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Abstract

[1] In this paper, we investigate the existence problem for positive solutions of the Yamabe type equation

$$\Delta_{\mathbb{H}^n} u + q(x)u - b(x)u^\sigma = 0, \qquad \sigma > 1, \tag{Y}$$

on the Heisenberg group $\mathbb{H}^n$, where $\Delta_{\mathbb{H}^n}$ is the Kohn-Spencer sublaplacian. The relevance of our results lies in the fact that $b(x)$ is allowed to change sign. The above PDE is tightly related to the CR Yamabe problem on the deformation of contact forms. We provide existence of a new family of solutions sharing some special asymptotic behaviour described in terms of the Koranyi distance $d(x)$ to the origin. Two proofs of our main Theorem, focused on different aspects, will be given. In particular, the second one relies on a function-theoretic approach that emphasizes the role of Green functions; such a method is suited to deal with more general settings, notably the Yamabe equation with sign-changing nonlinearity on non-parabolic manifolds, that will be investigated in the last part of this paper.

Introduction

Let $\mathbb{H}^n$ be the Heisenberg group of real dimension $2n+1$, that is, the nilpotent Lie group which, as a manifold, is the product $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ and whose group structure

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is given by

$$(z,t)\cdot(z',t') = (z+z', t+t'+2\mathrm{Im}(z,z')), \qquad \forall\,(z,t),(z',t')\in\mathbb{H}^n$$

where $(,)$ denotes the usual hermitian product in $\mathbb{C}^n$. A (real) basis for the Lie algebra of left invariant vector fields on $\mathbb{H}^n$ is given by

$$X_k = 2\mathrm{Re}\frac{\partial}{\partial z_k} + 2\mathrm{Im}z_k\frac{\partial}{\partial t}\ ,\ Y_k = 2\mathrm{Im}\frac{\partial}{\partial z_k} - 2\mathrm{Re}z_k\frac{\partial}{\partial t}\ ,\ \frac{\partial}{\partial t} \tag{0.1}$$

for $k = 1,\dots,n$. The above basis satisfies Heisenberg's canonical commutation relations

$$[X_j, Y_k] = -4\delta_{jk}\frac{\partial}{\partial t}, \tag{0.2}$$

all the other commutators being zero. It follows that the vector fields X_k, Y_k satisfy Hormander's condition and the Kohn-Spencer Laplacian defined as

$$\Delta_{\mathbb{H}^n} = \sum_{k=1}^{n}(X_k^2 + Y_k^2) \tag{0.3}$$

is hypoelliptic by Hormander's theorem [16]. A vector field in the span of $\{X_k, Y_k\}$ is called horizontal. In $\mathbb{H}^n$ one has a natural origin $o = (0,0)$ and a distinguished homogenous norm defined for $x = (z,t)$, by

$$d(x) = d(z,t) = \left(|z|^4 + t^2\right)^{\frac{1}{4}}, \tag{0.4}$$

where $|\cdot|$ is the norm in $\mathbb{C}^n$, which is homogenous of degree 1 with respect to the Heisenberg dilations $\delta_R:\ (z,t)\to(Rz, R^2t)$, $R>0$. This gives rise to the Koranyi distance $d(\cdot,\cdot)$ via the prescription

$$d(x,\hat{x}) = d(x^{-1}\cdot\hat{x}) \qquad \text{for } x,\hat{x}\in\mathbb{H}^n.$$

The Koranyi ball of radius R centered at some $q\in\mathbb{H}^n$ will be denoted with $B_R(q) = \{p\in\mathbb{H}^n : d(p,q) < R\}$. Defining the density function ψ with respect to o by

$$\psi(x) = \psi(z,t) = \frac{|z|^2}{d(z,t)^2} \qquad \text{for } x\neq 0, \tag{0.5}$$

we observe that $0\le\psi(x)\le 1$ on $\mathbb{H}^n\backslash\{o\}$ and that ψ is related to d by the next remarkable formulas:

$$\Delta_{\mathbb{H}^n} d = \frac{2n+1}{d}\psi, \tag{0.6}$$

$$|\nabla_{\mathbb{H}^n} d|^2 = \psi, \tag{0.7}$$

where $\nabla_{\mathbb{H}^n}$, the horizontal gradient, is the operator defined by

$$\nabla_{\mathbb{H}^n} u = \sum_{k=1}^{n}(X_k u)X_k + (Y_k u)Y_k \qquad \forall\, u\in C^1(\mathbb{H}^n),$$

so that $\nabla_{\mathbb{H}^n} u$ is a horizontal vector field and

$$\langle\nabla_{\mathbb{H}^n} u, \nabla_{\mathbb{H}^n} v\rangle = \sum_{k=1}^{n}(X_k u)(X_k v) + (Y_k u)(Y_k v), \qquad |\nabla_{\mathbb{H}^n} u|^2 = \sum_{k=1}^{n}(X_k u)^2 + (Y_k u)^2.$$

For the interior product on horizontal vector fields just defined and corresponding norm, we have the validity of the Cauchy-Schwarz inequality. Furthermore, for $f \in C^1(\mathbb{R})$

$$\nabla_{\mathbb{H}^n} f(u) = f'(u)\nabla_{\mathbb{H}^n} u$$

Finally, the horizontal divergenge div_o is defined, for horizontal vector fields $W = w_k X_k + \tilde{w}_k Y_k$ by

$$\mathrm{div}_o W = \sum_{k=1}^{n} [X_k(w_k) + Y_k(\tilde{w}_k)]$$

and it satisfies

$$\mathrm{div}_o(fW) = f\mathrm{div}_o W + \langle \nabla_{\mathbb{H}^n} f, W\rangle$$

so that

$$\Delta_{\mathbb{H}^n} u = \mathrm{div}_o(\nabla_{\mathbb{H}^n} u)$$

For future use, we also note that if u is a "radial function", that is, $u(z,t) = f(d(z,t))$ for some $f : \mathbb{R}_0^+ \to \mathbb{R}$ of class C^2 then we have

$$\Delta_{\mathbb{H}^n} u = \psi \left\{ f''(d) + \frac{2n+1}{d} f'(d) \right\} \tag{0.8}$$

In a previous paper, [7], we proved some existence results for positive solutions of the Yamabe-type equation

$$\Delta_{\mathbb{H}^n} u + q(x)u - b(x)u^{\sigma} = 0 \tag{0.9}$$

$\sigma > 1$, with $b(x) \geq 0$. This was motivated by the (generalized) CR Yamabe problem; see below for a more detailed discussion. The aim of this paper is to provide a new family of positive solutions when the coefficient $b(x)$ changes sign, a case that prevents the use of any of the techniques described in [7]. We achieve the goal with the aid of some recent results of ours, [4], on the usual (generalized) Yamabe problem on complete, non-compact manifolds. More precisely, with the techniques developed in [4], we shall provide sub- and supersolutions of (0.9) on $\mathbb{H}^n$ and then apply the monotone iteration scheme. This latter is well known in the elliptic contest; however it also works in the sub-elliptic case and the interested reader can find a fairly complete treatment in the Appendix of [7]. To the best of our knowledge, in the literature still little is known about the hypoelliptic Yamabe equation (0.9) when $b(x)$ is allowed to change sign, even with $q(x) \equiv 0$. A remarkable exception is [25], where an existence result very close to case $k = 0$ of our Theorem 1 below is proved. We postpone to Remark 4 the discussion on the relationship between the two theorems. At the end of this Introduction, we will briefly recall some further results on the existence problem for (0.9).

In what follows we shall denote with m the homogeneous dimension of $\mathbb{H}^n$, that is, $m = 2(n+1)$ (note that $m \geq 4$). Our main result is the following:

Theorem 1. *Assume $q(x) \in C^{\infty}(\mathbb{H}^n)$ and that*

$$\psi(x)A_1(d(x)) \leq q(x) \leq \psi(x)A_2(d(x)) \qquad \text{on } \mathbb{H}^n \backslash \{o\}, \tag{0.10}$$

for some $A_j \in C^0(\mathbb{R}_0^+)$, $j = 1, 2$ with $A_2(r) \leq n^2/r^2$ on $\mathbb{R}^+$. Suppose that, for some $k \in (-\infty, 1]$,

$$\begin{cases} r\log r \left[A_j(r) - \dfrac{n^2}{r^2}\right] \in L^1(+\infty) & \text{if } k = 1; \\ r\left[A_j(r) - k\dfrac{n^2}{r^2}\right] \in L^1(+\infty) & \text{if } k < 1, \end{cases} \tag{0.11}$$

for $j = 1, 2$. Let $b \in C^\infty(\mathbb{H}^n)$ and assume

$$|b(x)| \leq \psi(x)B(d(x)) \tag{0.12}$$

on $\mathbb{H}^n$, for some $B \in C^0(\mathbb{R}_0^+)$ for which, for the same k in (0.11) and some $\sigma > 1$, we have

$$\begin{cases} B(r)(\log r)^\sigma r^{1-n(\sigma-1)} \in L^1(+\infty) & \text{if } k = 1; \\ B(r) r^{1-n(1-\sqrt{1-k})(\sigma-1)} \in L^1(+\infty) & \text{if } k < 1. \end{cases} \tag{0.13}$$

Then the equation

$$\Delta_{\mathbb{H}^n} u + q(x)u - b(x)u^\sigma = 0 \tag{0.14}$$

has a family of positive solutions $u \in C^\infty(\mathbb{H}^n)$ such that, for some positive constant $C > 0$

$$\begin{cases} C^{-1} d(x)^{-n} \log d(x) \leq u(x) \leq C d(x)^{-n} \log d(x) & \text{if } k = 1; \\ C^{-1} d(x)^{-n(1-\sqrt{1-k})} \leq u(x) \leq C d(x)^{-n(1-\sqrt{1-k})} & \text{if } k < 1 \end{cases} \tag{0.15}$$

for $d(x) >> 1$. Moreover,

- *if $k \in [0, 1]$, $\|u\|_{L^\infty(\mathbb{H}^n)}$ can be chosen to be as small as we wish;*
- *if $k < 0$, for every compact set K and every $\varepsilon > 0$ we can find a solution u satisfying $\|u\|_{L^\infty(K)} \leq \varepsilon$.*

Remark 1. Note that the non-existence result given in Theorem 2.1 of [7] shows that assumption (0.13) is essentialy sharp.

Remark 2. It is worth to observe that for $k \in (0, 1]$ the above solutions are ground states for the equation (0.14). For $k = 0$ the solutions are bounded between two positive constants and finally for $k \in (-\infty, 0)$ they diverge (polynomially) at infinity. In particular, integrating the estimates in (0.15) we deduce that $u \notin L^2(\mathbb{H}^n)$ for each k, so these solutions seems to be hardly obtainable with the aid of variational techniques.

Remark 3. Via (0.6) and (0.8), we can "radialize" the problem and apply ordinary differential equations techniques to construct sub- and supersolutions. However, the presence of the factor $\psi(x)$ in (0.10) and (0.12) reflects, in some sense, the anisosotropic nature of the Heisenberg group: since $0 \leq \psi(x) \leq 1$ the occurence of $\psi(x)$ as a factor in the lower bound in (0.10) and in the upper bound in (0.10) and (0.12) is a genuine restriction, and since $\psi(x)$ vanishes for $x = (z, 0)$ it forces the corresponding coefficient to vanish along the t-axis. As a matter of fact, we note that for $f \in C^2(\mathbb{R}_0^+)$ a simple computation yields

$$\Delta_{\mathbb{H}^n} f(|z|) = \Delta_{\mathbb{R}^{2n}} f(|z|) = f''(|z|) + \frac{2n-1}{|z|} f'(|z|) \tag{0.16}$$

suggesting that one could realize the construction of the sub- and supersolutions by performing a "radialization" different from above, that is, for instance that suggested by (0.16) in which, the contrary to (0.8) the term $\psi(x)$ is not appearing. As it will become apparent from the proof of Theorem 1, although we can, in this way, avoid the presence of $\psi(x)$ in the assumptions (0.10) and (0.12), the unpleasant side effect of this procedure is that we have to strengthen the basic request $A_2(r) \leq \frac{n^2}{r^2}$ on $\mathbb{R}^+$ to

$$A_2(r) \leq \frac{(n-1)^2}{r^2}$$

this, in turn, implies a strenghthening of (0.11) and (0.13) where we have to substitute n with $n-1$.

Remark 4. The use of (0.16) instead of (0.8) to perform radializations has already been observed by F. Uguzzoni in [25]. It is worth to compare his main existence result, Theorem 1.3, to the appropriate modification of our Theorem 1 in the light of the different radialization process described in the previous remark. Towards this aim, we suppose that $|q(z,t)| \le A(|z|)$, $|b(z,t)| \le B(|z|)$, for some $A, B \in C^0(\mathbb{R}_0^+)$. By Theorem 1.3 in [25], if

$$\int_0^{+\infty} sA(s)\mathrm{d}s < 2n-2, \qquad rB(r) \in L^1(+\infty), \tag{0.17}$$

then for every $C > 0$ small enough there exists a positive solution $u(z,t)$ of (0.9) such that

$$u(z,t) \to C \qquad \text{as } |z| \to +\infty, \text{ uniformly with respect to } t. \tag{0.18}$$

On the other hand, by case $k = 0$ of our Theorem 1 (and the observations in Remark 3) solutions bounded from below and above by positive constants are shown to exist whenever

$$A(r) \le \frac{(n-1)^2}{r^2}, \qquad rA(r) \in L^1(+\infty), \qquad rB(r) \in L^1(+\infty). \tag{0.19}$$

As a matter of fact, slightly refining the proof of Theorem 1 (see Theorems 3 and 11 in [4], and Remark 27 therein), it is not hard to show that (0.19) implies the validity of the full asymptotic relation (0.18) for small enough C. Summarizing, conditions (0.17) and (0.19) are skew.

As we were mentioning above, the study of equation (0.9) is motivated, from the geometrical point of view, by the CR-Yamabe problem that we briefly describe. On $\mathbb{H}^n$ the vector fields $Z_k = X_k + iY_k$, $k = 1, ..., n$ span a subbundle $T_{1,0}$ of the complexified tangent bundle of $\mathbb{H}^n$ and give rise to its canonical CR structure with contact form Θ determined modulo the transformation

$$\tilde{\Theta} = u^{\frac{2}{n}}\Theta \tag{0.20}$$

for some $u \in C^\infty(\mathbb{H}^n)$, $u > 0$. The choice of Θ specifies a pseudohermitian structure on $\mathbb{H}^n$ and

$$\Theta_0 = \mathrm{d}t + i\sum_{k=1}^n \left(z^k \mathrm{d}\bar{z}^k - \bar{z}^k \mathrm{d}z^k\right)$$

defines the canonical structure.

A contact form Θ on a CR manifold M induces a scalar curvature, the Tanaka-Webster (TW for short) scalar curvature [26], R_Θ, which under the transformation (0.20) of the contact form Θ, transforms according to the equation

$$\frac{2n+2}{n}\Delta_\Theta u + R_\Theta u = R_{\tilde{\Theta}} u^{\frac{n+2}{n}} \tag{0.21}$$

where Δ_Θ is the hypoelliptic laplacian of the pseudohermitian manifold (M, Θ). The generalized CR-Yamabe problem, also called the prescribed TW curvature problem, consists in finding a deformation of type (0.20) of the contact form such that the new TW scalar curvature is an assigned function. This can be viewed as a generalization of the CR-Yamabe problem, where we require the new TW scalar curvature to be constant.

The TW scalar curvature of the canonical pseudohermitian structure Θ_0 on $\mathbb{H}^n$ is identically zero, and Δ_{Θ_0} is the operator defined in (0.3). Therefore the equation

$$\frac{2n+2}{n}\Delta_{\mathbb{H}^n} u = R_{\tilde{\Theta}_0} u^{\frac{n+2}{n}} \tag{0.22}$$

is the transformation law for the TW scalar curvature of $(\mathbb{H}^n, \Theta_0)$ under the change $\tilde{\Theta} = u^{\frac{2}{n}}\Theta_0$ of contact forms. Although the literature on the CR-Yamabe problem is vast, very few results are known on the prescribed TW curvature problem. In particular, existence for (0.21) with sign-changing $R_{\tilde{\Theta}}$ reveals to be a hard task, even on $\mathbb{H}^n$. To the best of our knowledge, results of this direction have been obtained in [19], [27] on the CR sphere, and more generally (but restricting to the real dimension 3), in [23], [9]. The papers [19] and [27] exploit a perturbative method and variational techniques to prove existence for (0.22) when $R_{\tilde{\Theta}_0}(x) = -1 + \varepsilon K(x)$, ε is sufficiently small and $K \in C^\infty(\mathbb{H}^n)$ is a Morse function satisfying a suitable non-degeneracy and a general index-counting formula. A different approach, based on the analysis of critical points at infinity, is the focus point of [23], [9]. However, an index condition on $R_{\tilde{\Theta}_0}$ is again required to guarantee the existence of positive solutions.
With the notations introduced above, as an immediate consequence of Theorem 1 and Remark 4 we recover the next theorem in [25].

Corollary 1 ([25], Theorem 1.3)**.** *Let $b(x) \in C^\infty(\mathbb{H}^n)$ satisfy*

$$|b(z,t)| \le B(|z|)$$

on $\mathbb{H}^n$ for some $B \in C(\mathbb{R}_0^+)$ such that

$$rB(r) \in L^1(+\infty)$$

Then, the canonical contact form Θ_0 of $\mathbb{H}^n$ can be conformally deformed to a new contact form Θ with Tanaka-Webster scalar curvature $b(x)$.

Remark 5. The equivalent of Corollary 1, in the setting of the Yamabe problem on $\mathbb{R}^m$, has been obtained in [21], see also [22]. Their sharp results are, to the best of our knowledge, the first successful attempt to solve the Yamabe problem with a sign-changing nonlinearity via radialization techniques and the monotone iteration scheme. However, it should be noticed that their approach, differently from our, strictly depends on the fact that $q(x) = 0$. This is one of the key motivations that lead us to introduce the new techniques described below and in [4]. Furthermore, in the elliptic setting and for Euclidean space $\mathbb{R}^m$, interesting existence results for $\Delta u + q(x)u - b(x)u^{\frac{m+2}{m-2}} = 0$ with sign-changing b have been obtained in [11] and [10] when the potential $q(x)$ is singular, more precisely

$$q(x) = \frac{\lambda}{r(x)^2}, \qquad \text{where} \qquad r(x) = \operatorname{dist}(x,o), \quad \lambda < \frac{(m-2)^2}{4}.$$

This constraint on λ is required to ensure that the singular Schrödinger operator $L = -\Delta_{\mathbb{H}^n} - \lambda/r(x)^2$ be non-negative and locally positive definite in the sense of quadratic forms, see also Subsection 0.1 below.

As a final observation, we underline that we shall give two proofs of Theorem 1 both based an a similar technical argument. However, the advantage of the second is that it is valid in a certain general setting of manifolds and operators with a "function theoretic" property. We refer to Section 2 for a more detailed discussion.

0.1 A few words on the spectral assumption $A_2(r) \le n^2/r^2$

It is worth to explain why the assumption

$$A_2(r) \le \frac{n^2}{r^2} = \frac{(m-2)^2}{4r^2} \tag{0.23}$$

is a basic request. Indeed, in the proof of the theorem we shall construct a supersolution of (0.14) by finding a solution γ of the Cauchy problem

$$\begin{cases} \left(r^{m-1}\gamma'\right)' + A_2(r)r^{m-1}\gamma = -B(r)r^{m-1}\gamma^{\sigma} & \text{on } \mathbb{R}^+, \\ \gamma(0) > 0, \qquad \gamma'(0) = 0. \end{cases}$$

Considering $y \in \mathbb{R}^m$ and setting $|y|$ for its Euclidean norm, $v(y) = \gamma(|y|)$ turns out to be a positive solution of

$$\Delta_{\mathbb{R}^m} v + A_2(r)v \le 0$$

so that, by a classical result ([2], [12], [20]), the spectral radius $\lambda_1^{L_2}(\mathbb{R}^m)$ of $L_2 = -\Delta_{\mathbb{R}^m} - A_2(r)$ acting on $C_c^\infty(\mathbb{R}^m)$ satisfies

$$\lambda_1^{L_2}(\mathbb{R}^m) \ge 0. \tag{0.24}$$

Via the classical Uncertainty Principle lemma (see also Theorem 5.2 of [3]), (0.23) is an optimal requirement to guarantee (0.24). In the setting of hypoelliptic operators, the result of [2], [12] and [20] is still valid and leads to the next

Proposition 1. *Let $q(x) \in C^\infty(\mathbb{H}^n)$, and consider $L_{\mathbb{H}^n} = -\Delta_{\mathbb{H}^n} - q(x)$. Then, the following assertions are equivalent:*

(i) There exists $w \in C^1(\mathbb{H}^n)$, $w \ge 0$, $w \not\equiv 0$ solving $\Delta_{\mathbb{H}^n} w + q(x)w \le 0$ weakly on $\mathbb{H}^n$, that is,

$$\int_{\mathbb{H}^n} \langle \nabla_{\mathbb{H}^n} w, \nabla_{\mathbb{H}^n} \varphi \rangle \ge \int_{\mathbb{H}^n} q(x) w \varphi \qquad \forall\, \varphi \in C_c^1(\mathbb{H}^n);$$

(ii) There exists $u \in C^\infty(\mathbb{H}^n)$, $u > 0$ solving $\Delta_{\mathbb{H}^n} u + q(x)u = 0$ on $\mathbb{H}^n$;

(iii) $\lambda_1^{L_{\mathbb{H}^n}}(\mathbb{H}^n) \ge 0$, that is,

$$\int_{\mathbb{H}^n} q(x)\varphi^2 \le \int |\nabla_{\mathbb{H}^n} \varphi|^2 \qquad \forall\, \varphi \in C_c^1(\mathbb{H}^n).$$

Remark 6. The above result is stated for elliptic operators in [12], [20] and [2] even under fairly weaker regularity assumptions on solutions and coefficients, and the proof carries over unchanged in a hypoelliptic setting. Indeed, the only ingredients are uniform local elliptic estimates (ensured, among others, by [16], [1], [17], [6]), the integration by parts formula

$$\int_{\mathbb{H}^n} \rho \Delta_{\mathbb{H}^n} \psi = -\int_{\mathbb{H}^n} \langle \nabla_{\mathbb{H}^n} \rho, \nabla_{\mathbb{H}^n} \psi \rangle \qquad \forall\, \rho \in C_c^1(\mathbb{H}^n),\ \psi \in C^2(\mathbb{H}^n), \tag{0.25}$$

which follows immediately from Green identities, and the maximum principle (which can be found in [5], [18]).

Because of the above proposition, suppose that we have a positive solution u of (0.14) with $b(x) \leq 0$, a possibility which is permitted by (0.12). Then, u solves

$$\Delta_{\mathbb{H}^n} u + q(x)u \leq 0 \qquad \text{on } \mathbb{H}^n \tag{0.26}$$

and thus $\lambda_1^{L_{\mathbb{H}^n}}(\mathbb{H}^n) \geq 0$. Therefore, if we are wishing to deal with sign-changing $b(x)$, the requirement $\lambda_1^{L_{\mathbb{H}^n}}(\mathbb{H}^n) \geq 0$ is substantially unavoidable. On the other hand, when $\lambda_1^{L_{\mathbb{H}^n}}(\mathbb{H}^n) < 0$, solutions of (0.14) can be produced in the case $b(x) \geq 0$, for example via the technique described in Theorem 3.1 of [7]. The spectral property $\lambda_1^{L_{\mathbb{H}^n}}(\mathbb{H}^n) < 0$ is guaranteed under the bound

$$q(x) \geq \psi(x) A_1(d(x))$$

provided that A_1 is large enough. Indeed, for sufficiently large A_1 the solution ξ of

$$\begin{cases} \xi'' + \frac{2n+1}{t}\xi + A_1 \xi = 0 \\ \xi(0) = 1, \qquad \xi'(0) = 0 \end{cases} \tag{0.27}$$

has a first zero at some $T > 0$, and thus $v(x) = \xi(d(x))$ solves

$$\begin{cases} \Delta_{\mathbb{H}^n} v + q(x)v \geq 0 \qquad \text{on } B_T(o) \\ v > 0 \quad \text{on } B_T(o), \qquad v = 0 \quad \text{on } \partial B_T(o) \end{cases}$$

In this case, it is easy to deduce that $\lambda_1^{L_{\mathbb{H}^n}}(\mathbb{H}^n) < 0$ via a simple comparison argument. Indeed, otherwise choose a smooth, positive solution u of $\Delta_{\mathbb{H}^n} u + q(x)u = 0$ on $\mathbb{H}^n$ ensured by Proposition 1. The function $w = v/u$ is then a solution of $u^{-2}\mathrm{div}_o(u^2 \nabla_o w) \geq 0$ on $B_T(o)$, $w = 0$ on $\partial B_T(o)$. By (a weighted version of) the comparison principle (see, for instance, [18]), the maximum of w is attained on $\partial B_T(o)$, and this forces w to be identically zero, contradiction.

According to Theorem 5.45 of [3], a sufficient condition for $A_1(r)$ to be large enough to produce a first zero of ξ is

$$A_1(r) \geq -\frac{c^2}{r^2}, \qquad \text{for some } c > 0 \text{ and for } \ r \geq r_0 >> 1, \text{ and} \tag{0.28}$$

$$\limsup_{r\to+\infty} \left\{ \int_{r_0}^{r} \sqrt{A_1(s) + \frac{c^2}{s^2}} \mathrm{d}s - \sqrt{n^2 + c^2} \log r \right\} = +\infty. \tag{0.29}$$

Indeed, the combination of (0.28) and (0.29) ensure that ξ is indeed oscillatory, that is, it has infinitely many zeroes.

Remark 7. The reader should be warned that Theorem 5.45 in [3] is stated for Schrödinger operators in the Euclidean setting. However, as it can be seen by inspecting the proof, the result depend on the oscillatory behaviour of the solution ξ of (0.27). Up to passing from n to $m = 2n+2$, (0.29) is the rephrasing of condition (5.135) in [3], which is a sufficient condition for ξ to be oscillatory. The interested reader is suggested to read the discussion preceeding Theorem 5.45 for a more detailed explanation.

Remark 8. Note that, with the aid of the oscillation criterion in (0.28), (0.29), we can improve Theorem 3.1 of [7] where in the proof we have been using the Hille-Nehari criterion for oscillation (see [24]). We leave the details to the interested reader suggesting him to consult [3], Section 5.7.

1 Proof of Theorem 1

In order to prove Theorem 1 we need to introduce some more notation and two recent results of ours, [4]. First of all, given $v \in C^0(\mathbb{R}_0^+)$ satisfying

$$v > 0 \text{ on } \mathbb{R}^+, \quad v \text{ is non-decreasing in a neighbourhood of } 0, \quad \frac{1}{v} \in L^1(+\infty) \qquad \text{(V)}$$

we introduce, following [3], the critical curve of v, $\chi(r) = \chi_v(r) \in C^0(\mathbb{R}^+)$ by setting

$$\chi(r) = \left(2v(r)\int_r^{+\infty}\frac{\mathrm{d}s}{v(s)}\right)^{-2} = \left[\left(-\frac{1}{2}\log\int_r^{+\infty}\frac{\mathrm{d}s}{v(s)}\right)'\right]^2 \qquad \text{on } \mathbb{R}^+. \tag{1.1}$$

A first integration immediately shows that

$$\sqrt{\chi(r)} \notin L^1(+\infty). \tag{1.2}$$

In case $v(r) = r^{m-1}$, $m \geq 3$, is the volume (up to constant) of the geodesic sphere of radius r in $\mathbb{R}^m$, we explicitly compute

$$\chi(r) = \frac{(m-2)^2}{4r^2} \tag{1.3}$$

It is also worth to introduce the functions $H_k(r)$, $k \in (-\infty, 1]$ defined by

$$\begin{cases} H_1(r) = -\sqrt{\displaystyle\int_r^{+\infty}\frac{\mathrm{d}s}{v(s)}}\log\int_r^{+\infty}\frac{\mathrm{d}s}{v(s)} & \text{for } k = 1; \\ H_k(r) = \left\{\displaystyle\int_r^{+\infty}\frac{\mathrm{d}s}{v(s)}\right\}^{\frac{1-\sqrt{1-k}}{2}} & \text{for } k < 1. \end{cases} \tag{1.4}$$

Note that H_1 is positive only for r sufficiently large. A computation shows that

$$\frac{1}{H_k^2 v} \in L^1(+\infty) \qquad \forall\, k \in (-\infty, 1], \tag{1.5}$$

and therefore, if $A \in L^\infty_{\mathrm{loc}}(\mathbb{R}_0^+)$ and $A(r) \leq \chi(r)$, the solution $h \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}_0^+)$ of the Cauchy problem

$$\begin{cases} (vh')' + Avh = 0 & \text{on } \mathbb{R}^+ \\ h(0) = 1, \qquad h'(0) = 0 \end{cases} \tag{1.6}$$

exists and it is unique, positive on $\mathbb{R}^+$ and satisfies

$$\frac{1}{h^2 v} \in L^1(+\infty). \tag{1.7}$$

Indeed, existence and uniqueness follow from Corollary 3.5 in [3]. Furthermore, by Theorem 5.2 of [3], $A \leq \chi$ ensures the positivity of h and the lower bound $h(r) \geq CH_1(r)$ for $r >> 1$ and some constant $C > 0$, so that

$$\frac{1}{h^2 v} \leq \frac{C}{H_1^2 v}.$$

The following results, Theorems 7 and 5 of [4], will be the main ingredients in the construction of the sub- and supersolutions.

Theorem 2 ([4], Theorem 7)**.** *Let $v \in C^0(\mathbb{R}_0^+)$ satisfying* (V)*, and consider $A \in L^\infty_{\rm loc}(\mathbb{R}_0^+)$ with the property that*

$$A(r) \le \chi(r) \qquad on\ \mathbb{R}^+.$$

Let $h \in \mathrm{Lip}_{\rm loc}(\mathbb{R}_0^+)$ be the positive solution of (1.6)*. Let $\sigma > 1$, $B \in L^\infty_{\rm loc}(\mathbb{R}_0^+)$ and suppose that*

$$\frac{B(r)h(r)^{\sigma-1}}{\sqrt{\chi_{h^2v}(r)}} \in L^1(+\infty). \tag{1.8}$$

Then, there exists a constant $\beta > 0$, depending on σ, A, B such that the following holds: for each $\gamma_\infty \in (0,\beta)$, there exists $0 < \gamma_0 \le \gamma_M$ and a positive solution γ of

$$\begin{cases} (v\gamma')' + Av\gamma = Bv\gamma^\sigma & \mathbb{R}^+ \\ \gamma(0) = \gamma_0, \qquad \gamma'(0) = 0 \end{cases} \tag{1.9}$$

such that

$$\gamma(r) \le \gamma_M h(r) \quad \text{on } \mathbb{R}^+, \qquad \frac{\gamma(r)}{h(r)} \to \gamma_\infty \text{ as } r \to \infty. \tag{1.10}$$

Moreover, $\gamma_M \to 0$ as $\gamma_\infty \to 0$.

Theorem 3 ([4], Theorem 5)**.** *Let $v \in C^0(\mathbb{R}_0^+)$ satisfying* (V)*, and consider $A \in L^\infty_{\rm loc}(\mathbb{R}_0^+)$ with the property that $A(r) \le \chi(r)$ on $\mathbb{R}^+$. Let $h \in \mathrm{Lip}_{\rm loc}(\mathbb{R}_0^+)$ be the positive solution of* (1.6)*. If, some $k \in (-\infty, 1]$, it holds*

$$\frac{A(r) - k\chi(r)}{\sqrt{\chi_{H_k^2v}(r)}} \in L^1(+\infty), \tag{1.11}$$

then $h(r) \sim CH_k(r)$ as $r \to +\infty$, for some constant $C > 0$.

Under the further assumption (1.11) on A, by Theorem 3 condition (1.8) can be rewritten as

$$\frac{B(r)H_k(r)^{\sigma-1}}{\sqrt{\chi_{H_k^2v}(r)}} \in L^1(+\infty) \tag{1.12}$$

(see also Remark 2.1 in [4]). Hereafter, we will consider the case $v(r) = r^{m-1}$, where $m = 2n+2$ is the homogeneous dimension of $\mathbb{H}^n$. Observe that $m \ge 4$, so that $v^{-1} \in L^1(+\infty)$ and the theorems are applicable. Via the definition of $H_k(r)$, we compute

$$\begin{cases} H_1(r) \sim Cr^{-\frac{m-2}{2}} \log r & \text{for } k = 1; \\ H_k(r) \sim Cr^{-\frac{m-2}{2}(1-\sqrt{1-k})} & \text{for } k < 1, \end{cases} \tag{1.13}$$

for some constant $C > 0$ and for $r \to +\infty$, thus

$$\begin{cases} \chi_{H_1^2v}(r) \sim \frac{1}{4r^2\log^2 r} & \text{for } k = 1; \\ \chi_{H_k^2v}(r) \sim \frac{(m-2)^2}{4r^2}(1-k) & \text{for } k < 1, \end{cases} \tag{1.14}$$

as $r \to +\infty$. It follows that conditions (1.11) and (1.12) can be respectively expressed as

$$\begin{cases} r\log r\left[A(r) - \dfrac{(m-2)^2}{4r^2}\right] \in L^1(+\infty) & \text{if } k = 1; \\ r\left[A(r) - k\dfrac{(m-2)^2}{4r^2}\right] \in L^1(+\infty) & \text{if } k < 1, \end{cases} \tag{1.15}$$

and
$$\begin{cases} B(r)(\log r)^{\sigma} r^{1-\frac{m-2}{2}(\sigma-1)} \in L^1(+\infty) & \text{if } k=1; \\ B(r) r^{1-\frac{m-2}{2}(1-\sqrt{1-k})(\sigma-1)} \in L^1(+\infty) & \text{if } k<1. \end{cases} \tag{1.16}$$

We are now ready for the

Proof of Theorem 1. First, note that $B(r)>0$ on $\mathbb{R}_0^+$. We let h_j, $j=1,2$, be the solutions of the linear problems

$$\begin{cases} (vh_j')' + A_j v h_j = 0 & \text{on } \mathbb{R}^+ \\ h_j(0)=1, \qquad h_j'(0)=0 \end{cases} \tag{1.17}$$

with $v(r)=r^{m-1}=r^{2n+1}$. Note that, because of (0.10) and the assumption on A_2 we have

$$A_1(r) \le A_2(r) \le \frac{n^2}{r^2} = \frac{(m-2)^2}{4r^2} \qquad \text{on } \mathbb{R}^+. \tag{1.18}$$

Indeed, if $\psi(x)\neq 0$ then from (0.10) obviously $A_1(d(x)) \le A_2(d(x))$. For $x=(z,t)$, $x\neq o$, $\psi(x)=0$ if and only if $z=0$ and by continuity $A_1(d(x)) \le A_2(d(x))$ in such points. The same holds at o. As we have already observed, because of (1.18) it follows that h_j are positive on $\mathbb{R}_0^+$. Furthermore, condition (0.11) coincides (1.15), thus by Theorem 3

$$h_j(r) \sim C_j H_k(r) \qquad \text{as } r\to+\infty \tag{1.19}$$

for some constants $C_j>0$.

To construct the supersolution to (0.14) we consider the problem

$$\begin{cases} (v\gamma')' + A_2 v\gamma = -Bv\gamma^{\sigma} & \text{on } \mathbb{R}^+ \\ \gamma(0)=\gamma_{2,0}>0, \qquad \gamma'(0)=0. \end{cases} \tag{1.20}$$

To apply Theorem 2 we observe that assumption (0.13) of Theorem 1 is exactly (1.16) which guarantees the validity of (1.12). Hence, there exists a constant $\beta_2>0$ and a function $\gamma_2>0$ satisfying (1.20), for each choice of $\gamma_{2,\infty}\in(0,\beta_2)$ and satisfying $0<\gamma_{2,0}\le\gamma_{2,M}$;

$$\gamma_2(r) \le \gamma_{2,M} h_2(r), \qquad \frac{\gamma_2(r)}{h_2(r)} \to \gamma_{2,\infty} \quad \text{as } r\to+\infty \tag{1.21}$$

with $\gamma_{2,M}\to 0^+$ as $\gamma_{2,\infty}\to 0^+$. We set $u_+(x)=\gamma_2(d(x))$. Then, using (0.8), the positivity of γ_2, the upper bound in (0.10) and (0.12) we have

$$\begin{aligned} \Delta_{\mathbb{H}^n} u_+ + q(x)u_+ &= \psi(x)\left\{\gamma_2''(d(x)) + \frac{2n+1}{d(x)}\gamma_2'(d(x))\right\} + q(x)\gamma_2 \\ &= \frac{\psi(x)}{v(d)}(v(d)\gamma_2')' + q(x)\gamma_2 \\ &= \frac{\psi(x)}{v(d)}\{-A_2(d)v(d)\gamma_2 - B(d)v(d)\gamma_2^{\sigma}\} + q(x)\gamma_2 \\ &= \Big[-\psi(x)A_2(d)+q(x)\Big]\gamma_2 - \psi(x)B(d)\gamma_2^{\sigma} \\ &\le -\psi(x)B(d)\gamma_2^{\sigma} \le b(x)u_+^{\sigma}, \end{aligned}$$

that is, $u_+(x)$ is a supersolution of (0.14). Furthermore, from (1.21) and (1.19) we infer that

$$u_+(x) \le \gamma_{2,M} h_2(d(x)) \quad \text{on } \mathbb{H}^n, \qquad u_+(x) \sim \gamma_{2,\infty} C_2 H_k(d(x)) \quad \text{as } d(x) \to +\infty \tag{1.22}$$

for some constant $C_2 > 0$.

Similarly we construct the subsolution $u_-(x)$ by considering the problem

$$\begin{cases} (v\gamma')' + A_1 v\gamma = Bv\gamma^\sigma & \text{on } \mathbb{R}^+ \\ \gamma(0) = \gamma_{1,0} > 0, \qquad \gamma'(0) = 0. \end{cases} \tag{1.23}$$

Again by assumption (0.13) of Theorem 1 we can apply Theorem 2 to find a constant $\beta_1 > 0$ and a positive solution γ_1 of (1.23) for each choice of $\gamma_{1,\infty} \in (0, \beta_1)$ and for appropriate $0 < \gamma_{1,0} \le \gamma_{1,M}$, with

$$\gamma_1(r) \le \gamma_{1,M} h_1(r) \quad \text{on } \mathbb{R}^+, \qquad \frac{\gamma_1(r)}{h_1(r)} \to \gamma_{1,\infty} \quad \text{as } r \to +\infty \tag{1.24}$$

and $\gamma_{1,M} \to 0^+$ as $\gamma_{1,\infty} \to 0^+$. We set $u_-(x) = \gamma_1(d(x))$ and using the assumptions of Theorem 1 in a way similar to what we did above we obtain

$$\Delta_{\mathbb{H}^n} u_- + q(x)u_-(x) \ge b(x)u_-^\sigma \qquad \text{on } \mathbb{H}^n,$$

that is $u_-(x)$ is a subsolution of (0.14). From (1.24) we have

$$u_-(x) \le \gamma_{1,M} h_1(d(x)) \quad \text{on } \mathbb{H}^n, \qquad u_-(x) \sim \gamma_{1,\infty} C_1 H_k(d(x)) \qquad \text{as } d(x) \to +\infty \tag{1.25}$$

for some constant $C_1 > 0$. Now, for each fixed $\gamma_{2,\infty} \in (0, \beta_2)$, using the second of (1.22) and (1.25) we can choose $\gamma_{1,\infty} \in (0, \beta_1)$ sufficiently small that $u_-(x) \le u_+(x)$ on $\mathbb{H}^n$. Applying the version of the monotone iteration scheme in the Appendix of [7], we deduce the existence of a positive solution u of (0.14) (smooth by hypoellipticity) satisfying the property

$$u_-(x) \le u(x) \le u_+(x) \le \gamma_{2,M} h_2(d(x)). \tag{1.26}$$

on $\mathbb{H}^n$. In particular, u is positive and using (1.22) and (1.25) we deduce the existence of positive constants $\Gamma_1 \le \Gamma_2$ such that

$$\Gamma_1 H_k(d(x)) \le u(x) \le \Gamma_2 H_k(d(x))$$

for $d(x)$ sufficiently large. Observe that, if $k \ge 0$, we can choose A_2 satisfying $A_2 \ge 0$, hence a first integration of (1.17) shows that h_2 is non-increasing, so that $h_2 \le 1$ on $\mathbb{R}^+$. From the upper bound in (1.26) and since $\gamma_{2,M} \to 0^+$ as $\gamma_{2,\infty} \to 0^+$, up to choosing $\gamma_{2,\infty}$ sufficiently small we can also satisfy the requirements on $\|u\|_{L^\infty(\mathbb{H}^n)}$ (for $k \in [0,1]$) and $\|u\|_{L^\infty(K)}$ (for $k < 0$), respectively. This completes the proof of Theorem 1. □

2 Another proof of Theorem 1 and a general result

The aim of this section is to give a second proof of Theorem 1 based on a general function-theoretic approach. Although the argument will reveal more involved than that given in Section 1, following this second proof the alert reader will realize the validity of Theorem 5 on a general Riemannian manifold. First of all we recall that if $G(z,t)$ is the fundamental solution of $\Delta_{\mathbb{H}^n}$ with singularity at the origin, then

$$G(x) = G(z,t) = \frac{C_{2n-2}}{d(x)^{2n}}, \tag{2.1}$$

for some constant C_{2n-2} only depending on n. This remarkable fact has been proved by Folland, [13]. Here, without loss of generality, we assume $C_{2n-2} = 1$. The positive function G thus verifies:

$$\Delta_{\mathbb{H}^n} G = 0 \qquad \text{on } \mathbb{H}^n \backslash \{o\}; \tag{2.2}$$

$$G(x) \to +\infty \qquad \text{as } x \to o; \tag{2.3}$$

$$\inf_{\mathbb{H}^n \backslash \{o\}} G = 0. \tag{2.4}$$

Note that a straightforward computation using (0.7) shows that

$$\frac{|\nabla_{\mathbb{H}^n} \log G|^2}{4} = \psi(x) \frac{n^2}{d(x)^2} \qquad \text{on } \mathbb{H}^n \backslash \{o\}. \tag{2.5}$$

We define

$$t(x) = -\frac{1}{2} \log G(x) \tag{2.6}$$

and, for $\eta \in C^2(\mathbb{R})$ we set

$$u(x) = e^{-t(x)} \eta(t(x)). \tag{2.7}$$

Note that, by (2.3) and (2.4), $t : \mathbb{R}^+ \to \mathbb{R}$ is surjective. A computation gives

$$\Delta_{\mathbb{H}^n} u + \left(1 - \frac{\ddot{\eta}}{\eta}(t(x))\right) \frac{|\nabla_{\mathbb{H}^n} \log G|^2}{4} u = \frac{1}{2} \frac{\Delta_{\mathbb{H}^n} G}{\sqrt{G}} \left[\eta(t(x)) - \dot{\eta}(t(x))\right] \tag{2.8}$$

therefore, using (2.2),

$$\Delta_{\mathbb{H}^n} u + \left(1 - \frac{\ddot{\eta}}{\eta}(t(x))\right) \frac{|\nabla_{\mathbb{H}^n} \log G|^2}{4} u = 0 \qquad \text{on } \mathbb{H}^n \backslash \{o\}. \tag{2.9}$$

We claim that assumption (0.10) of Theorem 1 is equivalent to

$$\hat{A}_1(t(x)) \frac{|\nabla_{\mathbb{H}^n} \log G|^2}{4} \leq q(x) \leq \hat{A}_2(t(x)) \frac{|\nabla_{\mathbb{H}^n} \log G|^2}{4} \tag{2.10}$$

with

$$\hat{A}_j(t) = \frac{1}{n^2} e^{\frac{2t}{n}} A_j\left(e^{\frac{t}{n}}\right) \qquad \text{for } j = 1, 2. \tag{2.11}$$

This immediately follows from the expression of $G(x)$ in (2.1) (with $C_{2n-2} = 1$) and from (2.5), (2.6). Similarly, (0.12) is equivalent to

$$|b(x)| \leq \hat{B}(t(x)) \frac{|\nabla_{\mathbb{H}^n} \log G|^2}{4} \tag{2.12}$$

with

$$\hat{B}(t) = \frac{1}{n^2} e^{\frac{2t}{n}} B\left(e^{\frac{t}{n}}\right). \tag{2.13}$$

From (2.9) and $\eta > 0$ we deduce

$$\begin{aligned}\Delta_{\mathbb{H}^n} u + q(x)u - b(x)u^\sigma &\geq \frac{u}{\eta(t(x))}\left\{\left[\hat{A}_1(t(x)) - 1\right]\eta + \ddot{\eta} - \hat{B}(t(x))e^{(1-\sigma)t(x)}\eta^\sigma\right\}\frac{|\nabla_{\mathbb{H}^n}\log G|^2}{4} \\ \Delta_{\mathbb{H}^n} u + q(x)u - b(x)u^\sigma &\leq \frac{u}{\eta(t(x))}\left\{\left[\hat{A}_2(t(x)) - 1\right]\eta + \ddot{\eta} + \hat{B}(t(x))e^{(1-\sigma)t(x)}\eta^\sigma\right\}\frac{|\nabla_{\mathbb{H}^n}\log G|^2}{4},\end{aligned} \tag{2.14}$$

thus to determine positive sub- and supersolutions $u_- \leq u_+$ of equation (0.14) on $\mathbb{H}^n$ of the form given in (2.7), it will be enough to determine positive solutions $\eta_- \leq \eta_+$ on the whole $\mathbb{R}$ of the differential inequalities

$$\ddot{\eta}_- + \left[\hat{A}_1(t) - 1\right]\eta_- - \hat{B}(t)e^{(1-\sigma)t}\eta_-^\sigma \geq 0 \tag{2.15}$$

$$\ddot{\eta}_+ + \left[\hat{A}_2(t) - 1\right]\eta_+ + \hat{B}(t)e^{(1-\sigma)t}\eta_+^\sigma \leq 0, \tag{2.16}$$

in such a way that u_-, u_+ can be extended at least in a C^1 way in o. To fix ideas, we concentrate on (2.16), and set for convenience $\eta = \eta_+$. We want to apply Theorem 2, and towards this aim we suppose that η is a positive solution of (2.16) on $\mathbb{R}$. For some fixed $\mu \geq 3$, we define a fake distance function ρ via

$$t = t(\rho) = \log\left(\sqrt{\mu - 2}\rho^{\frac{\mu-2}{2}}\right), \qquad \rho \in \mathbb{R}^+. \tag{2.17}$$

Note that, since t runs on the whole $\mathbb{R}$, ρ runs on the whole $\mathbb{R}^+$. Moreover,

$$t(0^+) = -\infty, \qquad t(+\infty) = +\infty, \qquad t'(\rho) = \frac{\mu - 2}{2\rho} > 0 \quad \text{on } \mathbb{R}^+. \tag{2.18}$$

We then define

$$z(\rho) = e^{-t(\rho)}\eta(t(\rho)). \tag{2.19}$$

Note the analogy between the definition of z and that of u in (2.7). We can thought of z to be a somewhat "radialized" version of u, as we shall explain below. From

$$\begin{aligned} z'(\rho) &= \tfrac{\sqrt{\mu-2}}{2}\rho^{-\frac{\mu}{2}}\left\{\dot{\eta}(t(\rho)) - \eta(t(\rho))\right\}; \\ z''(\rho) &= \tfrac{\sqrt{\mu-2}}{2}\rho^{-\frac{\mu}{2}-1}\left\{\tfrac{\mu-2}{2}\ddot{\eta}(t(\rho)) - (\mu-1)\dot{\eta}(t(\rho)) + \tfrac{\mu}{2}\eta(t(\rho))\right\}, \end{aligned} \tag{2.20}$$

since η is a solution of (2.16) we easily compute

$$(\rho^{\mu-1}z')' + \rho^{\mu-1}\bar{A}_2 z + \rho^{\mu-1}\bar{B}z^\sigma \leq 0 \qquad \text{on } \mathbb{R}^+, \tag{2.21}$$

where

$$\bar{A}_2(\rho) = \hat{A}_2(t(\rho))\chi(\rho), \qquad \bar{B}(\rho) = \hat{B}(t(\rho))\chi(\rho) \qquad \text{on } \mathbb{R}^+ \tag{2.22}$$

with

$$\chi(\rho) = \frac{(\mu-2)^2}{4\rho^2}$$

being the critical curve associated to $v(\rho) = \rho^{\mu-1}$. Similarly, if $\eta = \eta_-$ solves (2.15), then $z(\rho)$ defined in (2.19) satisfies

$$(\rho^{\mu-1}z')' + \rho^{\mu-1}\bar{A}_1 z - \rho^{\mu-1}\bar{B}z^\sigma \geq 0 \qquad \text{on } \mathbb{R}^+, \tag{2.23}$$

where
$$\bar{A}_1(\rho) = \hat{A}_1(t(\rho))\chi(\rho) \qquad \text{on } \mathbb{R}^+. \tag{2.24}$$
Next, we assume that $\bar{A}_j$, $j = 1, 2$ and $\bar{B}$ can be extended continuously to 0 so that $\bar{A}_j, B \in C^0(\mathbb{R}_0^+)$. This is equivalent to require the existence of

$$\lim_{\rho \to 0^+} \frac{\hat{A}_j(t(\rho))}{\rho^2} = \alpha_j \in \mathbb{R}, \quad j = 1, 2; \tag{2.25}$$

$$\lim_{\rho \to 0^+} \frac{\hat{B}(t(\rho))}{\rho^2} = \theta \in \mathbb{R}. \tag{2.26}$$

Because of (2.17),
$$\rho = (\mu - 2)^{-\frac{1}{\mu-2}} e^{\frac{2t}{\mu-2}}$$
and conditions (2.25), (2.26) are respectively equivalent to

$$\lim_{t \to -\infty} \hat{A}_j(t) e^{-\frac{4t}{\mu-2}} = \hat{\alpha}_j \in \mathbb{R}, \quad j = 1, 2; \tag{2.27}$$

$$\lim_{t \to -\infty} \hat{B}(t) e^{-\frac{4t}{\mu-2}} = \hat{\theta} \in \mathbb{R}. \tag{2.28}$$

Observe that, because of (2.11) and (2.13), since $A_j, B \in C^0(\mathbb{R}_0^+)$, (2.27) and (2.28) are satisfied for any choice of

$$\mu \geq 2(n+1) = m. \tag{2.29}$$

It is now worth to pause for a moment in order to discuss the core of the procedure that we are following. From the relation (2.7) between u and η, we can produce sub- and supersolution of (0.14) via solutions $\eta_\pm$ of the differential inequalities (2.15) and (2.16). These $\eta_\pm$ only depend on t, that is, on the level sets of G. The study of (2.15) and (2.16) reveals to be quite intricate, so we fix a simple non-parabolic model manifold (in our case $\mathbb{R}^\mu$), with distance function ρ and volume $v(\rho) = \rho^{\mu-1}$ of geodesic spheres, and we go "backward" from t to $\rho(t)$ and from η to a new function z. In this way, the function $\rho \circ t$ behaves like a "fake" distance function on our space, and we obtain the ODEs corresponding to (2.21) and (2.23), which are suited to apply Theorem 2. Due to the presence of the puncture at $x = o$, we shall be careful that the initial condition of z, once transferred to u, gives rise to C^1-functions, in order that u_+, u_- be actually (weak) sub- and supersolutions of (0.14) on the whole $\mathbb{H}^n$. This will be done in the last part of the proof.

With the aid of the technique described above, we have succeeded in "radializing" the CR Yamabe problem even without referring at any step to the Koranyi distance. Indeed, radialization is performed along the level sets of the Green function fixed at some point. On $\mathbb{H}^n$, by (2.1) level sets of G are a reparametrization of those of d, so we simply find another approach to prove Theorem 1. However, this second approach enables us to deal with much more general settings including every non-parabolic Riemannian manifold, regardless either of its geodesic completeness or of the fact that it has a pole (which is essential for the arguments in [4] to work). We thus consider the existence of solutions of the following Cauchy problems:

$$\begin{cases} (\rho^{\mu-1}\gamma_2')' + \rho^{\mu-1}\bar{A}_2\gamma_2 = -\rho^{\mu-1}\bar{B}\gamma_2^\sigma & \text{on } \mathbb{R}^+ \\ \gamma_2(0) = \gamma_{2,0} > 0, \qquad \gamma_2'(0) = 0. \end{cases} \tag{2.30}$$

and

$$\begin{cases} (\rho^{\mu-1}\gamma_1')' + \rho^{\mu-1}\bar{A}_1\gamma_1 = \rho^{\mu-1}\bar{B}\gamma_1^\sigma & \text{on } \mathbb{R}^+ \\ \gamma_1(0) = \gamma_{1,0} > 0, \qquad \gamma_1'(0) = 0 \end{cases} \tag{2.31}$$

under the assumptions described in Theorem 1. To do so, we shall check the validity of the requirements for the existence in Theorem 2. Let h_j, $j = 1, 2$ be the solutions of the linear problems

$$\begin{cases} (\rho^{\mu-1}h_j')' + \rho^{\mu-1}\bar{A}_j h_j = 0 & \text{on } \mathbb{R}^+, \\ h_j(0) = 1, \qquad h_j'(0) = 0. \end{cases} \tag{2.32}$$

Note that, from (2.11) and (2.24) and the assumption $A_2(r) \le n^2/r^2$ on $\mathbb{R}^+$ of Theorem 1, we deduce that

$$\bar{A}_1(\rho) \le \bar{A}_2(\rho) \le \chi(\rho) \qquad \text{on } \mathbb{R}^+. \tag{2.33}$$

Indeed, it is a simple matter to check that

$$\hat{A}_2(t) \le 1 \qquad \text{on } \mathbb{R}. \tag{2.34}$$

Hence, by Theorem 5.2 of [3] the solutions $h_j(\rho)$ of (2.32) are positive on $\mathbb{R}_0^+$. We determine their asymptotic behaviour as $\rho \to +\infty$ via Theorem 3. Towards this aim, for $k \in (-\infty, 1]$, we need to compute the asymptotics for $H_k(\rho)$ and $\chi_{H_k^2 v}(\rho)$, with $v(\rho) = \rho^{\mu-1}$: these are given by formulas (1.13) and (1.14), simply replacing m with μ and r with ρ:

$$\begin{cases} H_1(\rho) \sim C\rho^{-\frac{\mu-2}{2}} \log \rho & \text{for } k = 1; \\ H_k(\rho) \sim C\rho^{-\frac{\mu-2}{2}(1-\sqrt{1-k})} & \text{for } k < 1, \end{cases} \tag{2.35}$$

for some constant $C > 0$ and for $\rho \to +\infty$, and

$$\begin{cases} \chi_{H_1^2 v}(\rho) \sim \frac{1}{4\rho^2 \log^2 \rho} & \text{for } k = 1; \\ \chi_{H_k^2 v}(\rho) \sim \frac{(m-2)^2}{4\rho^2}(1-k) & \text{for } k < 1, \end{cases} \tag{2.36}$$

as $\rho \to +\infty$. Assumption (1.11) of Theorem 3 then becomes

$$\begin{cases} \left[\bar{A}_j(\rho) - \frac{(\mu-2)^2}{4\rho^2}\right] \rho \log \rho \in L^1(+\infty) & \text{if } k = 1; \\ \left[\bar{A}_j(\rho) - k\frac{(\mu-2)^2}{4\rho^2}\right] \rho \in L^1(+\infty) & \text{if } k < 1. \end{cases} \tag{2.37}$$

Using (2.24) we can easily express conditions (2.37) in terms of $\hat{A}_j(t)$ in the form

$$\begin{cases} t[\hat{A}_j(t) - 1] \in L^1(+\infty) & \text{if } k = 1; \\ \hat{A}_j(t) - k \in L^1(+\infty) & \text{if } k < 1. \end{cases} \tag{2.38}$$

Now, from (2.11) one can then write (2.38) in terms of A_j and verify that they are exactly conditions (0.11) of Theorem 1. Hence, from Theorem 3 we have that the two solutions h_j of (2.32) satisfy

$$h_j(\rho) \sim C_j H_k(\rho) \qquad \text{as } \rho \to +\infty, \tag{2.39}$$

for some positive constants $C_j > 0$ and where $H_k(\rho)$ has the asymptotic behaviour in (2.35).

We are left to check the validity of assumption (1.8) (with $m = \mu$, $B = \pm\bar{B}$ and $r = \rho$). Due to (2.39) and (2.35), (1.8) becomes

$$\begin{cases} \bar{B}(\rho)(\log\rho)^{\sigma}\rho^{1-\frac{\mu-2}{2}(\sigma-1)} \in L^1(+\infty) & \text{if } k=1; \\ \bar{B}(\rho)\rho^{1-\frac{\mu-2}{2}(1-\sqrt{1-k})(\sigma-1)} \in L^1(+\infty) & \text{if } k<1, \end{cases} \tag{2.40}$$

(as already computed in (1.16)), which via (2.24) can be expressed as

$$\begin{cases} \hat{B}(t)t^{\sigma-1}e^{t(1-\sigma)} \in L^1(+\infty) & \text{if } k=1; \\ \hat{B}(t)e^{t(1-\sqrt{1-k})(1-\sigma)} \in L^1(+\infty) & \text{if } k<1, \end{cases} \tag{2.41}$$

and using (2.13) one can check that these are exactly condition (0.13) of Theorem 1. Concluding, for each $j = 1, 2$, by Theorem 2 there exist $\beta_j > 0$ such that, for each $\gamma_{j,\infty} \in (0, \beta_j)$, there exist $0 < \gamma_{j,0} \leq \gamma_{j,M}$ and a positive solution γ_j (of (2.31) and (2.30), respectively) such that

$$\gamma_j(\rho) \leq \gamma_{j,M} h_j(\rho) \quad \text{on } \mathbb{R}^+, \qquad \frac{\gamma_j(\rho)}{h_j(\rho)} \to \gamma_{j,\infty} \quad \text{as } \rho \to +\infty.$$

Moreover, $\gamma_{j,M} \to 0^+$ as $\gamma_{j,\infty} \to 0+$. In particular, $\gamma_j(\rho) \sim \gamma_{j,\infty} C_j H_k(\rho)$ as $\rho \to +\infty$, for some constant $C_j > 0$. As in the final part of the first proof of Theorem 1, once fixed $\gamma_{2,\infty}$, we can choose $\gamma_{1,\infty}$ small enough that $\gamma_1(\rho) \leq \gamma_2(\rho)$ on $\mathbb{R}^+$. Setting $z_- = \gamma_1$, $z_+ = \gamma_2$, it holds $z_- \leq z_+$ and they respectively solve (2.23) and (2.21). Then,

$$\eta_-(t) = e^t z_-(\rho(t)), \qquad \eta_+(t) = e^t z_+(\rho(t))$$

solve, respectively, (2.15) and (2.16) with $\eta_- \leq \eta_+$ on $\mathbb{R}$. Finally,

$$u_-(x) = e^{-t(x)}\eta_-(t(x)), \qquad u_+(x) = e^{-t(x)}\eta_+(t(x))$$

are respectively a sub- and a supersolution of (0.14) on $\mathbb{H}^n \backslash \{o\}$ with $u_- \leq u_+$. It remains to show that u_-, u_+ extend to weak sub- and supersolutions to the whole $\mathbb{H}^n$. To prove this fact, it is enough to show that they extend in a C^1-way at the puncture $x = o$. Indeed, in this case the proof that u_-, u_+ are weak solutions on $\mathbb{H}^n$ follows from a simple integration by parts argument. To fix ideas let us consider the case of u_-. From (2.31) we have

$$\begin{cases} z_-(\rho) = \gamma_{1,0} + \frac{\alpha}{2}\rho^2 + o(\rho^2) \\ z_-'(\rho) = \alpha\rho + o(\rho) \end{cases} \tag{2.42}$$

as $\rho \to +\infty$ and with $\alpha = \mu^{-1}[\bar{B}(0)\gamma_{1,0}^{\sigma-1} - \bar{A}_1(0)]\gamma_{1,0}$. Consequently,

$$\beta_-(t) = \gamma_{1,0}e^t + \frac{\tilde{\alpha}}{2}e^{\frac{\mu+2}{\mu-2}t} + o\left(e^{\frac{\mu+2}{\mu-2}t}\right) \qquad \text{as } t \to -\infty,$$

where

$$\tilde{\alpha} = \alpha(\mu-2)^{-\frac{2}{\mu-2}}.$$

Thus, using the definition of u_- and that of $t(x)$ in (2.6) we obtain

$$u_-(x) = \gamma_{1,0} + \frac{\tilde{\alpha}}{2}G(x)^{-\frac{2}{\mu-2}} + o\left(G(x)^{-\frac{2}{\mu-2}}\right) \qquad \text{as } x \to o.$$

It follows, using (2.3), that

$$u_-(x) \to \gamma_{1,0} \qquad \text{as } x \to o. \tag{2.43}$$

Next,

$$\nabla_{\mathbb{H}^n} u_-(x) = e^{-t(x)}\{\dot\eta_-(t(x)) - \eta_-(t(x))\}\nabla_{\mathbb{H}^n} t(x) = \frac{\eta_-(t(x)) - \dot\eta_-(t(x))}{2}\frac{\nabla_{\mathbb{H}^n} G(x)}{\sqrt{G(x)}}. \tag{2.44}$$

Now,

$$\dot\eta_-(t) - \eta_-(t) = e^t z_-'(\rho(t))\dot\rho(t)$$

so that, using (2.42),

$$\dot\eta_-(t) - \eta_-(t) = \alpha(\mu-2)^{-2(\mu-2)-1} e^{\frac{\mu+2}{\mu-2}t} + o\left(e^{\frac{\mu+2}{\mu-2}t}\right) \qquad \text{as } t \to -\infty.$$

Substituting this latter into (2.44) and recalling (2.3) we obtain

$$|\nabla_{\mathbb{H}^n} u_-(x)| = \left|\alpha(\mu-2)^{-2(\mu-2)-1} G(x)^{-\frac{\mu}{\mu-2}} + o\left(G(x)^{-\frac{\mu}{\mu-2}}\right)\right| |\nabla_{\mathbb{H}^n} G(x)| \qquad \text{as } x \to o. \tag{2.45}$$

If we require that

$$G(x) \sim \frac{C}{d(x)^{\mu-2}}, \qquad |\nabla_{\mathbb{H}^n} G(x)| \sim \frac{\tilde C(\mu-2)\sqrt{\psi(x)}}{d(x)^{\mu-1}}, \tag{2.46}$$

for some positive constants $C, \tilde C$, then from (2.45) we would obtain

$$|\nabla_{\mathbb{H}^n} u_-(x)| \sim C d(x)\sqrt{\psi(x)} \to 0 \qquad \text{as } x \to o. \tag{2.47}$$

Note, by (2.1), the relations in (2.46) are satisfied for the choice $\mu = 2n+2 = m$, which is admissible by (2.29). Putting together (2.43) and (2.47) we obtain the desired requirements on u_-, and similarly on u_+. Concluding, the monotone iteration scheme yields the solution $u \in C^\infty(\mathbb{H}^n)$ of (0.14) with the required properties.

Summarizing, the above reasoning gives a second proof of Theorem 1. Despite the fact that this proof is longer and more involved than that given in Section 1, it is more general and it applies to different ambient spaces; notably, to ambiente spaces without special symmetries. Substituting (2.46) with

$$|\nabla_{\mathbb{H}^n} G(x)| G(x)^{-\frac{\mu}{\mu-2}} \to 0 \qquad \text{as } x \to o,$$

for some $\mu \ge 2(n+1) = m$, which still imply the validity of $|\nabla u_-(x)| \to 0^+$ as $x \to o$, the previous proof also implies the validity of the next

Theorem 4. *Let $(M, \langle\,,\,\rangle)$ be a Riemannian manifold of dimension $m \ge 3$, and let $q(x), b(x) \in C^{0,\alpha}_{\mathrm{loc}}(M)$, for some $\alpha > 0$. Let $o \in M$ be fixed, and suppose that there exists a positive function $G \in C^2(M\backslash\{o\}$ with the following properties:*

$$\begin{array}{ll} (i) & \Delta G = 0 \quad \text{on } M\backslash\{o\}, \\ (ii) & G(x) \to +\infty \quad \text{as } x \to o, \\ (iii) & \inf_{M\backslash\{o\}} G = 0, \\ (iv) & \text{for some } \mu \ge m,\ |\nabla G(x)| G(x)^{-\frac{\mu}{\mu-2}} \to 0^+ \quad \text{as } x \to o. \end{array} \tag{2.48}$$

Let $t(x) = -\frac{1}{2}\log G(x)$*. Suppose that, for some* $\hat{A}_j, \hat{B} \in C^0(\mathbb{R})$*,* $j = 1,2$ *we have*

$$\hat{A}_1(t(x))\frac{|\nabla \log G|^2}{4} \le q(x) \le \hat{A}_2(t(x))\frac{|\nabla \log G|^2}{4}$$
$$|b(x)| \le \hat{B}(t(x))\frac{|\nabla \log G|^2}{4}.$$

Assume the existence of

$$\lim_{t\to-\infty} \hat{A}_j(t)e^{-\frac{4t}{\mu-2}} = \hat{\alpha}_j \in \mathbb{R}, \quad j = 1,2; \tag{2.49}$$

$$\lim_{t\to-\infty} \hat{B}(t)e^{-\frac{4t}{\mu-2}} = \hat{\theta} \in \mathbb{R}, \tag{2.50}$$

and that $\hat{A}_2(t) \le 1$ *on* $\mathbb{R}$*. Furthermore, for some* $k \in (-\infty, 1]$*, suppose that*

$$\begin{cases} t[\hat{A}_j(t) - 1] \in L^1(+\infty) & \textit{if } k = 1; \\ \hat{A}_j(t) - k \in L^1(+\infty) & \textit{if } k < 1. \end{cases} \tag{2.51}$$

$$\begin{cases} \hat{B}(t)t^{\sigma-1}e^{t(1-\sigma)} \in L^1(+\infty) & \text{if } k = 1; \\ \hat{B}(t)e^{t(1-\sqrt{1-k})(1-\sigma)} \in L^1(+\infty) & \text{if } k < 1, \end{cases} \tag{2.52}$$

for some $\sigma > 1$*. Then, the equation*

$$\Delta u + q(x)u - b(x)u^\sigma = 0$$

has infinitely many positive solutions $u \in C^{2,\alpha}_{\mathrm{loc}}(M)$*.*

Of course, in the statement of Theorem 4, ∇ and Δ are the gradient and the Laplace-Beltrami operator with respect to the metric of M.

2.1 An alternative statement of Theorem 5

We conclude the paper with a simplified version of Theorem 4 which is, in some sense, an alternative formulation. Indeed, as it will become apparent, assumptions $(i), \ldots, (iv)$ in (2.48) force some rigidity on G. First, by (i), for sufficiently large $a > 0$ the function $G_a(x) = \min\{G(x), a\}$ is a bounded, non-constant weakly superharmonic function, thus M is necessarily non parabolic. On the other hand, if M is non-parabolic, an example of such a G is clearly given by $G(x) = \mathcal{G}(x, o)$, with $\mathcal{G}(x, y)$ the minimal positive Green function. Indeed, (iii) follows from minimality, for otherwise $\mathcal{G} - \inf_{M\setminus\{o\}} \mathcal{G}$ would be another Green kernel strictly smaller than $\mathcal{G}$, and (iv) with $\mu = m$ is a consequence of the estimates

$$\mathcal{G}(x, o) = \frac{r(x)^{2-m}}{(m-2)\omega_{m-1}} + \xi(x, o), \qquad |\nabla_x \mathcal{G}(x, o)| = \frac{r(x)^{1-m}}{\omega_{m-1}} + \eta(x, o) \qquad \text{for } m \ge 3, \tag{2.53}$$

where $r(x) = \mathrm{dist}(x, o)$, η, ξ are smooth on $M \times M$ and ω_{m-1} is the volume of the unit $(m-1)$-dimensional sphere (the reader can consult [14] for a proof of this fact). Next, we note that by standard comparison techniques the singularity of G is at most of the order of that of $\mathcal{G}(x, o)$. Indeed, suppose by contradiction that $\mathcal{G}(x, o) = o(G(x))$ as $x \to o$. Then, fix a smooth, relatively compact open set $\Omega \Subset M$ containing o, and let $\mathcal{G}_\Omega(x,o)$ be the Dirichlet Green function of Δ on Ω. Then, by standard theory, $\mathcal{G}_\Omega(x,o) \sim \mathcal{G}(x,o)$ as $x \to o$, thus $w = \mathcal{G}_\Omega(x,o)/G(x)$ would be a solution on $\Omega\backslash\{o\}$ of

$$\begin{cases} \operatorname{div}(G^2\nabla w) = 0 \qquad \text{on } \Omega\backslash\{o\} \\ w = 0 \qquad \text{on } \partial\big(\Omega\backslash\{o\}\big) \end{cases} \tag{2.54}$$

The maximum principle would then imply $w = 0$, a contradiction. In a similar way, the singularity of $G(x)$ cannot be milder than that of $\mathcal{G}(x,o)$, for otherwise G could extend harmonically across the puncture o (fix $\Omega \Subset M$, and prove that G coincides on Ω with the solution ξ of the Dirichlet problem with boundary data $G_{|\partial\Omega}$ by using the maximum principle on $w = (G-\xi)/\mathcal{G}$). Suppose now that $G(x)$ admits a Taylor expansion in a neighbourhood of o as a function of $r(x)$. Writing the first term in the form $Cr(x)^{2-\mu}$, for some $C > 0$ and $\mu > 2$, the above observations force $\mu = m$, which implies

$$\lim_{x\to o}\frac{G(x)}{\mathcal{G}(x,o)} = C > 0,$$

for some $C > 0$. The function $\xi(x) = G(x) - C\mathcal{G}(x,o)$ is thus a harmonic function on Ω, with a singularity milder than that of $\mathcal{G}(x,o)$ as $x \to o$, hence extends harmonically across o to a harmonic function $\xi \in C^\infty(M)$. Therefore,

$$G(x) = C\mathcal{G}(x,o) + \xi(x) \qquad \text{on } M. \tag{2.55}$$

Concluding, the structure of G satisfying (2.48) and admitting a Taylor expansion at $x = o$ is rigidly described by (2.55). For this reason, alternatively to (2.48) one can refer to the minimal positive Green function on a non-parabolic manifold, which yields to the next

Theorem 5. *Let $(M, \langle\,,\,\rangle)$ be a non-parabolic Riemannian manifold of dimension $m \geq 3$, with minimal positive Green kernel $\mathcal{G}(x,y)$. Let $q(x), b(x) \in C^{0,\alpha}_{\mathrm{loc}}(M)$, for some $\alpha > 0$. Fix $o \in M$ and consider $G(x) = \mathcal{G}(x,o)$. Let $t(x) = -\frac{1}{2}\log G(x)$. Suppose that, for some $\hat{A}_j, B \in C^0(\mathbb{R})$, $j = 1,2$ we have*

$$\begin{aligned} &\hat{A}_1(t(x))\tfrac{|\nabla \log G|^2}{4} \leq q(x) \leq \hat{A}_2(t(x))\tfrac{|\nabla \log G|^2}{4} \\ &|b(x)| \leq \hat{B}(t(x))\tfrac{|\nabla \log G|^2}{4}. \end{aligned} \tag{2.56}$$

Assume the existence of

$$\lim_{t\to-\infty}\hat{A}_j(t)e^{-\frac{4t}{m-2}} = \hat{\alpha}_j \in \mathbb{R}, \quad j = 1,2; \tag{2.57}$$

$$\lim_{t\to-\infty}\hat{B}(t)e^{-\frac{4t}{m-2}} = \hat{\theta} \in \mathbb{R}, \tag{2.58}$$

and that $\hat{A}_2(t) \leq 1$ on $\mathbb{R}$. Furthermore, for some $k \in (-\infty, 1]$, suppose that

$$\begin{cases} t[\hat{A}_j(t) - 1] \in L^1(+\infty) \qquad \text{if } k = 1; \\ \hat{A}_j(t) - k \in L^1(+\infty) \qquad \text{if } k < 1. \end{cases} \tag{2.59}$$

$$\begin{cases} \hat{B}(t)t^{\sigma-1}e^{t(1-\sigma)} \in L^1(+\infty) \qquad \text{if } k = 1; \\ \hat{B}(t)e^{t(1-\sqrt{1-k})(1-\sigma)} \in L^1(+\infty) \qquad \text{if } k < 1, \end{cases} \tag{2.60}$$

for some $\sigma > 1$. Then, the equation

$$\Delta u + q(x)u - b(x)u^\sigma = 0$$

has infinitely many positive solutions $u \in C^{2,\alpha}_{\mathrm{loc}}(M)$.

We conclude by observing that, if the topology of M is nontrivial, by Morse Theory the set $C(G)$ of stationary points of G is non-empty. Conditions (2.56) necessarily require $q, b \equiv 0$ on $C(G)$. On the other hand, one has the possibility to choose o in an arbitrary way in order for (2.56) to be met (together with the conditions on $\hat{A}_j, \hat{B}$). For this reason, knowing the structure of the set $C(G)$ is essential. As it has been shown in [15], its size is relatively small, as we have

$$\mathcal{H}^{m-2}(C(G) \cap K) < +\infty$$

for every compact set $K \Subset M \backslash \{o\}$ (a fortiori, by (2.53), for every $K \Subset M$), where $\mathcal{H}^{m-2}$ is the $(m-2)$-dimensional Hausdorff measure. In this respect, see also the very recent [8] for a sharp generalization and a quantitative improvement.

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