A note on Killing fields and CMC hypersurfaces

Published version:

DOI:10.1016/j.jmaa.2015.06.016

A NOTE ON KILLING FIELDS AND CMC HYPERSURFACES

L. Mari[1], P. Mastrolia[2] and M. Rigoli[3],

Abstract. In this note, we give some sufficient conditions for a CMC-hypersurface in a Riemannian manifold N to be invariant under the 1-parameter group of isometries generated by a Killing field on N. Our main result improves on previous ones by D. Hoffman-R. Osserman-R. Schoen and S. Fornari-J. Ripoll, and hinges on a new, simple existence theorem for a first zero of solutions of an ODE naturally associated to the problem. This theorem implies some classical oscillation criteria of W. Ambrose and R. Moore. Extension to constant higher-order mean curvature hypersurfaces are also presented.

1. Introduction

In 1982 D. Hoffman, R. Osserman and R. Schoen, [13], proved that if the (spherical) Gauss map ν of a complete, connected, oriented, constant mean curvature surface $F : M \to \mathbb{R}^3$ has image contained in a closed hemisphere $\overline{\Sigma}$ of S^2, then M is a circular cylinder or a plane. The conclusion is achieved by showing that M is invariant under a 1-parameter subgroup of translations of $\mathbb{R}^3$; indeed, they observe that, when $\nu(M)$ is contained in $\overline{\Sigma}$, there exists some unit vector $v \in S^2$ for which the function $u = \langle \nu, v\rangle$ is signed, and $u \equiv 0$ if and only if the vector field V of $\mathbb{R}^3$ obtained by translating v all over the space is tangent to M. In this case, the surface is therefore invariant by the 1-parameter subgroup generated by V.

This idea has been extended by S. Fornari and J. Ripoll, [10], to the case where V is any Killing vector field of the ambient space $(N, \langle\,,\rangle_N)$ and $F : M^m \to N^{m+1}$ is an isometric immersion of a connected, oriented, complete hypersurface with constant mean curvature H and unit normal vector field ν. Hereafter, and throughout all the paper, geometric entities on the ambient space N will be marked with a bar superscript. When $m = 2$, Fornari and Ripoll extend Hoffman-Osserman-Schoen result to the following (see Corollary 2 in [10]): suppose that the function $\langle \nu, V\rangle$ has constant sign, that

$$\overline{\mathrm{Ric}} \geq -2H^2 \langle\,,\rangle_N \tag{1.1}$$

in the sense of quadratic forms and, in case the universal covering of M is conformally the plane, assume $|V|$ is bounded on M. Then either M is invariant by the 1-parameter group of isometries of N generated by V or M is umbilic and $\overline{\mathrm{Ric}}(\nu,\nu) = 2H^2$ along the immersion.

The 2-dimensional case is of course special. First, and to simplify the writing, let us assume that M is simply connected; then by the Poincaré-Köbe uniformization theorem M is conformally either the sphere, the plane or the Poincaré disk. As we shall see below the function $u = \langle \nu, V\rangle$ satisfies the equation

$$\Delta u - 2Ku + \big(\overline{\mathrm{Ric}}(\nu,\nu) + 2\overline{K} + 4H^2\big)u = 0, \tag{1.2}$$

with K the Gaussian curvature of M and $\overline{K}$ the sectional curvature of N on the tangent plane of M (note that (1.2) coincides with the stability equation for constant mean curvature surfaces in N). Up to

Date: June 9, 2015.
2010 *Mathematics Subject Classification.* 53C20, 53C21.
Key words and phrases. Killing vector field; hypersurface; mean curvature; stability.
[1]Departamento de Matemática, Universidade Federal do Ceará, Fortaleza (Brazil). Email: mari@mat.ufc.br. Supported by the grant PRONEX - Núcleo de Análise Geométrica e Aplicacões Processo n PR2-0054-00009.01.00/11.
[2]Università degli Studi di Milano, Dipartimento di Matematica, Italy. Email: paolo.mastrolia@unimi.it. Supported by GNAMPA project "Analisi Globale ed Operatori Degeneri".
[3]Università degli Studi di Milano, Dipartimento di Matematica, Italy. Email: marco.rigoli@unimi.it.
The second author is member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

choosing ν appropriately, we can suppose that $u \geq 0$. Assumption (1.1) implies $\overline{\mathrm{Ric}}(\nu,\nu)+2\overline{K}+4H^2 \geq 0$, and (1.2) yields

$$\Delta u - 2Ku \leq 0, \tag{1.3}$$

hence, by classical results ([9, 17]) $-\Delta + 2K$ is non-negative in the spectral sense. A result of Fisher-Colbrie and Schoen [9] rules out the case $u > 0$ and M conformally equivalent to the disk. In other words, if M is conformally the disk, there exists $x_0 \in M$ such that $u(x_0) = 0$; by the maximum principle $u \equiv 0$ and M is invariant under the 1-parameter subgroup generated by V. We underline that this is a very special case related to dimension 2 and for which, in any case, we need the pointwise estimate (1.1).

In the remaining two case M is parabolic, and their argument is a straightforward application of this fact based on the pointwise bound (1.1) assigning a sign to the quantity $\overline{\mathrm{Ric}}(\nu,\nu) + 2H^2$ and on the further assumption of boundedness of $|V|$.

Remark 1.1. Since [9], the relation between the non-negativity of operators of the type $-\Delta + aK + V$ ($a \in \mathbb{R}^+$, $V \in L^1_{\mathrm{loc}}(M)$) on a surface M and the conformal type of M has been the subject of increasing interest. A beautiful and general result appeared in [2], and we refer to this paper also for an up-to-date account on the problem.

For the general dimension $m \geq 3$, in Corollary 1 of [10] the authors prove a corresponding result, but only under the stronger assumption that M is compact and that the pointwise bound corresponding to (1.1), that is

$$\overline{\mathrm{Ric}} \geq -mH^2 \langle\, ,\rangle_N \tag{1.4}$$

is satisfied.

As a final observation we note that there are plenty of manifolds supporting Killing fields, so that the above setting is really meaningful (see for instance Remark 2.3 below for some examples and observations).

From now on $o \in M$ is a fixed origin, $r(x) = \mathrm{dist}\,(x, o)$ is the Riemannian distance function from o and B_R, ∂B_R are respectively the geodesic ball of radius R centered at o and its boundary.

We prove the following results.

Theorem 1.2. *Let Y be a Killing field on the Riemannian manifold $(N, \langle\, ,\rangle_N)$ and let $F : M^m \to N^{m+1}$ be a complete, oriented, connected hypersurface with constant mean curvature H and unit normal ν. Suppose $u = \langle \nu, Y\rangle$ has constant sign, $\frac{1}{\mathrm{vol}\,(\partial B_R)} \notin L^1(+\infty)$ and*

$$\liminf_{R\to+\infty} \int_{B_R} \left[\overline{\mathrm{Ric}}(\nu,\nu) + mH^2\right] > 0. \tag{1.5}$$

Then M is invariant by the 1-parameter group of isometries of N generated by Y.

Note that the condition $\frac{1}{\mathrm{vol}\,(\partial B_R)} \notin L^1(+\infty)$ implies (and many times is equivalent to, for instance on model manifolds in the sense of Greene and Wu, [12]) the parabolicity of the manifold M, and is weaker than the more common sufficient condition

$$\frac{R}{\mathrm{vol}(B_R)} \notin L^1(+\infty),$$

see [21] or [20] for further details.

Companion to Theorem 1.2, we also have the following result.

Corollary 1.3. *Let $F : M^m \to N^{m+1}$ and Y be as in Theorem 1.2. Assume $u = \langle \nu, Y\rangle$ has constant sign outside a compact set $K \subset M$, $\frac{1}{\mathrm{vol}\,(\partial B_R)} \notin L^1(+\infty)$ and for some $a > 0$*

$$\lim_{R\to+\infty} \int_{B_R} \left[\overline{\mathrm{Ric}}(\nu,\nu) + mH^2\right] = +\infty. \tag{1.6}$$

Then M is invariant by the 1-parameter group of isometries of N generated by Y.

Note that in Corollary 1.3 we have compensated the relaxed request on the sign of u with assumption (1.6), clearly stronger than (1.5).

It is also important to observe that assumptions (1.5) and (1.6) are integral conditions, so that they allow the function

$$\overline{\mathrm{Ric}}(\nu,\nu) + mH^2$$

to be somewhere negative, even in a neighbourhood of infinity in M, which is a clear advantage on the pointwise conditions (1.1) or (1.4). In fact, in this case the latter trivially imply either the result or the fact that M is totally umbilical. Indeed, if

$$\overline{\mathrm{Ric}}(\nu,\nu) + mH^2 \geq 0, \tag{1.7}$$

the function u is a bounded below superharmonic function on the parabolic manifold M (this easily follows from Proposition 2.2 below) and must therefore be constant. If $u \neq 0$, from equation (2.7) of Section 2 we deduce $|\mathrm{II}|^2 + \overline{\mathrm{Ric}}(\nu,\nu) \equiv 0$, that together with (1.7) forces $|\mathrm{II}|^2 = mH^2$ and M is then totally umbilical (here II is the second fundamental tensor of the immersion, see the next section).

We also deal with the "non-parabolic" case $\frac{1}{\mathrm{vol}\,(\partial B_R)} \in L^1(+\infty)$, but we need to strengthen our assumptions requiring the average condition

$$A(r) \doteq \frac{1}{\mathrm{vol}(\partial B_r)} \int_{\partial B_r} \left[\overline{\mathrm{Ric}}(\nu,\nu) + mH^2\right] \geq 0 \qquad \text{for } r \in [R_0, +\infty), \tag{1.8}$$

for some $R_0 > 0$. Choose $f \in C^0(\mathbb{R}_0^+)$ satisfying

$$\mathrm{vol}\,(\partial B_r) \leq f(r) \qquad \text{and } \frac{1}{f} \in L^1(+\infty). \tag{1.9}$$

The search for f matching the first requirement of (1.9) can be made *via* the Laplacian comparison theorem (see for instance [15, 20]) by assigning a lower bound on the Ricci tensor of M, and since the behaviour at $+\infty$ of a carefully chosen f can be easily detected under this assumption, the second requirement in (1.9) turns out simple to check (we refer the reader to [4] for details). In view of (1.9), we can define the *critical curve*

$$\chi_f(r) = \left(2f(r)\int_r^{+\infty} \frac{\mathrm{d}s}{f(s)}\right)^{-2} = \left[\left(-\frac{1}{2}\log \int_r^{+\infty} \frac{\mathrm{d}s}{f(s)}\right)'\right]^2. \tag{1.10}$$

The critical curve, and its relationship with Green kernels, appears in a number of geometric problems, among them Yamabe-type equations on complete non-compact manifold (see [5, 3]). The interested reader is suggested to consult [4] for deepening. Under the assumptions (1.9), we can state the following result:

Theorem 1.4. *Let Y be a Killing vector field on the Riemannian manifold $(N, \langle\,,\rangle_N)$ and let $F : M^m \to N^{m+1}$ be a complete, connected, oriented hypersurface with constant mean curvature H and unit normal ν. Suppose that $u = \langle \nu, Y\rangle$ has constant sign outside a compact set $K \subset M$ and assume the validity of (1.8). Let $f \in C^0(\mathbb{R}_0^+)$ satisfying (1.9), and define χ_f according to (1.10). If*

$$\lim_{R\to+\infty} \int_{R_0}^{R} \left(\sqrt{A(s)} - \sqrt{\chi_f(s)}\right)\mathrm{d}s = +\infty, \tag{1.11}$$

then M is invariant by the 1-parameter group of isometries of N generated by Y.

Remark 1.5. Theorem 1.4 should be compared with Theorem 5.36 of [4], where (under a condition corresponding to (1.11)) the authors obtained a splitting and codimension reduction result very much in the spirit of the original Hoffman-Ossserman-Schoen's theorem.

The above results can be extended to constant higher order mean curvatures; we recall that, given the oriented hypersurface $F : M^m \to N^{m+1}$, the Weingarten operator in the direction of the normal ν

is the symmetruc operator $A : TM \to TM$ defined via the identity

$$\langle A(X), Y\rangle = \langle \mathrm{II}(X,Y), \nu\rangle \qquad \forall\, X, Y \in TM.$$

Let $\lambda_1, \dots, \lambda_m$ be its eigenvalues, that is, the principal curvatures of the immersion. We denote by S_j the j-th symmetric function in $\lambda_1, \dots, \lambda_m$ for $j = 0, \dots, m$:

$$S_0 \doteq 1, \qquad S_j = \sum_{1\le i_1 < \ldots < i_j \le m} \lambda_{i_1}\lambda_{i_2}\dots\lambda_{i_j}$$

The j-th mean curvature H_j is defined *via* the normalization

$$H_0 = 1, \qquad \binom{m}{j} H_j = S_j, \quad j = 1, \dots, m.$$

Thus H_1 is the mean curvature H, and H_m is the Gauss-Kronecker curvature of M.

Due to the complexity of formulas in the general case, let us assume that N has constant sectional curvature. Then, corresponding to Corollary 1.3 we have

Theorem 1.6. *Let Y be a Killing field on the Riemannian manifold $(N, \langle\ ,\ \rangle_N)$. Assume that N has constant sectional curvature $\alpha \in \mathbb{R}$ and let $F : M^m \to N^{m+1}$ be a complete, oriented, connected hypersurface with unit normal ν, constant $(j+1)$-th mean curvature $H_{j+1} \neq 0$ for some $j = 0, \dots, m-1$ and at least an elliptic point if $j \ge 1$. Suppose that $u = \langle \nu, Y\rangle$ has constant sign outside a compact set $K \subset M$ and that*

$$\frac{1}{\int_{\partial B_r} H_j} \notin L^1(+\infty). \tag{1.12}$$

If, for some $a > 0$,

$$\lim_{R\to+\infty} \int_{B_R} \left\{ \alpha(-1)^j (m-j)\binom{m}{j} H_j + \binom{m}{j+1}\left[m H_1 H_{j+1} - (m-j-1) H_{j+2}\right]\right\} = +\infty, \tag{1.13}$$

then M is invariant under the 1-parameter group of isometries of N generated by Y.

Remark 1.7. For $H_2 > 0$ constant, the theorem holds with no requirement on the existence of an elliptic point.

Remark 1.8. The same result holds if, instead of the existence of an elliptic point, we require $H_{j+1} \equiv 0$ for some $j = 0, \dots, m-1$, rank $(A) > j$ and there exists a point $p \in M$ satisfying $H_i(p) > 0$ for each $1 \le i \le j$.

Remark 1.9. In the assumptions of the theorem, $H_i > 0$ on M for $1 \le i \le j$ up to choosing ν in such a way that the second fundamental form is positive definite at the elliptic point (see condition (i) in Remark 2.5). Repeated applications of Newton inequalities $H_i^2 \ge H_{i-1}H_{i+1}$ give $H_1 H_{i+1} - H_{i+2} \ge 0$, $0 \le i \le j-1$. Using the latter we easily see that

$$\binom{m}{j+1}\left[m H_1 H_{j+1} - (m-j-1) H_{j+2}\right] \ge \binom{m}{j+1} (j+1) H_j H_{j+1}$$

so that (1.13) can be replaced by the stronger requirement

$$\lim_{R\to+\infty} \int_{B_R} \left\{ \alpha(-1)^j (m-j)\binom{m}{j} H_j + \binom{m}{j+1} (j+1) H_j H_{j+1}\right\} = +\infty. \tag{1.14}$$

For $j = 0$ the above coincide with (1.6) in case N has constant sectional curvature.

2. Preliminaries

We fix the index convention $1 \le a, b, \ldots \le m+1$ and $1 \le i, j, \ldots \le m$. Given the Riemannian manifold $\left(N^{m+1}, \langle\ ,\ \rangle\right)$, let $\{\theta^a\}$, $\{\theta^a_b\}$ be a local orthonormal coframe on N with dual frame $\{e_a\}$ and with corresponding Levi-Civita connection forms $\{\theta^a_b\}$. A vector field Y on N is a *Killing field* if and only if the Lie derivative of the metric in the direction of Y, $\mathcal{L}_Y \langle\ ,\ \rangle$, is identically null. Writing Y locally in the form $Y = Y^a e_a$, with the usual notation for covariant differentiation the condition $\mathcal{L}_Y \langle \, , \rangle = 0$ is equivalently expressed by

$$Y^a_b + Y^b_a \equiv 0 \quad \text{for each } a, b. \tag{2.1}$$

In the rest of the paper we let $\overline{R}_{abcd}$ denote the components of the $(0,4)$-Riemann curvature tensor of N. We have the following commutation rule:

Lemma 2.1. *For a general vector field Z on N it holds*

$$Z^a_{bc} = Z^a_{cb} + Z^d \overline{R}_{dabc}, \quad \text{for each } a, b, c. \tag{2.2}$$

The proof is a nice exercise using the moving frame formalism and it is therefore left to the interested reader (but for details, and for many more commutation rules, we refer to [6].

Given the oriented hypersurface $f : M^m \to N^{m+1}$ with unit normal vector field ν, we can choose the above coframe $\{\theta^a\}$ to be a Darboux frame along f, that is, $\nu = e_{m+1}$. In other words, getting rid of the pullback notation,

$$\theta^{m+1} = 0 \quad \text{on } M. \tag{2.3}$$

Differentiating (2.3), using the first structure equations $d\theta^a = -\theta^a_b \wedge \theta^b$ and Cartan's lemma (see again [15]) we have

$$\theta^{m+1}_i = h_{ij}\theta^j, \quad h_{ij} = h_{ji}, \tag{2.4}$$

where h_{ij} are the coefficients of the second fundamental tensor II of the immersion, that is

$$\mathrm{II} = h_{ij}\theta^i \otimes \theta^j \otimes \nu. \tag{2.5}$$

Codazzi equations in this formalism read as

$$h_{ijk} = h_{ikj} - \overline{R}^{m+1}_{ijk}. \tag{2.6}$$

The following result was proved in a special case by H. Rosenberg, [22], and in this setting by S. Fornari and J. Ripoll, [10]; we provide here a simple "moving frame" calculation that however will be essential for Proposition 2.7 below.

Proposition 2.2. *Let Y be a vector field on a Riemannian manifold $(N^{m+1}, \langle \, , \rangle)$, and let $f : M^m \to N^{m+1}$ be a two-sided isometric immersion. Choose a normal unit vector field ν and let A, H be the shape operator and the mean curvature of f in the direction of ν. Define $T = L_Y \langle \, , \rangle$. Then, the function $u = \langle Y, \nu \rangle$ satisfies*

$$\Delta u + \left(|II|^2 + \overline{\mathrm{Ric}}(\nu,\nu) \right) u = -h_{ik}T_{ik} - m \langle Y, \nabla H \rangle - T_{(m+1)ii} - \frac{1}{2} T_{ii(m+1)}. \tag{2.7}$$

In particular, if Y is conformal and letting $\eta \in C^\infty(N)$ be such that $L_Y \langle \, , \rangle = \eta \langle \, , \rangle$, then

$$\Delta u + \left(|II|^2 + \overline{\mathrm{Ric}}(\nu,\nu) \right) u = -\eta m H - m \langle Y, \nabla H \rangle - \frac{m}{2} \langle \overline{\nabla} \eta, \nu \rangle. \tag{2.8}$$

Proof. With our choice of frames we have $u = Y^{m+1}$, that is, u is the component of Y in the direction of e_{m+1} along f. Moreover, by its very definition, up to raising an index of T we have $T^a_b = Y^a_b + Y^b_a$. We compute

$$du = dY^{m+1} = Y^{m+1}_i \theta^i - Y^t \theta^{m+1}_t = \left(Y^{m+1}_i - Y^t h_{ti} \right) \theta^i = u_i \theta^i;$$

therefore,

$$\begin{aligned} u_{ij}\theta^j = du_i - u_k \theta^k_i &= d\left(Y^{m+1}_i - Y^t h_{ti} \right) - \left(Y^{m+1}_k - Y^t h_{tk} \right) \theta^k_i \\ &= \left(Y^{m+1}_{ij} - Y^t_i h_{tj} - Y^t_j h_{ti} - Y^{m+1} h_{ti} h_{tj} - Y^t h_{tij} \right) \theta^j, \end{aligned}$$

from which we determine the coefficients u_{ij} of the Hessian of u, Hess(u). Tracing with respect to i and j we get

$$\Delta u = Y^{m+1}_{ii} - 2h_{ti}Y^t_i - u|\mathrm{II}|^2 - Y^t h_{tii} = Y^{m+1}_{ii} - h_{ti}T_{ti} - u|\mathrm{II}|^2 - Y^t h_{tii} = Y^{m+1}_{ii} - h_{ti}T_{ti} - u|\mathrm{II}|^2 - Y^t h_{iit} + Y^t \overline{R}^{m+1}_{iti} \tag{2.9}$$

where the second equality follows by splitting ∇Y into its symmetric and skew-symmetric parts, and the third one follows from Codazzi's equations (2.6). Now, by (2.2),

$$\begin{aligned} Y^{m+1}_{ii} &= T^{m+1}_{ii} - Y^i_{(m+1)i} = T^{m+1}_{ii} - Y^i_{i(m+1)} - Y^j\overline{R}_{ji(m+1)i} - Y^{m+1}\overline{R}_{(m+1)i(m+1)i} \\ &= T^{m+1}_{ii} - Y^i_{i(m+1)} - Y^j\overline{R}_{ji(m+1)i} - u\overline{\mathrm{Ric}}(\nu,\nu) \\ &= T^{m+1}_{ii} - \frac{1}{2}T^i_{i(m+1)} - Y^j\overline{R}_{ji(m+1)i} - u\overline{\mathrm{Ric}}(\nu,\nu). \end{aligned}$$

Plugging into (2.9) gives (2.7). Equality (2.8) is then immediate by noting that $T_{ij} = \eta\delta_{ij}$.

□

Remark 2.3. As mentioned in the Introduction, there are plenty of manifolds supporting a Killing field, for instance (connected) locally symmetric Riemannian manifolds or often Riemannian manifolds expressed as homogeneous spaces under the action of a transitive Lie group, [18]. Another class of natural examples is that of Riemannian warped product manifolds $M \times_\mu \mathbb{R}$ with metric $\langle\ ,\ \rangle = \langle\ ,\ \rangle_M + \mu^2(x)dt^2$, $\mu : M \to \mathbb{R}^+$ smooth, with the choice $Y = \frac{\partial}{\partial t}$.

Remark 2.4. Formula (1.2) in the introduction is obtained from (2.7) for $\nabla H = 0$ and $m = 2$. In this case, by Gauss equations,

$$|\mathrm{II}|^2 = 4H^2 - 2\big(K - \overline{K}\big)$$

with the notation explained in (1.2).

We shall obtain a formula generalizing (2.7) to the case of higher order mean curvatures. Towards this aim we recall that the Newton operators $P_j : TM \to TM$, $j = 0,\dots,m$, are inductively defined by

$$P_0 = I, \qquad P_j = S_jI - A \circ P_{j-1}, \tag{2.10}$$

where I denotes the identity. To each one of them we associate a well-defined symmetric operator L_j acting, for instance, on $C^2(M)$ by the prescription

$$L_ju = \mathrm{tr}\,(P_j \circ \mathrm{hess}\,(u)), \tag{2.11}$$

where hess(u) is the $(1,1)$-version of the Hessian of u. If we assume that N has constant sectional curvature, a simple computation shows that div $P_j \equiv 0$, and L_j can thus be written in divergence form

$$L_ju = \mathrm{div}\,(P_j(\nabla u)). \tag{2.12}$$

Remark 2.5. In general L_j is not elliptic, but there are geometric sufficient conditions to guarantee this fact; we will be interested in the next two (see Proposition 6.27 in [4])

(i) If M has an elliptic point and $S_{j+1}(x) \neq 0\ \forall x \in M$, then each L_j, $1 \le i \le j$ is elliptic (in particular tr $P_i = (m-i)S_i > 0$ on M);
(ii) if $S_{j+1} \equiv 0$ then L_i is elliptic for each $1 \le i \le j$, provided rank$(A) > j$ and there exists apoint $p \in M$ satisfying $H_i(p) > 0$ for each $1 \le i \le j$.

In what follows we suppose that ν has been chosen in such a way that $H_i > 0$ on M for each $1 \le i \le j$, whenever one of the two cases of Remark 2.5 occurs.

For later use we recall the validity of the following result proved in [17] (see also [8]) generalizing [9].

Proposition 2.6. *Let Ω be an open set of a Riemannian manifold $(M, \langle\ ,\ \rangle)$. Let T be a linear, symmetric, elliptic operator and let $q \in C^0(\Omega)$. Define $L = T + q(x)$; then the following facts are equivalent:*

i) *there exists* $w \in C^1(\Omega)$, $w > 0$ *solving* $Lw = 0$ *weakly on* Ω;
ii) *there exists* $w \in H^{1,2}_{\mathrm{loc}}(\Omega)$, $w \geq 0$, $w \not\equiv 0$ *solving* $Lw \leq 0$ *weakly on* Ω;
iii) $\lambda_1^L(\Omega) \geq 0$.

Here $\lambda_1^L(\Omega)$ *is defined by the variational characterization*

$$\lambda_1^L(\Omega) = \inf_{\varphi \in C_c^\infty(\Omega), \varphi \not\equiv 0} \frac{\int_\Omega \left(-\varphi T\varphi - q(x)\varphi^2\right)}{\int_\Omega \varphi^2}. \tag{2.13}$$

Later we shall also use the fact that, in our case, L satisfies the unique continuation property (see [14] and the appendix in [20] for the case of the Laplacian). As a consequence of this fact and of the above definition we have

$$\lambda_1^L(\Omega_1) \geq \lambda_1^L(\Omega_2) \tag{2.14}$$

whenever $\Omega_1 \subseteq \Omega_2$, with strict inequality in case the interior of $\Omega_2 \setminus \Omega_1$ is nonempty.

Starting from Proposition 2.2 we readily generalize formula (2.7) to higher order mean curvatures.

Proposition 2.7. *Let* $F : M^m \to N^{m+1}$ *be an oriented isometrically immersed hypersurface with unit normal vector field* ν *and let* Y *be a Killing field on* N. *Suppose* N *has constant sectional curvature* α; *then the function* $u = \langle \nu, Y \rangle_N$ *satisfies*

(2.15)

$$L_j u = \left\{(-1)^{j-1}(m-j)\binom{m}{j}\alpha H_j - \binom{m}{j+1}\left[mH_1H_{j+1} - (m-j-1)H_{j+2}\right]\right\}u - \binom{m}{j+1}\langle \nabla H_{j+1}, Y\rangle,$$

$j = 0, \ldots, m$, *and where the* L_j*'s are the operators defined in* (2.12).

Proof. Using the expression for the coefficients u_{ij} of $\mathrm{Hess}\,(u)$ in Proposition 2.2, the fact that Y is Killing and N has constant sectional curvature α we obtain

$$u_{ij} = -Y^i_{m+1} - \alpha\delta_{ij}u - Y^t_i h_{tj} - Y^t_j h_{ti} - h_{it}h_{tj}u - Y^t h_{ijt}. \tag{2.16}$$

To simplify the writing let us replace the H_i with the corresponding S_i; in this case, formula (2.7) becomes

$$L_0 u = -m\alpha S_0 u - \left(S_1^2 - 2S_2\right)u - \langle \nabla S_1, Y\rangle.$$

This agrees, for $j = 0$, with the rewriting of (2.15) in the form

$$L_j u = \left\{(-1)^{j-1}(m-j)\alpha S_j - (S_1S_{j+1} - (j+2)S_{j+2})\right\}u - \langle \nabla S_{j+1}, Y\rangle. \tag{2.17}$$

To prove the general validity of the latter one may use an inductive procedure using (2.16) and the definition of the operator L_j. □

Our second main ingredient in our investigation is the next simple analytical step.

Theorem 2.8. *Let* $A, v \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+_0)$ *satisfy* $v > 0$ *on* $\mathbb{R}^+$, $v^{-1} \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+)$. *Assume that* $v^{-1} \notin L^1(+\infty)$, *and let* z *be a positive solution of*

$$\begin{cases} (v(r)z')' + A(r)v(r)z(r) = 0 & \text{on } \mathbb{R}^+ \\ z(0^+) = z_0 > 0, \quad v(0^+)z'(0^+) = \beta \in \mathbb{R}. \end{cases} \tag{2.18}$$

Then

$$\liminf_{r \to +\infty} \int_0^r A(t)v(t)\mathrm{d}t \leq \frac{\beta}{z_0}. \tag{2.19}$$

Proof. We reason by contradiction assuming the existence of $\Lambda > \frac{\beta}{z_0}$ and $R > 0$ sufficiently large such that, for all $r \geq R$,

$$\int_0^r A(t)v(t)\mathrm{d}t \geq \Lambda. \tag{2.20}$$

We define $y = -\frac{z'}{z}v$ so that, because of (2.18), y satisfies

$$\begin{cases} y' = A(r)v(r) + \frac{y^2}{v(r)} & \text{on } \mathbb{R}^+ \\ y(0^+) = -\frac{\beta}{z_0}. \end{cases} \tag{2.21}$$

Integrating the above equation on $[\varepsilon, r]$ for some $\varepsilon > 0$ sufficiently small we have

$$y(r) = y(\varepsilon) + \int_\varepsilon^r \frac{y^2(s)}{v(s)}\mathrm{d}s + \int_\varepsilon^r A(s)v(s)\mathrm{d}s.$$

By the monotone convergence theorem and the initial datum in (2.21) we obtain

$$y(r) = -\frac{\beta}{z_0} + \int_0^r \frac{y^2(s)}{v(s)}\mathrm{d}s + \int_0^r A(s)v(s)\mathrm{d}s. \tag{2.22}$$

From (2.20) we then deduce

$$y(r) \geq \Lambda - \frac{\beta}{z_0} + \int_0^r \frac{y^2(s)}{v(s)}\mathrm{d}s \geq \Lambda - \frac{\beta}{z_0} > 0$$

for $r \geq R$. Hence, for some $C > 0$ and $r \gg 1$,

$$\frac{y^2(r)}{v(r)} \geq \frac{C}{v(r)} \notin L^1(+\infty).$$

We set

$$G(r) = \int_0^r \frac{y^2(s)}{v(s)}\mathrm{d}s \tag{2.23}$$

so that $G(r) \to +\infty$ as $r \to +\infty$. Using (2.22) and (2.23) for $r \geq R$ we obtain

$$y(r) \geq \Lambda - \frac{\beta}{z_0} + G(r)$$

from which we infer the existence of $R_1 \geq R$ such that, for all $r \geq R_1$,

$$y(r) \geq \frac{1}{2}G(r).$$

Using the positivity of $G(r)$ we square the above inequality to get

$$y^2(r) \geq \frac{1}{4}G^2(r).$$

Since $G'(r) = \frac{y^2(r)}{v(r)}$, we thus have

$$\left(-\frac{1}{G(r)}\right)' \geq \frac{1}{4}\frac{1}{v(r)} \quad \text{on } [R_1, +\infty).$$

Integrating the latter on $[R_1, R_2]$, with $R_2 > R_1$, we deduce

$$\frac{1}{G(R_1)} \geq \frac{1}{G(R_1)} - \frac{1}{G(R_2)} \geq \frac{1}{4}\int_{R_1}^{R_2} \frac{ds}{v(s)},$$

and letting $R_2 \to +\infty$ we obtain a contradiction, proving the validity of (2.19). □

Corollary 2.9. *Let A, v be as in Theorem 2.8. Assume*

$$\liminf_{r\to+\infty} \int_0^r A(s)v(s)\mathrm{d}s > \frac{\beta}{z_0}, \; z_0 \neq 0. \tag{2.24}$$

Then any solution z of (2.18) with $z_0 \in \mathbb{R}\setminus\{0\}$ has a first zero. In particular, if for some $a \in \mathbb{R}_0^+$

$$\lim_{r\to+\infty} \int_a^r A(s)v(s)\mathrm{d}s = +\infty, \tag{2.25}$$

then any solution z of (2.18) is oscillatory.

Proof. The first part of the Corollary is obvious from Theorem 2.8. The second depends on the validity of the same on $[r_0, +\infty)$ for any $r_0 > 0$ with $z(r_0) \in \mathbb{R} \setminus \{0\}$ and the fact that (2.25) clearly implies

$$\lim_{r \to +\infty} \int_{r_0}^{r} A(s)v(s)\mathrm{d}s = +\infty$$

for any $r_0 \geq a$. □

Remark 2.10. We shall apply Corollary 2.9 by choosing $v(r) = \mathrm{vol}\,(\partial B_r)$, where B_r is a geodesic ball centered at $o \in M$. Due to the presence of the cut locus of o, v could possibly have some "jump" discontinuity. However, $v \in L^{\infty}_{\mathrm{loc}}(\mathbb{R})$ and $v^{-1} \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}^{+})$ by Proposition 1.6 of [4], and this is still enough to guarantee (when $\beta = 0$) the existence of a solution z of (2.18) with $z \in \mathrm{Lip}_{\mathrm{loc}}\left(\mathbb{R}^{+}_{0}\right)$ ([4], Proposition 4.2). Moreover, a non-identically null solution z of the equation in (2.18) has isolated zeros (if any) ([4], Proposition 4.6).

It is worth to observe that using Corollary 2.9 we can easily generalize two results of R. Moore [16] and W. Ambrose [1] respectively. To do this, we let $\frac{1}{v} \notin L^1(+\infty)$ and for some fixed $a > 0$ we set

$$w(r) = 1 + \int_{a}^{r} \frac{ds}{v(s)}; \tag{2.26}$$

finally we define

$$g(r) = z(r) w^{-\frac{\sigma}{2}}(r) \tag{2.27}$$

for some $0 \leq \sigma < 1$. Then the function g satisfies

$$((vw^{\sigma})g')' + \left(A(r) + \frac{\sigma(\sigma-2)}{4}\frac{1}{(vw)^2(r)}\right)(vw^{\sigma})g = 0.$$

Note that, since $\sigma < 1$,

$$\int_{a}^{r} \frac{ds}{v(s)w^{\sigma}(s)} = \frac{1}{1-\sigma} w^{1-\sigma}(s)\bigg|_{a}^{r} \to +\infty \quad \text{as } r \to +\infty.$$

Furthermore,

$$\int_{a}^{r} \left(A(s) + \frac{\sigma(\sigma-2)}{4(vw)^2(s)}\right) v(s)w^{\sigma}(s)\mathrm{d}s = \frac{\sigma(\sigma-2)}{4(\sigma-1)} w^{\sigma-1}(s)\bigg|_{a}^{r} + \int_{a}^{r} A(s)v(s)w^{\sigma}(s)\,\mathrm{d}s. \tag{2.28}$$

Since $w(r) \to +\infty$ as $r \to +\infty$ and $\sigma < 1$,

$$\frac{\sigma(\sigma-2)}{4(\sigma-1)} w^{\sigma-1} \to 0 \quad \text{as } r \to +\infty.$$

Hence, under the assumption

$$\int_{a}^{r} A(s)v(s)w^{\sigma}(s)\mathrm{d}s \to +\infty \quad \text{as } r \to +\infty$$

we have that the left-hand side of (2.28) diverges to $+\infty$ as $r \to +\infty$. Applying Corollary 2.9 we have therefore proved the following

Proposition 2.11. *Let A, v satisfy the assumptions of Theorem 2.8. Suppose that for some $a > 0$ and $0 \leq \sigma < 1$*

$$\lim_{r \to +\infty} \int_{a}^{r} v(s) \left(\int_{a}^{s} \frac{\mathrm{d}t}{v(t)} \right)^{\sigma} A(s)\,\mathrm{d}s = +\infty.$$

Then any solution z of

$$\begin{cases} (v(r)z')' + A(r)v(r)z = 0 & \text{on } [a, +\infty) \\ z(a) \in \mathbb{R} \setminus \{0\} \end{cases}$$

is oscillatory.

For $v(r) \equiv 1$ the above proposition recovers the result of Moore, [16], generalizing that of Ambrose, [1].

As it is well known for the case $v(r) \equiv 1$, the result is false for $\sigma = 1$ by considering Euler equation

$$z'' + \frac{1}{4(1+r)^2} z = 0. \tag{2.29}$$

In fact, the above equation has the explicit positive solution $z(r) = \sqrt{1+r}\log(1+r)$, hence by classical Sturm-Liouville theory no solution of (2.29) could be oscillatory.

3. Proof of the results

We begin with the

Proof. [of Theorem 1.2] Without loss of generality we can suppose $u = \langle \nu, Y\rangle \geq 0$ on M. According to Proposition 2.2, since the hypersurface has constant mean curvature and Y is Killing, u satisfies

$$\Delta u + \left(|\mathrm{II}|^2 + \overline{\mathrm{Ric}}(\nu,\nu)\right)u = 0. \tag{3.1}$$

If $u(x_0) = 0$ for some $x_0 \in M$, by the maximum principle (see [11], page 35) $u \equiv 0$, that is, Y is tangent to M which is therefore invariant by the 1-parameter group of isometries generated by Y. Otherwise $u > 0$ on M; since $|\mathrm{II}|^2 \geq mH^2$, from (3.1) we deduce

$$\Delta u + \left(\overline{\mathrm{Ric}}(\nu,\nu) + mH^2\right)u \leq 0 \quad \text{on } M. \tag{3.2}$$

Thus, by the aforementioned result of [9] or [17] (see also Proposition 2.6), the operator $L = \Delta + \overline{\mathrm{Ric}}(\nu,\nu) + mH^2$ is stable, in other words

$$\lambda_1^L(M) \geq 0. \tag{3.3}$$

We now show that this contradicts assumption (1.5). Indeed, define

$$A(r) = \frac{1}{v(r)}\int_{\partial B_r}\left[\overline{\mathrm{Ric}}(\nu,\nu) + mH^2\right], \quad v(r) = \mathrm{vol}\,(\partial B_r) \tag{3.4}$$

and consider the corresponding Cauchy problem (2.18). Choose $\beta = 0$, $z_0 > 0$ and let z be the corresponding solution, which exists by Remark 2.10. Using the coarea formula we have

$$\begin{aligned}\int_0^R A(s)v(s)\mathrm{d}s &= \int_0^R \left\{\frac{1}{v(s)}\int_{\partial B_s}\left[\overline{\mathrm{Ric}}(\nu,\nu) + mH^2\right]\right\}v(s)\mathrm{d}s \\ &= \int_0^R\int_{\partial B_s}\left[\overline{\mathrm{Ric}}(\nu,\nu) + mH^2\right]\mathrm{d}s = \int_{B_R}\left[\overline{\mathrm{Ric}}(\nu,\nu) + mH^2\right].\end{aligned}$$

Assumption (1.5) yields

$$\liminf_{R\to+\infty}\int_0^R A(s)v(s)\mathrm{d}s > 0 = \frac{\beta}{z_0},$$

that is, (2.24) of Corollary 2.9 is met. Thus z has a first zero at some $R_0 > 0$. We define $\varphi(x) = z(r(x)) \in \mathrm{Lip}_{\mathrm{loc}}(M)$. Using the Rayleigh variational characterization (2.13) of $\lambda_1^L(B_{R_0})$, (2.18) and the coarea formula, we have

$$\begin{aligned}\lambda_1^L(B_{R_0}) &\leq \frac{\int_{B_{R_0}}|\nabla\varphi|^2 - \left[\overline{\mathrm{Ric}}(\nu,\nu) + mH^2\right]\varphi^2}{\int_{B_{R_0}}\varphi^2} \\ &= \frac{\int_0^{R_0}\left\{|z'(s)|^2 v(s) - A(s)v(s)z^2(s)\right\}\mathrm{d}s}{\int_0^{R_0} z^2(s)v(s)\mathrm{d}s} = 0,\end{aligned}$$

as immediately seen by multiplying the equation in (2.18) by z, integrating by parts on $[0, R_0]$ and using $v(0^+) = 0$ and $(vz')(0) = \beta = 0$. Then, $\lambda_1^L(B_{R_0}) \leq 0$, and by the monotonicity property of eigenvalues we deduce $\lambda_1^L(M) < 0$, contradicting (3.3). This shows that necessarily $u \equiv 0$ and completes the proof of the Theorem. ∎

Proof. [of Corollary 1.3] Choose $a > 0$ sufficiently large that the compact set $K \subset B_a$ and consider the Cauchy problem, say (2.18)', given by (2.18) with initial data in a instead of 0 and with $A(r)$ and $v(r)$ defined in (3.4) (on all of $\mathbb{R}_0^+$). By assumption (1.6) and Corollary 2.9 any solution of (2.18)' is oscillatory. Fix two consecutive zeroes R_1 and R_2 of z and estimate $\lambda_1^L(B_{R_2} \setminus \overline{B}_{R_1})$ from above as before. As a result we get

$$\lambda_1^L(M \setminus \overline{B}_a) < \lambda_1^L(B_{R_2} \setminus \overline{B}_{R_1}) \leq 0. \tag{3.5}$$

However, if $u = \langle \nu, Y \rangle$ is (say) positive outside K, u is a positive solution of

$$\Delta u + \big(\overline{\text{Ric}}(\nu,\nu) + mH^2\big)u \leq 0 \quad \text{ on } \ M \setminus K,$$

so that the operator L has finite index, equivalently (by [19, 4, 7]) it is stable at infinity. Hence, up to have chosen $a > 0$ sufficiently large, $\lambda_1^L(M \setminus \overline{B}_a) \geq 0$, contradicting (3.5). It follows that we can find $x_0 \in M \setminus K$ such that $u(x_0) = 0$ and, by (3.1) and the maximum principle, $u \equiv 0$ on the connected component of the open set $M \setminus K$ containing x_0. However, u satisfies (3.1) on M, and by the unique continuation property $u \equiv 0$ on all of M, completing the proof of the Corollary. □

Proof. [of Theorem 1.4] The argument is exactly the one used in the proof of Corollary 1.3, once we observe that condition (1.11) together with (1.8) yield the validity of the oscillation result contained in Theorem 5.6 of [4]. □

Proof. [of Theorem 1.6] If $u \equiv 0$ we are done. Thus let $u \not\equiv 0$ and have a constant sign on $M \setminus \overline{B}_a$. By Proposition 2.7 u satisfies the equation

$$L_j u + q(x)u = 0 \qquad \text{on } M, \tag{3.6}$$

where

$$\begin{aligned} q(x) &= \left\{(-1)^j(m-j)\binom{m}{j}\alpha H_j + \binom{m}{j+1}[mH_1H_{j+1} - (m-j-1)H_{j+2}]\right\} \\ &= \left\{(-1)^j(m-j)\alpha S_j + (S_1S_{j+1} - (j+2)S_{j+2})\right\}. \end{aligned} \tag{3.7}$$

Choose the normal vector ν in such a way that $H_i > 0$ for $1 \leq i \leq j$. This is possible by Remark 2.5. Moreover, up to changing the sign of u, even if ν has already been fixed we can suppose $u \geq 0$ on $M \setminus B_a$. By the maximum principle, if $u(x_0) = 0$ for some $x_0 \in M \setminus \overline{B}_a$ then (3.6) implies $u \equiv 0$ on $M \setminus B_a$, and by the unique continuation property $u \equiv 0$ on M, which cannot be. Hence $u > 0$ on $M \setminus B_a$, and by Proposition 2.6 we deduce that

$$\lambda_1^{L_j+q}(M \setminus \overline{B}_a) \geq 0. \tag{3.8}$$

As before we shall now contradict (3.8). Towards this aim we let

$$v_j(r) = (m-j)\binom{m}{j}\int_{\partial B_r} H_j > 0 \tag{3.9}$$

v_j is nonnegative and positive for $r > 0$ by our choice of ν. We define

$$A(r) = \frac{1}{v_j(r)}\int_{\partial B_r} q(x) \tag{3.10}$$

and we consider the Cauchy problem

$$\begin{cases} (v_j(r)z')' + A(r)v_j(r)z = 0 & \text{on } \ (a, +\infty) \\ z(a) = z_a > 0. \end{cases} \tag{3.11}$$

Since $v_j > 0$ on $[a, +\infty)$ and $v_j, v_j^{-1} \in L^\infty_{\mathrm{loc}}([a, +\infty))$, by [4] we know the existence of a solution $z \in \mathrm{Lip}_{\mathrm{loc}}([a, +\infty))$; furthermore z has only isolated zeros. By the coarea formula we have

$$\lim_{R \to +\infty} \int_a^R A(s) v_j(s) = \lim_{R \to +\infty} \int_{B_R \setminus B_a} q(x) = +\infty$$

because of (1.13). This fact, together with (1.12) shows that the assumptions of Corollary 2.9 are satisfied and therefore z is oscillatory. Now let R_1 and R_2 be two consecutive zeros of z after a and define $\psi(x) = z(r(x))$ on the annulus $B_{R_2} \setminus B_{R_1}$ and zero on the complementary set. Now observe that, by the coarea formula and definition (3.10) of $A(r)$ we have

$$\int_M q(x)\psi^2 = \int_{R_1}^{R_2} z^2(s) A(s) v_j(s) \mathrm{d}s = (m-j) \int_M S_j A(r) \psi^2. \tag{3.12}$$

To study the sign of $\lambda_1^{L_j + q}\big(M \setminus \overline{B}_a\big)$ we evaluate $\int \big(-\psi L_j \psi - q(x)\psi^2\big)$. Using again the coarea formula, (3.12), integrating by parts and observing that $\mathrm{tr}\, P_j = (m-j)S_j$ we have

$$\begin{aligned}
\int \big(-\psi L_j \psi - q(x)\psi^2\big) &= \int_M \langle \nabla \psi, P_j(\nabla \psi)\rangle - q(x)\psi^2 \leq \int_M (\mathrm{tr}\, P_j)|\nabla \psi|^2 - q(x)\psi^2 \\
&= (m-j) \int_M S_j \Big[|\nabla \psi|^2 - A(r)\psi^2\Big] = \int_{R_1}^{R_2} \Big[\big(z'(s)\big)^2 - A(s) z^2(s)\Big] v_j(s) \mathrm{d}s \\
&- \int_{R_1}^{R_2} \Big[\big(v_j(s) z'\big)' + A(s) v_j(s) z\Big] z \mathrm{d}s = 0.
\end{aligned}$$

Therefore by domain monotonicity $\lambda_1^{L_j+q}\big(M \setminus \overline{B}_a\big) < 0$, contradicting (3.8). □

For the validity of Remark 1.8 simply observe that in its assumption, by Remark 2.5 (ii), the L_j's are elliptic for $1 \leq i \leq j$ and the above proof applies *verbatim*.

As for Remark 1.7, from $H_1^2 \geq H_2 > 0$ we have that H_1 can be chosen positive on M and L_1 is elliptic, so that the same argument applies.

References

[1] W. Ambrose. A theorem of Myers. *Duke Math. J.*, 24:345–348, 1957.

[2] P. Bérard and P. Castillon. Inverse spectral positivity for surfaces. *To appear on Rev. Mat. Iberoam.*

[3] B. Bianchini, L. Mari, and M. Rigoli. Quasilinear yamabe type equations with a sign-changing nonlinearity, and the prescribed curvature problem on non-compact manifolds. *available at arXiv:1404.3118.*

[4] B. Bianchini, L. Mari, and M. Rigoli. On some aspects of oscillation theory and geometry. *Mem. Amer. Math. Soc.*, 225(1056):vi+195, 2013.

[5] B. Bianchini, L. Mari, and M. Rigoli. Yamabe type equations with sign-changing nonlinearities on non-compact Riemannian manifolds. *J. Funct. Anal.*, 268(1):1–72, 2015.

[6] G. Catino, P. Mastrolia, D. D. Monticelli, and M. Rigoli. Conformal Ricci solitons and related integrability conditions. *arXiv:1405.3169v1 [math.DG]*, 2014.

[7] B. Devyver. On the finiteness of the Morse Index for Schrödinger operators. *Manuscripta Math.*, 139:249–271, 2012.

[8] M. F. Elbert. Constant positive 2-mean curvature hypersurfaces. *Illinois J. Math.*, 46(1):247–267, 2002.

[9] D. Fischer-Colbrie and R. Schoen. The structure of complete stable minimal surfaces in 3-manifolds of non negative scalar curvature. *Comm. Pure Appl. Math.*, XXXIII:199–211, 1980.

[10] S. Fornari and J. Ripoll. Killing fields, mean curvature, translation maps. *Illinois J. Math.*, 48(4):1385–1403, 2004.

[11] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[12] R. E. Greene and H. Wu. *Function theory on manifolds which possess a pole*, volume 699 of *Lecture Notes in Mathematics*. Springer, Berlin, 1979.

[13] D. A. Hoffman, R. Osserman, and R. Schoen. On the Gauss map of complete surfaces of constant mean curvature in $\mathbf{R}^3$ and $\mathbf{R}^4$. *Comment. Math. Helv.*, 57(4):519–531, 1982.

[14] J. L. Kazdan. Unique continuation in geometry. *Comm. Pure Appl. Math.*, 41(5):667–681, 1988.

[15] P. Mastrolia, M. Rigoli, and A. G. Setti. *Yamabe-type equations on complete, noncompact manifolds*, volume 302 of *Progress in Mathematics*. Birkhäuser/Springer Basel AG, Basel, 2012.

[16] R. A. Moore. The behavior of solutions of a linear differential equation of second order. *Pacific J. Math.*, 5:125–145, 1955.

[17] W. F. Moss and J. Piepenbrink. Positive solutions of elliptic equations. *Pacific J. Math.*, 75(1):219–226, 1978.

[18] K. Nomizu. On local and global existence of Killing vector fields. *Ann. of Math. (2)*, 72:105–120, 1960.

[19] J. Piepenbrink. Finiteness of the Lower Spectrum of Schrödinger Operators. *Math. Z.*, 140:29–40, 1974.

[20] S. Pigola, M. Rigoli, and A. G. Setti. *Vanishing and finiteness results in geometric analysis*, volume 266 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2008. A generalization of the Bochner technique.

[21] M. Rigoli and A. G. Setti. Liouville type theorems for ϕ-subharmonic functions. *Rev. Mat. Iberoamericana*, 17(3):471–520, 2001.

[22] H. Rosenberg. Hypersurfaces of constant curvature in space forms. *Bull. Sci. Math.*, 117(2):211–239, 1993.