Symmetry and rigidity for the hinged composite plate problem

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SYMMETRY AND RIGIDITY FOR THE HINGED COMPOSITE PLATE PROBLEM

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ABSTRACT. The composite plate problem is an eigenvalue optimization problem related to the fourth order operator $(-\Delta)^2$. In this paper we continue the study started in [10], focusing on symmetry and rigidity issues in the case of the *hinged composite plate problem*, a specific situation that allows us to exploit classical techniques like the moving plane method.

1. INTRODUCTION

The composite plate problem is an eigenvalue optimization problem that extends to the fourth order case the composite membrane problem extensively studied e.g., in [5–9, 11, 31], see also [20, 33] and references therein for related problems. One of the interesting situations is provided by the so called hinged composite plate, which can be described as follows: build a hinged plate, of prescribed mass and shape, out of different materials sharing the same elastic properties but having different densities, in such a way that its principal frequency is as low as possible.

We introduce below the above problem from a mathematical point of view. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $0 < h < H$ be two positive constants, and $M \in [h|\Omega|, H|\Omega|]$. We define the set of admissible densities as

$$P := \left\{ \rho : \Omega \to \mathbb{R} : \int_\Omega \rho(x)\, dx = M,\ h \le \rho \le H \text{ in } \Omega \right\}.$$

By *hinged composite plate problem* we mean the following minimization problem

$$(1.1) \qquad \Theta(h, H, M) := \inf_{\rho \in P}\ \inf_{u \in H^2(\Omega) \cap H_0^1(\Omega) \setminus \{0\}} \frac{\displaystyle\int_\Omega (\Delta u)^2}{\displaystyle\int_\Omega \rho\, u^2}.$$

A couple (u, ρ) which realizes the double infimum in (1.1) is called *optimal pair*.
To the best of our knowledge, the first results concerning problem (1.1) have been obtained in [10]. The Euler-Lagrange equation associated to (1.1) gives rise to the following fourth

order Navier problem

$$(1.2) \qquad \begin{cases} \Delta^2 u = \Theta \rho u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

A first crucial aspect when dealing with fourth order PDE's like (1.2) is the positivity of the solutions. This issue is closely related to the validity, or non validity, of a maximum principle, and it is one of the most studied topics in the literature of higher order PDE's. We refer to the monograph [15] for a wide and comprehensive treatment of the subject, cf. also [27, 28] and references therein for more results on polyharmonic problems, and [4, 32] and references therein for maximum principles for cooperative elliptic systems.

In our case the maximum principle holds essentially thanks to the fact that we work with Navier boundary conditions, and the proof of the positivity is slightly simplified because the solutions of (1.2) we are considering are *minimizers* of the variational problem (1.1), cf. also [10, Proposition 5.1]. This enables us to assume –without loss of generality– that if (u, ρ) is an optimal pair, u is of fixed sign, say positive, under suitable assumptions on the domain Ω, cf. Proposition 2.2. Furthermore, in [10, Theorems 1.3 and 1.4], we give an explicit representation of the optimal configuration ρ, namely if (u, ρ) is an optimal pair,

$$(1.3) \qquad \rho(x) = \rho_u(x) := h\,\chi_{\{u \le t\}}(x) + H\,\chi_{\{u > t\}}(x) \quad \text{for all } x \in \Omega$$

for a suitable $t = t(h, H, \Omega, u) > 0$, and we prove that, if $\partial\Omega$ is smooth enough, every optimal pair (u, ρ) satisfies

$$(1.4) \qquad u \in C^{3,\alpha}(\overline{\Omega}) \cap W^{4,q}(\Omega) \quad \text{for all } \alpha \in (0, 1) \text{ and } q \ge 1.$$

In particular, the combination of the regularity of u up to the boundary and the boundary condition $u = 0$ on $\partial\Omega$, yields that the set $\{u \le t\}$ contains a tubular neighborhood of $\partial\Omega$. We point out that the existence of optimal pairs as proved in [10] does not rely on the regularity of the boundary $\partial\Omega$. The only part where the regularity of the boundary is truly exploited is to get the *sharp* regularity up to the boundary. We refer to Section 5 for further comments on regularity in the case of convex domains. The eigenvalue optimization problem (1.1) presents many interesting features, among which the study of preservation of symmetry is quite relevant, especially in view of the results proved in [6] for the *second order problem*. In particular, symmetry preserving properties are necessary conditions for uniqueness of optimal configurations. Indeed, if an optimal pair (u, ρ) does not preserve the symmetry of the domain, it is always possible to realize the same double infimum Θ by taking the symmetric of (u, ρ) as a different optimal pair. Clearly, uniqueness of optimal pairs in this context has to be considered up to a multiplicative constant for u, since by zero degree homogeneity of (1.1) (w.r.t. u), if (u, ρ) is an optimal pair, for every constant $c \ne 0$, also (cu, ρ) is an optimal pair of the same problem. In this perspective, in [10] we proved that if $\Omega = B$ is a ball, there exists a unique optimal pair (u, ρ), and u is radial, positive and radially decreasing. As a consequence, the sublevel set $\{u \le t\}$ appearing in (1.3) is a ring-shaped region.

In this paper, continuing the study started in [10], we prove a rigidity result and a symmetry preservation property of optimal pairs for more general sets Ω.

Throughout the paper, given $x \in \partial\Omega$, $\nu(x)$ denotes the external unit normal to Ω at x.

We state first the rigidity result.

Theorem 1.1. *Let $\partial\Omega$ be C^4-smooth and let (u, ρ) be an optimal pair for (1.1). If u satisfies the additional condition*

$$(1.5) \qquad \frac{\partial u}{\partial \nu} = c \quad \text{on } \partial\Omega \quad \text{for some } c < 0,$$

then Ω is a ball and u is radially symmetric and radially decreasing.

Theorem 1.1 deals with an overdetermined problem in the spirit of Serrin [30] and Weinberger [35]. The techniques used by Serrin and Weinberger in their papers are very different: the first one is based on the moving plane method and maximum principle, while the second relies on integral identities and the construction of a suitable P-function.

Starting from the seminal papers [30, 35], many authors started studying possible generalizations of those results for different operators. In particular, there has been a certain attention devoted to fourth order overdetermined problems. Let us briefly survey part of the results already available in the literature. In [34], Troy proved a rigidity result for second order elliptic systems imposing as many extra conditions as the number of the equations, namely

$$
(1.6) \qquad \begin{cases} -\Delta u_i = f_i(u_1, \ldots, u_m) & \text{in } \Omega, \\ \quad\ \ u_i > 0 & \text{in } \Omega, \qquad i = 1, \ldots, m. \\ \quad\ \ u_i = 0 & \text{on } \partial\Omega, \end{cases}
$$

We observe that under suitable conditions on the f_i's, for $m = 2$, (1.6) can be formulated as a fourth order boundary value problem similar to the one we are interested in. In [1], Bennett considered a pure Dirichlet overdetermined problem, namely

$$
\begin{cases} \quad\ \Delta^2 u = -1 & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ \quad\ \ \Delta u = c & \text{on } \partial\Omega, \end{cases}
$$

and showed that the existence of a solution $u \in C^4(\overline{\Omega})$ implies that Ω is a ball, constructing an appropriate P-function in the spirit of Weinberger. In [24], Payne and Schaefer considered the Navier version of the problem considered by Bennett in [1], in the case of a planar domain Ω which is star-shaped with respect to the origin. The extension to any dimension for C^2 bounded domains was then provided in [25] by Philippin and Ragoub, and subsequently by Goyal and Schaefer in [18] with a different proof. In the present article, as in the aforementioned papers [24, 25, 34], the main tool for proving Theorem 1.1 is the moving plane method. We stress here that one of the keys that allows the use of such a classical technique is the monotonicity result provided by Lemma 2.3, which is heavily based on (1.3).

As already mentioned, the second main result concerns the preservation of symmetry; we extend to more general domains the results proved for the ball in [10]. In the next statement, we use the notation for the moving plane method recalled in Section 2.

Theorem 1.2. *Let Ω be symmetric and convex with respect to the hyperplane $\{x_1 = 0\}$, and with C^4-smooth boundary $\partial\Omega$. Let λ_1 be as in (2.3) and suppose that $\lambda_1 = 0$. If (u, ρ) is an optimal pair, then u is symmetric with respect to $\{x_1 = 0\}$ and strictly decreasing in x_1 for $x_1 > 0$. Consequently, ρ is symmetric with respect to the same hyperplane as well. Furthermore, the set $\{u > t\}$ is convex with respect to $\{x_1 = 0\}$.*

This theorem guarantees that optimal pairs preserve reflection symmetries of the domain, in presence of some convexity assumption. Since smooth strictly convex domains, and in particular balls, are covered by the assumptions of Theorem 1.2, we note that [10, Theorem 1.5] can be seen as a corollary of it.

Let us spend a few words concerning the hypotheses and the proof of Theorem 1.2. C^4-smoothness (actually even just convexity) of Ω is a sufficient condition to ensure that (1.2) can be *equivalently* written as the following second order *cooperative elliptic system* (we stress that the equivalence between weak solutions of (1.2) and weak solutions of (1.7) does

not always hold, see Section 5)

$$(1.7) \qquad \begin{cases} -\Delta u_1 = u_2 & \text{in } \Omega, \\ -\Delta u_2 = \Theta \rho u_1 & \text{in } \Omega, \\ \quad u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

Due to its cooperative structure, (1.7) –and consequently the fourth order problem (1.2)– inherits the strong maximum principle holding for second order elliptic equations. This allows us to apply the moving plane method for proving Theorem 1.2.
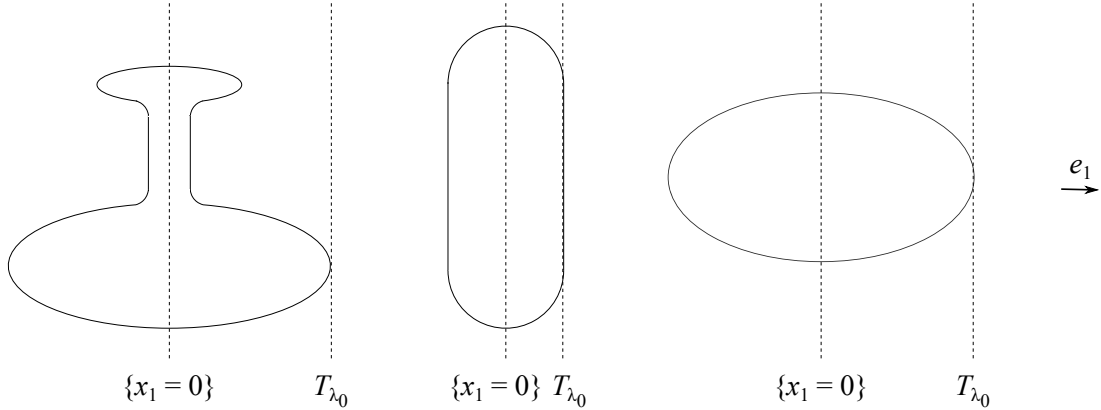

FIGURE 1. Qualitative representation of some planar domains covered by Theorem 1.2. They are meant to be C^4-smooth and symmetric and convex with respect to $\{x_1 = 0\}$. Note that points with horizontal tangent or flat parts in the e_1 direction –i.e., orthogonal to the symmetry axis $\{x_1 = 0\}$– are not allowed for $\partial\Omega \cap \{x_1 > 0\}$.

As already mentioned, the convexity of the domain is enough to have a maximum principle for (1.2), cf. Section 5; the main reason for requiring Ω to be C^4-smooth is to ensure C^1-regularity *up to the boundary* of both u and Δu. This is needed in a series of lemmas proved in Section 3, in the spirit of [34] by Troy and of [16] by Gidas-Ni-Nirenberg, which are the core of the proof of Theorem 1.2. On the other hand, the assumption $\lambda_1 = 0$ is essentially due to the lack of regularity of ρ_u, which prevents any use of the Mean Value Theorem for applying the maximum principle also to the auxiliary cooperative elliptic system (see for instance (5.4)) classically arising in the moving plane method. We want to stress that, while the hypothesis $\lambda_1 = 0$ seems to be mainly related to the technique used, a convexity hypothesis (at least w.r.t. $\{x_1 = 0\}$) in Theorem 1.2 seems to be crucial for the validity of the result. Indeed, some numerical evidence in [22] (cf. Fig. 6.9 therein) shows that symmetry breaking phenomena appear for thin annuli.

At this stage, it seems necessary to spend a few words regarding the possibility to relax the assumptions on Ω. In order to treat the case of more general domains, it would be quite natural to try to adapt the moving plane method as developed by Berestycki and Nirenberg in [2], for which it is enough to work with bounded sets. However, in our case, we already mentioned that we need some further assumption (e.g. convexity) to ensure the equivalence between the fourth order PDE and the second order elliptic system. In addition, we have some more considerations. Essentially, the technique of Berestycki and Nirenberg is composed by two parts: the starting procedure and the continuation. The first one is usually performed by means of *maximum principle in small domains*, in contrast with the smooth case, where one can use the Hopf Lemma. In our case, thanks to the

explicit expression (1.3) of the optimal configuration ρ_u, we can use the maximum principle in small domains for cooperative elliptic systems proved by de Figueiredo in [13], to let the moving plane method start. In principle, this technique would be the appropriate tool to deal with more general open sets Ω also in the crucial part of the *continuation of the moving plane method*, without any further restrictive requirements. However, in our case, due to the lack of regularity of ρ_u, it seems necessary to have more information on the *structure* of the set $\{u \leq t\}$ ((u, ρ_u) being optimal pair) or, better, its complementary in Ω. In particular, if we know that the set $\{u > t\}$ is symmetric and convex with respect to the hyperplane $\{x_1 = 0\}$, then we can adapt the proof of [2, Theorem 1.3] to our setting, including sets Ω which are merely convex. This partial symmetry preserving result for (possibly non-smooth) convex domains Ω is the content of Proposition 5.9. Without any structure information on $\{u > t\}$ we are not able to prove, with this technique, symmetry preserving properties for sets having $\lambda_1 \neq 0$ but $\lambda_2 = 0$ (defined in (2.4)), see Figure 2 and Remark 5.10.

The structure of the paper is the following: in Section 2 we fix the standing notation for implementing the moving plane technique and recall some basic results from the literature to make the paper self-contained. In Section 3 we state and prove some preliminary lemmas, while in Section 4 we prove Theorems 1.1 and 1.2. Finally, in Section 5, we treat the case of convex domains and prove Proposition 5.9.

2. NOTATION AND BACKGROUND

In this section we recall some known results, prove preliminary lemmas, and introduce definitions and notation which will be useful in the next sections. We stress that most of the results we are going to present in this section are valid for general convex sets, see Remark 5.6, but we state them in the less general context provided by Theorem 1.2.

Positivity. We prove below some positivity results for optimal pairs. For sake of completeness, we start with a maximum principle that slightly extends [14, Lemma 1].

Lemma 2.1. *Let Ω be a C^4-smooth bounded domain and set*

$$\mathcal{C}^+ := \{v \in H_0^1(\Omega) \cap H^2(\Omega) : v \geq 0 \text{ a.e. in } \Omega\}.$$

Suppose that $u \in H_0^1(\Omega) \cap H^2(\Omega)$ satisfies

$$(2.1) \qquad \int_\Omega \Delta u \Delta v \geq 0 \quad \text{for every } v \in \mathcal{C}^+.$$

Then, $u \in \mathcal{C}^+$. Furthermore, either $u \equiv 0$ or $u > 0$ a.e. in Ω.

Proof. Let $\varphi \in C_c^\infty(\Omega)$, $\varphi \geq 0$, and let $v_\varphi \in H_0^1(\Omega)$ be a solution of

$$-\Delta v_\varphi = \varphi \quad \text{in } \Omega.$$

By elliptic regularity theory, $v_\varphi \in H_0^1(\Omega) \cap H^2(\Omega)$ and by maximum principle [17, Theorem 8.1], $v_\varphi \geq 0$ in Ω. Hence, $v_\varphi \in \mathcal{C}^+$. Thus, using v_φ as test function in (2.1), we have

$$0 \leq \int_\Omega \Delta u \Delta v_\varphi = - \int_\Omega \Delta u \, \varphi.$$

By the arbitrariness of φ, we get again by [17, Theorem 8.1] that $u \geq 0$ a.e. in Ω. Finally, suppose that for some ball $B \Subset \Omega$

$$\inf_B u = \inf_\Omega u \geq 0.$$

Then, by [17, Theorem 8.19], $u \equiv \text{Const.}$, and so $u \equiv 0$ a.e. in Ω, being $u \in H_0^1(\Omega)$. $\square$

Using the previous lemma, it is possible to prove the following positivity result.

Proposition 2.2. *Let Ω be a C^4-smooth bounded domain, and (u, ρ) an optimal pair, then $u > 0$ and $\Delta u < 0$ a.e. in Ω.*

Proof. Proceeding as in [10, Proposition 5.1], with B replaced by Ω and using Lemma 2.1, it is possible to prove that $u > 0$ in Ω. Now, setting $u_1 := u$ and considering system (1.7) with $\rho = \rho_{u_1}$, we get for $u_2 := -\Delta u_1$

$$\begin{cases} -\Delta u_2 = \Theta \rho u_1 & \text{in } \Omega, \\ u_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence, by the strong maximum principle [17, Theorem 8.19], $-\Delta u_1 = u_2 > 0$ a.e. in Ω. $\quad\square$

Notation. We introduce now the notation for the moving plane technique. Given $x \in \mathbb{R}^n$, we denote by $x_1, \ldots, x_n$ its components. For a given unit vector $e \in \mathbb{R}^n$ and for $\lambda \in \mathbb{R}$, we define the hyperplane

$$T_\lambda := \{x \in \mathbb{R}^n : \langle e, x \rangle = \lambda\}.$$

From now on, unless otherwise stated, without loss of generality, we assume that $e = e_1$, i.e., the normal to T_λ is parallel to the x_1-direction. For every $\lambda \in \mathbb{R}$, we define the reflection with respect to T_λ as the function $\varphi_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$(2.2) \qquad \varphi_\lambda(x_1, x_2, \ldots, x_n) = (2\lambda - x_1, x_2, \ldots, x_n) \quad \text{for every } x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n.$$

We consider now the domain Ω where the problem is set. Given any $\lambda \in \mathbb{R}$, we introduce the (possibly empty) set

$$\Sigma_\lambda := \{x \in \Omega : x_1 > \lambda\}$$

and its reflection with respect to T_λ,

$$\Sigma'_\lambda := \{\varphi_\lambda(x) \in \mathbb{R}^n : x \in \Sigma_\lambda\}.$$

Since $\Omega \subset \mathbb{R}^n$ is bounded, we have that T_λ does not touch $\overline{\Omega}$ for sufficiently big λ. Decreasing the value of the parameter λ, we can find a value λ_0 such that T_{λ_0} touches $\overline{\Omega}$:

$$\lambda_0 := \sup\{\lambda \in \mathbb{R} : T_\lambda \cap \overline{\Omega} \neq \emptyset\}.$$

If we continue to lower the value of the parameter λ, we have that the hyperplane T_λ cuts off from Ω the portion Σ_λ. Clearly, at the beginning of the process, the reflection Σ'_λ of Σ_λ will be contained in Ω. Now, we define the value λ_1 as follows

$$(2.3) \qquad\qquad\qquad \lambda_1 := \sup\{\lambda < \lambda_0 : \text{(i) or (ii) is verified}\},$$

where

(i) Σ'_λ is internally tangent to the boundary $\partial\Omega$ at a certain point $P \notin T_\lambda$;
(ii) T_λ is orthogonal to the boundary $\partial\Omega$ at a certain point $Q \in T_\lambda \cap \partial\Omega$.

By construction,

$$\Sigma'_\lambda \subset \Omega \quad \text{for every } \lambda \in [\lambda_1, \lambda_0).$$

Nevertheless, by further decreasing the value of λ below λ_1, Σ'_λ might still be contained in Ω. Therefore we define the value

$$(2.4) \qquad\qquad\qquad \lambda_2 := \inf\{\lambda < \lambda_0 : \Sigma'_\lambda \subset \Omega\}.$$

For every $i = 1, 2$ and $\lambda \in [\lambda_2, \lambda_0)$, we can define

$$(2.5) \qquad\qquad\qquad w_i^{(\lambda)} := u_i \circ \varphi_\lambda - u_i \quad \text{in } \Sigma_\lambda.$$

We observe that, under the assumptions of Theorem 1.2, $\lambda_1 = \lambda_2 = 0$, and so the definition (2.5) is well-posed for every $\lambda \in [0, \lambda_0)$.

Now, let $x_0 \in \partial\Omega$ and $\epsilon > 0$: we define the following sets

$$\Omega_\epsilon := \{x \in \Omega \ : \ \|x - x_0\| < \epsilon\}, \tag{2.6}$$

and

$$S_\epsilon := \{x \in \partial\Omega \ : \ \|x - x_0\| < \epsilon\}. \tag{2.7}$$

We finally present an auxiliary result which will be useful in the next sections.

Lemma 2.3. *Let* $\lambda \in [\lambda_2, \lambda_0)$, (u, ρ_u) *be an optimal pair, and* $u \circ \varphi_\lambda - u \geq 0$ *in* Σ_λ. *Then*

$$(\rho_u \circ \varphi_\lambda)(u \circ \varphi_\lambda) - \rho_u u \geq 0 \quad \text{in } \Sigma_\lambda.$$

Proof. Taking into account the explicit representation (1.3) of ρ_u, pointwise there are four cases to treat.

- **Case I:** $x, \varphi_\lambda(x) \in \{u \leq t\} \cap \Sigma_\lambda$. In this case,

$$\rho_u(\varphi_\lambda(x))\, u(\varphi_\lambda(x)) - \rho_u(x)\, u(x) = h\,(u(\varphi_\lambda(x)) - u(x)) \geq 0.$$

- **Case II:** $x \in \{u \leq t\} \cap \Sigma_\lambda$ and $\varphi_\lambda(x) \in \{u > t\} \cap \Sigma_\lambda$. In this case,

$$\rho_u(\varphi_\lambda(x))\, u(\varphi_\lambda(x)) - \rho_u(x)\, u(x) = H\, u(\varphi_\lambda(x)) - h\, u(\varphi_\lambda(x))$$
$$\geq H(u(\varphi_\lambda(x)) - u(x)) \geq 0.$$

- **Case III:** $x \in \{u > t\} \cap \Sigma_\lambda$ and $\varphi_\lambda(x) \in \{u \leq t\} \cap \Sigma_\lambda$. This case cannot occur because $u(\varphi_\lambda(x)) \geq u$ in Σ_λ.
- **Case IV:** $x, \varphi_\lambda(x) \in \{u > t\} \cap \Sigma_\lambda$. In this case,

$$\rho_u(\varphi_\lambda(x))\, u(\varphi_\lambda(x)) - \rho_u(x)\, u(x) = H\,(u(\varphi_\lambda(x)) - u(x)) \geq 0.$$

Since these cases exhaust all possibilities, we are done. □

3. SOME PRELIMINARY LEMMAS

We start the present section by proving some technical lemmas which are the adaptation to our setting of those proved by Gidas, Ni and Nirenberg in [16] for second order problems, cf. also Troy [34, Lemma 4.1-4.3] for fourth order problems. Note that, with respect to the results by Troy, we work with less regular solutions, i.e., the u_i's satisfy the system in a weak sense. On the other hand, we can exploit the special form of the right hand side through the explicit representation of ρ given in (1.3).

Lemma 3.1. *Let* $\partial\Omega \in C^4$, $\lambda \in (\lambda_1, \lambda_0]$, $x_0 \in T_\lambda \cap \partial\Omega$, *and* (u_1, u_2) *be a weak solution of* (1.7) *with* $\rho = \rho_{u_1}$. *Fix* $\epsilon > 0$ *so small that* $\nu_1(x) > 0$ *for every* $x \in S_\epsilon$. *Then, there exists a positive constant* $\delta > 0$ *such that*

$$\frac{\partial u_i}{\partial x_1} < 0 \quad \text{in } \Omega_\delta \quad \text{for every } i = 1, 2.$$

Proof. We first observe that the existence of an ϵ as in the statement is ensured by the fact that $\nu_1(x_0) > 0$ being $\lambda \in (\lambda_1, \lambda_0]$, and by the regularity of Ω. By (1.4), $u_i \in C^1(\overline{\Omega_\epsilon})$, $u_i > 0$ in Ω_ϵ and $u_i = 0$ on S_ϵ for every $i = 1, 2$.

Let $x \in S_\epsilon$. Since for every $i = 1, 2$, $u_i > 0$ in Ω and $u_i = 0$ on $S_\epsilon \subset \partial\Omega$,

$$\frac{\partial u_i}{\partial x_1}(x) = \lim_{t \to 0^+} \frac{u_i(x) - u_i(x - te_1)}{t} \leq 0 \tag{3.1}$$

for every $x \in S_\epsilon$ and $i = 1, 2$. Arguing by contradiction, assume that there exists a sequence of points $\{x_k\}_{k \in \mathbb{N}} \subset \Omega_\epsilon$ such that

$$\lim_{k \to \infty} x_k = x_0 \quad \text{and} \quad \frac{\partial u_i}{\partial x_1}(x_k) \geq 0 \text{ for every } k \in \mathbb{N} \text{ and } i = 1, 2. \tag{3.2}$$

Now, fix $k \in \mathbb{N}$ and consider the interval in the positive x_1-direction having x_k as left bound. This interval must intersect S_ϵ at a certain point, denoted by a_k. In view of (3.1), $\frac{\partial u_i}{\partial x_1}(a_k) \leq 0$. Thus, by (3.2) and the regularity of the u_i's, we have

$$(3.3) \qquad\qquad \frac{\partial u_i}{\partial x_1}(x_0) = 0.$$

Since (u_1, u_2) solves weakly (1.7), we get

$$-\Delta u_1 = u_2 > 0 \quad \text{and} \quad -\Delta u_2 = \Theta \, \rho_{u_1} \, u_1 > 0 \quad \text{in } \Omega_\epsilon.$$

Finally, since $\partial \Omega \in C^4$, there exists an interior ball touching $\partial \Omega$ at x_0, hence by Hopf's Lemma (cf. [29], [3, Lemma 4]) we have $\frac{\partial u_i}{\partial \nu}(x_0) < 0$, and so

$$\frac{\partial u_i}{\partial x_1}(x_0) < 0,$$

which contradicts (3.3) and concludes the proof. $\qquad\qquad\qquad\qquad\qquad \square$

Lemma 3.2. *Let $\lambda \in [\lambda_1, \lambda_0)$ and (u_1, u_2) be a weak solution of (1.7) with $\rho = \rho_{u_1}$. Suppose that*

$$(3.4) \qquad u_i \leq u_i \circ \varphi_\lambda \quad but \quad u_i \not\equiv u_i \circ \varphi_\lambda \quad in \ \Sigma_\lambda \quad for \ some \ i = 1, 2.$$

Then,

$$u_i < u_i \circ \varphi_\lambda \quad in \ \Sigma_\lambda \quad for \ every \ i = 1, 2$$

and

$$\frac{\partial u_i}{\partial x_1} < 0 \quad on \ \Omega \cap T_\lambda \quad for \ every \ i = 1, 2.$$

Proof. Fix $\lambda \in [\lambda_1, \lambda_0)$. Since (u_1, u_2) solves weakly (1.7) we get

$$-\Delta(u_1 \circ \varphi_\lambda) = u_2 \circ \varphi_\lambda \quad \text{in } \Sigma_\lambda,$$

and

$$-\Delta(u_2 \circ \varphi_\lambda) = \Theta \, (\rho_{u_1} \circ \varphi_\lambda) \, (u_1 \circ \varphi_\lambda) \quad \text{in } \Sigma_\lambda,$$

where we have used the invariance of $-\Delta$ with respect to the reflection φ_λ. Now, by (2.5) and (1.7), we have

$$(3.5) \qquad\qquad -\Delta w_1^{(\lambda)} = u_2 \circ \varphi_\lambda - u_2 = w_2^{(\lambda)} \quad \text{in } \Sigma_\lambda,$$

and

$$(3.6) \qquad\qquad -\Delta w_2^{(\lambda)} = \Theta[(\rho_{u_1} \circ \varphi_\lambda) \, (u_1 \circ \varphi_\lambda) - \rho_{u_1} u_1] \quad \text{in } \Sigma_\lambda.$$

If (3.4) holds with $i = 2$, by (3.5) we get $-\Delta w_1^{(\lambda)} \geq 0$ in Σ_λ. Furthermore,

$$w_1^{(\lambda)} = \begin{cases} 0 & \text{on } T_\lambda, \\ u_1 \circ \varphi_\lambda & \text{on } \partial\Sigma_\lambda \cap \partial\Omega, \end{cases}$$

and so $w_1^{(\lambda)} \geq 0$ on $\partial\Sigma_\lambda$ being $\varphi_\lambda(\partial\Sigma_\lambda \cap \partial\Omega) \subset \overline{\Omega}$ and $u_1 \geq 0$ in $\overline{\Omega}$. Hence, by the weak maximum principle [17, Corollary 3.2], we get that $w_1^{(\lambda)} \geq 0$ in Σ_λ. If (3.4) holds with $i = 1$, $w_1^{(\lambda)} \geq 0$ in Σ_λ, and by Lemma 2.3, we know that the right-hand side of (3.6) is non-negative. Hence, $-\Delta w_2^{(\lambda)} \geq 0$ in Σ_λ, and arguing as in the previous case, we get $w_2^{(\lambda)} \geq 0$ by the weak maximum principle. Therefore, in both cases (3.4) holding with $i = 1$ and $i = 2$, we obtain

$$w_i^{(\lambda)} \geq 0 \quad \text{in } \Sigma_\lambda \quad \text{for every } i = 1, 2.$$

Thus, by the strong maximum principle [12, Theorem 2.2], we get that for every $i = 1, 2$

$$w_i^{(\lambda)} > 0 \quad \text{in } \Sigma_\lambda$$

and

$$0 > \frac{\partial w_i^{(\lambda)}}{\partial \nu} = -\frac{\partial w_i^{(\lambda)}}{\partial x_1} = 2\frac{\partial u_i}{\partial x_1} \quad \text{on } \Omega \cap T_\lambda,$$

where we have used the fact that $w_i^{(\lambda)} = 0$ on $\Omega \cap T_\lambda$. This concludes the proof. $\qquad\square$

Remark 3.3. Under the assumptions of Lemma 3.2, if there exist $i = 1, 2$ and a point $P \in \Omega \cap T_{\lambda_1}$ for which

$$\frac{\partial u_i}{\partial x_1}(P) = 0,$$

then, *for every $i = 1, 2$, u_i is symmetric with respect to the hyperplane* T_{λ_1}, and

$$\Omega = \Sigma_{\lambda_1} \cup \Sigma'_{\lambda_1} \cup (\Omega \cap T_{\lambda_1}).$$

Indeed, being $u_i \le u_i \circ \varphi_{\lambda_1}$ in Σ_{λ_1}, as a direct consequence of Lemma 3.2, we get

$$u_i \equiv u_i \circ \varphi_{\lambda_1} \quad \text{in } \Sigma_{\lambda_1} \quad \text{for every } i = 1, 2.$$

This means that both u_1 and u_2 are symmetric with respect to the hyperplane T_{λ_1}. Now, since $u_i > 0$ in Σ_{λ_1} and $u_i = 0$ on $\partial\Omega$, it must be

$$\Omega = \Sigma_{\lambda_1} \cup \Sigma'_{\lambda_1} \cup (\Omega \cap T_{\lambda_1}).$$

Otherwise, it would exist a point $Q \in (\Omega \cap \partial\Sigma'_{\lambda_1}) \setminus \left(\Sigma_{\lambda_1} \cup \Sigma'_{\lambda_1} \cup (\Omega \cap T_{\lambda_1})\right)$ where $u_i(Q) > 0$, being $Q \in \Omega$, and simultaneously $u_i(Q) = u_i(\varphi_{\lambda_1}(Q)) = 0$, being $\varphi_{\lambda_1}(Q) \in \partial\Sigma_{\lambda_1} \setminus T_{\lambda_1}$. This is impossible and concludes the proof.

Lemma 3.4. *Let $\partial\Omega \in C^4$, $\lambda \in (\lambda_1, \lambda_0)$, and (u_1, u_2) be a weak solution of* (1.7) *with $\rho = \rho_{u_1}$. Then*

$$\frac{\partial u_i}{\partial x_1} < 0 \quad and \quad u_i < u_i \circ \varphi_\lambda \quad in \ \Sigma_\lambda \quad for \ every \ i = 1, 2.$$

Proof. By Lemma 3.1 there exists $\delta > 0$ for which

$$\frac{\partial u_i}{\partial x_1} < 0 \quad \text{in } \Omega_\delta.$$

Hence, there exists $\bar\lambda \in (\lambda_1, \lambda_0)$ sufficiently close to λ_0 so that for both $i = 1, 2$

$$(3.7) \qquad\qquad \frac{\partial u_i}{\partial x_1} < 0 \quad \text{in } \Sigma_\lambda$$

for every $\lambda \in [\bar\lambda, \lambda_0)$. In particular, there exists $\lambda \in (\bar\lambda, \lambda_0)$ close enough to λ_0 for which $\Sigma'_\lambda \subset \Sigma_{\bar\lambda}$. Therefore, $\frac{\partial u_i}{\partial x_1} < 0$ in $\Sigma'_\lambda \cup \Sigma_\lambda \cup (T_\lambda \cap \Omega)$ for every $i = 1, 2$. Being the u_i's of class at least $C^1(\Omega)$, the Mean Value Theorem guarantees that for every $x \in \Sigma_\lambda$ there exists $\xi_x \in \{t\varphi_\lambda(x) + (1 - t)x \ : \ t \in (0, 1)\} \subset \Sigma'_\lambda \cup \Sigma_\lambda \cup (T_\lambda \cap \Omega)$ such that

$$0 > \frac{\partial u_i}{\partial x_1}(\xi_x) = \frac{u_i(x) - u_i(\varphi_\lambda(x))}{(x - \varphi_\lambda(x))_1}.$$

Hence,

$$(3.8) \qquad\qquad u_i < u_i \circ \varphi_\lambda \quad \text{in } \Sigma_\lambda \quad \text{for every } i = 1, 2$$

and for λ sufficiently close to λ_0. We need to gain some more space where (3.7) and (3.8) hold. To this aim, we lower λ until we reach the threshold $\mu \geq \lambda_1$, which is defined as follows:

$$(3.9) \quad \mu := \inf \left\{ \lambda \in (\lambda_1, \lambda_0) : \frac{\partial u_i}{\partial x_1} < 0 \text{ and } u_i < u_i \circ \varphi_\lambda \text{ hold in } \Sigma_\lambda \text{ for every } i = 1, 2 \right\}.$$

Thus, for $\lambda = \mu$ we have by continuity

$$(3.10) \qquad \frac{\partial u_i}{\partial x_1} < 0 \ \text{ and } \ u_i \leq u_i \circ \varphi_\mu \ \text{ in } \Sigma_\mu \quad \text{for every } i = 1, 2.$$

There are now two cases to treat. Either $\mu > \lambda_1$ or $\mu = \lambda_1$. Let us start with the case $\mu > \lambda_1$. We claim that, in this case

$$(3.11) \qquad\qquad u_i \not\equiv u_i \circ \varphi_\mu \ \text{ in } \Sigma_\mu \quad \text{for every } i = 1, 2.$$

Indeed, if $u_i \equiv u_i \circ \varphi_\mu$ in Σ_μ for some $i = 1, 2$, then by the continuity of u_i up to the boundary of Ω, we would have $u_i(x) = u_i(\varphi_\mu(x))$ for every $x \in \partial\Sigma_\mu \setminus T_\mu$. On the other hand, since $\mu > \lambda_1$, $\varphi_\mu(\partial\Sigma_\mu \setminus T_\mu) \subset \Omega$. Hence , by Propositions 2.2 and 5.5

$$u_i(x) = 0 < u_i(\varphi_\mu(x)) \quad \text{for every } x \in \partial\Sigma_\mu \setminus T_\mu,$$

which is absurd and proves the claim. Therefore, in view of conditions (3.10) and (3.11), we can apply Lemma 3.2 to get

$$(3.12) \qquad u_i < u_i \circ \varphi_\mu \text{ in } \Sigma_\mu \quad \text{and} \quad \frac{\partial u_i}{\partial x_1} < 0 \text{ in } \Omega \cap T_\mu \quad \text{for every } i = 1, 2.$$

By the definition (3.9) of μ, we know that at least one of the following two cases holds:

(a) there exist $\epsilon > 0$ and a sequence of points $(x_k) \subset \Sigma_{\mu-\epsilon} \setminus \Sigma_\mu$ converging to some point $x \in T_\mu$ such that

$$(3.13) \qquad \frac{\partial u_i}{\partial x_1}(x_k) \geq 0 \quad \text{for some } i = 1, 2 \text{ and for all } k \in \mathbb{N};$$

(b) there exist $\epsilon > 0$ and an increasing sequence of numbers $(\lambda_k)_{k \in \mathbb{N}} \subset (\mu - \epsilon, \mu)$ converging to μ, such that for every $k \in \mathbb{N}$, there is a point $x_k \in \Sigma_{\lambda_k} \setminus \Sigma_\mu$ for which

$$(3.14) \qquad u_i(x_k) \geq u_i(\varphi_{\lambda_k}(x_k)) \quad \text{for some } i = 1, 2 \text{ and for all } k \in \mathbb{N}.$$

If (a) holds, clearly $\frac{\partial u_i}{\partial x_1}(x) \geq 0$ for the same index i as in (3.13). Hence, by (3.12), $x \in \partial\Omega$. Since $\mu > \lambda_1$, T_μ is not orthogonal to $\partial\Omega$ at x and so $\nu_1(x) > 0$. In view of Lemma 3.1, this implies that $\frac{\partial u_i}{\partial x_1} < 0$ in a neighborhood of x, a contradiction.

If (b) holds, then, up to a subsequence, $x_k \to x \in \overline{\Sigma_\mu}$. Hence, $\varphi_{\lambda_k}(x_k) \to \varphi_\mu(x)$. Consequently, by the continuity of u_i,

$$u_i(x) \geq u_i(\varphi_\mu(x)) \quad \text{for the same index } i \text{ as in (3.14)}$$

which, in view of (3.12), gives $x \in \partial\Sigma_\mu$. If $x \in \partial\Sigma_\mu \setminus T_\mu$, then $\varphi_\mu(x) \in \Omega$, being $\mu > \lambda_1$. This forces

$$0 = u_i(x) < u_i(\varphi_\mu(x)),$$

which is absurd. Thus, it must be $x \in \partial\Sigma_\mu \cap T_\mu$. By the C^1-regularity of u_i and the Mean Value Theorem, for $k \in \mathbb{N}$ sufficiently large there exists

$$\xi_k \in \{t\varphi_{\lambda_k}(x_k) + (1 - t)x_k \ : \ t \in (0, 1)\} \subset \Sigma_{\mu-\epsilon} \setminus \Sigma_\mu$$

for which

$$\frac{\partial u_i}{\partial x_1}(\xi_k) = \frac{u_i(\varphi_{\lambda_k}(x_k)) - u_i(x_k)}{(\varphi_{\lambda_k}(x_k) - x_k)_1} \geq 0,$$

where in the last inequality we have used (3.14) and the fact that $x_k \in \Sigma_{\lambda_k}$. In this way, we have built a sequence (ξ_k) verifying (a), but we have already proved that this is not possible. This excludes the possibility that $\mu > \lambda_1$.

The only remaining possibility is that $\mu = \lambda_1$. Hence, for every $\lambda \in (\lambda_1, \lambda_0)$ the condition (3.8) holds. Reasoning as above, we obtain

$$u_i \not\equiv u_i \circ \varphi_\lambda \quad \text{in } \Sigma_\lambda \quad \text{for every } i = 1, 2,$$

for every $\lambda \in (\lambda_1, \lambda_0)$. So, in particular, by Lemma 3.2, $\frac{\partial u_i}{\partial x_1} < 0$ on $\Omega \cap T_\lambda$, which by the arbitrariness of $\lambda \in (\lambda_1, \lambda_0)$ gives that $\frac{\partial u_i}{\partial x_1} < 0$ in the whole Σ_{λ_1} for every $i = 1, 2$ and closes the proof. $\square$

4. Proofs of Theorems 1.1 and 1.2

In order to prove the rigidity result stated in Theorem 1.1, we take inspiration essentially from [25] by Philippin and Ragoub, and [34, Theorem 2] by Troy.

• *Proof of Theorem 1.1.* We start fixing an arbitrary direction, say e_1, along which we will move a hyperplane from infinity towards Ω. By the discussion in the Introduction, we know that $(u_1, u_2) := (u, -\Delta u)$ is a weak solution of (1.7) with $\rho = \rho_{u_1}$. We consider the functions $w_i^{(\lambda_1)}$ defined in (2.5), namely

$$w_i^{(\lambda_1)} = u_i \circ \varphi_{\lambda_1} - u_i \quad \text{in } \Sigma_{\lambda_1} \quad \text{for every } i = 1, 2.$$

By Lemma 3.4, we know that

$$w_i^{(\lambda_1)} \geq 0 \quad \text{in } \Sigma_{\lambda_1} \quad \text{for every } i = 1, 2.$$

Hence, by Lemma 3.2, either

$$(4.1) \qquad\qquad w_i^{(\lambda_1)} > 0 \quad \text{in } \Sigma_{\lambda_1} \quad \text{for every } i = 1, 2$$

or

$$(4.2) \qquad\qquad w_i^{(\lambda_1)} \equiv 0 \quad \text{in } \Sigma_{\lambda_1} \quad \text{for every } i = 1, 2.$$

Now, we observe that if the latter holds, the proof of the theorem is completed. Indeed, (4.2) implies in particular that $u = u_1$ and $-\Delta u = u_2$ are symmetric with respect to the hyperplane T_{λ_1}. Furthermore, being $u > 0$ in Ω by [10, Proposition 5.1], and $u = 0$ on $\partial\Omega$, it must be (cf. Remark 3.3)

$$\Omega = \Sigma_{\lambda_1} \cup \Sigma'_{\lambda_1} \cup (\Omega \cap T_{\lambda_1}),$$

that is Ω is symmetric with respect to T_{λ_1} as well. By the arbitrariness of the direction fixed, we infer that Ω must be a ball and u radially symmetric. The monotonicity property of the solution u then follows from [10, Theorem 1.5]. It remains to exclude the case (4.1).

To this aim, suppose first that we are in case (i) of (2.3), namely that Σ'_{λ_1} is internally tangent to $\partial\Omega$ at a point $P \notin T_{\lambda_1}$. It is clear that, $P' := \varphi_{\lambda_1}(P) \in \partial\Sigma_{\lambda_1} \setminus T_{\lambda_1}$, hence $P, P' \in \partial\Omega$, and so

$$u_1(P) = u_1(P') = 0.$$

Therefore, arguing as in the proof of Lemma 3.2, taking into account (4.1) and the fact that $w_1^{(\lambda_1)} \in C^2(\Sigma_{\lambda_1}) \cap C(\overline{\Sigma_{\lambda_1}})$, we deduce that $w_1^{(\lambda_1)}$ satisfies the following problem

$$\begin{cases} -\Delta w_1^{(\lambda_1)} = w_2^{(\lambda_1)} > 0 & \text{in } \Sigma_{\lambda_1}, \\ w_1^{(\lambda_1)} > 0 \text{ in } \Sigma_{\lambda_1}, \ w_1^{(\lambda_1)}(P') = 0. \end{cases}$$

Thus, by the Hopf Lemma [3, Lemma 4], we obtain

$$\frac{\partial w_1^{(\lambda_1)}}{\partial \nu}(P') < 0.$$

On the other hand, by straightforward calculations we have

$$\frac{\partial w_1^{(\lambda_1)}}{\partial \nu}(P') = 0.$$

Indeed, by (1.5)

$$\frac{\partial w_1^{(\lambda_1)}}{\partial \nu}(P') = \frac{\partial (u \circ \varphi_{\lambda_1})}{\partial \nu}(P') - \frac{\partial u}{\partial \nu}(P') = \frac{\partial u}{\partial \nu}(P) - \frac{\partial u}{\partial \nu}(P') = c - c = 0.$$

This excludes case (i).

Let us now assume that we are in case (ii) of (2.3). We want to show that the function $w_1^{(\lambda_1)}$ has a zero of order two at the point $Q \in \partial\Omega \cap T_{\lambda_1}$, namely that both the first and the second derivatives of $w_1^{(\lambda_1)}$ vanish at Q. This is achieved performing the same computations as in [30, pp. 307-308]. We stress that we can replicate that argument thanks to the $C^{3,\alpha}(\overline{\Omega})$-regularity of u_1, cf. (1.4). Actually, that argument is local in nature, so it is enough to note that in our situation we are even more smooth in a neighborhood of the boundary. We then reach a contradiction using the *corner lemma* of Serrin [30, Lemma 1]. This excludes the case (4.1) and closes the proof.   □

Remark 4.1. The techniques used to prove Theorem 1.1 actually allow to slightly relax the assumptions. Following [26, Theorem 1], we can prescribe the normal derivative $\frac{\partial u}{\partial \nu}$ to be constant on $\partial\Omega$ except on one point where either an interior or exterior sphere condition holds. Since the proof of [26, Theorem 1] is mainly based only on geometric considerations on the set Ω, that argument can be applied to our case as well.

• *Proof of Theorem 1.2.* As usual, being (u, ρ) an optimal pair, under any of the two conditions (i) or (ii) in the statement, we can consider the couple $(u_1, u_2) := (u, -\Delta u)$ as a weak solution of (1.7) with $\rho = \rho_{u_1}$. We start observing that, in order to prove the theorem, it is enough to show that

(4.3)   for every $\lambda \in (0, \lambda_0)$ it holds $u_i < u_i \circ \varphi_\lambda$ in Σ_λ for every $i = 1, 2$.

and

(4.4)   for every $-\lambda \in (-\lambda_0, 0)$ it holds $u_i < u_i \circ \varphi_{-\lambda}$ in $\varphi_0(\Sigma_\lambda)$ for every $i = 1, 2$.

Indeed, by (4.3) and by the definition (2.2) of φ_λ, for every $\lambda \in (0, \lambda_0)$, if we take $x = (x_1, y) \in \Sigma_\lambda$, i.e., $x \in \Omega$ with $x_1 > \lambda$, we get

$$u_i(x_1, y) < u_i(2\lambda - x_1, y) \quad \text{for every } i = 1, 2.$$

Hence, $(2\lambda - x_1, y) \to (-x_1, y)$ as $\lambda \searrow 0$, and by continuity

$$u_i(x_1, y) \le u_i(-x_1, y) \quad \text{for every } i = 1, 2.$$

On the other hand, for every $-\lambda \in (-\lambda_0, 0)$, if we take $x = (-x_1, y) \in \varphi_0(\Sigma_\lambda)$, i.e., $x \in \Omega$, with $-x_1 < -\lambda$, we get by (4.4)

$$u_i(-x_1, y) < u_i(-2\lambda + x_1, y) \quad \text{for every } i = 1, 2.$$

So, $(-2\lambda + x_1, y) \to (x_1, y)$ as $-\lambda \nearrow 0$, and by continuity

$$u_i(-x_1, y) \le u_i(x_1, y) \quad \text{for every } i = 1, 2.$$

Altogether, for any $x = (x_1, y) \in \Omega$

$$u_i(x_1, y) = u_i(-x_1, y) \quad \text{for every } i = 1, 2,$$

that is $u = u_1$ and $-\Delta u = u_2$ are symmetric with respect to $\{x_1 = 0\}$. Moreover, if for every $\lambda \in (0, \lambda_0)$ we define $w_i^{(\lambda)}$ as in (2.5), by (4.3) we know that $w_i^{(\lambda)} > 0$ in Σ_λ for every $i = 1, 2$. Therefore, for every $\lambda \in (0, \lambda_0)$ it holds

$$\begin{cases} -\Delta w_1^{(\lambda)} = w_2^{(\lambda)} > 0 & \text{in } \Sigma_\lambda, \\ \qquad\quad w_1^{(\lambda)} > 0 & \text{in } \Sigma_\lambda, \\ \qquad\quad w_1^{(\lambda)} = 0 & \text{on } T_\lambda. \end{cases}$$

Then, by Hopf's Lemma (cf. [3, Lemma 4]) we obtain

$$0 > \frac{\partial w_1^{(\lambda)}}{\partial \nu} = -\frac{\partial w_1^{(\lambda)}}{\partial x_1} = 2\frac{\partial u_1}{\partial x_1} \quad \text{on } T_\lambda,$$

where ν denotes the unit outward normal to $\partial \Sigma_\lambda$. This means that $\dfrac{\partial u}{\partial x_1} < 0$ for $x_1 > 0$, as required. The convexity of the set $\{u > t\}$ with respect to the hyperplane $\{x_1 = 0\}$ is then a consequence of the same property holding for the domain Ω and of the monotonicity of u.

By the assumptions on Ω, $\lambda_1 = 0$. Hence, the validity of (4.3) is guaranteed by Lemma 3.4. The case of (4.4) is analogous, due to the symmetry of the problem. $\qquad\square$

5. Convex domains

In this section we deal with convex sets Ω. In particular, this implies that $\partial \Omega$ is Lipschitz continuous, cf. [19, Corollary 1.2.2.3].

As already mentioned in the Introduction, setting $u_1 := u$, the equivalence between problem (1.2) and system (1.7) is a key tool for our purposes. Nevertheless, such equivalence is not straightforward when the domain is not smooth enough (i.e., C^4). When dealing with a fourth order problem of the form

$$(5.1) \qquad\qquad \begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial \Omega, \end{cases}$$

beyond the usual weak solutions, we can define the so-called *system solutions*, see e.g. [15, Section 2.7]. We recall below both definitions.

Definition 5.1. Let $f \in L^2(\Omega)$, we say that u is a *weak solution* of (5.1), if $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and

$$\int_\Omega \Delta u \, \Delta\varphi \, dx = \int_\Omega f\varphi \, dx \quad \text{for every } \varphi \in H^2(\Omega) \cap H_0^1(\Omega).$$

Definition 5.2. Let $f \in L^2(\Omega)$, we say that u is a *system solution* of (5.1), if $u \in H_0^1(\Omega)$, $\Delta u \in H_0^1(\Omega)$ and u solves (weakly) the system

$$(5.2) \qquad\qquad \begin{cases} -\Delta u = v & \text{in } \Omega, \\ -\Delta v = f & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial \Omega. \end{cases}$$

These two types of solutions are in general different, as it is shown by the Sapondzyan paradox or the Babuska paradox, see [15, Chapter 2, Section 7]. One common feature of these two paradoxes, is the fact that the set $\Omega \subset \mathbb{R}^n$ on which they are considered, admit concave corners. On the other hand, if one assumes Ω to be convex, these kinds of phenomena disappear, and one can prove that system solutions coincide with weak solutions.

Proposition 5.3 (Corollary 1.6 of [23])**.** *Let Ω be convex and $f \in L^2(\Omega)$. Then u is a weak solution of* (5.2) *if and only if u is a system solution of the same problem.*

Furthermore, the following result holds.

Proposition 5.4 ([21] and Theorem 1.2 of [23])**.** *Let Ω be convex. Then for every $f \in L^2(\Omega)$ the (unique) weak solution $u \in H_0^1(\Omega)$ of*

$$\begin{cases} -\Delta u = f & in \ \Omega, \\ u = 0 & on \ \partial\Omega \end{cases}$$

belongs to $H^2(\Omega)$.

Altogether, we can summarize the previous considerations as follows.

Proposition 5.5. *Let Ω be a convex bounded domain, then the following two sentences are equivalent*

    (i) *(u, ρ) is an optimal pair of* (1.1)*;*
    (ii) *$(u_1, u_2) := (u, -\Delta u)$ is a weak solution of* (1.7) *with $\rho = \rho_{u_1}$ as in* (1.3)*.*

Remark 5.6. We want to stress that, thanks to Proposition 5.4, Lemma 2.1 holds even for convex domains. Therefore this allows to follow verbatim the proof of Proposition 2.2, showing the positivity of u when Ω is a convex set without any further regularity assumption. Namely, the following statement holds true:

    Let Ω be a convex bounded domain and (u, ρ) an optimal pair, then $u > 0$ and $\Delta u < 0$ a.e. in Ω.

We recall below a particular instance of the maximum principle for cooperative systems in small domains proved for a more general situation by de Figueiredo in [13].

Lemma 5.7 (Proposition 1.1 of [13])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain,*

$$\mathcal{L} := \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} \quad and \quad A(x) := \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix},$$

with $a_{ij} \in L^\infty(\Omega)$ for $i, j = 1, 2$ and $a_{ij} \geq 0$ for $i \neq j$. Suppose that $U := (u_1, u_2)$ satisfies

$$\begin{cases} -(\mathcal{L} + A)U \geq 0 & in \ \Omega, \\ \liminf_{x \to \partial\Omega} U(x) \geq 0. \end{cases}$$

Then, there exists $\delta = \delta(n, \operatorname{diam}(\Omega)) > 0$ such that if $|\Omega| < \delta$, $U \geq 0$ in Ω (i.e, $u_1 \geq 0$ and $u_2 \geq 0$ in Ω).

We must now spend a few words regarding the regularity of u. We recall that if Ω is C^4, by [10, Theorem 1.3] we know that if (u, ρ) is an optimal pair, $u \in C^{3,\alpha}(\overline{\Omega}) \cap W^{4,q}(\Omega)$ for every $\alpha \in (0, 1)$ and $q \geq 1$. Since here Ω is required to be just convex, only Lipschitz regularity of the boundary is guaranteed. In particular, Ω satisfies an exterior cone condition and optimal interior regularity can be deduced as in [10, Theorem 1.3(a)], using a bootstrap argument and embedding theorems. We get in this way

$$u \in C^{3,\alpha}(\Omega) \cap W^{4,q}_{\text{loc}}(\Omega) \quad \text{for all } \alpha \in (0, 1) \text{ and } q \geq 1,$$

cf. [10, Remark 3.2].

We are interested also in the regularity *up to the boundary* of optimal pairs in the case of convex domains. In this regard, we can prove the following lemma.

Lemma 5.8. *Let Ω be convex and (u, ρ) be an optimal pair. Then, both u and Δu are of class $C^{0,\gamma}(\overline{\Omega})$ for some $\gamma \in (0, 1)$.*

Proof. We set as usual $u_1 := u$ and we consider the system (1.7) with $\rho = \rho_{u_1}$, i.e.,

$$\begin{cases} -\Delta u_1 = u_2 & \text{in } \Omega, \\ -\Delta u_2 = \Theta\rho u_1 & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $\Theta\rho u_1 \in L^2(\Omega)$, by Proposition 5.4 we know that $u_1, u_2 \in H^2(\Omega) \cap H_0^1(\Omega)$. Moreover, by Remark 5.6 and Proposition 5.5, we also know that $u_1, u_2 > 0$ a.e. in Ω. Now, without loss of generality, we can assume that $\Theta\rho \le K$ for some positive constant $K > 1$ and by (1.7) we get

(5.3)
$$\begin{cases} -\Delta u_1 < K u_2 & \text{in } \Omega, \\ -\Delta u_2 \le K u_1 & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

Let us now consider the function $\varphi := u_1 + u_2$. By (5.3), we obtain

$$\begin{cases} -\Delta\varphi \le K\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

Clearly, $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$, being u_1 and u_2 in the same space. Hence, by [17, Theorem 8.15] with $L = \Delta + K$, we get that

$$\sup_{\Omega} \varphi \le C\|\varphi\|_{L^2(\Omega)}$$

for some $C = C(n, K, |\Omega|) > 0$ independent of φ. Therefore, $\varphi \in L^\infty(\Omega)$, and since both $u_1 \le \varphi$ and $u_2 \le \varphi$, this gives

$$u_1, u_2 \in L^\infty(\Omega).$$

In particular, $u_1, u_2 \in L^q(\Omega)$, with $q > n/2$, and we can now exploit [17, Theorem 8.29] to get that

$$u_1, u_2 \in C^{0,\gamma}(\overline{\Omega}) \quad \text{for some } \gamma \in (0,1)$$

and conclude the proof. $\qquad\square$

We are now ready to prove our result for convex domains. If (u, ρ) is an optimal pair, by the explicit form (1.3) of $\rho = \rho_u$, we know that ρ_u inherits all symmetry properties of u. The following proposition states that, under suitable convexity assumptions, also the converse is true, namely also u inherits the symmetries of ρ_u. The argument used in the proof of the next proposition is in the spirit of the paper [2] by Berestycki and Nirenberg, where a second order problem was treated, by using a careful combination of the maximum principle for small domains, the strong maximum principle in its classical form, and the moving plane technique. In the following proposition we use again the notation related to the moving plane technique, introduced in Section 2.

Proposition 5.9. *Let Ω be a convex domain of $\mathbb{R}^n$ and suppose that Ω is symmetric with respect to the hyperplane $\{x_1 = 0\}$. Let (u, ρ) be an optimal pair such that the superlevel set $\{u > t\}$ is symmetric and convex with respect to $\{x_1 = 0\}$.*
Then, u is symmetric with respect to $\{x_1 = 0\}$ and strictly decreasing in x_1 for $x_1 > 0$.

Proof. As in the proof of Theorem 1.2, it is enough to prove (4.3). Let $w_i^{(\lambda)}$ be defined as in (2.5). For every $\lambda \in (\lambda_1, \lambda_0)$, we get

(5.4)
$$\begin{cases} -(\Delta w_1^{(\lambda)} + w_2^{(\lambda)}) = 0 & \text{in } \Sigma_\lambda, \\ -\{\Delta w_2^{(\lambda)} + \Theta[(\rho_{u_1} \circ \varphi_\lambda)(u_1 \circ \varphi_\lambda) - \rho_{u_1} u_1]\} = 0 & \text{in } \Sigma_\lambda, \\ w_1^{(\lambda)} \ge 0, \quad w_1^{(\lambda)} \not\equiv 0 & \text{on } \partial\Sigma_\lambda, \\ w_2^{(\lambda)} \ge 0, \quad w_2^{(\lambda)} \not\equiv 0 & \text{on } \partial\Sigma_\lambda. \end{cases}$$

Now, the second equation in (5.4) can be equivalently rewritten as

$$-[\Delta w_2^{(\lambda)} + \Theta(\rho_{u_1} \circ \varphi_\lambda)w_1^{(\lambda)}] = \Theta(\rho_{u_1} \circ \varphi_\lambda - \rho_{u_1})u_1 \quad \text{in } \Sigma_\lambda.$$

By condition $u_1 = 0$ on $\partial\Omega$ and the $C^{0,\gamma}(\overline{\Omega})$-regularity of u_1, we know that the set $\{u_1 \leq t\}$ contains a tubular neighborhood $\mathcal{U}$ of $\partial\Omega$. Hence, if λ is sufficiently close to λ_0, $\Sigma_\lambda \subset \mathcal{U}$ and by the expression (1.3) of ρ_{u_1}, we get $\rho_{u_1} \equiv h$ in Σ_λ. This implies

$$\rho_{u_1} \circ \varphi_\lambda - \rho_{u_1} \geq 0 \quad \text{in } \Sigma_\lambda.$$

Therefore, being $u_1 > 0$ in Ω by Remark 5.6 and Proposition 5.5, for $\lambda < \lambda_0$ sufficiently close to λ_0, we have

$$\begin{cases} -(\Delta w_1^{(\lambda)} + w_2^{(\lambda)}) = 0 & \text{in } \Sigma_\lambda, \\ -[\Delta w_2^{(\lambda)} + \Theta(\rho_{u_1} \circ \varphi_\lambda)w_1^{(\lambda)}] \geq 0 & \text{in } \Sigma_\lambda, \\ w_1^{(\lambda)} \geq 0, \quad w_1^{(\lambda)} \not\equiv 0 & \text{on } \partial\Sigma_\lambda, \\ w_2^{(\lambda)} \geq 0, \quad w_2^{(\lambda)} \not\equiv 0 & \text{on } \partial\Sigma_\lambda. \end{cases}$$

We can now apply the maximum principle in narrow domains for cooperative systems Lemma 5.7 in Σ_λ, with

$$A := \begin{pmatrix} 0 & 1 \\ \Theta\rho_{u_1} \circ \varphi_\lambda & 0 \end{pmatrix}$$

and $U = (w_1^{(\lambda)}, w_2^{(\lambda)})$, to get $w_i^{(\lambda)} \geq 0$ in Σ_λ for every $i = 1, 2$. Therefore, for λ close enough to λ_0, $w_1^{(\lambda)}$ and $w_2^{(\lambda)}$ satisfy respectively

$$\begin{cases} -\Delta w_1^{(\lambda)} = w_2^{(\lambda)} \geq 0 & \text{in } \Sigma_\lambda, \\ w_1^{(\lambda)} \geq 0, \quad w_1^{(\lambda)} \not\equiv 0 & \text{on } \partial\Sigma_\lambda \end{cases} \quad \text{and} \quad \begin{cases} -\Delta w_2^{(\lambda)} = \Theta(\rho_{u_1} \circ \varphi_\lambda)w_1^{(\lambda)} \geq 0 & \text{in } \Sigma_\lambda, \\ w_2^{(\lambda)} \geq 0, \quad w_2^{(\lambda)} \not\equiv 0 & \text{on } \partial\Sigma_\lambda. \end{cases}$$

Thus, by the strong maximum principle [17, Theorem 8.19], we get

$$(5.5) \qquad\qquad w_i^{(\lambda)} > 0 \quad \text{in } \Sigma_\lambda \quad \text{for every } i = 1, 2$$

and $\lambda < \lambda_0$ sufficiently close to λ_0.

We can now define μ_1 as

$$\mu_1 := \inf\big\{\lambda \in (0, \lambda_0) : u_i < u_i \circ \varphi_\lambda \text{ in } \Sigma_\lambda \text{ for every } i = 1, 2\big\}.$$

By (5.5), μ_1 is well defined and $\mu_1 \geq 0$. We want to prove that $\mu_1 = 0$. Suppose by contradiction that $\mu_1 > 0$. By continuity,

$$u_i \leq u_i \circ \varphi_{\mu_1} \text{ in } \Sigma_{\mu_1} \quad \text{for every } i = 1, 2.$$

Moreover, since $\mu_1 > 0$, there exists a point $x_1 \in \partial\Sigma_{\mu_1} \setminus T_{\mu_1}$ such that $\varphi_{\mu_1}(x_1) \in \Omega$, and hence $w_1^{(\mu_1)}(x_1) > 0$. This shows that $w_1^{(\mu_1)} \not\equiv 0$ on $\partial\Sigma_{\mu_1}$. Therefore, we also have that $w_1^{(\mu_1)}$ solves the following problem

$$\begin{cases} -\Delta w_1^{(\mu_1)} = w_2^{(\mu_1)} \geq 0 & \text{in } \Sigma_{\mu_1}, \\ w_1^{(\mu_1)} \geq 0, \quad w_1^{(\mu_1)} \not\equiv 0 & \text{on } \partial\Sigma_{\mu_1}. \end{cases}$$

Hence, the strong maximum principle [17, Theorem 8.19] guarantees that $w_1^{(\mu_1)} > 0$ in Σ_{μ_1}. Similarly, we get that $w_2^{(\mu_1)} > 0$ in Σ_{μ_1}.

We want to prove that

$$(5.6) \qquad\qquad w_i^{(\mu_1-\epsilon)} \geq 0 \quad \text{in } \Sigma_{\mu_1-\epsilon} \quad \text{for every } i = 1, 2.$$

This, together with the sign $w_i^{(\mu_1-\epsilon)} \geq 0$, $w_i^{(\mu_1-\epsilon)} \not\equiv 0$ on $\partial\Sigma_{\mu_1-\epsilon}$, would give by the strong maximum principle

$$w_i^{(\mu_1-\epsilon)} > 0 \quad \text{in } \Sigma_{\mu_1-\epsilon} \quad \text{for every } i = 1, 2,$$

and would contradict the minimality of μ_1, showing that actually $\mu_1 = 0$.

In order to prove (5.6), let $K \Subset \Sigma_{\mu_1}$ be a compact set, then $w_i^{(\mu_1)} > 0$ in K for every $i = 1, 2$. We now claim that there exits $\epsilon_0 > 0$ so small that for every $\epsilon \in (0, \epsilon_0)$:

$$(5.7) \qquad\qquad w_i^{(\mu_1-\epsilon)} > 0 \ \text{ in } K \quad \text{for every } i = 1, 2.$$

Indeed, let

$$M_i := \min_{x \in K} \left\{ u_i \circ \varphi_{\mu_1}(x) - u_i(x) \right\} > 0.$$

By continuity of the u_i's, there exists $\epsilon \ll 1$ such that

$$|u_i \circ \varphi_{\mu_1-\epsilon}(x) - u_i \circ \varphi_{\mu_1}(x)| \leq \tfrac{M_i}{2} \quad \text{for every } x \in K.$$

Hence, for every $x \in K$ and $i = 1, 2$,

$$w_i^{(\mu_1-\epsilon)}(x) = u_i \circ \varphi_{\mu_1-\epsilon}(x) - u_i \circ \varphi_{\mu_1}(x) + u_i \circ \varphi_{\mu_1}(x) - u_i(x) \geq \tfrac{M_i}{2} > 0,$$
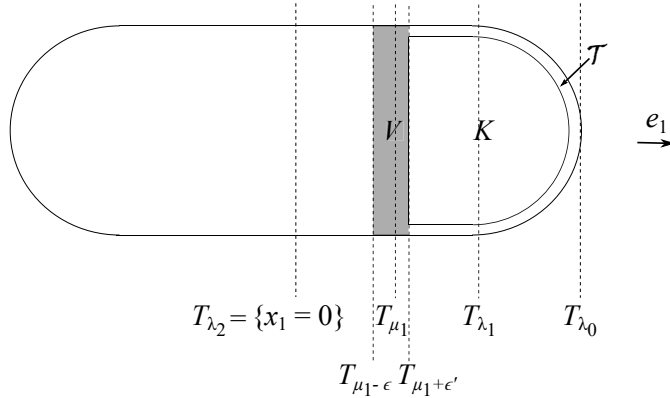
which proves the claim (5.7).

FIGURE 2. Qualitative representation of the domain Ω, its optimal cap Σ_{μ_1} in the assumption by contradiction, and its subsets K, $\mathcal{T}$ and V defined in the proof.

In particular, we choose K such that the complement of K in $\Sigma_{\mu_1+\epsilon'}$ is contained in a tubular neighborhood of $\partial\Omega$, more precisely

$$\mathcal{T} := \Sigma_{\mu_1+\epsilon'} \setminus K = \{ x \in \Sigma_{\mu_1+\epsilon'} : \text{dist}(x, \partial\Omega) < \epsilon' \}$$

for some $\epsilon' > 0$. Therefore, if we denote by $V := \Sigma_{\mu_1-\epsilon} \setminus \Sigma_{\mu_1+\epsilon'}$, we get

$$\Sigma_{\mu_1-\epsilon} = K \cup \mathcal{T} \cup V.$$

Our aim is to apply the maximum principle for narrow domains in $\Sigma_{\mu_1-\epsilon} \setminus K$, cf. Figure 2. Taking ϵ and ϵ' sufficiently small, we have $|\Sigma_{\mu_1-\epsilon} \setminus K| < \delta$, where $\delta = \delta(n, \text{diam}(\Sigma_{\mu_1-\epsilon} \setminus K)) > 0$ is given in Lemma 5.7.

In $\Sigma_{\mu_1-\epsilon} \setminus K$, $(w_1^{(\mu_1-\epsilon)}, w_2^{(\mu_1-\epsilon)})$ satisfies the following cooperative system

$$\begin{cases} -\left(\Delta w_1^{(\mu_1-\epsilon)} + w_2^{(\mu_1-\epsilon)}\right) = 0 & \text{in } \Sigma_{\mu_1-\epsilon} \setminus K, \\ -\left(\Delta w_2^{(\mu_1-\epsilon)} + \Theta(\rho_{u_1} \circ \varphi_{\mu_1-\epsilon}) w_1^{(\mu_1-\epsilon)}\right) = \Theta\left(\rho_{u_1} \circ \varphi_{\mu_1-\epsilon} - \rho_{u_1}\right) u_1 & \text{in } \Sigma_{\mu_1-\epsilon} \setminus K, \\ w_1^{(\mu_1-\epsilon)} \geq 0, \quad w_1^{(\mu_1-\epsilon)} \not\equiv 0 & \text{on } \partial\left(\Sigma_{\mu_1-\epsilon} \setminus K\right) \\ w_2^{(\mu_1-\epsilon)} \geq 0, \quad w_2^{(\mu_1-\epsilon)} \not\equiv 0 & \text{on } \partial\left(\Sigma_{\mu_1-\epsilon} \setminus K\right). \end{cases}$$

In order to apply Lemma 5.7, we need to prove that

$$(5.8) \qquad \rho_{u_1} \circ \varphi_{\mu_1-\epsilon} - \rho_{u_1} \geq 0 \quad \text{in } \Sigma_{\mu_1-\epsilon} \setminus K.$$

This is obviously true in $\mathcal{T}$, since for ϵ' sufficiently small, $\rho_{u_1} \equiv h$ in $\mathcal{T}$, by (1.3), being u_1 continuous in $\overline{\Omega}$ and $u_1 = 0$ on $\partial\Omega$. We now consider $x \in V$. If $u_1(x) \leq t$, $\rho_{u_1}(x) = h$ and so $\rho_{u_1}(\varphi_{\mu_1-\epsilon}(x)) - \rho_{u_1}(x) \geq 0$. If $u_1(x) > t$, we want to show that also $u_1(\varphi_{\mu_1-\epsilon}(x)) > t$. Indeed, since $x \in \{u_1 > t\}$ and $\{u_1 > t\}$ is symmetric and convex with respect to $\{x_1 = 0\}$,

$$\overline{xx^0} := \{\xi x + (1-\xi)\varphi_0(x) : \xi \in [0,1]\} \subset \{u_1 > t\}.$$

On the other hand, for ϵ sufficiently small, $\mu_1 - \epsilon > 0$, and so

$$|x - \varphi_{\mu_1-\epsilon}(x)| = 2\mathrm{dist}(x, T_{\mu_1-\epsilon}) = 2[x_1 - (\mu_1 - \epsilon)] < 2x_1 = |x - \varphi_0(x)|,$$

that is to say $\varphi_{\mu_1-\epsilon}(x)$ belongs to the segment $\overline{xx^0}$. Thus, $u_1(\varphi_{\mu_1-\epsilon}(x)) > t$, which gives $\rho_{u_1}(\varphi_{\mu_1-\epsilon}(x)) - \rho_{u_1}(x) = 0$. This proves (5.8) and by Lemma 5.7 gives

$$w_i^{(\mu_1-\epsilon)} \geq 0 \ \text{ in } \Sigma_{\mu_1-\epsilon} \setminus K \quad \text{for every } i = 1, 2.$$

Together with (5.7), we get (5.6), which concludes the proof of the proposition. $\square$

Remark 5.10. We stress that even for the composite membrane problem, the aforementioned convexity of $\{u > t\}$, provided Ω is convex as well, is proved with some extra assumptions in [6] and just conjectured for the general case. We believe it would be interesting to address this issue in a future work.

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