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(Article begins on next page)

Some geometric properties of hypersurfaces with constant r-mean curvature in Euclidean space

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Abstract

Let $f : M \to \mathbb{R}^{m+1}$ be an isometrically immersed hypersurface. In this paper, we exploit recent results due to the authors in [4] to analyze the stability of the differential operator L_r associated with the r-th Newton tensor of f. This appears in the Jacobi operator for the variational problem of minimizing the r-mean curvature H_r. Two natural applications are found. The first one ensures that, under a mild condition on the integral of H_r over geodesic spheres, the Gauss map meets each equator of $\mathbb{S}^m$ infinitely many times. The second one deals with hypersurfaces with zero $(r+1)$-mean curvature. Under similar growth assumptions, we prove that the affine tangent spaces f_*T_pM, $p \in M$, fill the whole $\mathbb{R}^{m+1}$.

1 Introduction

In what follows $f : M^m \to \mathbb{R}^{m+1}$ will always denote a connected, orientable, complete, non compact hypersurface of Euclidean space. We fix an origin $o \in M$ and let $r(x) = \mathrm{dist}(x, o)$, $x \in M$. We set B_r and ∂B_r for, respectively, the geodesic ball and the geodesic sphere centered at o with radius r. Moreover, let ν be the spherical Gauss map and denote with A both the second fundamental form and the shape operator in the orientation of ν. Associated with A we have the principal curvatures $k_1, \ldots, k_m$ and the set of symmetric functions S_j:

$$S_j = \sum_{i_1 < i_2 < \ldots < i_j} k_{i_1} \cdot k_{i_2} \cdot \ldots \cdot k_{i_j}, \qquad j \in \{1, \ldots, m\}, \quad S_0 = 1.$$

The j-mean curvature of f is defined

$$H_0 = 1, \qquad \binom{n}{j} H_j = S_j,$$

so that, for instance, H_1 is the mean curvature and H_m is the Gauss-Kronecker curvature of the hypersurface. Note that, when changing the orientation ν, the odd curvatures change sign, while the sign of the even curvatures is an invariant of the immersion. By Gauss equations and flatness of $\mathbb{R}^{m+1}$ it is easy to see that

$$H_2 = \binom{m}{2}^{-1} S_2 = \frac{1}{2}\binom{m}{2}^{-1} \mathrm{scal},$$

where scal is the scalar curvature of M. The j-mean curvatures satisfy the so-called Newton inequalities

$$H_j^2 \geq H_{j-1} H_{j+1},$$

equality holding if and only if p is an umbilical point (see [9]). We stress that no restriction is made on the sign of the H_i's.

Theorem 1.1. *Let $f : M \to \mathbb{R}^{m+1}$ be a hypersurface such that, for some $j \in \{0, m-2\}$, H_{j+1} is a non-zero constant. If $j \geq 1$, assume that there exists a point $p \in M$ at which the second fundamental form is definite. Set*

$$v_j(r) = \int_{\partial B_r} H_j, \qquad v_1(r) = \int_{\partial B_r} H_1. \tag{1}$$

where integration is with respect to the $(m-1)$-dimensional Hausdorff measure of ∂B_r. Fix an equator $E \subset \mathbb{S}^m$ and suppose that either

(*i*) $\displaystyle\int^{+\infty} \frac{\mathrm{d}r}{v_j(r)} = +\infty$ *and* $H_1 \notin L^1(M)$ *or*

(*ii*) $\displaystyle\int^{+\infty} \frac{\mathrm{d}r}{v_j(r)} < +\infty$ *and*

$$\limsup_{r\to+\infty} \sqrt{v_1(r)v_j(r)} \int_r^{+\infty} \frac{\mathrm{d}s}{v_j(s)} > \frac{1}{2}\left[(j+1)\binom{m+1}{j+2} H_{j+1}\right]^{-1/2}. \tag{2}$$

Then, there exists a divergent sequence $\{x_k\} \subset M$ such that $\nu(x_k) \in E$, where ν is the spherical Gauss map.

Remark 1.2. Up to changing the orientation of M, we can suppose that the second fundamental form at p is positive definite. As we will see later in more detail, this has the remarkable consequence that each H_i, $1 \leq i \leq n$, is strictly positive at every point of M. In particular, v_1 and v_j are both strictly positive and the requirements in (2) are meaningful.

Remark 1.3. When $j = 1$, the existence of an elliptic point $p \in M$ can be replaced by requiring H_2 to be a positive constant, see [6] for details. The case $j = 0$ has been considered in [4].

We clarify the role of (i) and (ii) with some examples. First, we deal with the case $j \neq 1$, and we assume that v_j is of order r^k (resp e^{kr}), for some $k > 0$. Then assumption (ii) requires that $v_1(r)$ is of order at least r^{k-2} (resp e^{kr}). Roughly speaking, v_1 has to be big enough with respect to the other integral curvature v_j. Under additional requirements on the intrinsic curvatures of M, standard volume comparisons allow to control the volume of ∂B_r and (ii) can be read as H_1 not decaying too fast at infinity. When $j = 1$, things are somewhat different. Indeed, (ii) implies that $v_1(r)$ does not *grow* too fast, that is, loosely speaking, it has at most exponential growth. This shows that two opposite effects balances in condition (ii). The same happens for (i) with $j = 1$, as a consequence of Cauchy-Schwartz inequality and coarea formula

$$\left(\int_R^r \frac{\mathrm{d}s}{v_1(s)}\right)\left(\int_{B_r \setminus B_R} H_1\right) \geq (r-R)^2.$$

Finally, we stress that (i) and (ii) are mild hypotheses as they only involve the integral of extrinsic curvatures. In other words, no pointwise control is required.

Up to identifying the image of the tangent space at $p \in M$ with an affine hyperplane of $\mathbb{R}^{m+1}$ in the standard way, we can also prove the following result:

Theorem 1.4. *Let $f : M \to \mathbb{R}^{m+1}$ be a hypersurface with $H_{j+1} \equiv 0$. If $j \geq 1$, assume* $\operatorname{rank}(A) > j$ *at every point. Define v_1, v_j as in* (1)*. Then, under assumptions* (2) (i) *or* (ii)*, for every compact set $K \subset M$ we have*

$$\bigcup_{p \in M \setminus K} T_pM \equiv \mathbb{R}^{m+1},$$

that is, the tangent envelope of $M \setminus K$ coincides with $\mathbb{R}^{m+1}$.

Remark 1.5. As we will see later, condition $\operatorname{rank}(A) > j$ implies that $H_i > 0$ for every $1 \leq i \leq j$.

2 Preliminaries

We start recalling the definition and some properties of the Newton tensors P_j, $j \in \{0, \dots, m\}$. They are inductively defined by

$$P_0 = I, \qquad P_j = S_j I - AP_{j-1}.$$

For future use, we state the following algebraic lemma. For a proof, see [3].

Lemma 2.1. *Let $\{e_i\}$ be the principal directions associated with A, $Ae_i = k_i e_i$, and let $S_j(A_i)$ be the j-th symmetric function of A restricted to the $(m-1)$-dimensional space $e_i^{\perp}$. Then, for each $1 \le j \le m-1$,*

(1) $AP_j = P_j A$;

(2) $P_j e_i = S_j(A_i) e_i$;

(3) $\mathrm{Tr}(P_j) = \sum_i S_j(A_i) = (m-j)S_j$;

(4) $\mathrm{Tr}(AP_j) = \sum_i k_i S_j(A_i) = (j+1)S_{j+1}$;

(5) $\mathrm{Tr}(A^2 P_j) = \sum_i k_i^2 S_j(A_i) = S_1 S_{j+1} - (j+2)S_{j+2}$.

It follows from (2) in the above lemma, and from the definition of P_m that $P_m = 0$. Related to the j-th Newton tensor there is a well defined, symmetric differential operator acting on $C_c^\infty(M)$:

$$L_j u = \mathrm{Tr}(P_j \mathrm{Hess}(u)) = \mathrm{div}(P_j \nabla u) \qquad \forall\, u \in C_c^\infty(M), \tag{3}$$

where the last equality is due to the fact that A is a Codazzi tensor in $\mathbb{R}^{m+1}$, see [5], [13]. L_j naturally appears when looking for stationary points of the curvature integral

$$\mathcal{A}_j(M) = \int_M S_j \mathrm{d}V_M,$$

for compactly supported volume preserving variations. These functionals can be viewed as a generalization of the volume functional. In fact, in [3] and [6] the stationary points of $\mathcal{A}_j$ are characterized as those immersions having constant S_{j+1}. In the above mentioned paper [6], M.F. Elbert computes the second variation of $\mathcal{A}_j$ in more general ambient spaces and obtains in the Euclidean setting the expression

$$T_j = L_j + \big(S_1 S_{j+1} - (j+2)S_{j+2}\big)$$

for the Jacobi operator. In what follows we are interested in the case of L_j elliptic. There are a number of different results giving sufficient conditions to guarantee this fact, and the next two fit the situation of our main theorems.

Proposition 2.2. *Let M be an m-dimensional connected, orientable hypersurface of some space form N. Then, L_i is elliptic for every $1 \le i \le j$ in each of the following cases:*

(i) M contains an elliptic point, that is, a point $p \in M$ at which A is definite (positive or negative), and $S_{j+1} \neq 0$ at every point of M. Note that, up to changing the orientation of M, we can assume A_p to be positive definite, and by continuity $S_{j+1} > 0$ on M.

(ii) $S_{j+1} \equiv 0$ and $\mathrm{rank}(A) > j$ at every point of M.

Moreover, in both cases, every i-mean curvature H_i is strictly positive on M, for $1 \leq i \leq j$.

For a proof of (i) see [3], while for (ii) see [10].

From the above proposition, the requirements on p and $\operatorname{rank}(A)$ in the main theorems ensure ellipticity. As stressed in Remark 1.3, when $j = 2$ in [6] it is shown that the sole requirement $H_2 > 0$ implies the ellipticity of L_1. In the assumptions of the above proposition, we can define the j-volume of some measurable subset $K \subset M$ as the integral

$$\mathcal{A}_j(K) = \int_K S_j \mathrm{d}V_M.$$

Hereafter, we restrict to the case L_j elliptic. Given the relatively compact domain $\Omega \subset M$, L_j is bounded from below on $C_c^\infty(\Omega)$ and, by Rellich theorem, for a sufficiently large λ, $(L_j - \lambda)$ is invertible with compact resolvent. By standard spectral theory, L_j is therefore essentially self-adjoint on $C_c^\infty(\Omega)$ (Theorem 3.3.2 in [12]). Essential self-adjointness implies that $C_c^\infty(\Omega)$ and $\mathrm{Lip}_0(\Omega)$ are cores for the quadratic form associated to L_j. The first eigenvalue $\lambda_1^{T_j}(\Omega)$, with Dirichlet boundary condition, is therefore defined by the Rayleigh characterization

$$\lambda_1^{T_j}(\Omega) = \inf_{\substack{\phi \in \mathrm{Lip}_0(\Omega) \\ \phi \neq 0}} \frac{\int_\Omega \langle P_j(\nabla\phi), \nabla\phi\rangle - \int_\Omega (S_1 S_{j+1} - (j+2)S_{j+2})\phi^2}{\int_\Omega \phi^2},$$

where $\mathrm{Lip}_0(\Omega)$ can be replaced with $C_c^\infty(\Omega)$. By the monotonicity property of eigenvalues (or, in other words, since L_j satisfies the unique continuation property, [2]), if Ω_1 is a domain with compact closure in Ω_2, and $\Omega_2 \backslash \Omega_1$ has nonempty interior, $\lambda_1^{T_j}(\Omega_1) > \lambda_1^{T_j}(\Omega_2)$. Hence, we deduce the existence of

$$\lambda_1^{T_j}(M) = \lim_{\mu \to +\infty} \lambda_1^{T_j}(\Omega_\mu),$$

where $\{\Omega_\mu\}$ is any exhaustion of M by means of increasing, relatively compact domains with smooth boundary. The next result is substantially an application of the result of Moss-Piepenbrink [11], slightly modified according to Fischer-Colbrie and Schoen [8] and Fischer-Colbrie [7] (consult also [12], Chapter 3 and, for the case of L_1, [6]).

Proposition 2.3. *Let M be a Riemannian manifold and let T_j be as above. The following statements are equivalent:*

(i) $\lambda_1^{T_j}(M) \geq 0$;

(ii) there exists $u \in C^\infty(M)$, $u > 0$ solution of $T_j u = 0$ on M.

Furthermore, there exists a compact set $K \subset M$ and $u \in C^\infty(M \backslash K)$, $u > 0$ solution of $T_j u = 0$ on $M \backslash K$ if and only if $\lambda_1^{T_j}(M \backslash K) \geq 0$.

Next, we shall need to consider the following Cauchy problem; here, as usual, $\mathbb{R}^+ = (0, +\infty)$ and $\mathbb{R}_0^+ = [0, +\infty)$.

$$\begin{cases} (v(t)z'(t))' + A(t)v(t)z(t) = 0 & \text{on } \mathbb{R}^+ \\ z'(t) = \mathrm{O}(1) \ \text{ as } t \downarrow 0^+, \ \ z(0^+) = z_0 > 0 \end{cases} \tag{4}$$

where $A(t)$ and $v(t)$ satisfy the following conditions:

(A1) $A(t) \in L^\infty_{\mathrm{loc}}(\mathbb{R}_0^+)$, $A(t) \geq 0$, $A \not\equiv 0$ in $\mathrm{L}^\infty_{\mathrm{loc}}$ sense;

(V1) $v(t) \in L^\infty_{\mathrm{loc}}(\mathbb{R}_0^+)$, $v(t) \geq 0$, $\dfrac{1}{v(t)} \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+)$;

(V2) there exists $a \in \mathbb{R}^+$ such that v is increasing on $(0, a)$ and $\lim_{t \to 0^+} v(t) = 0$.

Observe that $(V2)$ has to be interpreted as there exists a version of v which is increasing near 0 and whose limit as $t \to 0^+$ is 0.

By Proposition A.1 of [4] under the above assumptions (4) has a solution $z(t) \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}_0^+)$ (and condition $z'(t) = \mathrm{O}(1)$ as $t \downarrow 0^+$ is satisfied in an appropriate sense). Furthermore by Proposition A.3 of [4], $z(t)$ has only isolated zeros. In case $1/v \in L^1((1, +\infty))$, by Proposition 2.5 of [4] if, for some $T > 0$,

$$\limsup_{t \to \infty} \frac{\int_T^t \sqrt{A(s)} \mathrm{d}s}{-\frac{1}{2} \log \int_t^{+\infty} \frac{\mathrm{d}s}{v(s)}} > 1 \tag{5}$$

then, every solution of

$$\begin{cases} (v(t)z'(t))' + A(t)v(t)z(t) = 0 & \text{on } (t_0, +\infty), \ t_0 > 0 \\ z(t_0) = z_0 > 0 \end{cases} \tag{6}$$

has isolated zeros and is oscillatory. The same happens if

$$\int^{+\infty} \frac{\mathrm{d}t}{v(t)} = +\infty \qquad \text{and} \qquad \int^{+\infty} A(t)v(t)\mathrm{d}t = +\infty. \tag{7}$$

(see Corollary 2.4 of [4]).

A final result that we shall use is the following computation. (For a proof see [13] , [1]).

Proposition 2.4. *Let $f : M \to \mathbb{R}^{m+1}$ be an isometric immersion of an oriented hypersurface and $\nu : M \to \mathbb{S}^m$ its Gauss map. Fix $a \in \mathbb{S}^m$. Then*

$$\begin{aligned} L_j\langle a, \nu\rangle &= -\big(S_1 S_{j+1} - (j+2)S_{j+2}\big)\langle a, \nu\rangle - \langle \nabla S_{j+1}, a\rangle; \\ L_j\langle f, \nu\rangle &= -(j+1)S_{j+1} - \big(S_1 S_{j+1} - (j+2)S_{j+2}\big)\langle f, \nu\rangle - \langle \nabla S_{j+1}, f\rangle. \end{aligned} \tag{8}$$

where $\langle , \rangle$ stands for the scalar product of vectors in $\mathbb{S}^m \subset \mathbb{R}^{m+1}$.

In particular, if S_{j+1} is constant, we have $T_j\langle a,\nu\rangle = 0$. Moreover, if $S_{j+1}\equiv 0$, $T_j\langle f,\nu\rangle \equiv 0$.

3 Proof of Theorem 1.1

Fix an equator E and reason by contradiction: assume that there exists a sufficiently large geodesic ball B_R such that, outside B_R, ν does not meet E. In other words, $\nu(M\backslash B_R)$ is contained in the open spherical caps determined by E. Indicating with $a \in \mathbb{S}^m$ one of the two focal points of E, $\langle a,\nu(x)\rangle \neq 0$ on $M\backslash B_R$.

Let $\mathcal{C}$ be one of the (finitely many) connected components of $M\backslash B_R$; then $\nu(\mathcal{C})$ is contained in only one of the open spherical caps determined by E. Up to replacing a with $-a$, we can suppose $u = \langle a,\nu\rangle > 0$ on $\mathcal{C}$. Proceeding in the same way for each connected component we can construct a positive function u on $M\backslash B_R$. Since S_{j+1} is constant, by Proposition 2.4 we have that $u > 0$ satisfies

$$T_j u = L_j u + \big(S_1 S_{j+1} - (j+2)S_{j+2}\big)u = 0$$

on $M\backslash B_R$. Thus, by Proposition 2.3, $\lambda_1^{T_j}(M\backslash B_R) \geq 0$. We shall now show that the assumptions of the theorem contradict this fact. As already stressed, the existence of an elliptic point forces both H_j and H_{j+1} to be positive. Fix a radius $0 < R_0 < R$ and let K_j be a smooth positive function on M such that

$$K_j(x) = \begin{cases} 1 & \text{on } B_{R_0/2} \\ (m-j)S_j & \text{on } M\backslash B_{R_0} \end{cases} \tag{9}$$

Next, we define

$$v_j(t) = \int_{\partial B_t} K_j \tag{10}$$

Using Proposition 1.2 of [4] we see that $v_j(t)$ satisfies $(V1)$ with $v_j(t) > 0$ on $\mathbb{R}^+$ and $(V2)$. Next, we define

$$A(t) = \frac{1}{v_j(t)}\int_{\partial B_t} S_1 S_{j+1} - (j+2)S_{j+2}. \tag{11}$$

Then, repeated applications of Newton inequalities give

$$H_1 H_{j+1} - H_{j+2} \geq 0. \tag{12}$$

Thus, using (12)

$$\begin{aligned} &S_1S_{j+1} - (j+2)S_{j+2} = m\binom{m}{j+1}H_1H_{j+1} - (j+2)\binom{m}{j+2}H_{j+2} = \\ &= \binom{m}{j+1}(mH_1H_{j+1} - (m-j-1)H_{j+2}) \\ &\geq \binom{m}{j+1}\left[m - \frac{m-j-1}{j+2}\right]H_1H_{j+1} = (j+1)\binom{m+1}{j+2}H_1H_{j+1} \geq 0. \end{aligned} \tag{13}$$

This implies $A(t) \geq 0$, and

$$A(t)v_j(t) \geq (j+1)\binom{m+1}{j+2}H_{j+1}\int_{\partial B_t} H_1 = (j+1)\binom{m+1}{j+2}H_{j+1}v_1(t).$$

If $1/v_j \notin L^1((1,+\infty))$, then under (2) (i) and by the coarea formula we deduce $Av_j \notin L^1(\mathbb{R}^+)$. Hence, we can apply (7) to deduce that every solution of

$$\begin{cases} (v_j(t)z'(t))' + A(t)v_j(t)z(t) = 0 \quad \text{on } (t_0,+\infty),\ t_0 > 0 \\ z(t_0) = z_0 > 0 \end{cases} \tag{14}$$

is oscillatory. The same conclusion holds when $1/v_j \in L^1((1,+\infty))$. Indeed, from (11), (13)

$$\frac{\int_T^t \sqrt{A(s)}\mathrm{d}s}{-\frac{1}{2}\log\int_t^{+\infty}\frac{\mathrm{d}s}{v_j(s)}} \geq 2\sqrt{(j+1)\binom{m+1}{j+2}H_{j+1}}\frac{\int_T^t\sqrt{\frac{v_1(s)}{v_j(s)}}\mathrm{d}s}{-\log\int_t^{+\infty}\frac{\mathrm{d}s}{v_j(s)}}. \tag{15}$$

Using De l'Hopital rule and (2) (ii), (5) is met. Let now $R < T_1 < T_2$ be two consecutive zeros of $z(t)$ after R. Define

$$\psi(x) = \begin{cases} z(r(x)) & \text{on } \overline{B_{T_2}}\backslash B_{T_1} \\ 0 & \text{outside } \overline{B_{T_2}}\backslash B_{T_1}. \end{cases}$$

Note that $\psi \equiv 0$ on $\partial(\overline{B_{T_2}}\backslash B_{T_1})$, $\psi \in \mathrm{Lip}_0(M)$ and $\nabla\psi(x) = z'(r(x))\nabla r(x)$ where defined. Furthermore, by the coarea formula and the definition of $A(t)$ we have

$$\begin{aligned} \int_M (S_1S_{j+1} - (j+2)S_{j+2})\psi^2 &= \int_{T_1}^{T_2} z^2(t)\int_{\partial B_t}(S_1S_{j+1} - (j+2)S_{j+2})\mathrm{d}t = \\ &= \int_{T_1}^{T_2} z^2(t)A(t)v_j(t)\mathrm{d}t = (m-j)\int_M S_jA(r)\psi^2. \end{aligned}$$

Thus, using (4), the above identity and again the coarea formula

$$\begin{aligned}
&\int_M \langle P_j(\nabla\psi), \nabla\psi\rangle - (S_1S_{j+1} - (j+2)S_{j+2})\psi^2 \\
&\leq \int_M \mathrm{Tr}(P_j)|\nabla\psi|^2 - (S_1S_{j+1} - (j+2)S_{j+2})\psi^2 = \\
&= \int_M (m-j)S_j|\nabla\psi|^2 - (S_1S_{j+1} - (j+2)S_{j+2})\psi^2 \\
&= (m-j)\int_{\overline{B_{T_2}}\setminus B_{T_1}} S_j[(z')^2 - A(t)z^2] = \\
&= (m-j)\int_{T_1}^{T_2} [(z')^2 - A(t)z^2]v_j(t)\mathrm{d}t = \\
&= (m-j)\{z(t)z'(t)v_j(t)\big|_{T_1}^{T_2} - \int_{T_1}^{T_2}[(v_j(t)z'(t))' + A(t)v_j(t)z(t)]z(t)\mathrm{d}t = 0.
\end{aligned}$$

It follows that

$$\lambda_1^{T_j}(\overline{B_{T_2}}\setminus B_{T_1}) \leq \frac{1}{\int_M \psi^2}\left\{\int_M \langle P_j(\nabla\psi), \nabla\psi\rangle - (S_1S_{j+1} - (j+2)S_{j+2})\psi^2\right\} = 0.$$

As a consequence $\lambda_1^{T_j}(M\setminus B_R) < 0$, which gives the desired contradiction.

Remark 3.1. As a matter of fact, the orientability of M is not needed. If M is non orientable, ν is not globally defined. However, changing the sign of ν does not change either the assumptions or the conclusion of Theorem 1.1, since the antipodal map on $\mathbb{S}^m$ leaves each E fixed. If $\langle a, \nu\rangle \neq 0$ on $M\setminus B_R$, the normal field $X = \langle a, \nu\rangle\nu$ is nowhere vanishing and globally defined on $M\setminus B_R$. This shows that, in any case, every connected component of $M\setminus B_R$ is orientable.

4 Proof of Theorem 1.4

Assume that, for some K, the tangent envelope of $M\setminus K$ does not coincide with $\mathbb{R}^{m+1}$. By choosing cartesian coordinates appropriately, we can assume

$$0 \notin \bigcup_{p\in M\setminus K} T_pM.$$

Then, the function $u = \langle f, \nu\rangle$ is nowhere vanishing and smooth on $M\setminus K$. Up to changing the orientation, $u > 0$ on $M\setminus K$. By Proposition 2.4, $T_ju = -(j+1)S_{j+1}$=0. Note that here the assumption $H_{j+1} \equiv 0$ is essential. It follows that $\lambda_1^{T_j}(M\setminus K) \geq 0$. The rest of the proof is identical to that of Theorem 1.1. Again, according to Remark 3.1 we can drop the orientability assumption on M. Indeed, if the tangent envelope of $M\setminus K$ does not cover $\mathbb{R}^{m+1}$, the vector field $X = \langle f, \nu\rangle\nu$ is a globally defined, nowhere vanishing normal vector field on $M\setminus K$, hence $M\setminus K$ is orientable.

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