^{m,\mathbf{d}}(B) \\ & G & \longmapsto & \sigma(G). \end{array}$$

Note that the image $\sigma(G)$ under this map is indeed again a $\mathcal{P}(U)$-marked basis, as we are applying the functor $- \otimes_A B$ to the decomposition $(A[\mathbf{x}]_{\mathbf{d}}^m)_s = \langle G^{(s)} \rangle^A \oplus \langle \mathcal{N}(U)_s \rangle^A$ for every degree s.

Corollary 4.1. *Let $\mathcal{P}(U) \subset \mathbb{T}^m$ be the Pommaret basis of the quasi-stable module $U \subseteq A[\mathbf{x}]_{\mathbf{d}}^m$. Then every module $W \in \underline{\mathbf{Mf}}_{\mathcal{P}(U)}^{m,\mathbf{d}}(A)$ has the same Hilbert function as U.*

Proof. This is a simple reformulation of Remark 3.12. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

The above introduced functor turns out to be representable by an affine scheme that can be explicitly constructed by the following procedure. We consider the K-algebra $K[C]$ where C denotes the finite set of variables $\{C_{\alpha\eta kl} \mid x^\alpha e_k \in \mathcal{P}(U), x^\eta e_l \in \mathcal{N}(U), \deg(x^\eta e_l) = \deg(x^\alpha e_k)\}$ and construct the $\mathcal{P}(U)$-marked set $\mathcal{G} \subset K[C][\mathbf{x}]_{\mathbf{d}}^m$ consisting of all elements

$$F_\alpha^k = \left(x^\alpha - \sum_{x^\eta \in \mathcal{N}(J^{(k)})_{|\alpha|}} C_{\alpha\eta kk} x^\eta\right) e_k - \sum_{\substack{l \neq k, x^\eta e_l \in \mathcal{N}(J^{(l)})e_l \\ \deg(x^\eta e_l) = \deg(x^\alpha e_k)}} C_{\alpha\eta kl} x^\eta e_l \qquad (4.1)$$

with $x^\alpha e_k \in \mathcal{P}(U)$. Then, we compute all the complete reductions $x_i F_\alpha^k \xrightarrow{\mathcal{G}^{(s)}}_* L$ for every term $x^\alpha e_k \in \mathcal{P}(U)$ and every non-multiplicative variable $x_i \in \overline{\mathcal{X}}_{\mathcal{P}}(F_\alpha^k)$ and collect the coefficients of the monomials $x^\eta e_j \in \mathcal{N}(U)$ of all the reduced elements L in a set $\mathcal{R} \subset K[C]$.
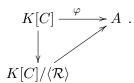
Theorem 4.2. *The functor $\underline{\mathbf{Mf}}_{\mathcal{P}(U)}^{m,\mathbf{d}}$ is represented by the scheme $\mathbf{Mf}_{\mathcal{P}(U)}^{m,\mathbf{d}} := \operatorname{Spec}(K[C]/\langle \mathcal{R} \rangle)$.*

Proof. We observe that each element f_α^k of a $\mathcal{P}(U)$-marked set G in $A[\mathbf{x}]_{\mathbf{d}}^m$ can be written in the following form:

$$f_\alpha^k = \left(x^\alpha - \sum_{x^\eta \in \mathcal{N}(J^{(k)})_{|\alpha|}} c_{\alpha\eta kk} x^\eta\right) e_k - \sum_{\substack{l \neq k, x^\eta e_l \in \mathcal{N}(J^{(l)})e_l \\ \deg(x^\eta e_l) = \deg(x^\alpha e_k)}} c_{\alpha\eta kl} x^\eta e_l, \quad c_{\alpha\eta kl} \in A.$$

Therefore, G can be obtained by specializing in $\mathcal{G}$ the variables $C_{\alpha\eta kl}$ to the constants $c_{\alpha\eta kl} \in A$.

Moreover, G is a $\mathcal{P}(U)$-marked basis if and only $x_i f_\alpha^k \xrightarrow{G^{(s)}}_* 0$ for every $x^\alpha e_k \in \mathcal{P}(U)$ and $x_i \in \overline{\mathcal{X}}_{\mathcal{P}}(f_\alpha^k)$. Equivalently, G is a $\mathcal{P}(U)$-marked basis, if and only if the evaluation morphism $\varphi : K[C] \to A$, $\varphi(C_{\alpha\eta kl}) = c_{\alpha\eta kl}$, factors through $K[C]/\langle \mathcal{R} \rangle$, namely, if and only if the following diagram commutes

$$\begin{array}{ccc} K[C] & \xrightarrow{\varphi} & A \\ \downarrow & \nearrow & \\ K[C]/\langle \mathcal{R} \rangle & & \end{array}$$

$\qquad \square$

Remark 4.3. The arguments presented in the proof of Theorem 4.2 generalize those presented in [8, 13] for ideals to our more general framework of modules.

As a consequence of this result we know that the scheme defined as $\mathrm{Spec}(K[C]/\langle\mathcal{R}\rangle)$ only depends on the submodule U and not on the possibly different procedures for constructing it: any other procedure that gives a set of "minimal" conditions on the coefficients C that are necessary and sufficient to guarantee that a $\mathcal{P}(U)$-marked set G is a $\mathcal{P}(U)$-marked basis generates an ideal $\mathcal{R}'$ such that $K[C]/\langle\mathcal{R}\rangle \simeq K[C]/\langle\mathcal{R}'\rangle$.

5. $\mathcal{P}(U)$-MARKED BASES AND SYZYGIES

We now study syzygies of a $\mathcal{P}(U)$-marked basis and we formulate a $\mathcal{P}(U)$-marked version of the involutive Schreyer theorem [18, Theorem 5.10]. For notational simplicity, this section is formulated for ideals in $A[\mathbf{x}]$, but it is straightforward to extend everything to submodules of free modules $A[\mathbf{x}]_{\mathbf{d}}^m$.

Let J be a quasi-stable monomial ideal in $A[\mathbf{x}]$ and I an ideal in $A[\mathbf{x}]$ generated by a $\mathcal{P}(J)$-marked basis G. Let m be the cardinality of $\mathcal{P}(J)$. We denote the terms in $\mathcal{P}(J)$ by $x^{\alpha(k)}$ and the polynomials in G by $f_{\alpha(k)}$, with $k \in \{1,\ldots,m\}$.

Lemma 5.1. Every polynomial $f \in I$ can be uniquely written in the form $f = \sum_{l=1}^m P_l f_{\alpha(l)}$ with $f_{\alpha(l)} \in G$ and $P_l \in A[\mathcal{X}_{\mathcal{P}}(f_{\alpha(l)})]$.

Proof. This is a consequence of Proposition 3.9 and Theorem 3.11 (vi). □

Take an arbitrary element $f_{\alpha(k)} \in G$ and choose an arbitrary non-multiplicative variable $x_i \in \overline{\mathcal{X}}_{\mathcal{P}}(f_{\alpha(k)})$ of it. We can determine, via the reduction process $\xrightarrow{G}$, for each $f_{\alpha(l)} \in G$ a unique polynomial $P_l^{k;i} \in A[\mathcal{X}_{\mathcal{P}}(f_{\alpha(l)})]$ such that $x_i f_{\alpha(k)} = \sum_{l=1}^m P_l^{k;i} f_{\alpha(l)}$. This relation corresponds to the syzygy

$$S_{k;i} = x_i e_k - \sum_{l=1}^m P_l^{k;i} e_l.$$

We denote the set of all thus obtained syzygies by

$$G_{\mathrm{Syz}} = \{S_{k;i} \mid k \in \{1,\ldots,m\},\ x_i \in \overline{\mathcal{X}}_{\mathcal{P}}(f_{\alpha(k)})\}.$$

We consider the syzygies in G_{Syz} as elements of $A[\mathbf{x}]_{\mathbf{d}}^m$ with $\mathbf{d} = (\deg(x^{\alpha(1)}),\ldots,\deg(x^{\alpha(m)}))$.

Lemma 5.2. Let $S = \sum_{l=1}^m S_l e_l$ be an arbitrary syzygy of the $\mathcal{P}(J)$-marked basis G with coefficients $S_l \in A[\mathbf{x}]$. Then $S_l \in A[\mathcal{X}_{\mathcal{P}}(f_{\alpha(l)})]$ for all $1 \leqslant l \leqslant m$ if and only if $S = 0_A^m$.

Proof. If $S \in \mathrm{Syz}(G)$, then $\sum_{l=1}^m S_l f_{\alpha(l)} = 0$. According to Lemma 5.1, each $f \in I$ can be uniquely written in the form $f = \sum_{l=1}^m P_l f_{\alpha(l)}$ with $f_{\alpha(l)} \in G$ and $P_l \in A[\mathcal{X}_{\mathcal{P}}(f_{\alpha(l)})]$. In particular, this holds for $0_A \in I$. Thus $0_A = S_l \in A[\mathcal{X}_{\mathcal{P}}(f_{\alpha(l)})]$ for all l and hence $S = 0_A^m$. □

Lemma 5.3. Let U be the monomial module $U = \oplus_{l=1}^m (\overline{\mathcal{X}}_{\mathcal{P}}(x^{\alpha(l)}))e_l$ where $(\overline{\mathcal{X}}_{\mathcal{P}}(x^{\alpha(l)}))$ is the ideal generated by $\overline{\mathcal{X}}_{\mathcal{P}}(f_{\alpha(l)})$ in $A[\mathbf{x}]$. Then U is a quasi-stable module with Pommaret basis $\mathcal{P}(U) = \{x_i e_l \mid 1 \leqslant l \leqslant m, x_i \in \overline{\mathcal{X}}_{\mathcal{P}}(f_{\alpha(l)})\}$ and G_{Syz} is a $\mathcal{P}(U)$-marked set in $A[\mathbf{x}]_{\mathbf{d}}^m$.

Proof. By [18, Lemma 5.9] we can immediately conclude that U is a quasi-stable module and that the set $\{x_i e_l \mid 1 \leqslant l \leqslant m, x_i \in \overline{\mathcal{X}}_{\mathcal{P}}(f_{\alpha(l)})\}$ is the Pommaret basis of U.

We define $\mathrm{Ht}(S_{i;l}) = x_i e_l$ and easily see that G_{Syz} is a $\mathcal{P}(U)$-marked-set: by definition of U, every term $x^\mu e_k$ in $\mathrm{Supp}(S_{l;i} - x_i e_l)$ belongs to $\mathcal{N}(U)$, because $x^\mu \in \mathcal{X}_{\mathcal{P}}(f_{\alpha(k)})$. □

Observe that for every $S_{k;i} \in G_{\mathrm{Syz}}$, $\mathcal{X}_{\mathcal{P}}(S_{k;i}) = \{x_0,\ldots,x_i\}$. As in Section 3, we define for every degree s the following set of polynomials in $\langle G_{\mathrm{Syz}}\rangle$:

$$G_{\mathrm{Syz}}^{(s)} = \{x^\delta S_{k;i} \mid S_{k;i} \in G_{\mathrm{Syz}}, x^\delta \in \mathcal{X}_{\mathcal{P}}(S_{k;i}), \deg(x^\delta S_{k;i}) = s\}.$$

Lemma 5.4. The set $G_{\mathrm{Syz}}^{(s)}$ generates the A-module $\mathrm{Syz}(G)_s$ for every s.

Proof. Let $S = \sum_{l=1}^{m} S_l e_l$ be an arbitrary non-vanishing syzygy in $\mathrm{Syz}(G)_s$. By Lemma 5.2, there is at least one index k such that the coefficient S_k contains a term x^μ depending on a non-multiplicative variable $x_i \in \overline{\mathcal{X}}_{\mathcal{P}}(f_{\alpha(k)})$. Among all such values of k and μ we choose the term $x^\mu e_k$ which is lexicographically maximal. Then, $x^\mu e_k$ belongs to the quasi-stable module U, hence there is $x^\delta S_{k;j} \in G_{\mathrm{Syz}}^{(s)}$ such that $x^\delta x_j = x^\mu$. We define $S' = S - \lambda x^\delta S_{k;j}$, where $\lambda \neq 0_A$ is the coefficient of $x^\mu e_k$ in S.

Now we have to show that for every x^ν which is contained in a term $\lambda x^\nu e_l$ in $\mathrm{Supp}(S') \cap U$, x^ν is lexicographically smaller than x^μ. The terms of $\mathrm{Supp}(S) \cap U$ contained in $\mathrm{Supp}(S')$ are by assumption lexicographically smaller than $x^\mu e_k$. Every other term arises from $x^\delta \sum_{l=1}^{m} P_l^{(k;j)} e_l$. We know that $x_j f_{\alpha(k)} = \sum_{l=1}^{m} P_l^{(k;j)} f_{\alpha(l)}$. In particular, a term $x^{\nu'}$ in $P_l^{(k;j)}$ is lexicographically smaller than x_j, by Lemma 3.9. Therefore every term in $x^\delta \sum_{l=1}^{m} P_l^{(k;j)} e_\beta$ is lexicographically smaller than $x^\delta x_j = x^\mu$. If $S' \neq 0$, again by Lemma 5.2, we iterate the procedure on a lexicographical maximal term of S' containing a nonmultiplicative variable. Since all new non-multiplicative terms introduced are lexicographically smaller, the reduction process must stop after a finite number of steps. As a result we get a representation $S' = \sum_{l=1}^{l} S'_l e_l$ such that $S'_l \in A[\mathcal{X}_{\mathcal{P}}(f_{\alpha(l)})]$ for all $1 \leqslant l \leqslant m$. But Lemma 5.2 says that this sum must be zero. $\qquad\square$

Theorem 5.5 ($\mathcal{P}(U)$-marked Schreyer Theorem). *Let $G = \{f_{\alpha(1)}, \ldots f_{\alpha(m)}\}$ be a $\mathcal{P}(J)$-marked basis. Then G_{Syz} is a $\mathcal{P}(U)$-marked basis of $\mathrm{Syz}(G)$, with U as in Lemma 5.3 .*

Proof. By Lemma 5.3, we know that G_{Syz} is a $\mathcal{P}(U)$-marked set. By Lemma 5.4, we know that $\langle G_{\mathrm{Syz}}^{(s)} \rangle^A = \langle \mathrm{Syz}(G)_s \rangle^A$ and we conclude by Theorem 3.11 (vi). $\qquad\square$

Iterating this result, we arrive at a (generally non-minimal) free resolution. In contrast to the classical Schreyer Theorem for Gröbner bases, we are able to determine the ranks of all appearing free modules without any further computations.

Theorem 5.6. *Let $G = \{f_{\alpha(1)}, \ldots, f_{\alpha(m)}\}$, $\deg(f_{\alpha(i)}) = d_i$, be a $\mathcal{P}(J)$-marked basis and I the ideal generated by G in $A[\mathbf{x}]$. We denote by $\beta_{0,j}^{(k)}$ the number of terms $x^\alpha \in \mathcal{P}(J)$ such that $\deg(x^\alpha) = j$ and $\min(x^\alpha) = x_k$ and set $D = \min_{x^\alpha \in \mathcal{P}(J)} \{i \mid x_i = \min(x^\alpha)\}$. Then I possesses a finite free resolution*

$$0 \longrightarrow \bigoplus A[\mathbf{x}](-j)^{r_{n-D,j}} \longrightarrow \cdots \longrightarrow \bigoplus A[\mathbf{x}](-j)^{r_{1,j}} \longrightarrow \bigoplus A[\mathbf{x}](-j)^{r_{0,j}} \longrightarrow I \longrightarrow 0 \quad (5.1)$$

of length $n - D$ where the ranks of the free modules are given by

$$r_{i,j} = \sum_{k=1}^{n-i} \binom{n-k}{i} \beta_{0,j-i}^{(k)}.$$

Proof. According to Theorem 5.5, G_{Syz} is a $\mathcal{P}(U)$-marked basis for the module $\mathrm{Syz}_1(I)$, with U as in Lemma 5.3. Applying the theorem again, we can construct a marked basis of the second syzygy module $\mathrm{Syz}_2(I)$ and so on. Recall that for every index $1 \leqslant l \leqslant m$ and for every non-multiplicative variable $x_k \in \overline{\mathcal{X}}_{\mathcal{P}}(f_{\alpha(l)})$ we have $\min(\mathrm{Ht}(S_{l;k})) = k > \min(\mathrm{Ht}(f_{\alpha(l)}))$. If D is the index of the minimal variable appearing in a head term in G, then the index of the minimal variable appearing in a head term in G_{Syz} is $D + 1$. This observation yields the length of the resolution (5.1). Furthermore $\deg(S_{k;i}) = \deg(f_{\alpha(i)})$, e. g. from the i-th to the $(i+1)$th module the degree from the basis element to the corresponding syzygies grows by one.

The ranks of the modules follow from a rather straightforward combinatorial calculation. Let $\beta_{i,j}^{(k)}$ denote the number of generators of degree j of the i-th syzygy module $\mathrm{Syz}_i(G)$ with minimal variable in the head term x_k. By definition of the generators $S_{l;k}$, we find

$$\beta_{i,j}^{(k)} = \sum_{t=1}^{k-1} \beta_{i-1,j-1}^{(n-t)}$$

as each generator with minimal variable smaller than k and degree $j-1$ in the marked basis of $\mathrm{Syz}_i(G)$ contributes one generator of minimal variable k and degree j to the marked basis of

$\mathrm{Syz}_i(G)$. A simple but lengthy induction allows us to express $\beta_{i,j}^{(k)}$ in terms of $\beta_{0,j}^{(k)}$:

$$\beta_{i,j}^{(k)} = \sum_{t=1}^{k-i} \binom{k-l-1}{i-1} \beta_{0,j-i}^{(t)}$$

Now we are able to compute the ranks of the free modules via

$$r_{i,j} = \sum_{k=1}^{n} \beta_{i,j}^{(k)} = \sum_{k=1}^{n} \sum_{t=1}^{k-i} \binom{k-t-1}{i-1} \beta_{0,j-i}^{(t)} = \sum_{t=1}^{n-i} \binom{n-k}{i} \beta_{0,j-i}^{(n-k)}.$$

The last equality follows from a classical identity for binomial coefficients. $\qquad\square$

Remark 5.7. Observe that the direct summands in the resolution (5.1) depend only on the Pommaret basis $\mathcal{P}(J)$ and not on the ideal I.

Corollary 5.8. *Let G be a $\mathcal{P}(J)$-marked basis and I the ideal generated by G in $A[\mathbf{x}]$. Define $r_{i,j}$ as in Theorem 5.6 and let $b_{i,j}$ be, as usual, the Betti numbers of I. Then*

- $b_{i,j} \leqslant r_{i,j}$ *for all* i,j;
- $\mathrm{reg}(I) \leqslant \mathrm{reg}(J)$;
- $\mathrm{pdim}(I) \leqslant \mathrm{pdim}(J)$.

Proof. The three inequalities follow from the free resolution (5.1) of I, recalling that $\mathrm{reg}(J) := \max_{x^\alpha \in \mathcal{P}(J)}\{\deg(x^\alpha)\}$ and $\mathrm{pdim}(J) = n - \min_{x^\alpha \in \mathcal{P}(J)}\{i \mid x_i = \min(x^\alpha)\}$. $\qquad\square$

If G is even a Pommaret basis for the reverse lexicographic term order, i.e. if J is the leading ideal ideal of I for this order, then we obtain the stronger results $\mathrm{reg}(I) = \mathrm{reg}(J)$ and $\mathrm{pdim}(I) = \mathrm{pdim}(J)$ (for other term orders we also get only estimates) [18, Corollaries 8.13, 9.5].

Example 5.9. Let $A[\mathbf{x}] = K[x_0, x_1, x_2]$, J the monomial ideal with Pommaret basis $\mathcal{P}(J) = \{x_2^3, x_2^2 x_1, x_2 x_1, x_1 x_0, x_1^2\}$ and I the polynomial ideal generated by $G = \{g_1 = x_2^3, g_2 = x_2^2 x_1, g_3 = x_2 x_1, g_4 = x_1 x_0 + x_2^2, g_5 = x_1^2\}$. One easily checks that G is a $\mathcal{P}(J)$-marked basis.

We explicit compute the multiplicative representations of $x_2 \cdot g_2$, $x_2 \cdot g_3$, $x_1 \cdot g_4$, $x_2 \cdot g_4$, $x_2 \cdot g_5$, that give the set $G_{\mathrm{Syz}} = \{S_{2;2}, S_{3;2}, S_{4;1}, S_{4;2}, S_{5;2}\} \subset A[\mathbf{x}]^5$:

$$
\begin{array}{ll}
x_2 \cdot g_2 = x_1 \cdot g_1\,, & S_{2;2} = x_2 \cdot e_2 - x_1 \cdot e_1\,, \\
x_2 \cdot g_3 = g_2\,, & S_{3;2} = x_2 \cdot e_3 - e_2\,, \\
x_1 \cdot g_4 = x_0 \cdot g_5 + g_2\,, & S_{4;1} = x_1 \cdot e_4 - x_0 \cdot e_5 - e_2\,, \\
x_2 \cdot g_4 = x_0 \cdot g_3 + g_1\,, & S_{4;2} = x_2 \cdot e_4 - x_0 \cdot e_3 - e_1\,, \\
x_2 \cdot g_5 = x_1 \cdot g_3\,, & S_{5;2} = x_2 \cdot e_5 - x_1 \cdot e_3\,.
\end{array}
$$

The only nonmultiplicative variable for G_{Syz} is $\overline{\mathcal{X}}_{\mathcal{P}}(S_{4;1}) = \{x_2\}$.

Therefore we have to compute the reduction of $x_2 S_{4;1}$ which is $x_2 S_{4;1} = x_1 S_{4;2} - S_{2;2} - x_0 S_{5;2}$ and hence $G_{\mathrm{Syz}_2} = \{x_2 e_3 - x_1 e_4 - e_1 - x_0 e_5\} \subset A[\mathbf{x}]^5$.

This leads to the following free resolution of I of length two:

$$0 \longrightarrow A[\mathbf{x}](-4) \xrightarrow{\delta_2} A[\mathbf{x}](-4) \oplus A[\mathbf{x}](-3)^4 \xrightarrow{\delta_1} A[\mathbf{x}](-3)^2 \oplus A[\mathbf{x}](-2)^3 \xrightarrow{\delta_0} I \longrightarrow 0,$$

where

$$\delta_0 = \begin{pmatrix} x_2^3 & x_2^2 x_1 & x_2 x_1 & x_1 x_0 + x_2^2 & x_1^2 \end{pmatrix},$$

$$\delta_1 = \begin{pmatrix} -x_1 & 0 & 0 & -1 & 0 \\ x_2 & -1 & -1 & 0 & 0 \\ 0 & x_2 & 0 & -x_0 & -x_1 \\ 0 & 0 & x_1 & x_2 & 0 \\ 0 & 0 & -x_0 & 0 & x_2 \end{pmatrix}, \quad \delta_2 = \begin{pmatrix} 1 \\ 0 \\ x_2 \\ -x_1 \\ x_0 \end{pmatrix}.$$

This free resolution is obviously not minimal. Minimizing the resolution leads to the minimal free resolution of I of length one:

$$0 \longrightarrow A[\mathbf{x}](-3)^2 \xrightarrow{\delta_1'} A[\mathbf{x}](-2)^3 \xrightarrow{\delta_0'} I \longrightarrow 0.$$

Hence in the present example, we have $1 = \mathrm{pdim}(I) < \mathrm{pdim}(J) = 2$ and $2 = \mathrm{reg}(I) < \mathrm{reg}(J) = 3$.

Example 5.10. Let $A[\mathbf{x}] = K[x_0, x_1, x_2]$, J the monomial ideal with Pommaret basis $\mathcal{P}(J) = \{x_2x_1, x_2^2x_1, x_2^3, x_1^3, x_2^2x_0, x_1^2x_0\}$ and I be the ideal generated by the $\mathcal{P}(J)$-marked basis $G = \{g_1 = x_2x_1 - x_2^2 - x_1^2, g_2 = x_2^2x_1, g_3 = x_2^3, g_4 = x_1^3, g_5 = x_2^2x_0, g_6 = x_1^2x_0\}$, where $\mathrm{Ht}(g_1) = x_2x_1$. Observe that G is not a Gröbner basis, for any term order, due to the terms in $x_2x_1 - g_1$.

By Theorem 5.6, we construct the following free resolution of I:

$$0 \longrightarrow A[\mathbf{x}](-5)^2 \xrightarrow{\ \delta_2\ } A[\mathbf{x}](-3) \oplus A[\mathbf{x}](-4)^6 \xrightarrow{\ \delta_1\ } A[\mathbf{x}](-2) \oplus A[\mathbf{x}](-3)^5 \xrightarrow{\ \delta_0\ } I \longrightarrow 0. \tag{5.2}$$

The above resolution is obviously not minimal. The minimal free resolution of I is

$$0 \longrightarrow A[\mathbf{x}](-5)^2 \xrightarrow{\ \delta_2'\ } A[\mathbf{x}](-4)^6 \xrightarrow{\ \delta_1'\ } A[\mathbf{x}](-2) \oplus A[\mathbf{x}](-3)^4 \xrightarrow{\ \delta_0'\ } I \longrightarrow 0.$$

In this case, although the resolution (5.2) is not minimal, the bounds on projective dimension and regularity given in Corollary 5.8 are sharp.

6. Marked Bases over Truncated Quasi-Stable Ideals

We now assume that the quasi-stable ideal J is saturated and we investigate for a degree $m \geqslant 0$ some special features of $\mathcal{P}(J_{\geqslant m})$-marked bases. Recall from Lemma 2.8 that for a saturated ideal the Pommaret basis $P(J)$ contains no term divisible by x_0. As a first tool, we relate $\mathcal{P}(J_{\geqslant m})$ to $\mathcal{P}(J)$.

Lemma 6.1. [5, Proposition 2.10] *Let J be a quasi-stable ideal. For every $m \geqslant 0$, we have*

$$\mathcal{P}(J_{\geqslant m}) = \mathcal{P}(J)_{\geqslant m+1} \cup \left(\bigcup_{x^\alpha \in \mathcal{P}(J)_{\leqslant m}} \mathcal{C}_{\mathcal{P}}(x^\alpha)_m \right). \tag{6.1}$$

We define for every $i \in \{1, \ldots, n\}$ the integer ρ_i as $\max\{\deg(x^\alpha) \mid x^\alpha \in \mathcal{P}(J), \alpha_i \geqslant 1\}$.

Proposition 6.2. *Assume $m \geqslant \max\{\rho_1, \ldots \rho_i\}$ for a fixed index $1 \leqslant i \leqslant n$ and consider the $\mathcal{P}(J_{\geqslant m})$-marked set $\mathcal{G}$ whose marked polynomials are defined as in (4.1). Consider a term $x^\alpha \in \mathcal{P}(J_{\geqslant m})$ such that $\min(x^\alpha) \leqslant x_{i-1}$. We have a unique representaiton*

$$x_i F_\alpha = \sum_{\alpha', j} p_{\alpha' j} x_j F_{\alpha'} + H_{i\alpha} \tag{6.2}$$

where $x_j \in \{x_0, \ldots, x_{i-1}\}$, $p_{\alpha' j} \in K[C]$, $x_j F_{\alpha'} \in \mathcal{G}^{(m+1)}$, $H_{i\alpha} \in \langle \mathcal{N}(J)_{m+1} \rangle^A$.

Proof. First, we observe that since $\min(x^\alpha) \leqslant x_{i-1}$, we find that $\deg(F_\alpha) = m$ by the assumption on m and Lemma 6.1. Furthermore, $\mathcal{N}(J_{\geqslant m})_s = \mathcal{N}(J)_s$ for every $s \geqslant m$.

We obtain a unique representation of $x_i F_\alpha$ in $\langle \mathcal{G}^{(m+1)} \rangle^A \oplus \langle \mathcal{N}(J)_{m+1} \rangle^A$ using the reduction relation $\xrightarrow{\ \mathcal{G}^{(m+1)}\ }$. We now prove that this representation has exactly the features of (6.2) under the made assumptions.

Consider any term $x_i x^\eta$ in $\mathrm{supp}(x_i F_\alpha) \cap J_{\geqslant m}$ ($x_i x^\alpha$ included): $x_i x^\eta \in \mathcal{C}_{\mathcal{P}}(x^{\alpha'})$ and $x_i x^\eta = x^{\alpha'} x^{\delta'}$ for some generator $x^{\alpha'} \in \mathcal{P}(J_{\geqslant m})$. If $x^{\delta'} = 1$, then $x^{\alpha'} \in \mathcal{P}(J_{\geqslant m})_{m+1} = \mathcal{P}(J)_{m+1}$ (by Lemma 6.1). This contradicts the fact that $m \geqslant \rho_i$. Hence $|\delta'| \geqslant 1$. If we consider $x_i x^\alpha$, then x_i divides $x^{\alpha'}$, since $x_i > \min(x^\alpha)$, and does not divide $x^{\delta'}$, as otherwise x^α would be a multiple of another term in $\mathcal{P}(J_{\geqslant m})$. Hence, $x_i >_{Lex} x^{\delta'}$. If we are considering $x_i x^\eta$ with $x^\eta \in \mathcal{N}(J)$, by Lemma 2.6 (vi), $x_i >_{Lex} x^{\delta'}$. Furthermore, since $\deg(F_\alpha) \geqslant m$ for every $F_\alpha \in \mathcal{G}$, we have that $\deg(x^{\delta'}) = 1$. Summing up, $x^{\delta'} = x_j \in \{x_0, \ldots, x_{i-1}\}$.

If $x_j(x^{\alpha'} - F_{\alpha'})$ belongs to $\langle \mathcal{N}(J_{\geqslant m}) \rangle^A$, there is nothing more to prove. If for some $x^{\eta'} \in \mathrm{supp}(x^{\alpha'} - F_{\alpha'})$, $x_j x^{\eta'} \in \mathcal{C}_{\mathcal{P}}(x^{\alpha''})$ and $x_j x^{\eta'} = x^{\alpha''} x^{\delta''} \in J_{\geqslant m}$, then $x_j >_{Lex} x^{\delta''}$, again by Lemma 2.6 (vi), and $x^{\delta''} \neq 1$ because $m \geqslant \max\{\rho_1, \ldots, \rho_i\} \geqslant \rho_j$ hence $x^{\delta''} \in \mathbb{T} \cap K[x_0, \ldots, x_{j-1}]$. Again observing that $\deg(F_\alpha) \geqslant m$ for every $F_\alpha \in \mathcal{G}$, we have $x^{\delta''} = x_{j'} \in \{x_0, \ldots, x_{j-1}\}$. We can iterate this argument, and obtain the representation (6.2). The uniqueness of this representation is a straightforward consequence of Theorem 3.11 (ii). $\qquad\square$

Corollary 6.3. *Assume $m \geqslant \operatorname{reg}(J)$ and consider the $\mathcal{P}(J_{\geqslant m})$-marked set $\mathcal{G}$ whose marked polynomials are defined as in (4.1). Consider $x^\alpha \in \mathcal{P}(J_{\geqslant m})$ and $x_i > \min(x^\alpha)$, we have a unique representation*

$$x_i F_\alpha = \sum P_{\alpha'\ell} x_\ell F_{\alpha'} + H_{i\alpha}, \qquad (6.3)$$

where $P_{\alpha'\ell} \in K[C]$, $x_\ell F_{\alpha'} \in \mathcal{G}^{(m+1)}$, $x_\ell \in \{x_0, \dots, x_{i-1}\}$, $H_{i\alpha} \in \langle \mathcal{N}(J_{\geqslant m}) \rangle^A$.

Theorem 6.4. *Assume $m \geqslant \operatorname{reg}(J)$ and G is a $\mathcal{P}(J_{\geqslant m})$-marked basis in $A[\mathbf{x}]$. Given any term $x^\alpha \in \mathcal{P}(J_{\geqslant m})$, we find for every term x^η in $\operatorname{Supp}(x^\alpha - f_\alpha)$ that $\min(x^\eta) \leqslant \min(x^\alpha)$.*

Proof. By Theorem 4.2, G is a $\mathcal{P}(J_{\geqslant m})$-marked basis if the values $\{c_{\alpha\eta}\} \subseteq A$ appearing as coefficients in the polynomials $f_\alpha \in G$ cancel the generators of the ideal $\mathfrak{U}$ defined at the beginning of Section 4. Hence, if we consider $x^\alpha \in \mathcal{P}(J_{\geqslant m})$ and we denote by x_j the smallest variable in x^α, by Corollary 6.3 applied with $x_{j+1} > \min(x^\alpha)$, we obtain the following representation:

$$x_{j+1} f_\alpha = \sum p_{\alpha'\ell} x_\ell f_{\alpha'}$$

with $x_\ell \in \{x_0, \dots \min(x^\alpha)\}$ and $x_\ell f_{\alpha'} \in G^{(m+1)}$. After cancellations on the right-hand side of the equality, every remaining term can be divided by x_{j+1}. After this division, we find

$$x^\alpha - f_\alpha = \sum p_{\alpha'\delta'} x_\ell x^{\eta'}$$

which implies our claim. □

Corollary 6.5. *Consider the $\mathcal{P}(J_{\geqslant m})$-marked set $\mathcal{G}$ whose marked polynomials are defined as in (4.1). Assume $m \geqslant \rho_1$.*

Consider $F_\alpha \in \mathcal{G}$ such that $\min(x^\alpha) = x_0$. We have a unique representation

$$x_1 F_\alpha = \sum p_{\alpha'} x_0 F_{\alpha'} + H_{1\alpha},$$

where $p_{\alpha'} \in K[C]$, $H_{1\alpha} \in \langle \mathcal{N}(J_{\geqslant m}) \rangle$.

Corollary 6.6. *Assume that $m \geqslant \rho_1$ and G is a $J_{\geqslant m}$-marked basis in $A[\mathbf{x}]$. If $x^\alpha \in \mathcal{P}(J_{\geqslant m})$ and $\min(x^\alpha) = x_0$, then x_0 divides f_α.*

Proof. This is a consequence of Corollary 6.5 arguing as in Theorem 6.4. □

7. Conclusions

In this paper, we defined and investigated properties of marked bases over a quasi-stable monomial module $U \subseteq A[\mathbf{x}]_{\mathbf{d}}^m$. The family of all modules generated by a marked basis over $\mathcal{P}(U)$ possesses a natural structure as an affine scheme (Theorem 4.2). In particular, we proved that the quasi-stable module U provides upper bounds on some homological invariants of any module generated by a $\mathcal{P}(U)$-marked basis such as Betti numbers, regularity or projective dimension (Corollary 5.8). Furthermore, we go into detail on the shape of marked bases on truncated quasi-stable ideals (Corollaries 6.3, 6.5, 6.6).

In a forthcoming paper we will exploit these properties and constructions to obtain local and global equations of Hilbert schemes and of special loci of them, such as those given by bounds on the invariants involved in Corollary 5.8. In this way, we will generalize to arbitrary characteristic and extend ideas and results in [1, 6]. Furthermore, we will also apply the techniques developed in the present paper to the investigation of Quot Schemes [15].

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