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Three solutions for a Neumann partial differential inclusion via nonsmooth Morse theory

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Abstract We study a partial differential inclusion, driven by the p-Laplacian operator, involving a p-superlinear nonsmooth potential, and subject to Neumann boundary conditions. By means of nonsmooth critical point theory, we prove the existence of at least two constant sign solutions (one positive, the other negative). Then, by applying the nonsmooth Morse identity, we find a third non-zero solution.

Keywords p-Laplacian · Partial differential inclusion · Morse theory

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1 Introduction

In the present paper we deal with the following partial differential inclusion, coupled with homogeneous Neumann boundary conditions:

$$\begin{cases} -\Delta_p u \in \partial j(x,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here and in what follows, $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain with C^2 boundary $\partial\Omega$, $p > 1$ and the p-Laplacian operator is defined as $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, while $\partial u/\partial \nu(x)$ denotes the outward normal derivative of u at $x \in \partial\Omega$. Finally, $\partial j(x,s)$ denotes the Clarke generalized subdifferential (with respect to s) of a potential $j : \Omega \times \mathbb{R} \to \mathbb{R}$ which is assumed to be measurable in x and locally Lipschitz continuous in s (see Section 2 for details).

Problem (1) can be seen as a set-valued analogous of the classical Neumann problem for a nonlinear elliptic equation of the following type:

$$\begin{cases} -\Delta_p u = f(x,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a (single-valued) Carathéodory mapping. One way to prove existence/multiplicity for the weak solutions of problem (2) (or related boundary value problems) is to apply variational methods, i.e., to seek solutions as critical points of an energy functional φ of class C^1, typically defined in the Sobolev space $W^{1,p}(\Omega)$. In particular, if the primitive of $f(x,\cdot)$ is p-superlinear at infinity, the functional φ can be unbounded from below, so one cannot apply global minimization and should expect a nontrivial behavior of $\varphi(u)$ as $\|u\| \to \infty$. Moreover, in the p-superlinear case the functional φ suffers from a lack of compactness, as Palais-Smale sequences are not bounded in general (to overcome such a difficulty, most authors use some kind of Ambrosetti-Rabinowitz condition). In such cases, Morse theory has proved to be a powerful tool to describe the topology of critical levels and hence to detect multiple critical points (see the monographs [2,9,26,31], the papers [1,4,5,16,21,24,27–30,33], and references therein).

We recall, in particular, one result of Wang [35], dealing with a Dirichlet problem for a semilinear ($p = 2$) elliptic equation with superlinear reaction. The author detected, beside the zero solution, two constant sign solutions (one positive, the other negative) as mountain pass-type critical points of the energy functional, then developed a Morse-theoretic argument based on the computation of critical groups at all critical points and at infinity, to prove existence of a third nontrivial solution (of undetermined sign).

On the set-valued side, the study of problems of the type (1) via variational methods dates back to the work of Chang [8], based on the nonsmooth critical point theory for locally Lipschitz continuous functionals introduced by Clarke [10]. Indeed, one can associate to problem (1) an energy functional $\varphi : W^{1,p}(\Omega) \to \mathbb{R}$ which is not differentiable, but only locally Lipschitz continuous, and look for critical points of φ. Such method was then developed by several authors (see the monographs [7,18], the papers [3, 19, 20, 22, 25], and references therein). Nonsmooth critical point theory embraces in fact a wider class of functionals, namely continuous functionals defined in metric spaces (see [12, 15, 17]) and in such very general framework Corvellec [11] also introduced the basic notions of a nonsmooth Morse theory.

The purpose of the present work is to contribute to the understanding of problem (1), especially in the case of a p-superlinear potential $j(x,\cdot)$, by applying the nonsmooth Morse theory of [11], conveniently adapted and developed. We will prove a nonsmooth analogue of the result of [35], ensuring existence of at least three nontrivial weak solutions of (1) (one positive, one negative, and the third of undetermined sign). To be more precise, let us list our hypotheses on the potential j:

H $j:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that $j(x,\cdot)$ is locally Lipschitz continuous for a.a. $x\in\Omega$, and $j(\cdot,0)\in L^1(\Omega)$; moreover, it satisfies

(i) there exist $a_0>0$ and $r\in(p,p^*)$ such that $|\xi|\le a_0(1+|s|^{r-1})$ for a.a. $x\in\Omega$ and all $s\in\mathbb{R}$, $\xi\in\partial j(x,s)$, where p^* denotes the Sobolev critical exponent

$$p^*=\begin{cases}Np/(N-p) & \text{if } p<N,\\ \infty & \text{if } p\geqslant N;\end{cases}$$

(ii) $\displaystyle\lim_{|s|\to\infty}\frac{j(x,s)}{|s|^p}=\infty$ uniformly for a.a. $x\in\Omega$;

(iii) there exists $q\in\left(\max\left\{(r-p)\frac{N}{p},1\right\},p^*\right)$ such that

$$\liminf_{|s|\to\infty}\min_{\xi\in\partial j(x,s)}\frac{\xi s-pj(x,s)}{|s|^q}>0 \text{ uniformly for a.a. } x\in\Omega;$$

(iv) there exists $\delta_0>0$ such that $j(x,s)\le j(x,0)$ for a.a. $x\in\Omega$ and all $|s|\le\delta_0$;

(v) there exists $c_0>0$ such that $\xi s\ge -c_0|s|^p$ for a.a. $x\in\Omega$ and all $s\in\mathbb{R}$, $\xi\in\partial j(x,s)$.

Hypothesis **H** (ii) defines our potential j as p-superlinear, while **H** (iii) is a nonsmooth version of the nonquadraticity condition of [14] (which holds, in particular, if j is smooth and satisfies the Ambrosetti-Rabinowitz condition). We present here two examples of potentials satisfying hypotheses **H**:

Example 1 Let $\sigma\in(p,p^*)$ and set for all $s\in\mathbb{R}$

$$j_1(s)=\begin{cases}0 & \text{if } s=0,\\ |s|^\sigma\ln|s|/\sigma & \text{if } 0<|s|<1,\\ |s|^p\ln|s|/p & \text{if } |s|\ge 1,\end{cases}$$

and

$$j_2(s)=\frac{|s|^\sigma}{\sigma}-\frac{|s|^p}{p}.$$

Then, both potentials $j_1,j_2:\mathbb{R}\to\mathbb{R}$ satisfy hypotheses **H**. Notice that j_2 is of class C^1 in $\mathbb{R}$, while j_1 is only Lipschitz continuous at $s=1$ (in the case of j_1 we can take $r<(Np+p^2)/N$ in **H** (i) and $q<p$ in **H** (iii)).

We provide problem (1) with a variational formulation. Our study involves two function spaces, the Sobolev space $W^{1,p}(\Omega)$ with norm $\|\cdot\| = (\|\nabla(\cdot)\|_p^p + \|\cdot\|_p^p)^{1/p}$ and $(C^1(\overline{\Omega}), \|\cdot\|_{C^1})$ (clearly $C^1(\overline{\Omega}) \subset W^{1,p}(\Omega)$). The dual space of $W^{1,p}(\Omega)$ is denoted by $(W^*, \|\cdot\|_*)$. Moreover, for all $t \in [1,\infty]$ we denote by $\|\cdot\|_t$ the norm of $L^t(\Omega)$. Both $W^{1,p}(\Omega)$ and $C^1(\overline{\Omega})$ are ordered Banach spaces with positive cones W_+ and C_+, respectively. We note that $\mathrm{int}(W_+) = \emptyset$, while

$$\mathrm{int}(C_+) = \left\{u \in C^1(\overline{\Omega}) : u(x) > 0 \text{ for all } x \in \overline{\Omega}\right\}.$$

By $A : W^{1,p}(\Omega) \to W^*$ we denote the p-Laplacian operator, i.e.

$$\langle A(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v\, dx \quad \text{for all } u, v \in W^{1,p}(\Omega).$$

We define the set-valued Nemytskij operator by setting for all $u \in L^r(\Omega)$

$$N(u) = \left\{w \in L^{r'}(\Omega) : w \in \partial j(x,u) \text{ a.e. in } \Omega\right\}$$

(where $r \in (p, p^*)$ is as in **H** (i)). Following a consolidated literature (see for instance [18]), we say that u is a *(smooth weak) solution* of (1) if $u \in C^1(\overline{\Omega})$ and there exists $w \in N(u)$ such that

$$A(u) = w \text{ in } W^*.$$

Now, for all $u \in W^{1,p}(\Omega)$ we set

$$\varphi(u) = \frac{\|\nabla u\|_p^p}{p} - \int_\Omega j(x,u)\, dx.$$

The functional $\varphi : W^{1,p}(\Omega) \to \mathbb{R}$ is locally Lipschitz continuous, and it is the energy functional of problem (1), in the sense that all critical points of φ are solutions of (1) (see Lemma 6 below).

Now we can state our main result:

Theorem 1 *If hypotheses* **H** *hold, then problem* (1) *admits at least three smooth non-zero solutions* $u_+, u_-, \tilde{u} \in C^1(\overline{\Omega})$ *such that* $u_\pm \in \pm\mathrm{int}(C_+)$.

To our knowledge, our result represents the first application of nonsmooth Morse theory to partial differential inclusions. We remark, also, that our three solutions theorem differs from most results of this type available in the literature (even in the smooth case): indeed, usually one first detects two local minimizers of the energy functional and then one further critical point of mountain pass type, while here (following [35]) we find two critical points of mountain pass type and one further critical point (of undetermined nature) via the Morse identity.

The paper has the following structure. In Section 2 we recall some basic features of nonsmooth critical point theory and prove a nonsmooth implicit function lemma (see Lemma 3). Section 3 is devoted to a presentation of the nonsmooth Morse theory. In Section 4, we use the nonsmooth mountain pass theorem to prove that (1) admits two constant sign smooth solutions (see Theorem 6). Finally, in Section 5, after computing the critical groups of φ at zero, at such constant sign solutions, and at infinity, we deduce the existence of a third smooth nontrivial solution (of undetermined sign) and conclude the proof of Theorem 1.

2 Preliminaries I: nonsmooth critical point theory

First, we introduce some notation. Throughout the paper, $(X, \|\cdot\|)$ denotes a reflexive Banach space, $(X^*, \|\cdot\|_*)$ its topological dual and $\langle\cdot,\cdot\rangle$ the duality between X^* and X. $B_\rho(u)$, $\overline{B}_\rho(u)$ and $\partial B_\rho(u)$ will denote the open and closed balls and the sphere in X centered at $u \in X$ with radius $\rho > 0$, respectively. For all $\varphi : X \to \mathbb{R}$ and all $c \in \mathbb{R}$ we set

$$\varphi^c = \{u \in X : \varphi(u) < c\}, \quad \overline{\varphi}^c = \{u \in X : \varphi(u) \le c\}$$

(we also set $\overline{\varphi}^\infty = X$). Finally, when estimates are considered, we shall denote by $c > 0$ positive constants, which are allowed to vary from line to line.

We recall some basic notions and results from nonsmooth critical point theory, referring to [10,18] for details. A functional $\varphi : X \to \mathbb{R}$ is said to be *locally Lipschitz continuous* if for every $u \in X$ there exist a neighborhood U of u and $L > 0$ such that

$$|\varphi(v) - \varphi(w)| \le L\|v - w\| \text{ for all } v, w \in U.$$

From now on we assume φ to be locally Lipschitz continuous. The *generalized directional derivative* of φ at u along $v \in X$ is

$$\varphi^\circ(u;v) = \limsup_{\substack{w \to u \\ t \to 0^+}} \frac{\varphi(w + tv) - \varphi(w)}{t}.$$

The *generalized subdifferential* of φ at u is the set

$$\partial\varphi(u) = \{u^* \in X^* : \langle u^*, v\rangle \le \varphi^\circ(u;v) \text{ for all } v \in X\}.$$

The following Lemmas display some basic properties of the tools introduced above, see [18, Propositions 1.3.7-1.3.12]:

Lemma 1 *If φ, $\psi : X \to \mathbb{R}$ are locally Lipschitz continuous, then*

(i) $\varphi^\circ(u;\cdot)$ *is positively homogeneous, sub-additive and continuous for all* $u \in X$;
(ii) $\varphi^\circ(u;-v) = (-\varphi)^\circ(u;v)$ *for all* $u, v \in X$;
(iii) *if* $\varphi \in C^1(X)$, *then* $\varphi^\circ(u;v) = \langle\varphi'(u), v\rangle$ *for all* $u, v \in X$;
(iv) $(\varphi + \psi)^\circ(u;v) \le \varphi^\circ(u;v) + \psi^\circ(u;v)$ *for all* $u, v \in X$.

Lemma 2 *If $\varphi, \psi : X \to \mathbb{R}$ are locally Lipschitz continuous, then*

(i) $\partial\varphi(u)$ *is convex, closed and weakly* compact for all* $u \in X$;
(ii) *the set-valued mapping* $\partial\varphi : X \to 2^{X^*}$ *is upper semicontinuous with respect to the weak* topology on* X^*;
(iii) *if* $\varphi \in C^1(X)$, *then* $\partial\varphi(u) = \{\varphi'(u)\}$ *for all* $u \in X$;
(iv) $\partial(\lambda\varphi)(u) = \lambda\partial\varphi(u)$ *for all* $\lambda \in \mathbb{R}$, $u \in X$;
(v) $\partial(\varphi + \psi)(u) \subseteq \partial\varphi(u) + \partial\psi(u)$ *for all* $u \in X$;
(vi) *for all* $u, v \in X$ *there exists* $u^* \in \partial\varphi(u)$ *such that* $\langle u^*, v\rangle = \varphi^\circ(u;v)$;
(vii) *if* $g \in C^1(\mathbb{R}, X)$, *then* $\varphi \circ g : \mathbb{R} \to \mathbb{R}$ *is locally Lipschitz, and*

$$\partial(\varphi \circ g)(t) \subseteq \left\{\langle u^*, g'(t)\rangle : u^* \in \partial\varphi(g(t))\right\}$$

for all $t \in \mathbb{R}$;

$(viii)$ *if u is a local minimizer (or maximizer) of φ, then* $0 \in \partial\varphi(u)$.

We also recall Lebourg's mean value theorem, see [18, Proposition 1.3.14]:

Theorem 2 *If $\varphi : X \to \mathbb{R}$ is locally Lipschitz and $u, v \in X$, then there exist $\tau \in (0,1)$ and $w^* \in \partial\varphi((1-\tau)u + \tau v)$ such that*

$$\varphi(v) - \varphi(u) = \langle w^*, v - u \rangle.$$

By Lemma 2 (i), we may define for all $u \in X$

$$m(u) = \min_{u^* \in \partial\varphi(u)} \|u^*\|_*. \tag{3}$$

We say that $u \in X$ is a *critical point* of φ if $m(u) = 0$ (i.e., $0 \in \partial\varphi(u)$). We denote by $K(\varphi)$ the set of critical points of φ and, for any $c \in \mathbb{R}$, we set

$$K_c(\varphi) = \{u \in K(\varphi) : \varphi(u) = c\},$$

and say that $c \in \mathbb{R}$ is a *critical value* of φ if $K_c(\varphi) \neq \emptyset$.

Now let us introduce some compactness conditions: we define a *Palais-Smale sequence* for the functional φ as a sequence $(u_n)_n$ in X such that $(\varphi(u_n))_n$ is bounded in $\mathbb{R}$ and $m(u_n) \to 0$. Then, φ is said so satisfy the Palais-Smale condition ((PS) for short), if every Palais-Smale sequence admits a (strongly) convergent subsequence.

Similarly, we define a *Cerami sequence* for the functional φ as a sequence $(u_n)_n$ in X such that $(\varphi(u_n))_n$ is bounded in $\mathbb{R}$ and $(1 + \|u_n\|)m(u_n) \to 0$. Then, φ is said so satisfy the Palais-Smale condition ((C) for short), if every Cerami sequence admits a (strongly) convergent subsequence. Clearly, (PS) implies (C), while (C) is more suitable for the non-coercive case.

We shall make use of the following nonsmooth version of the *mountain pass theorem*, see [18, Theorem 2.1.3].

Theorem 3 *Let $\varphi : X \to \mathbb{R}$ be locally Lipschitz continuous and satisfy (C), $u_0, u_1 \in X$, $u_0 \neq u_1$ and $\rho \in (0, \|u_1 - u_0\|)$ be such that*

$$\inf_{\partial B_\rho(u_0)} \varphi = \eta_\rho > \max\{\varphi(u_0), \varphi(u_1)\}.$$

Moreover, let

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}, \quad c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t)).$$

Then, $c \geq \eta_\rho$ and $K_c(\varphi) \neq \emptyset$.

In the proof of our main results we use the following technical lemma, which can be seen as a variant of the nonsmooth *implicit function theorem*, see [18, Theorem 1.3.8]:

Lemma 3 *Let $S \subseteq \partial B_1(0)$ be a nonempty set, let $\varphi : X \to \mathbb{R}$ be a locally Lipschitz continuous functional, $\mu < \inf_S \varphi$ be a real number, and suppose that the following conditions hold:*

(i) $\lim_{t\to\infty}\varphi(tu)=-\infty$ *for all* $u\in S$*;*

(ii) $\langle v^*,v\rangle<0$ *for all* $v\in\varphi^{-1}(\mu)$*,* $v^*\in\partial\varphi(v)$*.*

Then there exists a continuous mapping $\tau:S\to(1,\infty)$ *satisfying*

$$\varphi(tu)\begin{cases}>\mu & \text{if } t<\tau(u),\\ =\mu & \text{if } t=\tau(u),\\ <\mu & \text{if } t>\tau(u)\end{cases}$$

for all $u\in S$ *and* $t\geqslant 1$.

Proof Let us fix $u\in S$. By (i) there exists $t>1$ such that $\varphi(tu)=\mu$.
Claim 1: t is unique.
We argue by contradiction, assuming that there exist t_1,t_2 such that $1<t_1<t_2$ and $\varphi(t_iu)=\mu$ $(i=1,2)$. By Lemma 2 (vi), there exists $v^*\in\partial\varphi(t_1u)$ such that $\varphi^\circ(t_1u;u)=\langle v^*,u\rangle$. Then, by (ii), we have for some $\varepsilon>0$

$$\varphi^\circ(t_1u;u)=\frac{1}{t_1}\langle v^*,t_1u\rangle<-\varepsilon;$$

in particular

$$\limsup_{\substack{t\to t_1\\ h\to 0^+}}\frac{\varphi((t+h)u)-\varphi(tu)}{h}<-\varepsilon.$$

So there exists $\eta>0$ such that for all $t\in(t_1,t_1+\eta)$, $h\in(0,\eta]$

$$\varphi((t+h)u)<\varphi(tu)-h\varepsilon.$$

Letting $t\to t_1$ we get $\varphi((t_1+h)u)<\mu$ for all $h\in(0,\eta]$, hence $\eta<t_2-t_1$. We define the closed set

$$I=\{t>t_1:\ \varphi(tu)=\mu\}.$$

Then $t_2\in I$ and $I\subseteq(t_1+\eta,\infty)$, so there exists $\bar t=\min I$. Clearly $\varphi(tu)<\mu$ for all $t\in(t_1,\bar t)$, while arguing as above we can find $\eta'>0$ such that $\varphi(tu)<\mu$ for all $t\in(\bar t,\bar t+\eta)$. We see that $\bar t$ is a local maximizer of the locally Lipschitz continuous mapping $t\mapsto\varphi(tu)$. So, by Lemma 2 (vii)-$(viii)$ there exists $w^*\in\partial\varphi(\bar tu)$ such that

$$\langle w^*,u\rangle=0.$$

This in turn implies $\langle w^*,\bar tu\rangle=0$ with $\bar tu\in\varphi^{-1}(\mu)$, against (ii). This contradiction proves Claim 1.
For every $u\in S$, we denote by $\tau(u)$ the only number in $(1,\infty)$ satisfying $\varphi(\tau(u)u)=\mu$.
Claim 2: the mapping $\tau:S\to(1+\infty)$ is continuous.
Let $u_0\in S$, $\varepsilon\in(0,\tau(u_0)-1)$ and set $\tau_1=\tau(u_0)-\varepsilon/2$, $\tau_2=\tau(u_0)+\varepsilon/2$. Hence, by definition of τ,

$$\varphi(\tau_1u_0)>\mu>\varphi(\tau_2u_0).$$

Being φ continuous, we can find $\rho>0$ such that $\overline{B}_\rho(\tau_1u_0)\cup\overline{B}_\rho(\tau_2u_0)\subset B_\varepsilon(\tau(u_0)u_0)$ and $\varphi(u_1)>\mu>\varphi(u_2)$ for all $u_i\in\overline{B}_\rho(\tau_iu_0)$ $(i=1,2)$. There exists $\delta>0$ such that for all $u \in S \cap \overline{B}_\delta(u_0)$ we have $\tau_i u \in \overline{B}_\rho(\tau_i u_0)$ $(i = 1,2)$, so $\varphi(\tau_1 u) > \mu > \varphi(\tau_2 u)$. By Claim 1, this in turn implies $\tau(u) \in (\tau_1, \tau_2)$, which rephrases as

$$|\tau(u) - \tau(u_0)| < \varepsilon,$$

therefore τ is continuous, which yields Claim 2 and concludes the proof. □

In view of the forthcoming results, we also need to recall some basic features of the *metric critical point theory*, first introduced by Degiovanni & Marzocchi [17] (see also [12, 15]). Assume that (X,d) is a complete metric space and $\varphi : X \to \mathbb{R}$ is continuous. We define the *weak slope* of φ at $u \in X$ as the supremum $|d\varphi|(u)$ of all numbers $\sigma \in \mathbb{R}$ for which there exist $\delta > 0$ and a continuous mapping $H : [0,\delta] \times B_\delta(u) \to X$ such that

$$d(H(t,v),v) \leqslant t, \quad \varphi(H(t,v)) \leqslant \varphi(v) - \sigma t \quad \text{for all } (t,v) \in [0,\delta] \times B_\delta(u).$$

Thus, a *(metric) critical point* of φ is defined as a point $u \in X$ such that $|d\varphi|(u) = 0$. We set

$$\tilde{K}(\varphi) = \big\{u \in X : |d\varphi|(u) = 0\big\},$$

and for all $c \in \mathbb{R}$

$$\tilde{K}_c(\varphi) = \big\{u \in X : |d\varphi|(u) = 0,\, \varphi(u) = c\big\}.$$

Accordingly, we define a *(metric) Palais-Smale sequence* for φ as a sequence $(u_n)_n$ such that $(\varphi(u_n))_n$ is bounded in $\mathbb{R}$ and $|d\varphi|(u_n) \to 0$. The functional φ is said to satisfy the (metric) Palais-Smale condition (shortly, $(\widetilde{PS})$) if every Palais-Smale sequence admits a convergent subsequence in (X,d).

It is well known that, if $(X, \|\cdot\|)$ is a Banach space and $\varphi : X \to \mathbb{R}$ is locally Lipschitz continuous, then $\tilde{K}_c(\varphi) \subseteq K_c(\varphi)$ for all $c \in \mathbb{R}$, while the reverse inclusion does not hold in general (see [15, Theorem 3.9, Example 3.10]).

We conclude this section with a nonsmooth version of the *second deformation theorem* for locally Lipschitz continuous functionals, which was originally proved in the metric framework:

Theorem 4 *If $\varphi : X \to \mathbb{R}$ is locally Lipschitz continuous and satisfies (C), $a < b \leq \infty$ are such that $K_a(\varphi)$ is a finite set, while $K_c(\varphi) = \emptyset$ for all $c \in (a,b)$, then there exists a continuous mapping $h : [0,1] \times (\overline{\varphi}^b \setminus K_b(\varphi)) \to (\overline{\varphi}^b \setminus K_b(\varphi))$ such that*

(i) $h(0,u) = u$ *and* $h(1,u) \in \overline{\varphi}^a$ *for all* $u \in \overline{\varphi}^b \setminus K_b(\varphi)$;
(ii) $h(t,u) = u$ *for all* $(t,u) \in [0,1] \times \overline{\varphi}^a$;
(iii) $\varphi(h(t,u)) < \varphi(u)$ *for all* $(t,u) \in (0,1] \times (\overline{\varphi}^b \setminus K_b(\varphi))$.

In particular, $\overline{\varphi}^a$ is a strong deformation retract of $(\overline{\varphi}^b \setminus K_b(\varphi))$.

Proof We define on X the so-called *Cerami metric* by setting for all $u, v \in X$

$$d_C(u,v) = \inf_{\gamma \in \Gamma_{u,v}} \int_0^1 \frac{\|\gamma'(t)\|}{1 + \|\gamma(t)\|}\, dt,$$

where $\Gamma_{u,v}$ denotes the set of all piecewise C^1 paths joining u and v. The metric d_C induces the same topology as $\|\cdot\|$, while $\widetilde{(PS)}$-sequences in (X,d_C) coincide with (C)-sequences in $(X,\|\cdot\|)$ (see [12, Remark 4.2]). So, φ is continuous and satisfies $\widetilde{(PS)}$ in (X,d_C).

Moreover, by what observed above, $\tilde{K}_a(\varphi)$ is at most a finite set and $\tilde{K}_c(\varphi)=\emptyset$ for all $c\in(a,b)$. By [12, Theorem 5.3], there exists a deformation $h:[0,1]\times(\overline{\varphi}^b\setminus\tilde{K}_b(\varphi))\to(\overline{\varphi}^b\setminus\tilde{K}_b(\varphi))$ such that, for all $(t,u)\in[0,1]\times(\overline{\varphi}^b\setminus\tilde{K}_b(\varphi))$, the following conditions hold:

(a) if $h(t,u)\neq u$, then $\varphi(h(t,u))<\varphi(u)$;
(b) if $\varphi(u)\leqslant a$, then $h(t,u)=u$;
(c) if $\varphi(u)>a$, then $\varphi(h(1,u))=a$.

We deduce that for all $(t,u)\in[0,1]\times(\overline{\varphi}^b\setminus K_b(\varphi))$ we have $h(t,u)\in\overline{\varphi}^b\setminus K_b(\varphi)$. Otherwise, there would exist $(t,u)\in[0,1]\times(\overline{\varphi}^b\setminus K_b(\varphi))$ with $h(t,u)\in K_b(\varphi)\setminus\tilde{K}_b(\varphi)$; in particular, we would have $\varphi(h(t,u))=b=\varphi(u)$, so that, by (a), $h(t,u)=u$. Hence $u\in K_b(\varphi)$, a contradiction. In conclusion, we can restrict h to a deformation of $\overline{\varphi}^b\setminus K_b(\varphi)$ and we easily see that it satisfies (i)-(iii). □

3 Preliminaries II: nonsmooth Morse theory

In this section we discuss nonsmooth Morse theory, as established in [11], with minor adaptations to make it suitable to the study of problem (1). We refer to [26, Chapter 6] for the properties of singular homology.

For all $B\subseteq A\subseteq X$, $H_k(A,B)$ denotes the k-th singular homology group of a topological pair (A,B) (we choose $\mathbb{R}$ as the ring of coefficients, so $H_k(A,B)$ is a real linear space). Throughout this section we assume that $\varphi:X\to\mathbb{R}$ is locally Lipschitz continuous and satisfies (C). Let $u\in K_c(\varphi)$ ($c\in\mathbb{R}$) be an *isolated* critical point, i.e., there exists a neighborhood $U\subset X$ of u such that $K(\varphi)\cap U=\{u\}$. For all $k\in\mathbb{N}_0$ we define the *k-th critical group of φ at u* as

$$C_k(\varphi,u)=H_k(\overline{\varphi}^c\cap U,\overline{\varphi}^c\cap U\setminus\{u\}).$$

Due to Theorem 4 and the excision property of singular homology groups, $C_k(\varphi,u)$ is independent of the choice of U. We recall now a decomposition result for singular homology groups of sublevel sets of φ:

Lemma 4 *If $a<b\leqslant\infty$, $c\in(a,b)$ is the only critical value of φ in $[a,b]$, and $K_c(\varphi)$ is a finite set, then for all $k\in\mathbb{N}_0$*

$$H_k(\overline{\varphi}^b,\overline{\varphi}^a)=\bigoplus_{u\in K_c(\varphi)}C_k(\varphi,u).$$

Proof Assume that $K_c(\varphi)=\{u_1,\dots u_n\}$. We can find pairwise disjoint closed neigborhoods $U_1,\dots U_n\subset X$ of $u_1,\dots u_n$, respectively, and set $U=\bigcup_{i=1}^n U_i$. By Theorem 4,

$\overline{\varphi}^a$ and $\overline{\varphi}^c$ are strong deformation retracts of $\overline{\varphi}^c \setminus \{u_1, \dots u_n\}$ and $\overline{\varphi}^b$, respectively. By the excision property and [26, Corollary 6.15, Proposition 6.18], we have

$$\begin{aligned} H_k(\overline{\varphi}^b, \overline{\varphi}^a) &= H_k(\overline{\varphi}^c, \overline{\varphi}^c \setminus \{u_1, \dots u_n\}) = H_k(\overline{\varphi}^c \cap U, (\overline{\varphi}^c \cap U) \setminus \{u_1, \dots u_n\}) \\ &= \bigoplus_{i=1}^n H_k(\overline{\varphi}^c \cap U_i, (\overline{\varphi}^c \cap U_i) \setminus \{u_i\}) = \bigoplus_{i=1}^n C_k(\varphi, u_i), \end{aligned}$$

which proves our assertion. □

Assuming $\inf_{K(\varphi)} \varphi > -\infty$, we may define for all $k \in \mathbb{N}_0$ the *k-th critical group of φ at infinity* as

$$C_k(\varphi, \infty) = H_k(X, \overline{\varphi}^c),$$

where $c < \inf_{K(\varphi)} \varphi$. By Theorem 4 and [26, Corollary 6.15], $C_k(\varphi, \infty)$ is invariant with respect to c.

Henceforth, we will assume that $K(\varphi)$ is a *finite* set (in particular, all critical points are isolated). Now let $a < b$ be real numbers such that $\varphi(u) \notin \{a, b\}$ for all $u \in K(\varphi)$. For all $k \in \mathbb{N}_0$ we define the *k-th Morse type number* and the *k-th Betti type number* of the interval $[a, b]$ as

$$M_k(a, b) = \sum_{u \in K(\varphi) \cap \varphi^{-1}([a,b])} \dim C_k(\varphi, u), \quad \beta_k(a, b) = \dim H_k(\overline{\varphi}^b, \overline{\varphi}^a),$$

respectively. If $a < \varphi(u) < b$ for all $u \in K(\varphi)$, then again by Theorem 4 and [26, Corollary 6.15] we have

$$\beta_k(a, b) = \dim C_k(\varphi, \infty) \text{ for all } k \in \mathbb{N}_0. \tag{4}$$

Accordingly, we define two formal power series in the variable t, the *Morse polynomial* and the *Poincaré polynomial* as

$$M(a, b; t) = \sum_{k=0}^{\infty} M_k(a, b) t^k, \quad P(a, b; t) = \sum_{k=0}^{\infty} \beta_k(a, b) t^k,$$

respectively. The following identity, known as the nonsmooth *Morse identity*, will be the key tool in our study:

Theorem 5 *If $a < b$ are real numbers such that $\varphi(u) \in (a, b)$ for all $u \in K(\varphi)$, $M_k(a, b) < \infty$ for all $k \in \mathbb{N}_0$, and $M_k(a, b) = 0$ for k big enough, then there exists a polynomial with non-negative integer coefficients $Q(t)$ such that*

$$M(a, b; t) = P(a, b; t) + (1 + t) Q(t) \text{ for all } t \in \mathbb{R}.$$

Proof We first prove that for all $k \in \mathbb{N}_0$

$$\beta_k(a, b) \leqslant M_k(a, b). \tag{5}$$

Indeed, assume that $c_1, \dots, c_m$, with $a < c_1 < \dots < c_m < b$, are the critical values of φ in $[a,b]$. We fix $m+1$ real numbers $d_0, \dots, d_m$, such that $a = d_0 < c_1 < d_1 < c_2 < \dots < c_m < d_m = b$. By [26, Lemma 6.56 (a)] and Lemma 4, we have for all $k \in \mathbb{N}_0$

$$\dim H_k(\overline{\varphi}^b, \overline{\varphi}^a) \leqslant \sum_{j=1}^{m} \dim H_k(\overline{\varphi}^{d_j}, \overline{\varphi}^{d_{j-1}}) = \sum_{j=1}^{m} \sum_{u \in K_{c_j}(\varphi)} \dim C_k(\varphi, u),$$

which yields (5). In turn, (5) implies that $\beta_k(a,b)$ is finite and vanishes for k big enough. Now, again by Lemma 4 and [26, Lemma 6.56 (b)] we have for all $t \in \mathbb{R}$

$$\sum_{k=0}^{\infty} M_k(a,b)t^k = \sum_{j=1}^{m} \sum_{k=0}^{\infty} \beta_k(d_j, d_{j-1})t^k = \sum_{k=0}^{\infty} \beta_k(a,b)t^k + (1+t)Q(t)$$

for a convenient polynomial Q with coefficients in $\mathbb{N}_0$. This concludes the proof. $\square$

If in the Morse identity we choose $t = -1$, we obtain the nonsmooth *Poincaré-Hopf formula*:

$$\sum_{k=0}^{\infty} (-1)^k M_k(a,b) = \sum_{k=0}^{\infty} (-1)^k \beta_k(a,b). \tag{6}$$

4 Constant sign solutions

In this section we will prove the existence of a positive and a negative solution of (1). We keep the definitions of φ, N, and A given in the Introduction. Furthermore, we set $s^\pm = \max\{\pm s, 0\}$, and for all $(x,s) \in \Omega \times \mathbb{R}$ we set $j_\pm(x,s) = j(x, \pm s^\pm)$ (note that $j_\pm(x,\cdot)$ is locally Lipschitz continuous for a.a. $x \in \Omega$). We fix $\varepsilon > 0$ and for all $u \in W^{1,p}(\Omega)$ we define

$$\varphi_\pm^\varepsilon(u) = \frac{\|\nabla u\|_p^p}{p} + \frac{\varepsilon \|u^\mp\|_p^p}{p} - \int_\Omega j_\pm(x,u)\,dx$$

(note that φ and $\varphi_\pm^\varepsilon$ agree on $\pm W_+$, respectively). Moreover, for all $u \in L^r(\Omega)$ we set

$$N_\pm(u) = \left\{ w \in L^{r'}(\Omega) : w \in \partial j_\pm(x,u) \text{ a.e. in } \Omega \right\}.$$

Finally, $m_\pm^\varepsilon(u)$ are defined as in (3):

$$m_\pm^\varepsilon(u) = \min_{u^* \in \partial \varphi_\pm^\varepsilon(u)} \|u^*\|_*.$$

A fundamental property of the operator A (see [26, Proposition 2.72]) is:

Lemma 5 *The mapping $A : W^{1,p}(\Omega) \to W^*$ is continuous and has the $(S)_+$ property, i.e., if $(u_n)_n$ is a sequence in $W^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ and*

$$\limsup_{n \to \infty} \langle A(u_n), u_n - u \rangle \leq 0,$$

then $u_n \to u$ in $W^{1,p}(\Omega)$.

As said in the Introduction, the functional φ is the nonsmooth energy of problem (1):

Lemma 6 *If hypotheses* **H** *hold, then* $\varphi : W^{1,p}(\Omega) \to \mathbb{R}$ *is locally Lipschitz continuous and satisfies* (C)*. Moreover, if* $u \in K(\varphi)$*, then* $u \in C^1(\overline{\Omega})$ *is a solution of* (1).

Proof We assume $p < N$ (the argument for $p \geq N$ is easier). The functional $u \mapsto \|\nabla u\|_p^p/p$ is of class C^1 with derivative A. By **H** (i), the functional

$$u \mapsto \int_\Omega j(x,u)\,dx$$

is Lipschitz continuous on bounded sets of $L^r(\Omega)$ with generalized subdifferential contained in $N(\cdot)$ by the Aubin-Clarke theorem (see [18, Theorem 1.3.10]). By Lemma 2 (v) and the continuous embedding $W^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$, we deduce that φ is locally Lipschitz continuous in $W^{1,p}(\Omega)$ with

$$\partial\varphi(u) \subseteq A(u) - N(u) \text{ for all } u \in W^{1,p}(\Omega). \tag{7}$$

Now we prove that φ satisfies (C). Let $(u_n)_n$ be a sequence in $W^{1,p}(\Omega)$ such that $(\varphi(u_n))_n$ is bounded in $\mathbb{R}$ and $(1+\|u_n\|)m(u_n) \to 0$. We first prove that $(u_n)_n$ is bounded in $W^{1,p}(\Omega)$. Indeed, by Lemma 2 (i), (3) and (7), for all $n \in \mathbb{N}$ there exists $w_n \in N(u_n)$ such that $m(u_n) = \|A(u_n) - w_n\|_*$. So, for all $v \in W^{1,p}(\Omega)$ we have

$$\left|\langle A(u_n), v\rangle - \int_\Omega w_n v\,dx\right| \leqslant m(u_n)\|v\|. \tag{8}$$

Taking $v = u_n$, we get

$$-\|\nabla u_n\|_p^p + \int_\Omega w_n u_n\,dx \leqslant m(u_n)\|u_n\| = o(1). \tag{9}$$

Moreover, since $(\varphi(u_n))_n$ is bounded, we have

$$\|\nabla u_n\|_p^p - \int_\Omega p\,j(x,u_n)\,dx \leqslant c. \tag{10}$$

Adding (9) and (10) above, we obtain

$$\int_\Omega \big(w_n u_n - p\,j(x,u_n)\big)\,dx \leqslant c. \tag{11}$$

By **H** (iii) there exist $\beta, k > 0$ such that

$$\xi s - p\,j(x,s) \geq \beta|s|^q \text{ a.e. in } \Omega, \text{ for all } |s| > k,\ \xi \in \partial j(x,s). \tag{12}$$

So, from (11), (12) and **H** (i) we have

$$c \geqslant \int_{\{|u_n|>k\}} \beta|u_n|^q\,dx + \int_{\{|u_n|\leq k\}} \big(w_n u_n - p\,j(x,u_n)\big)\,dx \geqslant \beta\|u_n\|_q^q - c.$$

Thus, $(u_n)_n$ is bounded in $L^q(\Omega)$. Since $r < p^*$, we know that $(r-p)N/p < r$, so we may assume $q < r < p^*$ without any loss of generality. We set

$$\tau = \left(\frac{1}{q} - \frac{1}{r}\right)\left(\frac{1}{q} - \frac{1}{p^*}\right)^{-1} \in (0,1),$$

hence we have $1/r = (1-\tau)/q + \tau/p^*$ and $\tau r < p$. By the interpolation inequality (see [6, Remark 2, p. 93]) and the continuous embedding $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, we have

$$\|u_n\|_r \leqslant \|u_n\|_{p^*}^{\tau} \|u_n\|_q^{1-\tau} \leqslant c\|u_n\|^{\tau}.$$

By **H** (i), (8), Hölder inequality, and the inequality above, we obtain

$$\begin{aligned}\|u_n\|^p &\leqslant \int_\Omega a_0(|u_n| + |u_n|^r)\,dx + \|u_n\|_p^p + c \\ &\leqslant c\big(1 + \|u_n\|_1 + \|u_n\|_r^r + \|u_n\|_p^p\big) \\ &\leqslant c\big(1 + \|u_n\| + \|u_n\|^{\tau r} + \|u_n\|^{\tau p}\big).\end{aligned}$$

Recalling that $\max\{1, \tau r, \tau p\} < p$, we finally deduce that $(u_n)_n$ is bounded in $W^{1,p}(\Omega)$. Passing to a subsequence, we may assume that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ and $u_n \to u$ in $L^\sigma(\Omega)$ for every $\sigma \in [1, p^*)$. From (8) we have for all $n \in \mathbb{N}$

$$\begin{aligned}\langle A(u_n), u_n - u\rangle &\leqslant m(u_n)\|u_n - u\| + \int_\Omega |w_n(u_n - u)|\,dx \\ &\leqslant m(u_n)\|u_n - u\| + \|w_n\|_{r'}\|u_n - u\|_r = o(1).\end{aligned}$$

By the $(S)_+$ property of A in $W^{1,p}(\Omega)$ (see Lemma 5), we have that $u_n \to u$ in $W^{1,p}(\Omega)$. So (C) holds.

Finally, let $u \in K(\varphi)$. Then, by (7) there exists $w \in N(u)$ such that $A(u) = w$ in W^*. By [18, Theorem 1.5.5, Remark 1.5.9] we have that $u \in L^\infty(\Omega)$. Nonlinear regularity theory (see [23, Theorem 2]) then implies that $u \in C^1(\overline{\Omega})$. Thus, u is a solution of (1). □

The role of functionals $\varphi_\pm^\varepsilon$ is that of selecting strictly positive or negative solutions:

Lemma 7 *If hypotheses* **H** *hold, then* $\varphi_\pm^\varepsilon : W^{1,p}(\Omega) \to \mathbb{R}$ *are locally Lipschitz continuous and satisfy* (C). *Moreover, if* $u \in K(\varphi_\pm^\varepsilon) \setminus \{0\}$, *then* $u \in \pm\mathrm{int}(C_+)$ *is a solution of* (1).

Proof Let us consider φ_+^ε (the argument for φ_-^ε is analogous). For all $(x,s) \in \Omega \times \mathbb{R}$ we have

$$\partial j_+(x,s) \begin{cases} = \{0\} & \text{if } s < 0, \\ \subseteq \big\{\tau\xi : \xi \in \partial j(x,0),\ \tau \in [0,1]\big\} & \text{if } s = 0, \\ = \partial j(x,s) & \text{if } s > 0, \end{cases} \tag{13}$$

in particular $j_+ : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies **H** (i). Arguing as in the proof of Lemma 6 we deduce that φ_+^ε is locally Lipschitz and for all $u \in W^{1,p}(\Omega)$

$$\partial\varphi_+^\varepsilon(u) \subseteq A(u) - \varepsilon(u^-)^{p-1} - N_+(u). \tag{14}$$

We prove that φ_+^ε satisfies (C). Let $(u_n)_n$ be a sequence in $W^{1,p}(\Omega)$ such that $(\varphi_+^\varepsilon(u_n))_n$ is bounded in $\mathbb{R}$ and $(1 + \|u_n\|)m_+^\varepsilon(u_n) \to 0$. By Lemma 2 (i) and (14), for all $n \in \mathbb{N}$ there exists $w_n \in N_+(u_n)$ such that $m_+^\varepsilon(u_n) = \|A(u_n) - \varepsilon(u_n^-)^{p-1} - w_n\|_*$. So, for all $v \in W^{1,p}(\Omega)$ we get

$$\left| \langle A(u_n), v\rangle - \int_\Omega \big(\varepsilon(u_n^-)^{p-1} + w_n\big) v\, dx \right| \le m_+^\varepsilon(u_n)\|v\|. \tag{15}$$

We prove now that $(u_n)_n$ is bounded in $L^q(\Omega)$, considering separately the sequences $(u_n^+)_n$, $(u_n^-)_n$. Testing (15) with $v = u_n^+$, we have (note that $u_n^+ u_n^- = 0$ a.e. in Ω)

$$\|\nabla u_n^+\|_p^p - \int_\Omega w_n u_n^+\, dx = o(1),$$

while from the bound on $(\varphi(u_n))_n$ we have (note that $(u_n^+)^- = 0$ a.e. in Ω)

$$\|\nabla u_n^+\|_p^p - \int_\Omega p j(x, u_n^+)\, dx \leqslant c.$$

Applying **H** (i), (iii) as in Lemma 6, we see that $(u_n^+)_n$ is bounded in $L^q(\Omega)$. On the negative side, testing (15) with $v = -u_n^-$, we have

$$\|\nabla u_n^-\|_p^p + \varepsilon\|u_n^-\|_p^p = o(1),$$

hence $u_n^- \to 0$ in $W^{1,p}(\Omega)$. Reasoning as in Lemma 6, we have $\|u_n\|_r \leqslant c\|u_n\|^\tau$ for some $\tau \in [0, p/r)$. Testing (15) with $v = u_n$, we have by the estimates above and Hölder inequality

$$\begin{aligned} \|\nabla u_n\|_p^p &\leqslant \varepsilon\|u_n^-\|_p^p + \int_\Omega w_n u_n\, dx + o(1) \\ &\leqslant c\big(1 + \|u_n\|_1 + \|u_n\|_r^r\big) \\ &\leqslant c\big(1 + \|u_n\|_q + \|u_n\|^{\tau r}\big) \\ &\leqslant c\big(1 + \|\nabla u_n\|_p^{\tau r}\big), \end{aligned}$$

the last inequality following from the fact that $\|\nabla(\cdot)\|_p + \|\cdot\|_q$ is an equivalent norm on $W^{1,p}(\Omega)$ (see [6, Remark 15, p. 286]). Since $\tau r < p$, we easily deduce that $(u_n)_n$ is bounded in $W^{1,p}(\Omega)$. Now, we argue as in Lemma 6 and find a convergent subsequence.

Finally, let $u \in K(\varphi_+^\varepsilon) \setminus \{0\}$. Since $0 \in \partial\varphi_+^\varepsilon(u)$, by (14) we find $w \in N_+(u)$ such that

$$A(u) = \varepsilon(u^-)^{p-1} + w \text{ in } W^*. \tag{16}$$

Taking $v = u^-$, from (13) we have

$$\|\nabla u^-\|_p^p + \varepsilon\|u^-\|_p^p = 0,$$

hence $u \in W_+ \setminus \{0\}$. Arguing as in Lemma 6, we get $u \in C_+ \setminus \{0\}$. By **H** (v) we can apply the nonlinear maximum principle (see [34, Theorem 5] or [32, Theorem 1.1.1]) and we obtain $u \in \mathrm{int}(C_+)$. In particular, $u^- = 0$ and $N_+(u) = N(u)$, so by (16) u is a solution of (1). □

The next lemma deals with the zero solution:

Lemma 8 *If hypotheses* **H** *hold, then* 0 *is a local minimizer of both* φ *and* $\varphi_\pm^\varepsilon$*. In particular,* 0 *is a solution of* (1).

Proof We deal with φ (the argument for $\varphi_\pm^\varepsilon$ is analogous). Let $\delta_0 > 0$ be as in hypothesis **H** (iv) and set

$$B_{\delta_0}^C(0) = \left\{u \in C^1(\overline{\Omega}) : \|u\|_{C^1} < \delta_0\right\}.$$

In particular, for all $u \in B_{\delta_0}^C(0) \setminus \{0\}$ we have $\|u\|_\infty < \delta_0$, hence

$$\varphi(u) > -\int_\Omega j(x,u)\,dx \geqslant -\int_\Omega j(x,0)\,dx = \varphi(0).$$

So, 0 is a (strict) local minimizer of the restriction of φ to $C^1(\overline{\Omega})$. Reasoning as in [20, Proposition 3], we see that 0 is a (not necessarily strict) local minimizer in $W^{1,p}(\Omega)$ as well, hence by Lemma 2 $(viii)$ a critical point of φ. By Lemma 6, the function $u = 0$ solves (1). □

Now we are ready to introduce our two solutions result:

Theorem 6 *If hypotheses* **H** *hold, then problem* (1) *admits at least two non-zero solutions* $u_+ \in \mathrm{int}(C_+)$ *and* $u_- \in -\mathrm{int}(C_+)$.

Proof We focus on positive solutions, so we consider the functional φ_+^ε. From Lemma 8 we know that 0 is a local minimizer of φ_+^ε. If 0 is not a *strict* local minimizer, then we easily find another critical point $u_+ \in K(\varphi_+^\varepsilon)$, which by Lemma 7 turns out to be a positive solution of (1).
So, we assume that 0 is a strict local minimizer of φ_+^ε, i.e., there exists $\rho > 0$ such that $\varphi_+^\varepsilon(u) > \varphi_+^\varepsilon(0)$ for all $u \in \overline{B}_\rho(0) \setminus \{0\}$. We prove that in fact a stronger property holds:

$$\inf_{\partial B_\rho(0)} \varphi_+^\varepsilon =: \eta_\rho > \varphi_+^\varepsilon(0). \tag{17}$$

We argue by contradiction: assume that there exists a sequence $(u_n)_n$ in $\partial B_\rho(0)$ such that $\varphi_+^\varepsilon(u_n) \to \varphi_+^\varepsilon(0)$ as $n \to \infty$. Passing to a subsequence, we may assume that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ and $u_n \to u$ in $L^t(\Omega)$ for all $t \in [1, p^*)$. Clearly $u \in \overline{B}_\rho(0)$. It is easily seen that φ_+^ε is sequentially weakly lower semicontinuous in $W^{1,p}(\Omega)$, hence

$$\varphi_+^\varepsilon(u) \leq \liminf_n \varphi_+^\varepsilon(u_n) = \varphi_+^\varepsilon(0),$$

which in turn implies $u = 0$. Thus, by the relations above, we have

$$\frac{\|\nabla u_n\|_p^p}{p} + \frac{\varepsilon\|u_n\|_p^p}{p} = \varphi_+^\varepsilon(u_n) + \frac{\varepsilon\|u_n^+\|_p^p}{p} + \int_\Omega j_+(x,u_n)\,dx = o(1).$$

So $u_n \to 0$ in $W^{1,p}(\Omega)$, against the assumption that $u_n \in \partial B_\rho(0)$ for all $n \in \mathbb{N}$.
Now, by hypothesis **H** (ii) there exists $k \in \mathbb{R}$ such that

$$k > |\Omega|^{-\frac{1}{p}} \max\{\rho, |\eta_\rho|^{\frac{1}{p}}\}, \quad j(x,k) > k^p \text{ a.e. in } \Omega$$

(by $|\Omega|$ we denote the N-dimensional Lebesgue measure of Ω). Setting $\overline{u}(x) = k$ for all $x \in \Omega$, we have $\overline{u} \in W^{1,p}(\Omega)$ and $\|\overline{u}\| > \rho$. On the other hand,

$$\varphi_+^\varepsilon(\overline{u}) = -\int_\Omega j_+(x,k)\,dx < \eta_\rho.$$

We apply Theorem 3 with $u_0 = 0$, $u_1 = \overline{u}$, setting

$$\Gamma_+ = \left\{\gamma \in C([0,1], W^{1,p}(\Omega) : \gamma(0) = 0,\, \gamma(1) = \overline{u}\right\},$$

$$c_+ = \inf_{\gamma\in\Gamma_+} \max_{t\in[0,1]} \varphi_+^\varepsilon(\gamma(t)).$$

We find that that $c_+ \geq \eta_\rho$ and there exists $u_+ \in K_{c_+}(\varphi_+^\varepsilon)$. From (17) we know that $u_+ \neq 0$. Then, by Lemma 7, $u_+ \in \mathrm{int}(C_+)$ is a solution of (1).
A similar argument, applied to φ_-^ε, shows the existence of a solution $u_- \in -\mathrm{int}(C_+)$, which concludes the proof. □

5 Critical groups and the third solution

In this section we aim at proving the existence of a fourth critical point of the nonsmooth energy functional φ (which will turn out to be a third non-zero solution of problem (1)).
Avoiding trivial situations, we may assume that both $K(\varphi)$ and $K(\varphi_\pm^\varepsilon)$ are *finite* sets. In particular, then, every critical point of either φ or $\varphi_\pm^\varepsilon$ is isolated. We begin by computing the critical groups at infinity of φ (our result is the nonsmooth extension of [26, Proposition 6.64]):

Lemma 9 *If hypotheses* **H** *hold, then* $C_k(\varphi,\infty) = 0$ *for all* $k \in \mathbb{N}_0$.

Proof We prove that

$$\lim_{t\to\infty} \varphi(tu) = -\infty \quad \text{for all } u \in \partial B_1(0). \tag{18}$$

Indeed, fix $u \in \partial B_1(0)$ and choose $\sigma > 0$ (depending on u) such that $\sigma\|u\|_p^p > \|\nabla u\|_p^p/p$. By **H** (ii), there exists $k > 0$ such that $j(x,s) > \sigma|s|^p$ a.e. in Ω and for all $|s| > k$. We can find $t_0 > 1$ such that $t_0|u| > k$ on some subset of Ω with positive measure. So, for $t > t_0$ big enough, we have

$$\begin{aligned}
\varphi(tu) &= \frac{t^p\|\nabla u\|_p^p}{p} - \int_{\{|u|>k/t\}} j(x,tu)\,dx - \int_{\{|u|\leq k/t\}} j(x,tu)\,dx \\
&\leqslant \frac{t^p\|\nabla u\|_p^p}{p} - \sigma \int_{\{|u|>k/t\}} |tu|^p\,dx + c \\
&\leqslant t^p\Big(\frac{\|\nabla u\|_p^p}{p} - \sigma\|u\|_p^p\Big) + \sigma k^p|\Omega| + c,
\end{aligned}$$

and the last quantity tends to $-\infty$ as $t \to \infty$, which proves (18).

It is easily seen that φ is sequentially weakly lower semicontinuous is $W^{1,p}(\Omega)$, hence

$$\inf_{K(\varphi)\cup\overline{B}_1(0)} \varphi =: \theta > -\infty.$$

Now, we prove that there exists $\mu < \theta$ such that

$$\langle v^*, v\rangle < 0 \text{ for all } v \in \varphi^{-1}(\mu),\, v^* \in \partial\varphi(v). \tag{19}$$

First, by **H** (iii), we can find $\beta > 0, \tilde{k} > 0$ such that

$$\xi s - pj(x,s) > \beta|s|^q \text{ a.e. in } \Omega \text{ for all } |s| > \tilde{k},\, \xi \in \partial j(x,s).$$

For all $v \in W^{1,p}(\Omega)$ and $v^* \in \partial\varphi(v)$ there exists $w \in N(v)$ such that $v^* = A(v) - w$ in W^*. By **H** (i) we have

$$\begin{aligned}\int_\Omega \big(pj(x,v) - wv\big)\,dx &\le \int_{\{|v|\le\tilde{k}\}} \big(pj(x,v) - wv\big)\,dx - \beta\int_{\{|v|>\tilde{k}\}} |v|^q\,dx \\ &\le c - \beta\int_{\{|v|>\tilde{k}\}} |v|^q\,dx \leqslant \alpha,\end{aligned}$$

for some $\alpha > 0$ independent of v. Applying the inequality above, we have

$$\langle v^*, v\rangle = \|\nabla v\|_p^p - \int_\Omega wv\,dx \le p\varphi(v) + \alpha.$$

Let us choose

$$\mu < \min\left\{\theta, -\frac{\alpha}{p}\right\}. \tag{20}$$

For all $v \in \varphi^{-1}(\mu)$, $v^* \in \partial\varphi(v)$ we immediately get (19). Now Lemma 3 ensures the existence of a continuous mapping $\tau : \partial B_1(0) \to (1,\infty)$ such that for all $u \in \partial B_1(0)$, $t \ge 1$

$$\varphi(tu)\begin{cases} > \mu & \text{if } t < \tau(u), \\ = \mu & \text{if } t = \tau(u), \\ < \mu & \text{if } t > \tau(u). \end{cases}$$

Clearly, from the choice of μ we have

$$\overline{\varphi}^\mu = \big\{tu : u \in \partial B_1(0), t \ge \tau(u)\big\}.$$

We set

$$D = \big\{tu : u \in \partial B_1(0), t \ge 1\big\}$$

and define a continuous deformation $h : [0,1]\times D \to D$ by putting for all $(s,tu) \in [0,1]\times D$

$$h(s,tu) = \begin{cases} (1-s)tu + s\tau(u)u & \text{if } t < \tau(u), \\ tu & \text{if } t \ge \tau(u). \end{cases}$$

Then, for all $tu \in D$ we have $h(1,tu) \in \overline{\varphi}^\mu$. Moreover, we have $h(s,tu) = tu$ for all $s \in [0,1]$, $tu \in \overline{\varphi}^\mu$. Hence, $\overline{\varphi}^\mu$ is a strong deformation retract of D. Besides, we set for all $(s,tu) \in [0,1]\times D$

$$\tilde{h}(s,tu) = (1-s)tu + su,$$

and we see that $\partial B_1(0)$ is a strong deformation retract of D as well, by means of $\tilde{h}$. Applying [26, Corollary 6.15] twice, we have for all $k \in \mathbb{N}_0$

$$H_k(W^{1,p}(\Omega), \overline{\varphi}^\mu) = H_k(W^{1,p}(\Omega), D) = H_k(W^{1,p}(\Omega), \partial B_1(0)) = 0,$$

the last equality coming from [26, Propositions 6.24, 6.25] (recall that the sphere $\partial B_1(0)$ is contractible in itself, as $\dim(W^{1,p}(\Omega)) = \infty$). Finally, recalling (20) and the definition of critical group at infinity is enough to deduce $C_k(\varphi, \infty) = 0$ for all $k \in \mathbb{N}_0$, which concludes the proof. □

Similarly we compute the critical groups at infinity of $\varphi^\varepsilon_\pm$:

Lemma 10 *If hypotheses* **H** *hold, then* $C_k(\varphi^\varepsilon_\pm, \infty) = 0$ *for all* $k \in \mathbb{N}_0$.

Proof We deal with φ^ε_+ (the argument for φ^ε_- is similar). We set

$$S_+ = \{u \in \partial B_1(0) : u^+ \neq 0\},\ B_+ = \{tu : t \in [0,1], u \in S_+\}.$$

As in the proof of Lemma 9, using **H** (i) - (iii) we prove that

$$\lim_{t\to\infty} \varphi^\varepsilon_+(tu) = -\infty \quad \text{for all } u \in S_+,$$

and find

$$\mu < \inf_{K(\varphi^\varepsilon_+)\cup B_+} \varphi^\varepsilon_+$$

such that

$$\langle v^*, v\rangle < 0 \text{ for all } v \in (\varphi^\varepsilon_+)^{-1}(\mu),\ v^* \in \partial\varphi^\varepsilon_+(v).$$

Now, Lemma 3 ensures the existence of a continuous mapping $\tau_+ : S_+ \to (1,\infty)$ such that for all $u \in S_+$, $t \geq 1$

$$\varphi^\varepsilon_+(tu) \begin{cases} > \mu & \text{if } t < \tau_+(u), \\ = \mu & \text{if } t = \tau_+(u), \\ < \mu & \text{if } t > \tau_+(u). \end{cases}$$

Taking μ even smaller if necessary, we may assume

$$\mu < -\int_\Omega j(x,0)\,dx,$$

so that for all $u \in \overline{(\varphi^\varepsilon_+)}^\mu$ we have $u^+ \neq 0$. Therefore we have

$$\overline{(\varphi^\varepsilon_+)}^\mu = \{tu : u \in S_+, t \geqslant \tau_+(u)\}.$$

We set

$$D_+ = \{tu : u \in S_+, t \geqslant 1\}$$

and for all $(s,tu) \in [0,1] \times D_+$ we set

$$h_+(s,tu) = \begin{cases} (1-s)tu + s\tau_+(u)u & \text{if } t < \tau_+(u), \\ tu & \text{if } t \geq \tau_+(u), \end{cases}$$

$$\tilde{h}_+(s,tu) = (1-s)tu + su.$$

Therefore, we see that $\overline{(\varphi_+^\varepsilon)}^\mu$ and S_+ are strong deformation retracts of D_+ by means of h_+ and $\tilde{h}_+$, respectively. So we have for all $k \in \mathbb{N}_0$

$$H_k(W^{1,p}(\Omega), \overline{(\varphi_+^\varepsilon)}^\mu) = H_k(W^{1,p}(\Omega), D_+) = H_k(W^{1,p}(\Omega), S_+). \tag{21}$$

We set $u_0(x) = |\Omega|^{-1/p}$ for all $x \in \Omega$, hence $u_0 \in S_+$. Set for all $(s,u) \in [0,1] \times S_+$,

$$\hat{h}(s,u) = \frac{(1-s)u + su_0}{\|(1-s)u + su_0\|}$$

(note that $\hat{h}$ is well defined as $u^+ \neq 0$), so $\hat{h} : [0,1] \times S_+ \to S_+$ is a continuous deformation and $\{u_0\}$ turns out to be a strong deformation retract of S_+. So, S_+ is contractible in itself and by (21) and the definition of critical groups at infinity we have $C_k(\varphi_+^\varepsilon, \infty) = 0$ for all $k \in \mathbb{N}_0$. □

From Lemma 8 and Theorem 6 we know that φ has at least three critical points, namely 0, u_+, and u_-. We will conclude the proof of Theorem 1 by proving the existence of a fourth critical point of φ. We argue by contradiction, so in the following we assume

(A) 0, u_+, u_- are the only solutions of (1), in particular $K(\varphi) = \{0, u_+, u_-\}$

(hence, all critical points of φ are isolated). We aim at applying the nonsmooth Poincaré-Hopf formula (6), so we need to compute the critical groups of φ at all of its critical points (also the critical groups of $\varphi_\pm^\varepsilon$ will be involved in our argument). We begin by computing the critical groups at 0:

Lemma 11 *If hypotheses* **H** *and* (A) *hold, then* $C_k(\varphi, 0) = C_k(\varphi_\pm^\varepsilon, 0) = \delta_{k,0}\mathbb{R}$ *for all* $k \in \mathbb{N}_0$.

Proof from Lemma 8 and (A) we easily deduce that 0 is a strict local minimizer of both φ and $\varphi_\pm^\varepsilon$. So, the conclusion follows at once from [26, Axiom 7, Remark 6.10]. □

Computation of the critical groups at $u_\pm$ is a more delicate issue:

Lemma 12 *If hypotheses* **H** *and* (A) *hold, then* $C_k(\varphi, u_\pm) = \delta_{k,1}\mathbb{R}$ *for all* $k \in \mathbb{N}_0$.

Proof We consider u_+ (the argument for u_- is analogous). First we prove that

$$C_k(\varphi, u_+) = C_k(\varphi_+^\varepsilon, u_+). \tag{22}$$

For all $t \in [0,1]$ we set $\psi_t = (1-t)\varphi + t\varphi_+^\varepsilon$. Clearly, $\psi_t : W^{1,p}(\Omega) \to \mathbb{R}$ is locally Lipschitz continuous and satisfies (C) for all $t \in [0,1]$; moreover $\psi_0 = \varphi$ and $\psi_1 = \varphi_+^\varepsilon$.
Claim 1: $u_+ \in K(\psi_t)$ for all $t \in [0,1]$.
We need to restrict ourselves to the space $C^1(\overline{\Omega})$. So we set $\tilde{\varphi} = \varphi|_{C^1(\overline{\Omega})}$ and $\tilde{\psi}_t = \psi_t|_{C^1(\overline{\Omega})}$, and we easily see that $\tilde{\varphi}, \tilde{\psi}_t$ are locally Lipschitz continuous in $C^1(\overline{\Omega})$. Since $0 \in \partial\varphi(u_+)$, from the continuous embedding $C^1(\overline{\Omega}) \hookrightarrow W^{1,p}(\Omega)$, we see that $0 \in \partial\tilde{\varphi}(u_+)$, i.e., $u_+ \in K(\tilde{\varphi})$.

Now, we recall that $u_+ \in \mathrm{int}(C_+)$ and we note that $\tilde{\varphi} = \tilde{\psi}_t$ in C_+, so for all $v \in C^1(\overline{\Omega})$ we have

$$\begin{aligned}\tilde{\psi}_t^\circ(u_+;v) &= \limsup_{\substack{w\to u_+\\ \tau\to 0^+}} \frac{\tilde{\psi}_t(w+\tau v)-\tilde{\psi}_t(w)}{\tau}\\ &= \limsup_{\substack{w\to u_+\\ \tau\to 0^+}} \frac{\tilde{\varphi}(w+\tau v)-\tilde{\varphi}(w)}{\tau} = \tilde{\varphi}^\circ(u_+;v) \geqslant 0\end{aligned}$$

(note that $w+\tau v \in \mathrm{int}(C_+)$ as soon as τ is small enough and w is close enough to u_+ in $C^1(\overline{\Omega})$), hence $0 \in \partial\tilde{\psi}_t(u_+)$, i.e., $u_+ \in K(\tilde{\psi}_t)$. For all $v \in W^{1,p}(\Omega)$ there exists a sequence (v_n) in $C^1(\overline{\Omega})$ such that $v_n \to v$ in $W^{1,p}(\Omega)$, so by Lemma 1(i) we have

$$\psi_t^\circ(u_+;v) = \lim_n \psi_t^\circ(u_+;v_n) \geqslant \liminf_n \tilde{\psi}_t^\circ(u_+;v_n) \geqslant 0,$$

hence $u_+ \in K(\psi_t)$, which proves Claim 1.

Claim 2: u_+ is an isolated critical point of ψ_t, uniformly with respect to $t \in [0,1]$, i.e., there exists $\rho > 0$ such that $K(\psi_t)\cap B_\rho(u_+) = \{u_+\}$ for all $t \in [0,1]$.

We argue by contradiction, assuming that there exist sequences $(t_n)_n$ in $[0,1]$ and $(u_n)_n$ in $W^{1,p}(\Omega)\setminus\{u_+\}$ such that $u_n \in K(\psi_{t_n})$ for all $n \in \mathbb{N}$ and $u_n \to u_+$ in $W^{1,p}(\Omega)$ as $n \to \infty$. Reasoning as in Lemmas 6 and 7, for all $n \in \mathbb{N}$ we can find $w_n \in N(u_n)$ and $w_{n,+} \in N_+(u_n)$ such that u_n is a weak solution of the auxiliary problem

$$\begin{cases} -\Delta_p u = t_n\varepsilon(u_n^-)^{p-1} + (1-t_n)w_n + t_n w_{n,+} & \text{in } \Omega,\\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{23}$$

By $\mathbf{H}$ (i), we have for all $n \in \mathbb{N}$

$$|t_n\varepsilon(u_n^-)^{p-1} + (1-t_n)w_n + t_n w_{n,+}| \leqslant c(1+|u_n|^{r-1}) \text{ a.e. in } \Omega,$$

with a constant $c > 0$ independent of $n \in \mathbb{N}$. By [18, Theorem 1.5.5, Remark 1.5.9], the sequence $(u_n)_n$ turns out to be bounded in $L^\infty(\Omega)$, and by [23, Theorem 2] (u_n) is bounded also in $C^{1,\gamma}(\overline{\Omega})$ $(\gamma \in (0,1))$. By the compact embedding $C^{1,\gamma}(\overline{\Omega}) \hookrightarrow C^1(\overline{\Omega})$, passing if necessary to a subsequence, we have $u_n \to u_+$ in $C^1(\overline{\Omega})$. So, for $n \in \mathbb{N}$ big enough we have $u_n \in \mathrm{int}(C_+)$, in particular $u_n^- = 0$ and $N_+(u_n) = N(u_n)$. By the definition of $N(u_n)$ and convexity of the set $\partial j(x,u_n)$, we have

$$(1-t_n)w_n + t_n w_{n,+} \in N(u_n).$$

Thus the right-hand side of (23) is a selection of $\partial j(\cdot,u_n)$ a.e. in Ω and $u_n \in \mathrm{int}(C_+)\setminus\{u_+\}$ is a solution of (1), against our assumption (A). Thus, Claim 2 is proved.

Finally, we note that the mapping $t \mapsto \psi_t$ is continuous with respect to the norm $\|\cdot\|_{1,\infty}$ in a neighborhood of u_+. By Claims 1 and 2, we can apply the homotopy invariance of critical groups for non-smooth functionals (reasoning as in [13, Theorem 5.2]) and conclude that $C_k(\psi_t,u_+)$ is independent of t. In particular we have (22).

To conclude, let $a,b \in \mathbb{R}$ be such that $b < \varphi_+^\varepsilon(0) < a < \varphi_+^\varepsilon(u_+)$ (recall from the proof of Theorem 6 that $\varphi_+^\varepsilon(0) < \varphi_+^\varepsilon(u_+)$). Set $A = \overline{(\varphi_+^\varepsilon)}^a$, $B = \overline{(\varphi_+^\varepsilon)}^b$, so that $B \subset A$ and by [26, Proposition 6.14] the following long sequence is exact:

$$\cdots \to H_k(W^{1,p}(\Omega),B) \xrightarrow{j_*} H_k(W^{1,p}(\Omega),A) \xrightarrow{\partial_*} H_{k-1}(A,B) \xrightarrow{i_*} H_{k-1}(W^{1,p}(\Omega),B) \to \cdots$$

Here j_*, i_* are the homomorphisms induced by the inclusion mappings $j : (W^{1,p}(\Omega),B) \to (W^{1,p}(\Omega),A)$, $i : (A,B) \to (W^{1,p}(\Omega),B)$, respectively, and $\partial_* = \ell_* \circ \partial_k$, where $\ell : (A,\emptyset) \to (A,B)$ is the inclusion map and $\partial_k : H_k(W^{1,p}(\Omega),A) \to H_{k-1}(A)$ is the boundary homomorphism, see [26, Definition 6.9]. By (A) and Lemma 7, we have $K(\varphi_+^\varepsilon) = \{0,u_+\}$. So, Lemma 10 implies

$$H_k(W^{1,p}(\Omega),B) = C_k(\varphi_+^\varepsilon,\infty) = 0$$

(and the same obviously holds at order $k-1$). Moreover, Lemma 4 implies

$$H_k(W^{1,p}(\Omega),A) = C_k(\varphi_+^\varepsilon,u_+), \quad H_{k-1}(A,B) = C_{k-1}(\varphi_+^\varepsilon,0).$$

So, the sequence above rephrases as the shorter exact sequence

$$0 \to C_k(\varphi_+^\varepsilon,u_+) \xrightarrow{\partial_*} C_{k-1}(\varphi_+^\varepsilon,0) \to 0,$$

i.e., ∂_* is an isomorphism. Then, by (11), then, we have

$$C_k(\varphi_+^\varepsilon,u_+) = \delta_{k-1,0}\mathbb{R} = \delta_{k,1}\mathbb{R},$$

hence by (22) we get the conclusion. □

Finally, we conclude the proof our main result.

Proof of Theorem 1 From Lemma 8 and Theorem 6 we already know that (1) admits the solutions 0, u_+, and u_-. We argue by contradiction, assuming (A). We use Lemmas 9, 11, and 12 into (6), so we obtain

$$\sum_{k=0}^{\infty} (\delta_{k,0} + 2\delta_{k,1})(-1)^k = 0,$$

i.e., $-1 = 0$, a contradiction. Thus, (A) cannot hold, and there exists at least one more solution $\tilde{u} \in C^1(\overline{\Omega}) \setminus \{0,u_+,u_-\}$ of (1). Thus, the proof is complete. □

Remark 1 In the smooth case, Morse theory can be used in order to achieve further information on the solution set of a boundary value problem, for instance to produce a fourth non-zero solution if the equation is semilinear ($p = 2$) and the reaction term is asymptotically linear at infinity (see for instance [29, Section 6]). Nevertheless, such results typically require the use of the second order derivative of the energy functional, a notion which seems to have no natural counterpart within the framework of locally Lipschitz functionals on a Banach space.

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