Multivariate Marked Poisson Processes and Market Related Multidimensional Information Flows

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Abstract

The class of marked Poisson processes and its connection with subordinated Lévy processes allow us to propose propose a new interpretation of multidimensional information flows and their relation to market movements. The new approach provides a unified framework for multivariate asset return models in a Lévy economy. In fact, we are able to recover several processes commonly used to model asset returs as subcases. We consider a first application example using the Normal inverse Gaussian specification.

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Introduction

The notion of a stochastic change of time, interpreted as a measure of trading activity, dates back to Clark (1973) who was the first to link the deviation from normality of asset prices to the changes in the number of market orders in different time periods. Since then, subordination of Brownian motions was introduced to model asset returns, with the interpretation of the subordinator as a stochastic change of time. A subordinated Brownian motion is a process $Y(t)$ defined by the composition $Y(t) := B(\pi(t))$, where $\pi(t)$ is the subordinator and the Brownian motion $B(t)$ is called the subordinate process. A prominent example is the variance gamma process proposed by Madan and Seneta (1990) as an asset return model; the subordinator in this case is a gamma process.

The first subordinated multivariate model was constructed by considering a time change common to all assets represented by a univariate subordinator (see Madan and Seneta (1990) and Luciano and Schoutens (2006)). Unfortunately, the resulting models exhibited several shortcomings including the lack of independence between asset returns and a limited span of linear correlations. Furthermore, there is empirical evidence that trading activity is different across assets (e.g., Harris (1986)). From the theoretical perspective multivariate subordination allowing different assets to have different time-changes was introduced in the work of Barndorff-Nielsen et al. (2001). Semeraro (2008) and Luciano and Semeraro (2010) introduced the α-models, using a multivariate subordinator composed of a common component and an idiosyncratic component, named factor-based subordinator. However, to preserve the intuition of economic time, each asset return distribution is time changed by a one-dimensional subordinator. The subordinated process $\boldsymbol{Y}(t)$ is therefore given by the componentwise composition $\boldsymbol{Y}(t) = (B_1(\pi_1(t)), \dots, B_n(\pi_n(t)))$. We refer to this technique as componentwise subordination. Due to this constraint, the subordinate processes, i.e. the Brownian motions, must be independent. This means that returns conditional to trading activity are uncorrelated. As a consequence models constructed by componentwise subordination cannot span a wide correlation range.

To increase correlation flexibility Luciano and Semeraro (2010) extend the α-models dependence structure by introducing the $\rho\alpha$-models. A $\rho\alpha$-model is an $\mathbb{R}^n$-valued subordinated process $\{\boldsymbol{Y}(t), t > 0\}$ defined by

$$\boldsymbol{Y}(t) := (\boldsymbol{Y}^I(t) + \boldsymbol{Y}^\rho(t)) = (B_1(\pi_1(t)) + B_1^\rho(\pi^c(t)), \dots, B_n(\pi_n(t)) + B_n^\rho(\pi^c(t))), \tag{0.1}$$

where $\pi_j(t)$ and $\pi^c(t)$ are both mutually independent subordinators and independent of $B_j(t)$ and $B_j^\rho(t)$. There, $B_j(t)$ are independent Brownian motions and $B_j^\rho(t)$ are Brownian motions with correlations $\rho(B_i(t), B_j(t)) = \rho_{ij}, i, j \in \{1, \dots, n\}$. This preserves the intuition that each asset has its own subordinator, but includes the possibility of co-movements due to the correlated Brownian components.

Similarly, Ballotta and Bonfiglioli (2014) define the multivariate process using a common and an idiosyncratic component of returns. Specifically, they introduce an $\mathbb{R}^n$-valued subordinated process $\{\boldsymbol{Y}(t), t > 0\}$ defined by

$$\boldsymbol{Y}(t) := (X_1(t) + a_1 Z(t), \dots, X_n(t) + a_n Z(t)), \tag{0.2}$$

where $X_j(t)$ and $Z(t)$ are mutually independent Lévy processes. With the aim to preserve componentwise subordination, Buchmann et al. (2016) introduced the weak subordination, a new technique which allows to correlate Brownian motions and preserve the intuition of change of time.

The authors introduce a first example, the weak α - variance gamma process, which is between the α and the $\rho\alpha$-variance gamma processes.

An alternative approach to correlate the unit-time random variables resulting from univariate subordinated Brownian motions is proposed by Eberlein and Madan (2010). They model individual returns as one dimensional subordinated Brownian motions

$$Y_j(t) := \mu_j \pi_j(t) + \sigma_j W_j(\pi_j(t)), \;\; j = 1, ...n,$$

and assume that the subordinators are independent. They introduce dependence between returns at unit time by merely correlating the Brownian motions and keeping the subordinators independent. Therefore

$$\mathcal{L}(Y_j) = \mathcal{L}(\mu_j \pi_j + \sigma_j \sqrt{\pi_j} W_j), \;\; j = 1, ...n,$$

where W_j are standard normal variates with correlations ρ_{ij}^W.

Within this framework, we propose a new interpretation of trade activity which allows us to correlate asset returns similarly to Eberlein and Madan (2010) and remain in the Lévy setting. A link between subordinated Brownian motions and marked Poisson processes allows us to interpret the subordinator as the entire trade activity on a portfolio of assets up to a given time, measured by the corresponding Poisson random measure. By so doing, we introduce a new interpretation of the subordinator $\pi(t)$ which is still consistent with the intuition of economic time. By means of this new interpretation of trading activity, we propose to use multivariate subordination (not-componentwise) in Barndorff-Nielsen et al. (2001). We show that multivariate subordination processes provides a unified framework and includes as subcases most of the processes listed above. The new class of processes is fully characterized through its Lévy triplet, and the characteristic function is given in closed form.

The paper is organized as follows. Section 1 recalls preliminary results needed to introduce the model. Section 2 introduces the class of Lévy marked Poisson models and their link with subordinated Lévy processes. The main submodels are presented in Section 3. Section 4 specifies a flexible class of Lévy marked Poisson processes suitable to model stock returns. The characteristic function is provided in closed form as well as the linear correlation coefficient. This section also specifies marks and Poisson measure to find a multivariate version of the normal inverse Gaussian processes. In Section 5 we perform a calibration exercise of the NIG model to illustrate the flexibility of the model dependence structure.

1 Preliminaries

We refer to Sato (1999) for Lévy processes and subordination and to Çinlar (2011) for Poisson processes and their connection with Lévy processes.

Let Π be a Poisson random measure on a measurable space $(E, \mathcal{E})$ with a σ-finite mean measure μ_Π. By slight abuse of notation, with $\Pi = \{\Pi_i, i \in I\}$ we indicate both the random measure and the collection of its atoms indexed by some countable set I. Marked Poisson processes are constructed by attaching a random variable to each atom of the random measure Π. Formally, let $Z = \{Z_i, i \in I\}$ be a family of random variables (marks) on a measurable space $(F, \mathcal{F})$ indexed by the same countable set I. Assume that the variables Z_i are conditionally independent given Π

with distributions $Q(\Pi_i, \cdot)$, where $Q(\boldsymbol{s}, B)$ is a transition probability kernel from $(E, \mathcal{E})$ into $(F, \mathcal{F})$. Each variable Z_i can be considered as an indicator of some property associated with the atom Π_i. Then, as proved in Theorem 3.2 in Çinlar (2011), $\boldsymbol{M} := (\Pi, Z)$ forms a Poisson random measure on $(E \times F, \mathcal{E} \otimes \mathcal{F})$ with mean $\mu_\Pi \times Q$, where $(\mu_\Pi \times Q)(dx, dy) = \mu_\Pi(dx)Q(x, dy)$. The new measure $\boldsymbol{M}$ is called marked Poisson random measure.

Let us recall that the subordination of a Lévy process $\boldsymbol{L} = \{\boldsymbol{L}(t), t \geq 0\}$ by a univariate subordinator $\pi(t)$, i.e., a Lévy process on $\mathbb{R}_+ = [0, \infty)$ with increasing trajectories, defines a new process $\boldsymbol{X} = \{\boldsymbol{X}(t), t \geq 0\}$ by the composition $\boldsymbol{X}(t) := (L_1(\pi(t)), \ldots, L_n(\pi(t)))^T$. Theorem 30.1 in Sato (1999) characterizes the subordinated process $\boldsymbol{X}$ in terms of its Lévy triplet. Barndorff-Nielsen et al. (2001) generalize the above construction by allowing the introduction of multivariate subordinators, i.e., Lévy processes on $\mathbb{R}^n_+ = [0, \infty)^n$, whose trajectories are increasing in each coordinate. For purposes of introduction of multivariate subordination, the notion of $\mathbb{R}^d_+$-parameter process, as introduced in Barndorff-Nielsen et al. (2001), is required. Consider the multiparameter $\boldsymbol{s} = (s_1, ..., s_d)^T \in \mathbb{R}^d_+$ and the partial order on $\mathbb{R}^d_+$

$$\boldsymbol{s}^1 \preceq \boldsymbol{s}^2 \;\; \Leftrightarrow \;\; s^1_j \leq s^2_j, \, j = 1, \ldots, d.$$

Let now $\boldsymbol{L}(\boldsymbol{s}) = (L_1(\boldsymbol{s}), L_2(\boldsymbol{s}), \ldots, L_n(\boldsymbol{s}))^T$ be a process with parameters in $\mathbb{R}^d_+$ and values in $\mathbb{R}^n$. It is called an $\mathbb{R}^d_+$-parameter Lévy process on $\mathbb{R}^n$ if the following hold

- for any $m \geq 3$ and for any choice of $\boldsymbol{s}^1 \preceq \ldots \preceq \boldsymbol{s}^m$, $\boldsymbol{L}(\boldsymbol{s}^j) - \boldsymbol{L}(\boldsymbol{s}^{j-1})$, $j = 2, ..., m$, are independent,
- for any $\boldsymbol{s}^1 \preceq \boldsymbol{s}^2$ and $\boldsymbol{s}^3 \preceq \boldsymbol{s}^4$ satisfying $\boldsymbol{s}^2 - \boldsymbol{s}^1 = \boldsymbol{s}^4 - \boldsymbol{s}^3$, $\mathcal{L}(\boldsymbol{L}(\boldsymbol{s}^2) - \boldsymbol{L}(\boldsymbol{s}^1)) = \mathcal{L}(\boldsymbol{L}(\boldsymbol{s}^4) - \boldsymbol{L}(\boldsymbol{s}^3))$ where $\mathcal{L}(\cdot)$ denotes the law of the random variable,
- $\boldsymbol{L}(\boldsymbol{0}) = \boldsymbol{0}$ almost surely, and
- almost surely, $\boldsymbol{L}(\boldsymbol{s})$ is right continuous with left limits in $\boldsymbol{s}$ in the partial ordering of $\mathbb{R}^d_+$.

Let $\{\boldsymbol{L}(\boldsymbol{s}), \boldsymbol{s} \in \mathbb{R}^d_+\}$ be a multiparameter Lévy process on $\mathbb{R}^n$ with Lévy triplet $(\boldsymbol{\gamma}_{\boldsymbol{L}}, \Sigma_{\boldsymbol{L}}, \nu_{\boldsymbol{L}})$, and let $\boldsymbol{\pi}(t)$ be a d dimensional subordinator independent of $\boldsymbol{L}(\boldsymbol{s})$ having Lévy triplet $(\boldsymbol{\gamma}_{\boldsymbol{\tau}}, 0, \nu_{\boldsymbol{\pi}})$. The subordinated process $\boldsymbol{X} = \{\boldsymbol{X}(t), t \geq 0\}$ defined by

$$\boldsymbol{X}(t) := \boldsymbol{L}(\boldsymbol{\tau}(t)) = \begin{pmatrix} L_1(\pi_1(t), \ldots, \pi_d(t)) \\ \vdots \\ L_n(\pi_1(t), \ldots, \pi_d(t)) \end{pmatrix}, \; t \geq 0$$

is a Lévy process, as proved in Theorem 4.7 in Barndorff-Nielsen et al. (2001), who also provide its characteristic function and Lévy triplet.

2 Subordinated Brownian motion and marked Poisson process

In this section we introduce a multivariate model by multivariate subordination of a multiparameter Brownian motion and we provide the connection with marked Poisson processes and the implied new interpretation of trading activity.

The connection between multivariate subordinated Lévy processes and marked Poisson processes allows us to introduce a new interpretation of trading activity, which is consistent with the notion of stochastic time but more general. This connection lays the foundation of the use of (not-componentwise) multivariate subordination to construct asset return modeling. Advantages of multivariate subordination are the possibility to introduce correlation among prices conditional on trading activity and the characterization of the multivariate resulting process in terms of its characteristic function.

Let $\boldsymbol{B}_i(t)$ be independent Brownian motions on $\mathbb{R}^{n_i}$ with drift $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma_i}$, and let $\boldsymbol{B} = \{\boldsymbol{B}(\boldsymbol{s}), \boldsymbol{s} \in \mathbb{R}^d_+\}$, defined as $\boldsymbol{B}(\boldsymbol{s}) := (\boldsymbol{B}_1(s_1), \ldots, \boldsymbol{B}_d(s_d))^T$, be the associated multi-parameter Lévy process. Let $\boldsymbol{A}_i \in \mathcal{M}_{n\times n_i}(\mathbb{R})$. We can define the process $\boldsymbol{B_A} = \{\boldsymbol{B_A}(\boldsymbol{s}), \boldsymbol{s} \in \mathbb{R}^d_+\}$ as

$$\boldsymbol{B_A}(\boldsymbol{s}) = \boldsymbol{A}_1\boldsymbol{B}_1(s_1) + \ldots + \boldsymbol{A}_d\boldsymbol{B}_d(s_d) \;\; \boldsymbol{s} \in \mathbb{R}^d_+. \tag{2.1}$$

The process $\boldsymbol{B_A}$ is an $\mathbb{R}^d_+$-parameter Lévy process on $\mathbb{R}^n$, see Example 4.4 in Barndorff-Nielsen et al. (2001). Let $\boldsymbol{A} := (\boldsymbol{A}_1, \ldots, \boldsymbol{A}_d)$ and $\boldsymbol{B}$ be block matrices, we can write shortly $\boldsymbol{B_A}(\boldsymbol{s}) = \boldsymbol{A}\boldsymbol{B}(\boldsymbol{s})$. We call the $\mathbb{R}^d_+$-parameter Lévy process $\boldsymbol{B_A}(\boldsymbol{s})$ in (2.1) $\mathbb{R}^d_+$*-parameter Brownian motion.* At this point we can introduce the multiparameter Gaussian kernel corresponding to $\boldsymbol{B_A}(\boldsymbol{s})$.

Definition 2.1. *A process* $\boldsymbol{Y}$ *defined by*

$$\boldsymbol{Y}(t) := \boldsymbol{B_A}(\boldsymbol{\pi}(t)) = \begin{pmatrix} B_{\boldsymbol{A}_1}(\pi_1(t), \ldots, \pi_d(t)) \\ \vdots \\ B_{\boldsymbol{A}_n}(\pi_1(t), \ldots, \pi_d(t)) \end{pmatrix} \tag{2.2}$$

is a subordinated multi-parameter Brownian motion (SMBM), where $\boldsymbol{B_A}$ *is a multi-parameter Brownian motion and* $\boldsymbol{\pi}(t)$ *is a multivariate subordinator independent of* $\boldsymbol{B_A}$*.*

Let $\boldsymbol{\Pi}$ be the Poisson random measure on $(\mathbb{R}_+ \times \mathbb{R}^d_+, \mathcal{B}_{d+1})$ associated to the subordinator $\boldsymbol{\pi}(t)$, where $\mathcal{B}_{d+1}$ is the Borel σ-algebra. It holds

$$\boldsymbol{\pi}(t) := \int_{(0,t]\times\mathbb{R}^d_+} \boldsymbol{x}\boldsymbol{\Pi}(ds, d\boldsymbol{x}), \tag{2.3}$$

The atoms of $\boldsymbol{\Pi}$ are family of random variables $\boldsymbol{\Pi} = \{(\Pi_1, \boldsymbol{\Pi}_2) = \{(\Pi_{1i}, \boldsymbol{\Pi}_{2i}), i \in I\}\}$ on $\mathbb{R}_+ \times \mathbb{R}^d_+$, where Π_{1i} are the jump times and $\boldsymbol{\Pi}_{2i}$ are the jump sizes.

The following theorem essentially proved in Jevtić et al. (2017), provides a connection between marked Poisson processes and multivariate subordinated Lévy processes.

Theorem 2.1. *Let* $\boldsymbol{Y}(t)$ *be a SMBM, and let* $\boldsymbol{\Pi}$ *be the random measure associated to* $\boldsymbol{\pi}(t)$*. Then it exists a family of marks* $\boldsymbol{Z}$ *of* $\boldsymbol{\Pi}$ *on* $(\mathbb{R}^n, \mathcal{B}_n)$ *such that the family* $\boldsymbol{N} = (\Pi_1, \boldsymbol{Z})$ *forms a Poisson random measure* $\boldsymbol{N}$ *on* $(\mathbb{R}_+ \times \mathbb{R}^n, \mathcal{B}_{n+1})$ *and it holds*

$$\boldsymbol{Y}(t) =_{\mathcal{L}} \gamma_{\boldsymbol{Y}} t + \int_{(0,t]\times\mathbb{B}} \boldsymbol{y}[\boldsymbol{N}(ds, d\boldsymbol{y}) - \mu_{\boldsymbol{N}}(ds, d\boldsymbol{y})] + \int_{(0,t]\times\mathbb{B}^c} \boldsymbol{y}\boldsymbol{N}(ds, d\boldsymbol{y}), \tag{2.4}$$

where $\mathbb{B}$ *is the unit ball in* $\mathbb{R}^n$*.*

Proof. Let $\boldsymbol{Z} = \{\boldsymbol{Z}_i, i \in I\}$ be a family of marks of $\boldsymbol{\Pi} = \{(\Pi_{1i}, \boldsymbol{\Pi}_{2i}), i \in I\}$ on $(\mathbb{R}^n, \mathcal{B}_n)$, with distribution $\boldsymbol{G}(\boldsymbol{\Pi}_{2i}, \cdot)$, where $\boldsymbol{G}$ is a transition probability kernel from $(\mathbb{R}^d_+, \mathcal{B}_n)$ into $(\mathbb{R}^n, \mathcal{B}_n)$ defined by

$$\boldsymbol{G}(\boldsymbol{0}, B) := P(\boldsymbol{B}_A(\boldsymbol{0}) \in B) = \mathbb{1}_B(\boldsymbol{0}) \tag{2.5}$$

$$\boldsymbol{G}(\boldsymbol{s}, B) := P(\boldsymbol{B}_A(\boldsymbol{s}) \in B). \tag{2.6}$$

By Theorem 1.1 in Jevtić et al. (2017) the family $\boldsymbol{N} = (\Pi_1, \boldsymbol{Z})$ forms a Poisson random measure on $(\mathbb{R}_+ \times \mathbb{R}^n, \mathcal{B}_{n+1})$ with mean measure $\mu_{\boldsymbol{N}}(dt, d\boldsymbol{y}) = dt \int_{\mathbb{R}^d_+} \nu_{\boldsymbol{\Pi}}(d\boldsymbol{s}) Q(\boldsymbol{s}, d\boldsymbol{y})$ and, if $\boldsymbol{\gamma}_{\boldsymbol{Y}} = \int_{\mathbb{R}^d_+} \nu_{\boldsymbol{\Pi}}(d\boldsymbol{s}) \int_{\mathbb{B}} \boldsymbol{x} Q(\boldsymbol{s}, d\boldsymbol{x})$, the process $\boldsymbol{Y}(t)$ defined as

$$\boldsymbol{M}(t) := \gamma_{\boldsymbol{Y}} t + \int_{(0,t]\times\mathbb{B}} \boldsymbol{y}[\boldsymbol{N}(ds, d\boldsymbol{y}) - \mu_{\boldsymbol{N}}(ds, d\boldsymbol{y})] + \int_{(0,t]\times\mathbb{B}^c} \boldsymbol{y}\boldsymbol{N}(ds, d\boldsymbol{y}), \tag{2.7}$$

is (in law) the subordinated Lévy process $\boldsymbol{Y}(t)$.

□

Theorem 2.1 is stated in a slightly more generality than in Jevtić et al. (2017), since here each $\boldsymbol{B}_i(\boldsymbol{s}), i \in \{1, \ldots, d\}$ in (2.1) is a multivariate process with correlated margins. The links between a multi-parameter process and Gaussian marks and between subordinator $\boldsymbol{\pi}(t)$ and the Poisson random measure $\boldsymbol{\Pi}$ are proved in Jevtić et al. (2017). This connection provides a theoretical motivation to introduce multi-parameter Brownian motions as models of asset returns, by interpreting the multivariate subordinator $\boldsymbol{\pi}(t)$ as the whole information up to time t, measured by $\boldsymbol{\Pi}$.

Remark 1. *Notice that, if the subordinator is one dimensional the associate measure* $\boldsymbol{\Pi}$ *represents the market wide trade activity and by* (2.3) *we recover the intuition of market time.*

We now consider the componentwise subordination $\boldsymbol{Y}(t) = (B_1(\pi_1(t)), \ldots, B_n(\pi_n(t)))$, *where the Brownian motions must be independent. Then the associated marked Poisson process* $\boldsymbol{M}$ *in Theorem 2.1 must have marks with independent components. Also in this case* $\boldsymbol{\Pi}$ *is a measure of the whole trading activity, but each asset depends only on its own marginal (one-dimensional) trading activity. We recover the intuition of market time by equation* (2.3).

The model introduced above is general and includes several processes widely applied in financial modelling, as we are going to show in the next section.

We conclude this section by providing the characteristic function of $\boldsymbol{Y}$ in the following proposition, which is a straightforward derivation from Theorem 4.7 in Barndorff-Nielsen et al. (2001).

Proposition 2.1. *The characteristic function of a Gaussian-marked Poisson process on* $\mathbb{R}^n$ *has the following form*

$$\mathbb{E}[e^{i<\boldsymbol{z},\boldsymbol{Y}(t)>}] = \exp\{t\Psi_{\boldsymbol{\pi}}(\log\psi_{\tilde{\boldsymbol{B}}_1}(\boldsymbol{z}), \ldots, \log\psi_{\tilde{\boldsymbol{B}}_d}(\boldsymbol{z}))\}, \tag{2.8}$$

where $\Psi_{\boldsymbol{\pi}}$ *is the characterisctic exponent of* $\boldsymbol{\pi}(t)$, $\tilde{\boldsymbol{B}}(s_l) = \boldsymbol{A}_l\boldsymbol{B}_l(s_l)$ *and* $\tilde{\boldsymbol{B}} = \boldsymbol{A}_l\boldsymbol{B}_l(1)$.

Proof. Let $\boldsymbol{A}_i \in \mathcal{M}_{n\times n_i}(\mathbb{R})$ and let the process $\boldsymbol{B}_{\boldsymbol{A}}$ be defined as in (2.1). Have $\tilde{\boldsymbol{B}}(s_l) = \boldsymbol{A}_l\boldsymbol{B}_l(s_l)$. Then $\tilde{\boldsymbol{B}}(s_l)$ is a n-dimensional Brownian motion with parameters $\boldsymbol{\mu}_{\boldsymbol{A}} = \boldsymbol{A}_l\boldsymbol{\mu}_l$ and $\boldsymbol{\Sigma}_l = \boldsymbol{A}_l\boldsymbol{\Sigma}_l\boldsymbol{A}_l^T$. Thus

$$\boldsymbol{B}_{\boldsymbol{A}}(\boldsymbol{s}) = \boldsymbol{B}_{\boldsymbol{A}}(s_1, \ldots, s_d) = \boldsymbol{A}_1\boldsymbol{B}_1(s_1) + \ldots + \boldsymbol{A}_d\boldsymbol{B}_d(s_d) \;\; \boldsymbol{s} \in \mathbb{R}^d_+.$$

We have

$$\boldsymbol{B}_{\boldsymbol{A}}(\boldsymbol{\delta}_j) = \boldsymbol{B}_{\boldsymbol{A}}(0, \ldots, \underset{\text{j-th}}{\underbrace{1}}, \ldots, 0) = \boldsymbol{A}_j \boldsymbol{B}_j(1).$$

Thus

$$\begin{aligned}\psi_j(\boldsymbol{z}) &= \mathbb{E}[\exp\{i < \boldsymbol{B}_{\boldsymbol{A}}(\boldsymbol{\delta}_j), \boldsymbol{z} >\}] = \mathbb{E}[\exp\{i < \boldsymbol{A}_j \boldsymbol{B}_j(1), \boldsymbol{z} >\}] \\ &= \mathbb{E}\left[\exp\left\{i \sum_{l=1}^{n} \sum_{k=1}^{n_j} (\boldsymbol{A}_j)_{lk} B_{jk}(1) z_l\right\}\right] \\ &= \mathbb{E}[\exp\{i < \tilde{\boldsymbol{B}}_j, \boldsymbol{z} >\}] = \psi_{\tilde{\boldsymbol{B}}_j}(\boldsymbol{z}),\end{aligned}$$

where $\tilde{\boldsymbol{B}}_j := \tilde{\boldsymbol{B}}_j(1)$ has a normal distribution since it is a linear combination of normal distributions. Hence

$$\log(\psi_{\boldsymbol{B}^\rho}(\boldsymbol{z})) = (\log \psi_1(\boldsymbol{z}), \ldots, \log \psi_d(\boldsymbol{z})) = (\log \psi_{\tilde{\boldsymbol{B}}_1}(\boldsymbol{z}), \ldots, \log \psi_{\tilde{\boldsymbol{B}}_d}(\boldsymbol{z}))$$

giving

$$\psi_{\boldsymbol{Y}(t)}(\boldsymbol{z}) = \exp\{t\Psi_{\boldsymbol{\pi}}(\log \psi_{\boldsymbol{B}^\rho}(\boldsymbol{z}))\} = \exp\{t\Psi_{\boldsymbol{\pi}}(\log \psi_{\tilde{\boldsymbol{B}}_1}(\boldsymbol{z}), \ldots, \log \psi_{\tilde{\boldsymbol{B}}_d}(\boldsymbol{z}))\}.$$

□

3 Submodels

In this section we show that the class of SMBMs provides a unified framework, since it generalizes several processes widely used in finance for multiasset modelling.

3.1 α and $\rho\alpha$-models

Since α-models are a subcase of $\rho\alpha$-models obtained by setting $\rho_{ij} = 0$, for $i \neq j$, it suffices to show that $\rho\alpha$-models belong to the class SMBM. Let $\boldsymbol{B}$ be a n-dimensional Brownian motion with independent components and Lévy triplet $(\boldsymbol{\mu}, \boldsymbol{\Sigma}, 0)$ where $\boldsymbol{\mu} := (\mu_1, ..., \mu_n)$ and $\boldsymbol{\Sigma} := diag(\sigma_1^2, ..., \sigma_n^2)$. Have $\boldsymbol{B}^\rho$ to be a correlated n-dimensional Brownian motion, with correlations ρ_{ij}, marginal drifts $\boldsymbol{\mu}^\rho := (\mu_1\alpha_1, ..., \mu_n\alpha_n)$ and diffusion matrix $\boldsymbol{\Sigma}^\rho := (\rho_{ij}\sigma_i\sigma_j\sqrt{\alpha_i}\sqrt{\alpha_j})_{ij}$.

The $\mathbb{R}^n$-valued subordinated process $\boldsymbol{Y} = \{\boldsymbol{Y}(t), t > 0\}$ defined as

$$\boldsymbol{Y}(t) := \begin{pmatrix} B_1(\pi_1(t)) + B_1^\rho(\pi(t)) \\ .... \\ B_n(\pi_n(t)) + B_n^\rho(\pi(t)) \end{pmatrix}, \tag{3.1}$$

where π_j and π are independent subordinators, independent from $\boldsymbol{B}$ and $\boldsymbol{B}^\rho$ is a factor-based subordinated Brownian motion, also indicated as $\rho\alpha$-model.

Proposition 3.1. *Let $\boldsymbol{Y}^\rho$ be a $\rho\alpha$-model. Then $\boldsymbol{Y}$ belong to the class of MSMBs.*

Proof. Let us consider the following multiparameter Brownian motion

$$\boldsymbol{B_A} = \boldsymbol{A}(B_1(s_1), \dots, B_n(s_n), B_{n+1,1}(s_{n+1}), \dots B_{n+1,n}(s_{n+1}))^T,$$

where B_i, $i \in \{1, \dots, n\}$ are one dimensional Brownian motions with drift μ_i and standard deviation σ_i, $\boldsymbol{B}_{n+1}$ is a n-dimensional Brownian motion with parameters $\boldsymbol{\mu}^\rho, \boldsymbol{\Sigma}^\rho$ and $\boldsymbol{A} \in \mathcal{M}_{n\times 2n}$ such that

$$\boldsymbol{A} := \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{3.2}$$

Alternatively, $\boldsymbol{B_A} = \boldsymbol{A}_1 B_1(s_1) + \ldots + \boldsymbol{A}_n B_n(s_n) + \boldsymbol{A}_{n+1}\boldsymbol{B}_{n+1}(s_{n+1})$, where $\boldsymbol{A}_i = (0, \dots, 1, \dots, 0), i \in \{1, \dots, n\}$ and $\boldsymbol{A}_{n+1} := \boldsymbol{I}$, i.e. an identity matrix. Let $\boldsymbol{\pi}$ be a $(n+1)$-dimensional subordinator with independent components. We have

$$\boldsymbol{Y}(t) = \boldsymbol{B}_A(\boldsymbol{\pi}(t)) = \begin{pmatrix} B_1(\pi_1(t)) + B_{n+1,1}(\pi_{n+1}(t)) \\ \dots. \\ B_n(\pi_n(t)) + B_{n+1,n}(\pi_{n+1}(t)) \end{pmatrix} =_{\mathcal{L}} \begin{pmatrix} B_1(\pi_1(t)) + B_1^\rho(\pi_{n+1}(t)) \\ \dots. \\ B_n(\pi_n(t)) + B_n^\rho(\pi_{n+1}(t)) \end{pmatrix},$$

and the assert is proved. □

Remark 2. *The class of α-models are obtained by choosing $\boldsymbol{B}^\rho$ with independent components, in this case it can be proved that the $\mathbb{R}^n$-valued subordinated process $\boldsymbol{Y} = \{\boldsymbol{Y}(t), t > 0\}$ can be defined by*

$$\boldsymbol{Y}(t) := \begin{pmatrix} B_1(\tilde{\pi}_1(t)) \\ \dots. \\ B_n(\tilde{\pi}_n(t)) \end{pmatrix},$$

where $\tilde{\boldsymbol{\pi}} = (\tilde{\pi_1}, \dots, \tilde{\pi_n})$ is the factor-based subordinator defined by $\tilde{\pi}_i(t) = \pi_i(t) + \alpha_i \pi(t)$, $i = 1, \dots, n$. By so doing, we can obtain the α-models by choosing $\boldsymbol{A} = \boldsymbol{I}$. Another process that can be obtained by the same choice of $\boldsymbol{A}$ is the variance generalized gamma convolution (VGG) process introduced by Buchmann et al. (2017). The VGG process is obtained by subordination of independent Brownian motions with a multivariate subordinator belonging to the family of generalized gamma convolution. The correspondence stated in Theorem 2.1 implies that the VGG process can be constructed by marking the Poisson random measure associated to a subordinator of the generalized gamma convolution family with Gaussian marks with independent components.

3.2 Factor models

In this section we show that the factor model proposed in Ballotta and Bonfiglioli (2014) belongs to the family of Lévy marked Poisson processes. To construct this model, the Poisson random measure $\boldsymbol{\Pi}$ takes values on $\mathbb{R}_+ \times \mathbb{R}_+^{n+1}$, i.e. the subordinator takes values in $\mathbb{R}^{n+1}$, where n is the dimension of marks. Let $\boldsymbol{A} \in \mathcal{M}_{n\times(n+1)}$ be such that:

$$\boldsymbol{A} := \begin{pmatrix} 1 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & \cdots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & 1 & a_n \end{pmatrix}.$$

Define

$$Y^I(t) :=_{\mathcal{L}} B^I(X(t)) = \begin{pmatrix} B_1^I(\pi_1(t)) \\ .... \\ B_{n+1}^I(\pi_{n+1}(t)) \end{pmatrix}, \, t \geq 0.$$

Let now

$$Y(t) :=_{\mathcal{L}} \boldsymbol{A}B^I(X(t)) = \begin{pmatrix} Y_1^I(t) + a_1 Y_{n+1}^I(t) \\ .... \\ Y_n^I(t) + a_n Y_{n+1}^I(t) \end{pmatrix}, \, t \geq 0. \tag{3.3}$$

The process $\boldsymbol{Y}$ has the factor structure proposed in Ballotta and Bonfiglioli (2012). As an example, let $\boldsymbol{\pi}$ be a gamma subordinator with independent components and let $\mathcal{L}(\pi_j) = \Gamma(\frac{1}{\nu_j}, \frac{1}{\nu_j})$. In this case, the process $\boldsymbol{Y}^I$ has independent variance gamma margins with parameters (μ_j, σ_j, ν_j). Up to constraints on the parameters, process $\boldsymbol{Y}$ is in law the multivariate variance gamma process introduced in Ballotta and Bonfiglioli (2012).

The general framework constructed above is very rich and defines a broad class of models. Here we aim at showing that subordination of multi-parameter processes increases the ability to span a wide correlation range. We do this by analyzing the simplest model that incorporates correlation among Brownian motions, which is obtained by considering the Brownian motions in (2.1) one dimensional, as in the model introduced in Jevtić et al. (2017).

4 The model

As a particular case of (2.1), let us consider the $\mathbb{R}_+^d$-parameter Brownian motion

$$\boldsymbol{B}^\rho(\boldsymbol{s}) := \boldsymbol{B}^\rho(s_1, \ldots, s_d) = \boldsymbol{A}\boldsymbol{B}(\boldsymbol{s}) \;\; \boldsymbol{s} \in \mathbb{R}_+^d,$$

where $\boldsymbol{B}(\boldsymbol{s}) := (B_1(s_1), \ldots, B_d(s_d))$, where $\boldsymbol{A} \in \mathcal{M}_{n\times d}(\mathbb{R})$ and $B_i(s_i)$ are independent Brownian motions on $\mathbb{R}$, with drift μ_i and variance σ_i. To accommodate the cross section properties of trade we use a factor-based subordinator introduced in Semeraro (2008), defined by

$$\boldsymbol{\pi}(t) := \boldsymbol{\pi}^I(t) + \alpha\boldsymbol{\pi}^C(t) = (\pi_1^I(t) + \alpha_1\pi^C(t), \ldots, \pi_n^I(t) + \alpha_n\pi^C(t)), \tag{4.1}$$

where $\pi_j^I(t)$ and $\pi^C(t)$, for $j = 1, ..., n$, are independent subordinators with Lévy measures ν_j^I and ν^C respectively.

Remark 3. *The multivariate Poisson random measure $\boldsymbol{\Pi}$ associated to $\pi(t)$ is a Poisson random measure on $(\mathbb{R}_+ \times \mathbb{R}^n, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^n))$ with mean Leb $\times\, \nu_{\boldsymbol{\Pi}}$, which we call factor-based Poisson random measure. The measure $\nu_{\boldsymbol{\Pi}}$ and the characteristic exponent of $\boldsymbol{\pi}(t)$ are derived in Semeraro (2008) and recalled in Appendix A.*

We can now introduce the class of processes to model asset returns.

Definition 4.1. *Let $\boldsymbol{Y}(t)$ be defined by*

$$\boldsymbol{Y}(t) := \boldsymbol{B}^\rho(\boldsymbol{\pi}(t)) = \boldsymbol{A}\boldsymbol{B}(\boldsymbol{\pi}(t)), \tag{4.2}$$

where $\boldsymbol{\pi}(t)$ *is the factor-based subordinator in* (4.1) *and ut is independent of* $\boldsymbol{B}^\rho$. *The process* $\boldsymbol{Y}$ *defined in* (4.2) *is called* factor-based subordinated multi-parameter Brownian motion (factor-based SMBM).

Obviously Theorem 2.1 applies to factor-based SMBM processes. As a consequence they are Gaussian marked Poisson processes and we preserve the new interpretation of trading activity. By equation (2.8), where $\Psi_{\boldsymbol{\pi}}$ is deduced by (A), we can easily derive the characteristic function of factor-based SMBMs, which is

$$\begin{aligned}\psi_{\boldsymbol{Y}(t)}(\boldsymbol{z}) &= \exp\{t\Psi_{\boldsymbol{\pi}}(\log\psi_{\boldsymbol{B}^\rho}(\boldsymbol{z}))\} \qquad (4.3)\\ &= \prod_{j=1}^{n}\psi_{\pi_j^I(t)}\Big(i\sum_{i=1}^{n}a_{ij}\mu_j z_i - \frac{1}{2}\sigma_j^2\big(\sum_{i=1}^{n}a_{ij}z_i\big)^2\Big)\psi_{\pi^C(t)}\Big(\sum_{j=1}^{n}\alpha_j\Big[i\sum_{i=1}^{n}a_{ij}\mu_j z_i - \frac{1}{2}\sigma_j^2\big(\sum_{i=1}^{n}a_{ij}z_i\big)^2\Big]\Big),\end{aligned}$$

where $\psi_{\pi_j^I(t)}(w) = \exp\{t\Psi_{\pi_j^I}(w)\}$ and $\psi_{\pi^C(t)}(w) = \exp\{t\Psi_{\pi^C}(w)\}$. The marginal processes, which model individual asset returns, are multiparameter process defined on $\mathbb{R}^n$, in fact the k-th log-return is modeled as

$$B_k^\rho(\pi_1(t), \ldots, \pi_n(t)) = \sum_{i=1}^{n} a_{ki}B_i(\pi_i(t)),$$

having marginal characteristic function

$$\psi_{Y_k(t)}(z_k) = \prod_{j=1}^{n}\psi_{\pi_j^I(t)}\Big(ia_{kj}\mu_j z_k - \frac{1}{2}a_{kj}^2\sigma_j^2 z_k^2\Big)\psi_{\pi^C(t)}\Big(\sum_{j=1}^{n}\alpha_j\Big[ia_{kj}\mu_j z_k - \frac{1}{2}a_{kj}^2\sigma_j^2 z_k^2\Big]\Big). \qquad (4.4)$$

Notice that marginal distributions of returns depend on the joint distribution of $\boldsymbol{\pi}(t)$. The dependence of marginal returns on the trading activity of the entire collection of assets is now evident.

4.1 Linear correlation

This section shows that the class of factor-based SMBM allows to widen the correlations ranges of multivariate model widely used in financial applications. The correlation matrix $\boldsymbol{\rho} := \left(\rho_{m,l}^{\boldsymbol{Y}(t)}\right)_{n\times n}$ can be derived, as presented in Appendix B, by using the total covariance formula and has entries

$$\rho_{m,l}^{\boldsymbol{Y}(t)} = \frac{\sum_{i=1}^{n} a_{mi}a_{li}\sigma_i^2\mathbb{E}[\pi_i(t)] + \sum_{i=1}^{n}\sum_{j=1}^{n} a_{mi}a_{lj}\mu_i\mu_j\alpha_i\alpha_j\mathrm{Var}[\pi^C(t)]}{\sqrt{\mathrm{Var}[Y_m(t)]}\sqrt{\mathrm{Var}[Y_l(t)]}}, \qquad (4.5)$$

where

$$\mathrm{Var}[Y_k(t)] = \sum_{j=1}^{n} a_{kj}^2\mu_j^2\mathrm{Var}[\pi_j^I(t)] + \big(\sum_{j=1}^{n} a_{kj}\mu_j\alpha_j\big)^2\mathrm{Var}[\pi^C(t)] + \sum_{j=1}^{n} a_{kj}^2\sigma_j^2\mathbb{E}[\pi_j(t)].$$

Notice that, by infinite divisibility $\mathbb{E}[\pi_i(t)] = t\mathbb{E}[\pi_i(1)]$, $\mathrm{Var}[\pi^C(t)] = t\mathrm{Var}[\pi^C(1)]$ and $\mathrm{Var}[Y_i(t)] = t\mathrm{Var}[Y_i(1)]$, thus $\rho_{m,l}^{\boldsymbol{Y}(t)}$ is independent from t. The model correlations are flexible, since we can move independently return correlations and subordinator correlations. Furthermore, returns correlations are not bounded in absolute value neither from Brownian motions correlations nor from the subordinator correlations, as shown by considering the following limit cases

a. Consider the limit case of conditional independent Brownian motions $\boldsymbol{A} := \boldsymbol{I}$, $\mu_m, \mu_l > 0$ and positively correlated subordinators. In this case $\boldsymbol{B}^{\rho}(\boldsymbol{s})$ has independent components and $\rho_{m,l}^{\boldsymbol{Y}(t)} > 0$. Thus $\rho_{m,l}^{\boldsymbol{Y}(t)}$ is not bounded by the multiparameter Brownian motion correlations. The case of negatively correlated Brownian motions is similar.

b. Consider the case of the subordinator with independent components and positively correlated $\boldsymbol{B}^{\rho}(\boldsymbol{s})$, we have

$$\rho_{m,l}^{\boldsymbol{Y}(t)} = \frac{\sum_{i=1}^{n} a_{mi} a_{li} \sigma_i^2 \mathbb{E}[\pi_i(1)]}{\sqrt{\mathrm{Var}[Y_m(1)]}\sqrt{\mathrm{Var}[Y_l(1)]}},$$

which is positive. Thus $\rho_{m,l}^{\boldsymbol{Y}(t)} > 0$. The case of negatively correlated Gaussian marks is similar.

Notice that the model also exhibits nonlinear dependence. This fact is straightforward considering the subcase with conditional independent Gaussian-marks with zero means. Since $\mu_j = 0$ for all $j \in \{1, \ldots, n\}$ and A is diagonal the process has zero linear correlations (see Equation (4.5)) but it has dependent margins; indeed the Lévy measure of $\boldsymbol{Y}$ is given by

$$\nu_{\boldsymbol{Y}}(B) = \int_{\mathbb{R}_+^n} \boldsymbol{G}(\boldsymbol{s}, B) \nu_{\boldsymbol{\Pi}}(d\boldsymbol{s}).$$

From the expression of $\nu_{\boldsymbol{\Pi}}$, which has a common factor ν^C, it follows that the components of $\boldsymbol{Y}$ may jump together. In that, nonlinear dependence derives from the superimposition of either independent or correlated marks on the common factor of the Poisson measure.

Concerning the issues of negative dependence of asset returns modelled by factor-based SMBM, since the subordinator has always positively dependent components (see Semeraro (2008)), negative dependence is achieved by negative correlations of the multiparameter Brownian motion components. This model is also able to capture independence which occurs as a limit case when the multiparameter Brownian motion has independent components (A diagonal) and the common factor of the Poisson measure degenerates.

Here, to present a first application exercise, we introduce the NIG specification. The factor-based Poisson random measures in this case, inherit the subordinator parameters and the constraints on them, due to the subordinators convolution requirement as discussed in Luciano et al. (2013).

4.2 Normal inverse Gaussian SMBM

A *normal inverse Gaussian* (NIG) process with parameters $\gamma > 0$, $-\gamma < \beta < \gamma$, $\delta > 0$ is a Lévy process $X_{NIG} = \{X_{NIG}(t), t \geq 0\}$ with characteristic function

$$\psi_{NIG}(z) = \exp t(-\delta(\sqrt{\gamma^2 - (\beta + iu)^2} - \sqrt{\gamma^2 - \beta^2}).$$

Here, we construct a factor-based SMBM process of NIG type. Let $\boldsymbol{\pi}_{IG}(t)$ be a factor-based IG subordinator with parameters $(\gamma, \alpha_j, j \in \{1, \ldots, n\})$ introduced in Luciano and Semeraro (2010) and recalled in Appendix A.1. Notice that the assumption $\mathbb{E}[\pi_i(t)] = t$, which is usually assumed for the subordinator in the VG process, is not required. This is consistent with the interpretation of $\boldsymbol{\pi}_{IG}(t)$ as trading activity. Let $\gamma_j, \beta_j, \delta_j$ be such that

$$\gamma_j > 0, \quad -\gamma_j < \beta_j < \gamma_j, \quad \delta_j > 0.$$

Further, let

$$\frac{1}{\sqrt{\alpha_j}} = \delta_j \sqrt{\gamma_j^2 - \beta_j^2}. \tag{4.6}$$

Set $\mu_j = \beta_j \delta_j^2$ and $\sigma_j = \delta_j$. Under this assumption the process $\boldsymbol{Y}(t)$ defined in 2.1 is named *Normal Inverse Gaussian subordinated multiparameter Brownian motion* (NIG-SMBM). By means of equations (4.3), where $\Psi_{\boldsymbol{\pi}}$ is deduced by (A.1), the characteristic function of the NIG-SMBM process is

$$\begin{aligned} \psi_{\boldsymbol{Y}(t)}(\boldsymbol{z}) = \exp t\Big[\sum_{j=1}^{n} -(1-\sqrt{\alpha_j}\gamma)\Big(\sqrt{-2(i\mu_j \sum_{i=1}^{n} a_{ij} z_i - \frac{1}{2}\sigma_j^2 (\sum_{i=1}^{n} a_{ij} z_i)^2) + \frac{1}{\alpha_j}} - \frac{1}{\sqrt{\alpha_j}}\Big) \\ - \gamma\Big(\sqrt{-2\sum_{j=1}^{n} \alpha_j (i\mu_j \sum_{i=1}^{n} a_{ij} z_i - \frac{1}{2}\sigma_j^2 (\sum_{i=1}^{n} a_{ij} z_i)^2) + 1} - 1\Big)\Big] \end{aligned} \tag{4.7}$$

Hence, the k-th marginal characteristic function of $\boldsymbol{Y}(t)$ is

$$\begin{aligned} \psi^{NIG}_{Y_k(t)}(z_k) = \exp t\{-\sum_{j=1}^{n}(1-\alpha_j\gamma)\Big(\sqrt{-2(ia_{kj}\mu_j z_k - \frac{1}{2}a_{kj}^2\sigma_j^2 z_k^2) + \frac{1}{\alpha_j}} - \frac{1}{\sqrt{\alpha_j}}\Big) \\ - \gamma\Big(\sqrt{-2\sum_{j=1}^{n}(ia_{kj}\mu_j z_k - \frac{1}{2}a_{kj}^2\sigma_j^2 z_k^2)\alpha_j + 1} - 1\Big)\}, \end{aligned}$$

Notice that the marginal distributions depend on the common parameter γ, since they depend on the joint distribution of trading volume.

The $NIG - SMBM$ can be constructed as a linear transformation of the α-NIG process. In fact, from (4.2), we have:

$$\boldsymbol{Y}^{NIG} =_{\mathcal{L}} \boldsymbol{A}\boldsymbol{Y}_I^{NIG},$$

where $\boldsymbol{Y}_I^{NIG} := \boldsymbol{B}(\boldsymbol{\pi}(t))$, with the above specifications. We notice that $\boldsymbol{Y}_I^{NIG}$ is the α-NIG process in Luciano and Semeraro (2010), which has marginal distributions $NIG(\gamma_j, \beta_j, \delta_j)$. Thus, the marginal processes of a NIG-SMBM are linear combinations of dependent NIG processes. Obviously, if we further assume $\boldsymbol{A} := \boldsymbol{I}$, the process $\boldsymbol{Y}$ reduces to the α-NIG process $\boldsymbol{Y}_I^{NIG}$. Another interesting subcase is obtained by considering the limit case $\alpha_j = \alpha$ for each index j and $\gamma \to \frac{1}{\sqrt{\alpha}}$, the idiosyncratic components of $\boldsymbol{\pi}(t)$ degenerate and we find the model with only one GIG subordinator. On the opposite, if we assume that trading activities of assets are independent, thus the common component of the Poisson measure degenerates by having $\gamma = 0$. In this case $\boldsymbol{Y}(t)$ is a linear combination of independent NIG processes (see (4.2)). In this case the dependence structure is very similar to the one proposed in Eberlein and Madan (2010).

5 Empirical illustration

The aim of this section is to empirically explore the model flexibility in describing nonlinear and linear dependence and provide some intuition on the interaction of model parameters. To base our analysis on a meaningful parameter set, we estimate the model on time series data for a pair of stocks by a straightforward estimation procedure. Let $\boldsymbol{Y}$ be a bivariate $NIG-SMBM$ and define a $2-$dimensional price process, $\boldsymbol{S} = \{\boldsymbol{S}(t),\, t \geq 0\}$, such that

$$\boldsymbol{S}(t) = \boldsymbol{S}(0)\exp(\boldsymbol{c}t + \boldsymbol{Y}(t))$$

where $\boldsymbol{c} \in \mathbb{R}^2$ is the drift term (equivalently, $S_j(t) = S_j(0)\exp(c_j t + Y_j(t)),\, t \geq 0,\, j = 1, 2$). For the purpose of illustration, we consider daily log-returns on Goldman Sachs and Morgan Stanley US equity from January 3, 2011 to December 31st, 2015, with a total of 1258 observations. Daily logreturns are defined as

$$R_j = c_j + Y_j\,(1) = \log\frac{S_j\,(1)}{S_j\,(0)}, \quad j = 1, 2$$

The parameters of the process $\boldsymbol{Y}(t)$ in Definition 2.1 are $\boldsymbol{\mu}$, Σ, A, in addition to the parameters of the subordinator. In the NIG specification, we have that $\boldsymbol{Y}(t) = A\boldsymbol{Y}_I^{NIG}$, where $\boldsymbol{Y}_I^{NIG}$ is a bivariate α-NIG process, whose marginals are two dependent NIG factors. The model parameters defining the two NIG dependent factors can be directly introduced as $\gamma_1, \beta_1, \delta_1, \gamma_2, \beta_2, \delta_2$. Then we set $\mu_j = \beta_j \delta_j^2$, $\sigma_j = \delta_j$ and $\alpha_j = 1/\left(\delta_j^2\left(\gamma_j^2 - \beta_j^2\right)\right)$, $j = 1, 2$, as required by the NIG representation. The model in its general formulation is not parsimonious in terms of parameters. In fact, we have 13 parameters for the bivariate case. A reduction of the number of parameters can be made with additional assumptions which may depend on the application. The parameter vector $\boldsymbol{\theta}$ is

$$\boldsymbol{\theta} = [c_1, \gamma_1, \beta_1, \delta_1, c_2, \gamma_2, \beta_2, \delta_2, a_{11}, a_{12}, a_{21}, a_{22}, \gamma]\,.$$

Since $\Sigma^\rho = A\Sigma A^T$, without loss of generality, we can fix $\Sigma = diag(\sigma_1, \sigma_2)$. We start by estimating the α-NIG process by a two-step procedure. First, we calibrate the marginal NIG factors by maximum likelihood and then we estimate the remaining common parameter γ by matching historical and model asset correlations. The estimated parameters, reported in Table 1, allow for a reduced calibration approach. We recall that $\sigma_i = \delta_i,\;\; i = 1, 2$, thus we set $\delta_1 = 0.016$ and $\delta_2 = 0.017$. This choice reflects the estimate of the parameters δ_i obtained by fitting the two one-dimensional NIG processes on our asset return data. With these assumptions, the parameter vector $\boldsymbol{\theta}$ becomes

$$\boldsymbol{\theta} = [c_1, \gamma_1, \beta_1, c_2, \gamma_2, \beta_2, a_{11}, a_{12}, a_{21}, a_{22}, \gamma]\,,$$

for a total of 11 parameters.

The reduced model calibration is performed by the two-step GMM method, proposed by Hansen (1982), matching sample and model raw moments up to the fourth order. We have 4 marginal moments for asset 1 and 2, respectively, and 6 cross moments[1], for a total of $L = 14$ moment conditions. The model raw moments can be computed from the model joint characteristic function

$$E\left[R_1^{k_1} R_2^{k_2}\right] = (-i)^{k_1+k_2}\,\frac{\partial^{k_1+k_2}}{\partial z_1^{k_1}\partial z_2^{k_2}}\psi_{\mathbf{c}+\mathbf{Y}}\left(\mathbf{0}, \boldsymbol{\theta}\right), \quad k_1, k_2 \in \mathbb{N}.$$

[1] Specifically, we consider the following raw cross moments: $\mathbb{E}\left(R_1 R_2\right), \mathbb{E}\left(R_1^2 R_2\right), \mathbb{E}\left(R_1 R_2^2\right), \mathbb{E}\left(R_1^3 R_2\right), \mathbb{E}\left(R_1 R_2^3\right)$ and $\mathbb{E}\left(R_1^2 R_2^2\right)$.

We find $\boldsymbol{\theta}$ by solving

$$\widehat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \mathbf{G}'_T\left(\boldsymbol{\theta}\right) \mathbf{W} \mathbf{G}_T\left(\boldsymbol{\theta}\right)$$

where $\mathbf{G}_T\left(\boldsymbol{\theta}\right)$ is the $L \times 1$ vector of raw moment errors, T is the number of observations and $\mathbf{W}$ is a positive-definite weighting matrix. We set the weighting $\mathbf{W}$ equal to the identity matrix in the first step estimator and use a HAC estimator in the second step. Initial conditions are chosen by setting $A = I$, while the starting values of the remaining parameters are provided by the calibration of the α-NIG process presented in Table 1. Table 2 shows the model calibrated parameters, whereas sample and model standardized moments are reported in the Table 3. The standardized moments are defined as

$$m_{k_1 k_2} = \frac{\mathbb{E}\left(\left(R_1 - \mathbb{E}\left(R_1\right)\right)^{k_1}\left(R_2 - \mathbb{E}\left(R_2\right)\right)^{k_2}\right)}{\left(\mathbb{E}\left(\left(R_1 - \mathbb{E}\left(R_1\right)\right)^2\right)\right)^{k_1/2}\left(\mathbb{E}\left(\left(R_2 - \mathbb{E}\left(R_2\right)\right)^2\right)\right)^{k_2/2}}, \quad k_1, k_2 \in \mathbb{N}.$$

[Insert Table 2]

[Insert Table 3]

To test if the model is correctly specified, we perform the J-test for overidentifying restrictions. The J-statistic is 1.27, and has a p-value of 0.74 based on the chi-square distribution with three degree of freedom, thus providing support to the model specification.

We explore the features of the model changing the parameter γ and the matrix $\boldsymbol{A}$, i.e., changing the linear combination of the two α-NIG components, whose parameters are kept fixed. Figure 1 shows how the parameter γ drives nonlinear dependence. Here we can conclude that marginal and joint skewness and kurtosis move with γ, while linear correlation is not affected. Since the matrix $\boldsymbol{A}$ influence both the marginal processes and the dependence structure, we show the marginal and cross moments ranges spanned by changing $\boldsymbol{A}$. We change two parameters at a time, to have an accessible graphical representations. Firstly, we change parameters a_{11} and a_{22} and secondly we change parameters a_{12} and a_{21}. By so doing, we change the dependence structure by changing the weights of the two α-NIG factors in the model. Clearly asset one is is not affected by a_{21} and a_{22} and asset 2 is not affected by a_{11} and a_{12}.

Next, we examine the correlation range spanned by the model. For a given grid of target asset correlations ranging from -0.90 up to 0.90, we re-estimate the model by matching model and target asset correlations, under the constraint that the marginals moments stay unchanged. We have 11 parameters to match the asset correlation level, under 8 constraints provided by marginal moments. However, the aim of this exercise is to show that the model construction allows to span a large correlation range, and this is indeed the case, as illustrated in Table 4, which shows the cross moment conditions corresponding to the calibrated parameters which allow to match the target asset correlations and are reported in Table 5. In a general calibration procedure, all parameters are used to match not only linear dependence but also higher order cross moments and the extra degrees of freedom disappear.

Appendices

A Factor-based subordinators

We recall here the definition of factor-based subordinator introduced in Semeraro (2008). A factor-based subordinator $\boldsymbol{\pi}(t) = \boldsymbol{\pi}^I(t) + \boldsymbol{\pi}^C(t)$ is defined by

$$\boldsymbol{\pi}(t) = (\pi_1^I(t) + \alpha_1\pi^C(t), \ldots, \pi_n^I(t) + \alpha_n\pi^C(t)),$$

where $\pi_j^I(t)$ and $\pi^C(t)$, for $j = 1, ..., n$, are independent subordinators with Lévy measures ν_j^I and ν^C respectively. The multivariate Poisson random measure $\mathbf{\Pi}$ associated to $\pi(t)$ is a Poisson random measure on $(\mathbb{R}_+ \times \mathbb{R}^n, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^n))$ with mean $Leb \times \nu_{\mathbf{\Pi}}$, which we call factor-based Poisson random measure. We recall below the measure $\nu_{\mathbf{\Pi}}$, which is derived in Semeraro (2008). Consider a set $A \in \mathcal{B}(\mathbb{R}^n \setminus \{\mathbf{0}\})$ and $\Delta_\alpha = \{(\alpha_1 s, \ldots, \alpha_n s)^T : s \in \mathbb{R}_+\}$ where $\alpha_j \in \mathbb{R}$ for $j \in \{1, \ldots, n\}$, and $A_j^\alpha = Pr_j(A \cap \Delta_\alpha)$, having Pr_j be the projection of A on the j-th coordinate axes. Since $\frac{A_j^\alpha}{\alpha_j} = \{s \in \mathbb{R} : \alpha_j s \in A_j^\alpha\}$, and by construction $\frac{A_j^\alpha}{\alpha_j} = \frac{A_k^\alpha}{\alpha_k}$ for each $j, k \in \{1, \ldots, n\}$, we define $A_\Delta := \frac{A_j^\alpha}{\alpha_j}$ for each j. Finally, let $A_j := A \cap D_j$ having $D_j = \{\boldsymbol{x} \in \mathbb{R}^n : x_k = 0, k \neq j, k = 1, \ldots, n\}$. The Lévy measure $\nu_{\mathbf{\Pi}}$ is as follows

$$\nu_{\mathbf{\Pi}}(A) = \sum_{j=1}^{n} \nu_j^I(A_j) + \nu^C(A_\Delta), \;\; A \in \mathcal{B}(\mathbb{R}^n \setminus \{\mathbf{0}\}).$$

We finally recall the characteristic exponent of $\boldsymbol{\pi}(t)$. For any $\boldsymbol{w} = (w_1, ..., w_n)^T \in \mathbb{C}^n$ with $\Re(w_j) \leq 0,\; j = 1, ..., n$ $\Psi_{\boldsymbol{\pi}(t)}$ is given by

$$\Psi_{\boldsymbol{\pi}}(\boldsymbol{w}) = \sum_{j=1}^{n} \Psi_{\pi_j^I}(w_j) + \Psi_{\pi^C}\left(\sum_{j=1}^{n} \alpha_j w_j\right)$$

where for any $w \in \mathbb{C}$ with $Re(w) \leq 0,\; j = 1, ..., n,$ $\Psi_{\pi_j^I}(w) = \int_{\mathbb{R}_+}(e^{\langle w,s\rangle} - 1)\nu_{\pi_J^I}(ds)$ and $\Psi_{\pi^C}(w) = \int_{\mathbb{R}_+}(e^{\langle w,s\rangle} - 1)\nu_{\pi^C}(ds)$.

A.1 Factor-based inverse Gaussian subordinator

Let $\boldsymbol{\pi}(t)$ be a factor-based subordinator. We specify $\pi_i^I(t)$ and $\pi^C(t)$ in (4.1) to have IG marginal distributions with parameters α_j and γ, by defining

$$\pi_j^I \sim IG\left(1 - \gamma\sqrt{\alpha_j}, \tfrac{1}{\sqrt{\alpha_j}}\right), \; j = 1, ..., n$$
$$\pi^C \sim IG(\gamma, 1).$$

Using the closure properties of the IG distribution, we obtain that the process $\alpha_j\pi^C$ is $\alpha_j\pi^C \sim IG\left(\gamma\sqrt{\alpha_j}, \frac{1}{\sqrt{\alpha_j}}\right)$ and that its sum with π_j^I is still IG (see Luciano and Semeraro (2010)):

$$\pi_j^I + \alpha_j\pi^C \sim IG\left(1, \frac{1}{\sqrt{\alpha_j}}\right).$$

In order for the marginal distributions to have non negative parameters, the following constraints must be satisfied:

$$0 < \gamma < \frac{1}{\sqrt{\alpha_j}}, \quad j = 1, ..., n.$$

We call this subordinator factor-based IG with parameters $(\gamma, \alpha_j, j = 1, \ldots, n)$. The marginal subordinators $\pi_j(t)$ are IG:

$$\mathcal{L}(\pi_j(t)) = IG\left(t, \frac{1}{\sqrt{\alpha_j}}\right), \quad j = 1, ..., n.$$

We recall that for any $\boldsymbol{w} = (w_1, ..., w_n)^T \in \mathbb{C}^n$ with $\Re(w_j) \leq 0,\ j = 1, ..., n$ it easily follows from (A)

$$\Psi_{\boldsymbol{\pi}}(\boldsymbol{w}) = -\sum_{j=1}^{n}(1 - \sqrt{\alpha_j}\gamma)(\sqrt{-2w_j + \frac{1}{\alpha_j}} - \frac{1}{\sqrt{\alpha_j}}) - \gamma(\sqrt{-2\sum_{j=1}^{n} w_j\alpha_j + 1} - 1).$$

B Derivation of the Equation (4.5)

Given the law of total covariance we have

$$\begin{aligned}\mathrm{Cov}[B^\rho_m(\boldsymbol{\pi}(t)), B^\rho_l(\boldsymbol{\pi}(t))] &= \mathbb{E}[\mathrm{Cov}[B^\rho_m(\boldsymbol{\pi}(t)), B^\rho_l(\boldsymbol{\pi}(t)] \mid \boldsymbol{\pi}(t))] \\ &+ \mathrm{Cov}\big[\mathbb{E}(B^\rho_m(\boldsymbol{\pi}(t)) \mid \boldsymbol{\pi}(t)), \mathbb{E}(B^\rho_l(\boldsymbol{\pi}(t)) \mid \boldsymbol{\pi}(t))\big].\end{aligned}$$

Since we have

$$\begin{aligned}\mathbb{E}[B^\rho_m(\boldsymbol{\pi}(t)) \mid \boldsymbol{\pi}(t)] &= \mathbb{E}\left[\sum_{j=1}^{n} a_{mj} B_j(\pi_j(t)) \mid \boldsymbol{\pi}(t)\right] \\ &= \sum_{j=1}^{n} a_{mj}\mathbb{E}[B_j(\pi_j(t)) \mid \boldsymbol{\pi}(t)] = \sum_{j=1}^{n} a_{mj}\mu_j\pi_j(t)\end{aligned}$$

then

$$\begin{aligned}&\mathrm{Cov}\big[\mathbb{E}[B^\rho_m(\boldsymbol{\pi}(t)) \mid \boldsymbol{\pi}(t)], \mathbb{E}[B^\rho_l(\boldsymbol{\pi}(t)) \mid \boldsymbol{\pi}(t)]\big] \\ &= \mathrm{Cov}\left[\sum_{i=1}^{n} a_{mi}\mu_i\pi_i(t), \sum_{i=1}^{n} a_{li}\mu_i\pi_i(t)\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{mi}a_{lj}\mu_i\mu_j \mathrm{Cov}[\pi_i(t), \pi_j(t)].\end{aligned}$$

Now for a given realization of $\boldsymbol{\pi}(t) = \boldsymbol{s}$ we have

$$\mathcal{L}(\boldsymbol{B}^\rho(\boldsymbol{\pi}(t)) \mid_{\boldsymbol{\pi}(t)=\boldsymbol{s}}) \sim \mathcal{N}(\boldsymbol{\mu}^\rho(\boldsymbol{s}), \Sigma^\rho(\boldsymbol{s})), \quad \text{where} \ \ \Sigma^\rho(\boldsymbol{s}) = A\Sigma(\boldsymbol{s})A^T,$$

and $\Sigma(\boldsymbol{s}) = diag(\sigma_i^2 s_i)$, hence

$$\mathrm{Cov}[B^\rho_m(\boldsymbol{\pi}(t)), B^\rho_l(\boldsymbol{\pi}(t)) \mid \boldsymbol{\pi}(t)] = \Sigma^\rho(\boldsymbol{\pi}(t))_{m,l}$$

i.e. the m, l-th entry of $\Sigma^\rho(\boldsymbol{\pi}(t))$ matrix.

Hence

$$\mathbb{E}[\Sigma^\rho(\boldsymbol{\pi}(t))_{m,l}] = \sum_{i=1}^{n} a_{mi} a_{li} \sigma_i^2 \mathbb{E}[\pi_i(t)].$$

Finally,

$$\mathrm{Cov}[B_m^\rho(\boldsymbol{\pi}(t)), B_l^\rho(\boldsymbol{\pi}(t))] = \sum_{i=1}^{n} a_{mi} a_{li} \sigma_i^2 \mathbb{E}[\pi_i(t)] + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{mi} a_{lj} \mu_i \mu_j \mathrm{Cov}[\pi_i(t), \pi_j(t)]$$

giving

$$\rho_{m,l}^{\boldsymbol{Y}(t)} = \frac{\sum_{i=1}^{n} a_{mi} a_{li} \sigma_i^2 \mathbb{E}[\pi_i(t)] + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{mi} a_{lj} \mu_i \mu_j \mathrm{Cov}[\pi_i(t), \pi_j(t)]}{\sqrt{\mathrm{Var}[Y_m(t)]}\sqrt{\mathrm{Var}[Y_l(t)]}}$$

and therefore

$$\rho_{m,l}^{\boldsymbol{Y}(t)} = \frac{\sum_{i=1}^{n} a_{mi} a_{li} \sigma_i^2 \mathbb{E}[\pi_i(t)] + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{mi} a_{lj} \mu_i \mu_j \alpha_i \alpha_j \mathrm{Var}[\pi^C(t)]}{\sqrt{\mathrm{Var}[Y_m(t)]}\sqrt{\mathrm{Var}[Y_l(t)]}}.$$

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c_1	0.002
γ_1	56.462
β_1	-5.379
δ_1	0.016
c_2	0.001
γ_2	31.874
β_2	-1.079
δ_2	0.017
γ	0.551

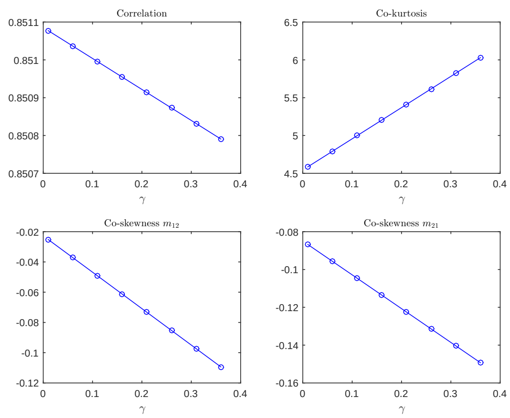
Table 1: α-NIG model. Estimated parameters.

c_1	0.001
γ_1	25.783
β_1	-0.017
c_2	0.001
γ_2	59.028
β_2	1.942
γ	0.395
a_{11}	0.348
a_{12}	-0.818
a_{21}	0.842
a_{22}	-0.604

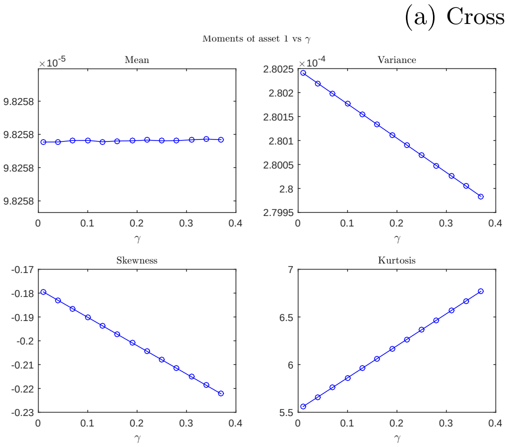
Table 2: NIG-SMBM model. Estimated parameters, setting $\delta_1 = 0.016$ and $\delta_2 = 0.017$.

Stand. moment	Sample	Model
m_{01}	0.0001	0.0001
m_{02}	0.0003	0.0003
m_{03}	-0.2164	-0.0840
m_{04}	6.8098	6.2692
m_{10}	0.0001	0.0002
m_{20}	0.0006	0.0005
m_{30}	-0.0624	-0.0457
m_{40}	9.0421	8.8435
m_{11}	0.8498	0.8476
m_{12}	-0.1146	-0.0537
m_{21}	-0.1638	-0.0625
m_{13}	6.9630	6.7198
m_{31}	6.0121	5.5999
m_{22}	6.1182	5.8008

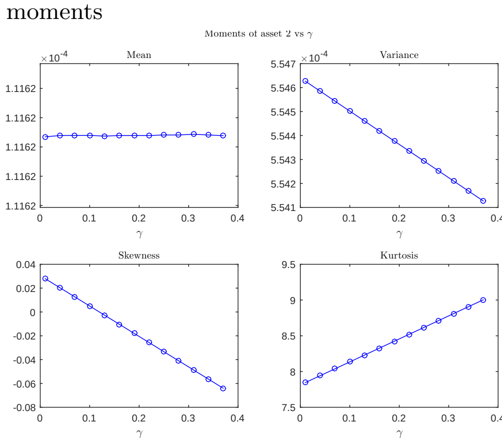
Table 3: Sample and model standardized moments.



(a) Cross moments



(b) Marginal moments of asset 1



(c) Marginal moments of asset 2

Figure 1: Marginal and cross moments vs γ

Target $\rho^{\boldsymbol{Y}}$	m$_{11}$	m$_{12}$	m$_{21}$	m$_{13}$	m$_{31}$	m$_{22}$
-0.90	-0.899	-0.029	0.112	-7.285	-6.255	6.435
-0.80	-0.800	-0.010	0.075	-6.652	-5.477	5.628
-0.70	-0.697	-0.013	0.006	-6.800	-5.391	5.936
-0.60	-0.601	-0.027	0.016	-6.032	-4.613	5.205
-0.50	-0.500	-0.034	0.018	-5.093	-3.699	4.375
-0.40	-0.400	-0.040	0.010	-4.241	-2.863	3.750
-0.30	-0.300	-0.043	0.000	-3.375	-2.023	3.198
-0.20	-0.200	-0.044	-0.013	-2.519	-1.187	2.726
-0.10	-0.100	-0.037	-0.028	-1.692	-0.310	2.240
0.00	0.000	-0.128	-0.017	2.007	0.680	3.494
0.10	0.103	-0.133	-0.050	3.167	1.995	3.935
0.20	0.200	-0.141	-0.052	4.162	2.840	4.520
0.30	0.301	-0.027	-0.052	1.570	1.809	1.618
0.40	0.403	-0.045	-0.067	2.460	2.406	2.138
0.50	0.503	-0.108	-0.087	5.341	4.141	4.702
0.60	0.595	-0.096	-0.109	4.622	4.063	3.988
0.70	0.701	-0.103	-0.104	6.775	5.551	5.937
0.80	0.800	-0.096	-0.114	7.274	6.073	6.422
0.90	0.900	-0.108	-0.150	7.574	6.507	6.789

Table 4: Target correlations and model standardized cross moments. Marginal moments are kept fixed.

Target $\rho^{\boldsymbol{Y}}$	c_1	γ_1	β_1	c_2	γ_2	β_2	γ	a_{11}	a_{12}	a_{21}	a_{22}
-0.90	0.0012	21.5693	-1.1763	-0.0004	43.9480	-5.2336	0.1878	-0.3459	0.6961	0.7495	-0.5881
-0.80	0.0011	28.5866	-0.8028	0.0000	38.7772	-4.4335	0.2133	-0.4418	0.6184	0.9629	-0.2793
-0.70	0.0017	29.9833	-0.2002	0.0008	43.4473	-8.3899	0.1532	-0.5778	0.4961	1.0073	0.1742
-0.60	0.0013	29.5902	-0.1257	0.0007	43.7197	-5.8600	0.2303	-0.5334	0.5621	0.9941	0.2223
-0.50	0.0012	29.7312	-0.1311	0.0006	43.8313	-4.4649	0.2997	-0.4743	0.6353	0.9957	0.2283
-0.40	0.0011	29.5550	-0.1197	0.0006	43.8854	-3.8778	0.3253	-0.4205	0.6858	0.9889	0.2510
-0.30	0.0011	29.4598	-0.1290	0.0006	43.8686	-3.4951	0.3346	-0.3612	0.7312	0.9852	0.2622
-0.20	0.0010	29.3724	-0.1557	0.0006	43.8958	-3.2441	0.3236	-0.2969	0.7709	0.9831	0.2660
-0.10	0.0010	29.3308	-0.2313	0.0006	43.9242	-3.0595	0.2627	-0.2206	0.8064	0.9851	0.2516
0.00	0.0011	20.1435	-0.8852	-0.0001	50.0664	-2.9803	0.2526	0.3256	0.7563	0.6971	-0.7007
0.10	0.0012	19.5589	-0.8564	-0.0002	62.6350	-3.8946	0.2781	0.3674	0.7866	0.6890	-0.7782
0.20	0.0013	19.6579	-0.7969	-0.0003	62.5734	-4.6991	0.1995	0.3970	0.7426	0.7055	-0.7362
0.30	0.0011	28.3429	-0.3600	0.0006	46.6894	-3.0166	0.0717	0.0564	0.8714	0.9655	0.2752
0.40	0.0010	27.5988	-0.2833	0.0007	45.8580	-2.9749	0.1311	0.1006	0.8572	0.9423	0.3243
0.50	0.0012	28.6710	-0.7961	0.0001	55.2121	-4.3440	0.3705	0.5014	0.6693	0.9624	-0.3449
0.60	0.0011	27.3207	0.1330	0.0006	47.1605	-3.1999	0.4212	0.1770	0.8491	0.9037	0.4573
0.70	0.0015	30.1855	-0.6654	0.0001	58.1677	-6.8726	0.2843	0.5775	0.5869	1.0118	-0.1943
0.80	0.0016	31.0546	-0.4776	0.0004	54.6971	-7.6637	0.2566	0.5952	0.5510	1.0368	-0.0177
0.90	0.0014	26.0858	0.3778	0.0009	58.1997	-6.6759	0.3769	0.4536	0.7190	0.8813	0.5116

Table 5: Calibrated parameters for target asset correlation levels, keeping the marginal moments fixed.