Keller–Osserman type conditions for differential inequalities with gradient terms on the Heisenberg group

DOI:10.1016/j.jde.2011.01.006

Keller-Osserman type conditions for differential inequalities with gradient terms on the Heisenberg group

Marco Magliaro*, Luciano Mari*, Paolo Mastrolia*, Marco Rigoli*

Dipartimento di Matematica, Università degli Studi di Milano, via Saldini 50, I-20133 Milano, ITALY

Abstract

We study the qualitative behaviour of non-negative entire solutions of differential inequalities with gradient terms on the Heisenberg group. We focus on two classes of inequalities: $\Delta^{\varphi} u \geq f(u)l(|\nabla u|)$ and $\Delta^{\varphi} u \geq f(u)-h(u)g(|\nabla u|)$, where f, l, h, g are non-negative continuous functions satisfying certain monotonicity properties. The operator Δ^{φ}, called the *φ-Laplacian*, generalizes the p-Laplace operator considered by various authors in this setting. We prove some Liouville theorems introducing two new Keller-Osserman type conditions, both extending the classical one which appeared long ago in the study of the prototype differential inequality $\Delta u \geq f(u)$ in $\mathbb{R}^m$. We show sharpness of our conditions when we specialize to the p-Laplacian. While proving these results we obtain a strong maximum principle for Δ^{φ} which, to the best of our knowledge, seems to be new. Our results continue to hold, with the obvious minor modifications, also for Euclidean space.

Keywords: Keller-Osserman; Heisenberg group; differential inequalities; gradient term.

1. Introduction and main results

Let H^m be the Heisenberg group of dimension $2m+1$, that is, the Lie group with underlying manifold $\mathbb{R}^{2m+1}$and group structure defined as follows: for all $q, q' \in H^m$, $q = (z,t) = (x_1, \ldots, x_m, y_1, \ldots, y_m, t)$, $q' = (z', t') =$

*Corresponding author. Tel. +39 0250316168

Email addresses: marco.magliaro@unimi.it (Marco Magliaro), lucio.mari@libero.it (Luciano Mari), paolo.mastrolia@gmail.com (Paolo Mastrolia), marco.rigoli@unimi.it (Marco Rigoli)

$(x'_1, \dots, x'_m, y'_1, \dots, y'_m, t')$,

$$q \circ q' = \left(z + z', t + t' + 2 \sum_{i=1}^{m} (y_i x'_i - x_i y'_i) \right).$$

A basis for the Lie algebra of left-invariant vector fields on H^m is given by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \qquad \frac{\partial}{\partial t} \tag{1}$$

for $j = 1, \dots, m$. This basis satisfies Heisenberg's canonical commutation relations for position and momentum,

$$[X_j, Y_k] = -4\delta_{jk} \frac{\partial}{\partial t}, \tag{2}$$

all other commutators being zero. It follows that the vector fields X_j, Y_k satisfy Hörmander's condition, and the *Kohn-Spencer Laplacian*, defined as

$$\Delta_{H^m} = \sum_{j=1}^{m} \left(X_j^2 + Y_j^2 \right) \tag{3}$$

is hypoelliptic by Hörmander's theorem (see [23]).
A vector field in the span of $\{X_j, Y_j\}$ will be called *horizontal*.
In H^m there are a "natural" origin $o = (0, 0)$ and a distinguished *homogeneous norm* defined, for $q = (z, t) \in H^m$, by

$$r(q) = r(z, t) = \left(|z|^4 + t^2 \right)^{1/4} \tag{4}$$

(where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^{2m}$), which is homogeneous of degree 1 with respect to the dilations $\delta_R : (z, t) \mapsto (Rz, R^2 t)$, $R > 0$. This gives rise to the *Koranyi distance*, defined by

$$d(q, q') = r(q^{-1} \circ q'), \qquad q, q' \in H^m. \tag{5}$$

We set

$$B_R(q_o) = \{q \in H^m : d(q, q_o) < R\}$$

to denote the (open) *Koranyi ball* of radius R centered at q_o. We simply use the notation B_R for balls centered at $q_o = o$. For $u \in C^1(H^m)$, we define the *horizontal gradient* $\nabla_{H^m} u$ by

$$\nabla_{H^m} u = \sum_{j=1}^{m} (X_j u) X_j + (Y_j u) Y_j, \tag{6}$$

so that, for for $f \in C^1(\mathbb{R})$, $\nabla_{H^m} f(u) = f'(u)\nabla_{H^m} u$, and observe that $\nabla_{H^m} u$ is a horizontal vector field. For such vector fields, a natural $\cdot$ product is given by the formula

$$W \cdot Z = \sum_{j=1}^{m} w^j z^j + \widetilde{w}^j \widetilde{z}^j, \tag{7}$$

where $W = w^j X_j + \widetilde{w}^j Y_j$, $Z = z^j X_j + \widetilde{z}^j Y_j$. By definition, $|\nabla_{H^m} u|^2_{H^m} = \nabla_{H^m} u \cdot \nabla_{H^m} u$, and we have the validity of the Cauchy-Schwarz inequality

$$|\nabla_{H^m} u \cdot \nabla_{H^m} v|_{H^m} \le |\nabla_{H^m} u|_{H^m} |\nabla_{H^m} v|_{H^m}. \tag{8}$$

In particular, we set

$$\psi(z,t) := |\nabla_{H^m} r|^2_{H^m} = \frac{|z|^2}{r^2(z,t)} \quad \text{for } (z,t) \neq o. \tag{9}$$

The function ψ is homogeneous of degree 0, hence bounded. It will be called the *density function* and in fact it is immediate to see that $0 \le \psi \le 1$ and that

$$\Delta_{H^m} r = \frac{2m+1}{r}\psi \qquad \text{in } H^m \backslash \{o\}. \tag{10}$$

Finally, the *horizontal divergence* is defined, for horizontal vector fields, by

$$\operatorname{div}_0 W = \sum_{j=1}^{m} \left[X_j(w^j) + Y_j(\widetilde{w}^j)\right] \tag{11}$$

and satisfies

$$\operatorname{div}_0(fW) = f \operatorname{div}_0(W) + \nabla_{H^m} f \cdot W, \tag{12}$$

so that

$$\Delta_{H^m} u = \operatorname{div}_0 \nabla_{H^m} u. \tag{13}$$

In the last few years some authors (see, for example, [15], [12], [9] and [4]) have studied a generalization of the Kohn Laplacian, defined, for $p \in [2, +\infty)$, by

$$\Delta^p_{H^m} u = \operatorname{div}_0 \left(|\nabla_{H^m} u|^{p-2}_{H^m} \nabla_{H^m} u \right) \tag{14}$$

which can be considered as a natural p-Laplace operator in the setting of the Heisenberg group.

In this paper we consider a further generalization, which we shall call φ-*Laplacian*, $\Delta^\varphi_{H^m}$, defined for $u \in C^2(H^m)$ as follows:

$$\Delta^\varphi_{H^m} u = \operatorname{div}_0 \left(\frac{\varphi(|\nabla_{H^m} u|_{H^m})}{|\nabla_{H^m} u|_{H^m}} \nabla_{H^m} u \right), \tag{15}$$

where φ satisfies the structural conditions

$$\begin{cases} \varphi \in C^0(\mathbb{R}_0^+) \cap C^1(\mathbb{R}^+), \quad \varphi(0) = 0, \\ \quad \varphi' > 0 \qquad \text{on } \mathbb{R}^+. \end{cases} \tag{Φ}$$

This family of operators, which includes all the p-Laplacians (obtained with the choice $\varphi(t) = t^{p-1}$, $p > 1$), has been recently studied in the context of Riemannian geometry (see, for example, [31] for motivations and further references). Although we shall focus our attention on this generalization, the main example we keep in mind is the p-Laplacian itself, to which an entire section is devoted.

The aim of this paper is to study weak (in the sense of Subsection 2.2 below) non-negative entire solutions of differential inequalities of the form

$$\Delta^{\varphi}_{H^m} u \geq f(u) l(|\nabla_{H^m} u|_{H^m}), \tag{16}$$

where f and l satisfy respectively the following conditions:

$$\begin{cases} f \in C^0(\mathbb{R}_0^+), \qquad f > 0 \text{ on } \mathbb{R}^+; \\ f \text{ is increasing on } \mathbb{R}_0^+; \end{cases} \tag{F}$$

$$\begin{cases} l \in C^0(\mathbb{R}_0^+), \quad l > 0 \text{ on } \mathbb{R}^+; \\ l \text{ is } C\text{-monotone non-decreasing on } \mathbb{R}_0^+; \end{cases} \tag{L}$$

We recall that l is said to be *C-monotone non decreasing* on $\mathbb{R}_0^+$ if, for some $C \geq 1$,

$$\sup_{s \in [0,\, t]} l(s) \leq C l(t), \qquad \forall\, t \in \mathbb{R}_0^+.$$

Clearly, if l is monotone non decreasing on $\mathbb{R}_0^+$, then it is 1-monotone non-decreasing on the same set; in fact the above condition allows a controlled oscillatory behaviour of l on $\mathbb{R}_0^+$. To express our next requests, from now on we assume that

$$\frac{t\varphi'(t)}{l(t)} \in L^1(0^+) \backslash L^1(+\infty), \quad \liminf_{t \to 0^+} \frac{\varphi(t)}{l(t)} = 0. \tag{Φ \& L}$$

Note that often (e.g. in the case of the p-Laplacian) the latter condition directly assures integrability at 0^+ in the former. We define

$$K(t) = \int_0^t \frac{s\varphi'(s)}{l(s)}\, \mathrm{d}s; \tag{17}$$

observe that $K : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a C^1-diffeomorphism with

$$K'(t) = \frac{t\varphi'(t)}{l(t)} > 0,$$

thus the existence of the increasing inverse $K^{-1} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$. Finally we set

$$F(t) = \int_0^t f(s)\,\mathrm{d}s.$$

Definition 1.1. *The* **generalized Keller-Osserman condition** *for inequality*

$$\Delta^{\varphi}_{H^m} u \geq f(u) l(|\nabla_{H^m} u|_{H^m})$$

is the request:

$$\frac{1}{K^{-1}(F(t))} \in L^1(+\infty). \tag{KO}$$

Note that, in the case of the p-Laplace operator and $l \equiv 1$, (KO) coincides with the well known Keller-Osserman condition for the p-Laplacian, that is, $\frac{1}{F(t)^{1/p}} \in L^1(+\infty)$.
In order to deal with the presence of the density function ψ in the version of our inequalities that we shall describe below, we need to assume two "relaxed homogeneity" requests on φ' and l:

$$s\varphi'(st) \leq D s^{\tau} \varphi'(t), \qquad \forall\ s \in [0,1],\ t \in \mathbb{R}_0^+, \tag{$\Phi 2$}$$

$$s^{1+\tau} l(t) \leq \Lambda\, l(st), \qquad \forall\ s \in [0,1],\ t \in \mathbb{R}_0^+, \tag{$L2$}$$

for some positive constants $D, \Lambda > 0$ and $\tau \geq 0$. We stress that $(L2)$ is a mild requirement: for example, it is satisfied by every $l(t)$ of the form

$$l(t) = \sum_{k=0}^{N} C_k t^{\nu_k}, \quad N \in \mathbb{N}, \qquad C_k \geq 0, \quad -\infty < \nu_k \leq 1+\tau \quad \text{for every } k.$$

Indeed, since $s \leq 1$ we have

$$l(st) = \sum_{k=0}^{N} C_k s^{\nu_k} t^{\nu_k} \geq \sum_{k=0}^{N} C_k s^{1+\tau} t^{\nu_k} = s^{1+\tau} l(t).$$

Note also that, if $(L2)$ is true for some τ_o, then it also holds for every $\tau \geq \tau_o$. This is interesting in the case of the p-Laplacian, which trivially satisfies $(\Phi 2)$ for every $0 \leq \tau \leq p-1$. In this case the choice $\tau = p-1$ is the least demanding on $l(t)$. We also observe that the coupling of $(\Phi 2)$ and $(L2)$ does not automatically imply the integrability at 0^+ in $(\Phi\ \&\ L)$. For instance if $\varphi(t) = t^{\tau}$ and $l(t) =$

$t^{\tau+1}$, then ($\Phi2$) and ($L2$) are satisfied, but $\frac{t\varphi'(t)}{l(t)} \notin L^1(0^+)$.

We shall prove the following Liouville-type result:

Theorem 1.1. *Let φ, f, l satisfy (Φ), (F), (L) and (Φ & L). Suppose also the validity of the relaxed homogeneity conditions ($\Phi2$), ($L2$). If the generalized Keller-Osserman condition (KO) holds, then every solution $0 \le u \in C^1(H^m)$ of*

$$\Delta^{\varphi}_{H^m} u \ge f(u)l(|\nabla_{H^m} u|_{H^m}) \qquad \text{on } H^m \tag{18}$$

is constant. Moreover, if $l(0) > 0$, then $u \equiv 0$.

The proof is achieved through the construction of a "radial" supersolution v of (18) (see the next section for the precise definition) on an annular region $B_T \backslash B_{t_0}$, $0 < t_0 < T$, which is small near ∂B_{t_0} and blows up at ∂B_T. A careful comparison between u and v allows us to conclude that u must necessarily be constant. As opposed to Osserman's approach (see [29]), in order to construct the supersolution we have not tried to solve the radialization of (18), since the presence of the gradient term may cause different behaviours near the first singular time. Roughly speaking, even if we could prove the local existence of a radial solution in a neighborhood of zero (which is not immediate due to the singularity of $1/r$ and possibly of φ' in 0), we cannot be sure that, in case the interval of definition is $[0, T)$, $T < +\infty$, the solution blows up at time T: *a priori*, it may even happen that the solution remains bounded, but the first derivative blows up, giving rise to some sort of cusp. The necessity of excluding this case led us to a different approach: a blowing-up supersolution is explicitly constructed, exploiting directly the Keller-Osserman condition. Beside being elementary, this alternative method also reveals the reason why (KO) is indeed natural as an optimal condition for the existence or non-existence of solutions.

As it will become apparent from the proof of Theorem 1.1 below, the result can be restated on the Euclidean space $\mathbb{R}^m$ getting rid of request ($\Phi2$) and (L2), which are related to the density function ψ. Indeed we have

Theorem 1.2. *Let φ, f, l satisfy (Φ), (F), (L), (Φ & L) and the generalized Keller-Osserman condition (KO). Let $u \in C^1(\mathbb{R}^m)$ be a non-negative solution of*

$$\Delta^{\varphi}_{\mathbb{R}^m} u = \operatorname{div}\left(|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u\right) \ge f(u)l(|\nabla u|) \qquad \text{on } \mathbb{R}^m. \tag{19}$$

Then u is constant. Moreover, if $l(0) > 0$, then $u \equiv 0$.

We mention that, for the particular case of the p-Laplace operator, this result has also appeared as the first part of Theorem 1.5 of [14], where the authors deal with a weighted version of (19). As the reader can easily check, if the weights are trivial, then the two theorems coincide; we observe, however, that the C-monotonicity of l in (L) is a slightly milder requirement than the one in [14].

To show the sharpness of (KO), we produce a global unbounded subsolution of (16) when (KO) is violated. For simplicity we only deal with the case of the p-Laplacian and we prove the following:

Theorem 1.3. *Assume the validity of* (F) *and* (L). *Suppose that*

$$\liminf_{t\to 0^+} \frac{t^{p-1}}{l(t)} = 0. \tag{p \& L}$$

Assume also the relaxed homogeneity condition

$$l(t)s^p \le \Lambda\, l(st) \qquad \forall\ s \in [0,1],\ t \in \mathbb{R}^+_0, \tag{$L2_p$}$$

for some $\Lambda > 0$. *Then, the following conditions are equivalent:*

i) there exists a non-negative, non-constant solution $u \in C^1(H^m)$ *of inequality* $\Delta^p_{H^m} u \ge f(u) l(|\nabla_{H^m} u|_{H^m})$;

ii) $\dfrac{1}{K^{-1}(F(t))} \notin L^1(+\infty)$.

As for Theorem 1.1, we can state the analogous result in Euclidean setting: in this latter case, assumption $(L2_p)$ is unnecessary. The particular case $l(t) = t^\theta$, for $\theta \in [0, p-1)$, has just appeared as Corollary 1.1 in [14]. We point out that such restriction on θ is required in order for $(p$ & $L)$ to be satisfied. Moreover, it should be observed that Theorem 2.3 in the aforementioned paper indeed ensures the existence of infinitely many positive, unbounded solutions of $\Delta_p u = f(u)|\nabla u|^\theta$ on $\mathbb{R}^m$, provided the Keller-Osserman condition does not hold. We would like to stress that the subsolution that we construct to prove $(ii) \Rightarrow (i)$ in Theorem 1.3 is unbounded as well. This fact is not accidental: indeed, slightly modifying our arguments, we shall prove that, under all the assumptions of Theorem 1.1 but (KO), bounded subsolutions still have to be constant.

Theorem 1.4. *Let* φ, f, l *satisfy* (Φ), (F), (L), $(\Phi$ & $L)$, $(\Phi 2)$ *and* $(L2)$. *Then every non-negative bounded* C^1*-solution* u *of*

$$\Delta^\varphi_{H^m} u \ge f(u) l(|\nabla_{H^m} u|_{H^m}) \qquad \text{on } H^m \tag{20}$$

is constant; moreover, if $l(0) > 0$, *then* $u \equiv 0$.

In the last part of the paper we show how the techniques introduced can be implemented to study differential inequalities of the form

$$\Delta^{\varphi}_{H^m} u \geq f(u) - h(u)g(|\nabla_{H^m} u|_{H^m}), \tag{21}$$

where the functions appearing in the RHS of the above are non-negative. The main results obtained are Theorem 5.3, that is, triviality of the solutions in the general setting under an appropriate Keller-Osserman condition, and Theorem 5.7 for the p-Laplace operator, where we show the sharpness of the condition in analogy with Theorem 1.3. Details appear in Section 5 below.

Differential inequalities of the type of (18) and (21) have been the subject of an increasing interest in the last years, mostly in the Euclidean setting (also for their connection with stochastic control theory, see for instance [25]). Among the literature on this topic we only cite some of the references. Most of the authors ([34], [35], [36], [24], [1], [17], [18], [19], [20], [2]) considered the prototype example

$$\Delta u = h(|x|)f(u) - \widetilde{h}(|x|)|\nabla u|^{\theta}, \quad \theta \geq 0, \quad h, \widetilde{h}, f \in C^0(\mathbb{R}),$$

on bounded and unbounded domains of $\mathbb{R}^m$, for particular choices of the weights $h, \widetilde{h}$ and of the function f. Generalization of the previous problem to the case of the p-Laplacian has been investigated by [21]. On the other hand, the problem

$$\Delta_p u \geq b(x)f(u)l(|\nabla u|)$$

has been studied, for instance, in [28], [30], [27]. Moreover, since the writing of this paper, a number of further contributions to the subject have appeared (see [14], [13], [11], [26]). Some of these use techniques which are similar, and in fact are in some sense sharp evolutions of the present arguments ([14], [13]), while others are based on completely different principles and are more tailored for the case of the Euclidean space or of structures, such as Carnot groups, which generalize the Heisenberg group ([11]; we also point out the earlier papers of A. Bonfiglioli and F. Uguzzoni [6] and of N. Garofalo and E. Lanconelli [16]). These last works, however, do not contemplate the possibility of including a gradient term in the RHS of the differential inequality. As for [14], the attention is focused on a weighted p-Laplace operator, and non-existence is also specially suited to show that the main existence results there are indeed sharp. It is worth spending a few words on the recent paper by L. D'Ambrosio and E. Mitidieri ([11]), which has been brought to our attention by the referee; in this work, refining the methods of [28], integral inequalities are exploited to derive *a priori* estimates, Liouville theorems and Harnack inequalities for a wide class of degenerate operators, which includes the φ-Laplacian on the Heisenberg group previously defined. Nonetheless, however interesting these results are, they do not seem to cover the cases that we treated in our main theorems. Furthermore, it should be noted that the approach of the present paper and of [13] can be successfully applied to the case of complete Riemannian manifolds (see [26]), an environment in which the techniques of [11] seem to be unable to yield sharp results.

The paper is organized as follows. In Section 2 we recall some analytical preliminaries and we prove the comparison and the strong maximum principle for the φ-Laplacian. Section 3 is devoted to the proof of both Theorems 1.1 and 1.4, whereas in Section 4 we show the equivalence in Theorem 1.3. Finally, in Section 5 we study inequality (21) and we give a Liouville theorem under a modified Keller-Osserman condition, together with a companion existence result for the particular case of the p-Laplace operator.

2. Preliminaries

The aim of this section is to introduce an explicit formula for the φ-Laplacian acting on radial functions and the appropriate notion of weak solution of differential inequalities of the type of (16) or, more generally, (21).

2.1. "Radialization" of the φ-Laplacian

Consider a *radial* function, that is, a function of the form

$$u(q) = \alpha(r(q)), \qquad q \in H^m, \tag{22}$$

where $\alpha : \mathbb{R}_0^+ \to \mathbb{R}$, $\alpha \in C^2(\mathbb{R}_0^+)$.

Now, a straightforward but somewhat lengthy computation yields the expression:

$$\Delta_{H^m}^{\varphi} u = \psi^{\frac{1}{2}} \left[\psi^{\frac{1}{2}} \varphi' \Big(|\alpha'(r)| \psi^{\frac{1}{2}} \Big) \alpha''(r) + \frac{2m+1}{r} \operatorname{sgn} \alpha'(r)\, \varphi \Big(|\alpha'(r)| \psi^{\frac{1}{2}} \Big) \right]. \tag{23}$$

It is worth to stress the following property, which allows us to shift the origin for the Koranyi distance from o to any other point q_0: if we denote with $\bar{r}(q) = d(q_0, q) = r(q_0^{-1} \circ q)$, a calculation shows that

$$[X_j(\bar{r})](q) = [X_j(r)](q_0^{-1} \circ q), \qquad [Y_j(\bar{r})](q) = [Y_j(r)](q_0^{-1} \circ q),$$

hence we obtain the invariance with respect to the left multiplication

$$\Delta_{H^m}^{\varphi} (\alpha \circ \bar{r})(q) = \Delta_{H^m}^{\varphi} (\alpha \circ r)(q_0^{-1} \circ q). \tag{24}$$

The above relation will come in handy in what follows.

2.2. Weak formulation

In order to simplify the notation, let us first introduce the function

$$A(t) = t^{-1}\varphi(t), \qquad A(t) \in C^0(\mathbb{R}^+). \tag{25}$$

With the help of the matrix $B = B(q)$ (see [8], pg. 294), defined by

$$B(q) = B(z,t) = \left(\begin{array}{c|c} I_{2m} & \begin{matrix} 2y \\ -2x \end{matrix} \\ \hline 2\,{}^t y \quad -2\,{}^t x & 4|z|^2 \end{array} \right),$$

where ${}^t x = (x_1, \ldots, x_m)$, and ${}^t y = (y_1, \ldots, y_m)$, we can write the φ-Laplacian in Euclidean divergence form. Indeed, indicating from now on with div, ∇ and $\langle \;,\; \rangle$ respectively the ordinary Euclidean divergence, gradient and scalar product in $\mathbb{R}^{2m+1}$, it is easy to see that $B\nabla u = \nabla_{H^m} u$, where with $B\nabla v$ we mean the vector in $\mathbb{R}^{2m+1}$ whose components in the standard basis $\left\{ \frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}, \frac{\partial}{\partial t} \right\}$ are given by the matrix multiplication of B with the components of ∇u in the same basis. Having made this precise, a standard check shows that

$$\langle \nabla u, B\nabla v \rangle = \nabla_{H^m} u \cdot \nabla_{H^m} v.$$

and we have

$$\Delta^{\varphi}_{H^m} u = \mathrm{div}_0 \left(A(|\nabla_{H^m} u|_{H^m}) \nabla_{H^m} u \right) = \mathrm{div} \left(A(|\nabla_{H^m} u|_{H^m}) B\nabla u \right),$$

which is the desired expression. Note that, when $\varphi(t) = t$, the above becomes the well-known formula (see, e.g., [5] and [8]) for the Kohn-Spencer Laplacian, that is, $\Delta^{\varphi}_{H^m} u = \mathrm{div}(B\nabla u)$. It follows that (16) can be interpreted in the weak sense as follows: for every $\zeta \in C^\infty_0(H^m)$, $\zeta \geq 0$, we have

$$\begin{aligned} \int_{\mathbb{R}^{2m+1}} \zeta \Delta^{\varphi}_{H^m} u &= \int_{\mathbb{R}^{2m+1}} \zeta \, \mathrm{div}(A(|\nabla_{H^m} u|_{H^m}) B\nabla u) = \\ &= - \int_{\mathbb{R}^{2m+1}} A(|\nabla_{H^m} u|_{H^m}) \, \langle B\nabla u, \nabla \zeta \rangle = \\ &= - \int_{\mathbb{R}^{2m+1}} A(|\nabla_{H^m} u|_{H^m}) \nabla_{H^m} u \cdot \nabla_{H^m} \zeta, \end{aligned}$$

and thus the weak form is

$$- \int_{\mathbb{R}^{2m+1}} A(|\nabla_{H^m} u|_{H^m}) \nabla_{H^m} u \cdot \nabla_{H^m} \zeta \geq \int_{\mathbb{R}^{2m+1}} f(u) l(|\nabla_{H^m} u|_{H^m}) \zeta \tag{26}$$

as expected. Hence, an *entire weak classical solution of* (16) is a function $u \in C^1(H^m)$ such that, for all $\zeta \in C_0^\infty(H^m), \zeta \geq 0$, (26) is satisfied. A similar definition of course holds for the differential inequality (21).

2.3. Comparison and strong maximum principle

In order to prove Theorem 1.1 we shall need a comparison theorem and a maximum principle which are well-known for the Kohn-Spencer Laplacian (see [7]). For viscosity solutions of fully nonlinear PDE, very general comparison principles have been established in [3] by extending the striking ideas in the paper of Crandall-Ishii-Lions [10] to the subelliptic case. Here we briefly prove the corresponding statements for the φ-Laplacian that we shall use below, basing on ideas taken from [32] and [33]. Throughout this subsection we shall assume (Φ) and $(\Phi2)$.

Proposition 2.1. *Let $\Omega \subset\subset H^m$ be a relatively compact domain with C^1 boundary. Let $u, v \in C^0(\overline{\Omega}) \cap C^1(\Omega)$ satisfy*

$$\begin{cases} \Delta^\varphi_{H^m} u \geq \Delta^\varphi_{H^m} v & \text{on } \Omega \\ u \leq v & \text{on } \partial\Omega. \end{cases} \tag{27}$$

Then $u \leq v$ on Ω.

Proof. The proof basically follows the one in [31] pp.85–86. However, we reproduce the steps for the sake of completeness. Let $w = v - u$. By contradiction assume that there exists $\bar{q} \in \Omega$ such that $w(\bar{q}) < 0$, and let $\varepsilon > 0$ be such that $w(\bar{q}) + \varepsilon < 0$. The function $w_\varepsilon = \min\{w + \varepsilon, 0\}$ has compact support in Ω, hence $-w_\varepsilon \geq 0$ is an admissible Lipschitz test function. The weak definition of (27), together with the divergence form of $\Delta^\varphi_{H^m}$, reads:

$$\begin{aligned} 0 &\geq \int_\Omega \left\langle |\nabla_{H^m} v|^{-1}_{H^m} \varphi(|\nabla_{H^m} v|_{H^m}) B\nabla v - |\nabla_{H^m} u|^{-1}_{H^m} \varphi(|\nabla_{H^m} u|_{H^m}) B\nabla u, \nabla w_\varepsilon \right\rangle = \\ &= \int_E \left\langle |\nabla_{H^m} v|^{-1}_{H^m} \varphi(|\nabla_{H^m} v|_{H^m}) B\nabla v - |\nabla_{H^m} u|^{-1}_{H^m} \varphi(|\nabla_{H^m} u|_{H^m}) B\nabla u, \nabla(v-u) \right\rangle, \end{aligned} \tag{28}$$

where $E = \{q : w(q) < -\varepsilon\}$. We denote by h the integrand in (28). With the aid of the Cauchy-Schwarz inequality we have

$$h \geq [\varphi(|\nabla_{H^m} v|_{H^m}) - \varphi(|\nabla_{H^m} u|_{H^m})](|\nabla_{H^m} v|_{H^m} - |\nabla_{H^m} u|_{H^m}) \geq 0, \tag{29}$$

where the latter inequality is due to the monotonicity of φ.
It follows from (28) and (29) that $0 \geq \int_\Omega h \geq 0$, hence $h = 0$ a.e. on Ω.

This implies that $|\nabla_{H^m} u|_{H^m} = |\nabla_{H^m} v|_{H^m}$ on E, and therefore

$$\begin{aligned} 0 =& h = |\nabla_{H^m} u|_{H^m}^{-1} \varphi(|\nabla_{H^m} u|_{H^m}) \langle B\nabla(v-u), \nabla(v-u)\rangle = \\ =& |\nabla_{H^m} u|_{H^m}^{-1} \varphi(|\nabla_{H^m} u|_{H^m}) |\nabla_{H^m}(v-u)|_{H^m}^2 . \end{aligned}$$

This shows that

$$|\nabla_{H^m}(w_\varepsilon)|_{H^m}^2 = 0, \tag{30}$$

whence w_ε is constant. Indeed, from (30) we have $X_j(w_\varepsilon) = Y_j(w_\varepsilon) = 0$ for every $j = 1 \dots m$, and using the commutation law (2) we also have $\partial w_\varepsilon / \partial t = 0$; recalling the definition of X_j and Y_j, all the components of the Euclidean gradient of w_ε vanish, proving the constancy of w_ε. Since $w_\varepsilon(\bar{q}) < 0 = w_{\varepsilon|\partial\Omega}$ we reach the desired contradiction. □

The next key tool to prove our main non-existence result is the strong maximum principle for $\Delta^\varphi_{H^m}$. Since this operator is nonlinear and everywhere degenerate, classical maximum principles such as, for instance, those in [33] cannot be directly applied. Furthermore, as far as we know, no strong maximum principle for such operators has been obtained as of yet. However, in the particular case of the p-Laplace operator, the strong maximum principle for continuous p-harmonic functions is a consequence of the full Harnack inequality that has been proved, for instance, in Section 4 of [22]. For this reason, we provide a simple, direct proof which, in the very special case of the Kohn-Spencer Laplacian, represents an approach to the problem which is alternative to Bony's classical method (see [7]). We wish to thank Professor L. Brandolini for his helpful suggestions in dealing with this problem.

Proposition 2.2. *Let $\Omega \subset H^m$ be a domain. Let $u \in C^0(\overline{\Omega}) \cap C^1(\Omega)$ satisfy*

$$\Delta^\varphi_{H^m} u \geq 0 \quad \textit{in } \Omega \tag{31}$$

and let $u^* = \sup_\Omega u$. If $u(q_M) = u^*$ for some $q_M \in \Omega$, then $u \equiv u^*$.

Proof. By contradiction, assume the existence of a solution u of (31) and of $q_M \in \Omega$ such that $u(q_M) = u^*$, but $u \not\equiv u^*$. Set $\Gamma = \{q \in \Omega : u(q) = u^*\}$. Let $\delta > 0$ and define

$$\Omega^+ = \{q \in \Omega : u^* - \delta < u(q) < u^*\}; \qquad \Gamma_\delta = \{q \in \Omega : u(q) = u^* - \delta\}; \tag{32}$$

note that $\partial\Omega^+ \cap \Omega = \Gamma \cup \Gamma_\delta$. Let $q' \in \Omega^+$ be such that

$$d(q', \Gamma) < d(q', \Gamma_\delta), \qquad d(q', \Gamma) < d(q', \partial\Omega) \tag{33}$$

(this is possible up to choosing q' sufficiently close to q_M). Let $B_R(q')$ be the largest Koranyi ball centered at q' and contained in Ω^+. Then, by construction $u < u^*$ in $B_R(q')$ while $u(q_0) = u^*$ for some $q_0 \in \partial B_R(q')$. Since q_0 is an absolute maximum for u in Ω, we have $\nabla u(q_0) = 0$.

Now we construct an auxiliary function. Towards this aim, we consider the annular region

$$E_R(q') = \overline{B_R(q')} \setminus B_{R/2}(q') \subset \overline{\Omega^+}; \tag{34}$$

we fix $a \in (u^* - \delta, u^*)$ to be determined later and consider the following problem

$$\begin{cases} z'' + \frac{c}{t} z' \leq 0 & \text{in } (R/2, R) \\ z(R/2) = a, \quad z(R) = u^* \\ u^* - \delta < z \leq u^*, \quad z' > 0 & \text{in } [R/2, R], \end{cases} \tag{35}$$

where $c = \frac{D(2m+1)}{\tau}$ is positive and D and τ are the constants appearing in condition (Φ2). Note that, for example, the function

$$z(t) = \frac{(u^* - a)R^{c-1}}{1 - 2^{c-1}} \left(\frac{1}{t^{c-1}} - \left(\frac{2}{R} \right)^{c-1} \right) + a$$

satisfies (35) when $c \neq 1$, while

$$z(t) = \frac{u^* - a}{\log 2} \log \frac{t}{R} + u^*$$

is a solution when $c = 1$.
Using the invariance property (24), such a function gives rise to a C^2-solution $v(q) = z(\bar{r}(q))$, where $\bar{r}(q) = r(q'^{-1} \circ q)$, of

$$\begin{cases} \Delta^{\varphi}_{H^m} v \leq 0 & \text{in } E_R(q') \\ v = a \quad \text{on } \partial B_{R/2}(q'), \quad v = u^* & \text{on } \partial B_R(q') \\ u^* - \delta < v \leq u^*. \end{cases} \tag{36}$$

Indeed hypothesis (Φ2) yields, for $s > 0$,

$$\varphi'(st) \leq D s^{\tau - 1} \varphi'(t)$$

and integrating in the variable s between $\varepsilon > 0$ and 1 we find

$$\frac{\varphi(t) - \varphi(\varepsilon t)}{t} \leq D \varphi'(t) \frac{1 - \varepsilon^{\tau}}{\tau},$$

which, as $\varepsilon \to 0$ gives the inequality

$$\varphi(t) \leq \frac{D}{\tau} t\varphi'(t).$$

Using this we have

$$\begin{aligned}\Delta^{\varphi}_{H^m} v &\leq \psi\varphi'\left(z'\psi^{\frac{1}{2}}\right)z'' + \frac{D}{\tau}\left(z'\psi^{\frac{1}{2}}\right)\varphi'\left(z'\psi^{\frac{1}{2}}\right)\frac{(2m+1)}{t}\psi^{\frac{1}{2}} = \\ &= \psi\varphi'\left(z'\psi^{\frac{1}{2}}\right)\left\{z'' + \frac{c}{t}z'\right\} \leq 0.\end{aligned}$$

It is important to point out that there exists a positive constant $\lambda > 0$ such that

$$\langle \nabla v, \nabla \bar{r} \rangle = z'(\bar{r})|\nabla \bar{r}|^2 \geq \lambda > 0 \quad \text{on} \quad \partial E_R(q'); \tag{37}$$

this follows since $\bar{r}$ differs from r by a translation of the Heisenberg group (that is, a diffeomorphism), and $|\nabla r|^2 = \frac{1}{r^6}\left(|z|^6 + \frac{t^2}{4}\right)$ only vanishes at the origin o. Next we choose $a \in (u^* - \delta, u^*)$ close enough to u^* so that $u \leq v$ on $\partial B_{R/2}(q')$: this is possible since $\partial B_{R/2}(q') \subset\subset \Omega^+$ and thus $\max_{\partial B_{R/2}(q')} u < u^*$. Now $u, v \in C^0(\overline{E_R(q')}) \cap C^1(E_R(q'))$ and, since $v \equiv u^*$ on $\partial B_R(q')$, they satisfy

$$\begin{cases} \Delta^{\varphi}_{H^m} u \geq \Delta^{\varphi}_{H^m} v & \text{on } E_R(q') \\ u \leq v & \text{on } \partial E_R(q'). \end{cases} \tag{38}$$

Then by Proposition 2.1 we have $u \leq v$ on $E_R(q')$.
Let us consider the function $v - u$: it satisfies $v - u \geq 0$ on $E_R(q')$ and $v(q_0) - u(q_0) = u^* - u^* = 0$, so that $\langle \nabla(v-u), \nabla \bar{r} \rangle (q_0) \leq 0$. Therefore

$$0 = \langle \nabla u, \nabla \bar{r} \rangle (q_0) \geq \langle \nabla v, \nabla \bar{r} \rangle (q_0) > 0, \tag{39}$$

a contradiction. □

Remark 2.3. Obviously, one can state an analogous minimum principle using the substitution $v(q) = -u(q)$; however, a direct proof of the minimum principle following the above steps reveals some further difficulties due to the density function, which is not bounded from below away from zero.

Remark 2.4. Because of the use of radial functions, the proof of the strong maximum principle can be rephrased on a general Carnot group (that is, a nilpotent, stratified Lie group) provided its homogeneous norm r arising from the fundamental solution of the sublaplacian has good properties. Namely, we need that r is ∞-harmonic, that is,

$$\nabla_0(|\nabla_0 r|^2) \cdot \nabla_0 r = 0,$$

where ∇_0 is the horizontal gradient. Such groups include all the groups of Heisenberg type. For a detailed introduction on Carnot groups, we refer the interested reader to [5].

3. Proof of Theorems 1.1 and 1.4

The strategy for proving Theorem 1.1 and Theorem 1.4 is essentially the same and, as already mentioned, it is based on the construction of a radially symmetric supersolution v of (16) defined on a suitable annular region $B_T \backslash B_{r_0}$. In order for the desired Liouville theorem to follow from the comparison principle (i.e. Proposition 2.1), we need v to be sufficiently large on ∂B_T. In particular, to ensure the constancy of a priori unbounded solutions, we shall be forced to require that v blow up as $r \to T^-$. The existence of such v is granted via the Keller-Osserman condition.

3.1. Construction of the supersolution

The construction of the radial supersolution requires the next technical Lemma, which also appears as Lemma 5.2 in [14]. We refer to the Introduction for notation and properties.

Lemma 3.1. *Let $\sigma \in (0, 1]$; then the generalized Keller-Osserman condition (KO) implies*

$$\frac{1}{K^{-1}(\sigma F(t))} \in L^1(+\infty). \tag{40}$$

Proof. We perform the change of variables $t = s\sigma$ to have

$$\int^{+\infty} \frac{\mathrm{d}s}{K^{-1}(\sigma F(s))} = \sigma^{-1} \int^{+\infty} \frac{\mathrm{d}t}{K^{-1}(\sigma F(\sigma^{-1}t))}.$$

Since f and K^{-1} are increasing by assumption, we get

$$F(\sigma^{-1}t) = \int_0^{\sigma^{-1}t} f(z)\,\mathrm{d}z = \sigma^{-1} \int_0^t f(\sigma^{-1}\xi)\,\mathrm{d}\xi \geq \sigma^{-1} \int_0^t f(\xi)\,\mathrm{d}\xi = \sigma^{-1}F(t)$$

and

$$K^{-1}(\sigma F(\sigma^{-1}t)) \geq K^{-1}(F(t)),$$

thus

$$\int^{+\infty} \frac{\mathrm{d}s}{K^{-1}(\sigma F(s))} \leq \sigma^{-1} \int^{+\infty} \frac{\mathrm{d}t}{K^{-1}(F(t))} < +\infty. \tag{41}$$

□

The construction of the supersolution relies on the technique described in the following

Lemma 3.2. *Assume the validity of* (Φ), (F), (L), $(\Phi\ \&\ L)$ *Fix*

$$0 < \varepsilon < \eta < A, \qquad and \qquad 0 < t_0 < t_1.$$

Then, for every $\widetilde{B} > 0$ *there exist* $T > t_1$ *and a strictly increasing, convex function*

$$\alpha : [t_0, T) \to [\varepsilon, A)$$

satisfying

$$\begin{cases} (\varphi(\alpha'))' + \frac{2m+1}{t}\varphi(\alpha') \leq \widetilde{B} f(\alpha) l(\alpha'); \\ \alpha(t_0) = \varepsilon, \quad \alpha(t_1) \leq \eta; \\ \alpha(t) \uparrow A \text{ as } t \to T^-. \end{cases} \tag{42}$$

If, furthermore, the Keller-Osserman condition (KO) *is satisfied, then* A *can be replaced by* $+\infty$*, that is,* α *diverges as* $t \to T^-$.

Proof. Consider $\sigma \in (0, 1]$ to be determined later and choose $T_\sigma > t_0$ such that

$$T_\sigma - t_0 = \int_\varepsilon^A \frac{\mathrm{d}s}{K^{-1}(\sigma F(s))}.$$

Note that, when $A = +\infty$ and (KO) holds, the RHS is well defined by Lemma 3.1. Moreover, since the RHS diverges as $\sigma \to 0^+$, up to choosing σ sufficiently small we can shift T_σ in such a way that $T_\sigma > t_1$. We implicitly define the C^2-function $\alpha(t)$ by requiring

$$T_\sigma - t = \int_{\alpha(t)}^A \frac{\mathrm{d}s}{K^{-1}(\sigma F(s))} \qquad \text{on } [t_0, T_\sigma).$$

We observe that, by construction, $\alpha(t_0) = \varepsilon$ and, since $K^{-1} > 0$, $\alpha(t) \uparrow A$ as $t \to T_\sigma$. A first differentiation yields

$$\frac{\alpha'}{K^{-1}(\sigma F(\alpha))} = 1,$$

hence α is monotone increasing and $\sigma F(\alpha) = K(\alpha')$. Differentiating once more we deduce

$$\sigma f(\alpha)\alpha' = K'(\alpha')\alpha'' = \frac{\alpha'\varphi'(\alpha')}{l(\alpha')}\alpha''.$$

Cancelling α' throughout we obtain

$$[\varphi(\alpha')]' = \varphi'(\alpha')\alpha'' = \sigma f(\alpha) l(\alpha');$$

thus, integrating on $[t_0, t]$,

$$\varphi(\alpha'(t)) = \varphi(\alpha'(t_0)) + \sigma \int_{t_0}^{t} f(\alpha(s))l(\alpha'(s))\,\mathrm{d}s.$$

Using (F) and (L) we deduce the following chain of inequalities:

$$\begin{aligned}
&[\varphi(\alpha')]' + \frac{2m+1}{t}\varphi(\alpha') = \\
&= \sigma f(\alpha)l(\alpha') + \frac{2m+1}{t}\varphi(\alpha'(t_0)) + \frac{2m+1}{t}\sigma\int_{t_0}^{t} f(\alpha(s))l(\alpha'(s))\,\mathrm{d}s = \\
&= \left[\sigma + \frac{2m+1}{t}\frac{\varphi(\alpha'(t_0))}{f(\alpha(t))l(\alpha'(t))} + \frac{2m+1}{t}\frac{\sigma\int_{t_0}^{t} f(\alpha(s))l(\alpha'(s))\,\mathrm{d}s}{f(\alpha(t))l(\alpha'(t))}\right] f(\alpha(t))l(\alpha'(t)) \leq \\
&\leq \left[\sigma + \frac{2m+1}{t}\frac{C\varphi(\alpha'(t_0))}{f(\alpha(t_0))l(\alpha'(t_0))} + \frac{2m+1}{t}\frac{\sigma C f(\alpha(t))l(\alpha'(t))(t-t_0)}{f(\alpha(t))l(\alpha'(t))}\right] f(\alpha(t))l(\alpha'(t)),
\end{aligned}$$

that is,

$$[\varphi(\alpha')]' + \frac{2m+1}{t}\varphi(\alpha') \leq \widetilde{C}\left[\frac{\varphi(\alpha'(t_0))}{t_0 f(\alpha(t_0))l(\alpha'(t_0))} + \sigma\right] f(\alpha(t))l(\alpha'(t)), \tag{43}$$

for some uniform constant $\widetilde{C}$. Since $K(0) = 0$, $\alpha(t_0) = \varepsilon$ and $\alpha'(t_0) = K^{-1}(\sigma F(\varepsilon)) \to 0$ as $\sigma \to 0$, and using $(\Phi \ \&\ L)$, choosing σ small enough on an appropriate sequence we can estimate the whole square bracket with $\widetilde{B}$ to show the validity of the first of (42).
It remains to prove that, possibly with a further reduction of σ, $\alpha(t_1) \leq \eta$. From the trivial identity

$$\int_{\alpha(t_1)}^{A} \frac{\mathrm{d}s}{K^{-1}(\sigma F(s))} = T_\sigma - t_1 = (T_\sigma - t_0) + (t_0 - t_1) = \int_{\varepsilon}^{A} \frac{\mathrm{d}s}{K^{-1}(\sigma F(s))} + (t_0 - t_1)$$

we deduce

$$\int_{\varepsilon}^{\alpha(t_1)} \frac{\mathrm{d}s}{K^{-1}(\sigma F(s))} = t_1 - t_0.$$

It suffices to choose σ such that $\int_{\varepsilon}^{\eta} \frac{\mathrm{d}s}{K^{-1}(\sigma F(s))} > t_1 - t_0$; then obviously $\alpha(t_1) \leq \eta$.
This completes the proof of the Lemma. □

3.2. Last step of the proofs

We first prove Theorem 1.4 under the assumptions (Φ), (F), (L), $(\Phi \ \&\ L)$, $(\Phi 2)$ and $(L2)$. Later on, under the additional hypothesis (KO), we also prove the constancy of possibly unbounded solutions u of

$$\Delta^{\varphi}_{H^m} u \geq f(u)l(|\nabla_{H^m} u|_{H^m}) \qquad \text{on} \quad H^m. \tag{44}$$

Therefore, we denote by $u^* = \sup u$ and we first assume that $u^* < +\infty$. We reason by contradiction and assume $u \not\equiv u^*$; by Proposition 2.2 $u < u^*$ on H^m. Choose $r_0 > 0$ and define

$$u_0^* = \sup_{\overline{B}_{r_0}} u < u^*.$$

Fix $\eta > 0$ sufficiently small such that $u^* - u_0^* > 2\eta$, and choose $\widetilde{q} \in H^m \setminus \overline{B}_{r_0}$ such that $u(\widetilde{q}) > u^* - \eta$. Choose also A in such a way that $A > 2\eta + \varepsilon$. We then set $\widetilde{r} = r(\widetilde{q})$ and, for our choice of $\widetilde{r}, r_0, A, \varepsilon, \eta$ we construct the radial function $v(q) = \alpha(r(q))$ on $B_T \setminus B_{r_0}$, with α and $T > \widetilde{r}$ as in Lemma 3.2, $\widetilde{B} = 1/(\Lambda D)$, and satisfying the further requirement:

$$\varepsilon \leq v \leq \eta \qquad \text{on } B_{\widetilde{r}} \setminus \overline{B}_{r_0}.$$

We observe that v is a supersolution for (44). In order to show this, first we note that by integration, (Φ) and $s \in [0,1]$, $(\Phi 2)$ implies the inequality

$$\varphi(st) \leq D s^{\tau} \varphi(t), \quad t \in \mathbb{R}_0^+, \; s \in [0,1]. \tag{45}$$

Next, considering the radial expression (23), using (L), $(\Phi 2)$, (45) and Lemma 3.2 we have

$$\begin{aligned}
\Delta_{H^m}^{\varphi} \alpha(r(q)) &= \psi^{\frac{1}{2}} \left[\psi^{\frac{1}{2}} \varphi'\Big(\alpha'(r)\psi^{\frac{1}{2}}\Big)\alpha''(r) + \frac{2m+1}{r} \varphi\Big(\alpha'(r)\psi^{\frac{1}{2}}\Big) \right] \leq \\
&\leq \psi^{\frac{1+\tau}{2}} D \left[\varphi'(\alpha'(r))\alpha''(r) + \frac{2m+1}{r} \varphi(\alpha'(r)) \right] \leq \\
&\leq \psi^{\frac{1+\tau}{2}} D \left[\frac{1}{\Lambda D} f(\alpha(r)) l(\alpha'(r)) \right] \leq \\
&\leq f(\alpha(r)) l(\alpha'(r)\psi^{\frac{1}{2}}) = f(\alpha(r)) l(|\nabla_{H^m} \alpha(r)|_{H^m}).
\end{aligned}$$

Moreover

$$u(\widetilde{q}) - v(\widetilde{q}) > u^* - \eta - \eta = u^* - 2\eta,$$

and, on ∂B_{r_0},

$$u(q) - v(q) \leq u_0^* - \varepsilon < u^* - 2\eta - \varepsilon.$$

Since also

$$u(q) - v(q) \leq u^* - A < u^* - 2\eta - \varepsilon \qquad \text{for } \; q \in \partial B_T.$$

Thus, the difference $u - v$ attains a positive maximum μ in $B_T \setminus \overline{B}_{r_0}$. Let Γ_μ be a connected component of

$$\big\{ q \in B_T \setminus \overline{B}_{r_0} : u(q) - v(q) = \mu \big\}.$$

Let $\xi \in \Gamma_\mu$ and note that $u(\xi) > v(\xi)$ and $|\nabla_{H^m} u(\xi)|_{H^m} = |\nabla_{H^m} v(\xi)|_{H^m}$. As a consequence, since f is strictly increasing,

$$\Delta^\varphi_{H^m} u(\xi) \geq f(u(\xi)) l(|\nabla_{H^m} u(\xi)|_{H^m}) > f(v(\xi)) l(|\nabla_{H^m} v(\xi)|_{H^m}) \geq \Delta^\varphi_{H^m} v(\xi).$$

By continuity, there exists an open set $V \supset \Gamma_\mu$ such that

$$\Delta^\varphi_{H^m} u \geq \Delta^\varphi_{H^m} v \qquad \text{on } V. \tag{46}$$

Fix now $\xi \in \Gamma_\mu$ and a parameter $0 < \rho < \mu$; let $\Omega_{\xi,\rho}$ be the connected component containing ξ of the set

$$\left\{q \in B_T \backslash \overline{B}_{r_0} : u(q) > v(q) + \rho\right\}.$$

We observe that $\xi \in \Omega_{\xi,\,\rho}$ for every ρ and that $\Omega_{\xi,\,\rho}$ is a nested sequence as ρ converges to μ. We claim that if ρ is close to μ, then $\overline{\Omega}_{\xi,\,\rho} \subset V$. This can be shown by a compactness argument such as the following: since Γ_μ is closed and bounded, there exists $\varepsilon > 0$ such that $d(V^c, \Gamma_\mu) \geq \varepsilon$. Suppose, by contradiction, that there exist sequences $\rho_n \uparrow \mu$ and $\{q_n\}$ such that $q_n \in \Omega_{\xi,\rho_n}$ and $d(q_n, \Gamma_\mu) > \varepsilon$. Then, we can assume that the sequence is contained in Ω_{ξ,ρ_0} which, by construction, has compact closure; passing to a subsequence converging to some $\overline{q}$, we have by continuity

$$d(\overline{q}, \Gamma_\mu) \geq \varepsilon, \tag{47}$$

but, on the other hand, $(u - v)(\overline{q}) = \lim_n (u - v)(q_n) \geq \lim_n \rho_n = \mu$, hence $\overline{q} \in \Gamma_\mu$ and this contradicts (47). Therefore, $d(\partial\Omega_{\xi,\rho}, \Gamma_\mu) \to 0$ as $\rho \to \mu$, and the claim is proved.

On $\partial\Omega_{\xi,\,\rho}$ we have $u(q) = v(q) + \rho$; since $v(q) + \rho$ solves

$$\Delta^\varphi_{H^m}(v + \rho) = \Delta^\varphi_{H^m} v \leq f(v) l(|\nabla_{H^m} v|_{H^m}) \leq f(v + \rho) l(|\nabla_{H^m}(v + \rho)|_{H^m}),$$

by Proposition 2.1,

$$u(q) \leq v(q) + \rho.$$

But $u(\xi) = v(\xi) + \mu$ and $\xi \in \Omega_{\xi,\,\rho}$, a contradiction. This shows that $u \equiv c$, where c is a non-negative constant; in case $l(0) > 0$ we have $0 = \Delta^\varphi_{H^m} c \geq f(c) l(0)$. This implies $f(c) = 0$, hence $c = 0$.

Assume now the validity of the Keller-Osserman condition (KO), and suppose that u is a solution of (44). By the previous arguments, if u is not constant then necessarily $u^* = +\infty$. Again, fix $r_0 > 0$ such that $u \not\equiv 0$ on B_{r_0}, and define $u^*_0 = \sup_{\overline{B}_{r_0}} u$. Choose $\widetilde{q}, \eta, \varepsilon$ in such a way that $u(\widetilde{q}) > 2u^*_0$, $0 < \varepsilon < \eta < u^*_0$, and consider the function α defined as in Lemma (3.2) with $A = +\infty$. Then, $v(q) = \alpha(r(q))$ is a supersolution of (44) and

$$\begin{aligned} &u(q) - v(q) \leq u_0^* - \varepsilon && \text{on } \partial B_{r_0}, \\ &u(\widetilde{q}) - v(\widetilde{q}) > 2u_0^* - \eta > u_0^* \\ &u(q) - v(q) \to -\infty && \text{as } r(q) \to T^-. \end{aligned}$$

Hence, $u - v$ attains a positive maximum in $B_T \backslash \overline{B}_{r_0}$. The proof now proceeds in the same way as in the previous case.

4. Proof of Theorem 1.3

This section is devoted to proving the result stated in Theorem 1.3; first of all we observe that the sufficiency of the Keller-Osserman condition, i.e. implication $i) \Rightarrow ii)$, follows from Theorem 1.1. In particular, it is easy to see that (p & L) implies (Φ & L) and that ($L2_p$) implies ($L2$). This latter follows since $\Delta^p_{H^m}$ satisfies ($\Phi2$) for every $0 \leq \tau \leq p-1$ (as we have already pointed out), and $\tau = p - 1$ is the best choice. Our aim is therefore to provide existence of unbounded C^1-solutions of inequality (18) under the assumption that (KO) is not satisfied; this will be achieved by pasting two subsolutions defined on complementary sets. Such solutions will be "*radial stationary functions*", that is, functions of the form

$$v(q) = w(|z|), \qquad q = (z,t) \in H^m,$$

where $w : \mathbb{R}_0^+ \to \mathbb{R}$, $w \in C^2(\mathbb{R}_0^+)$. Performing computations very similar to those in Subsection 2.1, we obtain the following identities:

$$|\nabla_{H^m}|z||_{H^m} \equiv 1, \qquad \Delta_{H^m}|z| = \frac{2m-1}{|z|}, \tag{48}$$

and thus the expression of the φ-Laplacian for a radial stationary function is

$$\Delta^{\varphi}_{H^m} v = \varphi'(|w'(|z|)|)w''(|z|) + \frac{2m-1}{|z|}\operatorname{sgn}(w'(|z|))\varphi(|w'(|z|)|). \tag{49}$$

This shows that radial stationary functions in the Heisenberg group behave as Euclidean radial ones, and this fact allows us to avoid dealing with the density function. Define implicitly the function w on $\mathbb{R}_0^+$ by setting

$$t = \int_1^{w(t)} \frac{\mathrm{d}s}{K^{-1}(F(s))}. \tag{50}$$

Note that w is well defined, $w(0) = 1$ and, by Lemma 3.1 and since the Keller-Osserman condition does not hold, $w(t) \to +\infty$ as $t \to +\infty$. Differentiating (50) yields

$$w' = K^{-1}(F(w(t))) > 0, \tag{51}$$

and a further differentiation gives

$$(p-1)(w')^{p-2}w'' = f(w)l(w'). \tag{52}$$

We fix $\bar{z} > 0$ to be specified later and set $A_{\bar{z}} = \{(z,t) \in H^m : |z| < \bar{z}\}$, and let $u_1(z,t)$ be the radial stationary function defined on $H^m \setminus A_{\bar{z}}$ by the formula $u_1(z,t) = w(|z|)$. Since, by (48), $|\nabla_{H^m} u_1|_{H^m} = w'$, using (49) and (52) we conclude that u_1 satisfies

$$\Delta^p_{H^m} u_1 = (p-1)(w')^{p-2}w'' + \frac{2m-1}{|z|}(w')^{p-1} \geq f(u_1)l(|\nabla_{H^m} u_1|_{H^m}) \tag{53}$$

on $H^m \setminus A_{\bar{z}}$. To produce a subsolution u_2 on $A_{\bar{z}}$, we fix constants β_o, $\Theta > 0$, and, denoting with p' the conjugate exponent of p, we let

$$\beta(t) = \frac{\Theta}{p'} t^{p'} + \beta_o.$$

Noting that $\beta'(0) = 0$, we deduce that the function $u_2(z,t) = \beta(|z|)$ is C^1 on H^m, and an easy computation with the aid of (49) shows that

$$\Delta^p_{H^m} u_2 = 2m\Theta^{p-1}. \tag{54}$$

Since $\beta', \beta'' \geq 0$, and by the monotonicity of f and l, it follows that, if

$$2m\Theta^{p-1} \geq Cf(\beta(\bar{z}))l(\beta'(\bar{z})), \tag{55}$$

where C is the constant of the C-monotonicity of l, then

$$\Delta^p_{H^m} u_2 \geq f(u_2)l(|\nabla_{H^m} u_2|_{H^m}) \quad \text{on } A_{\bar{z}}. \tag{56}$$

To join u_1 and u_2 so that the resulting function u is C^1, we shall choose the parameters $\bar{z}$, Θ, β_o, in such a way that (55) and

$$\begin{cases} \beta(\bar{z}) = w(\bar{z}) \\ \beta'(\bar{z}) = w'(\bar{z}) \end{cases} \tag{57}$$

are satisfied. Towards this aim, we define

$$\bar{z} = \int_1^{\mu} \frac{\mathrm{d}s}{K^{-1}(F(s))} > 0, \tag{58}$$

where $1 < \mu \leq 2$. Note that, by definition, $w(\bar{z}) = \mu$, $w'(\bar{z}) = K^{-1}(F(\mu))$, $\bar{z} \to 0$ as $\mu \to 1^+$ and, by the monotonicity of K^{-1} and F

$$\frac{\mu - 1}{K^{-1}(F(2))} \leq \bar{z} \leq \frac{\mu - 1}{K^{-1}(F(1))}, \tag{59}$$

Putting together (55) and (57) and recalling the relevant definitions we need to show that the following system of inequalities has a solution:

$$\begin{aligned} &(i) \quad K^{-1}(F(\mu))\bar{z}/p' + \beta_o = \mu \\ &(ii) \quad \Theta \bar{z}^{p'-1} = K^{-1}(F(\mu)) \\ &(iii) \quad \Theta^{p-1} \geq \tfrac{C}{2m} f(\mu) l(K^{-1}(F(\mu))). \end{aligned} \tag{60}$$

Since, by (59),

$$K^{-1}(F(\mu))\frac{\bar{z}}{p'} \leq \frac{1}{p'}\frac{K^{-1}(F(2))}{K^{-1}(F(1))}(\mu - 1)$$

for μ sufficiently close to 1 the first summand on the left hand side of (i) is strictly less that 1, and therefore we may choose $\beta_o > 0$ in such a way that (i) holds. Next we let Θ be defined by (ii), and note that, by (59),

$$\Theta = K^{-1}(F(\mu))\bar{z}^{1-p'} \geq \frac{\left[K^{-1}(F(1))\right]^{p'}}{(\mu - 1)^{p'-1}} \to +\infty \quad \text{as } \mu \to 1^+.$$

Therefore, since

$$f(\mu) l(K^{-1}(F(\mu))) \leq C f(2) l(K^{-1}(F(2))),$$

if μ is close enough to 1 then (iii) is also satisfied. Summing up, if μ is sufficiently close to 1, the function

$$u(x) = \begin{cases} u_1(x) & \text{on } H^m \setminus A_{\bar{z}} \\ u_2(x) & \text{on } A_{\bar{z}} \end{cases} \tag{61}$$

is a classical weak solution of $\Delta^p_{H^m} u \geq f(u) l(|\nabla_{H^m} u|_{H^m})$. Indeed, the weak inequality (26) follows easily from the C^1-regularity of u on $\partial A_{\bar{z}}$. This concludes the proof.

5. More differential inequalities

The aim of this section is to show that the method used so far allows us to treat other types of inequalities; in particular, we focus our attention on (21), that is,

$$\Delta^{\varphi}_{H^m} u \geq f(u) - h(u)g(|\nabla_{H^m} u|_{H^m}).$$

As a matter of fact, the most interesting case arises when $h \geq 0$ and $g \geq 0$, that is, when we have the action of two terms of opposite sign and when the standard comparison arguments do not apply. Indeed, as we shall see, in the generalized Keller-Osserman condition the terms h and f play very different roles.

5.1. Basic assumptions and a new adapted Keller-Osserman condition

We collect the following further set of hypotheses:

$$h \in C^0(\mathbb{R}^+),\ h(t) \geq 0 \text{ on } \mathbb{R}^+,\ h \in L^1(0^+),\ h \text{ monotone non-increasing;} \quad (H)$$

$$t\varphi'(t) \in L^1(0^+); \qquad (\Phi 0)$$

$$\exists\, B > 0,\, \theta \in (-\infty, 2) : \varphi'(ts) \geq B\varphi'(t)s^{-\theta} \ \forall\, t \in \mathbb{R}^+, \forall\, s \in [1, +\infty). \qquad (\Phi 3)$$

Integrating, it is easy to deduce that the following condition is implied by (Φ3):

$$\varphi(ts) \geq B\varphi(t)s^{1-\theta} \ \forall\, t \in \mathbb{R}^+, \forall\, s \in [1, +\infty), \qquad (62)$$

Note that

$$\text{when } \varphi(t) = t^{p-1},\ p > 1, \quad (\Phi 3) \text{ is met with } B = 1,\ \theta \in [2-p, 2). \qquad (63)$$

Again, by way of example, if

$$\varphi(t) = \int_0^t \frac{\mathrm{d}s}{P(s)},$$

where $P(s)$ is a polynomial with degree at most θ, non-negative coefficients and such that $P'(0) > 0$, then φ satisfies (Φ3). We would also like to stress that conditions (Φ3) and (Φ2) are compatible, as it is apparent, for instance, for the p-Laplacian .

As in the previous theorems, the necessity of dealing with the density function leads us to require a relaxed homogeneity also on g, as expressed by the following inequality:

$$g(st) \leq \widetilde{D} s^{\tau+1} t^2 \varphi'(t) \quad \forall\, s \in [0,1],\ t \in \mathbb{R}^+ \qquad (G)$$

where τ is as in (Φ2) and $\widetilde{D}$ is a positive constant; this bound on g is also due to a structural constraint which comes from the construction of the supersolution. Unfortunately, for the p-Laplacian this turns out to be quite restrictive. For example, if $g(t) = Dt^\nu$, for some $0 \le \nu$ and some constant $D > 0$, it is not hard to see that (G) holds if and only if $\nu = p$. However, since (21) is an inequality, solving for this g will solve for any other smaller g.
We now examine the steps leading to the definition of the Keller-Osserman condition adapted to inequality (21). Setting $t = 1$ in (Φ3) we have

$$\varphi'(s) \ge B\varphi'(1)s^{-\theta},$$

and since $\varphi'(1) > 0$ we deduce, integrating and using $\theta < 2$,

$$t\varphi'(t) \notin L^1(+\infty).$$

In the present case, $l \equiv 1$ and the definition of K given in (17) becomes

$$K(t) = \int_0^t s\varphi'(s)\,\mathrm{d}s.$$

It follows that (Φ3) with $\theta \le 2$ implies that K is a C^1-diffeomorphism from $\mathbb{R}_0^+$ onto itself. From (Φ3) we also have, for $s \in \mathbb{R}^+, y \in [1, +\infty)$,

$$\int_0^t sy\varphi'(sy)\,\mathrm{d}s \ge By^{1-\theta}\int_0^t s\varphi'(s)\,\mathrm{d}s,$$

so that

$$K(ty) \ge By^{2-\theta}K(t) \qquad \forall\, t \in \mathbb{R}^+,\ \forall\, y \in [1, +\infty). \tag{64}$$

Next, we define

$$\widehat{F}(t) = \int_0^t f(s)e^{(2-\theta)\int_0^s h(x)\,\mathrm{d}x}\,\mathrm{d}s.$$

For $s \in \mathbb{R}^+$ we let

$$t = K^{-1}\Big(\sigma\widehat{F}(s)\Big).$$

Since K^{-1} is non-decreasing we get

$$y = \frac{K^{-1}\Big(\widehat{F}(s)\Big)}{K^{-1}\Big(\sigma\widehat{F}(s)\Big)} \ge 1,$$

and applying inequality (64) we deduce

$$K\Big(K^{-1}\Big(\widehat{F}(s)\Big)\Big) \ge BK\Big(K^{-1}\Big(\sigma\widehat{F}(s)\Big)\Big)\left[\frac{K^{-1}\Big(\widehat{F}(s)\Big)}{K^{-1}\Big(\sigma\widehat{F}(s)\Big)}\right]^{2-\theta}.$$

Hence we obtain

$$\left[\frac{K^{-1}\Big(\widehat{F}(s)\Big)}{K^{-1}\Big(\sigma\widehat{F}(s)\Big)}\right]^{2-\theta} \leq \frac{1}{B\sigma}. \tag{65}$$

Since $\theta < 2$ this can be written as

$$\frac{\sigma^{\frac{1}{2-\theta}}}{K^{-1}\Big(\sigma\widehat{F}(s)\Big)} \leq \frac{B^{-\frac{1}{2-\theta}}}{K^{-1}\Big(\widehat{F}(s)\Big)}, \quad s \in \mathbb{R}^+. \tag{66}$$

In conclusion, the following inequality holds:

$$\int^{+\infty} \frac{e^{\int_0^s h(x)\,\mathrm{d}x}}{K^{-1}\Big(\sigma\widehat{F}(s)\Big)}\,\mathrm{d}s \leq \left(\frac{1}{B\sigma}\right)^{\frac{1}{2-\theta}} \int^{+\infty} \frac{e^{\int_0^s h(x)\,\mathrm{d}x}}{K^{-1}\Big(\widehat{F}(s)\Big)}\,\mathrm{d}s. \tag{67}$$

We are now ready to introduce the further generalized Keller-Osserman condition in the form

Definition 5.1. *The* **generalized Keller-Osserman condition** *for inequality*

$$\Delta^{\varphi}_{H^m} u \geq f(u) - h(u)g(|\nabla_{H^m} u|_{H^m})$$

is the request:

$$\frac{e^{\int_0^t h(x)\,\mathrm{d}x}}{K^{-1}\Big(\widehat{F}(t)\Big)} \in L^1(+\infty). \tag{$\widehat{KO}$}$$

As we have already mentioned, the roles of f and h in the above condition are far from being specular. In particular, h has two contrasting effects: on the one hand the explicit term $e^{\int_0^t h(x)\,\mathrm{d}x}$ supports the non-integrability, hence the existence, on the other hand its presence in the expression for $\widehat{F}(t)$ supports integrability.

We observe that, under assumptions (H) and $(\Phi 3)$, inequality (67) implies that, if $(\widehat{KO})$ holds, then for every $\sigma \in (0, 1]$

$$\frac{e^{\int_0^t h(x)\,\mathrm{d}x}}{K^{-1}\Big(\sigma\widehat{F}(t)\Big)} \in L^1(+\infty). \tag{68}$$

A particular case arises when $h \in L^1(+\infty)$. We are going to see that, independently of the sign of h, condition $(\widehat{KO})$ and KO are indeed equivalent:

Proposition 5.1. *Assume* (Φ), (F), $(\Phi 3)$ *and suppose that* $h : \mathbb{R}^+_0 \to \mathbb{R}$ *is a* *continuous function such that* $h \in L^1(+\infty)$. *Then*

$$\frac{e^{\int_0^t h(x)\,\mathrm{d}x}}{K^{-1}\left(\widehat{F}(t)\right)} \in L^1(+\infty) \quad \textit{if and only if} \quad \frac{1}{K^{-1}(F(t))} \in L^1(+\infty).$$

Proof. First of all we observe that, since $\theta < 2$,

$$\widehat{F}(t) = \int_0^t f(s) e^{(2-\theta)\int_0^s h(x)\,\mathrm{d}x}\,\mathrm{d}s \leq e^{(2-\theta)\|h\|_{L^1}} \int_0^t f(s)\,\mathrm{d}s = \Lambda_1 F(t)$$

with $\Lambda_1 \geq 1$. Similarly $F(t) \leq \Lambda_2 \widehat{F}(t)$ with $\Lambda_2 \geq 1$.
Thus, since K^{-1} is non-decreasing

$$\int^{+\infty} \frac{\mathrm{d}s}{K^{-1}(F(s))} \leq \int^{+\infty} \frac{\mathrm{d}s}{K^{-1}\left(\Lambda_1^{-1}\widehat{F}(s)\right)}. \tag{69}$$

We now perform the change of variables $t = s\Lambda_1^{-1}$. Thus

$$\int^{+\infty} \frac{\mathrm{d}s}{K^{-1}\left(\Lambda_1^{-1}\widehat{F}(s)\right)} \leq \Lambda_1 \int^{+\infty} \frac{\mathrm{d}t}{K^{-1}\left(\Lambda_1^{-1}\widehat{F}(\Lambda_1 t)\right)}. \tag{70}$$

Since $\Lambda_1 \geq 1$, denoting with $a(s) = f(s)e^{(2-\theta)\int_0^s h(x)\,\mathrm{d}x}$ we have

$$\widehat{F}(\Lambda_1 t) = \int_0^{\Lambda_1 t} a(y)\,\mathrm{d}y = \Lambda_1 \int_0^t a(\Lambda_1 x)\,\mathrm{d}x \geq \Lambda_1 e^{-(2-\theta)\|h\|_{L^1}} \int_0^t a(z)\,\mathrm{d}z = \Lambda \widehat{F}(t)$$

for some constant $0 < \Lambda \leq \Lambda_1$. Hence $\Lambda_1^{-1}\widehat{F}(\Lambda_1 t) \geq \sigma \widehat{F}(t)$, where $\sigma = \Lambda\Lambda_1^{-1} \leq 1$. Using (69), (70), the monotonicity of K^{-1} and Lemma 3.1 (in particular inequality (41)) we show that

$$\begin{aligned} \int^{+\infty} \frac{\mathrm{d}s}{K^{-1}(F(s))} &\leq \int^{+\infty} \frac{\mathrm{d}s}{K^{-1}\left(\Lambda_1^{-1}\widehat{F}(s)\right)} \leq \\ &\leq \Lambda_1 \int^{+\infty} \frac{\mathrm{d}s}{K^{-1}(\sigma F(s))} \leq \frac{\Lambda_1}{\sigma} \int^{+\infty} \frac{\mathrm{d}s}{K^{-1}(F(s))}. \end{aligned} \tag{71}$$

Therefore, $h \in L^1(\mathbb{R}^+)$ and (71) immediately imply that

$$\frac{e^{\int_0^t h(x)\,\mathrm{d}x}}{K^{-1}\left(\widehat{F}(t)\right)} \in L^1(+\infty) \qquad \text{if and only if} \qquad \frac{1}{K^{-1}(F(t))} \in L^1(+\infty).$$

□

5.2. Construction of the supersolution and final steps

Now we proceed with the construction of the supersolution; the idea follows the lines of Lemma 3.2, but we briefly reproduce the main steps.

Lemma 5.2. *Assume the validity of* (Φ), (F), (H), $(\Phi 3)$ *and of the Keller-Osserman assumption* $(\widehat{KO})$. *Fix* $0 < \varepsilon < \eta$, $0 < t_0 < t_1$. *Then there exists* $\sigma \in (0, 1]$, $T_\sigma > t_1$ *and* $\alpha : [t_0, T_\sigma) \to [\varepsilon, +\infty)$ *satisfying*

$$\begin{cases} (\varphi(\alpha'))' + \frac{2m+1}{t}\varphi(\alpha') \le \frac{1}{D} f(\alpha) - h(\alpha)(\alpha')^2\varphi'(\alpha'); \\ \alpha' > 0, \quad \alpha(t) \uparrow +\infty \quad \text{as} \quad t \to T_\sigma^-, \\ \alpha(t_0) = \varepsilon \quad \text{and} \quad \alpha(t) \le \eta \quad \text{on } [t_0, t_1]. \end{cases} \tag{72}$$

Proof. First of all we observe that, using $(\widehat{KO})$ and (68) we have that

$$\int_\varepsilon^{+\infty} \frac{e^{\int_0^s h(x)\,\mathrm{d}x}}{K^{-1}\Big(\sigma\widehat{F}(s)\Big)}\,\mathrm{d}s \uparrow +\infty \quad \text{as } \sigma \downarrow 0^+.$$

We thus fix $\sigma_0 \in (0, 1]$ so that, for every $\sigma \in (0, \sigma_0]$

$$T_\sigma = t_0 + \int_\varepsilon^{+\infty} \frac{e^{\int_0^s h(x)\,\mathrm{d}x}}{K^{-1}\Big(\sigma\widehat{F}(s)\Big)}\,\mathrm{d}s > t_1. \tag{73}$$

Implicitly define the C^2-function $\alpha : [t_0, T_\sigma) \to [\varepsilon, +\infty)$ by setting

$$T_\sigma - t = \int_{\alpha(t)}^{+\infty} \frac{e^{\int_0^s h(x)\,\mathrm{d}x}}{K^{-1}\Big(\sigma\widehat{F}(s)\Big)}\,\mathrm{d}s. \tag{74}$$

By construction, $\alpha(t_0) = \varepsilon$ and $\alpha(t) \to +\infty$ as $t \to T_\sigma^-$. We differentiate (74) a first time to obtain

$$K^{-1}\Big(\sigma\widehat{F}(\alpha)\Big) = \alpha' e^{\int_0^\alpha h} \tag{75}$$

so that $\alpha' > 0$. Transforming the above into $\sigma\widehat{F}(\alpha) = K\Big(\alpha' e^{\int_0^\alpha h}\Big)$, differentiating once more and using the definition of $\widehat{F}$ and K we arrive at

$$\sigma f(\alpha) e^{(2-\theta)\int_0^\alpha h}\alpha' = \alpha' e^{2\int_0^\alpha h}\varphi'\Big(\alpha' e^{\int_0^\alpha h}\Big)\Big[\alpha'' + (\alpha')^2 h(\alpha)\Big].$$

We use $(\Phi 3)$ and $\alpha' > 0$ to deduce

$$\sigma f(\alpha) \ge B\varphi'(\alpha')\Big[\alpha'' + (\alpha')^2 h(\alpha)\Big]$$

and thus

$$\varphi'(\alpha')\alpha'' \le \frac{\sigma}{B} f(\alpha) - (\alpha')^2\varphi'(\alpha')h(\alpha). \tag{76}$$

Integrating (76) on $[t_0, t]$ and using $\alpha' > 0$, $\varphi' \geq 0$, (F) and (H) we obtain

$$\varphi(\alpha'(t)) \leq \varphi(\alpha'(t_0)) + \frac{\sigma}{B} t f(\alpha(t)). \tag{77}$$

Putting together (76) and (77) and using (F)

$$\begin{aligned} &\varphi'(\alpha')\alpha'' + \frac{2m+1}{t}\varphi(\alpha') \leq \\ &\leq f(\alpha)\left[\frac{\sigma}{B}2(m+1) + \frac{2m+1}{t_0}\frac{\varphi(\alpha'(t_0))}{f(\alpha(t_0))}\right] - (\alpha')^2 h(\alpha)\varphi'(\alpha'). \end{aligned} \tag{78}$$

From (75)

$$\alpha'(t_0) = K^{-1}\left(\sigma\widehat{F}(\varepsilon)\right)e^{-\int_0^\varepsilon h(x)\,\mathrm{d}x}.$$

Therefore, since $\varphi(t) \to 0$ as $t \to 0^+$, choosing $\sigma \in (0, \sigma_0]$ sufficiently small, (78) yields

$$\varphi'(\alpha')\alpha'' + \frac{2m+1}{t}\varphi(\alpha') \leq \frac{1}{D}f(\alpha) - h(\alpha)(\alpha')^2\varphi'(\alpha')$$

on $[t_0, T_\sigma)$. To prove that $\alpha(t) \leq \eta$ on $[t_0, t_1]$ we observe that

$$t_1 - t_0 = T_\sigma - t_0 + t_1 - T_\sigma = \int_\varepsilon^{\alpha(t_1)} \frac{e^{\int_0^s h(x)\,\mathrm{d}x}}{K^{-1}\left(\sigma\widehat{F}(s)\right)}\,\mathrm{d}s.$$

Hence, since the integrand goes monotonically to $+\infty$ as $\sigma \to 0^+$, we need to have $\alpha(t_1) \to \varepsilon$ as $\sigma \to 0^+$. Since $\alpha' > 0$ this proves the desired property. □

We are now ready to state the non-existence result for inequality (21). The proof is a minor modification of the one given for Theorem 1.1, therefore we only sketch the main points referring to subsection 3.2 for definitions and notations.

Theorem 5.3. *Let φ, f, h, g satisfy (Φ), (F), (H), (G), $(\Phi 0)$, $(\Phi 2)$, $(\Phi 3)$, and $(\widehat{KO})$. Let u be a non-negative C^1-solution of*

$$\Delta^\varphi_{H^m} u \geq f(u) - h(u)g(|\nabla_{H^m} u|_{H^m}) \qquad \text{on } H^m. \tag{79}$$

Then $u \equiv 0$.

Proof. First of all, note that it is sufficient to prove that u is equal to a constant c; indeed, by assumption (G), $0 = \Delta^\varphi_{H^m} c \geq f(c) - h(c)g(0) = f(c)$ and the conclusion follows from (F). Now we prove that a maximum principle holds for equation (21) on a domain Ω; indeed, if we assume $u(\widetilde{q}) = u^*$ for some $\widetilde{q} \in \Omega$, then there exists a neighbourhood $U_{\widetilde{q}} \subseteq \Omega$ such that, for every $\varepsilon > 0$, $g(|\nabla_{H^m} u|_{H^m}) < \varepsilon$ on $U_{\widetilde{q}}$. This implies, up to choosing ε sufficiently small, $\Delta^\varphi_{H^m} u \geq f(u) - h(u^*)\varepsilon \geq 0$ on $U_{\widetilde{q}}$. Then, by Theorem 2.2, $u \equiv u^*$ on such neighbourhood, and thus the set $\{q \in \Omega : u(q) = u^*\}$ is non-empty, open and closed in Ω; therefore, $u \equiv u^*$ in Ω.

Eventually, to prove the constancy of u, assume, by contradiction, that there exists $q_0 \in H^m$ such that $u(q_0) < u^*$; then, by the maximum principle, $u < u^*$ on H^m. We now proceed as in the proof of Theorem 1.1 and define r_0, η, $\widetilde{q}$, $\widetilde{r}$ in the same way. Then, we construct the function $v(q) = \alpha(r(q))$, with α as in Lemma 5.2. A calculation shows that

$$\begin{aligned}\Delta^{\varphi}_{H^m} v &= \psi^{\frac{1}{2}}\left[\psi^{\frac{1}{2}}\varphi'\Big(\alpha'(r)\psi^{\frac{1}{2}}\Big)\alpha''(r) + \frac{2m+1}{r}\varphi\Big(\alpha'(r)\psi^{\frac{1}{2}}\Big)\right] \leq \\ &\leq \psi^{\frac{1+\tau}{2}} D\left[\varphi'(\alpha'(r))\alpha''(r) + \frac{2m+1}{r}\varphi(\alpha'(r))\right] \leq \\ &\leq \psi^{\frac{1+\tau}{2}} D\left[\frac{1}{D}f(\alpha) - h(\alpha)(\alpha')^2\varphi'(\alpha')\right] \leq \\ &\leq f(\alpha(r)) - \frac{D}{\widetilde{D}}h(\alpha(r))g(\alpha'(r)\psi^{\frac{1}{2}}) \leq f(v) - h(v)g(|\nabla_{H^m} v|_{H^m}),\end{aligned}$$

where in the last inequality we have used (G) and we have chosen D in $(\Phi 2)$ big enough to ensure $D \geq \bar{D}$.
If ξ lies in the connected component Γ_μ, using (F), (H) and $|\nabla_{H^m} u(\xi)|_{H^m} = |\nabla_{H^m} v(\xi)|_{H^m}$ we obtain

$$\begin{aligned}\Delta^{\varphi}_{H^m} u(\xi) &\geq f(u(\xi)) - h(u(\xi))g(|\nabla_{H^m} u(\xi)|_{H^m}) > && (80)\\ &> f(v(\xi)) - h(v(\xi))g(|\nabla_{H^m} v(\xi)|_{H^m}) \geq \Delta^{\varphi}_{H^m} v(\xi). && (81)\end{aligned}$$

The rest of the proof follows the same lines of the proof of Theorem 1.3. □

Remark 5.4. We note that the maximum principle is indeed unnecessary for the proof of the final steps in Theorems 1.1 and 5.3. If we assume that u is not constant, we can consider a point q_0 such that $u(q_0) < u^*$ and, by continuity, a small radius r_o such that $u|_{\partial B_{r_0}}(q_0) < u^*$. Using the invariance property, we can consider q_0 as the origin for the Koranyi distance, and proceed analogously to the end.

As for Theorem 1.1, we can state the Euclidean counterpart of Theorem 5.3 substituting assumption (G) with the request

$$g(t) \leq Dt^2\varphi'(t) \quad \text{on } (0, +\infty). \qquad (\widetilde{G})$$

We have:

Theorem 5.5. *Let φ, f, h, g satisfy* (Φ), (F), (H), $(\widetilde{G})$, $(\Phi 0)$, $(\Phi 3)$, *and* $(\widehat{KO})$.

Let $u \in C^1(\mathbb{R}^m)$ be a non-negative solution of

$$\Delta^{\varphi}_{\mathbb{R}^m} u \geq f(u) - h(u)g(|\nabla u|) \qquad \text{on } \mathbb{R}^m. \tag{82}$$

Then $u \equiv 0$.

Remark 5.6. We observe that, in Theorem 1.1 of [13], the authors deal with the equation (in their notation)

$$\operatorname{div}\left(g(|x|)|\nabla u|^{p-2}\nabla u\right) \geq h(|x|)f(u) - \widetilde{h}(|x|)l(|\nabla u|), \tag{83}$$

and provide non-existence under a modified Keller-Osserman condition which they call (ρKO) (see p. 690). Therefore, a comparison with our Theorem 5.5 is due. Although the two equations are made essentially different by the presence of $h(u)$ in (82), in the very special case $\varphi(t) = t^{p-1}, h(u) \equiv 1$ in (82) and $h = \widetilde{h} = g \equiv 1$ in (83), that is (in our notation),

$$\Delta_p u \geq f(u) - g(|\nabla u|), \tag{84}$$

we can check the mutual relation between the two Keller-Osserman conditions. In the setting of (84) ($\widehat{KO}$) reads

$$\frac{e^{\int_0^t h(s)\mathrm{d}s}}{\left[\widehat{F}(t)\right]^{1/p}} \in L^1(+\infty), \tag{85}$$

where

$$\widehat{F}(t) = \int_0^t f(s)e^{(2-\theta)\int_0^s h(x)\mathrm{d}x}\mathrm{d}s;$$

under our assumptions on φ, ($\widetilde{G}$) becomes $g(t) \leq Ct^p$ on $\mathbb{R}^+$, for some $C > 0$, and, by (63), (Φ3) is met for every $\theta \in [2-p, 2)$. In fact, $\theta = 2-p$ is the best choice, and for such value we get

$$\widehat{F}(t) = \int_0^t f(s)e^{ps}\mathrm{d}s. \tag{86}$$

On the other hand, combining ($\mathscr{L}$) and (1.5) in [13], the requirement the authors are forced to make translates, in our notation, into $g(t) \leq t^p$, and their assumption (H)' becomes $p \geq m$, which we do not need. The condition (ρKO) is

$$\frac{e^{-\int_0^t \rho}}{F_\rho(t)^{1/p}} \in L^1(+\infty), \tag{87}$$

with

$$F_\rho(t) = \int_0^t f(s)e^{-p\int_0^s \rho(\sigma)\mathrm{d}\sigma},$$

for some weight $\rho \in C^0(\mathbb{R}_0^+)$ such that, by their condition (1.5), $\rho(t) \leq -\frac{1}{p-1}$ on $\mathbb{R}_0^+$. Since $p \geq m$, the choice $\rho(t) \equiv -1$ for the weight is admissible and gives rise exactly to (85). In conclusion, when the theorems overlap, our assumptions are less demanding than those in [13].

5.3. Another existence result for the p-Laplacian

As a quick application of Lemma 5.1 and Theorem 1.3, we can deduce that the modified Keller-Osserman condition $(\widehat{KO})$ is optimal in the case of the p-Laplacian.

Theorem 5.7. *Let f, h, g satisfy (F), (H), (G), $(\Phi 2)$ and $(\Phi 3)$ with $\tau = 0$. Furthermore suppose that $h \in L^1(\mathbb{R}^+)$. Then, the following conditions are equivalent:*

i) there exists a non-negative, non-constant solution $u \in C^1(H^m)$ of inequality $\Delta^p_{H^m} u \geq f(u) - h(u)g(|\nabla_{H^m} u|_{H^m})$;

ii) $\dfrac{1}{K^{-1}(F(t))} \notin L^1(+\infty)$.

Proof. First, we deduce from the assumptions and from Proposition 5.1 the equivalence between (KO) and $(\widehat{KO})$. We have already pointed out that the p-Laplacian satisfies $(\Phi 2)$ for every $0 \leq \tau \leq p-1$: as it can be checked, the choice of $\tau = 0$ is the least stringent on (G). Furthermore, $(\Phi 0)$ is authomatic. This shows that $i) \Rightarrow ii)$ is an immediate application of Theorem 5.3. As for the other implication, set $l(t) \equiv 1$ and apply the existence part of Theorem 1.3 (note that all the assumptions are satisfied), to get a solution of

$$\Delta^p_{H^m} u \geq f(u).$$

Since the RHS is trivially greater than $f(u) - h(u)g(|\nabla_{H^m} u|_{H^m})$ we have the desired conclusion.

□

Acknowledgements: We would like to thank Professor L. Brandolini for his kind interest in the paper and his helpful suggestions that, among other things, led to a correct proof of the strong maximum principle.

[1] C. Bandle and E. Giarrusso, *Boundary blow up for semilinear elliptic equations with nonlinear gradient terms*, Adv. Differential Equations **1** (1996), no. 1, 133–150.

[2] C. Bandle, A. Greco, and G. Porru, *Large solutions of quasilinear elliptic equations: existence and qualitative properties*, Boll. Un. Mat. Ital. B (7) **11** (1997), no. 1, 227–252.

[3] F. H. Beatrous, T. J. Bieske, and J. J. Manfredi, *The maximum principle for vector fields*, The p-harmonic equation and recent advances in analysis, Contemp. Math., vol. **370**, Amer. Math. Soc., Providence, RI, 2005, pp. 1–9.

[4] M. Biroli, C. Picard, and N. A. Tchou, *Homogenization of the p-Laplacian associated with the Heisenberg group*, Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5) **22** (1998), 23–42.

[5] A. Bonfiglioli, E. Lanconelli, and F. Uguzzoni, *Stratified Lie groups and potential theory for their sub-Laplacians*, Springer Monographs in Mathematics, Springer, Berlin, 2007.

[6] A. Bonfiglioli and F. Uguzzoni, *Nonlinear Liouville theorems for some critical problems on H-type groups*, J. Funct. Anal. **207** (2004), no. 1, 161–215.

[7] J. M. Bony, *Principe du maximum, inégalité de Harnack et unicité du problème de cauchy pour les opérateurs elliptiques dégénérés*, Ann. Inst. Fourier (Grenoble) **19** (1969), no. 1, 277–304.

[8] L. Brandolini, M. Rigoli, and A. G. Setti, *Positive solutions of Yamabe-type equations on the Heisenberg group*, Duke Math. J. **91** (1998), no. 2, 241–295.

[9] L. Capogna, D. Danielli, and N. Garofalo, *Capacitary estimates and the local behavior of solutions of nonlinear subelliptic equations*, Amer. J. Math. **118** (1997), 1153–1196.

[10] M. G. Crandall, H. Ishii, and P. L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.) **27** (1992), no. 1, 1–67.

[11] L. D'Ambrosio and E. Mitidieri, *A priori estimates, positivity results, and nonexistence theorems for quasilinear degenerate elliptic inequalities*, Adv. Math. **224** (2010), no. 3, 967–1020.

[12] A. Domokos, *Differentiability of solutions for the non-degenerate p-Laplacian in the Heisenberg group*, J. Differential Equations **204** (2004), 439–470.

[13] R. Filippucci, P. Pucci, and M. Rigoli, *On entire solutions of degenerate elliptic differential inequalities with nonlinear gradient terms*, J. Math. Anal. Appl. **356** (2009), no. 2, 689–697.

[14] ———, *Nonlinear weighted p-laplacian elliptic inequalities with gradient terms*, Commun. Contemp. Math. **12** (2010), no. 3, 501–535.

[15] N. Garofalo, *Gradient bounds for the horizontal p-Laplacian on a Carnot group and some applications*, Manuscripta Math. **130** (2009), no. 3, 375–385.

[16] N. Garofalo and E. Lanconelli, *Existence and nonexistence results for semilinear equations on the Heisenberg group*, Indiana Univ. Math. J. **41** (1992), no. 1, 71–98.

[17] M. Ghergu, C. Niculescu, and V. Rădulescu, *Explosive solutions of elliptic equations with absorption and non-linear gradient term*, Proc. Indian Acad. Sci. Math. Sci. **112** (2002), no. 3, 441–451.

[18] M. Ghergu and V. Rădulescu, *Explosive solutions of semilinear elliptic systems with gradient term*, RACSAM Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. **97** (2003), no. 3, 467–475.

[19] ———, *Nonradial blow-up solutions of sublinear elliptic equations with gradient term*, Commun. Pure Appl. Anal. **3** (2004), no. 3, 465–474.

[20] E. Giarrusso and G. Porru, *Problems for elliptic singular equations with a gradient term*, Nonlinear Anal. **65** (2006), no. 1, 107–128.

[21] H. A. Hamid and M. F. Bidaut-Véron, *Correlation between two quasilinear elliptic problems with a source term involving the function or its gradient*, C. R. Math. Acad. Sci. Paris **346** (2008), no. 23-24, 1251–1256.

[22] J. Heinonen and I. Holopainen, *Quasiregular maps on Carnot groups*, J. Geom. Anal. **7** (1997), no. 1, 109–148.

[23] L. Hörmander, *Hypoelliptic second order differential equations*, Acta Math. **119** (1967), 147–171.

[24] A. V. Lair and A. W. Wood, *Large solutions of semilinear elliptic equations with nonlinear gradient terms*, Int. J. Math. Math. Sci. **22** (1999), no. 4, 869–883.

[25] J.-M. Lasry and P.-L. Lions, *Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints. I. The model problem*, Math. Ann. **283** (1989), no. 4, 583–630.

[26] L. Mari, M. Rigoli, and A. G. Setti, *Keller-Osserman conditions for diffusion-type operators on Riemannian manifolds*, J. Funct. Anal. **258** (2010), 665–712.

[27] O. Martio and G. Porru, *Large solutions of quasilinear elliptic equations in the degenerate case*, Complex analysis and differential equations (Uppsala, 1997), Acta Univ. Upsaliensis Skr. Uppsala Univ. C Organ. Hist., vol. 64, Uppsala Univ., Uppsala, 1999, pp. 225–241.

[28] E. Mitidieri and S. I. Pokhozhaev, *A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities*, Tr. Mat. Inst. Steklova **234** (2001), 1–384.

[29] R. Osserman, *On the inequality* $\triangle u \geq f(u)$, Pacific J. Math. **7** (1957), 1641–1647.

[30] S. Pigola, M. Rigoli, and A . G. Setti, *Maximum principles and singular elliptic inequalities*, J. Funct. Anal. **193** (2002), no. 2, 224–260.

[31] S. Pigola, M. Rigoli, and A. G. Setti, *Maximum principles on Riemannian manifolds and applications*, Mem. Amer. Math. Soc., vol. **822**, Amer. Math. Soc., Providence, RI, 2005.

[32] P. Pucci, M. Rigoli, and J. Serrin, *Qualitative properties for solutions of singular elliptic inequalities on complete manifolds*, J. Differential Equations **234** (2007), no. 2, 507–543.

[33] P. Pucci and J. Serrin, *The strong maximum principle revisited*, J. Differential Equations **196** (2004), 1–66.

[34] Z. Zhang, *The asymptotic behaviour of solutions with boundary blow-up for semilinear elliptic equations with nonlinear gradient terms*, Nonlinear Anal. **62** (2005), no. 6, 1137–1148.

[35] ———, *Boundary blow-up elliptic problems with nonlinear gradient terms*, J. Differential Equations **228** (2006), no. 2, 661–684.

[36] ———, *Existence of large solutions for a semilinear elliptic problem via explosive sub-supersolutions*, Electron. J. Differential Equations (2006), No. 2, 8 pp. (electronic).