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Dynamics and Welfare in Recombinant Growth Models with Intellectual Property Rights: a Computational Method*

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Abstract

We extend the continuous time Tsur and Zemel's (2007) endogenous recombinant growth model with both physical capital and knowledge accumulation to allow for a basic IPRs system driven by public intervention as in Marchese et al. (2018). We analyze the effect of different policy regimes on social welfare comparing the outcomes in a decentralized and centralized frameworks. This requires to carefully study the transition dynamics associated to different values of the policy parameter. To this aim, we present a computational approach, extending the method developed by Privileggi (2011, 2015), to provide a novel procedure capable of approximating the transition dynamic paths and performing Skiba-point analysis even in our complex framework. Our numerical analysis shows that stricter policy regimes generate higher welfare levels and that a strictly positive tax level may be needed to avoid stagnation.

Keywords: Recombinant Growth; Transition Dynamics; Skiba Point; Welfare
JEL Codes: C61; C63; C68; O31; O41

1 Introduction

Knowledge is by far one of the most important determinants of long-run economic growth; thus, in the economics literature great emphasis is placed on assessing the impact of different types of policies on knowledge accumulation. In such a framework, the role played by intellectual property rights (IPRs) policy is still controversial since two opposite effects need to be balanced. On the one hand, a tighter IPRs policy allows stronger incentives for economic agents to engage in knowledge creation activities; on the other hand, the same policy increases the cost borne by the public domain to exploit the newly created knowledge in order to generate further innovation. The net impact of these two opposite effects determines whether tighter IPRs policy regimes might lead to higher or lower levels of social welfare, thus indicating whether they might be desirable for the society as a whole.

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The main contributions of this paper are twofold: (i) we provide a non-trivial extension of Tsur and Zemel's [16] model based on physical capital and knowledge accumulation to introduce a basic IPRs system driven by public intervention in order to compare the decentralized and the centralized outcomes; (ii) and we present a numerical approach to compute social welfare in such a complex framework in order to analyze how it is affected by different policy regimes. In our setup, the government completely finances the production of knowledge by relying upon a tax-subsidy scheme, and once knowledge has been produced it is immediately released for free in the public domain, as in [5]. This allows us to analyze how different degrees of public intervention affect the overall macroeconomic outcome both in the short and in the long run, quantifying the associated welfare effects. In order to do so, our computational procedure yields the approximation of the complete transition path of our endogenous growth model with IPRs, where knowledge does not evolve only because of profit seeking behavior (as traditionally discussed in the growth literature [2, 14]), but also because of externalities that characterize knowledge advances during the Weitzman's [18] combinatorial nature of knowledge accumulation. This description of the complexity underlying knowledge dynamics is consistent with some empirical evidence [1] and allows us to better understand the relationship between policy, growth and welfare in real world economies.

To evaluate the effect of policy on welfare, we perform a comparative dynamics exercise analyzing how different values of the policy instrument will be reflected in the evolution of consumption over time, and thus on the level of social welfare. A critical aspect of such an approach consists of computing the value of consumption along the whole transitional dynamic path. We need to distinguish between three different kind of trajectories: those occurring along a characteristic curve labeled as 'turnpike', those outside the turnpike but eventually converging to the turnpike, and those never converging to the turnpike but ending up in stagnation. Because such types of transitional dynamics are tough objects to deal with, we rely on a wide range of numerical techniques in order to quantitatively assess different consumption paths and the social welfare they generate. The method we adopt in this paper provides a non-trivial extension of previous works by Privileggi [10, 11, 12], who has developed a reliable approach to study the transitional dynamics in continuous time recombinant growth models á-la Tsur and Zemel [16]. Specifically, we first apply a suitable transformation to the ODE defining the optimal transition dynamics in order to study their associated 'detrended' system; such a transformation is a first novel technical contribution of the paper and leads to a system of ODEs [system (29)] that can be treated numerically. Next, a numerical method based on [11] is applied to such system in order to approximate the optimal policy along the (transitory) turnpike, while techniques extending those discussed in [12] are employed to approximate the transition dynamics starting outside the turnpike and leading toward the turnpike, as well as those describing the path toward a steady state. A bisection algorithm (Algorithm 1), providing a key step in the construction of the whole time-path trajectories starting off the turnpike and leading toward sustained growth, constitutes the second major original technical contribution. The resulting approximated trajectories allow to perform welfare analysis and comparative dynamics. The numerical procedure we develop in this paper can be applied also in other frameworks in order to numerically identify the stable arm leading to the equilibrium sustaining persistent growth — versus all smooth trajectories leading to a stagnation trap — arising from a dynamic optimization bang-bang problem, in which the whole growth enhancing stable arm is composed by multiple but joined (continuous) trajectories.

The paper proceeds as follows. Section 2 introduces our extended recombinant growth model and discusses short and long-run equilibria and the eventual convergence towards an asymptotic balanced growth path equilibrium. Sections 3–5 are the core of this paper, where we present a computational approach to fully analyze all types of transition dynamics associated to different policy regimes. Specifically, in Section 3, after introducing a suitable detrendization of variables, we characterize and compute the optimal consumption path along the turnpike, while in Section 4 we focus on trajectories that start off (above) the turnpike. We develop an algorithm (based on a bisection routine) to identify the intersection point between paths starting above the turnpike and their continuation along the turnpike itself, together with the optimal policies along the early transition, so that we can build the whole optimal consumption paths as piecewise functions by joining the early trajectories with their continuation along the turnpike. In Section 5 a similar kind of analysis is performed for studying trajectories not converging to the turnpike but leading to stagnation. All these findings allow us to assess the impact of different policy regimes on welfare, thus understanding how economic policy should be used in order to promote improvements in social welfare. Section 6 presents a specific illustrative example to test the performances of our approach and shows that welfare increases with the policy parameter and that a strictly positive tax level may be required to avoid stagnation. Section 7, as usual, concludes and discusses directions of future research.

2 Model and Asymptotic Equilibria

The model we consider is a continuous time Ramsey [13]-type model of endogenous growth with optimal knowledge creation; the setup is similar to [16] with the exception that the IPR system and government intervention follows [5]. The social planner maximizes social welfare by choosing the level of consumption, $c(t)$, and government expenditure, $G(t)$, taking into account the dynamic evolution of capital and knowledge. Social welfare is defined as the infinite discounted (ρ is the pure rate of time preference) sum of instantaneous utilities; the instantaneous utility function takes the following iso-elastic form: $u(c) = (c^{1-\sigma} - 1) / (1 - \sigma)$, where $\sigma \geq 1$ is the inverse of the intertemporal elasticity of substitution.[1] A unique final consumption good is competitively produced in the economy according to a Cobb-Douglas production technology combining physical capital, $k(t)$, and the stock of knowledge, $A(t)$: $y(t) = F[k(t), A(t)] = \theta k(t)^{\alpha} A(t)^{1-\alpha}$ where $0 < \alpha < 1$ is the capital share and $\theta > 0$ a scale parameter measuring the total factor productivity. Apart from this consumption good, in the economy also knowledge is competitively produced by R&D-firms which sell the newly created knowledge to the government, which then provides to make it freely available in the society. Output can be allocated to consumption, capital investment (for simplicity, no depreciation is assumed), $\dot{k}(t)$, or government spending, $G(t)$; thus capital evolves according to the following law of motion: $\dot{k}(t) = y(t) - c(t) - G(t)$.

As in [5], the government relies upon a tax-subsidy scheme to finance the decentralized production of knowledge, in which taxes on knowledge-producing firms' rents are used to subsidize knowledge production. Public expenditure is then used to purchase new knowledge, which evolves according to a recombinant rule defined by:

$$\dot{A}(t) = \frac{G(t)}{\phi_{\tau}[A(t)]}, \tag{1}$$

where $\phi_{\tau}(A)$ represents the expected price of new knowledge paid by the government; as in [5] it is a function of the stock of knowledge A and depends on a tax parameter, τ, as it is an equilibrium price that includes a tax on pure R&D profits determined by the rate τ. The tax parameter $0 \leq \tau < 1$ represents the policy instrument available to the government to implement

[1]For $\sigma = 1$ $u(c)$ boils down to logarithmic utility.

tighter of softer IPRs regimes, represented by smaller or larger values of τ respectively. Indeed, a higher value of τ implies a weaker appropriability (measured by $1-\tau$) of the profits generated by new knowledge creation, corresponding to a weaker protection of intellectual property (softer IPRs regimes). In the following we refer to τ as the policy parameter and we talk about different policy regimes in order to stress that the intensity of the IPR policy is (inversely) related to the degree of public intervention. Note that $\tau = 1$ represents a situation in which the decentralized knowledge production system is completely equivalent to the centralized production system as in [16]; such implementation is to a large extent impossible in reality because it would require the government to hold full information about individual agent's behavior, a task generally deemed outside the reach of the government in a decentralized economy.

We assume that only pair of ideas can be matched and the probability of success in creating new ideas is described by a hyperbolic function (increasing and concave as in [18]). Specifically, following [18]'s notation, we set $m = 2$ where m represents the number of possible ideas that can be matched, and $\pi(x) = \beta x / (\beta x + 1)$, where $\pi(x)$ represents the probability of success in matching ideas and $\beta > 0$ measures the efficiency of such a matching process. The last assumption implies the following expression for the unit costs of knowledge production, which represents the cost in terms of output of producing one unit of knowledge (see equation (33) in [5]):

$$\phi_\tau(A) = \frac{1}{\beta}\left[(1-\tau)\left(\frac{2A-1}{2A-3}\right)^2 + \tau\left(\frac{2A-1}{2A-3}\right)\right], \tag{2}$$

which comes from our specification of $\pi(\cdot)$ and the fact that $m = 2$.

Hence, the planner's problem can be formulated as:

$$\max_{[c(t),G(t)]_{t=t_0}^{\infty}} \int_0^\infty \frac{c^{1-\sigma}-1}{1-\sigma} e^{-\rho t}\,\mathrm{dt} \tag{3}$$

$$\text{subject to} \begin{cases} \dot{k}(t) = F[k(t), A(t)] - c(t) - G(t) \\ \dot{A}(t) = \dfrac{G(t)}{\phi_\tau(A)} \end{cases} \tag{4}$$

where $\phi_\tau(A)$ is defined by (2) and with the additional constraints $G(t) \leq y(t)$, $c(t) \leq k(t) + y(t)$, the usual non-negativity constraints, and given the initial conditions $k(0)$ and $A(0)$. Suppressing the time argument, the current-value Hamiltonian associated to the above problem reads as:

$$H(k, A, G, c, \lambda, \delta) = u(c) + \lambda[F(k, A) - c - G] + \delta\frac{G}{\phi_\tau(A)}, \tag{5}$$

where λ and δ are the costate variables associated with k and A respectively. Along with the transversality condition and the state equations (4), first order necessary conditions are:

$$u'(c) = \lambda \tag{6}$$

$$G = \begin{cases} 0 & \text{if } \delta/\phi_\tau(A) < \lambda \\ \tilde{G} & \text{if } \delta/\phi_\tau(A) = \lambda \\ F(k, A) & \text{if } \delta/\phi_\tau(A) > \lambda \end{cases} \tag{7}$$

$$\dot{\lambda} = \rho\lambda - \lambda F_k(k, A) \tag{8}$$

$$\dot{\delta} = \rho\delta - \lambda F_A(k, A) + \delta\frac{G\phi'_\tau(A)}{\phi_\tau(A)^2}, \tag{9}$$

where $\tilde{G}$ will be defined by (13) later.

Remark 1 *While the costates $\lambda(t)$ and $\delta(t)$ are continuous functions of time,*[2] *clearly conditions (7) envisage a discontinuous optimal R&D financing (a* 'bang-bang' *solution) due to linearity of the Hamiltonian in the variable G. On the other hand, through (6) and the continuity of $u'(\cdot)$, continuity of $\lambda(t)$ implies that* the optimal trajectory of consumption, $c(t)$, must be a continuous function of time *as well.*

As in [16], three curves in the (A, k) space, that is the (transitory) turnpike, the asymptotic turnpike and the stagnation curves, are useful for characterizing the solutions of the social planner problem in our regulated economy. The (transitory) turnpike defines the short run developing path followed by growing economies, the asymptotic turnpike their long run growth path, and the stagnation curve the path followed by non-growing economies doomed to stagnation (see [16]). These curves can be defined as follows.

1. The locus satisfying $F_k(k, A) = F_A(k, A)/\phi_\tau(A)$, which defines the *(transitory) turnpike* curve:

$$\tilde{k}_\tau(A) = \frac{\alpha}{1-\alpha}\phi_\tau(A)\,A = \frac{\alpha}{\beta(1-\alpha)}\left[(1-\tau)\left(\frac{2A-1}{2A-3}\right)^2 + \tau\left(\frac{2A-1}{2A-3}\right)\right]A. \tag{10}$$

2. The locus satisfying $F_k(k, A) = F_A(k, A)/\phi_\tau(A)$ for large A, which defines the *asymptotic turnpike* curve:

$$\tilde{k}_\tau^\infty(A) = \frac{\alpha}{\beta(1-\alpha)}(A + 2 - \tau). \tag{11}$$

3. The locus $F_k(k, A) = \rho$, which defines the *stagnation line*:

$$\hat{k}(A) = \left(\frac{\theta\alpha}{\rho}\right)^{1/(1-\alpha)} A. \tag{12}$$

Differentiating $\tilde{k}_\tau(A)$ in (10) with respect to time and substituting into both equations forming the dynamic constraint (4), yields

$$\tilde{G}(t) = \frac{[\tilde{y}_\tau(t) - \tilde{c}(t)]\,\phi_\tau[A(t)]}{\tilde{k}'_\tau[A(t)] + \phi_\tau[A(t)]}, \tag{13}$$

where $\tilde{y}_\tau(t) = \theta\left\{\tilde{k}_\tau[A(t)]\right\}^\alpha A(t)^{1-\alpha}$. Condition (13) establishes a relationship between the optimal investment in R&D, $\tilde{G}(t)$, as a function of the other control variable, the optimal consumption $\tilde{c}(t)$, when the economy is constrained to grow along the curve $\tilde{k}_\tau(A)$ defined in (10); that is, in view of (7), when $\delta(t)/\phi_\tau[A(t)] = \lambda(t)$ holds.

By following the same approach as in [16], it is possible to characterize the long run equilibrium, which depends on whether the stagnation line lies above or below the asymptotic turnpike. This is summarized in the following proposition.

[2] More precisely, they are *continuous and piecewise continuously differentiable* [3].

Proposition 1

i) *For any tax policy parameter value $0 \le \tau < 1$, the economy has the potential to sustain long-run growth if the stagnation line lies above the asymptotic turnpike for sufficiently large A, that is, if*

$$\theta\alpha\left[\frac{\beta(1-\alpha)}{\alpha}\right]^{1-\alpha} > \rho. \tag{14}$$

Conversely, if (14) does not hold then the economy eventually reaches a steady (stagnation) point on the line $\hat{k}(A)$ corresponding to zero growth.

ii) *Under (14), for a given tax scheme $0 \le \tau < 1$ and initial knowledge stock $A_0 > 3/2$ there is a corresponding threshold capital stock $k_\tau^{sk}(A_0) \ge 0$ such that whenever $k_0 \ge k_\tau^{sk}(A_0)$ the economy—possibly after an initial transition outside the turnpike—first reaches the turnpike $\tilde{k}_\tau(A)$ in a finite time, and then continues to grow along it as time elapses until the asymptotic turnpike $\tilde{k}_\tau^\infty(A)$ is reached in the long-run. Along $\tilde{k}_\tau^\infty(A)$ the economy follows an asymptotic balanced growth path (ABGP) characterized by a* common constant growth rate *of output, knowledge, capital and consumption given by*

$$\gamma = \frac{1}{\sigma}\left\{\theta\alpha\left[\frac{\beta(1-\alpha)}{\alpha}\right]^{1-\alpha} - \rho\right\}, \tag{15}$$

which is independent of the tax parameter τ. *If $k_0 < k_\tau^{sk}(A_0)$ the economy eventually stagnates.*

It is straightforward to show that the limit of the unit cost of knowledge in (2) converges to the unit cost of knowledge used by Tsur and Zemel [16] for $A \to \infty$, that is, for $t \to \infty$ when there is sustained growth. Therefore, the proof of Proposition 1 follows the same steps as the (long and involved) proof in the Appendix A of [16], which employs a direct approach based on scanning all types of feasible trajectories to test whether they possibly exhibit optimality properties. Accordingly, the (optimal) trajectories leading either to sustained growth along the asymptotic turnpike (11) or toward stagnation envisaged by (i) or (ii) in Proposition 1 are all and only trajectories that satisfy the necessary conditions (6)–(9) together with the following transversality condition due to Michel [6]:

$$\lim_{t\to\infty} H\left[k(t), A(t), G(t), c(t), \lambda(t), \delta(t)\right] e^{-\rho t} = 0, \tag{16}$$

where $H(k, A, G, c, \lambda, \delta)$ is the current-value Hamiltonian defined in (5). All other feasible trajectories either do not satisfy condition (16) or turn out to be suboptimal.[3]

Proposition 1 establishes that if (14) holds and k_0 is sufficiently large with respect to the initial knowledge stock, A_0, the economy grows along a turnpike path which, in the long-run, converges to an asymptotic balanced growth path[4] (ABGP) with knowledge and capital growing at the same constant rate. Note that even if the growth rate is independent of τ,

[3] Note that such a direct method does not rely on sufficient conditions for optimality; as a matter of fact, neither of the standard Arrow or Mangasarian conditions hold for the current-value Hamiltonian defined in (5), as the latter turns out to be convex in the state variable A. However, as trajectories violating condition (16) are clearly suboptimal (only capital or knowledge can grow at a rate larger than the discount rate ρ in the feasible region, not consumption), optimality conditions, including condition (16), guarantee that the objective function in (3) is bounded along all feasible trajectories.

[4] An ABGP is a solution of an optimal growth model in which all variables grow at asymptotically constant growth rates (see [9] for a discussion of BGP and ABGP equilibria).

the point $k_\tau^{sk}(A_0)$, where the superscript 'sk' is used to refer a Skiba-type [15] point (*i.e.*, a point separating the basins of attraction of two different equilibria), the turnpike $\tilde{k}_\tau(A)$ and the asymptotic turnpike $\tilde{k}_\tau^\infty(A)$ do depend on τ. This implies that the degree of tax intervention does affect the transitional dynamics and therefore the level of social welfare. In order to understand how welfare is related to different tax regimes, we need to analyze how the whole transitional dynamic path is affected by the tax parameter. Specifically, different turnpike behaviors due to different policy parameter values are characterized in the next proposition, whose proof is straightforward.

Proposition 2

i) *The turnpike curves are monotonic with respect to the tax parameter τ for fixed A:*

$$\begin{aligned} \tau_1 < \tau_2 \quad &\Longleftrightarrow \quad \tilde{k}_{\tau_1}(A) > \tilde{k}_{\tau_2}(A) \qquad \textit{for all } A < \infty \\ \tau_1 < \tau_2 \quad &\Longleftrightarrow \quad \tilde{k}_{\tau_1}^\infty(A) > \tilde{k}_{\tau_2}^\infty(A) \qquad \textit{for all } A, \end{aligned}$$

with $\tilde{k}_\tau(A)$ and $\tilde{k}_\tau^\infty(A)$ defined in (10) and (11) respectively;

ii) *on the turnpikes also output levels, $y_\tau(A) = \theta\left[\tilde{k}_\tau(A)\right]^\alpha A^{1-\alpha}$ are monotonic with respect to the tax parameter τ for fixed A: $y_{\tau_1}(A) > y_{\tau_2}(A)$ for $\tau_1 < \tau_2$ and fixed A.*

Proposition 2 states that economies with smaller degrees of public intervention (*i.e.*, with smaller τ), require larger capital/knowledge and output/knowledge ratios in order to sustain growth. The intuition about this result is that a lower degree of intervention (τ close to zero) distorts the capital/knowledge ratio in favor of capital; such a distortion involves a larger than optimal exploitation of physical inputs, a fact that might have negative consequences on the entire macroeconomic environment. Another distortion is that the smaller the corrective public intervention the higher the threshold capital stock required for starting the growth process, $k_\tau^{sk}(A_0)$, that is, the more demanding becomes the growth path in terms of initial wealth and consumer's patience. The tax-subsidy scheme mitigates these effects, by reducing the capital/knowledge distortion.

In order to study how a given economy reacts to different IPRs policies (different values of $0 \leq \tau < 1$, assumed to be constant over time) we perform comparative dynamics by changing the value of parameter τ while keeping constant all other parameters' values; specifically, we wish to characterize how the (unique) Skiba-type point $k_\tau^{sk}(A_0)$ on the turnpike changes for different τ-values and to compare the social welfare associated to the optimal trajectories corresponding to different policy regimes for an economy starting from the same initial pair (k_0, A_0) in $t = 0$. Provided that condition (14) holds and that $k_0 \geq k_\tau^{sk}(A_0)$, the turnpike $\tilde{k}_\tau(A)$ is 'trapping', *i.e.*, the economy keeps growing along it after it is reached. Note that we need to distinguish between two types of transitions: one driving the system toward the turnpike starting from outside it, and another characterizing the optimal path along $\tilde{k}_\tau(A)$ after it has been entered. Since there exists only one turnpike that crosses the initial point (k_0, A_0), corresponding to a specific value for parameter τ, then there is one single value for τ such that the associated turnpike $\tilde{k}_\tau(A)$ satisfies $\tilde{k}_\tau(A_0) = k_0$. Thus, in order to analyze the welfare implications of optimal trajectories corresponding to different policies starting from the same initial condition (k_0, A_0), besides characterizing the optimal dynamics along their corresponding turnpikes, we must also study their optimal dynamics outside such turnpikes for the initial time interval required to reach them. The computational approach to study these different dynamics is presented in the next two sections, where we first numerically compute the optimal consumption $\tilde{c}(t)$ along the turnpike $\tilde{k}_\tau(A)$, and then we select the unique optimal trajectory joining (k_0, A_0) (outside the turnpike) in $t = 0$ with the turnpike itself at some instant $t_0 > 0$ (when the turnpike is entered).

3 Dynamics Along the Turnpike

Let $t_0 \geq 0$ be the instant at which the turnpike is reached [if $k_0 = \tilde{k}_\tau(A_0)$ then $t_0 = 0$]. Under condition (14) and assuming that $k(t_0) \geq k_\tau^{sk}[A(t_0)]$, for $t \geq t_0$ the relevant variables are bound to move along the turnpike $\tilde{k}_\tau(A)$ and the planner problem (3), (4) reduces into a simpler optimization problem in one state variable, A, and one control, c, and one dynamic constraint with reference to the initial instant t_0:[5]

$$\tilde{V}_\tau[A(t_0)] = \max_{[c(t)]_{t=t_0}^{\infty}} \int_{t_0}^{\infty} \frac{c^{1-\sigma}-1}{1-\sigma} e^{-\rho t}\,\mathrm{dt} \tag{17}$$

$$\text{subject to} \begin{cases} \dot{A} = \dfrac{\tilde{y}_\tau(A) - c}{\tilde{k}'_\tau(A) + \phi_\tau(A)} \\ k(t_0) = \tilde{k}_\tau[A(t_0)], \end{cases}$$

with the additional constraint $0 \leq c \leq \tilde{k}_\tau(A) + \tilde{y}_\tau(A)$, where the time argument has been dropped for simplicity, $\tilde{y}_\tau(A) = \theta \tilde{k}_\tau(A)^\alpha A^{1-\alpha}$ is the output as a function of the sole variable A on the turnpike $\tilde{k}_\tau(A)$ as defined in (10), $\tilde{k}'_\tau(A) = (\partial/\partial A)\tilde{k}_\tau(A)$, and $\phi_\tau(A)$ is given by (2). It is easily checked that necessary conditions on the current-value Hamiltonian for problem (17) yield the following system of ordinary differential equations (ODEs) defining the optimal dynamics for A and c along the turnpike:

$$\begin{cases} \dot{A} = \dfrac{\theta \tilde{k}_\tau(A)^\alpha A^{1-\alpha} - c}{\tilde{k}'_\tau(A) + \phi_\tau(A)} \\ \dot{c} = \dfrac{c}{\sigma}\left\{\theta\alpha\left[\dfrac{\tilde{k}_\tau(A)}{A}\right]^{\alpha-1} - \rho\right\}. \end{cases} \tag{18}$$

As the stock of knowledge A cannot be depleted and, as the optimal investment in R&D must be positive—*i.e.*, $G(t) > 0$ for all t—along the turnpike, A must grow: $\dot{A}(t) > 0$ for all $t \geq t_0$. Some characteristics of the turnpike are summarized in the next proposition.

Proposition 3

i) *For all* $0 < \alpha < 1$ *and all* $0 \leq \tau < 1$ *the graph of* $\tilde{k}_\tau(A)$ *is a U-shaped curve on* $(3/2, +\infty)$*, reaching its unique minimum on a unique interior point* $\underline{A} > 3/2$.

ii) *Moreover, for all* $0 < \alpha < 1$ *and all* $0 \leq \tau < 1$ *the denominator on the RHS of the first equation in (18),* $\tilde{k}'_\tau(A) + \phi_\tau(A)$*, vanishes on a unique interior point* $A^s > 3/2$*, and* $\tilde{k}'_\tau(A) + \phi_\tau(A) < 0$ *for* $3/2 < A < A^s$*, while* $\tilde{k}'_\tau(A) + \phi_\tau(A) > 0$ *for* $A > A^s$*;* A^s *satisfies* $3/2 < A^s < \underline{A}$.

[5] The state variable k and the control G become functions of A and c according to (10) and (13) respectively.

Proof. i) Differentiating $\tilde{k}_\tau(A)$ in (10) with respect to A one gets

$$\tilde{k}'_\tau(A) = \frac{\alpha\left[8A^3 - 36A^2 + 2(11+10\tau)A - 3 - 6\tau\right]}{\beta(1-\alpha)(2A-3)^3}, \tag{19}$$

where the denominator is positive for all $A > 3/2$, so that the sign of $\tilde{k}'_\tau(A)$ depends on the sign of the numerator, which is a 3^{rd}-degree polynomial in A with positive coefficient on A^3. As its value on $A = 3/2$ is negative, equal to $-24(1-\tau) < 0$, and its derivative with respect to A is negative as well on $A = 3/2$, equal to $-32 + 20\tau < 0$, the polynomial is negative and decreasing in $A = 3/2$, so that there is a unique real zero $\underline{A} > 3/2$ such that $\tilde{k}'_\tau(\underline{A}) = 0$, which is the unique interior minimum for $\tilde{k}_\tau(A)$.
ii) Using (19) and (2) we can write

$$\tilde{k}'_\tau(A) + \phi_\tau(A) = \frac{8A^3 - (20 + 16\alpha + 8\tau - 8\alpha\tau)A^2 + (14 + 8\alpha + 16\tau + 4\alpha\tau)A - 3 - 6\tau}{\beta(1-\alpha)(2A-3)^3}$$

where, again, in the right-hand side term the denominator is positive for all $A > 3/2$ and the sign of $\tilde{k}'_\tau(A) + \phi_\tau(A)$ depends on the sign of the numerator, which is a 3^{rd}-degree polynomial in A with positive coefficient on A^3. Its value on $A = 3/2$ is negative, equal to $-24\alpha(1-\tau) < 0$, while its second derivative, equal to $48A - 40 + 16(2-\tau)(1-\alpha)$, is positive for all $A > 3/2$, so that the polynomial is negative on $A = 3/2$ and convex for all $A > 3/2$; therefore, there is a unique real zero $A^s > 3/2$ such that $\tilde{k}'_\tau(A^s) + \phi_\tau(A^s) = 0$, $\tilde{k}'_\tau(A) + \phi_\tau(A) < 0$ for $3/2 < A < A^s$ and $\tilde{k}'_\tau(A) + \phi_\tau(A) > 0$ for $A > A^s$. Note that, because $\phi_\tau(A^s) > 0$, $\tilde{k}'_\tau(A^s) + \phi_\tau(A^s) = 0$ implies $\tilde{k}'_\tau(A^s) < 0$, and thus $3/2 < A^s < \underline{A}$ must hold. ∎

Proposition 3(i) states that the graph of $\tilde{k}_\tau(A)$ is a U-shaped curve, so that capital $\tilde{k}(t)$ decreases when t is small and increases for larger t along the turnpike. Note that, as $\dot{A}(t) > 0$ for all $t \geq t_0$, the whole ratio on the RHS of the first equation in (18) must be positive for all $t \geq t_0$, that is, the numerator, $\theta\tilde{k}_\tau(A)^\alpha A^{1-\alpha} - c$, must have the same sign of the denominator, $\tilde{k}'_\tau(A) + \phi_\tau(A)$, and must vanish on a unique interior point, A^s. Moreover, as $\theta\tilde{k}_\tau(A)^\alpha A^{1-\alpha} = \tilde{y}_\tau(A)$, Proposition 3(ii) implies that the optimal consumption $\tilde{c}_\tau$ must satisfy $\tilde{c}_\tau > \tilde{y}_\tau(A)$ for $A < A^s$, $\tilde{c}_\tau < \tilde{y}_\tau(A)$ for $A > A^s$, and $\tilde{c}_\tau = \tilde{y}_\tau(A)$ for $A = A^s$. We thus conclude that in early times spent along the turnpike it is optimal to take away physical capital from the output-producing sector both for investment in R&D and consumption.

3.1 Detrended Dynamics

As system (18) diverges in the long-run, we transform the state variable A and the control c in a state-like variable, μ, and a control-like variable, χ, defined respectively by

$$\mu = \frac{\tilde{k}_\tau(A)}{A} = \frac{\alpha}{1-\alpha}\phi_\tau(A) \tag{20}$$

$$\chi = \frac{c}{A}, \tag{21}$$

where in the second equality of (20) we used (10) and $\phi_\tau(A)$ is defined in (2).

As $\phi_\tau(\cdot)$ is strictly decreasing, there is a one-to-one relationship between A and μ; in order to find it explicitly, we first set

$$\omega(A) = \frac{1}{\beta}\left(\frac{2A-1}{2A-3}\right), \tag{22}$$

so that the knowledge price can conveniently be rewritten as $\phi_\tau(A) = \beta(1-\tau)\omega(A)^2 + \tau\omega(A)$, and then write (20) in implicit form:

$$\beta(1-\tau)\omega(A)^2 + \tau\omega(A) - \frac{1-\alpha}{\alpha}\mu = 0. \tag{23}$$

Hence, two solutions for $\omega(A)$ in (23) are found; ruling out the negative one, we can express ω satisfying (23) as a function of the detrended variable μ:

$$\omega(\mu) = \frac{R(\mu) - \tau}{2\beta(1-\tau)}, \tag{24}$$

where

$$R(\mu) = \sqrt{\tau^2 + 4\left[\frac{1-\alpha}{\alpha}\beta(1-\tau)\right]\mu}. \tag{25}$$

Clearly, with $R(\mu)$ defined as in (25) $\omega(\mu)$ in (24) turns out to be strictly increasing in μ. Then, using (22), we can write A as a strictly decreasing function of ω: $A = 1/(\beta\omega - 1) + 3/2$; finally, substituting ω as in (24) we obtain:

$$A = \frac{2(1-\tau)}{R(\mu) - 2 + \tau} + \frac{3}{2}, \tag{26}$$

which establishes a one-to one, strictly decreasing relationship between A and μ.

By dividing the first equation in (18) by A, differentiating (20) with respect to time so that $\dot{\mu} = \left[\tilde{k}'_\tau(A) - \mu\right]\left(\dot{A}/A\right)$, substituting $\dot{A}/A$ accordingly, and using (21), we get

$$\dot{\mu} = \frac{\left[\tilde{k}'_\tau(A) - \mu\right](\theta\mu^\alpha - \chi)}{\tilde{k}'_\tau(A) + \phi_\tau(A)}. \tag{27}$$

In order to rewrite (27) entirely in terms of μ and χ, the key step is to write $\tilde{k}'_\tau(A)$ as a function of μ. Differentiating (10) with respect to A one gets $\tilde{k}'_\tau(A) = [\alpha/(1-\alpha)][\phi'_\tau(A)A + \phi_\tau(A)]$, so that we must compute $\phi'_\tau(A)$ first. Using (22) in the expression of $\phi_\tau(A)$ and differentiating with respect to A we get $\phi'_\tau(A) = \omega'(A)[2\beta(1-\tau)\omega(A) + \tau]$. Noting that $\omega'(A) = -4/\left[\beta(2A-3)^2\right] = -(1/\beta)[2/(2A-3)]^2 = -(1/\beta)[\beta\omega(A) - 1]^2$, where in the last equality again we used (22), we can write

$$\begin{aligned}
\phi'_\tau(A) &= \omega'(A)[2\beta(1-\tau)\omega(A) + \tau] \\
&= -\frac{1}{\beta}[\beta\omega(A) - 1]^2[2\beta(1-\tau)\omega(A) + \tau] \\
&= -\frac{1}{\beta}\left[\frac{R(\mu) - \tau}{2(1-\tau)} - 1\right]^2[R(\mu) - \tau + \tau] \\
&= -\frac{R(\mu)}{\beta}\left[\frac{R(\mu) + \tau - 2}{2(1-\tau)}\right]^2,
\end{aligned}$$

where in the third equality we used (24) to replace $\omega(A)$ with $\omega(\mu)$—the positive solution of the implicit form (23) as a function of μ—and $R(\mu)$ is given by (25). Substituting the expression above in $\tilde{k}'_\tau(A) = [\alpha/(1-\alpha)][\phi'_\tau(A)A + \phi_\tau(A)]$ and using the second equality in (20), $\phi_\tau(A) = [(1-\alpha)/\alpha]\mu$, after a fair amount of algebra $\tilde{k}'(A)$ becomes

$$\tilde{k}'_\tau(A) = \mu - \frac{\alpha R(\mu)\left[3R(\mu)^2 - 2(4-\tau)R(\mu) + 4 - \tau^2\right]}{8\beta(1-\alpha)(1-\tau)^2}, \tag{28}$$

which, when substituted in (27) and, again, using (20), yields

$$\dot{\mu} = \frac{\tilde{k}_\tau'(A) - \mu}{\tilde{k}_\tau'(A) + \phi_\tau(A)}(\theta\mu^\alpha - \chi) = \left[1 - \frac{\mu}{\alpha\tilde{k}_\tau'(A) + (1-\alpha)\mu}\right](\theta\mu^\alpha - \chi)$$
$$= \left\{1 - \frac{8\beta(1-\alpha)(1-\tau)^2\mu}{8\beta(1-\alpha)(1-\tau)^2\mu - \alpha^2 R(\mu)\left[3R(\mu)^2 - 2(4-\tau)R(\mu) + 4 - \tau^2\right]}\right\}(\theta\mu^\alpha - \chi).$$

Differentiating (21) with respect to time, using the second equality in (18) and substituting $\dot{A}/A$ from the first equation in (18), we get

$$\dot{\chi} = \left[\frac{\theta\alpha\mu^{\alpha-1} - \rho}{\sigma} - \frac{\theta\mu^\alpha - \chi}{\tilde{k}_\tau'(A) + \phi_\tau(A)}\right]\chi,$$

which, by replacing $\tilde{k}_\tau'(A)$ as in (28) and $\phi_\tau(A) = [(1-\alpha)/\alpha]\mu$, yields the following ODE for the control-like variable χ:

$$\dot{\chi} = \left[\frac{\theta\alpha\mu^{\alpha-1} - \rho}{\sigma} - \frac{8\alpha\beta(1-\alpha)(1-\tau)^2(\theta\mu^\alpha - \chi)}{8\beta(1-\alpha)(1-\tau)^2\mu - \alpha^2 R(\mu)\left[3R(\mu)^2 - 2(4-\tau)R(\mu) + 4 - \tau^2\right]}\right]\chi.$$

Hence, we have built the following system of ODEs describing the transition optimal dynamics in the detrended variables μ (state) and χ (control), conjugate to the true (diverging) system (18):

$$\begin{cases} \dot{\mu} = \left[1 - \dfrac{8\beta(1-\alpha)(1-\tau)^2\mu}{Q(\mu)}\right](\theta\mu^\alpha - \chi) \\ \dot{\chi} = \left[\dfrac{\theta\alpha\mu^{\alpha-1} - \rho}{\sigma} - \dfrac{8\alpha\beta(1-\alpha)(1-\tau)^2(\theta\mu^\alpha - \chi)}{Q(\mu)}\right]\chi, \end{cases} \tag{29}$$

where, to simplify notation, we have set

$$Q(\mu) = 8\beta(1-\alpha)(1-\tau)^2\mu - \alpha^2 R(\mu)\left[3R(\mu)^2 - 2(4-\tau)R(\mu) + 4 - \tau^2\right], \tag{30}$$

with $R(\mu)$ defined in (25)—*i.e.*, $R(\mu) = \sqrt{\tau^2 + 4[\beta(1-\alpha)(1-\tau)/\alpha]\mu}$.

Besides the more algebraic complexity of function $Q(\mu)$ in (30)—including $R(\mu)$ as defined in (25)—system (29) somewhat resembles system (56) in [10] (or, equivalently, system (22) in [11]). As a matter of fact, the whole qualitative behavior of the dynamics described by (29) turns out to be similar to that in those models, as it will be briefly illustrated below.

3.2 Steady States and Phase Diagram

The state-like variable μ has range $[\mu^*, +\infty)$ with $\mu^* = \alpha/[\beta(1-\alpha)]$. To see this note that μ^* in (25) yields $R(\mu^*) = 2 - \tau$, which, when plugged into (26), leads to the upper bound of A, $A \to +\infty$; while, again using (25) in (26), $\mu \to +\infty$ corresponds to $A = 3/2$, the lower bound of the A range. System (29) has *three steady states* in the (μ, χ) *phase diagram.* From the first equation in (29) two loci on which $\dot{\mu} = 0$ on the $[\mu^*, +\infty) \times \mathbb{R}_{++}$ plane are found:

$$\text{the curve } \chi = \theta\mu^\alpha \text{ and the vertical line } \mu \equiv \mu^*. \tag{31}$$

The former vanishes the second factor in the RHS of the first equation in (29), while μ^* is the largest (and only feasible) solution of $Q(\mu) - 8\beta(1-\alpha)(1-\tau)^2\mu = 0$, vanishing the first factor in the RHS of the same equation. By (30), such equation is equivalent to $3R(\mu)^2 - 2(4-\tau)R(\mu) + 4 - \tau^2 = 0$, which admits the only feasible solution[6] $R(\mu^*) = 2 - \tau$. From the second equation in (29) the unique locus on which $\dot{\chi} = 0$ is given by:

$$\chi = \theta\mu^\alpha - \frac{Q(\mu)\left(\theta\alpha\mu^{\alpha-1} - \rho\right)}{8\alpha\beta\sigma(1-\alpha)(1-\tau)^2}. \tag{32}$$

Note that the necessary condition for growth (14) is equivalent to

$$\theta\alpha(\mu^*)^{\alpha-1} > \rho. \tag{33}$$

1. The first steady state is thus (μ^*, χ^*) defined by

$$\mu^* = \frac{\alpha}{\beta(1-\alpha)} \qquad \text{and} \qquad \chi^* = \theta\left[\frac{\alpha}{\beta(1-\alpha)}\right]^\alpha\left(1 - \frac{1}{\sigma}\right) + \frac{\rho}{\beta\sigma(1-\alpha)}, \tag{34}$$

   where χ^* is (32) evaluated at $\mu = \mu^*$, which is the intersection point between the second locus in (31), $\mu \equiv \mu^*$, and the curve (32). We shall see in Proposition 4 below that (μ^*, χ^*) is the saddle-path stable steady state to which system (29) converges in the long-run. Indeed, μ^* corresponds to the *capital/knowledge ratio* along the asymptotic turnpike $\tilde{k}_\tau^\infty(A)$, *i.e.*, the slope of $\tilde{k}_\tau^\infty(A)$ in (11), while χ^* is the *consumption/knowledge ratio*, *i.e.*, the asymptotic slope of the optimal policy $\tilde{c}(A)$, when consumption steadily grows at the constant rate γ defined in (15). Note that (μ^*, χ^*) is independent of the policy parameter τ.

2. As $Q(\mu^*) - 8\beta(1-\alpha)(1-\tau)^2\mu^* = 0$ implies $Q(\mu^*) > 0$, (33) implies that on $\mu = \mu^*$ the locus (32) lies strictly below the first locus in (31), $\chi = \theta\mu^\alpha$. However, as $0 < \alpha < 1$, $\theta\alpha\mu^{\alpha-1}$ is a decreasing function of μ; hence, there is a unique value $\hat{\mu} > \mu^*$ such that $\left[\theta\alpha(\hat{\mu})^{\alpha-1} - \rho\right] = 0$. It is clear from the second term in the RHS of (32) that the locus (32) and the first locus in (31) intersect in $\mu = \hat{\mu}$; hence, the point $(\hat{\mu}, \hat{\chi})$, with coordinates

$$\hat{\mu} = \left(\frac{\theta\alpha}{\rho}\right)^{\frac{1}{1-\alpha}} \qquad \text{and} \qquad \hat{\chi} = \theta\left(\frac{\theta\alpha}{\rho}\right)^{\frac{\alpha}{1-\alpha}}, \tag{35}$$

   is another steady state for (29). Also this point is independent of τ. From (12), it is immediately seen that the (unique) steady state $\hat{\mu}$ in (35) corresponds to any steady state on the stagnation line $\hat{k}(A)$ in the (A, k) space. Using $\hat{\mu}$ into (26), we get $\hat{A}_\tau = 2(1-\tau)/[R(\hat{\mu}) + \tau - 2] + 3/2$, where $\hat{A}_\tau$ denotes the (unique) value at which the turnpike $\tilde{k}_\tau(A)$ intersects the stagnation line $\hat{k}(A)$ in the original model.

3. The last steady state corresponds to the second intersection point between the locus (32) and the first locus in (31), which is identified by a zero of the function $Q(\mu)$, again vanishing the second term in the RHS of (32). As $R(\mu)$ in (25) is strictly increasing in μ, it is invertible, so that $\mu = \alpha(R^2 - \tau^2)/[4\beta(1-\alpha)(1-\tau)]$; after substituting into (30), we can rewrite $Q(\mu)$ as a 3^{rd}-degree polynomial in R with negative coefficient on the term R^3: $\overline{Q}(R) = -3\alpha^2R^3 + 2\alpha[1 + 4\alpha - (1+\alpha)\tau]R^2 - \alpha^2(4 - \tau^2)R - 2\alpha(1-\tau)\tau^2$. Recall that $R^* = R(\mu^*) = 2 - \tau$ is the left endpoint of the range for R, $[R^*, +\infty)$, corresponding to the range $[\mu^*, +\infty)$ for μ. As $Q(\mu^*) = \overline{Q}(R^*) > 0$ and its derivative is positive as well on R^*, $\overline{Q}'(R^*) = 4(1-\alpha)(\tau^2 - 3\tau + 2) > 0$ for all $0 \le \tau < 1$, there is a unique

[6] The other solution, $R(\mu) = (2+\tau)/3$, when plugged into (26), yields $A = 0$, which is unfeasible.

feasible real zero, call it $R^s > R^*$, such that $\overline{Q}(R^s) = 0$, $\overline{Q}(R) > 0$ for $R^* \leq R < R^s$ and $\overline{Q}(R) < 0$ for $R > R^s$. To R^s corresponds μ_τ^s which, according to (26), defines our third and last steady state $(\mu_\tau^s, \chi_\tau^s)$ as

$$\mu_\tau^s = \frac{\alpha\left[(R^s)^2 - \tau^2\right]}{4\beta(1-\alpha)(1-\tau)} \qquad \text{and} \qquad \chi_\tau^s = \theta(\mu_\tau^s)^\alpha . \tag{36}$$

Note that μ_τ^s is defined as a function of R^s, which cannot be explicitly computed as function of parameters α and τ; however, for our purposes it will be sufficient to calculate it whenever some values for α and τ are chosen. Thus, $\mu_\tau^s > \mu^*$ is the largest (and only admissible) zero of the function $Q(\mu)$ defined in (30), with $Q(\mu) > 0$ for $\mu^* \leq \mu < \mu_\tau^s$ and $Q(\mu) < 0$ for $\mu > \mu_\tau^s$. The value μ_τ^s corresponds to the critical point A^s on which $c = \tilde{y}_\tau(A^s)$ in the (A, c) space discussed in Proposition 3(ii). The steady state $(\mu_\tau^s, \chi_\tau^s)$ defined in (36) is the only one depending on the policy parameter τ in our model.

While the singular point $(\mu_\tau^s, \chi_\tau^s)$ lies north-east of the long-run steady state (μ^*, χ^*) for all admissible parameters' values, the position of $(\hat{\mu}, \hat{\chi})$ depends on the magnitude of the discount rate ρ with respect to parameters α, θ and β. As in [10, 11, 12], we shall assume that

$$\theta\alpha(\mu_\tau^s)^{\alpha-1} < \rho < \theta\alpha(\mu^*)^{\alpha-1} , \tag{37}$$

envisaging a phase diagram in which $(\hat{\mu}, \hat{\chi})$ lies north-east of (μ^*, χ^*) and south-west of $(\mu_\tau^s, \chi_\tau^s)$. Note that the RHS in (37) equals the necessary condition (33) for long-run growth.

Proposition 4 *Provided that (37) holds, the two fixed points (μ^*, χ^*) and $(\hat{\mu}, \hat{\chi})$, defined in (34) and (35) respectively, can be classified as follows.*

i) (μ^*, χ^*) *is* saddle-path stable*, with the stable arm converging to it from north-east whenever the initial values $(\mu(t_0), \chi(t_0))$ are suitably chosen.*

ii) $(\hat{\mu}, \hat{\chi})$*, with coordinates defined in (35), is an unstable clockwise-rotating spiral.*

Proof. By studying both the phase diagram and/or the Jacobian of (29) evaluated at (μ^*, χ^*) and $(\hat{\mu}, \hat{\chi})$ the result is readily shown; we omit the cumbersome calculations for brevity, which, at any rate, follow steps similar to those in the proof of Proposition 4 in [10]. ■

The critical point $(\mu_\tau^s, \chi_\tau^s)$ defined in (36) is a 'supersingular' steady state whose Jacobian contains elements diverging to infinity, so that its stability/instability properties cannot be classified analytically. It corresponds to the value A^s on which the RHS of the first equation in (18) is not defined (both the numerator and the denominator vanish) as discussed in Proposition 3(ii). It is crossed by the stable arm of the saddle-path at low values of the stock of knowledge A, that is, in early times (in proximity of the very beginning of the economy's dynamics). Proposition 4(ii) implies that the steady state $(\hat{\mu}, \hat{\chi})$ is irrelevant for our analysis, as the optimal trajectory keeps well apart from it. The qualitative phase diagram associated to (29) is reported in Figure 1, where all loci are drawn and stability/instability properties of the three steady states are illustrated; it clearly resembles that in Figure 1 on p. 266 in [10].

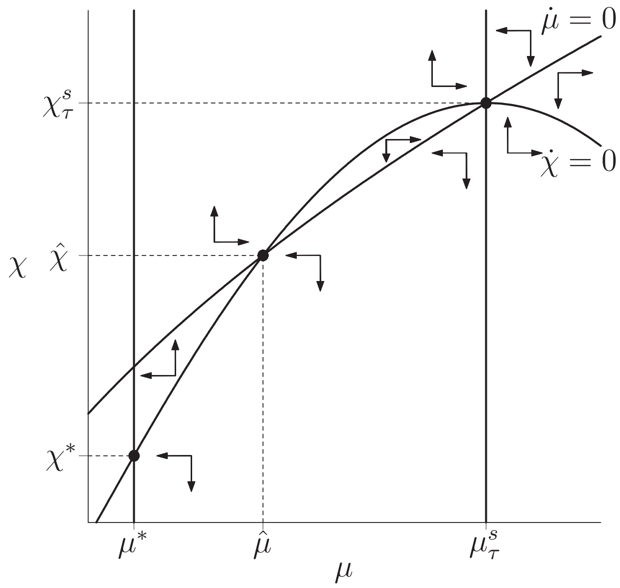

FIGURE 1: phase diagram of system (29) when $\theta\alpha\left(\mu^s_\tau\right)^{\alpha-1} < \rho < \theta\alpha\left(\mu^*\right)^{\alpha-1}$.

3.3 Optimal Policy, Welfare and Time-path Trajectories

Remember that $\tilde{k}_\tau(A) > \tilde{k}^\infty_\tau(A)$ for all A (and thus for all t); this is consistent with $\mu(t) > \mu^*$ for all t. Because, by Proposition 4(i), the stable arm $\chi(\mu)$—which is the *optimal policy expressed in terms of state-like and control-like variables*—approaches (μ^*, χ^*) from north-east, along the turnpike both ratios $\mu(t) = \tilde{k}_\tau[A(t)]/A(t)$ and $\chi(t) = \tilde{c}(t)/A(t)$ decline in time when they are approaching the asymptotic turnpike $\tilde{k}^\infty_\tau(A)$ corresponding to (μ^*, χ^*).

In order to study the policy function $\chi(\mu)$—which is the conjugate of $\tilde{c}(A)$ in the original model—we apply the technique developed by [7] and tackle the unique ODE given by the ratio between the two equations in (29):

$$\chi'(\mu) = \frac{\left[\left(\alpha\theta\mu^{\alpha-1} - \rho\right)/\sigma\right] Q(\mu) - 8\alpha\beta(1-\alpha)(1-\tau)^2\left[\theta\mu^\alpha - \chi(\mu)\right]}{\left[Q(\mu) - 8\beta(1-\alpha)(1-\tau)^2\mu\right]\left[\theta\mu^\alpha - \chi(\mu)\right]}\chi(\mu), \tag{38}$$

where $Q(\mu)$ is defined in (30). To numerically approximate the solution, $\chi_\tau(\mu)$, of (38) we apply a *Projection method* based on *OLS* of the residual function associated to equation (38), assuming that the approximate solution is a linear combination of *Chebyshev polynomials* up to *degree n* translated on an interval $\left[\underline{\mu}_\tau, \overline{\mu}_\tau\right]$—whose endpoints $\underline{\mu}_\tau$ and $\overline{\mu}_\tau$ and further details will be discussed in Subsection 6.1—containing the two relevant steady states, (μ^*, χ^*) and $(\mu^s_\tau, \chi^s_\tau)$, which are used to set the initial condition for the OLS algorithm. Such function $\chi_\tau(\mu)$ is then used together with (20) and (21) to get the *optimal consumption policies* for problem (17) according to

$$\tilde{c}_\tau(A) = \chi_\tau(\mu)A = \chi_\tau\left[\frac{\alpha}{\beta(1-\alpha)}\phi_\tau(A)\right]A, \tag{39}$$

where $\phi_\tau(A)$ is defined in (2).

For parameter values [see (51) in Section 6] satisfying condition (48) of Proposition 4 in [12], we can compute the value functions associated to (17)—yielding social welfare along the turnpikes and toward the ABGP as functions of the initial stock of knowledge A and independently of the starting instant t_0, for any τ-value in $[0, 1)$—by means of the Hamilton-

Jacobi-Bellman (HJB) equation as (see equation (47) in [12])

$$\tilde{V}_\tau(A) = \frac{1}{\rho}\left[\frac{\tilde{c}_\tau(A)^{1-\sigma}-1}{1-\sigma} + \frac{\theta\tilde{k}_\tau(A)^\alpha A^{1-\alpha} - \tilde{c}_\tau(A)}{\tilde{c}_\tau(A)^\sigma}\right], \tag{40}$$

where $\tilde{c}_\tau(A)$ is given by (39).

To get the time-path trajectories $\mu_\tau(t)$ for any fixed value $\tau \in [0,1)$ we substitute the approximated optimal policies $\chi_\tau(\mu)$ into the first equation of (29), which is then numerically solved through the standard *Fehlberg fourth-fifth order Runge-Kutta method with degree four interpolant* available in Maple, using the initial conditions given by the upper bounds $\overline{\mu}_\tau$. The time-path trajectories for the detrended controls are thus computed as $\chi_\tau[\mu_\tau(t)]$. The time-path trajectories of the stock of knowledge, $\tilde{A}_\tau(t)$, and capital, $\tilde{k}_\tau(t)$, along the turnpikes are obtained using $\mu_\tau(t)$ in (26) and then computing $\tilde{k}_\tau(t) = \tilde{k}_\tau\left[\tilde{A}_\tau(t)\right]$ from the definition of turnpike in (10). Similarly, the time-path trajectories of output along the turnpikes are given by $\tilde{y}_\tau(t) = \theta\tilde{A}_\tau(t)\left\{\tilde{k}_\tau\left[\tilde{A}_\tau(t)\right]/\tilde{A}_\tau(t)\right\}^\alpha$, while the time-path trajectories of the optimal consumption along the turnpikes, $\tilde{c}_\tau(t)$, are obtained using trajectories $\chi_\tau(t)$ and $\tilde{A}_\tau(t)$ in (21).

4 Dynamics and Welfare Above the Turnpike

We noted at the end of Section 2 that also optimal trajectories starting 'outside' the turnpike at $t=0$ and entering the turnpike at some later instant $t_0 > 0$ must be considered. As observed in [12], trajectories starting from initial conditions (A_0, k_0) that lie 'above' the turnpike at $t=0$ are easier to handle; therefore, we assume that $k(0) = k_0 \geq \tilde{k}_\tau[A(0)] = \tilde{k}_\tau(A_0)$. When $k_0 > \tilde{k}_\tau(A_0)$ we must study the optimal trajectories from (A_0, k_0) in $t=0$ to $\left(A(t_0), \tilde{k}_\tau[A(t_0)]\right)$ in $t = t_0 > 0$. Any optimal trajectory above the turnpike must satisfy the last necessary condition in (7), $\delta/\phi_\tau(A) > \lambda$, corresponding to the largest possible investment in R&D activities by the government:[7] $G = \theta k^\alpha A^{1-\alpha}$. In other words, from the social planner point of view it is optimal to invest all the output into the production of new knowledge along such early-transition trajectories. Hence, problem (3), (4) simplifies into one in two interlinked state variables, A and k, and one control variable, c:

$$\tilde{V}_\tau^{ab}[A_0, A(t_0)] = \max_{[c(t)]_{t=0}^{t_0}} \int_0^{t_0} \frac{c^{1-\sigma}-1}{1-\sigma} e^{-\rho t}\,\mathrm{dt} \tag{41}$$

$$\text{subject to } \begin{cases} \dot{k} = -c \\ \dot{A} = \dfrac{\theta k^\alpha A^{1-\alpha}}{\phi_\tau(A)} \\ k(0) = k_0,\ A(0) = A_0 \\ k(t_0) = \tilde{k}_\tau[A(t_0)],\ c(t_0) = \tilde{c}_\tau[A(t_0)], \end{cases}$$

where the superscript '*ab*' is used to refer any function related to dynamics *above* the turnpike, with the additional constraint $0 \leq c \leq k$, where again the time argument has been dropped for simplicity. The terminal conditions $k(t_0) = \tilde{k}_\tau[A(t_0)]$ and $c(t_0) = \tilde{c}_\tau[A(t_0)]$ bound our trajectory to reach the turnpike $\tilde{k}_\tau(A)$ at time $t = t_0$ with the same consumption value as the optimal consumption $\tilde{c}_\tau[A(t_0)]$ on the turnpike evaluated according to (39) on $A = A(t_0)$. The

[7] See Proposition 1 on p. 3464 in [16].

latter condition holds because the control c—the optimal consumption—is continuous for all $t \geq 0$, as noted in Remark 1.

Necessary conditions on the current-value Hamiltonian associated to (41) yield the following optimal dynamics:

$$\begin{cases} \dot{k} = -c \\ \dot{A} = \dfrac{\theta k^{\alpha} A^{1-\alpha}}{\phi_{\tau}(A)} \\ \dot{c} = \dfrac{c}{\sigma}\left[\theta\alpha\left(\dfrac{k}{A}\right)^{\alpha-1} - \rho\right], \end{cases} \tag{42}$$

which is a boundary problem in three variables, k, A, c, and one unknown, the terminal instant at which the turnpike is entered, t_0, with four boundary conditions—the initial and terminal conditions in problem (41).

To approximate the solution of (41) we take the ratios $\dot{k}/\dot{A}$ and $\dot{c}/\dot{A}$ in system (42) and study the following system of ODEs in the functions $k(A)$ and $c(A)$ (see [7]):

$$\begin{cases} k'(A) = -\dfrac{c(A)\,\phi_{\tau}(A)}{\theta k(A)^{\alpha} A^{1-\alpha}} \\ c'(A) = \dfrac{\left\{\theta\alpha\left[k(A)/A\right]^{\alpha-1} - \rho\right\} c(A)\,\phi_{\tau}(A)}{\theta k(A)^{\alpha} A^{1-\alpha}}. \end{cases} \tag{43}$$

Provided that we know the value $A_{\tau}^{r} = A(t_0)$ on which at instant t_0 the trajectories starting from (A_0, k_0) in $t = 0$ and defined by the dynamics above stop and switch regime becoming the optimal dynamics along the turnpike $\tilde{k}_{\tau}(A)$ discussed in the previous subsection, to solve (43) we apply a *Projection method* based on *Chebyshev Orthogonal Collocation* on n collocation points over the (compact) interval $[A_0, A_{\tau}^{r}]$ applied to the two residual functions—one for each policy $k_{\tau}^{ab}(A)$ and $c_{\tau}^{ab}(A)$ to be estimated—built upon approximation functions which are linear combinations of n Chebyshev polynomials. As initial condition for the *Maple 'fsolve' routine* used to numerically solve the system of $2n+2$ equations setting the two residual functions equal to zero on each collocation node plus the two terminal conditions $k_{\tau}^{ab}(A) = \tilde{k}_{\tau}(A_r)$ and $c_{\tau}^{ab}(A) = \tilde{c}_{\tau}(A_r)$ available from the calculations in the previous subsection, we use a *Chebyshev regression of order* n on the lines crossing the pairs of points $\left(A_0, \tilde{k}_{\tau}(A_0)\right)$, $\left(A_r, \tilde{k}_{\tau}(A_r)\right)$ and $(A_0, \tilde{c}_{\tau}(A_0))$, $(A_r, \tilde{c}_{\tau}(A_r))$ for the $k_{\tau}^{ab}(A)$ and $c_{\tau}^{ab}(A)$ policies respectively. Further details can be found in Section 5 of [12].

Note that the terminal conditions $k_{\tau}^{ab}(A) = \tilde{k}_{\tau}(A_r)$ and $c_{\tau}^{ab}(A) = \tilde{c}_{\tau}(A_r)$—and thus the whole policies $k_{\tau}^{ab}(A)$ and $c_{\tau}^{ab}(A)$ over $[A_0, A_{\tau}^{r}]$—crucially depend on the number A_{τ}^{r}. As a matter of fact, both the value A_{τ}^{r} and the instant t_0 at which A_{τ}^{r} is reached are unknown. The following Algorithm 1, based on a bisection routine, aims at finding A_{τ}^{r} by searching the unique zero of the function

$$f(A_{\tau}^{r}) = k_{\tau}^{ab}(A_0) - k_0, \tag{44}$$

where $k_{\tau}^{ab}(A_0)$ is the initial capital level corresponding to A_0 at $t = 0$ along the backward-in-time trajectory from the point $\left(A_{\tau}^{r}, \tilde{k}_{\tau}(A_{\tau}^{r})\right)$ on the τ-valued turnpike according to the capital optimal policy $k_{\tau}^{ab}(A)$ solving (43), and k_0 is the initial capital value set in problem (41). In other words, the algorithm runs several estimates of the solution of system (43)—corresponding to different intersection points $\left(A_{\tau}^{r}, \tilde{k}_{\tau}(A_{\tau}^{r})\right)$—and each estimated policy $k_{\tau}^{ab}(A)$ is then evaluated (backward-in-time) at the initial knowledge value A_0 until it matches the true initial capital value k_0. Besides yielding A_τ^r, clearly Algorithm 1 provides also the associated optimal policies $k_\tau^{ab}(A)$ and $c_\tau^{ab}(A)$ over $[A_0, A_\tau^r]$ as its output.

Because, by construction, we are considering initial capital endowments that lie 'above' the turnpike at $t = 0$, $k_0 > \tilde{k}_\tau(A_0)$, so that $f(A_0) < 0$, A_0 clearly plays a useful role as left endpoint of the initial interval bracketing the unique zero of f in (44). As far as the right endpoint of such bracket is concerned, we shall see from the analysis in Subsection 6.2 (see Table 1) that the knowledge value corresponding to the Skiba-point on each turnpike considered in Section 6 turns out to be larger than our choice of initial stock of knowledge, $\tilde{A}_\tau^{sk} > A_0$, for the τ-values there considered. Therefore, we can start from the known value $\tilde{A}_\tau^{sk}$ and then add subsequent increments until a knowledge point A_R is found such that $f(A_R) > 0$, then we set $[A_0, A_R]$ as the initial bracketing interval for the zero of f we are looking for.

Algorithm 1 (Finds intersection point A_τ^r and policies $k_\tau^{ab}(A)$, $c_\tau^{ab}(A)$ over $[A_0, A_\tau^r]$)

Step 1 (Initialization): *Choose $0 \le \tau < 1$ and set $[A_L, A_R] = \left[A_0, \tilde{A}_\tau^{sk}\right]$, with $\tilde{A}_\tau^{sk}$ being the knowledge stock corresponding to the (unique) Skiba-point on the turnpike $\tilde{k}_\tau(A)$, as the initial interval for searching the interval bracketing the zero of f in (44), set a (switch) variable $B = 1$, choose an increment $\epsilon > 0$, choose stopping rule parameters $0 < \varepsilon, \eta < 1$, and set (fake) initial values $f(A_\tau^r) = f(A_R) = 1 > \eta$.*

Step 2 (Bisection Loop): *While $A_R - A_L > \varepsilon$ and $|f(A_\tau^r)| > \eta$ do:*

1. *if $B = 1$ then set $A_R = A_R + \epsilon$ (increase right bound) and $A_\tau^r = A_R$, else set $A_\tau^r = (A_R - A_L)/2$ (compute midpoint),*
2. *approximate policies $k_\tau^{ab}(A)$ and $c_\tau^{ab}(A)$ over $[A_0, A_\tau^r]$ by solving (43) through the Collocation-Projection method described above,*
3. *compute $k_\tau^{ab}(A_0)$ evaluating the policy $k_\tau^{ab}(A)$ found in the previous step on $A = A_0$,*
4. *update $f(A_\tau^r)$ in (44) by setting $f(A_\tau^r) = k_\tau^{ab}(A_0) - k_0$,*
5. *if $B = 1$ and $f(A_\tau^r) < 0$ then (keep searching for bracket right endpoint) go to step 2, else (bisection loop)*
   - if $B = 1$ set $B = 0$ (stop searching for bracket),
   - refine the bounds: if $f(A_\tau^r) f(A_R) < 0$ then set $A_L = A_r$, else set $A_R = A_r$ and update $f(A_R)$ by setting $f(A_R) = f(A_\tau^r)$.

Step 3: *Report the intersection point value A_τ^r from step 2.1 and optimal policies $k_\tau^{ab}(A)$ and $c_\tau^{ab}(A)$ over $[A_0, A_\tau^r]$ from step 2.2.*

Once we have the functions $k_\tau^{ab}(A)$ and $c_\tau^{ab}(A)$ over $[A_0, A_\tau^r]$ we can numerically compute the optimal time-path trajectories $A_\tau^{ab}(t)$, $k_\tau^{ab}(t)$, $y_\tau^{ab}(t)$ and $c_\tau^{ab}(t)$ between $t = 0$ and the instant t_0 at which each turnpike is reached. To get the optimal time-path trajectory of the stock of knowledge $A_\tau^{ab}(t)$ along this early transition dynamic for the economy, $k_\tau^{ab}(A)$ is substituted into the first equation of (42) so to obtain a ODE with respect to time which can be numerically solved through the standard *Fehlberg fourth-fifth order Runge-Kutta method with degree four interpolant* method available in , using the initial condition $A_\tau^{ab}(0) = A_0$ in $t = 0$ over a tentative time range of $[0, t_{\max}]$, with the right endpoint larger than the intersection instant t_0, that is, such that $t_{\max} > t_0$. The choice of $t_{\max}$ comes from a guess-and-try approach and is delicate issue as it must be larger than t_0—which is still unknown—but not too large to prevent the

Runge-Kutta algorithm to collapse (see also Remark 4 in [12]). Next, we eventually calculate t_0 by solving $A_\tau^{ab}(t) = A_r$ with respect to t through the *'fsolve' routine* over $[0, t_{\max}]$. The other optimal time-path trajectories over $[0, t_0]$ are then computed as $k_\tau^{ab}(t) = k_\tau^{ab}\left[A_\tau^{ab}(t)\right]$, $y_\tau^{ab}(t) = G_\tau^{ab}(t) = \theta k_\tau^{ab}(t)^\alpha A_\tau^{ab}(t)^{1-\alpha}$ and $c_\tau^{ab}(t) = c_\tau^{ab}\left[A_\tau^{ab}(t)\right]$ respectively.

The whole optimal time-path trajectories, $A_\tau(t)$, $k_\tau(t)$, $y_\tau(t)$, $c_\tau(t)$ and $G_\tau(t)$, for all $t \geq 0$ are then built as *piecewise functions* by joining each trajectory above the turnpike over $[0, t_0]$ with its 'continuation' along the turnpike over $(t_0, +\infty)$ at the instant $t = t_0$, the latter being obtained through the procedure discussed at the end of Subsection 3.3 and shifted forward in time up to the instant at which the knowledge stock value A_τ^r is reached. As $G(t) = y(t)$ for $0 \leq t < t_0$ while $G(t) = \tilde{G}(t) < y(t)$ for $t \geq t_0$, with $\tilde{G}(t)$ given by (13), we expect to observe a discontinuity 'jump' for the control G at the instant $t = t_0$ [see necessary conditions (7)], while all other trajectories must exhibit a kink on $t = t_0$, where they are not differentiable (see Figures 6 and 7 in Section 6).

We shall see in Subsection 6.3 that the speed of the economy's growth is heavily affected by whether the initial condition (A_0, k_0) lies either 'on' or 'above' the turnpike at $t = 0$. Because, when k lies above the turnpike, for the social planner it is optimal to invest all output in R&D activities, when $k_0 > \tilde{k}_\tau(A_0)$ the time period, t_0, required to reach the turnpike is much shorter than the time period necessary to reach the same point on the turnpike when starting already on the turnpike, along which it is optimal to invest only a fraction of the output in R&D activities.

Finally, to estimate the social welfare associated to the whole piecewise-built time-path trajectory toward the ABGP starting at $t = 0$ from (A_0, k_0), we apply the Principle of Optimality and again Proposition 4 in [12]. Specifically, after having computed the intersection instant t_0, we can exploit the early optimal consumption trajectory $c_\tau^{ab}(t)$ just calculated and conveniently define the welfare as the sum of two terms:

$$\tilde{V}_\tau^{ab}(A_0, A_\tau^r) = \int_0^{t_0} e^{-\rho t} \frac{c_\tau^{ab}(t)^{1-\sigma} - 1}{1-\sigma} \, \mathrm{d}t + e^{-\rho t_0} \tilde{V}_\tau(A_\tau^r), \tag{45}$$

where $\tilde{V}_\tau(A_\tau^r)$ is the value function of problem (17) according to (40) evaluated at the intersection point A_τ^r obtained from Algorithm 1. That is, at $t = t_0$ we consider the welfare generated along the turnpike when it starts with initial stock of knowledge A_τ^r, and discount this value in $t = 0$. The first integral on the RHS of (45) is approximated through a *Gauss-Legendre quadrature* routine on a large number of nodes over the time range $[0, t_0]$, using the time-path trajectory value of optimal consumption, $c_\tau^{ab}(t)$, defined before on each node.

5 Dynamics and Welfare Toward Stagnation

Finally, we must also evaluate social welfare toward the steady state $\left(A_0, \hat{k}(A_0)\right)$ on the stagnation line starting from the initial point (A_0, k_0) in $t = 0$. This value must be compared with those obtained from trajectories leading to the ABGP as results of the analysis in Subsection 3.3 and Section 4 in order to check whether the Skiba condition is satisfied. According to condition $\delta/\phi_\tau(A) < \lambda$ in (7), the optimal dynamics characterizing this scenario follow a constant zero-R&D investment policy, $G(t) \equiv 0$, and are just standard saddle-path stable trajectories of a typical Ramsey [13] model in which the level A_0 of knowledge stock remains constant through time. That is, they are solutions of the following problem in the two variables k (state) and c

(control), and the usual dynamic constraint:

$$\bar{V}(A_0, k_0) = \max_{[c(t)]_{t=0}^{t_0}} \int_0^\infty \frac{c^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} \, \mathrm{d}t \tag{46}$$

$$\text{subject to } \begin{cases} \dot{k} = \theta k^\alpha A_0^{1-\alpha} - c, \\ k(0) = k_0. \end{cases}$$

Necessary conditions on the current-value Hamiltonian associated to (46) easily lead to the following well-known conditions:

$$\begin{cases} \dot{k} = \theta k^\alpha A_0^{1-\alpha} - c \\ \dot{c} = \dfrac{c}{\sigma}\left[\theta\alpha\left(\dfrac{k}{A_0}\right)^{\alpha-1} - \rho\right]. \end{cases} \tag{47}$$

Rescaling the variables k and c in system (47) by the ratios $\mu = k/A_0$ and $\chi = c/A_0$ and keeping A_0 constant, we can evaluate the optimal policy associated to (46) in the same 'detrended' (μ, χ) space that contains the optimal policy $\tilde{\chi}_\tau(\mu)$ of model (17) given by the solution of (38) obtained in Subsection 3.3. A constant stock of knowledge $A \equiv A_0$ implies $\dot{A} \equiv 0$, which allows to rewrite system (47) as

$$\begin{cases} \dot{\mu} = \theta\mu^\alpha - \chi \\ \dot{\chi} = \dfrac{\chi}{\sigma}\left(\theta\alpha\mu^{\alpha-1} - \rho\right). \end{cases}$$

Again we take the ratio of equations above and study the unique ODE characterizing the optimal policy in this scenario:

$$\chi'(\mu) = \frac{\left(\theta\alpha\mu^{\alpha-1} - \rho\right)\chi(\mu)}{\sigma\left[\theta\mu^\alpha - \chi(\mu)\right]}. \tag{48}$$

We approximate the solution of (48) through a *Projection method* based on *Chebyshev Orthogonal Collocation* on n collocation points applied to a residual function built upon an approximation function which is a linear combination of n Chebyshev polynomials translated over the (compact) interval $[\hat{\mu}, \mu_{\max}]$, where $\hat{\mu} = \hat{k}(A)/A$ is the steady state defined in (35), which happens to be neutral with respect to the optimal dynamics on the turnpike discussed in Section 3, representing all points on the stagnation line in the detrended space[8] (μ, χ), for all A, while $\mu_{\max}$ is the upper-bound choice for the (global) analysis along the turnpike that will be discussed in Subsection 6.1. Such choice for the right endpoint $\mu_{\max}$ of the range over which a solution of (48) is being approximated allows for a direct comparison between trajectories that start on or above the turnpike and either diverge toward the ABGP or converge to stagnation, as will become clear in Subsection 6.2. See also footnote 10 in [12] for further technical details.

The *optimal consumption policy* for problem (46), which is a function of the only variable k, is then obtained as

$$\bar{c}(A, k) = \bar{\chi}\left(\frac{k}{A}\right) A, \tag{49}$$

where its dependence on the initial stock of knowledge A—which remains constant with respect to optimal dynamics—has been emphasized in order to eventually reach a formulation for a

[8]See footnotes 13 and 14 in [12].

value function that is a function of A. To approximate the value function $\bar{V}(A,k)$ of problem (46) again we exploit the HJB equation as (see equation (45) in [12])

$$\bar{V}(A,k) = \frac{1}{\rho}\left[\frac{\bar{c}(A,k)^{1-\sigma}-1}{1-\sigma} + \frac{\theta k^{\alpha}A^{1-\alpha}-\bar{c}(A,k)}{\bar{c}(A,k)^{\sigma}}\right], \tag{50}$$

where $\bar{c}(A,k)$ is given by (49).

6 Simulations

We assume the following values for the fundamentals parameters in our economy, which are common in the macroeconomic literature (see, *e.g.*, [8, 10, 11, 12]) and satisfy both the necessary growth condition (14) and condition (48) of Proposition 4 in [12]:[9]

$$\alpha = 0.5, \qquad \rho = 0.04, \qquad \theta = 1, \qquad \sigma = 1 \text{ (log utility)}, \qquad \beta = 0.01429. \tag{51}$$

Our goal is to perform comparative dynamics analysis among different optimal transition trajectories characterized by the same parameters' values as in (51) and starting from the same values of initial stock of knowledge, A_0, and physical capital, k_0, under different values of the policy parameter, $0 \leq \tau < 1$, which are assumed to be constant over time. Specifically, we consider the following three values:

$$\tau = 0, 0.5, 0.9. \tag{52}$$

From Proposition 2 we know that to each τ-value in (52) corresponds a different (transitory) turnpike $\tilde{k}_{\tau}(A)$ as defined in (10), each lying one below the other for increasing values of the parameter τ, with $\tilde{k}_0(A)$—*i.e.*, the turnpike corresponding to $\tau = 0$—on top, characterizing a scenario in which the largest capital/knowledge ratio is optimal to sustain growth, and $\tilde{k}_{0.9}(A)$ at the bottom, envisaging an equilibrium with smaller capital/knowledge ratio and closer to the the first-best solution. Each curve converges to its own linear asymptotic turnpike, $\tilde{k}_{\tau}^{\infty}(A)$, as defined in (11), which are parallel lines corresponding to long-run balanced growth with the same constant growth rate given by (15):

$$\gamma = 0.0198. \tag{53}$$

Figure 2 depicts in the (A,k) space the stagnation line $\hat{k}(A) = (156.25)\,A$ in dark gray, and three turnpike curves in *light gray*, *dark gray* and *black*, corresponding to $\tau = 0$, $\tau = 0.5$ and $\tau = 0.9$ respectively. We shall identify with these three colors all relevant curves related to these three examples in subsequent figures. To the value for the initial stock of knowledge, A_0, common to all three policy regimes, correspond three capital values on each turnpike: $\tilde{k}_0(A_0) > \tilde{k}_{0.5}(A_0) > \tilde{k}_{0.9}(A_0)$. To simplify our analysis we assume that the initial condition common to all policy scenarios is the point $(A_0,k_0) = \left(A_0, \tilde{k}_0(A_0)\right)$ on the 'highest' turnpike among the three, corresponding to the softer policy regime ($\tau = 0$ under full decentralization); specifically, we set[10] $A_0 = 2.2190$ and $k_0 = 887.6010$. Note that this choice for the A_0-value is such that the corresponding initial capital values on each turnpike lie all above the capital value $\hat{k}(A_0) = 346.7191$ on the stagnation line (the point on the bottom left corner in Figure 2), for all τ-values in (52); that is, $\tilde{k}_{\tau}(A_0) > \hat{k}(A_0)$ for all $\tau = 0, 0.5, 0.9$.

[9]The β value has been chosen in order to contain the error in the simulation of the optimal policy in all three policy regimes considered in (52).

[10]See the discussion after conditions (54) and (55) below.

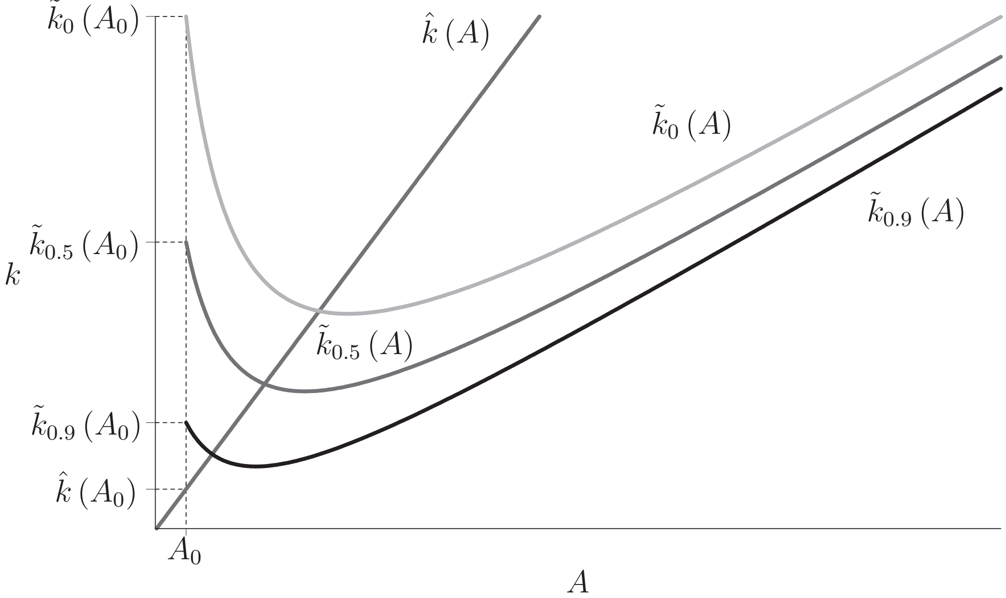

FIGURE 2: the stagnation line and three turnpikes, for $\tau \in \{0, 0.5, 0.9\}$.

6.1 Trajectories Along the Turnpikes

For the parameters' values in (51) we are able to produce satisfactory approximations of the optimal trajectories for our economy in all policy scenarios considered in (52). We exploit the value of the unique steady state $(\mu^*, \chi^*) = (69.9790, 5.5983)$ defined in (34) plus those of all different steady states $(\mu_\tau^s, \chi_\tau^s)$ defined in (36), each corresponding to a τ-value in (52) and satisfying the left inequality in condition (37) for our parameters' values as in (51); specifically, for $\tau = 0$, $\tau = 0.5$ and $\tau = 0.9$ we have $(\mu_0^s, \chi_0^s) = (230.9069, 15.1956)$, $(\mu_{0.5}^s, \chi_{0.5}^s) = (203.9620, 14.2815)$ and $(\mu_{0.9}^s, \chi_{0.9}^s) = (181.5192, 13.4729)$.

In order to allow the Projection algorithm to work fine in all scenarios we establish the initial stock of knowledge, A_0, by choosing a μ-value larger than the largest value for the μ_τ^ss considered, corresponding to the $\tau = 0$ (full decentralization) regime, $\mu_0^s = 230.9069$; specifically, we set $\mu_{\max} = 400$. Using (26) with $\tau = 0$, we get

$$A_0 = \frac{2}{R(\mu_{\max}) - 2} + \frac{3}{2} = 2.2190, \tag{54}$$

to which corresponds the initial capital value

$$k_0 = \tilde{k}_0(A_0) = 887.6010 \tag{55}$$

on the highest turnpike (the light gray curve in Figure 2). A large upper bound for μ, $\mu_{\max} = 400$, has been chosen because it allows each upper bound $\overline{\mu}_\tau = \tilde{k}_\tau(A_0)/A_0$ in the Projection method to be larger than μ_τ^s for all τ-values in (52). Insofar, we obtain policy simulations on the (compact) intervals $\left[\underline{\mu}_\tau, \overline{\mu}_\tau\right]$ given by $[\mu^*, \overline{\mu}_0] = [\mu^*, \mu_{\max}] = [69.9790, 400]$, $[\mu^*, \overline{\mu}_{0.5}] = [69.9790, 283.6535]$ and $[\mu^*, \overline{\mu}_{0.9}] = [69.9790, 190.5762]$, all including the steady states abscissae μ_0^s, $\mu_{0.5}^s$ and $\mu_{0.9}^s$.

Thus, the whole range of the analysis is $[\mu_{\min}, \mu_{\max}] = [\mu^*, \overline{\mu}_0] = [69.9790, 400]$. In all policy scenarios we set $n = 7$ as the largest degree of the Chebyshev polynomials for the OLS-Projection method, while the integral of the squares of the residual function associated to equation (38) is approximated through Gauss-Chebyshev quadrature over 57 nodes on each interval $[\mu^*, \overline{\mu}_\tau]$. We apply a *constrained optimization routine* using the two equality constraints provided by the steady state pairs (μ^*, χ^*) and $(\mu^s_\tau, \chi^s_\tau)$ for each τ-regime. Specifically, we use the *nonlinear programming (NLP) solver* with the *sequential quadratic programming (sqp)* method, exploiting the vector given by a *Chebyshev regression of order* 7 on the segment crossing the two steady states (μ^*, χ^*) and $(\mu^s_\tau, \chi^s_\tau)$ as initial condition for the algorithm. See [11, 12] for all caveats related to such a procedure.

Figure 3 plots in the (μ, χ) space the approximated optimal policies $\chi_\tau(\mu)$ for the three examples considered, $\chi_0(\mu)$, $\chi_{0.5}(\mu)$ and $\chi_{0.9}(\mu)$ (in light gray, dark gray and black respectively), obtained through our procedure together with loci and steady states of all three cases (the latter are the balls colored light gray to black). Accuracy tests show that our results exhibit a worst (largest) maximum error of 0.015, corresponding to the highest peak of the residual function[11] for the $\tau = 0$ case. The other τ-values yield better results, with the best (smallest) maximum error of 0.001 when $\tau = 0.9$. Like in similar works, residual functions do not really oscillate around zero, exhibiting a qualitative pattern similar to that in Figure 3 of [11], while the 8 coefficients associated to each Chebyshev polynomial of the approximation function follow a uniformly decreasing pattern at least up to the fifth coefficient, after which they start to oscillate. All in all, these results exhibit a slightly better performance than those from the analogous computations in [11], while they are definitely worse than those in [12].

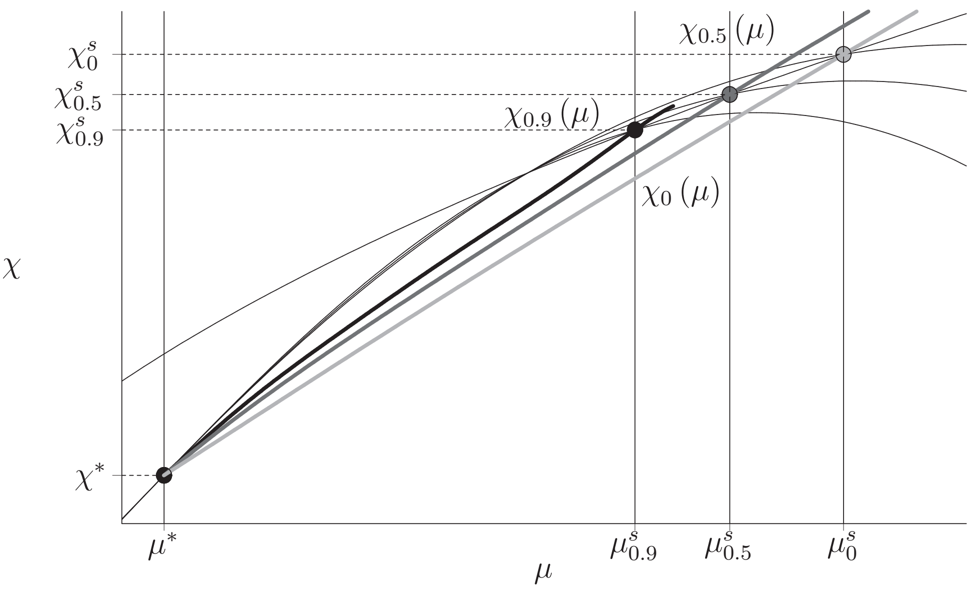

FIGURE 3: phase diagrams, steady states and approximate detrended policies for $\tau \in \{0, 0.5, 0.9\}$.

According to Subsection 3.3, the analysis proceeds by numerically approximating the optimal consumption policies $\tilde{c}_\tau(A)$ in (39) starting from $\left(A_0, \tilde{k}_\tau(A_0)\right)$ in $t = 0$ on each turnpike and then evolving along the turnpike itself for each policy regime envisaged in (52). These policies are then employed in the HJB equation (40) to determine the value function $\tilde{V}_\tau(A)$ of problem (17), yielding social welfare as a function of the stock of knowledge $A \geq A_0$ when the economy evolves along each turnpike toward the ABGP. Such results will be used later either together with welfare estimations when the economy ends up in stagnation to evaluate Skiba-type points on each turnpike in Subsection 6.2, or, according to (45), together with welfare estimations along early trajectories outside the turnpike in Subsection 6.4.

[11]Recall that the (absolute value of the) residual function tell us how far from the true policy our approximation is in each policy scenario: the larger the residual, the worse the approximation.

Applying the method described at the end of Subsection 3.3 we obtain all approximate time-path trajectories: Figures 4(a) and 4(b) report $\mu_\tau(t)$ and $\chi_\tau(t)$ for $0 \le t \le 400$ and $\tau \in \{0, 0.5, 0.9\}$, Figure 4(c) draws the $\tilde{A}_\tau(t)$ trajectories all starting from $A_0 = 2.2190$ in $t = 0$, while Figures 4(d), 4(e) and 4(f) show the corresponding optimal capital, $\tilde{k}_\tau(t)$, output, $\tilde{y}_\tau(t)$, and consumption, $\tilde{c}_\tau(t)$, trajectories.

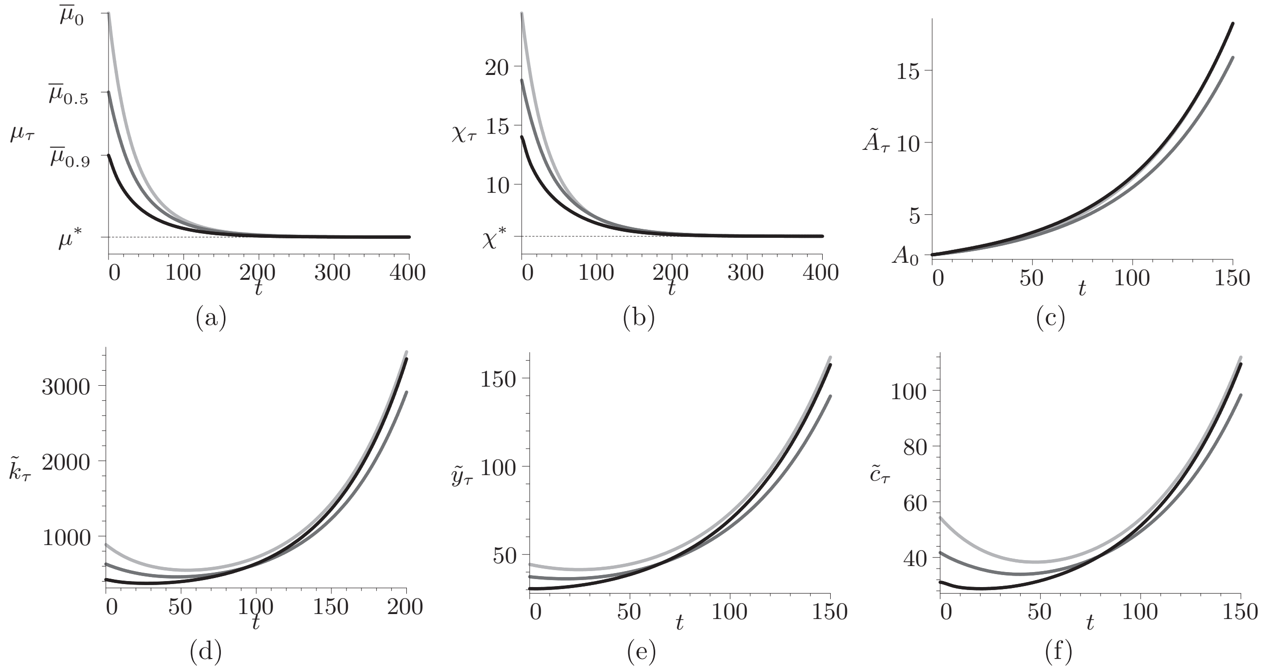

FIGURE 4: time-path trajectories of (a) $\mu_\tau(t)$, (b) $\chi_\tau(t)$, (c) $\tilde{A}_\tau(t)$, (d) $\tilde{k}_\tau(t)$, (e) $\tilde{y}_\tau(t)$ and (f) $\tilde{c}_\tau(t)$ along the turnpikes, for $\tau = 0$ (light gray), $\tau = 0.5$ (dark gray) and $\tau = 0.9$ (black).

Note that, because here our goal was to approximate the optimal time-path trajectories along each turnpike independently of other conditions, we have assumed that $t_0 = 0$ in all three scenarios. Therefore, in Figure 4(d) the initial capital values in $t = 0$ are all different from each other, as they correspond to their own turnpike value on the same initial stock of knowledge $A_0 = 2.2190$: $\tilde{k}_0(A_0) = k_0 = 887.6010 > \tilde{k}_{0.5}(A_0) > \tilde{k}_{0.9}(A_0)$ (see Figure 2). Similarly, also the initial output and consumption values in Figures 4(e) and 4(f) are different. Hence, such trajectories do not provide the correct information aimed at performing comparative dynamics, as, unlike assuming identical initial conditions as in the following Subsections 6.3 and 6.4, they start from different initial capital endowments.

6.2 Skiba-Points On the Turnpikes

Our next goal is to approximate the function $\bar{c}(A, k)$ according to (49) in Section 5 yielding the optimal policy toward the stagnation point $\left(A, \hat{k}(A)\right) = (A, (156.25)A)$ on the stagnation line, starting from any initial pair (A, k) such that $A \ge A_0$ and with k/A ratio values in the range[12] $\left[\hat{k}(A)/A, k_0/A_0\right]$, that is, as $k_0/A_0 = 887.6010/2.2190 = 400$, for $A \ge A_0$ and $k/A \in [156.25, 400]$. Recall that in all such computations A is being kept constant; also note that this

[12]This range is justified by the assumption that all initial capital values considered in our simulation lie above the steady state value $\hat{k}(A) = (156.25)A$ on the stagnation line and are equal or less than the upper bound $k_0 = 887.6010$.

routine can be run just once for all policy regimes, as the optimal policy toward stagnation, in its detrended variables version, turns out to be the same for all scenarios. Hence, following Section 5, we numerically approximate the solution of the ODE (48) through a Collocation-Projection method based on $n = 14$ collocation points over the (compact) interval $[\hat{\mu}, \mu_{\max}] = [156.25, 400]$, where $\hat{\mu} = \hat{k}(A)/A$ is defined in (35) while $\mu_{\max} = k_0/A_0$ is the upper-bound already chosen in Subsection 6.1. Such a choice for the range $[\hat{\mu}, \mu_{\max}]$—specifically, for the left endpoint $\hat{\mu}$—is motivated by our assumptions that the trajectories in all scenarios here considered start from points above the stagnation line, implying that $\mu(0) > \hat{k}(A)/A$ always hold. The resulting approximated solution of (48), that we label $\bar{\chi}(\mu)$, exhibits an outstanding performance with a maximum error of 10^{-9} and a residual function symmetrically oscillating around zero, while the 15 coefficients associated to each Chebyshev polynomial of the approximation function uniformly decrease from $a_0 = 19.9320$ to $a_{14} = 8.7 \times 10^{-10}$.

Using the function $\bar{\chi}(\mu)$ just obtained in (49) we get the optimal policy $\bar{c}(A, k)$ that, in turn, allows for the approximation of the value function $\bar{V}(A, k)$ of problem (46), yielding social welfare when the economy evolves toward stagnation as a function of any initial stock of knowledge $A \geq A_0$ and initial capital k such that $k/A \in [156.25, 400]$, through the HJB equation (50).

With both value functions $\tilde{V}_\tau(A)$ and $\bar{V}(A, k)$ obtained so far, we can evaluate the unique Skiba-point on each turnpike corresponding to the values of τ considered in (52). Specifically, because our choice of parameters' values in (51) satisfies condition (48) of Proposition 4 in [12], after reformulating each turnpike according to (10) and using the value functions $\tilde{V}_\tau(A)$ and $\bar{V}(A, k)$ just calculated according to (40) and (50) respectively, we apply the analogous of Algorithm 1 in [12] to approximate the unique $\left(\tilde{A}_\tau^{sk}, \tilde{k}_\tau^{sk}\right) = \left(\tilde{A}_\tau^{sk}, \tilde{k}_\tau\left(\tilde{A}_\tau^{sk}\right)\right)$ pair that satisfies $\tilde{V}_\tau(A) = \bar{V}\left[A, \tilde{k}_\tau(A)\right]$, *i.e.*, that equates the social welfare yield by the trajectory starting on $\left(\tilde{A}_\tau^{sk}, \tilde{k}_\tau^{sk}\right)$ and then following its optimal path along the turnpike toward the ABGP to the welfare generated by the optimal path leading to stagnation.

These findings are listed in Table 1: it is remarkable the decreasing pattern of both the initial stock of knowledge, $\tilde{A}_\tau^{sk}$, and stock of capital, $\tilde{k}_\tau^{sk}$, required to allow the economy to take off toward sustained growth when starting on the turnpike, with respect to increasing τ-values. In other words, larger values of parameter τ relieve the initial conditions required for the economy to grow in the long-run, allowing poorer countries to undertake a growth path. It is also noticeable from the second column in Table 1 that, as $\tilde{A}_\tau^{sk} > A_0 = 2.2190$ for all τ-values in (52), if the economy were supposed to start with an initial capital endowment corresponding to $\tilde{k}_\tau(A_0)$ on each turnpike, none of the scenarios considered in our study would allow a sustained growth pattern under our choice of initial stock of knowledge $A_0 = 2.2190$, as the social planner would find optimal to lead the economy to eventual stagnation, regardless the policy regime chosen.

τ	$\tilde{A}_\tau^{sk}$	$\tilde{k}_\tau^{sk}$
0	3.0149	581.4472
0.5	2.7154	488.8517
0.9	2.3462	400.5483

TABLE 1: Skiba-points in terms of initial knowledge, $\tilde{A}_\tau^{sk}$, and capital, $\tilde{k}_\tau^{sk}$, on each turnpike.

6.3 Trajectories that Start Above the Turnpikes

To complete our study, from now on we assume that the economy starts from the same initial condition on the highest turnpike, that is on the pair $(A_0, k_0) = (2.2190, 887.6010)$—corresponding to the left endpoint on the light gray curve on the top left corner of Figure 2—and perform a truly comparative dynamics analysis across the different values of the policy parameter τ listed in (52). As the value $k_0 = \tilde{k}_0(A_0) = 887.6010$ lies on the turnpike defined by the value $\tau = 0$, only for the last two τ-values, $\tau = 0.5, 0.9$, we must consider an early period of time, from $t = 0$ to $t = t_0 > 0$, during which the optimal time-path trajectories follow the pattern discussed in Section 4 starting from the initial condition (A_0, k_0)—which lies strictly 'above' all values $\tilde{k}_\tau(A_0)$ for $\tau > 0$—before entering their own turnpike at $t = t_0$ and keep growing along it thereafter.

Running the bisection Algorithm 1 presented in Section 4 for $\tau = 0.5, 0.9$ we get two values for the stock of knowledge, $A_\tau^r > A_0$, which are reported in the second column of Table 2, and two optimal policies $k_\tau^{ab}(A)$ and $c_\tau^{ab}(A)$. We used the the last two Skiba values $\tilde{A}_\tau^{sk}$ in the second column of Table 1 as starting point for the search of the right endpoint of the initial bracketing interval $[A_0, A_R]$ for the zero of f in step 1 of Algorithm 1. We actually ran Algorithm 1 for several τ-values other than 0.5 and 0.9. Depending on the τ-value, we set n (the number of nodes and the largest degree of the Chebyshev polynomials in the Collocation-Projection method) between 21 for smaller τ and 24 for larger τ; similarly the step increment ϵ has been set equal to 1 for smaller τ and equal to 2 for larger τ, while the stopping rules has been set at $\varepsilon = \eta = 10^{-7}$ for all τ-values. All policies estimations exhibit quite reasonable maximum errors of order 10^{-6}, corresponding to the highest peak of the residual functions for the $k_\tau^{ab}(A)$ policies [maximum errors for the $c_\tau^{ab}(A)$ policies are on average 10^{-2} smaller], residual functions symmetrically oscillating around zero and all $n + 1$ coefficients associated to each Chebyshev polynomial of the approximation function uniformly decreasing for all τ-values. The time elapsed for running Algorithm 1 varies from 158 seconds when $\tau = 0.3$ to 229 seconds when $\tau = 0.8$ on an Intel Dual-Core CPU machine with 4GB RAM, with numbers of iterations ranging from 27 to 29.

Figure 5(a) shows the functions $k_{0.5}^{ab}(A)$ and $k_{0.9}^{ab}(A)$ representing the optimal capital associated to the stock of knowledge for $A_0 \leq A \leq A_{0.5}^r$ and $A_0 \leq A \leq A_{0.9}^r$, respectively, before entering the turnpikes $\tilde{k}_{0.5}(A)$ and $\tilde{k}_{0.9}(A)$, according to the last two rows in the second column of Table 2, at the points $A_{0.5}^r = 5.2342$ and $A_{0.9}^r = 6.1187$ (thick part of the curves).[13] Similarly, Figure 5(b) reports the functions $c_{0.5}^{ab}(A)$ and $c_{0.9}^{ab}(A)$ defining the optimal consumption associated to the stock of knowledge for $A_0 \leq A \leq A_{0.5}^r$ and $A_0 \leq A \leq A_{0.9}^r$ respectively, before entering their turnpike consumption policies $\tilde{c}_{0.5}(A)$ and $\tilde{c}_{0.9}(A)$ as computed in Subsection 6.1, at the points $A_{0.5}^r$ and $A_{0.9}^r$ (thick part of the curves). Note the kinks at the points A_τ^r in both figures.

Following the steps described after Algorithm 1 in Section 4, we first numerically compute the optimal time-path trajectories $A_\tau^{ab}(t)$, $k_\tau^{ab}(t)$, $y_\tau^{ab}(t)$ and $c_\tau^{ab}(t)$ between $t = 0$ and some instant $t_{\max} > 0$, the latter to be determined through some guess-and-try, being careful to choose a value larger than the (still unknown) instant t_0 at which each turnpike is reached. A value of $t_{\max}$ around 10 worked well in all our simulations. Next, by solving $A_\tau^{ab}(t) = A_r$ with respect to t over $[0, t_{\max}]$, the corresponding instants $t_0 > 0$ at which the optimal trajectories from above intersect their respective turnpikes are found; they are reported in the last two rows of the third column of Table 2 [in the first row $A_0^r = A_0$ and $t_0 = 0$ because, for $\tau = 0$,

[13]The whole turnpikes $\tilde{k}_{0.5}(A)$ and $\tilde{k}_{0.9}(A)$ for $A \geq A_0$ are the union of the thin curves on the left with the thick ones to the right in the figure, corresponding to the dark gray and black curves in Figure 2.

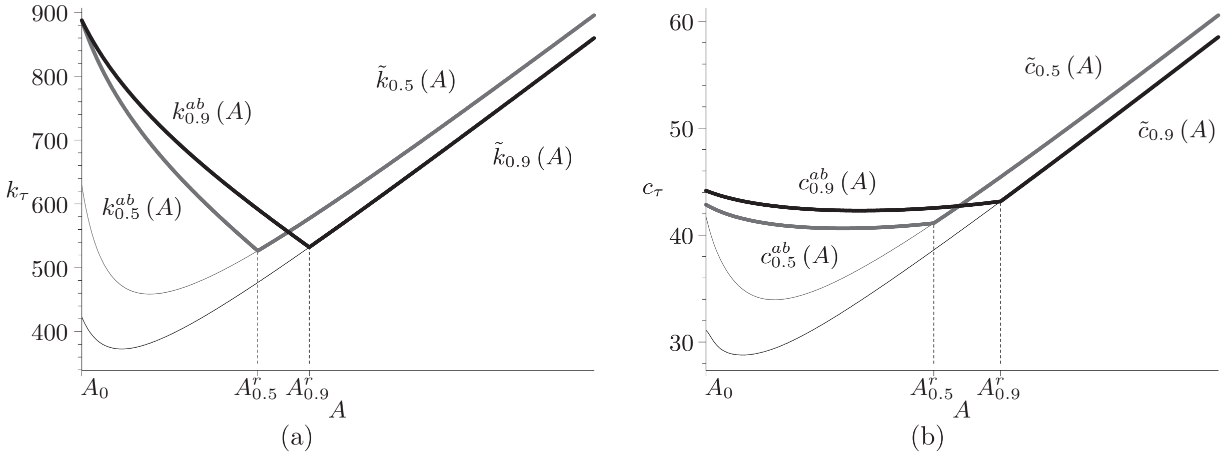

FIGURE 5: functions (a) $k_\tau(A)$ and (b) $c_\tau(A)$ representing optimal capital and consumption before hitting their turnpike values at A^r_τ, after which they become turnpike themselves, for $\tau = 0.5$ (dark gray) and $\tau = 0.9$ (black).

the initial knowledge-capital pair lies exactly on the highest turnpike, $\tilde{k}_0(A_0)$]. Finally, the whole optimal time-path trajectories $A_\tau(t)$, $k_\tau(t)$, $y_\tau(t)$, $c_\tau(t)$ and $G_\tau(t)$ over $(0, +\infty)$ when the economy starts at $t = 0$ from $(A_0, k_0) = (2.2190, 887.6010)$ are obtained by joining the trajectories above the turnpike over $[0, t_0]$ just calculated with their 'continuation' along the turnpike over $(t_0, +\infty)$ provided by the analysis of Subsection 6.1. Figure 6 plots $A_\tau(t)$, $k_\tau(t)$, $y_\tau(t)$ and $c_\tau(t)$: the light gray curves, corresponding to the zero policy parameter regime $\tau = 0$, denote trajectories starting already on the $\tilde{k}_0(A)$ turnpike at $t = 0$ which keep moving along it thereafter, while the dark gray and black curves, corresponding to the positive policy parameter regimes $\tau = 0.5$ and $\tau = 0.9$, are the piecewise trajectories obtained by joining[14] at $t = t_0$ the trajectories above the turnpikes for $0 \le t \le t_0$ with their continuation along the turnpikes for $t \ge t_0$.

It turns out that along the time-path trajectories $\tilde{A}_\tau(t)$, $\tilde{k}_\tau(t)$, $\tilde{y}_\tau(t)$ and $\tilde{c}_\tau(t)$ along each turnpike as computed in Subsection 6.1 for $\tau = 0.5, 0.9$, if the economy were to start already on the turnpike at $t = 0$ (rather than from the capital level $k_0 = 887.6010$ strictly above each turnpike), it would take around ten times longer than t_0 to reach the same intersection points A^r_τ. In other words, the time period t_0 required to enter the turnpikes *starting from above the turnpikes* is around *ten times shorter* than the time period needed to reach the same points when *starting already on the turnpikes*. Indeed, an initial capital endowment k_0 which is is strictly larger than the $\tilde{k}_\tau(A_0)$ level on the turnpike is beneficial in that it implies that it is optimal to invest the maximum amount allowed in the production of new knowledge, which equals output, $G(t) = y(t)$, a policy that translates into steeper $A(t)$, $y(t)$ trajectories for $0 \le t \le t_0$ [dark gray and black curves in Figures 6(a) and 6(c)], indicating a high boost to growth provided during the initial paths covered 'above' the turnpikes. Such initial boost, in turn, explains the better performances of all $A_\tau(t)$, $k_\tau(t)$, $y_\tau(t)$ and $c_\tau(t)$ trajectories under the $\tau > 0$ regimes (*e.g.*, those in dark gray and black in Figure 6) than their counterparts for $\tau = 0$ (in light gray in Figure 6) when t becomes large.

Note the kinks of all the trajectories for $\tau = 0.5$ and $\tau = 0.9$ at the instant t_0 in Figure 6 [especially $A_\tau(t)$, $k_\tau(t)$ in Figures 6(a) and 6(b)], corresponding to the kinks at the points A^r_τ

[14]From the third column in Table 2 we see that all intersection points are reached at instants t_0 that are very close to each other in different policy regimes.

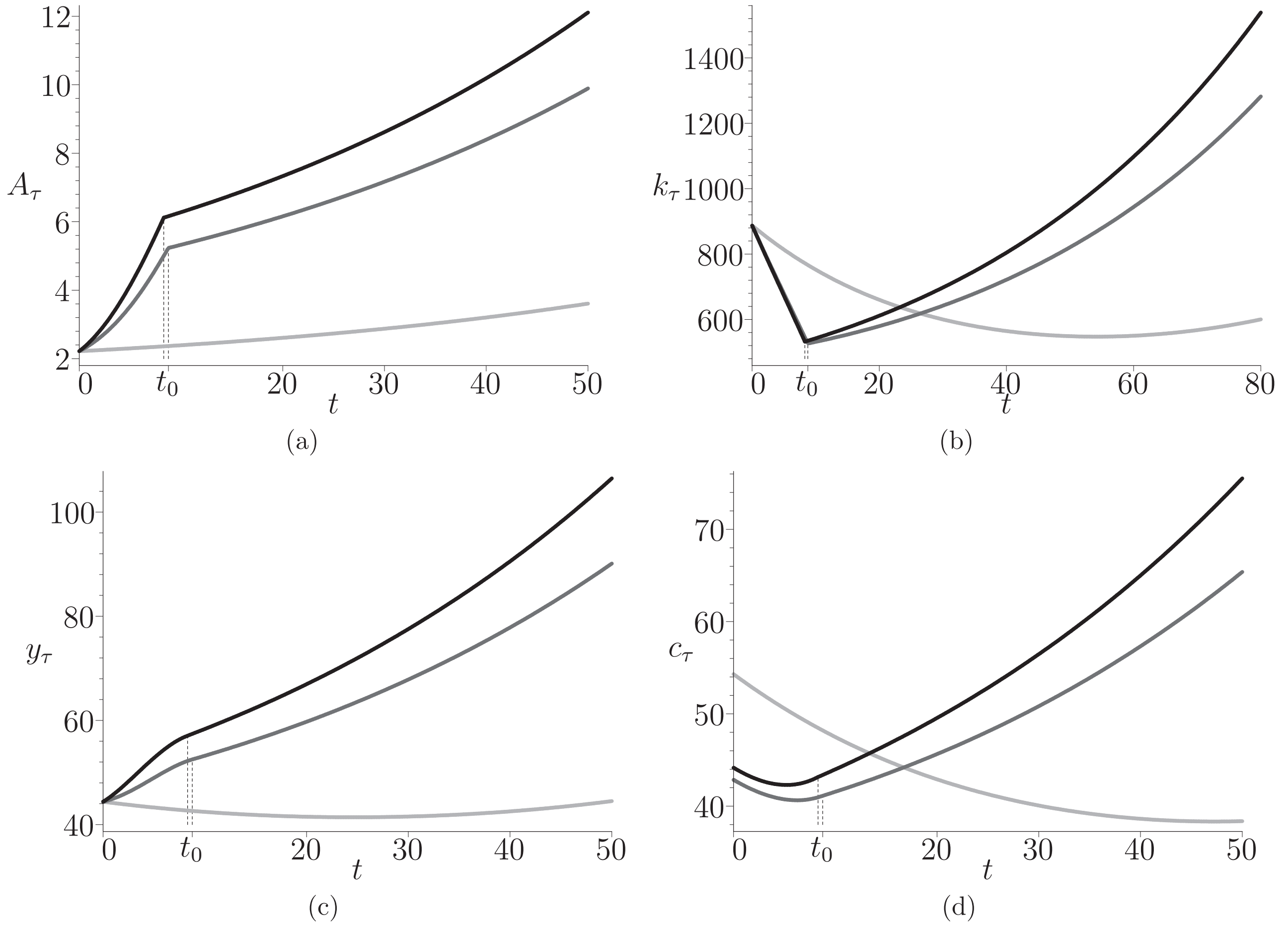

FIGURE 6: whole optimal transition time-path trajectories, (a) $A_\tau(t)$, (b) $k_\tau(t)$, (c) $y_\tau(t)$ and (d) $c_\tau(t)$, under the three policy regimes, $\tau = 0$ (full decentralization) in light gray, $\tau = 0.5$ in dark gray and $\tau = 0.9$ in black, all starting from $A_0 = 2.2190$ and $k_0 = 887.6010$ in $t = 0$.

in Figure 5: on t_0 such trajectories are not differentiable due to the discontinuity jump of the optimal investment in R&D activities time-path trajectory, $G_\tau(t)$, as it is evident from Figure 7.

From figure 6(d), showing the whole time-path trajectories of optimal consumption in the three policy parameter scenarios, however, it is not clear what the effect of the different policies on welfare may be, as in early times the $\tau = 0.5$ and $\tau = 0.9$ regimes envisage a lower consumption than in the $\tau = 0$ regime, only to catch up and rapidly overcome later on. As a matter of fact, the former trajectories may fail to deliver a welfare higher than that yield by the latter if discounting assigns less weight to later consumption than that assigned to early consumption.

6.4 Welfare Estimates

The last two columns of Table 2 report our main welfare results. For the initial condition $(A_0, k_0) = (2.2190, 887.6010)$ common to all τ-values in (52), the social welfare estimates in the fifth column of Table 2 are computed through (45) at the end of Section 4, where, whenever $t_0 > 0$, the first integral on the RHS is approximated through a *Gauss-Legendre quadrature* routine on 1000 nodes over each time range $[0, t_0]$, using the time-path trajectories approximations

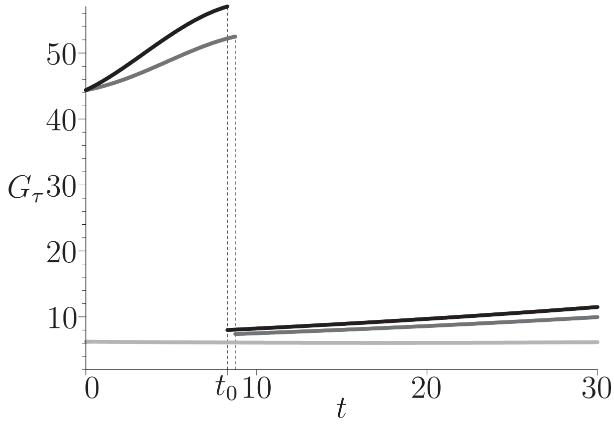

FIGURE 7: whole transition time-path trajectories of optimal R&D financing, $G_\tau(t)$, under the three policy regimes, $\tau = 0$ (full decentralization) in light gray, $\tau = 0.5$ in dark gray and $\tau = 0.9$ in black, all starting from $A_0 = 2.2190$ and $k_0 = 887.6010$ in $t = 0$.

of optimal consumption obtained in the previous subsection, $c_\tau^{ab}(t)$, on each node. Note that the first scenario, corresponding to $\tau = 0$, implies $t_0 = 0$ so that the social welfare along the turnpike and toward the ABGP would actually be given by (40) at the end of Subsection 3.3, yielding $\tilde{V}_0^{ab}(A_0, A_0) = \tilde{V}_0(A_0) = 95.2964$. However, from Table 1 we know that when $\tau = 0$ and $(A_0, k_0) = (2.2190, 887.6010)$ the economy does not satisfy the Skiba condition; this is confirmed by the first value in the fourth column of Table 2, and implies that the first number in the fifth column comes from the value function toward stagnation estimated through (50) at the end of Section 5, $\bar{V}(A_0, k_0) = \bar{V}(2.2190, 887.6010) = 95.8745$, rather than the former smaller estimate $\tilde{V}_0(A_0) = 95.2964$. The other two welfare values in the fifth column, estimated through the value function $\tilde{V}_\tau^{ab}(A_0, A_\tau^r)$ as properly defined in (45), are larger than $\bar{V}(A_0, k_0) = 95.8745$; consistently, $(A_0, k_0) = (2.2190, 887.6010)$ satisfies the Skiba condition when $\tau = 0.5, 0.9$, as confirmed by the last two rows in the fourth column.

The last two rows of Table 2 clearly indicate that the initial push deriving from starting above any turnpike, if on one hand entails sacrificing consumption in early times as shown by Figure 6(d), on the other hand it turns out to be sufficiently strong to 1) let the economy escape the stagnation trap otherwise forecast by Table 1 for all regimes, including the active-policies characterized by $\tau > 0$, and 2) yield a social welfare that is strictly increasing in the τ parameter values, as it is apparent from the fifth column of Table 2 and is confirmed by all other simulations we have run. Indeed, consistently with the first row in Table 1, when $\tau = 0$ our economy happens to be born on the highest turnpike $\tilde{k}_0(A_0)$ and doomed to renounce growth in the long-run as the Skiba condition is not satisfied. Conversely, positive policy regimes implemented by policymakers may let the same economy take off toward long-run sustained growth, as shown in the last two rows of the fourth column of Table 2.

We can show that even by considering a wider set of τ values the results above illustrated are robust. We can thus conclude that the approximated values found through the whole procedure discussed so far lead to the following results:

i) under *a no tax regime* ($\tau = 0$), *the economy ends up in stagnation*; while

ii) already under a *'mild' positive tax regime*($\tau \geq 0.1$), *the economy grows along an ABGP in the long-run*;

iii) *social welfare is strictly increasing with the tax parameter* τ.

τ	A^r_τ	t_0	Toward	Social welfare
0	2.2190 ($= A_0$)	0	stagnation	95.8745
0.5	5.2342	8.76	ABGP	97.8152
0.9	6.1187	8.30	ABGP	99.8471

Table 2: Intersection points A^r_τ between each trajectory from above and its turnpike, instants t_0 at which the intersection occurs, type of equilibrium (toward either stagnation or sustained growth along an ABGP) and social welfare for the common initial condition $(A_0, k_0) = (2.2190, 887.6010)$ in all policy regimes.

This result is somewhat surprising, since it implies that capital crowding out has positive effects on welfare. However, this is explained by the power of recombinant technological progress, which at early times makes it convenient to take away some capital from the output producing sector to be used for consumption, thus freeing resources coming from the current output to invest them into knowledge advances. Note that similar results have been derived by [5], who analyze a similar but simpler setting in which capital is an intermediate non-durable good. The fact that the welfare effects associated with different IPR regimes are qualitatively similar even under different model's formulations suggests that our results are robust, and the main driver of our conclusions is represented by the recombinant nature of technological progress.

7 Concluding Remarks

In this paper we provide a numerical method to assess the impact of different policy regimes on social welfare in a continuous time endogenous recombinant growth model presented á-la [16] extended to allow a decentralized production of knowledge as in [5]. The wide range of techniques used include Projection methods, Gauss-Chebyshev and Gauss-Legendre quadrature, and standard Runge-Kutta type algorithms. Thus, this work represents a further step forward in the analysis of the transitional dynamics of recombinant growth models [10, 11, 12] providing some interesting insights into the evaluation of alternative policies to promote economic growth and ultimately improvements in social welfare.

In order to quantitatively assess welfare effects through a comparative dynamic analysis, we need to compute the entire transitional dynamics of the optimal consumption associated to different policy regimes. This requires to distinguish between trajectories that starts on the turnpike and trajectories starting outside the turnpike, further distinguished between those converging to the turnpike and those ending up in stagnation. We develop an algorithm (based on a bisection routine) to identify the intersection point between paths starting above the turnpike and the turnpike itself in order to build the whole optimal consumption path as a piecewise function of time by joining each trajectory above the turnpike with its continuation along the turnpike. This allows to compute social welfare and thus compare the welfare levels associated to different policy parameters, for a better understanding of which policy regime might be more desirable for our society as a whole. Our simulation based on a certain parametrization of the model allows us to conclude that welfare increases with the policy parameter, thus stricter policy regimes should be preferred in order to maximize social welfare.

By analyzing the effects of different IPRs policy regimes on social welfare to understand whether the decentralized and centralized outcomes differ and by developing a numerical approach to quantify the extent to which they do, our paper fills some existing gaps in the literature on endogenous recombinant growth. However, still little is known about the implications of recombinant knowledge production on economic growth patterns, thus our analysis can be extended along several directions. In particular, the assumption underlying all endogenous recombinant growth models is that output is used to produce new knowledge which clearly makes the Hamiltonian function linear in one control variable; however, knowledge is more realistically produced by using some input (*i.e.*, labor) which, by being optimally allocated across two sectors as in [4] and [17], will allow to relax the bang-bang features of the dynamic optimization problem, eventually yielding simpler and more intuitive transitional dynamics. Also, technological progress is often the result of research and development activities performed by profit-maximizing firms, thus, embedding the knowledge recombinant production function into a framework of monopolistic competition á-la [14] will allow to shed some more light on the long run growth patterns in economies driven by recombinant knowledge production. These further issues are left for future research.

Supplementary Material

The Maple code for all numeric procedures described in this paper, including the detailed code for Algorithm 1 as well as for the computations performed in section 6, is available from the authors upon request.

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