On the rate of convergence to equilibrium for the linear Boltzmann equation with soft potentials

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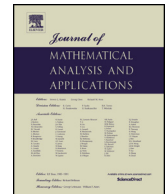
On the rate of convergence to equilibrium for the linear Boltzmann equation with soft potentials

José A. Cañizo [a,1], Amit Einav [b,c,2], Bertrand Lods [d,*]

[a] *Departamento de Matemática Aplicada, Universidad de Granada, Av. Fuentenueva S/N, 18071 Granada, Spain*
[b] *Institut für Analyisis und Scientific Computing, Technische Universität Wien, Austria* [3]
[c] *Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, UK*
[d] *Departement of Economics and Statistics & Collegio Carlo Alberto, Università degli Studi di Torino, Corso Unione Sovietica, 218/bis, 10134 Torino, Italy*

A R T I C L E   I N F O

A B S T R A C T

In this work we present several quantitative results of convergence to equilibrium for the linear Boltzmann operator with soft potentials under Grad's angular cut-off assumption. This is done by an adaptation of the famous entropy method and its variants, resulting in explicit algebraic, or even stretched exponential, rates of convergence to equilibrium under appropriate assumptions. The novelty in our approach is that it involves functional inequalities relating the entropy to its production rate, which have independent applications to equations with mixed linear and non-linear terms. We also briefly discuss some properties of the equation in the non-cut-off case and conjecture what we believe to be the right rate of convergence in that case.

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* Corresponding author.

E-mail addresses: canizo@ugr.es (J.A. Cañizo), aeinav@asc.tuwien.ac.at (A. Einav), bertrand.lods@unito.it (B. Lods).

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1. Introduction

This work is concerned with the asymptotic behaviour of the linear homogeneous Boltzmann equation in the less explored case of soft potential interactions, and with a cut-off assumption (the precise definition of all the above will be given shortly). We are interested in the application of entropy techniques to study the approach to equilibrium in the relative entropy sense, and in the application of entropy inequalities to estimate its rate. Our results complement a previous work by two of the authors [5], where the case of hard potentials was studied following the same techniques.

Our motivation comes partly from the study of the linear Boltzmann equation itself, which is a basic model in kinetic theory describing the collisional interaction of a set of particles with a thermal bath at a fixed temperature. Interactions among the particles themselves are neglected, and thus the equation is linear. Various versions of the linear Boltzmann equation are used to model phenomena such as neutron scattering [27,28], radiative transfer [1] and cometary flows [16] (we refer to [13, Chapter XXI] for a detailed presentation of the mathematical theory of linear collisional kinetic equations), and appears in some non-linear models as a background interaction term [4,10,17]. On the other hand, a technical motivation for our results is that inequalities relating the logarithmic entropy to its production rate are interesting by themselves, and are helpful in the study of non-linear models involving a linear Boltzmann term. These inequalities are intriguing and have been studied in [5] in the case of hard potentials; we intend to complete these ideas by looking at the case of soft potentials. Our strategy of proof is close to that in [11] (which applies to the non-linear Boltzmann equation), and is based on this type of inequalities.

The linear Boltzmann equation we consider here has been studied in several previous works [5,23,24,31]. Its spectral gap properties are understood since [19], with constructive estimates on the size of the spectral gap in $L^2(M^{-1})$ (where M is the equilibrium) for hard potentials given in [24]. Semigroup techniques were used in [23,27] to obtain convergence to equilibrium for all initial conditions in L^1, without explicit rates. An important related equation is the *linearised* Boltzmann equation, which has been treated for example in [3,8,22,29,30]. Roughly speaking, the spectral gap properties of both equations (linear and linearised) are now understood in a variety of spaces. The difference in our present approach is that it is based on functional inequalities for the logarithmic entropy, which have their own interest and are more robust when applied to models with mixed linear and non-linear terms [4,10].

Similar questions for the non-linear space-homogeneous Boltzmann equation have also been considered in the literature, and we refer to [15] for an overview and to [11] for convergence results with soft potentials. Mathematical questions are more involved in the non-linear setting, and of course the picture becomes more complete in the linear case. However, the question remains open regarding the validity of some functional inequalities in the non-cutoff case; we comment on this at the end of this introduction.

1.1. The linear Boltzmann operator

In this work we will be interested in properties of the solution to the following spatially homogeneous Boltzmann equation

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$$\begin{cases} \partial_t f(t, v) & = \boldsymbol{L}_\gamma f(t, v) := \mathcal{Q}_\gamma(f(t, \cdot), \mathcal{M})(v) \qquad t \geq 0 \\ f(0, v) & = f_0(v) \end{cases} \tag{1.1}$$

where $\mathcal{M}$ is the Maxwellian with the same mass as f_0, and $\mathcal{Q}_\gamma(f, g)$ denotes the bilinear Boltzmann collision operator

$$\mathcal{Q}_\gamma(f, g) = \int\limits_{\mathbb{R}^d \times \mathbb{S}^{d-1}} B_\gamma(v - v_*, \cos \theta) \left(f(v')g(v'_*) - f(v)g(v_*) \right) \mathrm{d}v_* \, \mathrm{d}\sigma \tag{1.2}$$

associated to a given interaction kernel of the form

$$B_\gamma(v - v_*, \cos \theta) = |v - v_*|^\gamma \, b(\cos \theta) \tag{1.3}$$

with $\gamma \in (-d, 0)$, and a given even nonnegative function on $[-1, 1]$, b, that satisfies

$$\|b\|_1 = \int\limits_{\mathbb{S}^{d-1}} b(\cos \theta) \, \mathrm{d}\sigma = |\mathbb{S}^{d-2}| \int\limits_{-1}^{1} b(s) \left(1 - s^2\right)^{\frac{d-3}{2}} \, \mathrm{d}s < \infty \tag{1.4}$$

(the so-called Grad's angular cut-off assumption). For simplicity, we will assume that $\|b\|_1 = 1$. The linear Boltzmann operator is then defined by

$$\boldsymbol{L}_\gamma f = \mathcal{Q}_\gamma(f, \mathcal{M}).$$

In the above, v' and v'_* are the pre-collisional velocities which result, respectively, in the velocities v and v_* after the elastic collision, expressed by the equation

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \qquad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma, \tag{1.5}$$

for a random unit vector σ. The deviation angle, which appears in (1.3), is defined by

$$\cos \theta = \frac{(v'_* - v') \cdot (v_* - v)}{|v_* - v|^2} = \frac{v - v_*}{|v - v_*|} \cdot \sigma.$$

The function f considered in (1.2) is assumed to be a non-negative function with unit mass. As such, the associated normalised Maxwellian is given by

$$\mathcal{M}(v) = \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{|v|^2}{2}\right), \qquad v \in \mathbb{R}^d. \tag{1.6}$$

Our study concerns itself with collision kernels of the form (1.3) with $\gamma \in (-d, 0)$. We will use the following, well known, terminology:

1) If $\gamma > 0$ and b satisfies (1.4), we are in the case of *hard potentials* with angular cut-off.
2) If $\gamma = 0$ and b satisfies (1.4), we are in the case of *Maxwell interactions* with angular cut-off.
3) If $-d < \gamma < 0$ and b satisfies (1.4), we are in the case of *soft potentials* with angular cut-off.

Our quantitative investigation of the rate of decay to equilibrium of solutions to equation (1.1) uses the so-called entropy method. The study of this method for the case of hard potentials has been explored in [5], and the goal of this work is to extend this study to the soft potentials case.

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Before we present the main result of our work we recall in the next section a few known facts about the linear Boltzmann equation.

1.2. Known properties of the linear Boltzmann equation

Basic results regarding equation (1.1) are its well-posedness and the long time behaviour of the Cauchy problem (see for instance [23]):

Proposition 1.1. *Assume that $B(v - v_*, \sigma) = |v - v_*|^\gamma b(\cos\theta)$ where $\gamma \in (-d, 0)$ and $b : [-1, 1] \to \mathbb{R}^+$ is an even function that satisfies* (1.4). *Then, the operator $\boldsymbol{L}_\gamma$ is a bounded operator in $L^1(\mathbb{R}^d)$ and, as such, generates a C_0-semigroup $(U(t))_{t \geq 0}$ of positive operators in $L^1(\mathbb{R}^d)$. Consequently, for any non-negative $f_0 \in L^1(\mathbb{R}^d)$ there exists a unique (mild) solution $f(t, \cdot)$ to* (1.1) *with $f(0, \cdot) = f_0$, given by $f(t) = U(t)f_0$. Moreover, $(U(t))_{t \geq 0}$ is a* stochastic semigroup, *i.e.*

$$\int_{\mathbb{R}^d} U(t)f_0(v)\, \mathrm{d}v = \int_{\mathbb{R}^d} f(t, v)\, \mathrm{d}v = \int_{\mathbb{R}^d} f_0(v)\, \mathrm{d}v \qquad \forall t \geq 0,$$

and for any $f_0 \in L^1(\mathbb{R}^d)$

$$\lim_{t \to \infty} \|U(t)f_0 - \varrho_0 \mathcal{M}\|_{L^1(\mathbb{R}^d)} = 0$$

where $\varrho_0 = \displaystyle\int_{\mathbb{R}^d} f_0(v)\, \mathrm{d}v$.

Notice that the above long-time behaviour of the solution to (1.1) does not require *any additional assumption* on the initial datum. However, it does not provide any kind of rate of convergence for such general initial datum. In fact, we will show in the Appendix B that, without additional assumptions on the initial datum, the rate of convergence can be *arbitrarily slow*.

From this point onwards, unless stated otherwise, we will assume that

$$\varrho_0 = \int_{\mathbb{R}^d} f_0(v)\, \mathrm{d}v = 1.$$

The first important observation in the study of the rate of convergence to equilibrium is the fact that linear Boltzmann equation (1.1) admits infinitely many Lyapunov functionals.

Lemma 1.2. *Let $\boldsymbol{\Phi} \colon \mathbb{R}^+ \to \mathbb{R}^+$ be a convex function and let $f(t, v)$ be non-negative solution to* (1.1). *Then the functional*

$$H_{\boldsymbol{\Phi}}(f(t)|\mathcal{M}) = \int_{\mathbb{R}^d} \mathcal{M}(v)\boldsymbol{\Phi}\left(\frac{f(t, v)}{\mathcal{M}(v)}\right)\, \mathrm{d}v$$

is non-increasing.

We refer to the Appendix A for a formal proof of that property which is a general property of stochastic semigroups (see [25]). For the particular choice $\boldsymbol{\Phi}(x) = x \log x - x + 1$, one recovers the famous Boltzmann relative entropy, which we will denote by $H(f|\mathcal{M})$ and concludes the H-Theorem:

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$$\frac{\mathrm{d}}{\mathrm{d}t} H(f(t)|\mathcal{M}) \leq 0 \qquad \forall t \geq 0$$

for any solution $f(t,v)$ to (1.1) with unit mass and initial data in an appropriate weighted space. The rate at which the relative entropy decreases is fundamental for the understanding of the large time behaviour of $f(t,\cdot)$. Defining the *entropy production* as:

$$\mathscr{D}_\gamma(f) = -\int_{\mathbb{R}^d} \boldsymbol{L}_\gamma(f) \log\left(\frac{f(v)}{\mathcal{M}(v)}\right) \mathrm{d}v,$$

which is obtained by the minus of the formal derivative of the entropy under the flow of the equation, the entropy method seeks to find a general functional inequality that connects the entropy and the entropy production. Such inequality is transformed into a differential inequality along the flow of the equation, from which a concrete rate of convergence to equilibrium can be obtained.

The definition of $\mathscr{D}_\gamma$ can easily be extended to any linear Boltzmann operator $\boldsymbol{L}_\gamma$ with $\gamma > -d$. More generally, we will denote the entropy production associated to a linear operator $\mathcal{Q}(f, \mathcal{M})$ with collision kernel $B(v - v_*, \sigma)$. by $\mathscr{D}_B(f)$, and an easy computation shows that

$$\mathscr{D}_B(f) = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} B(v - v_*, \sigma) \mathcal{M}(v) \mathcal{M}(v_*) \left(h(v') - h(v)\right) \log \frac{h(v')}{h(v)} \, \mathrm{d}v \, \mathrm{d}v_* \, \mathrm{d}\sigma \qquad (1.7)$$

where $h = \frac{f}{\mathcal{M}}$. In particular, as expected, $\mathscr{D}_B(f) \geq 0$.

The study of the entropy method is more developed for the Maxwellian and hard potentials case. In particular, we state the following theorem from [5], which will play an important role in our own study:

Theorem 1.3. *Consider a collision kernel $B(v - v_*, \sigma)$ associated to Maxwell interactions*

$$B(v - v_*, \sigma) = b(\cos\theta),$$

where $b : [-1, 1] \to \mathbb{R}$ is an even function satisfying (1.4). Then, there exists $\lambda_0 > 0$, depending only on b such that

$$\mathscr{D}_0(f) \geq \lambda_0 H\left(f|\mathcal{M}\right), \qquad (1.8)$$

for any non-negative f with unit mass such that

$$\int_{\mathbb{R}^d} \left(1 + |v|^2\right) f(v) \left|\log f(v)\right| \mathrm{d}v < \infty.$$

In general, we don't expect a linear inequality like (1.8) relating the entropy production to the relative entropy in the case of soft potentials. Indeed, such an inequality would imply the existence of a positive spectral gap in the space $L^2(\mathcal{M}^{-1})$ for the operator $\boldsymbol{L}_\gamma$, which is known to be false (see [8] for the linearised case and [23] for the linear case). This is since the essential spectrum of $\boldsymbol{L}_\gamma$ can be shown to contain a whole interval of the type $[-\nu_0, 0]$ (see Remark 2.2 for more details and references on this topic).

The next type of functional inequality one may explore is the following weaker inequality:

$$\mathscr{D}_\gamma(f) \geq C_\delta H(f|\mathcal{M})^{1+\delta} \qquad (1.9)$$

for some large class of probability densities f and for some explicit $\delta > 0$ and $C_\delta > 0$. In fact, to quantify the long time behaviour of the linear Boltzmann equation, it is enough for an inequality of the form (1.9) to be valid along the flow of solutions to (1.1).

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Next, we describe the main result of the present work.

1.3. Main results

Before stating our main results we will introduce some convenient notation. Given a non-negative measurable function f, we denote the k-th moment, and generalised k-th moment, of f by

$$m_k(f) = \int_{\mathbb{R}^d} |v|^k f(v) \, \mathrm{d}v \qquad M_k(f) = \int_{\mathbb{R}^d} \langle v \rangle^k f(v) \, \mathrm{d}v \qquad \forall k \in \mathbb{R}$$

where $\langle v \rangle = \sqrt{1 + |v|^2}$ for any $v \in \mathbb{R}^d$. Moreover, given $s \in \mathbb{R}$ and $p \in (1, \infty)$, we set

$$M_{s,p}(f) = M_s(|f|^p),$$

and notice that $M_{0,p}(f) = \|f\|_p$. For a given $s \geq 0$ we denote by

$$\|f\|_{L_s^1} = M_s(|f|)$$

and

$$\|f\|_{L_s^1 \log L} = \int_{\mathbb{R}^d} \langle v \rangle^s |f(v)| \, |\log|f(v)|| \, \mathrm{d}v,$$

and define the function spaces

$$L_s^1 = L_s^1(\mathbb{R}^d) = \left\{ f \colon \mathbb{R}^d \to \mathbb{R} \mid f \text{ measurable and } \|f\|_{L_s^1} < \infty \right\}$$
$$L_s^1 \log L = \left\{ f \colon \mathbb{R}^d \to \mathbb{R} \mid f \text{ is measurable and } \|f\|_{L_s^1 \log L} < \infty \right\}.$$

Even if $\|\cdot\|_{L_s^1 \log L}$ is not a norm, this notation is commonly seen in the literature.

We are now ready to state our first main result.

Theorem 1.4. *Take $p > 1$ and $-d < \gamma < 0$, and let $f_0 \in L_s^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ be a non-negative function with unit mass, for $s \geq s_{p,d,\gamma}$, where $s_{p,d,\gamma} > 2 + |\gamma|$ is an explicit constant that depends only on p, d and γ. Let $f = f(t)$ be the solution to equation* (1.1) *with a bounded angular kernel b. Then for any*

$$\sigma < -1 + \frac{s - 2}{|\gamma|}$$

there exists a uniform constant $C_0 > 0$ depending only on $d, \gamma, p, s, \sigma, \|f_0\|_{L_s^1}, \|f_0\|_p$ and $H(f_0|\mathcal{M})$ such that

$$H(f(t)|\mathcal{M}) \leq C_0 (1 + t)^{-\sigma}, \tag{1.10}$$

for all $t \geq 0$.

The strategy of the proof is to obtain the inequality

$$\mathscr{D}_\gamma(f) \geq \mathscr{D}_0(f)^{\frac{\mu - \gamma}{\mu}} \mathscr{D}_\mu(f)^{\frac{\gamma}{\mu}}$$

for some $\gamma < 0 < \mu$ by means of interpolation estimates and deduce from it the inequality

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$$\mathscr{D}_\gamma(f) \geq C(f)^{\frac{\gamma}{\mu}} H(f|\mathcal{M})^{1-\frac{\gamma}{\mu}} \tag{1.11}$$

where $C(f)$ is an explicit functional involving norms of f in appropriate $L^1_{\kappa_1}$ and $L^1_{\kappa_2} \log L$ spaces, for a suitable κ_1, κ_2. To use this inequality to deduce Theorem 1.4 one needs to control $C(f)$ along the flow of the equation. This is achieved by obtaining the following:

  (i) Explicit time dependent upper bound on the moments $m_s(f(t))$ of the solutions.
 (ii) Explicit time dependent upper bound on the L^p-norms $\|f(t)\|_p$.
(iii) Pointwise Gaussian lower bounds for the solutions $f(t, v)$.

The methods we use to obtain the above estimations are inspired by the works [11] and [32], that deal with a similar problem related to the non-linear Boltzmann equation.

Our second main result concerns the decay of the solution to (1.1) for a more restrictive class of initial datum satisfying a strong Gaussian control of the form

$$\int_{\mathbb{R}^d} \mathcal{M}(v)^{1-p} f_0(v)^p \, dv < \infty$$

for some $p > 1$. In this case, one can obtain a better rate of decay – one of the form of a stretched exponential:

Theorem 1.5. *Let $f_0 \in L^1(\mathbb{R}^d)$ be a non-negative function such that $f_0 \in L^p(\mathbb{R}^d)$ for some $p > 1$ and where $d \geq 2$. Assume that the angular kernel, b, is bounded and satisfies*

$$b(x) \geq b_0 \left(1 - x^2\right)^{\frac{\nu}{2}}, \tag{1.12}$$

for some $b_0 > 0$, $0 \leq \nu \leq 1$. Then, if

$$H_p(f_0) = \int_{\mathbb{R}^d} \mathcal{M}(v)^{1-p} f_0(v)^p \, dv < \infty$$

we have that for any $t_0 > 0$ there exist two uniform constants $C^{(1)}_{t_0}, C^{(2)}_{t_0} > 0$ depending only on $d, \gamma, p, b_0, \nu, t_0$ and $H_p(f_0)$ such that any non-negative solution to (1.1) with initial data f_0, $f(t)$, satisfies

$$\mathscr{D}_\gamma(f(t)) \geq \frac{C^{(1)}_{t_0} H\left(f(t)|\mathcal{M}\right)}{\left|\log\left(C^{(2)}_{t_0} H\left(f(t)|\mathcal{M}\right)\right)\right|^{\frac{|\gamma|}{2}}}, \quad \forall t \geq t_0. \tag{1.13}$$

As a consequence we can find appropriate constants $C_1, \lambda_1 > 0$ depending on d, γ, p, b_0 such that

$$H\left(f(t)|\mathcal{M}\right) \leq C_1 \exp\left(-\lambda_1 t^{\frac{2}{2+|\gamma|}}\right). \tag{1.14}$$

The above decay rate is similar to that obtained for the linearised Boltzmann equation in [8], yet with a less restrictive condition on the initial datum. Indeed, the condition in [8] involves a *pointwise* Gaussian decay of the type

$$\sup_{v \in \mathbb{R}^d} \exp\left(a|v|^2\right) |f(v)| < \infty$$

for some $a \in (0, 1/4)$.

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The proof of Theorem 1.5 uses a suitable improvement of the interpolation inequality between $\mathscr{D}_\gamma$ and $\mathscr{D}_0$, which involves now the entropy production associated to some (non-physical) interaction kernel of the form

$$B(v - v_*, \sigma) = \exp(a|v - v_*|) \, b(\cos \theta).$$

This is reminiscent of a similar approach used in the study of the entropy production associated to the Becker–Döring equation in [9]. An additional ingredient of the proof is the instantaneous generation of a Maxwellian lower bound to the solutions of (1.1), which is the reason why assumption (1.12) is needed.

We also consider the non-cutoff case briefly in Section 6. If one assumes that

$$c_0 |\theta|^{-(d-1)-\nu} \leq b(\cos \theta) \leq c_1 |\theta|^{-(d-1)-\nu}, \qquad \nu \in (0, 2) \tag{1.15}$$

for certain positive constants $c_1 \geq c_0 > 0$ then the cutoff assumption (1.4) is not satisfied. The spectral gap properties of the *linearised* Boltzmann equation are well-understood also in this case [30,20], and by following the technique in [30] we show a analogous result for the linear Boltzmann equation: if $\gamma + \nu > 0$ the operator $\boldsymbol{L}_\gamma$ has a spectral gap in the space $L^2(\mathcal{M}^{-1})$ (see Proposition 6.1). Since we are interested in inequalities involving the logarithmic entropy, we may wonder whether a similar linear inequality holds true for the entropy production

$$\mathscr{D}(f) = -\int_{\mathbb{R}^d} \boldsymbol{L} f \log \left(\frac{f}{\mathcal{M}} \right) \, \mathrm{d}v.$$

While we have not been able to prove this, we conjecture that it is indeed the case. More precisely:

Conjecture 1. *For a non cut-off collision kernel $B(v - v_*, \sigma) = |v - v_*|^\gamma b(\cos \theta)$ with $\gamma \in (-d, 0)$ and $b(\cdot)$ satisfying (1.15) such that*

$$\gamma + \nu > 0$$

there exists $\lambda_{\gamma,b} > 0$ such that

$$\mathscr{D}(f) \geq \lambda_{\gamma,b} H(f|\mathcal{M}) \tag{1.16}$$

for all $f \geq 0$ with unit mass.

A linear inequality like (1.16) is usually refer to as a *modified Logarithmic Sobolev inequality* and is known to be equivalent to the exponential decay of $H(f(t)|\mathcal{M})$ along the flow of solutions to the Boltzmann equation

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t, v) = \boldsymbol{L} f(t, v), \qquad f(0, \cdot) = f_0 \in L_2^1 \log L \tag{1.17}$$

(see for instance [6]). Such a modified Logarithmic Sobolev inequality would imply the spectral gap inequality (6.2) with

$$\lambda \geq \frac{\lambda_{\gamma,b}}{2}$$

(but is not equivalent to it).

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1.4. Organization of the paper

The structure of the paper is as follows: Section 2 is dedicated to the main entropy–entropy production inequality of the type (1.11) and to the investigation of points (i)–(iii), leading to the proof of our first main result in Section 3. In Section 4 we show the creation of pointwise Maxwellian lower bounds under certain restrictions on the angular kernel. This will not only give an alternative to point (iii) (which will not improve the rate of convergence for Theorem 1.4), but will be crucial in the proof of Theorem 1.5, which we will give in Section 5. In Section 6, we discuss the case of the linear Boltzmann equation with soft potential *without the cut-off* assumption and show the existence of a spectral gap for a certain range of the parameters. This is done by an adaptation of similar results from [30]. The last pages of the paper are dedicated to several Appendices that provide additional details that we felt would hinder the flow of the main work.

2. The entropy inequality and technical estimates

The goal of this section is to find an appropriate entropy–entropy production inequality associated to $\boldsymbol{L}_\gamma$, from which we will be able to obtain a quantitative estimation on the rate of convergence to equilibrium.

In order to achieve this we start by rewriting the operator $\boldsymbol{L}_\gamma$ as the sum of a gain and a loss operators. Due to the cut-off assumption (1.4) the operator $\boldsymbol{L}_\gamma$ can be decomposed in the following way:

$$\boldsymbol{L}_\gamma f(v) = \boldsymbol{K}_\gamma f(v) - \Sigma_\gamma(v) f(v),$$

where

$$\boldsymbol{K}_\gamma f(v) = \int\limits_{\mathbb{R}^d \times \mathbb{S}^{d-1}} |v - v_*|^\gamma \, b\,(\cos\theta)\, f\,(v')\, \mathcal{M}\,(v'_*)\, \mathrm{d}v_*\, \mathrm{d}\sigma \qquad (2.1)$$

and the collision frequency Σ_γ is given by

$$\Sigma_\gamma(v) = \int\limits_{\mathbb{R}^d \times \mathbb{S}^{d-1}} b(\cos\theta)\, |v - v_*|^\gamma\, \mathcal{M}\,(v_*)\, \mathrm{d}v_*\, \mathrm{d}\sigma = \int\limits_{\mathbb{R}^d} |v - v_*|^\gamma\, \mathcal{M}\,(v_*)\, \mathrm{d}v_*. \qquad (2.2)$$

The loss operator, of a simpler nature, satisfies the following (see [8] or [11, Lemma 6.1] for a detailed proof):

Lemma 2.1. *For any $\gamma \in \mathbb{R}$ there exists explicit constants $C_1, C_2 > 0$, depending only on γ, d and $\|b\|_1$ such that*

$$C_1 \,(1 + |v|)^\gamma \leq \Sigma_\gamma(v) \leq C_2 \,(1 + |v|)^\gamma. \qquad (2.3)$$

Remark 2.2. From the above estimate one can easily infer that the range of the mapping $-\Sigma_\gamma$ is given by $[-\nu_0, 0)$ for $\nu_0 = \inf_{v \in \mathbb{R}^d} \Sigma_\gamma(v)$. Using the fact that $\boldsymbol{K}_\gamma$ is a compact operator in the space $L^2(\mathcal{M}^{-1})$ (see for instance [8] or [21,22] for the proof in the linearised setting), one deduces that the essential spectrum of $\boldsymbol{L}_\gamma$ in that space contains $[-\nu_0, 0)$. In particular, $\boldsymbol{L}_\gamma$ does not exhibit a spectral gap in that space.

We are now ready to state our main entropy inequality.

Theorem 2.3. *Let $\gamma \in (-d, 0)$ and let $f \in L^1_\mu \log L\left(\mathbb{R}^d\right) \cap L^1_{\mu+2}(\mathbb{R}^d)$ for some $\mu > 0$, be a non-negative function with unit mass. Then*

$$\mathscr{D}_\gamma(f) \geq \mathscr{D}_0(f)^{\frac{\mu-\gamma}{\mu}} \mathscr{D}_\mu(f)^{\frac{\gamma}{\mu}} \geq \lambda_0^{1-\frac{\gamma}{\mu}} \mathscr{D}_\mu(f)^{\frac{\gamma}{\mu}} H\,(f|\mathcal{M})^{1-\frac{\gamma}{\mu}}, \qquad (2.4)$$

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and it also holds that

$$\mathscr{D}_\mu(f) \geq C_{s,d}\left(\int_{\mathbb{R}^d} (1+|v|)^\mu f(v)\log f(v)\,\mathrm{d}v + \int_{\mathbb{R}^d} (1+|v|)^{\mu+2} f(v)\,\mathrm{d}v - \int_{\mathbb{R}^d} \boldsymbol{K}_\mu(f)(v)\log f(v)\,\mathrm{d}v\right) \quad (2.5)$$

where $C_{\mu,d}$ is a universal constant that depends only on μ and d, and λ_0 is the positive parameter (depending on b) appearing in Theorem 1.3 for Maxwell molecules.

Proof. Recall that, for all $\alpha > -d$

$$\mathscr{D}_\alpha(f) = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} |v - v_*|^\alpha b(\cos\theta)\mathcal{M}(v)\mathcal{M}(v_*)\left(h(v') - h(v)\right)\log\frac{h(v')}{h(v)}\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma.$$

Introducing the measure $\mathrm{d}\nu(v, v_*, \sigma) = \frac{1}{2}b(\cos\theta)\mathcal{M}(v)\mathcal{M}(v_*)\left(h(v') - h(v)\right)\log\frac{h(v')}{h(v)}\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma$ on $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}$ one has

$$\mathscr{D}_\alpha(f) = \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} |v - v_*|^\alpha \,\mathrm{d}\nu(v, v_*, \sigma)$$

and, using Hölder's inequality on

$$\mathscr{D}_0(f) = \int_{\mathbb{R}^d} |v - v_*|^{\frac{\mu\gamma}{\mu-\gamma}}|v - v_*|^{\frac{-\mu\gamma}{\mu-\gamma}}\,\mathrm{d}\nu(v, v_*, \sigma)$$

with $p = \frac{\mu-\gamma}{\mu}$, $q = -\frac{\mu-\gamma}{\gamma}$ we get

$$\mathscr{D}_\gamma(f) \geq \mathscr{D}_0(f)^{\frac{\mu-\gamma}{\mu}}\mathscr{D}_\mu(f)^{\frac{\gamma}{\mu}}. \quad (2.6)$$

Next, as $\mathscr{D}_\mu(f) = -\int_{\mathbb{R}^d} \mathcal{L}_\mu(f)\log(f/\mathcal{M})\,\mathrm{d}v$ we have that

$$\mathscr{D}_\mu(f) \leq \int_{\mathbb{R}^d} \Sigma_\mu(v)f(v)\log f(v)\,\mathrm{d}v + \int_{\mathbb{R}^d} \boldsymbol{K}_\mu(f)(v)\log\mathcal{M}(v)\,\mathrm{d}v$$

$$- \int_{\mathbb{R}^d} \Sigma_\mu(v)(f)\log\mathcal{M}(v)\,\mathrm{d}v - \int_{\mathbb{R}^d} \boldsymbol{K}_\mu(f)(v)\log f(v)\,\mathrm{d}v.$$

Since $\log(\mathcal{M}(v)) = -\frac{d}{2}\log(2\pi) - |v|^2/2 < 0$ and $\boldsymbol{K}_\mu(f)(v) \geq 0$ when f is non-negative we conclude that $\int_{\mathbb{R}^d} \boldsymbol{K}_\mu(f)(v)\log\mathcal{M}(v)\,\mathrm{d}v \leq 0$. Moreover, using Lemma 2.1 we find $C_1, C_2 > 0$, depending only on d and μ such that

$$\mathscr{D}_\mu(f) \leq C_2 \int_{\mathbb{R}^d} (1+|v|)^\mu f(v)\log f(v)\,\mathrm{d}v + \frac{d}{2}\log(2\pi)C_2 \int_{\mathbb{R}^d} (1+|v|)^\mu f(v)\,\mathrm{d}v$$

$$+ \frac{C_2}{2}\int_{\mathbb{R}^d} (1+|v|)^{\mu+2} f(v)\,\mathrm{d}v - \int_{\mathbb{R}^d} \boldsymbol{K}_\mu(f)(v)\log(f(v))\,\mathrm{d}v.$$

The above, together with (2.6) and Theorem 1.3 complete the proof. $\square$

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The above Theorem is the reason for us to investigate the evolution of moments and L^p norms of f, as well as pointwise lower bounds of f. These are the topics of the following subsections. From now on, we shall always assume that $\gamma \in (-d, 0)$.

2.1. The evolution of moments

The study of moments and their time evolution is fundamental in many kinetic equations (and other PDEs where "energy methods" are applicable). In the case of the Boltzmann equation, the study of creation and propagation of moments for soft and hard potentials with angular cut-off is radically different. The linear Boltzmann equation we study here exhibits properties that reflect a similar moment growth as its non-linear counterpart.

We recall that notations for the moments of f have been introduced in subsection 1.3. To simplify writing, we will denote by

$$m_s(t) = m_s(f(t, \cdot)), \qquad M_s(t) = M_s(f(t, \cdot)) \qquad \forall t \geq 0, \quad s \in \mathbb{R}.$$

For a given $s \in \mathbb{R}$ we define the function

$$\boldsymbol{w}_s(v) = \int_{\mathbb{R}^d} |v - v_*|^\gamma |v_*|^s \mathcal{M}(v_*) \, dv_*, \tag{2.7}$$

which will play an important role in the sequel and which satisfies the following estimate (similar in nature to Lemma 2.1):

Lemma 2.4. *For any $s \geq 0$ and $\gamma > -d$, $\boldsymbol{w}_s$ is a bounded function. That is,*

$$\sup_{v \in \mathbb{R}^d} \boldsymbol{w}_s(v) := \|\boldsymbol{w}_s\|_\infty < \infty. \tag{2.8}$$

The main theorem we prove in this section is the following:

Theorem 2.5. *Let $f_0 \in L^1_s(\mathbb{R}^d)$ for $s = 2$ or $s > 2 \max(|\gamma|, 1)$ such that f_0 has unit mass and let $f(t)$ be the unique solution for (1.1). If $b \in L^\infty(\mathbb{S}^{d-1})$ and $\|b\|_1 = 1$ there exists a constant C_s, depending only on s, γ, d, the collision kernel and $\|f_0\|_{L^1_s(\mathbb{R}^d)}$ such that*

$$m_s(t) \leq C_s(1 + t), \qquad \forall t \geq 0. \tag{2.9}$$

In order to prove the above theorem we will need to use the so-called Povzner's Lemma (see [26]). The version we present here can be found in greater generality in [7].

Lemma 2.6. *Assume that the angular kernel $b(\cdot)$ is a bounded function and let $s > 2$. Setting*

$$I_s(v, v_*) = \int_{\mathbb{S}^{d-1}} b(\cos\theta) \left(|v'|^s + |v'_*|^s - |v|^s - |v_*|^s \right) d\sigma, \qquad v, v_* \in \mathbb{R}^d$$

we have that

$$I_s(v, v_*) \leq C_s^{(1)} |v|^{\frac{s}{2}} |v_*|^{\frac{s}{2}} - C_s^{(2)} \left(|v|^s + |v_*|^s \right) \left(1 - \mathbf{1}_{\frac{|v|}{2} \leq |v_*| \leq 2|v|}(v, v_*) \right), \tag{2.10}$$

where $C_s^{(1)}, C_s^{(2)}$ are positive constants that depend only on d, s and the angular kernel, and where $\mathbf{1}_A$ is the indicator function of the set A.

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The reason we use this version of Povzner's Lemma rather than others (such as the one in [18]) is due to the fact that it gives a minimal order of growth in terms of $|v|$. As any order of growth in $|v_*|$ is absorbed by the Maxwellian, the result obtained with this version of the lemma is optimal in our setting.

Proof of Theorem 2.5. We start with considering the case $s = 2$ as it doesn't require the boundedness of b, due to the special geometry in this case. Using the natural pre-post collision change of variables we find that

$$\frac{\mathrm{d}}{\mathrm{d}t} m_2(t) = \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} |v - v_*|^\gamma \, b\left(\cos\theta\right) f(t, v) \mathcal{M}(v_*) \left(|v'|^2 - |v|^2\right) \mathrm{d}v \, \mathrm{d}v_* \, \mathrm{d}\sigma.$$

As

$$|v'|^2 - |v|^2 = |v_*|^2 - |v_*'|^2$$

and $\|b\|_1 = 1$, we find that

$$\frac{\mathrm{d}}{\mathrm{d}t} m_2(t) \leq \int_{\mathbb{R}^d} f(t, v) \boldsymbol{w}_2(v) \, \mathrm{d}v \leq \|\boldsymbol{w}_2\|_\infty,$$

where we have used the mass conservation property of the equation. Thus

$$m_2(t) \leq \max\left(m_2(0), \|\boldsymbol{w}_2\|_\infty\right)(1 + t). \tag{2.11}$$

Next, we consider $s > 2$. Similar to the above we find that

$$\frac{\mathrm{d}}{\mathrm{d}t} m_s(t) = \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} |v - v_*|^\gamma \, b\left(\cos\theta\right) f(t, v) \mathcal{M}(v_*) \left(|v'|^s - |v|^s\right) \mathrm{d}v \, \mathrm{d}v_* \, \mathrm{d}\sigma$$

$$= \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} |v - v_*|^\gamma \, b\left(\cos\theta\right) f(t, v) \mathcal{M}(v_*) \left(|v'|^s + |v_*'|^s - |v|^s - |v_*|^s\right) \mathrm{d}v \, \mathrm{d}v_* \, \mathrm{d}\sigma$$

$$+ \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} |v - v_*|^\gamma \, b\left(\cos\theta\right) f(t, v) \mathcal{M}(v_*) \left(|v_*|^s - |v_*'|^s\right) \mathrm{d}v \, \mathrm{d}v_* \, \mathrm{d}\sigma$$

$$\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - v_*|^\gamma \, f(t, v) \mathcal{M}(v_*) I_s(v, v_*) \, \mathrm{d}v \, \mathrm{d}v_* + \|\boldsymbol{w}_s\|_\infty.$$

Using Lemma 2.6, together with the fact that

$$\left(|v|^s + |v_*|^s\right) \mathbf{1}_{\frac{|v|}{2} \leq |v_*| \leq 2|v|}(v, v_*) \leq (2^s + 1) |v_*|^s,$$

we conclude that there exist appropriate universal constants C_i $(i = 1, 2, 3)$ that depends only on γ, d and the angular kernel such that

$$\frac{\mathrm{d}}{\mathrm{d}t} m_s(t) \leq C_s^{(1)} \|\boldsymbol{w}_{\frac{s}{2}}\|_\infty m_{\frac{s}{2}}(t) - C_s^{(2)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - v_*|^\gamma \, |v|^s \, f(t, v) \mathcal{M}(v_*) \, \mathrm{d}v \, \mathrm{d}v_*$$

$$+ \left(C_s^{(2)} (2^s + 1) + 1\right) \|\boldsymbol{w}_s\|_\infty$$

$$\leq C_1 m_{\frac{s}{2}}(t) - C_2 \int_{\mathbb{R}^d} \Sigma_\gamma(v) |v|^s \, f(t, v) \, \mathrm{d}v + C_3.$$

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Using Lemma 2.1 we find that, using abusive notations for the constants,

$$\frac{\mathrm{d}}{\mathrm{d}t} m_s(t) \le C_1 m_{\frac{s}{2}}(t) - C_2 \int_{\mathbb{R}^d} (1 + |v|)^\gamma |v|^s f(t, v) \, \mathrm{d}v + C_3$$

$$= C_1 m_{\frac{s}{2}}(t) - C_2 \int_{\mathbb{R}^d} \langle v \rangle^\gamma (1 + |v|^s) f(t, v) \, \mathrm{d}v + C_2 \int_{\mathbb{R}^d} \langle v \rangle^\gamma f(t, v) \, \mathrm{d}v + C_3$$

i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t} m_s(t) \le C_1 m_{\frac{s}{2}}(t) - C_2 M_{s+\gamma}(t) + C_3.$$

Since $s > 2|\gamma|$ we see that $s + \gamma > s/2$ and as such

$$m_{\frac{s}{2}}(t) \le M_{\frac{s}{2}}(t) \le M_{s+\gamma}(t)^{\frac{s}{2(s+\gamma)}}.$$

Thus, in our settings,

$$\frac{\mathrm{d}}{\mathrm{d}t} m_s(t) \le M_{s+\gamma}(t)^{-\frac{s+2\gamma}{2(s+\gamma)}} \left(C_1 M_{s+\gamma}(t) - C_2 M_{s+\gamma}(t)^{1+\frac{s+2\gamma}{2(s+\gamma)}} \right) + C_3.$$

Since for any $\delta > 0$ the exists a constant $C(a, b, \delta) > 0$ such that

$$\sup_{x>0} \left(ax - bx^{1+\delta} \right) \le C(a, b, \delta),$$

and since $s + 2\gamma > 0$ and $M_s(f) \ge m_0(f)$, we conclude that there exists appropriate constants such that

$$\frac{\mathrm{d}}{\mathrm{d}t} m_s(t) \le C_1 M_{s+\gamma}(t)^{-\frac{s+2\gamma}{2(s+\gamma)}} + C_3 \le C_s,$$

completing the proof. $\quad\square$

Theorem 2.5 gives us the tools to improve any growth estimation of a given moment, as long as the initial data has higher moments.

Corollary 2.7. *Let $s_1 = 2$ or $s_1 > 2 \max(1, |\gamma|)$, and let $s_2 \ge s_1$. Then, if $f_0 \in L^1_{s_2}(\mathbb{R}^d)$ with a unit mass, and $b \in L^\infty(\mathbb{S}^{d-1})$ such that $\|b\|_1 = 1$, we have that*

$$m_{s_1}(t) \le C_{s_2} (1 + t)^{\frac{s_1}{s_2}}, \qquad \forall t \ge 0, \tag{2.12}$$

for some constant C_{s_2}, depending only on s_2, γ, d, the collision kernel $b(\cdot)$ and $\|f_0\|_{L^1_{s_2}(\mathbb{R}^d)}$.

Proof. This follows from simple interpolation. $\quad\square$

2.2. L^p estimates

The goal of this subsection is to show the propagation of L^p bounds – as long as one has enough moments. The approach we present here follows that of [32]. The main result we will show is:

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Theorem 2.8. *Let $f_0 \in L^1\left(\mathbb{R}^d\right) \cap L^p\left(\mathbb{R}^d\right)$ for some $p > 1$ and let $f = f(t,v)$ be a non-negative solution to* (1.1)*. There exists a constant $C_{p,d,\gamma}$ depending only on p, d and γ and $\boldsymbol{r} > 1$ such that*

$$\|f(t,\cdot)\|_p^p \leq \|f_0\|_p^p + C_{p,d,\gamma} \int_0^t M_{\boldsymbol{r}}^p(\tau)\, d\tau. \tag{2.13}$$

Remark 2.9. Under the assumptions of the above Theorem, setting

$$\eta_0 := \min\left(p-1, \frac{(p-1)^2}{(2-p)^+}, \frac{p^2}{\sqrt{\frac{1}{(p-1)^2} + p^2 + \frac{1}{p-1}}}, \frac{p^2}{\sqrt{\frac{1}{(1+\frac{\gamma}{d})^2} + p^2 + \frac{1}{1+\frac{\gamma}{d}}}}\right)$$

where $a^+ = \max(a, 0)$, we can deduce from the proof that, for any $0 < \eta < \eta_0$, one can chose

$$\boldsymbol{r} = \frac{|\gamma|}{\eta}\max\left(p-1-\eta, (p+\eta)(p-2)+1\right). \tag{2.14}$$

It is easy to see that $p - 1 - \eta < (p+\eta)(p-2) + 1$ if and only if $\eta > 2 - p$. This means that whenever $p \geq 2$ we have

$$\boldsymbol{r} = -\frac{\gamma}{\eta}\left((p+\eta)(p-2)+1\right).$$

In addition, we notice that if $p \to 1^+$, one has $\eta_0 \simeq \frac{1}{2}(p-1)^2$. Choosing $\eta \simeq \eta_0$ we get

$$\boldsymbol{r} \simeq_{p\to 1^+} -\frac{2\gamma}{(p-1)^2}\left(p-1-\frac{(p-1)^2}{2}\right) \simeq -\frac{2\gamma}{p-1}.$$

Remark 2.10. Before we set the stage for the proof of Theorem 2.8, we would like to note an important difference between the study of the linear Boltzmann equation and the fully non-linear equation in this setting. The work [32] deals with collision kernels of the form $B(v - v_*, \sigma) = \Phi(|v - v_*|)b(\cos\theta)$ where

$$(1+r)^\gamma \leq \Phi(r) \leq c_2(1+r)^\gamma$$

for some $c_1, c_2 > 0$. Our version of the L^p bound propagation, however, deals directly with $\phi(r) = r^\gamma$. The reason we are able to do that is the presence of the Maxwellian in the collision operator, acting as a mollifier to the singularity.

The proof of this Theorem relies on the following integrability property of $\boldsymbol{K}_\gamma$ that can be found in [2]:

Theorem 2.11. *Let $\gamma \in (-d, 0)$ and $1 < r, q, \ell < \infty$ with*

$$\frac{1}{q} + \frac{1}{\ell} = 1 + \frac{\gamma}{d} + \frac{1}{r}. \tag{2.15}$$

Then the gain operator $\boldsymbol{K}_\gamma$ satisfies

$$\|\boldsymbol{K}_\gamma(f)\|_r \leq C\|f\|_q\|\mathcal{M}\|_\ell \tag{2.16}$$

where $C = C(r, q, \ell, \gamma, d) > 0$ is a uniform constant that depends only on r, q, ℓ, γ and d.

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Proof of Theorem 2.8. We recall the notation $M_{p,s}(f) = M_s\left(|f|^p\right)$ and, as before, set $M_{p,s}(t) = M_{p,s}(f(t,\cdot))$ for any $t \geq 0$, where $f(t,v)$ is the unique solution to (1.1). We have that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{p}\|f(t)\|_p^p = \int_{\mathbb{R}^d} \boldsymbol{K}_\gamma(f(t))(v)f^{p-1}(t,v)\,\mathrm{d}v - \int_{\mathbb{R}^d} \Sigma_\gamma(v)f^p(t,v)\,\mathrm{d}v.$$

Using Lemma 2.1, the equivalence of $(1+|v|)^\gamma$ and $\langle v \rangle^\gamma$, and taking $r, r' > 1$ to be Hölder conjugates (with r to be determined shortly) we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{p}\|f(t)\|_p^p \leq \|\boldsymbol{K}_\gamma(f(t))\|_r \|f(t)\|_{r'(p-1)}^{p-1} - c_\gamma M_{p,\gamma}(t),$$

for some appropriate constant c_γ. Using Theorem 2.11 with $q, \ell > 1$ to be fixed later, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{p}\|f(t)\|_p^p \leq C\|f(t)\|_q \|f(t)\|_{r'(p-1)}^{p-1} - c_\gamma M_{p,\gamma}(t), \tag{2.17}$$

for some uniform constant C, depending only on r, q, ℓ, γ and d satisfying (2.15). As our goal is to control the L^p growth by a high enough moment, we will now focus our attention on showing that $\|f(t)\|_{r'(p-1)}$ and $\|f(t)\|_q$ can both be controlled by powers of $M_{p,\gamma}(t)$ and $M_{\boldsymbol{r}}(t)$, for some explicit $\boldsymbol{r}$, in a certain range of parameters. For any $1 < q < p$ we define

$$\alpha = -\gamma\frac{q-1}{p-1} < -\gamma,$$

and find that, for any measurable function f:

$$\int_{\mathbb{R}^d} |f(v)|^q\,\mathrm{d}v = \int_{\mathbb{R}^d} \frac{\langle v \rangle^\alpha}{\langle v \rangle^\alpha} |f(v)|^{-\frac{\alpha p}{\gamma}} |f(v)|^{\frac{\alpha+\gamma}{\gamma}}\,\mathrm{d}v$$

$$\leq \left(\int_{\mathbb{R}^d} \langle v \rangle^\gamma |f(v)|^p\,\mathrm{d}v\right)^{-\frac{\alpha}{\gamma}} \left(\int_{\mathbb{R}^d} \langle v \rangle^{\frac{\alpha\gamma}{\alpha+\gamma}} |f(v)|\,\mathrm{d}v\right)^{\frac{\alpha+\gamma}{\gamma}}.$$

Thus, introducing the notation $\mathfrak{a} = -\gamma\frac{q-1}{p-q}$

$$\|f(t)\|_q \leq M_{p,\gamma}(t)^{\frac{q-1}{q(p-1)}} M_{\mathfrak{a}}(t)^{\frac{p-q}{q(p-1)}}. \tag{2.18}$$

We would like to explore the special case $q = r'(p-1)$, which we need to verify is possible. We notice that if $r > p$ then $r'(p-1) < p$. Also, in order for $r'(p-1)$ to be greater than 1 we only need that

$$r < \begin{cases} \frac{1}{2-p} & 1 < p < 2 \\ \infty & p \geq 2. \end{cases}$$

Since for $1 < p < 2$ we always have that $p < 1/(2-p)$, a choice of $r > p$ such that $1 < r'(p-1) < p$ is indeed always possible. With this choice, setting $\mathfrak{b} = -\gamma\frac{r(p-2)+1}{r-p}$, we find that

$$\|f(t)\|_{r'(p-1)}^{p-1} \leq M_{p,\gamma}(t)^{1-\frac{r-1}{r(p-1)}} M_{\mathfrak{b}}(t)^{\frac{r-p}{r(p-1)}}. \tag{2.19}$$

Plugging (2.18), (2.19) in (2.17) we find that

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$$\frac{d}{dt}\frac{1}{p}\|f(t)\|_p^p \le C\, M_{p,\gamma}(t)^{1-\frac{r-q}{rq(p-1)}} M_{\mathfrak{a}}(t)^{\frac{p-q}{q(p-1)}} M_{\mathfrak{b}}(t)^{\frac{r-p}{r(p-1)}} - c_\gamma M_{p,\gamma}(t). \tag{2.20}$$

Since for any $0 < \delta < 1$ and $a, b > 0$, we have that $\sup_{x>0}\left(ax^{1-\delta} - bx\right) = \left(\frac{1-\delta}{b}a\right)^{\frac{1}{\delta}}$, we see that if $0 < \frac{r-q}{rq(p-1)} < 1$

$$\frac{d}{dt}\frac{1}{p}\|f(t)\|_p^p \le C_{p,r,q,\gamma}\left(M_{\mathfrak{a}}(t)^{\frac{p-q}{q(p-1)}} M_{\mathfrak{b}}(t)^{\frac{r-p}{r(p-1)}}\right)^{\frac{rq(p-1)}{r-q}} \le C_{p,r,q,\gamma} M_{\max(\mathfrak{a},\mathfrak{b})}(t)^p. \tag{2.21}$$

This will give us the desired result as long as we can choose $1 < r, q, \ell < \infty$ such that

$$q < p < r, \quad (2-p)r < 1, \quad \frac{1}{q} + \frac{1}{\ell} = 1 + \frac{\gamma}{d} + \frac{1}{r}, \quad \text{and} \quad \frac{r-q}{rq(p-1)} < 1.$$

As ℓ only appears in (2.16) in the norm $\|\mathcal{M}\|_\ell$, we can choose it to be as large as we want. In particular, for our setting, we can replace (2.15) with

$$\frac{1}{q} < 1 + \frac{\gamma}{d} + \frac{1}{r}, \tag{2.22}$$

and then choose ℓ accordingly. We will choose $r = p + \eta$, $q = p - \eta$ and see what conditions we must have for $\eta > 0$:

- For $q > 1$ we require that $\eta < p - 1$.
- The condition $(2-p)r < 1$ is only valid when $1 < p < 2$. In that case any $\eta < \frac{(p-1)^2}{2-p}$ will do.
- Demanding that $r - q < rq(p-1)$ in this setting is equivalent to $2\eta < \left(p^2 - \eta^2\right)(p-1)$ which is valid when

$$\eta < \frac{p^2}{\sqrt{\frac{1}{(p-1)^2} + p^2} + \frac{1}{p-1}}.$$

- Lastly, inequality (2.22) is equivalent to $2\eta < \left(1 + \frac{\gamma}{d}\right)\left(p^2 - \eta^2\right)$, which is valid when

$$\eta < \frac{p^2}{\sqrt{\frac{1}{(1+\frac{\gamma}{d})^2} + p^2} + \frac{1}{1+\frac{\gamma}{d}}}.$$

This concludes the proof with the choice $r = \max(\mathfrak{a}, \mathfrak{b})$. $\quad\square$

2.3. Lower bounds – modification of the solution

In most studies connected to the entropy method for kinetic equations, a lower bound on the function is needed. In this subsection we will adapt a method from [11] to achieve such a bound by a forced modification of the solution. We will then investigate the relation between the entropy and entropy production of it and the original solution.

Given a non-negative and measurable function f and $0 < \delta < 1$ we define

$$f_\delta(v) = (1 - \delta)\, f(v) + \delta \mathcal{M}(v). \tag{2.23}$$

A simple, yet important observation is that if f is integrable and of unit mass, the same occurs for f_δ. Moreover, $\boldsymbol{L}_\gamma f_\delta = (1 - \delta)\boldsymbol{L}_\gamma f$ since $\boldsymbol{L}_\gamma(\mathcal{M}) = 0$.

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Lemma 2.12. *Let $f \in L^1 \left(\mathbb{R}^d \right)$ be of unit mass. For all $0 < \delta < 1$ we have that*

$$H \left(f | \mathcal{M} \right) \leq \frac{H \left(f_\delta | \mathcal{M} \right)}{1 - \delta} + \frac{\delta}{1 - \delta} \left(\log \left(\frac{1}{\delta} \right) - \frac{(1 - \delta) \log (1 - \delta)}{\delta} \right). \tag{2.24}$$

Proof. We start by noticing that since the function $\phi(x) = x \log x$ satisfies

$$\phi(x + y) \geq \phi(x) + \phi(y),$$

for any $x, y > 0$, we have that

$$H \left(f_\delta \right) \geq H \left((1 - \delta) f \right) + H(\delta \mathcal{M}) = (1 - \delta) \log (1 - \delta) + \delta \log \delta + (1 - \delta) H(f) + \delta H(\mathcal{M}), \tag{2.25}$$

where $H(f) = \int_{\mathbb{R}^d} f(v) \log f(v) dv$. On the other hand as

$$H \left(f | \mathcal{M} \right) = H(f) - \int_{\mathbb{R}^d} f(v) \log \mathcal{M}(v) \, dv = H(f) - H(\mathcal{M}) - \int_{\mathbb{R}^d} (f(v) - \mathcal{M}(v)) \log \mathcal{M}(v) \, dv \tag{2.26}$$

we find that

$$(1 - \delta) H(f) + \delta H(\mathcal{M}) = (1 - \delta) H \left(f | \mathcal{M} \right) + H \left(\mathcal{M} \right) + (1 - \delta) \int_{\mathbb{R}^d} (f(v) - \mathcal{M}(v)) \log \mathcal{M}(v) \, dv$$
$$= (1 - \delta) H \left(f | \mathcal{M} \right) + (1 - \delta) \int_{\mathbb{R}^d} f(v) \log \mathcal{M}(v) \, dv + \delta H \left(\mathcal{M} \right). \tag{2.27}$$

Using (2.26) again for f_δ yields

$$H \left(f_\delta \right) = H \left(f_\delta | \mathcal{M} \right) + (1 - \delta) \int_{\mathbb{R}^d} f(v) \log \mathcal{M}(v) \, dv + \delta H \left(\mathcal{M} \right). \tag{2.28}$$

Thus, combining (2.27) and (2.28) yields

$$(1 - \delta) H(f) + \delta H(\mathcal{M}) = (1 - \delta) H \left(f | \mathcal{M} \right) + H \left(f_\delta \right) - H \left(f_\delta | \mathcal{M} \right).$$

Plugging this into (2.25) we find that

$$(1 - \delta) H \left(f | \mathcal{M} \right) \leq H \left(f_\delta | \mathcal{M} \right) - (1 - \delta) \log (1 - \delta) - \delta \log \delta$$

from which the result follows. □

The next step in our study will be to understand how the entropy dissipation of $f_\delta(t)$ behaves with respect to the entropy dissipation of $f(t)$. From this point onwards we will assume that $\delta = \delta(t)$ is a smooth function of t.

Lemma 2.13. *Let $f(t, v)$ be a non-negative solution to (1.1) that has a unit mass and let $g(t, \cdot) = f_{\delta(t)}(t, \cdot)$ be defined as in (2.23). Then, if $\delta(t)$ is a non-increasing function, we have that*

$$\frac{\mathrm{d}}{\mathrm{d}t} H \left(g(t) \right) \leq -\mathscr{D}_\gamma \left(g(t) \right) - \delta'(t) H \left(f(t) | \mathcal{M} \right) + \frac{\delta'(t) \log \left(\delta(t) \right)}{1 - \delta(t)} \tag{2.29}$$

where $\delta'(t) = \frac{\mathrm{d}}{\mathrm{d}t} \delta(t)$.

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Proof. To begin with, we notice that $g(t, \cdot)$ solves the following equation:

$$\partial_t g(t, v) = \boldsymbol{L}_\gamma g(t, v) - \frac{\delta'(t)}{1 - \delta(t)} \left(g(t, v) - \mathcal{M}(v) \right). \tag{2.30}$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t} H\left(g(t) | \mathcal{M} \right) = -\mathscr{D}_\gamma \left(g(t) \right) - \frac{\delta'(t)}{1 - \delta(t)} \int_{\mathbb{R}^d} \left(g(t, v) - \mathcal{M}(v) \right) \log \left(\frac{g(t, v)}{\mathcal{M}(v)} \right) \mathrm{d}v \tag{2.31}$$

since $g(t, v)$ has a unit mass for all t. Using the convexity of the relative entropy we see that

$$\int_{\mathbb{R}^d} g(t, v) \log \left(\frac{g(t, v)}{\mathcal{M}(v)} \right) \mathrm{d}v = H\left(g(t) | \mathcal{M} \right) \le (1 - \delta(t)) H\left(f(t) | \mathcal{M} \right).$$

Also, since $g(t, v) \ge \delta(t) \mathcal{M}(v)$

$$- \int_{\mathbb{R}^d} \mathcal{M}(v) \log \left(\frac{g(t, v)}{\mathcal{M}(v)} \right) \mathrm{d}v \le \log \left(\frac{1}{\delta(t)} \right).$$

Combining the above with (2.31) and using the fact that $\delta'(t) \le 0$ we conclude the result. $\quad\square$

We now have all the tools we need to prove Theorem 1.4.

3. Algebraic rate of convergence to equilibrium

The key to proving Theorem 1.4 is the entropy inequality (2.5). We start the section with a couple of simple lemmas that evaluate the terms in that inequality.

Lemma 3.1. *Let f be a non-negative function of unit mass.*

1) *Let $\mu > 0$ and $p > 1$. Then for any $\epsilon > 0$ there exists a uniform constant, $C_{\mu, d, p, \epsilon} > 0$, depending only on μ, d, p and ϵ such that*

$$\int_{f(t, v) \ge 1} (1 + |v|)^\mu f(v) \log f(v) \, \mathrm{d}v \le C_{\mu, d, p, \epsilon} \left(1 + m_{(1+\epsilon)\mu}(f) \right)^{\frac{1}{1+\epsilon}} \|f\|_p^{\frac{p\epsilon}{1+\epsilon}}.$$

2) *For any $\mu > 0$, it holds*

$$\int_{\mathbb{R}^d} (1 + |v|)^{\mu+2} f(v) \, \mathrm{d}v \le 2^{\mu+1} \left(1 + m_{\mu+2}(f) \right).$$

3) *If $f(v) \ge A \exp \left(-B |v|^2 \right)$ for some $A, B > 0$ we have that*

$$- \int_{\mathbb{R}^d} \boldsymbol{K}_\mu f(v) \log f(v) \, \mathrm{d}v \le C_{\mu, d, \gamma} \left(|\log A| \left(1 + m_\mu(f) \right) + B(1 + m_{\mu+2}(f)) \right),$$

where $C_{\mu, d, \gamma}$ is a uniform constant depending only on μ, d and γ.

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Proof. To prove **1**) we notice that by Hölder's inequality

$$
\int\limits_{f(t,v)\geq 1} (1+|v|)^{\mu}\, f(v) \log f(v)\, dv \leq \left(\int\limits_{\mathbb{R}^d} (1+|v|)^{(1+\epsilon)\mu}\, f(v)\, dv \right)^{\frac{1}{1+\epsilon}} \left(\int\limits_{f\geq 1} (\log f(v))^{\frac{1+\epsilon}{\epsilon}}\, f(v)\, dv \right)^{\frac{\epsilon}{1+\epsilon}}
$$

$$
\leq 2^{\mu} \left(\sup_{x>1} |\log x|^{(1+\epsilon)/\epsilon}\, x^{1-p} \right)^{\frac{\epsilon}{1+\epsilon}} \left(1 + m_{(1+\epsilon)\mu}(f) \right)^{\frac{1}{1+\epsilon}} \|f\|_p^{\frac{p\epsilon}{1+\epsilon}},
$$

showing the desired result. The second point **2**) is obvious, and to show **3**) we notice that under the condition on f one has that

$$
-\int\limits_{\mathbb{R}^d} \boldsymbol{K}_{\mu}(f)(v) \log f(v)\, dv \leq -\log A \int\limits_{\mathbb{R}^d} \boldsymbol{K}_{\mu} f(v)\, dv + B \int\limits_{\mathbb{R}^d} |v|^2\, \boldsymbol{K}_{\mu} f(v)\, dv.
$$

Now, as

$$
\int\limits_{\mathbb{R}^d} \phi(v) \boldsymbol{K}_{\mu}(v)\, dv = \int\limits_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} |v - v_*|^{\mu}\, b\,(\cos\theta)\, f(v) \mathcal{M}(v_*) \phi\,(v')\, dv\, dv_*\, d\sigma
$$

we find that

$$
\int\limits_{\mathbb{R}^d} \boldsymbol{K}_{\mu} f(v)\, dv = \int\limits_{\mathbb{R}^d} \Sigma_{\mu}(v) f(v)\, dv \leq C(1 + m_{\mu}(f)),
$$

for some uniform constant C, due to Lemma 2.1. For $\phi(v) = |v|^2$ we use the fact that $|v'|^2 \leq |v|^2 + |v_*|^2$ and conclude that

$$
\int\limits_{\mathbb{R}^d} |v|^2\, \boldsymbol{K}_{\mu}(v)\, dv \leq C\left(1 + m_{\mu+2}(f)\right).
$$

This completes the proof. □

Lemma 3.2. *Assume that $b \in L^{\infty}(\mathbb{S}^{d-1})$ with $\|b\|_1 = 1$. Let $f_0 \in L^1(\mathbb{R}^d)$ be a non-negative function with unit mass. Assume in addition that there exists $p > 1$ such that $f_0 \in L^p(\mathbb{R}^d) \cap L^1_s(\mathbb{R}^d)$ for some $s > 2\max(1, |\gamma|)$ such that $s \geq \boldsymbol{r}$, with $\boldsymbol{r}$ as in* (2.14).

Consider $\mu, \epsilon > 0$ such that $(1+\epsilon)\mu \leq s$ and $\mu + 2 \leq s$, and let $f(t,v)$ be a non-negative solution to (1.1). *Define*

$$
g(t,v) = f_{\delta(t)}(t,v) = (1 - \delta(t)) f(t,v) + \delta(t)\mathcal{M}(v)
$$

where $0 \leq \delta(t) \leq 1$ is a smooth decreasing function.

(i) *Then there exists a uniform constant C_0, depending only on $d, \gamma, p, s, \mu, \epsilon$ and $\|f_0\|_p$ and $\|f_0\|_{L^1_s}$ such that*

$$
\int\limits_{\mathbb{R}^d} (1+|v|)^{\mu}\, g(t,v) \log g(t,v)\, dv + \int\limits_{\mathbb{R}^d} (1+|v|)^{\mu+2}\, g(t,v)\, dv \leq C_0\,(1+t)^{\ell} \tag{3.1}
$$

where $\ell := \frac{\mu}{s} + \max\left(\frac{2}{s}, \frac{\epsilon}{1+\epsilon}\left(\frac{rp}{s} + 1\right)\right)$.

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(ii) *There exists a uniform constant C_1, depending only on d, γ, s and $\|f_0\|_{L_s^1}$ such that*

$$-\int_{\mathbb{R}^d} \boldsymbol{K}_\mu g(t, v) \log g(t, v) \, \mathrm{d}v \leq C_1 \left(-\log\left(\delta(t)\right) + (1 + t)^{\frac{\mu+2}{s}} \right). \tag{3.2}$$

Proof. Using Corollary 2.7, Theorem 2.8 and Lemma 3.1 and the conditions on η and s we find that there exists a universal constant C, depending on the appropriate parameters and norms, such that

$$\int_{f(t,v) \geq 1} (1 + |v|)^\mu f(t, v) \log f(t, v) \, \mathrm{d}v + \int_{\mathbb{R}^d} (1 + |v|)^{\mu+2} f(t, v) \, \mathrm{d}v$$

$$\leq C_1 (1 + t)^{\frac{\mu}{s}} (1 + t)^{\frac{\epsilon}{1+\epsilon}\left(\frac{rp}{s}+1\right)} + C_2 (1 + t)^{\frac{\mu+2}{s}}$$

showing that (3.1) holds for the solution $f(t, v)$. Since $\phi(x) = x \log x$ is convex on $\mathbb{R}_+$

$$g(t) \log g(t) \leq (1 - \delta(t)) f(t) \log f(t) + \delta(t) \mathcal{M} \log \mathcal{M}.$$

Thus

$$\int_{\mathbb{R}^d} (1 + |v|)^\mu g(t, v) \log g(t, v) \, \mathrm{d}v + \int_{\mathbb{R}^d} (1 + |v|)^{\mu+2} g(t, v) \, \mathrm{d}v$$

$$\leq \int_{f(t,v) \geq 1} (1 + |v|)^\mu f(t, v) \log f(t, v) \, \mathrm{d}v + \int_{\mathbb{R}^d} (1 + |v|)^{\mu+2} f(t, v) \, \mathrm{d}v + C_\mathcal{M},$$

with $C_\mathcal{M}$ independent of δ or t, concluding the proof of (i).

To show (ii) we remind ourselves that $g(t, v) \geq \delta(t) \mathcal{M}(v)$, and using part **3**) of Lemma 3.1 we find that

$$-\int_{\mathbb{R}^d} \boldsymbol{K}_\mu g(t, v) \log g(t, v) \, \mathrm{d}v \leq C_{\mu,d,\gamma} \left(-\log\left(\delta(t)\right) + m_{\mu+2}\left(g(t)\right) \right)$$

$$= C_{\mu,d,\gamma} \left(-\log\left(\delta(t)\right) + (1 - \delta(t)) m_{\mu+2}\left(f(t)\right) + \delta(t) \right) \leq C_{\mu,d,\gamma} \left(-\log\left(\delta(t)\right) + (1 + t)^{\frac{2+\mu}{s}} \right). \quad \square$$

To complete the proof of Theorem 1.4, we need the following Lemma which is reminiscent of [11, Lemma 7.2]. As the proof is an easy adaptation, we omit the details here.

Lemma 3.3. *Let $\alpha, \beta > 0$ such that $\alpha < 1$. Consider the differential inequality*

$$u'(t) \leq -C (1 + t)^{-\alpha} u(t)^{1+\beta} + \xi(t), \qquad t \geq 0.$$

If $u(t)$ is an absolutely continuous function satisfying the above, and if

$$C_\xi = \sup_{t \geq 0} (1 + t)^{\frac{\beta+1-\alpha}{\beta}} \xi(t) < \infty$$

then

$$u(t) \leq \max\left(1, u(0), \left(\frac{1 - \alpha + \beta C_\xi}{\beta C}\right)^{\frac{1}{\beta}} \right) (1 + t)^{-\frac{1-\alpha}{\beta}}.$$

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We are finally ready to prove our main theorem.

Proof of Theorem 1.4. As the previous lemmas indicate, we start by identifying $s_{p,d,\gamma}$ in the theorem as r, defined in (2.14). We start by choosing $\epsilon, \mu > 0$ small enough such that $(1 + \epsilon)\mu \le s - 2$ and

$$\frac{\epsilon}{1+\epsilon}\left(\frac{rp}{s} + 1\right) \le \frac{2}{s}.$$

In such a case, $\ell = \frac{\mu+2}{2}$ where ℓ is defined in Lemma 3.2. Consider the function

$$\delta(t) = \frac{1}{2}\exp\left(-(1+t)^{\frac{\mu+2}{s}}\right).$$

As before, set $g(t,v) = f_\delta(t,v) = (1 - \delta(t))f(t,v) + \delta(t)\mathcal{M}(v)$. Using Theorem 2.3 with Lemma 3.2, we find that there exists a uniform constant, $C = C(f_0, d, \gamma, p, s, \mu)$ that depends on the appropriate parameters and norms, as well as λ_0 from Theorem 1.3, such that

$$\mathscr{D}_\gamma(g(t)) \ge C\left((1+t)^{\frac{\mu+2}{s}} + (1+t)^{\frac{2}{s}}\right)^{\frac{\gamma}{\mu}} H(g(t)|\mathcal{M})^{1-\frac{\gamma}{\mu}}$$

$$\ge 2^{\frac{\gamma}{\mu}} C(1+t)^{\frac{\mu+2}{s}\frac{\gamma}{\mu}} H(g(t)|\mathcal{M})^{1-\frac{\gamma}{\mu}}.$$

Combining the above with Lemma 2.13, and using the fact that $H(f(t)|\mathcal{M}) \le H(f_0|\mathcal{M})$ for any $t \ge 0$, we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}H(g(t)|\mathcal{M}) \le -C_{0,d,\gamma,p,s,\mu}(1+t)^{\frac{\mu+2}{s}\frac{\gamma}{\mu}} H(g(t)|\mathcal{M})^{1-\frac{\gamma}{\mu}} + \xi(t)$$

where we introduced

$$\xi(t) = \frac{\mu+2}{2s}(1+t)^{\frac{\mu+2}{s}-1}\exp\left(-(1+t)^{\frac{\mu+2}{s}}\right)\left(H(f_0|\mathcal{M}) + 2(1+t)^{\frac{(\mu+2)}{s}}\right).$$

From the above differential inequality, applying Lemma 3.3 with

$$\alpha = -\frac{\mu+2}{s}\frac{\gamma}{\mu}, \qquad \beta = -\frac{\gamma}{\mu}$$

we see that, provided $\alpha < 1$, there exists a constant $\mathbf{C} > 0$ that depends only on the appropriate parameters such that

$$H(g(t)|\mathcal{M}) \le \mathbf{C}(1+t)^{-\frac{1-\alpha}{\beta}} \qquad \forall t \ge 0.$$

Choosing

$$\mu = \frac{s-2}{1+\epsilon},$$

for an appropriate ϵ sufficiently small, in order to maximise the convergence rate, we see that $\alpha = \frac{|\gamma|}{s-2} + 2\epsilon\frac{|\gamma|}{s(s-2)} < 1$ provided that $|\gamma| < s - 2$ and ϵ small enough. This is indeed valid in our setting and In that case, $-\frac{1-\alpha}{\beta} = \frac{s-2}{(1+\epsilon)\gamma} + \frac{s+2\epsilon}{(1+\epsilon)s}$ and

$$H(g(t)|\mathcal{M}) \le \mathbf{C}(1+t)^{\frac{s-2}{(1+\epsilon)\gamma} + \frac{s+2\epsilon}{(1+\epsilon)s}} \qquad \forall t \ge 0$$

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for some constant $\mathbf{C} > 0$ depending only on $\|f_0\|_p$, $\|f_0\|_{L^1_s}$ and the parameters. Using the above with (2.24) we conclude that

$$H\left(f(t)|\mathcal{M}\right) \leq 2\mathbf{C}\left(1+t\right)^{\frac{s-2}{(1+\epsilon)\gamma}+\frac{s+2\epsilon}{(1+\epsilon)s}} + \exp\left(-\left(1+t\right)^{\frac{s+2\epsilon}{(1+\epsilon)s}}\right)\left(\left(1+t\right)^{\frac{s+2\epsilon}{(1+\epsilon)s}} + \sup_{0<x<1}\frac{(x-1)\log\left(1-x\right)}{x}\right),$$

concluding the proof. $\square$

Using the decay rate in Theorem 1.4 one can easily obtain by interpolation the boundedness of moments and L^p norms, using a technique sometimes known as "slowly growing *a priori* bounds" (see for example [14,32]). We start by showing that moments are uniformly bounded in time, if a sufficiently high moment is initially bounded:

Theorem 3.4. *Take $p > 1$ and $-d < \gamma < 0$, and let $f_0 \in L^1_s\left(\mathbb{R}^d\right) \cap L^p(\mathbb{R}^d)$ be a non-negative function with unit mass, for some $s \geq s_{p,d,\gamma}$ (where $s_{p,d,\gamma}$ is the constant from Theorem 1.4). Let $f = f(t)$ be the solution to equation (1.1) with a bounded angular kernel b.*

Given $k > 0$, there exists $\beta > k$ depending only on k, d, s, γ such that if additionally we have $M_\beta(f_0) < +\infty$ then it holds that

$$M_k(f(t)) \leq C_k \qquad \text{for all } t \geq 0,$$

for some constant $C_k > 0$ that depends only on k, p, s, d, γ, $M_\beta(f_0)$ and $\|f_0\|_p$.

Proof. According to Theorem 2.5 we have

$$M_\beta(f(t)) \leq C_\beta(1+t) \qquad \text{for } t \geq 0.$$

On the other hand, fixing $0 < \sigma < -1 + \frac{s-2}{|\gamma|}$ (for definiteness, take $\sigma := -\frac{1}{2} + \frac{s-2}{2|\gamma|}$), we can apply Theorem 1.4 and the Csiszár–Kullback inequality to get

$$\|f(t) - \mathcal{M}\|_1^2 \leq H(f(t)|\mathcal{M}) \leq C_0(1+t)^{-\sigma} \qquad \text{for } t \geq 0. \tag{3.3}$$

Now, by interpolation, for $\theta \in (0,1)$ given by $\beta(1-\theta) = k$, that is

$$\theta = \frac{\beta - k}{\beta},$$

we have

$$M_k(|f(t) - \mathcal{M}|) \leq \|f(t) - \mathcal{M}\|_1^\theta \, M_\beta(|f(t) - \mathcal{M}|)^{1-\theta}$$
$$\leq C_0^{\frac{\theta}{2}}(1+t)^{-\frac{\theta\sigma}{2}}\left(M_\beta(f(t)) + M_\beta(\mathcal{M})\right)^{1-\theta}$$
$$\leq C(1+t)^{-\frac{\theta\sigma}{2}}(1+t)^{1-\theta} = C(1+t)^{-\frac{\theta\sigma}{2}+1-\theta},$$

for some $C > 0$ depending on the allowed quantities. Taking β large, θ becomes close to 1 and we can choose β so that

$$-\frac{\theta\sigma}{2} + 1 - \theta < 0,$$

which corresponds to β satisfying $\beta > \frac{k(2+\sigma)}{\sigma}$. This gives

$$M_k(|f(t) - \mathcal{M}|) \leq C \qquad \text{for all } t \geq 0,$$

which gives the result since $M_k(f(t)) \leq M_k(|f(t) - \mathcal{M}|) + M_k(\mathcal{M}) \leq C + M_k(\mathcal{M})$. $\square$

We turn now to the boundedness of L^p norms:

Theorem 3.5. *Take $p > 1$ and $-d < \gamma < 0$, and let $f_0 \in L^1_s(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ be a non-negative function with unit mass, for some $s \geq s_{p,d,\gamma}$ (where $s_{p,d,\gamma}$ is the constant from Theorem 1.4). Let $f = f(t)$ be the solution to equation (1.1) with a bounded angular kernel b.*

Given $q > 0$, there exists $r > 1$ depending only on q, d, s, γ such that if additionally we have $\|f_0\|_r < +\infty$ then it holds that

$$\|f(t)\|_q \leq C_q \qquad \text{for all } t \geq 0,$$

for some constant $C_q > 0$ that depends only on q, p, s, d, γ and $\|f_0\|_r$.

Proof. The proof is similar to the previous one. Using Theorems 2.8 and 2.5 we have

$$\|f(t)\|_r \leq C(1+t)^2 \qquad \text{for } t \geq 0,$$

for some $C > 0$ depending on the allowed quantities. By interpolation, for $\theta \in (0,1)$ given by

$$\theta = \frac{q - r}{q(r - 1)},$$

we have, using also (3.3) (with the same choice of σ),

$$
\begin{aligned}
\|f(t) - \mathcal{M}\|_q &\leq \|f(t) - \mathcal{M}\|_1^\theta \|f(t) - \mathcal{M}\|_r^{1-\theta} \\
&\leq C_0^{\frac{\theta}{2}}(1+t)^{-\frac{\theta\sigma}{2}} \left(\|f(t)\|_r + \|\mathcal{M}\|_r\right)^{1-\theta} \\
&\leq C(1+t)^{-\frac{\theta\sigma}{2}}(1+t)^{2(1-\theta)} = C(1+t)^{-\frac{\theta\sigma}{2}+2(1-\theta)},
\end{aligned}
$$

for some $C > 0$ depending on the allowed quantities only. Taking r large, θ approaches 1 so we can choose r so that

$$-\frac{\theta\sigma}{2} + 2(1-\theta) < 0.$$

With this choice,

$$\|f(t) - \mathcal{M}\|_q \leq C \qquad \text{for all } t \geq 0,$$

which proves the result by noticing that $\|f(t)\|_q \leq \|f(t) - \mathcal{M}\|_q + \|\mathcal{M}\|_q$. $\square$

Remark 3.6. The previous bounds can be now used in the proof of Theorem 1.4 to improve the decay exponent. We do not give the details of this improvement since we do not believe it to be optimal, and the exponent σ depends anyway on $s_{p,d,\gamma}$, which has a complicated explicit expression.

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4. Instantaneous generation of Maxwellian lower bounds

In this Section we will investigate the phenomena of instantaneous creation of a Maxwellian lower bound to the solution of our linear Boltzmann equation, a property that is well understood for the fully non-linear Boltzmann equation. We arrive at this result by a careful investigation of the gain operator, $\boldsymbol{K}_\gamma$. The following Lemma, whose proof is left to Appendix A.1, is the first step in this direction.

Lemma 4.1. *For a collision kernel of the form* (1.3), *the gain part operator* $\boldsymbol{K}_\gamma = \mathcal{Q}^+(\cdot, \mathcal{M})$ *admits the following representation:*

$$\boldsymbol{K}_\gamma f(v) = \int_{\mathbb{R}^d} k_\gamma(v, w) f(w) \, \mathrm{d}w, \qquad v, w \in \mathbb{R}^d,$$

with

$$k_\gamma(v, w) = \frac{2^{d-1}}{(2\pi)^{\frac{d}{2}} |v-w|} \exp\left(-\frac{1}{8}\left(|v-w| + \frac{|v|^2 - |w|^2}{|v-w|}\right)^2\right) \int_{(v-w)^\perp} \xi_{b,\gamma}(z, v, w) \, \mathrm{d}z \qquad (4.1)$$

where

$$\xi_{b,\gamma}(z, v, w) = \exp\left(-\frac{|V_\perp + z|^2}{2}\right) b\left(\frac{|z|^2 - |v-w|^2}{|z|^2 + |v-w|^2}\right) |z - (v-w)|^{\gamma - (d-2)}, \qquad (4.2)$$

with $V_\perp$ *being the projection of* $V = \frac{v+w}{2}$ *on the subspace that is perpendicular to* $v - w$.

In what follows, we will assume that $\gamma \in (-d, 0]$ is given. The key ingredient in establishing the creation of a lower bound is in estimating the term

$$\mathcal{I}_b(v, w) = \int_{(v-w)^\perp} \xi_{b,\gamma}(z, v, w) \, \mathrm{d}z, \qquad (4.3)$$

which will be the purpose of our next lemma. For $b = 1$, we simply use the notation $\mathcal{I}(v, w)$ to denote $\mathcal{I}_b(v, w)$.

Lemma 4.2. *Consider* $\mathcal{I}_b(v, w)$ *as defined in* (4.3). *Then*

(i) *If* $b = 1$ *and* $\beta \leq 0$ *then*

$$|v-w|^\beta \, \mathcal{I}(v, w) \geq C_{d,\gamma,\beta} \exp\left(-\left(|v|^2 + |w|^2\right)\right), \qquad (4.4)$$

   where $C_{d,\gamma,\beta} > 0$ *is a universal constant depending only on* d, γ *and* β.

(ii) *If* $b(x) \geq b_0 \left(1 - |x|^2\right)^{\frac{\nu}{2}}$ *for some* $b_0 > 0$, $0 \leq \nu \leq 1$, *and if* $d \geq 2$ *then*

$$|v-w|^{-1} \, \mathcal{I}_b(v, w) \geq C_{d,\gamma,b_0,\nu} \exp\left(-\frac{(2\nu + d - \gamma - 2)\left(|v|^2 + |w|^2\right)}{(d - \gamma - 2)}\right), \qquad (4.5)$$

   where $C_{d,\gamma,b_0,\nu} > 0$ *is a universal constant depending only on* d, γ, b_0 *and* ν.

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Proof. As $z \perp v - w$ we have that

$$|v - w|^\beta \geq \left(|z|^2 + |v - w|^2\right)^{\frac{\beta}{2}} = |z - (v - w)|^\beta.$$

Since $|V_\perp| \leq |V| \leq \frac{|v| + |w|}{2}$, and since $|x \pm y|^2 \leq 2\left(|x|^2 + |y|^2\right)$ we find that

$$\exp\left(-\frac{|V_\perp + z|^2}{2}\right) \geq \exp\left(-\frac{|v|^2 + |w|^2}{2}\right) \exp\left(-|z|^2\right).$$

As such

$$|v - w|^\beta \mathcal{I}(v, w) \geq \exp\left(-\frac{|v|^2 + |w|^2}{2}\right) \int\limits_{(v-w)^\perp} \exp\left(-|z|^2\right) \left(|z|^2 + |v - w|^2\right)^{\frac{\gamma - d + 2 + \beta}{2}} \, dz.$$

If $\beta \geq d - 2 - \gamma$ then

$$|v - w|^\beta \mathcal{I}(v, w) \geq \exp\left(-\frac{|v|^2 + |w|^2}{2}\right) \int\limits_{\mathbb{R}^{d-1}} |z|^{\gamma - d + 2 + \beta} \exp\left(-|z|^2\right) \, dz,$$

while if $\beta < d - 2 - \gamma$ then, for a given $\epsilon > 0$, we can find a universal constant $C_{\epsilon, d, \gamma, \beta}$ such that

$$\left(|z|^2 + |v - w|^2\right)^{\frac{\gamma - d + 2 + \beta}{2}} \geq C_{\epsilon, d, \gamma, \beta} \exp\left(-\epsilon\left(|z|^2 + |v - w|^2\right)\right),$$

from which we find that

$$|v - w|^\beta \mathcal{I}(v, w) \geq C_{\epsilon, d, \gamma, \beta} \exp\left(-\frac{|v|^2 + |w|^2}{2}\right) \exp\left(-\epsilon |v - w|^2\right) \int\limits_{\mathbb{R}^{d-1}} \exp\left(-(1 + \epsilon) |z|^2\right) \, dz,$$

completing the proof of (i) with the choice of $\epsilon = \frac{1}{4}$.

To show (ii) we start by noticing that

$$b\left(\frac{|z|^2 - |v - w|^2}{|z|^2 + |v - w|^2}\right) \geq 2^\nu b_0 \frac{|z|^\nu |v - w|^\nu}{\left(|z|^2 + |v - w|^2\right)^\nu}.$$

Next, using Hölder inequality we find that for any $0 < \alpha < 1$, to be chosen at a later stage, one has that

$$\mathcal{I}(v, w) = \int\limits_{(v-w)^\perp} \xi_\gamma(z, v, w) b\left(\frac{|z|^2 - |v - w|^2}{|z|^2 + |v - w|^2}\right)^\alpha b\left(\frac{|z|^2 - |v - w|^2}{|z|^2 + |v - w|^2}\right)^{-\alpha} \, dz \tag{4.6}$$

$$\leq \mathcal{I}_b(v, w)^\alpha \mathcal{I}_{b^{-\alpha/(1-\alpha)}}(v, w)^{1-\alpha}.$$

Due to the lower bound on b we have that

$$\mathcal{I}_{b^{-\alpha/(1-\alpha)}}(v, w) \leq \left(2^\nu b_0\right)^{-\frac{\alpha}{1-\alpha}} |v - w|^{-\frac{\alpha \nu}{1-\alpha}}$$

$$\int\limits_{(v-w)^\perp} \exp\left(-\frac{|V_\perp + z|^2}{2}\right) \left(|z|^2 + |v - w|^2\right)^{\frac{\gamma - d + 2}{2} + \frac{\alpha \nu}{1-\alpha}} |z|^{-\frac{\alpha \nu}{1-\alpha}} \, dz.$$

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We now choose

$$\alpha = \frac{d-2-\gamma}{2\nu + d - 2 - \gamma}, \tag{4.7}$$

which satisfies that $0 < \alpha < 1$ as well as $\frac{\alpha\nu}{1-\alpha} = \frac{d-2-\gamma}{2}$. With this in hand we get that

$$\mathcal{I}_{b^{-\alpha/(1-\alpha)}}(v,w) \leq (2^\nu b_0)^{-\frac{\alpha}{1-\alpha}} |v-w|^{-\frac{\alpha\nu}{1-\alpha}} \int\limits_{(v-w)^\perp} \exp\left(-\frac{|V_\perp + z|^2}{2}\right) |z|^{-\frac{d-2-\gamma}{2}} \, dz.$$

Splitting the integral according to $|z| > 1$ or $|z| \leq 1$, it is easy to see that

$$\sup_{v,w} \int\limits_{(v-w)^\perp} \exp\left(-\frac{|V_\perp + z|^2}{2}\right) |z|^{-\frac{d-2-\gamma}{2}} \, dz \leq C_{d,\gamma}$$

for some positive constant $C_{d,\gamma} > 0$ depending only on d and $\gamma > -d$. Then, there is some positive constant C (depending on d, γ, ν and b_0) such that

$$\mathcal{I}_{b^{-\alpha/(1-\alpha)}}(v,w) \leq C \, |v-w|^{-\frac{\alpha\nu}{1-\alpha}} \,.$$

Going back to (4.6), we find that

$$|v-w|^{-1} \mathcal{I}_b(v,w) \geq C_{d,\gamma,\nu,b_0} |v-w|^{\nu-1} \mathcal{I}(v,w)^{\frac{1}{\alpha}}$$

for some $C_{d,\gamma,\nu,b_0} > 0$. The result now follows from (i) if $\nu \leq 1$ where we recall that α is given by (4.7). $\square$

Remark 4.3. It is interesting to notice that the above constant C_{d,γ,ν,b_0} can be written as $C_{d,\gamma,\nu,b_0} = C_{d,\gamma,\nu} b_0$ for some universal constant $C_{d,\gamma,\nu}$ depending only on d, γ and $\nu \in [0,1]$.

Corollary 4.4. *Assume that $d \geq 2$ and that the angular kernel $b(\cdot)$ satisfies*

$$b(x) \geq b_0 \left(1 - |x|^2\right)^{\frac{\nu}{2}}, \qquad x \in (-1,1)$$

for some $b_0 > 0$ and $0 \leq \nu \leq 1$. Then, for all $v, w \in \mathbb{R}^d$,

$$k_\gamma(v,w) \geq C_{d,\gamma,b_0,\nu} \exp\left(-\lambda_1 |v|^2\right) \exp\left(-\lambda_2 |w|^2\right),$$

where $C_{d,\gamma,b_0,\nu} > 0$ is a universal constant depending only on d, γ, b_0 and ν and

$$\lambda_1 = \frac{3}{4} + \frac{2\nu + d - \gamma - 2}{d - \gamma - 2} > 0, \qquad \lambda_2 = \frac{1}{4} + \frac{2\nu + d - \gamma - 2}{d - \gamma - 2} > 0.$$

Proof. We start by noticing that $|v-w|^2 \leq 2\left(|v|^2 + |w|^2\right)$, and

$$\frac{\left(|v|^2 - |w|^2\right)^2}{|v-w|^2} = \left(\frac{|(v-w)(v+w)|}{|v-w|}\right)^2 \leq |v+w|^2 \leq 2\left(|v|^2 + |w|^2\right).$$

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As such

$$\frac{1}{4}\left(|v-w|+\frac{|v|^2-|w|^2}{|v-w|}\right)^2 = \frac{\left(|v|^2-|w|^2\right)^2}{4|v-w|^2} + \frac{|v|^2-|w|^2}{2} + \frac{|v-w|^2}{4}$$

$$\leq |v|^2 + |w|^2 + \frac{|v|^2-|w|^2}{2} = \frac{3|v|^2}{2} + \frac{|w|^2}{2}.$$

The result now follows from (4.1) and Lemma 4.2. $\square$

We are now ready to prove the main theorem of this section:

Theorem 4.5. *Let $f_0 \in L^1\left(\mathbb{R}^d\right)$ be a non-negative function with unit mass and finite second moment. Let $f(t,v)$ be a non-negative solution to (1.1) with angular kernel b that satisfies*

$$b(x) \geq b_0\left(1-|x|^2\right)^{\frac{\nu}{2}}, \qquad x \in (-1,1)$$

for some $b_0 > 0$ and $0 \leq \nu \leq 1$. Then, if $d \geq 2$, there exists a constant $C_{d,\gamma,b_0,\nu} > 0$, depending only on d, γ, b_0 and ν, such that for any $s > 0$, $v \in \mathbb{R}^d$ and $t > t_0 > 0$ we have that

$$f(t,v) \geq C_{d,\gamma,b_0,\nu}\left(1-\exp\left(-\|\Sigma_\gamma\|_\infty t_0\right)\right)\exp\left(-\lambda_1\left(|v|^2 + \sup_{\tau \leq t}(2m_s(\tau))^{\frac{2}{s}}\right)\right), \qquad (4.8)$$

where $C_{d,\gamma,b_0,\nu}$ is a constant that depends only on d, γ, b_0 and ν, and λ_1 is defined in Corollary 4.4.

Proof. As $f(t,v)$ is the solution to (1.1) we find that

$$\partial_t f(t,v) + \Sigma_\gamma(v) f(t,v) = \boldsymbol{K}_\gamma(f)(v).$$

Using Lemma 4.1 and Corollary 4.4 we can conclude that

$$\partial_t f(t,v) + \|\Sigma_\gamma\|_\infty f(t,v) \geq C_{d,\gamma,b_0,\nu}\exp\left(-\lambda_1|v|^2\right)\int_{\mathbb{R}^d}\exp\left(-\lambda_2|w|^2\right)f(t,w)\,\mathrm{d}w$$

$$\geq C_{d,\gamma,b_0,\nu}\exp\left(-\lambda_1\left(|v|^2+R^2\right)\right)\int_{|w|<R}f(t,w)\,\mathrm{d}w,$$

for any $R > 0$. For any $s > 0$, we know that

$$\int_{|w|<R}f(t,w)\,\mathrm{d}w = 1 - \int_{|w|\geq R}f(t,w)\,\mathrm{d}w \geq 1 - \frac{1}{R^s}\int_{|w|>R}|w|^s f(t,w)\,\mathrm{d}w \geq 1 - \frac{m_s(t)}{R^s}.$$

Using the above, and choosing $R = (2m_s(t))^{\frac{1}{s}}$, we find that for any $s > 0$

$$\partial_t f(t,v) + \|\Sigma_\gamma\|_\infty f(t,v) \geq \frac{C_{d,\gamma,b_0,\nu}}{2}\exp\left(-\lambda_1\left(|v|^2+(2m_s(t))^{\frac{2}{s}}\right)\right). \qquad (4.9)$$

Solving the above inequality and using that f_0 is nonnegative yields the result. $\square$

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Remark 4.6. Note that if there exists $a, B > 0$ such that $f_0(v) \geq B \exp\left(-a\,|v|^2\right)$ then solving the differential inequality (4.9) yields now

$$f(t,v) \geq C \exp\left(-\lambda\left(|v|^2 + \sup_{\xi \leq t}\left(2m_s(\xi)\right)^{\frac{2}{s}}\right)\right)$$

for some explicit $C = C(a, B, d, \gamma, b_0, \nu)$ and $\lambda = \lambda(d, \gamma, \nu, a)$ and all $t \geq 0$.

A simple consequence of the above estimate is the following

Corollary 4.7. *Under the assumption of Theorem 4.5, the nonnegative solution $f(t,v)$ to (1.1) with a bounded angular kernel b is such that, for any $t_0 > 0$ and $\mu > 0$,*

$$-\int_{\mathbb{R}^d} \boldsymbol{K}_\mu f(t,v) \log f(t,v)\, \mathrm{d}v \leq C_2\left(1+t\right)^{\frac{2+\mu}{s}},\tag{4.10}$$

where C_2 is a uniform constant depending only on d, γ, b_0, ν, s and t_0. If there exists $A, B > 0$ such that $f_0(v) \geq A \exp\left(-B\,|v|^2\right)$ then the above is valid from $t_0 = 0$ and the constant will also depend on A and B.

Proof. The proof follows immediately from Theorem 2.5, Lemma 3.1 and Theorem 4.5. $\square$

5. Stretched-exponential rate of convergence to equilibrium

At this section we will investigate the rate of decay for equilibrium under the additional assumption of having an exponential moment. We start by noticing the following simple result which we deduce from Lemma 1.2 for the convex function $\boldsymbol{\Phi}(x) = x^p$.

Proposition 5.1. *Let $p > 1$ and consider the functional*

$$H_p(f) = \int_{\mathbb{R}^d} \mathcal{M}(v)^{1-p}\, |f(v)|^p\, \mathrm{d}v.$$

Then, if $H_p(f_0) < \infty$ we have that any non-negative solution $f(t,v)$ to (1.1) with initial data f_0 satisfies

$$H_p(f(t)) \leq H_p(f_0) < \infty \qquad \forall t \geq 0.$$

We will now want to explore how the above H_p can improves our rate of convergence to equilibrium. We start by improving the interpolation inequality between $\mathscr{D}_\gamma$ and $\mathscr{D}_0$ provided by inequality (2.6):

Lemma 5.2. *For a given $a > 0$ and $q \geq 1$ define*

$$\boldsymbol{\Gamma}_{a,q}(f) = \frac{1}{2}\int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} b\left(\cos\theta\right)\exp\left(a\,|v - v_*|^q\right)\mathcal{M}(v)\mathcal{M}(v_*)\Psi\left(h(v), h\left(v'\right)\right) \mathrm{d}v\, \mathrm{d}v_*\, \mathrm{d}\sigma,$$

with $\Psi(x,y) = (x - y)\log\left(x/y\right)$ and $h = \frac{f}{\mathcal{M}}$. Then for any $\gamma < 0$ one has that

$$\mathscr{D}_\gamma(f) \geq \frac{a^{\frac{|\gamma|}{q}}}{2}\mathscr{D}_0(f)\left(\log\frac{2\boldsymbol{\Gamma}_{a,q}(f)}{\mathscr{D}_0(f)}\right)^{\frac{\gamma}{q}}.\tag{5.1}$$

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Proof. For a given $R > 0$ we set $\mathcal{Z}_{a,R} = \left\{ (v, v_*) \in \mathbb{R}^d \times \mathbb{R}^d \; ; \; |v - v_*| \leq \left(\frac{R}{a}\right)^{\frac{1}{q}} \right\}$ and denote by $\mathcal{Z}_{a,R}^c$ its complementary in $\mathbb{R}^{2d}$. We have that

$$
\mathscr{D}_0(f) = \frac{1}{2} \int_{\mathcal{Z}_{a,R} \times \mathbb{S}^{d-1}} b\left(\cos\theta\right) |v - v_*|^{|\gamma|} |v - v_*|^{\gamma} \mathcal{M}(v) \mathcal{M}(v_*) \Psi\left(h(v), h(v')\right) \, \mathrm{d}v \, \mathrm{d}v_* \, \mathrm{d}\sigma
$$

$$
+ \frac{1}{2} \int_{\mathcal{Z}_{a,R}^c \times \mathbb{S}^{d-1}} b\left(\cos\theta\right) \exp\left(-a |v - v_*|^q\right) \exp\left(a |v - v_*|^q\right) \mathcal{M}(v) \mathcal{M}(v_*) \Psi\left(h(v), h(v')\right) \, \mathrm{d}v \, \mathrm{d}v_* \, \mathrm{d}\sigma
$$

$$
\leq \left(\frac{R}{a}\right)^{\frac{|\gamma|}{q}} \mathscr{D}_\gamma(f) + \exp(-R) \mathbf{\Gamma}_{a,q}(f).
$$

We also notice that for any $a, q > 0$ we have that $1 \leq \exp\left(a |v - v_*|^q\right)$ and as such $\mathscr{D}_0(f) \leq \mathbf{\Gamma}_{a,q}(f)$. Thus, the choice

$$
R = \log\left(\frac{2\mathbf{\Gamma}_{a,q}(f)}{\mathscr{D}_0(f)}\right) > \log 2 > 0,
$$

is valid and yields

$$
\frac{\mathscr{D}_0(f)}{2} \leq a^{\frac{\gamma}{q}} \left(\log\left(\frac{2\mathbf{\Gamma}_{a,q}(f)}{\mathscr{D}_0(f)}\right)\right)^{\frac{|\gamma|}{q}} \mathscr{D}_\gamma(f)
$$

which completes the proof. $\square$

Corollary 5.3. *Under the same conditions of Lemma 5.2 we have that if $f(t, v)$ is a non-negative solution to (1.1) such that*

$$
\mathbf{\Gamma}_{a,q}^* = \sup_{t \geq 0} \mathbf{\Gamma}_{a,q}(f(t)) < \infty
$$

then

$$
\mathscr{D}_\gamma(f(t)) \geq \frac{a^{\frac{|\gamma|}{q}} \lambda_0 H(f(t)|\mathcal{M})}{2\left(\log\left(\frac{2\mathbf{\Gamma}_{a,q}^*}{\lambda_0 H(f(t)|\mathcal{M})}\right)\right)^{\frac{|\gamma|}{q}}} \tag{5.2}
$$

Proof. This follows immediately from Theorem 1.3 and Lemma 5.2. $\square$

In order to be able to conclude the desired rate of convergence to equilibrium we will need to connect $H_p(f(t))$ and $\mathbf{\Gamma}_{a,b}(f(t))$. To do so we notice the following:

Lemma 5.4. *Let $a > 0$, $p > 1$ and $1 < q \leq 2$ (with the additional assumption that, $a < 1/4$ whenever $q = 2$). Then, for any non-negative function $f(v)$ we have that*

$$
\mathbf{\Gamma}_{a,q}(f) \leq C_{a,q,p,d} \int_{\mathbb{R}^d} \exp\left(2^q a |v|^q\right) f(v)^p \, \mathrm{d}v
$$

$$
- \|b\|_\infty \int_{f(v') \leq 1} \exp\left(2^{q-1} a |v|^q\right) f(v) \exp\left(2^{q-1} a |v_*|^q\right) \mathcal{M}(v_*) \log\left(f(v')\right) \, \mathrm{d}v \, \mathrm{d}v_* \, \mathrm{d}\sigma \tag{5.3}
$$

for a uniform constant $C_{a,q,p,d} > 0$ that depends only on a, q, p and d.

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Proof. We start by noticing that since $|v - v_*|^q \leq 2^{q-1} (|v|^q + |v_*|^q)$ and $|v - v_*| = |v' - v'_*|$ we have that, for any $v, v_*, \sigma \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}$:

$$\exp\left(a\,|v - v_*|^q\right) \leq \min\left(\exp\left(2^{q-1}a\left(|v|^q + |v_*|^q\right)\right), \exp\left(2^{q-1}a\left(|v'|^q + |v'_*|^q\right)\right)\right).$$

Next, since

$$\mathbf{\Gamma}_{a,q}(f) \leq \|b\|_\infty \int \exp\left(a\,|v - v_*|^q\right) f(v) \mathcal{M}(v_*) \left(\log\left(\frac{f(v)}{\mathcal{M}(v)}\right) - \log\left(\frac{f(v')}{\mathcal{M}(v')}\right)\right) \mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma$$

we see that

$$\mathbf{\Gamma}_{a,q}(f) \leq \|b\|_\infty \left|\mathbb{S}^{d-1}\right| C_p \int_{f(v) \geq 1} \exp\left(2^{q-1}a\,|v|^q\right) f(v)^p \exp\left(2^{q-1}a\,|v_*|^q\right) \mathcal{M}(v_*) \,\mathrm{d}v\,\mathrm{d}v_*$$

$$+ \|b\|_\infty \left|\mathbb{S}^{d-1}\right| \int_{\mathbb{R}^{2d}} f(v) \left(\frac{d\log(2\pi)}{2} + \frac{|v|^2}{2}\right) \exp\left(2^{q-1}a\left(|v_*|^q + |v|^q\right)\right) \mathcal{M}(v_*) \,\mathrm{d}v\,\mathrm{d}v_*$$

$$- \|b\|_\infty \int_{f(v') \leq 1} f(v) \exp\left(2^{q-1}a\left(|v_*|^q + |v|^q\right)\right) \mathcal{M}(v_*) \log\left(f(v')\right) \mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma$$

where we discarded the term involving $\log \mathcal{M}(v')$ which is nonpositive. Under the additional requirement that $2^{q-1}a < \frac{1}{2}$ if $q = 2$ we see that we can find a constant $C_{a,q,p,d} > 0$ that depends only on a, q, p and d such that

$$\mathbf{\Gamma}_{a,q}(f) \leq C_{a,q,p,d} \int_{f(v) \geq 1} \exp\left(2^q a\,|v|^q\right) \left(f(v)^p + f(v)\right) \mathrm{d}v$$

$$- \|b\|_\infty \int_{f(v') \leq 1} \exp\left(2^{q-1}a\,|v|^q\right) f(v) \exp\left(2^{q-1}a\,|v_*|^q\right) \mathcal{M}(v_*) \log\left(f(v')\right) \mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma$$

$$\leq 2C_{a,q,p,d} \int_{\mathbb{R}^d} \exp\left(2^q a\,|v|^q\right) f(v)^p \,\mathrm{d}v$$

$$- \|b\|_\infty \int_{f(v') \leq 1} \exp\left(2^{q-1}a\,|v|^q\right) f(v) \exp\left(2^{q-1}a\,|v_*|^q\right) \mathcal{M}(v_*) \log\left(f(v')\right) \mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma$$

which concludes the proof. $\square$

Corollary 5.5. *Let $p > 1$ and $0 < a < \min\left(\frac{1}{8}, \frac{p-1}{8p}\right)$. Then for any non-negative f such that*

$$f(v) \geq A \exp\left(-B\,|v|^2\right),$$

for some $A, B > 0$ we have that

$$\mathbf{\Gamma}_{a,2}(f) \leq C_{a,p,d} \left(H_p(f) + \left(|\log A| + 2B\right) H_p(f)^{\frac{1}{p}}\right), \tag{5.4}$$

for a uniform constant $C_{a,p,d}$ that depends only on $a, p, \|b\|_\infty$ and d.

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Proof. As $a < \frac{p-1}{8}$ we find that

$$\int_{\mathbb{R}^d} \exp\left(4a\,|v|^2\right) f(v)^p \,\mathrm{d}v \leq (2\pi)^{\frac{d}{2}}\, H_p(f).$$

Next, due to the lower bound on f we find that

$$-\log f\,(v') \leq -\log A + B\,|v'|^2 \leq |\log A| + B\left(|v|^2 + |v_*|^2\right)$$

and as such

$$-\int_{f(v')\leq 1} \exp\left(2a\,|v|^2\right) f(v) \exp\left(2a\,|v_*|^2\right) \mathcal{M}(v_*) \log f\,(v')\, \mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma$$

$$\leq (|\log A| + 2B)\, C_{a,d} \int_{R^d \times \mathbb{R}^d} \exp\left(4a\,|v|^2\right) f(v) \exp\left(4a\,|v_*|^2\right) \mathcal{M}(v_*)\, \mathrm{d}v\,\mathrm{d}v_*$$

$$= (|\log A| + 2B)\, C_{a,d} \int_{\mathbb{R}^d} \exp\left(4a\,|v|^2\right) \mathcal{M}(v)^{\frac{p-1}{p}} \mathcal{M}(v)^{\frac{1-p}{p}} f(v)\, \mathrm{d}v$$

$$\leq (|\log A| + 2B)\, C_{a,d} \left(\int_{\mathbb{R}^d} \exp\left(\frac{4ap}{p-1}\,|v|^2\right) \mathcal{M}(v)\, \mathrm{d}v\right)^{\frac{p-1}{p}} H_p(f)^{\frac{1}{p}}.$$

The result follows from Lemma 5.4 since $a < \frac{p-1}{8p}$. $\quad\square$

Lastly, before proving Theorem 1.5, we show the following simple lemma:

Lemma 5.6. *Let f be a non-negative function and let $s \geq 0$. Then*

$$m_s(f) \leq m_{\frac{sp}{p-1}}\,(\mathcal{M})^{\frac{p-1}{p}}\, H_p(f)^{\frac{1}{p}}. \tag{5.5}$$

Proof. We have that

$$m_s(f) = \int_{\mathbb{R}^d} |v|^s\, \mathcal{M}(v)^{\frac{p-1}{p}} \mathcal{M}(v)^{\frac{1-p}{p}} f(v)\, \mathrm{d}v \leq \left(\int_{\mathbb{R}^d} |v|^{\frac{sp}{p-1}}\, \mathcal{M}(v)\, \mathrm{d}v\right)^{\frac{p-1}{p}} H_p(f)^{\frac{1}{p}},$$

completing the proof. $\quad\square$

Proof of Theorem 1.5. Since $H_p(f_0) < \infty$ we know, due to Corollary 5.1 that

$$H_p(f(t)) \leq H_p(f_0) < \infty.$$

This implies, by Lemma 5.6 that $f(t, v)$ has bounded moments of any order. Using this together with Theorem 4.5 we conclude that for any $t_0 > 0$ we can find appropriate constants such that

$$f(t, v) \geq A_1 \exp\left(-B_1\,|v|^2\right).$$

This, together with Corollaries 5.3 and 5.5 with the choice of $a = \frac{1-p}{16p}$ shows inequality (1.13). As $\mathscr{D}_\gamma(f(t)) = -\frac{\mathrm{d}}{\mathrm{d}t} H\left(f(t)|\mathcal{M}\right)$ the aforementioned inequality implies the desired convergence for $t \geq t_0$.

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We are only left to show the correct rate of decay for $t < t_0$. Since all the moments exist, we can use Theorem 1.4 (since $H_p(f)$ controls $\|f\|_p$) to find that for $t \leq t_0$.

$$H\left(f(t)|\mathcal{M}\right) \leq C_3 \left(1+t\right)^{-1} \leq C_3 \left(\sup_{t \leq t_0} \frac{\exp\left(\lambda t^{\frac{1}{1+\frac{|\gamma|}{2}}}\right)}{1+t} \right) \exp\left(-\lambda t^{\frac{1}{1+\frac{|\gamma|}{2}}}\right)$$

form some constant C_3. This, together with our rate of decay for $t > t_0$, concludes the proof. $\square$

6. About the non cut-off case

In this final section we aim to discuss a few preliminary results for the linear Boltzmann equation with soft potential *and without angular cut off assumption*. More precisely, we will assume that there exist two positive constants $c_1 \geq c_0 > 0$ such that

$$c_0|\theta|^{-(d-1)-\nu} \leq b(\cos\theta) \leq c_1|\theta|^{-(d-1)-\nu}, \qquad \nu \in (0,2). \tag{6.1}$$

In this case, it is simple to check that

$$\int_{\mathbb{S}^{d-1}} b(\cos\theta)\,\mathrm{d}\sigma = \infty.$$

The divergence of the above integral means that we are not able to split our linear operator into a gain and loss parts.

However, the study of the non-linear Boltzmann equation for soft potentials without cut-off [20], and in particular the spectral analysis its linearised version (see for instance [30]), suggests that the long-time behaviour of the linear Boltzmann equation should, for some range of the parameters γ, ν, be similar to the one of the Boltzmann equation for *hard potentials*. In particular, we will show in the next subsection the existence of a spectral gap as soon as $\gamma + \nu > 0$.

6.1. Existence of a spectral gap

We still assume here that $b(\cdot)$ satisfies (6.1) and we denote by $\mathbb{D}(f)$ the Dirichlet form associated to the linear Boltzmann operator, $\boldsymbol{L}_B$:

$$\mathbb{D}(f) = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} b(\cos\theta)|v - v_*|^\gamma \mathcal{M}(v)\mathcal{M}(v_*)\left(h(v) - h(v')\right)^2 \,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma$$

where $h = \frac{f}{\mathcal{M}}$.

An adaptation of the approach appearing in [30] yields the following:

Proposition 6.1. *For any $\varepsilon > 0$ there is an explicit constant $C = C(B, \varepsilon) > 0$ such that*

$$\mathbb{D}(f) \geq C \left\| f - \varrho_f \mathcal{M} \right\|^2_{L^2(\langle v \rangle^{\gamma+\nu-\varepsilon}\mathcal{M}^{-1})}.$$

In particular, if $\gamma + \nu > 0$ there is $\lambda > 0$ so that

$$\mathbb{D}(f) \geq \lambda \|f - \varrho_f \mathcal{M}\|^2_{L^2(\mathcal{M}^{-1})}, \tag{6.2}$$

i.e. $\boldsymbol{L}_B$ admits a spectral gap of size λ in the space $L^2(\mathcal{M}^{-1})$.

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Proof. As was mentioned earlier, this result is a direct adaptation of [30, Proposition 3.1]. We sketch the proof here for completion.

Using the fact that $\mathbb{D}(f - \varrho_f \mathcal{M}) = \mathbb{D}(f)$, one can assume without loss of generality that $\varrho_f = 0$. Since, from (6.1), there exists $c_0 > 0$ such that

$$b(\cos\theta) \geq c_0 (\sin\theta/2)^{-(d-1)-\nu}$$

it suffices to prove the result for

$$B(v - v_*, \sigma) = |v - v_*|^\gamma (\sin\theta/2)^{-(d-1)-\nu}. \tag{6.3}$$

For a given $v, v_* \in \mathbb{R}^{2d}$, and for $0 < \beta < d - 1 + \nu$ to be chosen later, we define the set

$$\mathbf{C}_\beta = \mathbf{C}_\beta(v, v_*) = \left\{ \sigma \in \mathbb{S}^{d-1} ; \ (\sin\theta/2)^{-(d-1)-\nu} \geq |v - v_*|^\beta \right\}.$$

Since the set $\mathbf{C}_\beta$ is invariant under the transformation $\sigma \to -\sigma$ and $(v, v_*) \to (v', v'_*)$

$$
\begin{aligned}
\mathbb{D}(f) &\geq \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} dv\, dv_* \int_{\mathbf{C}_\beta(v, v_*)} b(\cos\theta) |v - v_*|^\gamma \mathcal{M}(v) \mathcal{M}(v_*) \left(h(v) - h(v') \right)^2 dv\, dv_*\, d\sigma \\
&\geq \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbf{C}_\beta} |v - v_*|^{\gamma + \beta} \mathcal{M}(v) \mathcal{M}(v_*) \left(h(v) - h(v') \right)^2 dv\, dv_*\, d\sigma \\
&= \int_{\mathbb{R}^{2d} \times \mathbf{C}_\beta} |v - v_*|^{\gamma + \beta} \mathcal{M}(v) \mathcal{M}(v_*) h^2(v)\, dv\, dv_*\, d\sigma \\
&\quad - \int_{\mathbb{R}^{2d} \times \mathbf{C}_\beta} |v - v_*|^{\gamma + \beta} \mathcal{M}(v) \mathcal{M}(v_*) h(v) h(v')\, dv\, dv_*\, d\sigma = \mathcal{D}_1 - \mathcal{D}_2.
\end{aligned}
$$

Now,

$$
\begin{aligned}
\mathcal{D}_1 &= \int_{\mathbb{R}^{2d} \times \mathbf{C}_\beta} |v - v_*|^{\gamma + \beta} \mathcal{M}(v) \mathcal{M}(v_*) h^2(v)\, dv\, dv_*\, d\sigma \\
&= \int_{\mathbb{R}^d} f^2(v) \mathcal{M}^{-1}(v)\, dv \int_{\mathbb{R}^d} \mathcal{M}(v_*) |v - v_*|^{\gamma + \beta}\, dv_* \int_{\mathbf{C}_\beta} d\sigma.
\end{aligned}
$$

As (see [30]) there is some universal constant $c = c_d > 0$ such that

$$\int_{\mathbf{C}_\beta} d\sigma \geq c\, |v - v_*|^{-\frac{\beta(d-1)}{\nu + d - 1}}$$

for any given $v, v_* \in \mathbb{R}^d$, we find that

$$\mathcal{D}_1 \geq c \int_{\mathbb{R}^{2d}} f^2(v)\, dv \int_{\mathbb{R}^d} \mathcal{M}(v_*) |v - v_*|^{\gamma + \frac{\beta\nu}{\nu + d - 1}}\, dv_*$$

which, according to Lemma 2.1 yields the existence of some explicit constant $C_{\nu,\beta} > 0$ such that

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$$\mathcal{D}_1 \geq C_{\nu,\beta} \int_{\mathbb{R}^d} f^2(v)\langle v\rangle^{\gamma + \frac{\beta\nu}{\nu+d-1}}\, dv.$$

Next, we notice that

$$|\mathcal{D}_2| \leq \int_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} |v-v_*|^{\gamma+\beta}\mathcal{M}(v)\mathcal{M}(v_*)h(v)h(v')\, dv\, dv_*$$

$$= \int_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} |v-v_*|^{\gamma+\beta}f(v)\mathcal{M}(v_*)\mathcal{M}^{-1}(v')f(v')\, dv\, dv_*$$

$$= \int_{\mathbb{R}^d} f(v)\mathcal{M}^{-1}(v)\, \boldsymbol{K}_{\gamma+\beta}f(v)\, dv$$

where $\boldsymbol{K}_{\gamma+\beta}$ is the gain operator of the linear Boltzmann operator associated to the (cut-off) kernel $B(v-v_*,\cos\theta) = |v-v_*|^{\gamma+\beta}$. Recalling that (see Lemma 4.1)

$$\boldsymbol{K}_{\gamma+\beta}f(v) = \int_{\mathbb{R}^d} k_{\gamma+\beta}(v,w)f(w)\, dw,$$

we find that

$$|\mathcal{D}_2| \leq \int_{\mathbb{R}^{2d}} f(v)\mathcal{M}^{-1}(v)k_{\gamma+\beta}(v,w)f(w)\, dv\, dw.$$

Following [30] again, one can show that

$$|\mathcal{D}_2| \leq C\int_{\mathbb{R}^d} f(v)^2\langle v\rangle^{\gamma+\beta-(d-1)}\mathcal{M}^{-1}(v)\, dv$$

as soon as $d-1 < \beta < d-1+\nu$. Since this condition on β implies that

$$\gamma < \gamma+\beta-(d-1) < \gamma + \frac{\beta\nu}{\nu+d-1}$$

we find that for any $\delta > 0$ there exists $C_\delta > 0$ such that

$$|\mathcal{D}_2| \leq C_\delta \int_{\mathbb{R}^d} f^2(v)\langle v\rangle^\gamma\, \mathcal{M}^{-1}(v)\, dv + \delta \int_{\mathbb{R}^d} f^2(v)\langle v\rangle^{\gamma+\frac{\beta\nu}{\nu+d-1}}\, \mathcal{M}^{-1}(v)\, dv.$$

Therefore, choosing $\delta > 0$ small enough, one gets

$$\mathbb{D}(f) \geq \mathcal{D}_1 - \mathcal{D}_2 \geq C_1 \int_{\mathbb{R}^d} f^2(v)\langle v\rangle^{\gamma+\frac{\beta\nu}{\nu+d-1}}\mathcal{M}^{-1}(v)\, dv - C_2 \int_{\mathbb{R}^d} f^2(v)\langle v\rangle^\gamma\, \mathcal{M}^{-1}(v)\, dv$$

for some $C_1, C_2 > 0$. Since in addition, one can show that (again, see [30])

$$\mathbb{D}(f) \geq C_\gamma \|f\|^2_{L^2(\langle v\rangle^\gamma \mathcal{M}^{-1})},$$

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we conclude that

$$\mathbb{D}(f) \geq C_3 \int_{\mathbb{R}^d} f^2(v) \langle v \rangle^{\gamma + \frac{\beta\nu}{\nu+d-1}} \mathcal{M}^{-1}(v) \, \mathrm{d}v$$

for some explicit constant C_3 depending on γ, β, ν. At this point we will choose

$$\beta = (\nu + d - 1)\left(1 - \frac{\epsilon}{\nu}\right)$$

for $\epsilon > 0$ small enough, and conclude the desired result. $\quad\square$

Appendix A. Basic properties of the linear Boltzmann equation

We collect here some of the technical properties of the linear Boltzmann operator used in the core of the text. We begin with the proof of Lemma 1.2 given in the Introduction:

Proof of Lemma 1.2. The fact that $H_{\boldsymbol{\Phi}}(\cdot|\mathcal{M})$ is a Lyapunov functional of (1.1) for *any* convex function $\boldsymbol{\Phi}$ is a general property of stochastic semigroups. A rigorous proof can be found in [25]. We will only provide a formal proof of this property. Differentiating $H_{\boldsymbol{\Phi}}(f(t)|\mathcal{M})$ under the flow of the equation and denoting by $h = f/\mathcal{M}$, we find that

$$\frac{\mathrm{d}}{\mathrm{d}t} H_{\boldsymbol{\Phi}}(f(t)|\mathcal{M}) = \int_{\mathbb{R}^d} \partial_t f(t, v) \boldsymbol{\Phi}'\left(h(t, v)\right) \, \mathrm{d}v$$

where $\boldsymbol{\Phi}'$ denotes the derivative of $\boldsymbol{\Phi}$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} H_{\boldsymbol{\Phi}}(f(t)|\mathcal{M}) = -\int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} B\left(|v - v_*|, \sigma\right) \mathcal{M}(v)\mathcal{M}(v_*)\left(h(t, v) - h\left(t, v'\right)\right) \boldsymbol{\Phi}'\left(h(t, v)\right) \, \mathrm{d}v \, \mathrm{d}v_* \, \mathrm{d}\sigma$$

where we have used the fact that $\mathcal{M}(v)\mathcal{M}(v_*) = \mathcal{M}(v')\mathcal{M}(v'_*)$. Using the usual pre-post collision change of variables yields

$$\frac{\mathrm{d}}{\mathrm{d}t} H_{\boldsymbol{\Phi}}(f(t)|\mathcal{M}) = -\frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} B\left(|v - v_*|, \sigma\right) \mathcal{M}(v)\mathcal{M}(v_*) \times$$
$$\times \left(h(t, v) - h(t, v')\right)\left(\boldsymbol{\Phi}'\left(h(t, v)\right) - \boldsymbol{\Phi}'\left(h\left(t, v'\right)\right)\right) \, \mathrm{d}v \, \mathrm{d}v_* \, \mathrm{d}\sigma.$$

The latter is nonnegative due to the convexity of $\boldsymbol{\Phi}$. $\quad\square$

A.1. Carleman's representation

We now recall the Carleman's representation (see [12,33]) of the gain operator for general interactions which we used in Section 4:

Lemma A.1. *For any $\alpha \in \mathbb{R}$ the gain operator $\boldsymbol{K}_\alpha = \mathcal{Q}_\alpha^+(\cdot, \mathcal{M})$ can be written as*

$$\boldsymbol{K}_\alpha f(v) = \int_{\mathbb{R}^d} k_\alpha(v, w) f(w) \, \mathrm{d}w, \tag{A.1}$$

where, for any $v, w \in \mathbb{R}^{2d}$,

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$$k_\alpha(v,w) = 2^{d-1} |v-w|^{-1} \int\limits_{(v-w)^\perp} |z-(v-w)|^{\alpha-(d-2)} \, b\left(\frac{|z|^2 - |v-w|^2}{|z|^2 + |v-w|^2} \right) \mathcal{M}(z+v) \, d\pi(z) \qquad (A.2)$$

with $(v-w)^\perp$ denoting the hyperplane orthogonal to $(v-w)$ and $d\pi(z)$ is the Lebesgue measure on that hyperplane. Moreover,

$$k_\alpha(v,w)\mathcal{M}(w) = k_\alpha(w,v)\mathcal{M}(v) \qquad \forall v,w \in \mathbb{R}^d \times \mathbb{R}^d.$$

Proof. We start by recalling Carleman representation (see [18, Appendix C] for the derivation of the present expression): for a given interaction kernel $B(v - v_*, \sigma)$ and given measurable functions f, g

$$\int\limits_{\mathbb{R}^d \times \mathbb{S}^{d-1}} B(v - v_*, \sigma) \, f(v'_*) \, g(v') \, dv_* \, d\sigma = 2^{d-1} \int\limits_{\mathbb{R}^d} \frac{f(w)}{|v-w|} \, dw \int\limits_{E_{v,w}} \frac{B\left(2v - z - w, \frac{z-w}{|z-w|}\right) g(z)}{|2v - z - w|^{d-2}} \, d\pi(z),$$

$$(A.3)$$

where $E_{v,w}$ is the hyperplane that passes through v and is perpendicular to $v - w$. Applying this to $B(v - v_\star, \sigma) = |v - v_*|^\alpha \, b(\cos\theta)$, one notes that, due to symmetry (recall that b is even), it holds

$$\boldsymbol{K}_\alpha f(v) = 2^{d-1} \int\limits_{\mathbb{R}^d} \frac{f(w)}{|v-w|} \, dw \int\limits_{E_{v,w}} \frac{B\left(2v - z - w, -\frac{z-w}{|z-w|}\right) \mathcal{M}(z)}{|2v - z - w|^{d-2}} \, d\pi(z)$$

$$= 2^{d-1} \int\limits_{\mathbb{R}^d} \frac{f(w)}{|v-w|} \, dw \int\limits_{(v-w)^\perp} \frac{|v - z - w|^\alpha \, b\left(\frac{v-z-w}{|v-z-w|} \cdot \frac{w-z-v}{|w-z-v|}\right) \mathcal{M}(z+v)}{|v - z - w|^{d-2}} \, d\pi(z)$$

$$= 2^{d-1} \int\limits_{\mathbb{R}^d} \frac{f(w)}{|v-w|} \, dw \int\limits_{(v-w)^\perp} |v - z - w|^{\alpha-(d-2)} \, b\left(\frac{|z|^2 - |v-w|^2}{|z|^2 + |v-w|^2}\right) \mathcal{M}(z+v) \, d\pi(z),$$

where we used the fact that $z \perp (v-w)$ in the one before last expression.

This proves (A.1) and (A.2). In addition, for any $z \in (v-w)^\perp$, we have $\langle z, v \rangle = \langle z, w \rangle$, which implies that

$$|z+v|^2 + |w|^2 = |z+w|^2 + |v|^2.$$

Thus, on $(v-w)^\perp$ we have that $\mathcal{M}(z+w)\mathcal{M}(v) = M(z+v)\mathcal{M}(w)$. This, together with (A.2), shows that $k_\alpha(v,w)\mathcal{M}(w) = k_\alpha(w,v)\mathcal{M}(v)$. $\square$

Remark A.2. We would like to point out at this point that the above representation of the gain part allows to establish an alternative form of the entropy production associated to a convex mapping $\boldsymbol{\Phi} : \mathbb{R}^+ \mapsto \mathbb{R}^+$. Indeed, for any $\alpha > -d$, let $\mathscr{D}_\alpha^{\boldsymbol{\Phi}}$ be the associated $\boldsymbol{\Phi}$-entropy production of $\boldsymbol{L}_\alpha$:

$$\mathscr{D}_\alpha^{\boldsymbol{\Phi}}(f) = -\int\limits_{\mathbb{R}^d} \boldsymbol{L}_\alpha f(v) \boldsymbol{\Phi}'\left(\frac{f(v)}{\mathcal{M}(v)}\right) dv.$$

Then, one can proves easily that

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$$\mathscr{D}_\alpha^\Phi(f) = \frac{1}{2} \int\limits_{\mathbb{R}^d \times \mathbb{R}^d} k_\alpha(v, w) \mathcal{M}(w) \left(h(v) - h(w) \right) \left(\Phi'(h(v)) - \Phi'(h(w)) \right) \, dv \, dw$$

where $h = f/\mathcal{M}$ and Φ' denotes the derivative of Φ.

As the above expression is actually valid for any $\alpha \in \mathbb{R}$, one can use it to give an alternative proof to the interpolation inequality (2.4) by showing that for $\gamma \in (-d, 0)$ and $\mu > 0$,

$$k_0(v, w) \le k_\gamma(v, w)^{\frac{\mu}{\mu - \gamma}} k_\mu(v, w)^{-\frac{\gamma}{\mu - \gamma}},$$

holds for a.e. $v, w \in \mathbb{R}^d$.

With the representation of k_γ at hand, we can now show Lemma 4.1. The proof is a simple adaptation of a similar study in [12,23].

Proof of Lemma 4.1. We start by writing $V = \frac{v + \omega}{2}$ as $V = V_0 + V_\perp$, where $V_\perp$ is the projection of V on $(v - \omega)^\perp$ and V_0 is parallel to $v - w$. Then $v = V + \frac{v - w}{2}$ and for any $z \in (v - w)^\perp$

$$|v + z|^2 = \left| \left(V_0 + \frac{v - w}{2} \right) + (V_\perp + z) \right|^2 = \left| V_0 + \frac{v - w}{2} \right|^2 + |V_\perp + z|^2$$

$$= |V_0|^2 + V_0 \cdot (v - w) + \frac{|v - w|^2}{4} + |V_\perp + z|^2.$$

As

$$\frac{|v|^2 - |w|^2}{2} = V \cdot (v - w) = V_0 \cdot (v - w) = \pm |V_0| \, |v - w|$$

we can conclude that

$$|V_0|^2 = \frac{\left(|v|^2 - |w|^2 \right)^2}{4 \, |v - w|^2}.$$

Thus,

$$|v + z|^2 = \frac{\left(|v|^2 - |w|^2 \right)^2}{4 \, |v - w|^2} + \frac{|v|^2 - |w|^2}{2} + \frac{|v - w|^2}{4} + |V_\perp + z|^2$$

$$= \frac{1}{4} \left(|v - w| + \frac{|v|^2 - |w|^2}{|v - w|} \right)^2 + |V_\perp + z|^2,$$

which completes the proof. $\square$

Appendix B. Slow convergence to equilibrium

In this Appendix we show that the rate of convergence to equilibrium in (1.1) is naturally prescribed by the tails of the initial datum f_0. Our main result is a simple adaptation of the analogue Theorem from [11] for the non-linear Boltzmann equation:

Theorem B.1. *Let $f_0 \in L^1(\mathbb{R}^d)$ be a non-negative initial datum with unit mass and let $f(t, \cdot)$ denotes the solution to* (1.1). *For any $k \geq 0$, there exist explicit constants $C_1 > 0$ and $C_{2,k} > 0$ such that*

$$\|f(t) - \mathcal{M}\|_{L^1_k} \geq C_1 \int\limits_{|v| > t^{\frac{1}{|\gamma|}}} \langle v \rangle^k f_0(v) \, \mathrm{d}v - C_{2,k} \exp\left(-\frac{t^{\frac{2}{|\gamma|}}}{4}\right) \qquad \forall t \geq 0.$$

Proof. Using Duhamel's formula, one has, for a given $t > 0$,

$$f(t, v) = \exp\left(-\Sigma_\gamma(v) t\right) f_0(v) + \int\limits_0^t \boldsymbol{K}_\gamma f(s, v) \exp\left(-\Sigma_\gamma(v)(t - s)\right) \, \mathrm{d}s \qquad \text{for a.e. } v \in \mathbb{R}^d.$$

In particular, since $f(t, \cdot)$ is nonnegative

$$f(t, v) \geq \exp\left(-\Sigma_\gamma(v) t\right) f_0(v) \qquad \text{for a.e. } v \in \mathbb{R}^d, \qquad t > 0.$$

Using the fact that $\Sigma_\gamma(v) \leq c_\gamma(1 + |v|)^\gamma \leq c_\gamma |v|^\gamma$ for any $v \in \mathbb{R}^d$, one gets

$$f(t, v) \geq \exp\left(-c_\gamma |v|^\gamma t\right) f_0(v) \qquad \text{for a.e. } v \in \mathbb{R}^d, \qquad t > 0$$

and, in particular, setting $\alpha = 1/|\gamma|$, one sees that

$$f(t, v) \geq \exp\left(-c_\gamma\right) f_0(v) \qquad \text{for a.e. } |v| > t^\alpha.$$

Consequently,

$$\|f(t) - \mathcal{M}\|_{L^1_k} \geq \int\limits_{|v| > t^\alpha} |f(t, v) - \mathcal{M}(v)| \langle v \rangle^k \, \mathrm{d}v \geq \int\limits_{|v| > t^\alpha} \langle v \rangle^k f(t, v) \, \mathrm{d}v - \int\limits_{|v| > t^\alpha} \langle v \rangle^k \mathcal{M}(v) \, \mathrm{d}v$$

$$\geq \exp\left(-c_\gamma\right) \int\limits_{|v| > t^\alpha} \langle v \rangle^k f_0(v) \, \mathrm{d}v - (2\pi)^{-\frac{d}{2}} \int\limits_{|v| > t^\alpha} \langle v \rangle^k \exp\left(-\frac{|v|^2}{2}\right) \, \mathrm{d}v.$$

Since

$$\int\limits_{|v| > t^\alpha} \langle v \rangle^k \exp\left(-\frac{|v|^2}{2}\right) \, dv \leq \exp\left(-\frac{t^{2\alpha}}{4}\right) \int\limits_{\mathbb{R}^d} \langle v \rangle^k \exp\left(-\frac{|v|^2}{4}\right) \, dv$$

the proof is complete. $\quad\square$

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