TOWARDS THE MODELING OF NEURONAL FIRING BY GAUSSIAN PROCESSES

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Abstract. This paper focuses on the outline of some computational methods for the approximate solution of the integral equations for the neuronal firing probability density and an algorithm for the generation of sample-paths in order to construct histograms estimating the firing densities. Our results originate from the study of non-Markov stationary Gaussian neuronal models with the aim to determine the neuron's firing probability density function. A parallel algorithm has been implemented in order to simulate large numbers of sample paths of Gaussian processes characterized by damped oscillatory covariances in the presence of time dependent boundaries. The analysis based on the simulation procedure provides an alternative research tool when closed-form results or analytic evaluation of the neuronal firing densities are not available.

1 Introduction

This contribution deals with the implementation of procedures and methods, worked out in our group during the last few years, in order to provide algorithmic solutions to the problem of determining the first passage time (FPT) probability density function (pdf) and its relevant statistics for continuous state-space and continuous parameter Gaussian processes describing the stochastic modeling of a single neuron's activity.

In most modeling approaches, it is customary to assume that a neuron is subject to input pulses occurring randomly in time, (see, for instance, [13] and references therein). As a consequence of the received stimulations, it reacts by producing a response that consists of a spike train. The reproduction of the statistical features of such spike trains has been the goal of many researches who have focused the attention on the analysis of the interspike intervals. Indeed, the importance of the interspike intervals is due to the generally accepted hypothesis that the information transferred within the nervous system is usually encoded by the timing of occurrence of neuronal spikes.

To describe the dynamics of the neuronal firing we consider a stochastic process $X(t)$ representing the change in the neuron membrane potential between two consecutive spikes (cf., for instance, [9]). In this context, the threshold voltage is viewed as a deterministic function $S(t)$ and the instant when the membrane potential crosses $S(t)$ as a FPT random variable.

The modeling of a single neuron's activity by means of a stochastic process has been the object of numerous investigations during the last four decades. A milestone contribution in this direction is the much celebrated paper by Gerstein and Mandelbrot [7] in which a random walk and its continuous diffusion limit (the Wiener process) was proposed with the aim of describing a possible, highly schematized, spike generation mechanism. However, despite the excellent fitting of a variety of data, this model has been the target of severe criticism on the base of its extreme idealization in contrast with some electrophysiological

evidence: for example, this model does not take into account the spontaneous exponential decay of the neuron membrane potential. An improved model is the so called Ornstein-Uhlenbeck (OU) model, that embodies the presence of such exponential decay. However, the OU model does not allow to obtain any closed form expression for the firing pdf, except for some very particular cases of no interest within the neuronal modeling context. Rather cumbersome computations are thus required to obtain evaluations of the statistics of the firing time. Successively, alternative neuronal models have been proposed, that include more physiologically features. The literature on this subject is too vast to be recalled here. We limit ourselves to mentioning that a review of most significant neuronal models can be found in [10], [13] and in the references therein. In particular, in [10] it is presented an outline of appropriate mathematical techniques by which to approach the FPT problem in the neuronal context.

We shall now formally define the firing pdf for a model based on a stochastic process $X(t)$ with continuous sample paths. First, assume $P[X(t_0) = x_0] = 1$, with $x_0 < S(t_0)$, i.e. we view the sample paths of $X(t)$ as originating at a preassigned state x_0 at initial time t_0. Then,

$$T_{x_0} = \inf_{t \geq t_0} \big\{ t : X(t) > S(t) \big\}, \qquad x_0 < S(t_0)$$

is the FPT of $X(t)$ through $S(t)$, and

$$g[S(t), t | x_0, t_0] = \frac{\partial}{\partial t} P(T_{x_0} < t)$$

is its pdf.

Henceforth, the FPT pdf $g[S(t), t | x_0, t_0]$ will be identified with the firing pdf of a neuron whose membrane potential is modeled by $X(t)$ and whose firing threshold is $S(t)$.

Throughout this paper, we shall focus our attention on neuronal models rooted on diffusion and Gaussian processes, partially motivated by the generally accepted hypothesis that in numerous instances the neuronal firing is caused by the superposition of a very large number of synaptic input pulses which is suggestive of the generation of Gaussian distributions by virtue of some sort of central limit theorems.

It must be explicitly pointed out that models based on diffusion processes are characterized by the "lack of memory" as a consequence of the underlying Markov property. However, in the realistic presence of correlated input stimulations, the Markov assumption breaks down and one faces the problem of considering more general stochastic models, for which the literature on FPT problem appears scarce and fragmentary. Simulation procedures then provide possible alternative investigation tools especially if they can be implemented on parallel computers, (see [3]). The goal of a typical simulation procedure is to sample N values of the FPT by a suitable construction of N time-discrete sample paths of the process and then to record the instants when such sample paths first cross the boundary. In such a way, one is led to obtain estimates of the firing pdf and of its statistics, that may be implemented for data fitting purposes.

The aim of this paper is to outline numerical and theoretical methods to characterize the FPT pdf for Gaussian processes. Attention will be focused on Markov models in Section 2, and on non-Markov models in Section 3. Finally, Section 4 will be devoted to the description of some computational results.

2   Gauss-Markov processes

We start briefly reviewing some essential properties of Gauss-Markov processes. Let $\{X(t), t \in I\}$, where I is a continuous parameter set, be a real continuous Gauss-Markov process with the following properties (cf. [8]):

(i) $m(t) := E[X(t)]$ is continuous in I;

(ii) the covariance $c(s,t) := E\left\{[X(s) - m(s)][X(t) - m(t)]\right\}$ is continuous in $I \times I$;

(iii) $X(t)$ is non-singular, except possibly at the end points of I where it could be equal to $m(t)$ with probability one.

A Gaussian process is Markov if and only if its covariance satisfies

(1)
$$c(s, u) = \frac{c(s,t)\, c(t,u)}{c(t,t)} \quad \forall s, t, u \in I, s \le t \le u.$$

It is well known [8], that well-behaved solutions of (1) are of the form

(2)
$$c(s,t) = h_1(s)\, h_2(t), \qquad s \le t,$$

where

(3)
$$r(t) := \frac{h_1(t)}{h_2(t)}$$

is a monotonically increasing function by virtue of the Cauchy-Schwarz inequality, and $h_1(t)\, h_2(t) > 0$ because of the assumed nonsingularity of the process on I. The conditional pdf $f(x, t | x_0, t_0)$ of $X(t)$ is a normal density characterized respectively by conditional mean and variance

$$M(t|t_0) = m(t) + \frac{h_2(t)}{h_2(t_0)}\, [x_0 - m(t_0)]$$

$$V(t|t_0) = h_2(t)\left[h_1(t) - \frac{h_2(t)}{h_2(t_0)}\, h_1(t_0) \right],$$

with $t, t_0 \in I$, $t_0 < t$. It satisfies the Fokker-Planck equation and the associated initial condition

$$\frac{\partial f(x, t | y, \tau)}{\partial t} = - \frac{\partial}{\partial x}\, [A_1(x, t)\, f(x, t | y, \tau)] + \frac{1}{2}\, \frac{\partial^2}{\partial x^2}\, [A_2(t)\, f(x, t | y, \tau)],$$

$$\lim_{\tau \uparrow t} f(x, t | y, \tau) = \delta(x - y),$$

with $A_1(x, t)$ and $A_2(t)$ given by

$$A_1(x, t) = m'(t) + [x - m(t)]\, \frac{h_2'(t)}{h_2(t)}, \qquad A_2(t) = h_2^2(t)\, r'(t),$$

the prime denoting derivative with respect to the argument.

The class of the Gauss-Markov processes $\{X(t), t \in [0, +\infty)\}$, such that $f(x, t | y, \tau) \equiv f(x, t - \tau | y)$, is characterized by means and covariances of the following two forms:

$$m(t) = \beta_1 t + c, \quad c(s, t) = \sigma^2 s + c_1$$
$$(0 \le s \le t < +\infty,\ \beta_1, c \in \mathbf{R},\ c_1 \ge 0,\ \sigma \ne 0)$$

or

$$m(t) = -\frac{\beta_1}{\beta_2} + c\, e^{\beta_2 t}, \quad c(s, t) = c_1\, e^{\beta_2 t}\left[c_2\, e^{\beta_2 s} - \frac{\sigma^2}{2 c_1 \beta_2}\, e^{-\beta_2 s} \right]$$
$$\left(0 \le s \le t < +\infty,\ \beta_1, c, c_2 \in \mathbf{R},\ \sigma \ne 0,\ c_1 \ne 0,\ \beta_2 \ne 0,\ c_1 c_2 - \frac{\sigma^2}{2\beta_2} \ge 0 \right).$$

The first type includes the Wiener process, used in [7] to describe the neuronal firing, while the second type includes the Ornstein–Uhlenbeck process that has often been invoked as a model for neuronal activity (see, for instance, [13]).

Any Gaussian process with covariance as in (2) can be represented in terms of the standard Wiener process $\{W(t), t \geq 0\}$ as

$$(4) \qquad\qquad X(t) = m(t) + h_2(t)\, W\big[r(t)\big],$$

and is therefore Markov. This last equation suggests the way to construct the FPT pdf of a Gauss-Markov process $X(t)$ in terms of the FPT pdf of the standard Wiener process $W(t)$. As an example, from (4) for the conditioned FPT pdf one has:

$$(5) \qquad\qquad g[S(t), t|x_0, t_0] = \frac{dr(t)}{dt}\, g_W\big\{S^*[r(t)], r(t)|x_0^*, r(t_0)\big\},$$

where $r(t)$ is defined in (3) and $g_W[S^*(\vartheta), \vartheta|x_0^*, \vartheta_0]$ is the FPT pdf of $W(\vartheta)$ from x_0^* at time ϑ_0 to the continuous boundary $S^*(\vartheta)$, with

$$x_0^* = \frac{x_0 - m[r^{-1}(\vartheta_0)]}{h_2[r^{-1}(\vartheta_0)]}, \qquad S^*(\vartheta) = \frac{S[r^{-1}(\vartheta)] - m[r^{-1}(\vartheta)]}{h_2[r^{-1}(\vartheta)]}.$$

Results on the FPT pdf for the standard Wiener process can thus in principle be used via (5) to obtain the FPT pdf of any continuous Gauss-Markov process. For instance, if $S^*(\vartheta)$ is linear in ϑ, $g_W[S^*(\vartheta), \vartheta|x_0^*, \vartheta_0]$ is known and $g[S(t), t|x_0, t_0]$ can be obtained via (5). Instead, if $g_W[S^*(\vartheta), \vartheta|x_0^*, \vartheta_0]$ is not known, a numerical algorithm, or a simulation procedure, should be used for the standard Wiener process and, after that, $g[S(t), t|x_0, t_0]$ can be obtained via the indicated transformation.

The above procedure often exhibits the serious drawback of ensuing unacceptable time dilations (see [4]). For instance, exponentially large times are involved when transforming the Ornstein-Uhlenbeck process to the Wiener process, which makes such a method hardly viable. Hence, it is desirable to dispose of a direct and efficient computational method to obtain evaluation of the firing pdf. Along such a direction, in [4] it has been proved that the conditioned FPT density of a Gauss-Markov process can be obtained by solving the non-singular Volterra second kind integral equation

$$g[S(t), t|x_0, t_0] = -2\Psi[S(t), t|x_0, t_0] + 2\int_{t_0}^t g[S(\tau), \tau|x_0, t_0]\, \Psi[S(t), t|S(\tau), \tau]\, d\tau$$

$$(6) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x_0 < S(t_0))$$

with $S(t), m(t), h_1(t), h_2(t) \in C^1(I)$ and

$$\Psi[S(t), t\,|\,y, \tau] = \left\{ \frac{S'(t) - m'(t)}{2} - \frac{S(t) - m(t)}{2}\frac{h_1'(t)h_2(\tau) - h_2'(t)h_1(\tau)}{h_1(t)h_2(\tau) - h_2(t)h_1(\tau)} \right.$$

$$(7) \qquad\qquad \left. - \frac{y - m(\tau)}{2}\frac{h_2'(t)h_1(t) - h_2(t)h_1'(t)}{h_1(t)h_2(\tau) - h_2(t)h_1(\tau)} \right\} f[S(t), t\,|\,y, \tau]$$

where $f[x, t|y, \tau]$ is the transition pdf of $X(t)$. Closed form solutions of (6) are available in [4] for different families of boundaries.

By making use of this result, in [4] an efficient numerical procedure based on a repeated Simpson's rule has been proposed to evaluate FPT densities of Gauss-Markov processes, that can be implemented to obtain reliable evaluations of firing densities for neuronal models based on Gauss-Markov processes.

3   Gauss non-Markov processes

The methods proposed in the previous Section rest on the strong Markov assumption on the stochastic process modeling the neuron's membrane potential, which grants the possibility of making use of the mentioned analytic and computational methods for FPT pdf evaluations. This is not the case when the stochastic process used to model the neuron's firing mechanism is non-Markov. Here we shall focus our attention on Gauss non-Markov processes. However, thus doing we face the lack of effective analytical methods for obtaining manageable closed-form expressions for the FPT pdf, although some preliminary analytical results have been obtained by Ricciardi and Sato in [11] for a class of stationary Gaussian processes.

Indeed, if $X(t)$ is one-dimensional non-singular stationary Gaussian process mean square differentiable, a series expansion for the FPT pdf was derived (see, [12]). In the following, for convenience, we shall take $t_0 = 0$ as initial time and, without loss of generality, assume that $E[X(t)] = 0$ and $P[X(t_0) = x_0] = 1$, with x_0 an arbitrarily specified initial state. Furthermore, the covariance function $E[X(t)X(\tau)] := \gamma(t - \tau)$ will be assumed to be such that $\gamma(0) = 1, \dot{\gamma}(0) = 0$ and $\ddot{\gamma}(0) < 0$ (this last assumptions being equivalent to the mean square differentiable property). The FPT pdf of $X(t)$ through $S(t)$ is then given by the following expression

$$(8) \qquad g[S(t), t|x_0] = W_1(t|x_0) + \sum_{i=1}^{\infty} (-1)^i \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{i-1}}^t dt_i W_{i+1}(t_1, \ldots, t_i, t|x_0),$$

with

$$W_n(t_1, \ldots, t_n|x_0)$$
$$= \int_{\dot{S}(t_1)}^{\infty} dz_1 \cdots \int_{\dot{S}(t_n)}^{\infty} dz_n \prod_{i=1}^{n} [z_i - \dot{S}(t_i)] \, p_{2n}[S(t_1), \ldots, S(t_n); z_1, \ldots, z_n|x_0],$$

where $p_{2n}(x_1, \ldots, x_n; z_1, \ldots, z_n|x_0)$ is the joint pdf of $\{X(t_1), \ldots, X(t_n), Z(t_1) = \dot{X}(t_1),$ $\ldots, Z(t_n) = \dot{X}(t_n)\}$ conditional upon $X(0) = x_0$. Due to the great complexity of the involved multiple integrals, expression (8) does not appear to be manageable for practical uses, even though it has recently been shown that it allows to obtain some interesting asymptotic results [5]. Since (8) is a Leibnitz series for each fixed $t > 0$, estimates of the FPT pdf can in principle be obtained since its partial sum of order n provides a lower or an upper bound to g depending on whether n is even or odd. However, also the evaluation of such partial sums is extremely cumbersome.

In conclusion, at the present time for this class of Gaussian processes, no effective analytical methods, nor viable numerical algorithms are available to evaluate the FPT pdf. A simulation procedure seems to be the only residual way of approach.

To this aim, we have restored and updated an algorithm due to Franklin [6] in order to construct sample paths of a stationary Gaussian process with spectral density of a rational type and deterministic starting point. The idea is the following. Let us consider the linear filter

$$(9) \qquad X(t) = \int_0^{\infty} h(s) \, W(t - s) \, ds$$

where $h(t)$ is the impulse response function and $W(t)$ is the input signal. By Fourier transformation, (9) yields

$$(10) \qquad \Gamma_X(\omega) = |H(\omega)|^2 \, \Gamma_W(\omega)$$
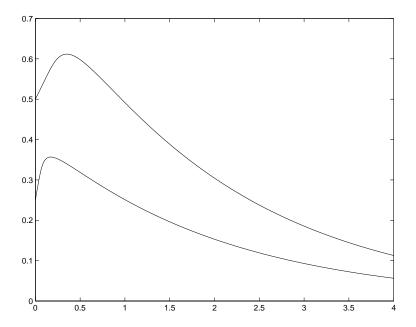
Figure 1: Plot of the boundary $S(t)$ given in (13) for $\beta = 0.5$ and $d = 0.25, 0.50$ (bottom to top).

where $\Gamma_W(\omega)$ and $\Gamma_X(\omega)$ are the spectral densities of input $W(t)$ and output $X(t)$, respectively, and where $H(\omega)$ denotes the Fourier transform of $h(t)$. Equation (10) is suggestive of a method to construct a Gaussian process $X(t)$ having a preassigned spectral density $\Gamma_X(\omega) \equiv \Gamma(\omega)$. It is indeed sufficient to consider a white noise $\Lambda(t)$, having spectral density $\Gamma_W(\omega) \equiv 1$, as the input signal and then select $h(t)$ in such a way that $|H(\omega)|^2 = \Gamma(\omega)$. If the spectral density of $X(t)$ is of rational type, namely if

$$(11) \qquad \Gamma(\omega) := \int_{-\infty}^{\infty} \gamma(t)\, e^{-i\,\omega\, t}\, dt = \left| \frac{P(i\,\omega)}{Q(i\,\omega)} \right|^2$$

where P and Q are polynomials with $deg(P) < deg(Q)$, setting $H(\omega) = P(i\,\omega)/Q(i\,\omega)$, from (10) it follows

$$X(t) = \frac{P(D)}{Q(D)}\, \Lambda(t)$$

where $D = d/dt$. To calculate the output signal $X(t)$ it is thus necessary to solve first the differential equation $Q(D)\,\phi(t) = \Lambda(t)$ to obtain the stationary solution $\phi(t)$, and then evaluate $X(t) = P(D)\,\phi(t)$. The simulation procedure is designed to construct sample paths of the process $X(t)$ at the instants $t = 0, \Delta t, 2\Delta t, \dots$ where Δt is a positive constant time increment. The underlying idea can be applied to any Gaussian process having spectral densities of a rational type and, since the sample paths of the simulated process are generated independently of one another, this simulation procedure is particularly suited for implementation on supercomputers.

Extensive computations have been performed on parallel computers to explore the different shapes of the FPT pdf as induced by the oscillatory behaviors of covariances and thresholds (cf., for instance, [1] and [2]).

4   Numerical comparisons

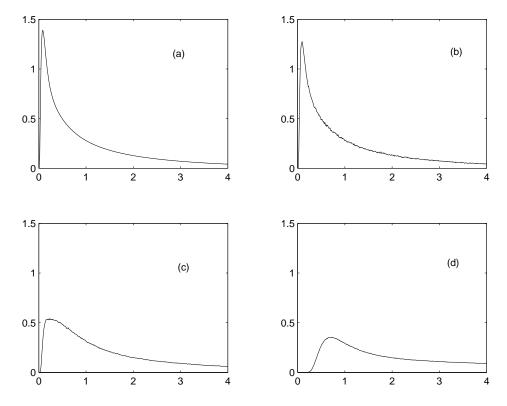The aim of this Section is to compare the behavior of the FPT pdf's among Gauss-Markov

Figure 2: Plots refer to FPT pdf $g(t)$ through the boundary (13) with $\beta = 0.5$ and $d = 0.25$ for a zero-mean Gaussian process characterized by correlation function (12). In Figure 2(a) $g(t)$ given by (14) has been plotted. The estimated FPT pdf $\tilde{g}(t)$ with $\alpha = 10^{-10}$ is shown in Figure 2(b), with $\alpha = 0.25$ in Figure 2(c) and with $\alpha = 0.5$ in Figure 2(d).

processes and Gaussian non-Markov processes, in order to analyze how the lack of memory affects the shape of the density, also with reference to the specified type of correlation function. For simplicity, set $x_0 = 0$ and $P[X(0) = 0] = 1$, so that in the following we shall consider the FPT pdf $g(t) := g[S(t), t|0, 0]$.

To be specific, we consider a stationary Gaussian process $X(t)$ with zero mean and correlation function

$$(12) \qquad \gamma(t) := e^{-\beta |t|} \cos(\alpha t), \qquad \beta \in I\!\!R^+$$

which is the simplest type of correlation having a concrete engineering significance [14]. When the correlation function is of type (12), $X(t)$ is not mean square differentiable, since $\dot{\gamma}(0) \neq 0$. Thus the series expansion (8) does not hold. However, specific assumptions on the parameter α help us characterize the shape of the FPT pdf.

We start assuming $\alpha = 0$, so that the correlation function (12) can be factorized as

$$\gamma(t) = e^{-\beta \tau} e^{-\beta (t-\tau)} \qquad \beta \in \mathbf{R}^+, \, 0 < \tau < t.$$

Hence, choosing $h_1(t) = e^{\beta t}$ and $h_2(t) = e^{-\beta t}$ in (2), $X(t)$ becomes Gauss-Markov. Therefore, for any boundary $S(t)$, the FPT pdf $g(t)$ can be numerically evaluated by solving the
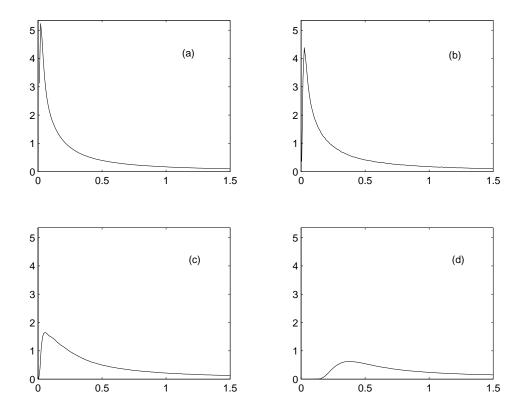
Figure 3: Same as in Figura 2 with $d = 0.5$.

integral equation (6). In the following, we consider boundaries of the form

$$(13) \qquad S(t) = d\, e^{-\beta t} \left\{ 1 - \frac{e^{2\beta t} - 1}{2\, d^2} \ln\left[\frac{1}{4} + \frac{1}{4}\sqrt{1 + 8\exp\left(-\frac{4\, d^2}{e^{2\beta t} - 1}\right)} \ \right] \right\},$$

with $d > 0$. It is evident that $\lim_{t \to 0} S(t) = d$ and that $S(t)$ tends to 0 as t increases. Furthermore, as d decreases, the boundary becomes flatter. In Figure 1, the boundary $S(t)$ given in (13) is plotted for $\beta = 0.5$ and for two choices of the parameter d, i.e. $d = 0.25$ and $d = 0.5$.

As proved in [4] for boundaries of the form (13), the FPT pdf $g(t)$ of a Gauss-Markov process admits the following closed form:

$$(14) \qquad g(t) = \frac{4\, d\, \beta\, e^{\beta t}}{e^{2\beta t} - 1} \ \frac{\sqrt{1 + 8\,\exp\left(-\dfrac{4 d^2}{e^{2\beta t} - 1}\right)}}{1 + \sqrt{1 + 8\,\exp\left(-\dfrac{4 d^2}{e^{2\beta t} - 1}\right)}} \ f[S(t), t | 0, 0],$$

where $f[S(t), t | 0, 0]$ is the transition pdf of the Gauss-Markov process $X(t)$.

For a zero-mean Gauss-Markov process characterized by the correlation function (12) with $\beta = 0.5$ and $\alpha = 0$, the FPT pdf $g(t)$ given by (14) through the boundary (13) is plotted in Figure 2(a) for $d = 0.25$ and in Figure 3(a) for $d = 0.5$. Note that as d increases the mode increase, whereas the corresponding ordinate decrease.

Setting $\alpha \neq 0$ in (12), the Gaussian process $X(t)$ is no longer Markov and its spectral density is given by

$$(15) \qquad \Gamma(\omega) = \frac{2\,\beta\,(\omega^2 + \alpha^2 + \beta^2)}{\omega^4 + 2\,\omega^2\,(\beta^2 - \alpha^2) + (\beta^2 + \alpha^2)^2}\,,$$

thus being of a rational type. Since in (15) the degree of the numerator is less than the degree of the denominator, it is possible to apply the simulation algorithm described in Section 3 in order to estimate the FPT pdf $\tilde{g}(t)$ of the process.

The simulation procedure has been implemented by a parallel FORTRAN 90 code on a 128-processor IBM SP4 supercomputer, based on MPI language for parallel processing. The number of simulated sample paths has been set equal to 10^7. The estimated FPT pdf $\tilde{g}(t)$ through the boundary (13) with $\beta = 0.5$ and $d = 0.25$ are plotted in Figures 2(b)÷2(d) for Gaussian processes with correlation function (12) having $\alpha = 10^{-10}, 0.25, 0.5$, respectively. For the same processes, Figures 3(b)÷3(d) show the estimated FPT pdf $\tilde{g}(t)$ through the boundary (13) with $\beta = 0.5$ and $d = 0.5$. Note that as α increases, the shape of the FPT pdf $\tilde{g}(t)$ becomes flatter and the related mode increases. Furthermore, as Figures 2(a)-2(b) and Figures 3(a)-3(b) show, $\tilde{g}(t)$ is very similar to $g(t)$ for small values of α.

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