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**If the primes are finite, then all of them divide the number one**

We propose a novel proof of the infinitude of the primes based on elementary considerations of Legendre’s function  $\phi$ , defined in [1, p. 153] as

$$\phi(x, y) = |\{1 \leq n \leq x : \text{integer } n \text{ has no prime factors } \leq y\}|,$$

where  $x$  and  $y$  are positive integers. The reader can see that

$$\pi(x) = \pi(\sqrt{x}) + \phi(x, \sqrt{x}) - 1,$$

where  $\pi(\cdot)$  is the prime-counting function. Let  $p_1, \dots, p_s$  be the prime numbers less than or equal to  $y$ . Using the inclusion-exclusion principle, it can be proved that

$$\phi(x, y) = x - \sum_{1 \leq i \leq s} \left\lfloor \frac{x}{p_i} \right\rfloor + \sum_{1 \leq i, j \leq s} \left\lfloor \frac{x}{p_i p_j} \right\rfloor + \dots + (-1)^s \left\lfloor \frac{x}{p_1 \cdots p_s} \right\rfloor,$$

where  $\lfloor \cdot \rfloor$  is the floor function. Pinasco [2] also used this principle for proving the infinitude of the primes, but his proof is remarkably different from ours.

Suppose that  $\{p_1, \dots, p_s\}$  is the set of all prime numbers. Consider  $N = p_1 \cdots p_s$ . Then  $\phi(N^2, N) = 1$ . On the other hand, we have

$$\phi(N^2, N) = N^2 - \sum_{1 \leq i \leq s} \left\lfloor \frac{N^2}{p_i} \right\rfloor + \sum_{1 \leq i, j \leq s} \left\lfloor \frac{N^2}{p_i p_j} \right\rfloor + \dots + (-1)^s N.$$

Hence,  $\phi(N^2, N) = mN$  for some integer  $m$ , i.e., every prime number divides 1. This means that all primes, and, consequently, all non-zero natural numbers, are invertible in  $\mathbb{N}$ , i.e., we find that  $\mathbb{N}$  is a field. This completes the proof.

REFERENCES

1. R. Crandall, C. Pomerance, *Prime numbers. A computational perspective*, Springer–Verlag, New York, 2nd Edition, 2005.
2. J. P. Pinasco, New proofs of Euclid’s and Euler’s theorems, *Amer. Math. Monthly* **116** (2009) 172–174.

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