Yamabe type equations with a sign-changing nonlinearity, and the prescribed curvature problem

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YAMABE TYPE EQUATIONS WITH A SIGN-CHANGING NONLINEARITY, AND THE PRESCRIBED CURVATURE PROBLEM

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ABSTRACT. In this paper, we investigate the prescribed scalar curvature problem on a non-compact Riemannian manifold $(M, \langle \, , \, \rangle)$, namely the existence of a conformal deformation of the metric $\langle \, , \, \rangle$ realizing a given function $\widetilde{s}(x)$ as its scalar curvature. In particular, the work focuses on the case when $\widetilde{s}(x)$ changes sign. Our main achievement are two new existence results requiring minimal assumptions on the underlying manifold, and ensuring a control on the stretching factor of the conformal deformation in such a way that the conformally deformed metric be bi-Lipschitz equivalent to the original one. The topological-geometrical requirements we need are all encoded in the spectral properties of the standard and conformal Laplacians of M. Our techniques can be extended to investigate the existence of entire positive solutions of quasilinear equations of the type

$$\Delta_p u + a(x)u^{p-1} - b(x)u^{\sigma} = 0$$

where Δ_p is the p-Laplacian, $\sigma > p - 1 > 0$, $a, b \in L^{\infty}_{\mathrm{loc}}(M)$ and b changes sign, and in the process of collecting the material for the proof of our theorems, we have the opportunity to give some new insight on the subcriticality theory for the Schrödinger type operator

$$Q'_V \; : \; \varphi \longmapsto -\Delta_p \varphi - a(x)|\varphi|^{p-2}\varphi.$$

In particular, we prove sharp Hardy-type inequalities in some geometrically relevant cases, notably for minimal submanifolds of the hyperbolic space.

CONTENTS

1. INTRODUCTION, I: EXISTENCE FOR THE GENERALIZED YAMABE PROBLEM

Generalizations of the classical Yamabe problem on a Riemannian manifold have been the focus of an active area of research over the past 30 years. Among these, the prescribed scalar curvature problem over non-compact manifolds appears to be challenging: briefly, given a non-compact Riemannian manifold $(M^m, \langle\,,\,\rangle)$ with scalar curvature $s(x)$ and a smooth function $\widetilde{s} \in C^\infty(M)$, the problem asks under which conditions there exists a conformal deformation of $\langle\,,\,\rangle$,

$$\text{(1.1)} \qquad\qquad \widetilde{\langle\,,\,\rangle} = \varphi^2 \langle\,,\,\rangle, \qquad 0 < \varphi \in C^\infty(M),$$

realizing $\widetilde{s}(x)$ as its scalar curvature. When the dimension m of M is at least 3, writing $\varphi = u^{\frac{2}{m-2}}$, the problem becomes equivalent to determining a positive solution $u \in C^\infty(M)$ of the Yamabe equation

$$\text{(1.2)} \qquad \Delta u - \frac{s(x)}{c_m} u + \frac{\widetilde{s}(x)}{c_m} u^{\frac{m+2}{m-2}} = 0, \qquad c_m = \frac{4(m-1)}{m-2}.$$

Here, Δ is the Laplace-Beltrami operator of the background metric $\langle\,,\,\rangle$. For $m = 2$, setting $\varphi = e^u$ one substitutes (1.2) with

$$2\Delta u - s(x) + \widetilde{s}(x)e^{2u} = 0$$

where now $u \in C^\infty(M)$ may change sign (see [43]). Hereafter, we will confine ourselves to dimension $m \geq 3$, and M will always be assumed to be connected. Agreeing with the literature, we will call the linear operator in (1.2):

$$\text{(1.3)} \qquad\qquad L_{\langle\,,\,\rangle} \doteq -\Delta - \frac{s(x)}{c_m} = -\Delta + \frac{m-2}{4(m-1)} s(x)$$

the conformal Laplacian of $(M, \langle\,,\,\rangle)$.

The original Yamabe problem is a special case of the prescribed scalar curvature problem, namely that when $\widetilde{s}(x)$ is a constant, and for this reason, in the literature, the prescribed scalar curvature problem is often called the generalized Yamabe problem. Besides establishing existence of a positive solution u of (1.2), it is also useful to investigate its qualitative behaviour since this reflects into properties of $\widetilde{\langle\,,\,\rangle}$. For instance, $u \in L^{\frac{2m}{m-2}}(M)$ is equivalent to the fact that $\widetilde{\langle\,,\,\rangle}$ has finite volume. Also, if u is bounded between two positive constants, the identity map

$$\text{(1.4)} \qquad\qquad i \;\; : \;\; (M, \langle\,,\,\rangle) \longrightarrow (M, \widetilde{\langle\,,\,\rangle})$$

is globally bi-Lipschitz, and thus $\widetilde{\langle\,,\,\rangle}$ inherits some fundamental properties of $\langle\,,\,\rangle$. For instance, geodesic completeness, parabolicity, Gromov-hyperbolicity, etc. (see [36, 35]). Agreeing with the literature, when $C^{-1}\langle\,,\,\rangle \leq \widetilde{\langle\,,\,\rangle} \leq C\langle\,,\,\rangle$ for some constant $C > 0$ we will say that $\widetilde{\langle\,,\,\rangle}$ and $\langle\,,\,\rangle$ are uniformly equivalent.

Given the generality of the geometrical setting, it is reasonable to expect that existence or non-existence of the desired conformal deformation heavily depends on the topological and metric properties of M and their relations with $\widetilde{s}(x)$. As we shall explain in awhile, a particularly intriguing (and difficult) case is when $\widetilde{s}(x)$ is allowed to change sign. In this situation, with the exception of a few special cases, a satisfactory answer to the prescribed curvature problem is still missing. To properly put our results into perspective, first we describe some of the main technical problems that arise when looking for solutions of (1.2) for sign-changing $\widetilde{s}(x)$. Then, we briefly comment on some classical and more recent approaches. In particular, we pause to describe in detail four

results that allow us to grasp the situation in the relevant examples of Euclidean and hyperbolic spaces and to underline the key features of our new achievements. We stress that, when $\widetilde{s}(x) \leq 0$, there is a vast literature and the interaction between topology and geometry is better understood. Among the various references on the existence problem, we refer the reader to [8, 72, 71, 16, 48].

If $\widetilde{s}(x)$ is positive somewhere, basic tools to produce solutions are in general missing. More precisely, uniform L^∞-estimates fail to hold on regions where $\widetilde{s}(x)$ is non-negative, and comparison theorems are not valid where $\widetilde{s}(x)$ is positive. This suggests why, in the literature, equation (1.2) in a non-compact ambient space has mainly been studied via variational and concentration-compactness techniques ([37, 80]) or radialization techniques ([57, 56, 42, 8]). We also quote the interesting method developed in [72, 71, 9].

To the best of our knowledge, up to now there have been few attempts to adapt the variational approach to (non-compact, of course) spaces other than $\mathbb{R}^m$, [37, 80]. In this respect, a particularly interesting result is the next one due to Q.S. Zhang [80].

Theorem 1.1 ([80], Thm. 1.1). *Let $(M^m, \langle\,,\,\rangle)$ be a complete manifold with dimension $m \geq 3$ and scalar curvature $s(x) \geq 0$. Suppose that $\mathrm{vol}(B_r(x)) \leq Cr^m$ for some uniform C independent of x, and that M has positive Yamabe invariant $Y(M)$:*

$$(1.5) \qquad Y(M) = \inf\left\{\int_M \left[|\nabla\phi|^2 + \frac{s(x)}{c_m}\phi^2\right] \ : \ \phi \in \mathrm{Lip}_c(M), \ \int_M \phi^{\frac{2m}{m-2}} = 1\right\},$$

c_m as in (1.2). Assume further that

- *$\widetilde{s}(x) \geq 0, \not\equiv 0$ and $\widetilde{s}(x) \to 0$ as $r(x) \to +\infty$,*
- *$\widetilde{s}(x)$ is sufficiently flat around at least one of its maximum points.*

Then, there exists a solution $u \in L^{\frac{2m}{m-2}}(M)$ of (1.2) such that

$$(1.6) \qquad\qquad\qquad\qquad u \leq C\big(1 + r(x)\big)^{-\frac{m-2}{2}},$$

for some $C > 0$. In particular, $\widetilde{\langle\,,\,\rangle} = u^{\frac{4}{m-2}}\langle\,,\,\rangle$ has finite volume and is geodesically incomplete.

Remark 1.1. The flatness condition above is the one usually required for the compact Yamabe problem, see [28, 29].

The above theorem is not, indeed, the most general statement of Zhang's result, but however the version here is a good compromise between generality and simplicity, and it is enough for the sake of comparison with our main theorems. On the positive side, topological conditions on M are not so demanding. However, we underline that the polynomial volume growth assumption is essential for Zhang's method to work, hence this excludes the case of negatively curved manifolds like the hyperbolic space $\mathbb{H}_\kappa^m$ of sectional curvature $-\kappa^2$. On the contrary, as the recent [13] highlights, the radialization methods developed by W.M. Ni, M. Naito and N. Kawano in [57, 56, 42] on $\mathbb{R}^m$, and by P. Aviles and R. McOwen in [8] for $\mathbb{H}^m$ are very flexible with respect to curvature control on M, but on the other hand they require M to possess a pole (that is, a point o for which the exponential map $\exp_o$ is a diffeomorphism), a quite restrictive topological assumption. We quote the two results, starting from Ni-Naito-Kawano's theorem.

Theorem 1.2 ([57, 56, 42])**.** *Let* $\widetilde{s}(x) \in C^{\infty}(\mathbb{R}^m)$, $m \geq 3$, *and suppose that there exists* $B \in C^0(\mathbb{R})$ *such that*

$$(1.7) \qquad |\widetilde{s}(x)| \leq B\big(r(x)\big) \qquad and \qquad rB(r) \in L^1(+\infty).$$

Then, there exists a small $\gamma_0 > 0$ *such that, for each* $\gamma \in (0, \gamma_0)$, *there exists a conformal deformation* $\widetilde{\langle\,,\,\rangle}$ *of the flat metric* $\langle\,,\,\rangle$ *such that*

$$(1.8) \qquad \widetilde{\langle\,,\,\rangle}_x \to \gamma\langle\,,\,\rangle_x \qquad as \ r(x) \to +\infty.$$

Theorem 1.3 ([8], Thm 4)**.** *Let* $(M^m, \langle\,,\,\rangle)$ *be a complete manifold with a pole and dimension* $m \geq 3$, *and suppose that there exist constants* $\bar{\kappa} \geq \kappa > 0$ *such that sectional curvature* K *of* M *be pinched as follows:*

$$(1.9) \qquad -\bar{\kappa}^2 \leq K \leq -\kappa^2 < 0, \qquad with \quad \bar{\kappa}^2 < \frac{(m-1)^2}{m(m-2)}\kappa^2.$$

Suppose also that $\widetilde{s}(x) \in C^{\infty}(M)$ *satisfies*

$$(1.10) \qquad -C_1 \leq \widetilde{s}(x) \leq -C_2 < 0 \qquad outside \ of \ a \ compact \ set,$$

for some constants $C_1, C_2 > 0$. *Then, there exists* $\delta > 0$ *sufficiently small such that, if*

$$(1.11) \qquad \widetilde{s}(x) \leq \delta \qquad on \ M,$$

there exists a conformal deformation $\widetilde{\langle\,,\,\rangle}$ *realizing* $\widetilde{s}(x)$ *and satisfying*

$$(1.12) \qquad C^{-1}\langle\,,\,\rangle \leq \widetilde{\langle\,,\,\rangle} \leq C\langle\,,\,\rangle \qquad on \ M,$$

for some positive constant C.

Remark 1.2. Theorem 1.3 has later been improved in [72] with a different technique: however, the main Theorem 0.1 in [72] still requires (1.11) and a couple of conditions on the curvatures of M that, though more general than (1.9), nevertheless are more demanding than (1.23), (1.24) appearing in our Corollary 1.2 below.

Remark 1.3. For the special case of the Hyperbolic space, in [71] (see Theorem 1.1 therein) the authors were able to guarantee the existence of a solution for the Yamabe equation giving rise to a complete metric even when (1.10) is replaced by the weaker

$$-Cr(x)^2 \leq \widetilde{s}(x) < 0 \qquad outside \ of \ a \ compact \ set.$$

The counterpart of this improvement is that a control of the type (1.12) is no longer available. We remark that, in Theorem 1.1 of [71], condition (1.11) still appears.

Inspired by Ni-Naito-Kawano's approach, in [13] we have obtained sharp existence theorems for (1.2) (and, more generally, for (2.1) below) on manifolds possessing a pole o via mild assumptions on the radial sectional curvature K_{rad} (the sectional curvature restricted to 2-planes containing ∇r, with $r(\cdot) = \mathrm{dist}(\cdot, o)$). In the particular case of manifolds close to the hyperbolic space, our outcome has been the following result. Observe that condition 1.15 below guarantees the existence of solutions even when $\widetilde{s}(x)$ is strongly oscillating. On the other hand, (1.16) implies that the conformally deformed metric is incomplete and has finite volume.

Theorem 1.4 ([13], Thm 2)**.** *Let* $(M, \langle\,,\,\rangle)$ *be a complete manifold of dimension* $m \geq 3$, *with a pole* o *and sectional curvature* K *satisfying*

$$(1.13) \qquad -\kappa^2 - \mathcal{K}\big(r(x)\big) \leq K(x) \leq -\kappa^2,$$

for some constant $\kappa > 0$ and some non-negative $\mathcal{K} \in C^0(\mathbb{R}_0^+) \cap L^1(\mathbb{R}^+)$. Suppose that the scalar curvature $s(x)$ of M is such that

$$(1.14) \qquad s(x) \geq -\frac{(m-1)^3\kappa^2}{m-2} \qquad on\ M.$$

Then, for each $\widetilde{s}(x) \in C^\infty(M)$ satisfying, for some $B \in C^0(\mathbb{R}_0^+)$,

$$(1.15) \qquad |\widetilde{s}(x)| \leq B(r(x)), \qquad e^{-2\kappa r}B(r) \in L^1(+\infty),$$

the metric $\langle\,,\,\rangle$ can be conformally deformed to a smooth metric $\widetilde{\langle\,,\,\rangle}$ of scalar curvature $\widetilde{s}(x)$, satisfying

$$(1.16) \qquad \Gamma_1 e^{-2\kappa r(x)}\langle\,,\,\rangle_x \leq \widetilde{\langle\,,\,\rangle}_x \leq \Gamma_2 e^{-2\kappa r(x)}\langle\,,\,\rangle_x \qquad \forall\, x \in M,$$

for some $0 < \Gamma_1 \leq \Gamma_2$. In particular, $\widetilde{\langle\,,\,\rangle}$ is incomplete and has finite volume. Furthermore, Γ_2 and consequently Γ_1 can be chosen to be as small as we wish.

Remark 1.4. The growth conditions (1.7) and (1.15) are sharp: it is proved in [21] (for $\mathbb{R}^m$) and [15] (for $\mathbb{H}_\kappa^m$) that no conformal deformation exists whenever $\widetilde{s}(x) \leq 0$ on $\mathbb{R}^m$ (respectively, on $\mathbb{H}_\kappa^m$) and

$$\widetilde{s}(x) \leq -\frac{C}{r(x)^2 \log r(x)} \qquad \left(\text{respectively,} \quad \widetilde{s}(x) \leq -\frac{Ce^{2\kappa r(x)}}{r(x)\log r(x)}\right)$$

for some $C > 0$ and large $r(x)$.

The above four results are, to the best of our knowledge, an up-to-date account of what is known on the prescribed scalar curvature problem, in dimension $m \geq 3$ and with sign-changing $\widetilde{s}(x)$, on non-compact manifolds close to $\mathbb{R}^m$ and $\mathbb{H}_\kappa^m$. Figures 1 and 2 below summarize Theorems 1.1 to (1.4) when assumptions overlap.
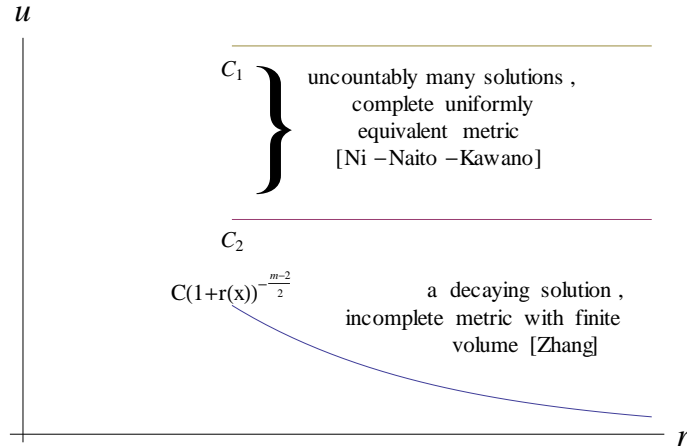
FIGURE 1. Euclidean space, (1.7) and Zhang's assumptions on $\widetilde{s}(x)$ in force.

A first step in the direction of removing the pole requirement has been taken in [14] by adapting some ideas of [13] via the use of Green functions. Unfortunately, even though the requirements on $s(x)$ and $\widetilde{s}(x)$ in Theorem 5 of [14] are sharp, they express in a form that is generally difficult to check. In summary, the task of obtaining results
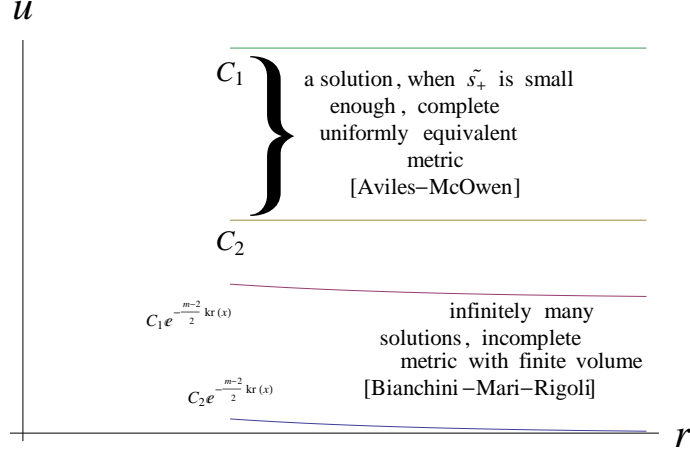
FIGURE 2. Manifolds close to $\mathbb{H}^m_\kappa$, $-C_1 \leq \widetilde{s}(x) \leq -C_2 < 0$ for large x.

of the type above but with a substantial weakening of the geometric assumptions calls for new ideas, and this is the objective of the present work. More precisely, we have a twofold concern in this paper. First, we aim to produce an existence theorem for sign-changing $\widetilde{s}(x)$ where topological and geometrical conditions are confined to a minimum. Second, we also want to keep control on the conformally deformed metric, in particular in such a way that $\langle\,,\,\rangle$ and $\widetilde{\langle\,,\,\rangle}$ are uniformly equivalent. Our contributions are Theorems 1.5, 1.6 and 2.3 below, a special case of Theorems 2.1, 2.2 which we are going to describe in awhile.

Notation. Hereafter, given $b \in L^\infty_{\mathrm{loc}}(M)$, we respectively denote with b_+ and b_- its positive and negative parts, so that $b = b_+ - b_-$. For $a \in L^\infty_{\mathrm{loc}}(M)$, we will write

$$a(x) = O\big(b(x)\big) \qquad \big(\text{respectively,} \quad a(x) \asymp b(x)\,\big) \quad \text{as } x \text{ diverges}$$

to indicate that there exists a constant $C > 0$ and a compact set Ω such that

$$a(x) \leq Cb(x) \qquad \big(\text{respectively,} \quad C^{-1}b(x) \leq a(x) \leq Cb(x)\,\big)$$

on $M \backslash \Omega$.

Our first result deals with the case of non-parabolic manifolds with non-negative scalar curvature. We recall that a manifold M is said to be non-parabolic if it admits a positive, non-constant solution of $\Delta u \leq 0$. The notion of non-parabolicity will be recalled later in a more general setting (see Proposition 2.1 and the subsequent discussion), here we limit to refer the interested reader to [34] for deepening. The following theorem shall be compared to Theorems 1.1 and 1.2. In particular, to compare Theorem 1.5 below with Zhang's Theorem 1.1 we need some tools that will be defined in the next introduction, and therefore we postpone the analysis to Remark 2.3.

Theorem 1.5. *Let* $(M, \langle\,,\,\rangle)$ *be a non-parabolic manifold of dimension* $m \geq 3$ *and scalar curvature* $s(x)$ *satisfying*

$$(1.17) \qquad\qquad s(x) \geq 0 \text{ on } M, \qquad s \in L^1(M),$$

and let $\widetilde{s} \in C^\infty(M)$ with the following properties:

$$(1.18) \qquad \widetilde{s}_+ \ \text{has compact support}, \qquad \widetilde{s} \in L^1(M).$$

Then, for each constant $C > 0$, $\langle \, , \, \rangle$ can be pointwise conformally deformed to a new metric $\widetilde{\langle \, , \, \rangle}$ of scalar curvature $\widetilde{s}(x)$ such that $\widetilde{\langle \, , \, \rangle} \leq C\langle \, , \, \rangle$. Moreover, if s and $\widetilde{s}$ have compact support, each such $\widetilde{\langle \, , \, \rangle}$ can be chosen to be uniformly equivalent to $\langle \, , \, \rangle$.

Remark 1.5. Non-parabolicity is a very mild requirement, and it is necessary to guarantee existence in all the cases investigated in Theorem 1.5. In fact, if M is scalar flat and taking $\widetilde{s}(x)$ to be compactly supported, non-negative and not identically zero, any eventual solution u of the Yamabe equation (1.2) would be a (non-constant) positive solution of $\Delta u = -\widetilde{s}(x)/c_m u^{\frac{m+2}{m-2}} \leq 0$, showing that M must be non-parabolic.

Theorem 1.5 applies, for instance, to the physically relevant setting of asymptotically flat spaces. According to [47], $(M^m, \langle \, , \, \rangle)$ is called asymptotically flat if

- its scalar curvature $s(x)$ satisfies (1.17),
- there exists a compact set $K \subset M$ such that each connected component U_j of $M \backslash K$ has a global chart $\Psi_j : (\mathbb{R}^m \backslash B_R(0), \langle \, , \, \rangle_{\text{can}}) \to U_j$ for which the local expression g_{ij} of $\langle \, , \, \rangle$ satisfies

$$(1.19) \qquad |g_{ij} - \delta_{ij}| = O(r^{-p}), \quad |\partial_k g_{ij}| = O(r^{-p-1}), \quad |\partial^2_{kl} g_{ij}| = O(r^{-p-2})$$

as $r(x) = |x| \to +\infty$, for some $p > (m-2)/2$ and for each $1 \leq i, j, k, l \leq m$.

Corollary 1.1. *Let $(M, \langle \, , \, \rangle)$ be an asymptotically flat manifold of dimension $m \geq 3$. Then, for each smooth function $\widetilde{s}(x)$ satisfying*

$$(1.20) \qquad \widetilde{s}_+ \ \text{has compact support} \qquad \text{and} \qquad \widetilde{s} \in L^1(M),$$

and for each constant $C > 0$, $\widetilde{s}(x)$ is realizable via a conformal deformation $\widetilde{\langle \, , \, \rangle}$ of $\langle \, , \, \rangle$ satisfying $\widetilde{\langle \, , \, \rangle} \leq C\langle \, , \, \rangle$. Furthermore, if $\widetilde{s} \equiv 0$ outside some compact set, then $\widetilde{\langle \, , \, \rangle}$ can be chosen to be uniformly equivalent to $\langle \, , \, \rangle$.

Now, we deal with manifolds whose original scalar curvature can be somewhere negative.

Theorem 1.6. *Let $(M^m, \langle \, , \, \rangle)$ be a non-parabolic Riemannian manifold of dimension $m \geq 3$ with scalar curvature $s(x)$. Suppose that the conformal Laplacian (1.3) admits a positive Green function on M.*

Let $\widetilde{s}(x) \in C^\infty(M)$ be such that

$$(1.21) \qquad \widetilde{s}_+ \ \text{has compact support}, \qquad \widetilde{s}(x) \asymp s(x) \ \text{ as } x \text{ diverges}.$$

Then, there exists $\delta > 0$ such that if

$$(1.22) \qquad \widetilde{s}(x) \leq \delta \qquad \text{on } M,$$

then $\widetilde{s}$ is realizable via a uniformly equivalent, conformal deformation $\widetilde{\langle \, , \, \rangle}$ of $\langle \, , \, \rangle$.

Remark 1.6. Observe that (1.21) implies that the original scalar curvature $s(x)$ is non-positive outside a compact set.

Note that Theorems 1.5 and 1.6 seem to be new even in the simpler case $\widetilde{s}(x) \leq 0$ on M. In this respect, these are skew with the main theorem in [48] and with Theorem 2.30 of [12].

The requirement (1.3) shows the central role played by the conformal Laplacian $L_{\langle \, , \, \rangle}$ for the Yamabe equation. The relevance of $L_{\langle , \rangle}$ for the original Yamabe problem is well-known and highlighted, for instance, in the comprehensive [47]. We will spend a considerable part of the paper to discuss on assumptions like (1.3). Clearly, Theorem 1.6 is tightly related to Aviles-McOwen's Theorem 1.3, a parallel which is even more evident in view of the next

Corollary 1.2. *Let $(M, \langle \, , \, \rangle)$ be a complete manifold of dimension $m \geq 3$ with a pole o and sectional curvature K satisfying*

$$K \leq -\kappa^2, \tag{1.23}$$

for some constant $H > 0$. Suppose further that

$$s(x) \geq -\frac{(m-1)^3}{(m-2)}\kappa^2 \tag{1.24}$$

on M. Let $\widetilde{s}(x) \in C^\infty(M)$ satisfying

$$-C_1 \leq \widetilde{s}(x) \leq -C_2 < 0 \qquad \text{outside some compact set,} \tag{1.25}$$

for some constants $0 < C_2 < C_1$. Then, there exists $\delta > 0$ such that if

$$\widetilde{s}(x) \leq \delta \qquad \text{on } M, \tag{1.26}$$

then $\widetilde{s}(x)$ is realizable by a conformal deformation $\widetilde{\langle \, , \, \rangle}$ of $\langle \, , \, \rangle$ which is uniformly equivalent to $\langle \, , \, \rangle$.

Remark 1.7. Corollary 1.2 improves on Theorem 1.3, since requirement (1.9) in Theorem 1.3 implies (1.23), (1.24). In particular, the hyperbolic space $\mathbb{H}_\kappa^m$ of sectional curvature $-\kappa^2$ satisfies all the assumptions of Corollary 1.2, being $s(x) = -m(m-1)\kappa^2$. Moreover, as the proof in [8] shows, (1.9) is essential to ensure the existence of $\delta > 0$ in (1.11); the case of equality in (1.24) seems, therefore, hardly obtainable with the approach described in [8]. It is worth to observe that the existence of a pole and the pinching assumption (1.9) on the sectional curvature are needed in [8] to apply both the Laplacian comparison theorems (from above and below) for the distance function $r(x) = \mathrm{dist}(x, o)$ in order to find suitable radial sub- and supersolutions. On the contrary, here the weaker (1.23) and (1.24) are just used to ensure that the conformal Laplacian has a positive Green function.

We pause for a moment to comment on assumption (1.22). Both Theorems 1.5 and 1.6 will be consequences of Theorem 2.3 below, and thus they will be proved via a common technique. The reason why assumption (1.22) is required in Theorem 1.6 but not in Theorem 1.5 can be summarized in the existence, in the second case, of a global, positive supersolution for the conformal Laplacian (that is, a solution w of $L_{\langle , \rangle}w \geq 0$ on M) which is bounded both from below and from above by positive constants; one can take, for instance, $u \equiv 1$. Such a function is not possible to construct in the general setting of Theorem 1.6 (see Remark 6.2 below for deepening). We stress that, unfortunately, the value of δ in Corollary 1.2 is not explicit: indeed, it depends on a uniform L^∞ bound for solutions of some suitable PDEs, which is shown to exist via an indirect method.
The need of (1.22) to obtain existence for $\widetilde{\langle , \rangle}$ is investigated in Remark 6.4. It is very interesting that the same condition (1.22) appears both in our theorem and in Aviles-McOwen's one, as well as in Theorems 0.1 in [72] and 1.1 in [71], although the techniques to prove them are different. This may suggest that, in general, (1.22) could

not be removable. However, at present we still have no counterexample showing that (1.22) is necessary. For future work, we thus feel interesting to investigate the next

Question. *Can assumption* (1.22) *in Theorem 1.6 be removed, even without expecting the new metric to be uniformly equivalent to* $\langle\,,\,\rangle$?

2. INTRODUCTION, II: OUR MAIN RESULTS IN THEIR GENERAL SETTING

Although the prescribed scalar curvature problem is the main focus of our investigation, the techniques developed here allow us to study more general classes of PDEs, namely nonlinear extensions (described in (2.4)) of the equation

$$(2.1) \qquad \Delta u + a(x)u - b(x)u^{\sigma} = 0 \qquad \text{on } M, \qquad u > 0 \ \text{ on } M$$

with $\sigma > 1$, $a, b \in C^{\infty}(M)$ and sign-changing b. Note that the signs of a, b are reversed with respect to those of $s, \widetilde{s}$ in (1.2), and that σ can be greater than $\frac{m+2}{m-2}$, preventing a direct use of variational techniques. However, when $\sigma \leq \frac{m+2}{m-2}$ and $b(x) < 0$ on M, the investigation of (2.1) on Euclidean space is still the core of a very active area of research. In this respect, we quote the seminal [17] and, for sign-changing $b(x)$ (and singular $a(x)$), the recent [30].

As a matter of fact, even for (2.1) the spectral properties of the linear part $L = -\Delta - a(x)$ play a prominent role, in particular the analysis of the fundamental tone $\lambda_1^L(M)$ of the Friedrichs extension of $(L, C_c^{\infty}(M))$. We recall that $\lambda_1^L(M)$ is characterized via the Rayleigh quotient as follows:

$$(2.2) \qquad \lambda_1^L(M) = \inf_{0 \neq \varphi \in \mathrm{Lip}_c(M)} \frac{\int_M \left[|\nabla\varphi|^2 - a(x)\varphi^2 \right] \mathrm{d}x}{\|\varphi\|_{L^2(M)}^2}.$$

For instance, if $\lambda_1^L(M) < 0$, the situation is somewhat rigid:

(a) if $b(x) \leq 0$, then (2.1) has no positive solutions. This follows from a direct spectral argument[1];

(b) if $b(x) \geq 0$ and the zero set of b is small in a suitable spectral sense, then there always exist a minimal and a maximal (possibly coinciding) positive solutions of (2.1); see [48, 63] and Section 2.4 in [12].

It is important to underline that, in both cases, the geometry of M only reveals via the spectral properties of L. In other words, no *a-priori* assumptions of completeness of M, nor curvature nor topological requests are made. As suggested by (a) and (b) above, it seems that the subtler case is that of investigating existence under the assumption $\lambda_1^L(M) \geq 0$. This condition is often implicitly met in the literature and it is automatically satisfied in many geometric situations. This happens, for instance, for Theorems 1.1 to 1.4.

There is another aspect of the above picture which is worth mentioning. Partial differential equations similar to (2.1) are of interest even for quasilinear operators more general than the Laplacian. Just to give an example, of a certain importance in Physics, we can consider the general equation for radiative cooling

$$(2.3) \qquad \kappa^{-1}\mathrm{div}\big(\kappa|\nabla u|^{p-2}\nabla u\big) - \big(\tau\kappa^{-1}\big)u^4 = 0 \qquad \text{on } \mathbb{R}^m,$$

where $\kappa > 0$ is the coefficient of the heat conduction and τ is a function describing the radiation (see [70], p.9). The existence problem for this type of quasilinear PDEs when

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[1]Indeed, assume the contrary and let $u > 0$ solves (2.1); then, u is a positive solution of $Lu \geq 0$, and a result of [32, 54, 3] implies that $\lambda_1^L(M) \geq 0$, contradicting our assumption.

the coefficient of the nonlinearity changes sign seems to be quite open. This suggests to extend our investigation to the existence of positive solutions to the quasilinear, Yamabe-type equation

(2.4) $$\Delta_{p,f}u + a(x)u^{p-1} - b(x)F(u) = 0 \qquad \text{on } M,$$

where $f \in C^\infty(M)$,

(2.5) $$\Delta_{p,f}u = e^f \text{div}\big(e^{-f}|\nabla u|^{p-2}\nabla u\big)$$

and $F(u)$ is a nonlinearity satisfying the following assumptions:

(2.6) $$\begin{cases} F \in C^0(\mathbb{R}), \quad F(0) = 0, \quad F > 0 \text{ on } \mathbb{R}^+, \\[2mm] \dfrac{F(t)}{t^{p-1}} \text{ is strictly increasing on } \mathbb{R}^+, \\[2mm] \lim_{t\to 0^+} \dfrac{F(t)}{t^{p-1}} = 0, \qquad \lim_{t\to+\infty} \dfrac{F(t)}{t^{p-1}} = +\infty \end{cases}$$

The prototype example of $F(t)$ is $F(t) = t^\sigma$ for $\sigma > p - 1$ and $t \geq 0$. Of course (2.1) is recovered by choosing $p = 2$ and f constant. We underline that, even for (2.4) in the Euclidean space and with $F(t) = t^\sigma$, there seems to be no result covering the cases described in Theorems 2.1 and 2.2. The family of operators in (2.5) above encompasses two relevant geometrical cases: $\Delta_{p,0}$ the standard p-Laplacian Δ_p, and $\Delta_{2,f}$ the drifted Laplacian, Δ_f, appearing, for instance, in the analysis of Ricci solitons and quasi-Einstein manifolds. Note that the radiative cooling equation is of type (2.4) provided $1 < p \leq 5$. Note also that, since the definition of $\Delta_{p,f}$ is intended in the weak sense, solutions will be, in general, only of class $C^{1,\mu}_{\text{loc}}(M)$ by [78].

Notation. Hereafter, with a slight abuse of notation, with $C^{1,\mu}_{\text{loc}}(M)$ we mean that for each relatively compact open set $\Omega \subset M$, there exists $\mu = \mu(\Omega) \in (0, 1)$ such that $u \in C^{1,\mu}(\overline{\Omega})$.

Remark 2.1. We stress that, with possibly the exception of Theorem 1.1 when $F(t) = t^\sigma$ and $\sigma \leq p^* - 1$, the techniques used to prove Theorems 1.1 to 1.4 seem hard to extend to deal with (2.5) for nonradial f, even for $p = 2$. For constant f, it seems also very difficult to adapt them to investigate (2.5) when $p \neq 2$. In particular, the transformation performed in [13] for $p = 2$ to absorb the linear term $a(x)u$ can only be applied when the driving operator is linear.

To state our main result, Theorem 2.1 below, we need to introduce some terminology. Let $d\mu_f$ be the weighted measure $e^{-f}dx$, with dx the Riemannian volume element on M. For $V \in L^\infty_{\text{loc}}(M)$, we consider the functional Q_V defined on $\text{Lip}_c(M)$ by

(2.7) $$Q_V(\varphi) = \frac{1}{p}\left[\int_M |\nabla\varphi|^p d\mu_f - \int_M V|\varphi|^p d\mu_f\right].$$

Its Gateaux derivative Q'_V is given by

(2.8) $$Q'_V(w) = -\Delta_{p,f}w - V|w|^{p-2}w \qquad \text{for } w \in W^{1,p}_{\text{loc}}(M).$$

When $M = \mathbb{R}^m$, the spectral properties of Q_V have been investigated in [5, 4, 33] and in a series of papers by Y. Pinchover and K. Tintarev (see in particular [67, 68]). From now on, we follow the notation and terminology in [68]. In the linear case $p = 2$, $f \equiv 0$, that is, for the Schrödinger operator $Q'_V = -\Delta - V$, we refer the reader to [54, 3, 60, 59, 55].

To begin with, and according to [55, 67], we recall the following:

Definition 2.1. *For $V \in L^\infty_{\text{loc}}(M)$, define Q_V as in (2.7) and let $\Omega \subseteq M$ be an open set.*

    i) *Q_V is said to be **non-negative** on Ω (shortly, $Q_V \geq 0$) if and only if $Q_V(\varphi) \geq 0$ for each $\varphi \in \text{Lip}_c(\Omega)$, that is, if and only if the Hardy type inequality*

$$(2.9) \qquad \int_M V(x)|\varphi|^p \mathrm{d}\mu_f \leq \int_M |\nabla\varphi|^p \mathrm{d}\mu_f \qquad \forall\, \varphi \in \text{Lip}_c(\Omega).$$

    holds.

    ii) *Q_V is said to be **subcritical** (or **non-parabolic**) on Ω if and only if $Q_V \geq 0$ there, and there exists $w \in L^1_{\text{loc}}(\Omega)$, $w \geq 0$, $w \not\equiv 0$ on Ω, such that*

$$(2.10) \qquad \int_M w(x)|\varphi|^p \mathrm{d}\mu_f \leq Q_V(\varphi) \qquad \forall\, \varphi \in \text{Lip}_c(\Omega).$$

Sometimes, especially in dealing with the prescribed scalar curvature problem and when no possible confusion arises, we also say that Q'_V, and not Q_V, is non-negative (or subcritical). The term "non-parabolic" is justified by the following statement for Q_0 (that is, Q_V with $V(x) \equiv 0$), which is part of Proposition 4.4 below:

Proposition 2.1. *Let $(M, \langle\,,\,\rangle)$ be Riemannian, $f \in C^0(M)$ and $p > 1$. Then, Q_0 is subcritical on M if and only if there exists a non-constant, positive weak solution $g \in C^0(M) \cap W^{1,p}_{\text{loc}}(M)$ of $\Delta_{p,f}\, g \leq 0$.*

According to the literature, the existence of such g is one of the equivalent conditions that characterize M as being not p-parabolic; there are various other characterizations of p-parabolicity, given in terms of Green kernels, p-capacity of compact sets, Ahlfors' type maximum principles, and so on. We refer to the survey [34] for deepening in the linear case $p = 2$, and to (see [79, 64, 41, 40, 38]) for $p \neq 2$. The equivalence in Proposition 2.1 has been observed, in the linear setting, by [7, 19, 49], and for $p \neq 2$ it has also recently been proved in [22] with a technique different from our.

In fact, all of these characterizations of the non-parabolicity of $-\Delta_{p,f}$ can be seen as a special case of a theory developed in [55, 66] (when $p = 2$) and in [67, 68] for operators Q_V with potential. In Section 4, we recall the main result in [67, 68], the ground state alternative, and we give a proof of it by including a further equivalent condition, see Theorem 4.1 below; as a corollary, we prove Propositions 4.4 and 2.1.

Remark 2.2. In the prescribed scalar curvature problem, the role of $\langle\,,\,\rangle$ and $\widetilde{\langle\,,\,\rangle}$ can be exchanged. Such a symmetry suggests that those geometric conditions which are invariant with respect to a conformal change of the metric turn out to be more appropriate to deal with the Yamabe equation. This is the case for the non-negativity and the subcriticality of the conformal Laplacian $L_{\langle\,,\,\rangle}$ of $(M, \langle\,,\,\rangle)$ in (1.3). In fact, the covariance of L with respect to the conformal deformation $\widetilde{\langle\,,\,\rangle} = u^{\frac{4}{m-2}}\langle\,,\,\rangle$ of the metric:

$$L_{\widetilde{\langle\,,\,\rangle}}(\cdot) = u^{-\frac{m+2}{m-2}} L_{\langle\,,\,\rangle}(u\cdot)$$

implies that, for each $0 \leq w \in L^1_{\text{loc}}(M)$ and $\varphi \in \text{Lip}_c(M)$,

$$\int_M \left[\|\widetilde\nabla\varphi\|^2 + \frac{\widetilde s}{c_m}\varphi^2\right]\mathrm{d}\widetilde x - \int_M \widetilde w\varphi^2\mathrm{d}\widetilde x = \int_M \left[|\nabla(u\varphi)|^2 + \frac{s}{c_m}(u\varphi)^2\right]\mathrm{d}x - \int_M w(u\varphi)^2\mathrm{d}x,$$

where $\sim$ superscript indicates quantities referred to $\widetilde{\langle\,,\,\rangle}$, $\|\cdot\|$ is its induced norm, and

$$\widetilde w = wu^{-\frac{4}{m-2}} \in L^1_{\text{loc}}(M), \quad \widetilde w \geq 0.$$

Consequently, $L_{\langle\,,\rangle}$ is non-negative (resp. subcritical) if and only if so is $L_{\widetilde{\langle,\rangle}}$.

Remark 2.3. As a direct consequence of the ground state alternative, the positivity of the Yamabe invariant $Y(M)$ in Zhang's Theorem 1.1 implies that $L_{\langle\,,\rangle}$ is subcritical (see Remark 4.3). On the other hand, in our Theorem 1.5 the subcriticality of $L_{\langle\,,\rangle}$ follows combining the non-parabolicity of M and $s(x) \geq 0$. Although, in general, the positivity of $Y(M)$ might not imply the non-parabolicity of M, this is so if M is scalar flat outside a compact set and $\mathrm{vol}(M) = +\infty$. Indeed, if $s(x) \equiv 0$ outside a compact set K, then $Y(M) > 0$ gives the validity of an L^2-Sobolev inequality on $M\backslash K$, and coupling with $\mathrm{vol}(M) = +\infty$ the non-parabolicity of M follows by a result in [20, 64]. We underline that, in the same assumptions, again by [20, 64] property $\mathrm{vol}(M) = +\infty$ is automatic when M is geodesically complete. Summarizing, if the manifold in Zhang's Theorem 1.1 is scalar flat near infinity, the geometric requirements there properly contain those of our Theorem 1.5.

We are now ready to state

Theorem 2.1. *Let M^m be a Riemannian manifold, $f \in C^\infty(M)$ and $p \in (1, +\infty)$. Suppose that Q_0 is subcritical on M, and let $a \in L^\infty_{\mathrm{loc}}(M)$ be such that Q_a is subcritical on M. Consider $b \in L^\infty_{\mathrm{loc}}(M)$, and assume*

    i) $b_-(x)$ has compact support;

    ii) $a(x) = O\big(b(x)\big)$ as x diverges;

    iii) for some $\theta > 0$, $\big(a(x) - \theta b_+(x)\big)_- \in L^1(M, \mathrm{d}\mu_f)$.

Fix a nonlinearity $F(t)$ satisfying (2.6). Then, there exists $\delta > 0$ such that if

$$(2.11) \qquad\qquad\qquad b(x) \geq -\delta \qquad \text{on } M,$$

there exists a weak solution $u \in C^{1,\mu}_{\mathrm{loc}}(M)$ of

$$(2.12) \qquad \begin{cases} \Delta_{p,f} u + a(x)u^{p-1} - b(x)F(u) = 0 & \text{on } M \\ 0 < u \leq \|u\|_{L^\infty(M)} < +\infty. \end{cases}$$

If we replace ii) and iii) by the stronger condition

$$iv) \qquad b_+(x) \asymp a(x) \qquad \text{as } x \text{ diverges,}$$

and we keep the validity of (2.11), then u can also be chosen to satisfy

$$(2.13) \qquad\qquad\qquad \inf_M u > 0.$$

In the next theorem, we remove requirement (2.11); see also Remark 6.4 for a related discussion.

Theorem 2.2. *Let M^m be a Riemannian manifold, $f \in C^\infty(M)$ and $p \in (1, +\infty)$. Suppose that Q_0 is subcritical on M and let $a \in L^\infty_{\mathrm{loc}}(M)$ be such that Q_a is subcritical on M. Consider $b \in L^\infty_{\mathrm{loc}}(M)$, and assume*

    i) $b_-(x)$ has compact support;

    ii') $a(x) \leq 0$ outside a compact set;

    iii') $a(x), b(x) \in L^1(M, \mathrm{d}\mu_f)$.

Fix a nonlinearity $F(t)$ satisfying (2.6). Then, there exists a sequence $\{u_k\} \subset C^{1,\mu}_{\text{loc}}(M)$ of distinct weak solutions of

$$(2.14) \qquad \begin{cases} \Delta_{p,f} u_k + a(x)u_k^{p-1} - b(x)F(u_k) = 0 & \text{on } M \\ 0 < u_k \leq \|u_k\|_{L^\infty(M)} < +\infty, \end{cases}$$

such that $\|u_k\|_{L^\infty(M)} \to 0$ as $k \to +\infty$. If we replace $ii')$ and $iii')$ by the stronger condition

$$iv') \qquad a(x), b(x) \quad \text{have compact support,}$$

then each u_k also satisfies $\inf_M u_k > 0$.

One of the main features in the proof of Theorems 2.1 and 2.2 above is a new flexible technique, which is based on a direct use of the non-negativity and subcriticality assumptions on Q_a and Q_0. Consequently, all the geometric information needed on M is encoded in the spectral behaviour of Q_0 and Q_a. For this reason, in Sections 4 and 5 we concentrate on operators Q_V to show that the assumption on Q_0, Q_a in Theorems 2.1 and 2.2 can be made explicit and easily verifiable in various relevant cases.

We now come to the strategy to prove Theorems 2.1 and 2.2. The lack of tools to produce solutions in the present generality forces us to proceed along very simple, general schemes. In particular, the argument can be roughly divided into three parts:

(1) For a big relatively compact domain Ω, we solve locally (2.4) with boundary condition 1 and $b(x)$ replaced by $b_+(x)$. Call z_Ω the solution. This is the easiest part, and is addressed in Lemma 6.1.

(2) We find uniform L^∞ estimates from below and above for z_Ω, independent of Ω. According to our geometric assumptions, these estimates can be on the whole M or on a relatively compact set Λ. The proof of this step combines Lemmas 6.2 and 6.3, and Proposition 6.1.

(3) Making use of the results in Step (2), we "place" b_- in the Dirichlet problem for (2.4) on a domain Ω via an iterative procedure, to produce a local solution of (2.4) that possesses uniform upper and lower bounds. The desired global solution is then obtained by passing to the limit along an exhaustion $\{\Omega_j\}$. Note that this is the point where a distinction between Theorems 2.1 and 2.2 appears.

Among the lemmas, which are of independent interest, we underline and briefly comment on the next uniform L^∞-estimate, Lemma 6.3. This result is a cornerstone both for steps (2) and (3).

Lemma 2.1 (Uniform L^∞-estimate). *Let M be a Riemannian manifold, $f \in C^\infty(M)$, $p \in (1, +\infty)$. Let $A, B \in L^\infty_{\text{loc}}(M)$ with $B \geq 0$ a.e. on M. Assume that either*

(i) $B \equiv 0$ and Q_A is subcritical, or

(ii) $B \not\equiv 0$ and Q_A is non-negative.

Suppose that there exist a smooth, relatively compact open set $\Lambda \Subset M$ and a constant $c > 0$ such that

$$(2.15) \qquad A \leq cB \qquad \text{a.e. on } M \backslash \Lambda,$$

and fix a smooth, relatively compact open set Λ' such that $\Lambda \Subset \Lambda'$, and a nonlinearity $F(t)$ satisfying (2.6).

Then, there exists a constant $C_\Lambda > 0$ such that, for each smooth, relatively compact open set Ω with $\Lambda' \Subset \Omega$, the solution $0 < z \in C^{1,\mu}(\overline{\Omega})$ of

$$(2.16) \qquad \begin{cases} \Delta_{p,f} z + A(x) z^{p-1} - B(x) F(z) = 0 & on\ \Omega, \\ z = 1 & on\ \partial\Omega. \end{cases}$$

satisfies

$$(2.17) \qquad\qquad z \leq C_\Lambda \qquad on\ \Omega.$$

The proof of the above estimate is accomplished by using non-negativity (resp. subcriticality) of Q_A alone. As far as we know, the argument in the proof seems to be new and applicable beyond the present setting. Clearly, when z is C^2 and $B \in C^0(M)$, $B > 0$ on M, possibly evaluating (2.16) at a interior maximum point x_0 we get

$$(2.18) \qquad\qquad \frac{F(z)}{z^{p-1}}(x_0) \leq \sup_M \left(\frac{A_+}{B} \right),$$

whence by (2.6) $z(x_0)$ is uniformly bounded from above. Taking into account the boundary condition for z, in this case the uniform L^∞ estimate is trivial with no assumption on Q_A. On the other hand, even a single point at which $B(x) = 0$ makes this simple argument to fail, and actually Lemma 2.1 will be applied in cases when we have no control at all on the zero-set of B. Observe that (2.15) is just assumed to hold outside of a compact set, hence A is not required to be non-positive on the set where $B = 0$. This suggests that the validity of (2.18) cannot be recovered "in the limit" by using approximating positive functions B_ε for B and related solutions z_ε for z. Note also that, when $B \equiv 0$, Proposition 3.4 below shows that Q_A is necessarily non-negative, for otherwise z might not exist for sufficiently large Ω's. Therefore, in the present generality at least the non-negativity of Q_A on the whole M needs to be assumed in any case.

We pause for a moment to comment on the subcriticality of Q_V. By its very definition, a sufficient condition for Q_V to be subcritical is the coupling of the following two:

- Q_0 is subcritical, thus there exists $w \in L^1_{\mathrm{loc}}(M)$, $w \geq 0$, $w \not\equiv 0$ such that

$$(2.19) \qquad \int_M w(x)|\varphi|^p \mathrm{d}\mu_f \leq \int_M |\nabla\varphi|^p \mathrm{d}\mu_f \qquad \forall\, \varphi \in \mathrm{Lip}_c(M), \quad \text{and}$$

- $V \leq w$, $V \not\equiv w$.

Therefore, when Q_0 is subcritical, we can state simple, explicit conditions guaranteeing the subcriticality of Q_V provided that we know explicit $w \in L^1_{\mathrm{loc}}(M)$, $w \geq 0$, $w \not\equiv 0$ satisfying (2.19). We define each of these w a Hardy weight for $\Delta_{p,f}$.

In the literature, there are conditions to imply the subcriticality of Q_0 that involve curvature bounds, volume growths, doubling properties and Sobolev type inequalities. For example, when $f \equiv 0$, in [41] it is proved that a complete, non-compact manifold M with non-negative Ricci curvature outside a compact set is not p-parabolic (i.e. Q_0 is subcritical) if and only if $p < m$. The interested reader can also consult [40, 73]. However, it seems challenging to obtain explicit Hardy weights in the setting of [41, 40, 73]. Nevertheless, Hardy weights have been found in some interesting cases, starting with the famous Hardy type inequality for Euclidean space

$$(2.20) \qquad \left(\frac{m-p}{p} \right)^p \int_{\mathbb{R}^m} \frac{|\varphi|^p}{r^p} \mathrm{d}x \leq \int_{\mathbb{R}^m} |\nabla\varphi|^p \mathrm{d}x \qquad \forall\, \varphi \in \mathrm{Lip}_c(\mathbb{R}^m),$$

where $r(x) = |x|$ and $m > p$. In recent years ([19, 49, 12, 10, 1, 22, 23, 24]) it has been observed how Hardy weights are related to positive Green kernels for $\Delta_{p,f}$. By exploiting the link established in Proposition 4.4 below, we will devote Section 5 to produce explicit Hardy weights in various geometrically relevant cases, see Theorems 5.1, 5.2, 5.3 below: in fact, a typical construction of Hardy weights via the Green kernel is compatible with comparison results for the Laplacian of the distance function, and thus Hardy weights can be transplanted from model manifolds to general manifolds, as observed in [12], Theorem 4.15 and subsequent discussion. Moreover, the set of Hardy weights is convex in $L^1_{\mathrm{loc}}(M)$, thus via simple procedures one can produce new weights, such as multipole Hardy weights or weights blowing up along a fixed submanifold of M. Hardy weights can also be transplanted to submanifolds, but this procedure is more delicate and requires extra care. Let N^n be a Cartan-Hadamard manifold (i.e. a simply connected, complete manifold of non-positive sectional curvature), and let M^m be a minimal submanifold of N^n. Suppose that the sectional curvature $\bar{K}$ of N satisfies $\bar{K} \le -\kappa^2$, for some constant $\kappa \ge 0$. In [19, 49], the authors proved the following Hardy type inequality:

$$(2.21) \qquad \left(\frac{m-2}{2}\right)^2 \int_M \frac{\varphi^2}{\rho^2}\mathrm{d}x \le \int_M |\nabla\varphi|^2\mathrm{d}x \qquad \forall\, \varphi \in \mathrm{Lip}_c(M),$$

$\rho(x)$ being the extrinsic distance in N from a fixed origin $o \in N$. The Hardy weight in (2.21) is sharp if $\kappa = 0$ (in particular, if N is the Euclidean space), but not if $\kappa > 0$. Here, we will prove (2.21) as a particular case of Theorem 5.3 below, which also strengthen (2.21) to a sharp inequality when $\kappa > 0$, in particular for minimal submanifolds of hyperbolic spaces. We stress that our Hardy weight for $\kappa > 0$ is skew with the one found in [49].

Using the Hardy inequalities mentioned before, we can rewrite the subcriticality assumption for Q_0 and Q_V in Theorems 2.1, 2.2 in simple form for a wide class of manifolds; by a way of example, see Corollary 5.1 in Section 7. We conclude by rephrasing Theorems 2.1, 2.2 in the setting of the generalized Yamabe problem.

Theorem 2.3. *Let $(M, \langle\,,\,\rangle)$ be a non-parabolic Riemannian manifold of dimension $m \ge 3$ and scalar curvature $s(x)$. Suppose that the conformal Laplacian $L_{\langle\,,\,\rangle}$ in (1.3) is subcritical, and let $\widetilde{s} \in C^\infty(M)$.*

(I) Assume that

    i) $\widetilde{s}_+$ has compact support;

    ii) $s_-(x) = O\big(\widetilde{s}_-(x)\big)$ as x diverges;

    iii) for some $\theta > 0$, $\big(\theta\widetilde{s}_-(x) - s_-(x)\big)_+ \in L^1(M)$.

    Then, there exists $\delta > 0$ such that if

$$(2.22) \qquad\qquad\qquad \widetilde{s}(x) \le \delta,$$

the metric $\langle\,,\,\rangle$ can be pointwise conformally deformed to a new metric $\widetilde{\langle\,,\,\rangle}$ with scalar curvature $\widetilde{s}(x)$ and satisfying

$$(2.23) \qquad\qquad\qquad \widetilde{\langle\,,\,\rangle} \le C\langle\,,\,\rangle \qquad on\ M,$$

for some constant $C > 0$. Moreover, if ii) and iii) are replaced by the stronger

    iv) $s(x) \asymp \widetilde{s}(x)$ as x diverges,

> *then (under the validity of* (2.22)*) there exists a pointwise conformal deformation $\widetilde{\langle\,,\,\rangle}$ of $\langle\,,\,\rangle$ as above and satisfying*

$$(2.24) \qquad\qquad C_1\langle\,,\,\rangle \leq \widetilde{\langle\,,\,\rangle} \leq C_2\langle\,,\,\rangle \qquad \textit{on } M,$$

> *for some constants $0 < C_1 \leq C_2$. In particular, $\widetilde{\langle\,,\,\rangle}$ is non-parabolic, and it is complete whenever $\langle\,,\,\rangle$ is complete.*

(*II*) *If ii) and iii) are replaced with*

    ii') $s(x) \geq 0$ outside a compact set;

    iii') $s(x), \widetilde{s}(x) \in L^1(M)$,

> *then the existence of the desired conformal deformation is guaranteed without the requirement* (2.22)*, and moreover the constant C in* (2.23) *can be chosen as small as we wish (so that, indeed, there exist infinitely many conformal deformations realizing $\widetilde{s}$). If ii') and iii') are replaced with*

$$iv') \qquad s(x), \widetilde{s}(x) \qquad \textit{have compact support,}$$

> *each of these conformally deformed metrics $\widetilde{\langle\,,\,\rangle}$ satisfies* (2.24)*.*

The paper is organized as follows. In Section 3 we collect some basic material on Q_V and Q'_V. Section 4 will then be devoted to the criticality theory for Q_V, its link with Hardy weights and with a Q_V-capacity theory. In Section 5, we use comparison geometry to produce sharp Hardy inequalities. Section 6 contains the proof of Lemma 2.1 and of our main Theorems 2.1, 2.2. Then, in Section 7 we derive our geometric corollaries, and we place them among the existing literature. Finally, in the Appendix we give a full proof of the pasting lemma, an important technical result for the Q_V-capacity theory. Besides the presence of new results, a major concern of Sections 3 to 5 is to help the reader to get familiar with various aspects of the theory of Schrödinger type operators Q'_V. For this reason, the experienced reader may possibly skip them and go directly to Section 6.

3. Preliminaries

In this section we recall some general facts for the operators $\Delta_{p,f}, Q_V, Q'_V$ that are extensions, to a quasilinear setting, of some classical results of spectral theory (see [55, 66, 54, 3]). The interested reader may consult [5, 4, 33, 67, 68] for further information.

Notation. Hereafter, given two open subsets Ω, U, with $\Omega \Subset U$ we indicate that Ω has compact closure contained in U. We say that $\{\Omega_j\}$ is an exhaustion of M if it is a sequence of relatively compact, connected open sets Ω_j with smooth boundary and such that $\Omega_j \Subset \Omega_{j+1} \Subset M$, $M = \bigcup_j \Omega_j$. The symbol 1_U denotes the characteristic function of a set U, and the symbol $\doteq$ is used to define an object.

Definition 3.1. *Let $\Omega \subset M$ be an open set and, for $V \in L^\infty_{\mathrm{loc}}(\Omega)$, let Q_V, Q'_V be as in* (2.7)*,* (2.8)*. We say that $w \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a supersolution on Ω (respectively, subsolution, solution) if $Q'_V(w) \geq 0$ weakly on Ω (resp. ≤ 0, $= 0$) that is, if*

$$\int_\Omega |\nabla w|^{p-2}\langle \nabla w, \nabla\varphi\rangle \mathrm{d}\mu_f - \int_\Omega V|w|^{p-2}w\varphi \geq 0 \qquad (\textit{resp. } \leq 0, = 0)$$

for each non-negative $\varphi \in \mathrm{Lip}_c(\Omega)$.

The basic technical material that is necessary for our purposes is summarized in the following

Theorem 3.1. *Let $\Omega \Subset M$ be a relatively compact, open domain with $C^{1,\alpha}$ boundary for some $0 < \alpha < 1$. Let $f \in C^\infty(M)$, $p \in (1, +\infty)$, $V \in L^\infty_{\mathrm{loc}}(M)$ and define Q_V, Q'_V as in (2.7), (2.8). Let $g \in L^\infty(\Omega)$, $\xi \in C^{1,\alpha}(\partial\Omega)$ and suppose that $u \in W^{1,p}(\Omega)$ is a solution of*

$$
(3.1) \qquad \begin{cases} Q'_V(u) = g & on \ \Omega, \\ u = \xi & on \ \partial\Omega. \end{cases}
$$

Then,

(1) *[Boundedness] $u \in L^\infty_{\mathrm{loc}}(\Omega)$, and for any relatively compact, open domains $B \Subset B' \Subset \Omega$ there exists a positive constant $C = C(p, f, m, g, \xi, \Omega, \|u\|_{L^p(B', \mathrm{d}\mu_f)})$ such that*

$$\|u\|_{L^\infty(B)} \le C.$$

If $\xi \in C^{2,\alpha}(\partial\Omega)$, C can be chosen globally on Ω, and thus $u \in L^\infty(\Omega)$.

(2) *[$C^{1,\mu}$-regularity] When $u \in L^\infty(\Omega)$, there exists $\mu \in (0,1)$ depending on p, f, m, g, α and on upper bounds for $\|u\|_{L^\infty}, \|g\|_{L^\infty}, \|\xi\|_{C^{1,\alpha}}, \|V\|_{L^\infty}$ on Ω such that*

$$\|u\|_{C^{1,\mu}(\overline\Omega)} \le C$$

for some constant C depending on α, p, the geometry of Ω and upper bounds for $\|u\|_{L^\infty}, \|g\|_{L^\infty}, \|\xi\|_{C^{1,\alpha}}, \|V\|_{L^\infty}$ on Ω.

(3) *[Harnack inequality]. For any relatively compact open sets $B \Subset B' \Subset \Omega$ there exists $C = C(f, p, m, B, B') > 0$ such that, for each $u \ge 0$, $u \in W^{1,p}(\Omega)$ solution of $Q'_V(u) = 0$ on Ω,*

$$
(3.2) \qquad \sup_B u \le C \inf_{B'} u.
$$

In particular, either $u > 0$ on Ω or $u \equiv 0$ on Ω.

(3a) *[Half-Harnack inequalities] For any relatively compact, open sets $B \Subset B' \Subset \Omega$ the following holds:*

   (Subsolutions) *for each $s > p-1$, there exists $C = C(f, p, m, B, B', V, s) > 0$ such that for each $u \ge 0$, $u \in W^{1,p}(\Omega)$ solution of $Q'_V(u) \le 0$ on Ω*

$$
(3.3) \qquad \sup_B u \le C\|u\|_{L^s(B')};
$$

   (Supersolutions) *for each*

$$s \in \left(0, \frac{(p-1)m}{m-p}\right) \quad if \ p < m, \qquad s \in (0, +\infty) \quad if \ p \ge m,$$

   there exists $C = C(f, p, m, B, B', V, s) > 0$ such that for each $u \ge 0$, $u \in W^{1,p}(\Omega)$ solution of $Q'_V(u) \ge 0$ on Ω

$$
(3.4) \qquad \|u\|_{L^s(B')} \le C \inf_B u.
$$

(4) *[Hopf lemma] Suppose that $\xi \ge 0$, $g \ge 0$ and let $u \in C^1(\overline\Omega)$ be a solution of (3.1) with $u \ge 0$, $u \not\equiv 0$. If $x \in \partial\Omega$ is such that $u(x) = \xi(x) = 0$, then, indicating with ν the inward unit normal vector to $\partial\Omega$ at x we have $\langle \nabla u, \nu \rangle(x) > 0$.*

Remark 3.1.

(1) The local boundedness of u is a particular case of Serrin's theorem, see [70], Theorem 7.1.1, and does not need the boundary condition. When $\xi \in C^{2,\alpha}(\partial\Omega)$, global boundedness can be reached via a reflection technique described at page 54 of [33], see also [77, 11].

(2) is a global version, Theorem 1 of [50], of a local regularity result in [78] and [27].

(3) is due to J. Serrin, see Theorem 7.2.1 in [70] for $p < m$, the discussion at the beginning of Section 7.4 therein for $p = m$, and Theorem 7.4.1 for $p > m$.

(3a) The half-Harnack for subsolutions can be found in Theorem 7.1.1 of [70], the one for supersolutions in the subsequent Theorems 7.1.2 (case $p < m$) and 7.4.1 (case $p > m$) of [70]. Again, see the discussion at the beginning of Section 7.4 of [70].

(4) The Hopf lemma can be found in Corollary 5.5 of [69].

An important tool for our investigation is the following Lagrangian representation in [67].

Proposition 3.1. *For each $\varphi, g \in W^{1,p}_{\text{loc}}(M)$, with φ/g a.e. finite on M, the Lagrangian*

$$(3.5) \qquad \mathcal{L}(\varphi, g) = |\nabla\varphi|^p + (p-1)\left(\frac{\varphi}{g}\right)^p |\nabla g|^p - p\left(\frac{\varphi}{g}\right)^{p-1} |\nabla g|^{p-2}\langle\nabla g, \nabla\varphi\rangle$$

satisfies $\mathcal{L}(\varphi, g) \geq 0$ on M, and $\mathcal{L}(\varphi, g) \equiv 0$ on some connected open set U if and only if φ is a constant multiple of g on U.
Moreover, suppose that $g \in W^{1,p}_{\text{loc}}(M)$ is a positive solution of $Q'_V(g) = 0$ (resp. $Q'_V(g) \geq 0$) on M. Then, for $\varphi \in L^\infty_c(M) \cap W^{1,p}(M)$, $\varphi \geq 0$ it holds

$$(3.6) \qquad\qquad Q_V(\varphi) = \int_M \mathcal{L}(\varphi, g)\mathrm{d}\mu_f \qquad (resp, \ \geq).$$

Proof. The non-negativity of $\mathcal{L}(\varphi, g)$ follows by applying Cauchy-Schwarz and Young inequalities on the third addendum in (3.5), and analyzing the equality case, $\mathcal{L}(\varphi, g) \equiv 0$ if and only if $\varphi = cg$ on M, for some constant $c \in \mathbb{R}$.
We now prove the integral (in)equality in (3.6). By Harnack inequality, g is locally essentially bounded from below on M. This, combined with our regularity requirement on φ, guarantees that $\varphi^p/g^{p-1} \in W^{1,p}(M)$ and is compactly supported. Thus, we integrate on M the pointwise identity

$$|\nabla\varphi|^p - |\nabla g|^{p-2}\langle\nabla g, \nabla\left(\frac{\varphi^p}{g^{p-1}}\right)\rangle = \mathcal{L}(\varphi, g),$$

and couple with the weak definition of $Q'_V(g) = 0$ (resp. ≥ 0) applied to the test function φ^p/g^{p-1}:

$$0 = \int_M |\nabla g|^{p-2}\langle\nabla g, \nabla\left(\frac{\varphi^p}{g^{p-1}}\right)\rangle\mathrm{d}\mu_f - \int_M Vg^{p-1}\left(\frac{\varphi^p}{g^{p-1}}\right)\mathrm{d}\mu_f \qquad (resp. \ \leq)$$

to deduce (3.6). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Box$

Rewriting the expression of $\mathcal{L}(\varphi, g)$ we deduce the next useful Picone type inequality due to [4, 26, 6].

Proposition 3.2. *Let M be a Riemannian manifold, and let $\Omega \Subset M$ be a relatively compact, connected open set. Then, the functional*

(3.7)
$$I(w,z) = \int_\Omega |\nabla w|^{p-2} \left\langle \nabla w, \nabla \frac{w^p - z^p}{w^{p-1}} \right\rangle \mathrm{d}\mu_f - \int_\Omega |\nabla z|^{p-2} \left\langle \nabla z, \nabla \frac{w^p - z^p}{z^{p-1}} \right\rangle \mathrm{d}\mu_f$$

is non-negative on the set

$$\mathcal{D}_\Omega = \left\{ (w,z) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega) \ : \ w, z \geq 0 \text{ on } \Omega \ , \ \frac{w}{z}, \frac{z}{w} \in L^\infty(\Omega) \right\}.$$

Furthermore, $I(w,z) = 0$ if and only if $w = Cz$ on Ω, for some constant $C > 0$.

Proof. Since $w, z \in \mathcal{D}_\Omega$, it is easy to see that $\frac{w^p}{z^{p-1}}, \frac{z^p}{w^{p-1}} \in W^{1,p}(\Omega)$. We can therefore expand the integrand in (3.7) and rearrange to deduce that

$$I(w,z) = \int_\Omega \left[\mathcal{L}(w,z) + \mathcal{L}(z,w) \right] \mathrm{d}\mu_f,$$

with $\mathcal{L}$ as in (3.5). The first part of previous proposition then gives the desired inequality. $\qquad\square$

Now, we investigate property $Q_V \geq 0$ and its consequences. By its very definition, $Q_V \geq 0$ on an open set $\Omega \subset M$ is equivalent to the non-negativity of the fundamental tone

(3.8)
$$\lambda_V(\Omega) = \inf_{0 \not\equiv \varphi \in \mathrm{Lip}_c(\Omega)} \frac{pQ_V(\varphi)}{\|\varphi\|^p_{L^p(\Omega, \mathrm{d}\mu_f)}}.$$

If Ω is a relatively compact domain with smooth boundary, then it is well-known that the infimum (3.8) is attained by a first eigenfunction $\phi \not\equiv 0$ solving Euler-Lagrange equation

$$\begin{cases} Q'_V(\phi) = \lambda_V(\Omega)|\phi|^{p-2}\phi & \text{on } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \end{cases}$$

and $\phi > 0$ on Ω up to changing its sign [2]. Furthermore, by Harnack inequality, if $\Omega \subset \Omega'$ are two relatively compact open sets and $\Omega' \backslash \Omega$ has non-empty interior, then $\lambda_V(\Omega) > \lambda_V(\Omega')$.

The next comparison result will be used throughout the paper, and improves on Theorem 5 of [33].

Proposition 3.3. *Let M^m, p, f be as above and, for $A \in L^\infty_{\mathrm{loc}}(M)$, define Q_A, Q'_A as in (2.7), (2.8) with $V(x) = A(x)$. Consider a relatively compact, open set $\Omega \Subset M$ with smooth boundary, and let $u_1, u_2 \in C^{1,\mu}(\overline{\Omega})$, for some $\mu \in (0,1)$. Furthermore, suppose that, for some non-negative $B \in L^\infty(\Omega)$ and a nonlinearity $F(t)$ satisfying (2.6),*

(3.9)
$$\begin{cases} \Delta_{p,f} u_1 + A|u_1|^{p-2}u_1 - BF(u_1) \geq 0, \\ \Delta_{p,f} u_2 + A|u_2|^{p-2}u_2 - BF(u_2) \leq 0, \\ u_1 \leq u_2 \ \text{on } \partial\Omega, \qquad u_1 \geq 0, \, u_2 > 0 \ \text{on } \Omega. \end{cases}$$

Then, either

    i) $u_1 \leq u_2$ on Ω, or

    ii) $B(x) \equiv 0$ on Ω, u_2 satisfies $Q'_A(u_2) = 0$, $u_2 \equiv 0$ on $\partial\Omega$, and $\lambda_A(\Omega) = 0$.

---

[2]Briefly, $|\phi|$ still minimizes the Rayleigh quotient in (3.8), thus it satisfies the Euler-Lagrange equation $Q'_V(|\phi|) = \lambda_V(\Omega)|\phi|^{p-1}$, hence $|\phi| > 0$ on Ω by Harnack inequality in Theorem 3.1, (3).

Proof. We let $\xi = u_{2|\partial\Omega} \in C^{1,\mu}(\partial\Omega)$ and let

$$V = A(x) - B(x)\frac{F(u_2)}{u_2^{p-1}}.$$

Note that $V \in L^\infty(\Omega)$ by (2.6). For $x \in \partial\Omega$ such that $u_2(x) = 0$, we let ν be the inward unit normal to $\partial\Omega$ at x. Then, applying Theorem 3.1 (4) we deduce that

$$\langle\nabla u_2(x), \nu\rangle(x) > 0,$$

by continuity, there thus exists a constant $C > 0$ such that

(3.10) $$u_1 \le Cu_2 \qquad \text{on } T(\partial\Omega),$$

for some tubular neighbourhood $T(\partial\Omega)$ of $\partial\Omega$. Using assumption $u_2 > 0$ on Ω we can suppose that (3.10) is true on all of $\overline{\Omega}$ with $C > 1$. Because of (3.9) and since $B(x) \ge 0$, $C > 1$ and, by (2.6), $F(t)/t^{p-1}$ is increasing on $\mathbb{R}^+$, Cu_2 is still a supersolution:

$$\Delta_{p,f}(Cu_2) + A(Cu_2)^{p-1} - BF(Cu_2) = C^{p-1}\left[\Delta_{p,f}u_2 + Au_2^{p-1}\right] - BF(Cu_2)$$

$$\le C^{p-1}BF(u_2) - BF(Cu_2) \le B(Cu_2)^{p-1}\left[\frac{F(u_2)}{u_2^{p-1}} - \frac{F(Cu_2)}{(Cu_2)^{p-1}}\right] \le 0.$$

Using $u_1 \ge 0$ as a subsolution, by (3.10) and applying the method of sub- and super-solutions, see Theorem 4.14, page 272, in [25], we find a solution v of

(3.11) $$\begin{cases} \Delta_{p,f}v + A|v|^{p-2}v - BF(v) = 0, \\ v = u_2 \qquad \text{on } \partial\Omega, \end{cases}$$

satisfying

(3.12) $$u_1 \le v \le Cu_2 \qquad \text{on } \Omega$$

By the $C^{1,\mu}$-regularity of Theorem 3.1, $v \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$. If we show that $v \le u_2$ then (3.12) implies $u_1 \le u_2$ on Ω which is the conclusion $i)$ of the Proposition.

Suppose that this is not the case, that is, assume that the open set $U = \{v > u_2\}$ is non-empty. We are going to prove that $ii)$ holds. Since $v \ge u_1 \ge 0$ and v is positive on U, then $v > 0$ on Ω as a consequence of the Harnack inequality in Theorem 3.1 (use $V = A(x) - B(x)F(v)/v^{p-1}$, which by (2.6) is bounded on Ω). Alternatively, one can use the version of the strong maximum principle in Theorem 5.4.1 in [70]. Now, again by the Hopf Lemma of Theorem 3.1, $\langle\nabla v(x), \nu(x)\rangle > 0$, ν the inward unit normal to $\partial\Omega$ at x, at each point $x \in \partial\Omega$ where $v(x) = u_2(x) = 0$. Hence, the ratio u_2/v is well defined at x along the half line determined by ν. This shows that u_2/v and similarly v/u_2 are in $L^\infty(\Omega)$. Applying Proposition 3.2 on U we deduce $I(u_2, v) \ge 0$, and $I(u_2, v) = 0$ if and only if u_2 and v are proportional on U. However, the positivity of the test function $(v^p - u_2^p)/u_2^{p-1} > 0$ on U implies, by (3.9) and (3.11), that

$$0 \le I(u_2, v) \le -\int_U B(u_2^p - v^p)\left(\frac{F(u_2)}{u_2^{p-1}} - \frac{F(v)}{v^{p-1}}\right)d\mu_f.$$

Being $F(t)/t^{p-1}$ strictly increasing on $\mathbb{R}^+$ and $B \ge 0$, we deduce

(3.13) $$B(u_2^p - v^p)\left(\frac{F(u_2)}{u_2^{p-1}} - \frac{F(v)}{v^{p-1}}\right) \ge 0$$

on U, whence $I(u_2, v) = 0$. We therefore conclude that $u_2 = cv$ on U, for some constant c which, because of the definition of U, satisfies $c > 1$. Using that $v = u_2$ on ∂U, we

necessarily have $v = u_2 = 0$ on ∂U, hence $U \equiv \Omega$. Substituting $u_2 = cv$ on Ω into (3.13) we deduce

$$B\left(c^p - 1\right) v^{p-1} \left(\frac{F(cv)}{(cv)^{p-1}} - \frac{F(v)}{v^{p-1}} \right) \equiv 0.$$

Since $v > 0$ on Ω and $F(t)/t^{p-1}$ is strictly increasing, $B \equiv 0$ and, from (3.11), $v > 0$ solves

$$\begin{cases} \Delta_{p,f} v + A(x)|v|^{p-2}v = 0 & \text{on } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Consequently, 0 admits a positive eigenfunction of Q_A'. By a result in [6], $\lambda_A(\Omega) = 0$, showing the validity of $ii)$. $\qquad\square$

Remark 3.2. We underline that, in the above proposition, the non-negativity of Q_A is not required. However, if $B \equiv 0$, u_2 turns out to be a positive solution of $Q_A(u) \geq 0$, and using Proposition 3.4 below we automatically have $\lambda_A(\Omega) \geq 0$.

In what follows we shall frequently use the next formula: for $I \subset \mathbb{R}$ and $\alpha \in C^2(I)$ with $\alpha' > 0$ on I, and for $u \in C^0(M) \cap W^{1,p}_{\mathrm{loc}}(M)$ with $u(M) \subset I$ we have, weakly on M,

$$(3.14) \qquad \Delta_{p,f}\alpha(u) = \alpha'(u)\left|\alpha'(u)\right|^{p-2}\Delta_{p,f}u + (p-1)\alpha''(u)\left|\alpha'(u)\right|^{p-2}\left|\nabla u\right|^p.$$

A second ingredient is the following existence result that goes under the name of the Allegretto-Piepenbrink theorem, see [5, 4, 33]. We include a proof of the next slightly more general version, for the sake of completeness.

Proposition 3.4. *Let $(M, \langle\,,\,\rangle)$ be a non-compact Riemannian manifold, $f \in C^\infty(M)$, $p \in (1, +\infty)$ and, for $V \in L^\infty_{\mathrm{loc}}(M)$, set Q_V', Q_V as in (2.8) and (2.7). Then, the following statements are equivalent:*

i) There exists $w \in C^0(M) \cap W^{1,p}_{\mathrm{loc}}(M)$, $w > 0$ weak solution of

$$(3.15) \qquad\qquad Q_V'(w) \geq 0 \qquad \text{on } M;$$

ii) There exists $u \in C^{1,\mu}_{\mathrm{loc}}(M)$, $u > 0$ weak solution of

$$(3.16) \qquad\qquad Q_V'(u) = 0 \qquad \text{on } M;$$

iii) $Q_V \geq 0$ on M.

iv) For each relatively compact domain $\Omega \Subset M$ with $C^{1,\alpha}$ boundary for some $\alpha \in (0,1)$, and for each $\xi \in C^{1,\alpha}(\partial\Omega)$, $\xi \geq 0$, there exists a unique solution $\varphi \in C^{1,\mu}(\overline{\Omega})$ of

$$(3.17) \qquad\qquad \begin{cases} Q_V'(\varphi) = 0 & \text{on } \Omega, \\ \varphi = \xi & \text{on } \partial\Omega \end{cases}$$

satisfying $\varphi \geq 0$ on Ω. Moreover, if $\xi \not\equiv 0$, then $\varphi > 0$ on Ω.

Proof. The scheme of proof is $ii) \Rightarrow i) \Rightarrow iii) \Rightarrow iv) \Rightarrow ii)$; Note that the first implication is trivial.

$i) \Rightarrow iii)$. It follows immediately from Proposition 3.1 and the non-negativity of $\mathcal{L}$.

$iii) \Rightarrow iv)$. By assumption $\lambda_V(M) \geq 0$; it follows that, for Ω as in $iv)$, by the monotonicity property for eigenvalues $\lambda_V(\Omega) > 0$. Hence, the variational problem associated to (3.17) is coercive and sequentially weakly lower-semicontinuous (see also Theorem 7.1 in Appendix). Therefore (3.17) admits a weak solution $\varphi \in W^{1,p}(\Omega)$. By the $C^{1,\mu}$-regularity of Theorem 3.1 we have that $\varphi \in C^{1,\mu}(\overline{\Omega})$ for some $\mu \in (0,1)$.

Moreover, by the local Harnack inequality of item (3), $\varphi > 0$ on $\overline{\Omega}$ whenever $\phi \geq 0$ on Ω, unless $\varphi \equiv 0$.

By contradiction suppose that φ is somewhere negative in Ω. Since $\xi \geq 0$, on $\partial\Omega$, $\varphi_- = -\min\{\varphi, 0\} \in \mathrm{Lip}_0(\Omega)$ and it is thus an admissible test function for (3.17) on Ω. We have

$$0 = Q'_V(\varphi)[-\varphi_-] \doteq -\int_\Omega \left\{ |\nabla\varphi|^{p-2}\langle\nabla\varphi, \nabla\varphi_-\rangle - V|\varphi|^{p-2}\varphi\varphi_- \right\} \mathrm{d}\mu_f \equiv pQ_V(\varphi_-)$$

and therefore $\lambda_V(\Omega) \leq 0$, a contradiction.

$iv) \Rightarrow ii)$. Choose an exhaustion $\{\Omega_j\}$ of M. Let $u_j > 0$, $u_j \in C^{1,\mu_j}(\overline{\Omega}_j) \subset W^{1,p}(\Omega_j)$ be a solution of

$$(3.18) \qquad \begin{cases} Q'_V(u_j) = 0 & \text{on } \Omega_j \\ u_j = 1 & \text{on } \partial\Omega_j. \end{cases}$$

Fix $x_0 \in \Omega_1$ and rescale u_j in such a way that $u_j(x_0) = 1$ for every j. By Theorem 3.1 3), $\{u_j\}$ is uniformly locally bounded in Ω, thus by Theorem 3.1 2) $\{u_j\}$ is uniformly locally bounded in $C^{1,\mu}(\Omega)$. It follows that $\{u_j\}$ has a subsequence converging weakly and pointwise to a weak solution $u \in C^{1,\mu}_{\mathrm{loc}}(M)$ of

$$\begin{cases} Q'_V(u) = 0 & \text{on } M \\ u(x_0) = 1. \end{cases}$$

Since $u \geq 0$ and $u \not\equiv 0$, again by 3) of Theorem 3.1 we deduce $u > 0$ on M. This shows the validity of $ii)$. $\qquad\square$

Next, we need a gluing result which we will call the pasting lemma. Although for $V \equiv 0$ this is somehow standard (a simple proof can be given by adapting Lemma 2.4 in [65]), the presence of a nonzero V makes things more delicate. First, we introduce some definitions. We recall that, given an open subset $\Omega \subset M$ possibly with non-compact closure, the space $W^{1,p}_{\mathrm{loc}}(\overline{\Omega})$ is the set of all functions u on Ω such that, for every relatively compact open set $U \Subset M$ with $U \cap \Omega \neq \emptyset$, $u \in W^{1,p}(\Omega \cap U)$. A function u in this space is thus well behaved on relatively compact portions of $\partial\Omega$, while no global control is assumed on the $W^{1,p}$ norm of u.

Lemma 3.1 (The pasting lemma). *Let $(M, \langle,\rangle)$ be a Riemannian manifold, $f \in C^\infty(M)$, $p \in (1, \infty)$, $V \in L^\infty_{\mathrm{loc}}(M)$. Let Ω_1, Ω_2 be open sets such that $\Omega_1 \subset \Omega_2$. For $j = 1, 2$, let $u_j \in C^0(\overline{\Omega}_j) \cap W^{1,p}_{\mathrm{loc}}(\overline{\Omega}_j)$ be a positive supersolution of Q'_V on Ω_j, that is, $Q'_V(u_j) \geq 0$ on Ω_j. If*

$$(3.19) \qquad\qquad u_2 \leq u_1 \qquad \text{on } \partial\Omega_1 \cap \Omega_2,$$

then the positive function

$$(3.20) \qquad\qquad u \doteq \begin{cases} \min\{u_1, u_2\} & \text{on } \overline{\Omega}_1 \\ u_2 & \text{on } \Omega_2 \backslash \Omega_1 \end{cases}$$

is in $W^{1,p}_{\mathrm{loc}}(\overline{\Omega}_2)$ and it satisfies $Q'_V(u) \geq 0$ on Ω_2.

When $\Omega_1 \equiv \Omega_2$, a general result of V.K. Le, [46], guarantees that $\min\{u_1, u_2\}$ is a supersolution. The pasting lemma can then be deduced by an approximation argument, along the lines described in [2], and we leave the details to the interested reader. In the Appendix below, we give a quite different proof by using the obstacle

problem for Q_V and the minimizing properties of its solutions, that might have an independent interest.

4. Criticality theory for Q_V, capacity and Hardy weights

The criticality theory for Q_V reveals an interesting scenario, and extends in a non-trivial way the parabolicity theory for the standard Laplacian and for the p-Laplacian (developed, among others, in [34, 64, 79]). Although a thorough description goes beyond the scope of this paper, nevertheless the validity of the pasting lemma gives us the opportunity to complement known results (especially those in [67, 68]) by relating them to a capacity theory for Q_V, see Theorem 4.1 below. We underline that, although the Q_V-capacity theory is investigated by following the same lines as those for the standard p-Laplacian, as in the previous results the presence of a nontrivial V makes things subtler.

Let $Q_V \geq 0$ on M, and fix a positive *supersolution* $g \in C^0(M) \cap W_{\mathrm{loc}}^{1,p}(M)$ of Q'_V, that is, a solution of

$$(4.1) \qquad\qquad\qquad\qquad Q'_V(g) \geq 0.$$

For each $K \Subset \Omega \Subset M$, K compact, Ω open, let

$$\mathcal{D}(K, \Omega, g) = \left\{ \varphi \in C^0(\overline{\Omega}) \cap W_0^{1,p}(\Omega) \; : \; \varphi \geq g \;\text{ in a neighbourhood of } K \right\}$$

and define the Q_V-**capacity**

$$\operatorname{cap}_{Q_V}(K, \Omega, g) \doteq \inf_{\varphi \in \mathcal{D}(K, \Omega, g)} Q_V(\varphi)$$

Clearly, $\operatorname{cap}_{Q_V}(K, \Omega, g)$ grows if we decrease Ω, as well as if we increase K. If $V \equiv 0$, it is customary to choose $g \equiv 1$ as solution of $\Delta_{p,f} g = 0$, and we recover the classical definition of capacity. We however underline that, for the next arguments to work, it is essential that the fixed g solves $Q'_V(g) \geq 0$ on M, for otherwise the basic properties needed in the next results could not hold.

Proposition 4.1. *Let K be the closure of an open domain and suppose that $\partial\Omega$, ∂K are of class $C^{1,\alpha}$ for some $\alpha \in (0,1)$. Then*

$$(4.2) \qquad\qquad\qquad\qquad \operatorname{cap}_{Q_V}(K, \Omega, g) = Q_V(u)$$

where u is the unique positive solution $u \in C^0(\overline{\Omega}) \cap W_0^{1,p}(\Omega) \cap C^{1,\mu}(\overline{\Omega} \backslash K)$ of

$$(4.3) \qquad\qquad \begin{cases} Q'_V(u) = 0 & \text{on } \Omega \backslash K, \\[2mm] u = g \quad \text{on } K, \qquad u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

We call such a solution u the Q_V-capacitor of (K, Ω, g).

Remark 4.1. The existence, uniqueness and positivity of u is granted by $iv)$ in Proposition 3.4. As for regularity, the interior estimate $u \in C_{\mathrm{loc}}^{1,\mu}(\overline{\Omega} \backslash K)$ follows by [78, 27], and the boundary continuity at $\partial K \cup \partial\Omega$ by Theorem 5.4, page 235 in [52]. The fact that $u \in W^{1,p}(\Omega)$ follows by standard theory of Sobolev functions[3].

---

[3]In fact, $u - g \in W^{1,p}(\Omega \backslash K)$ has zero trace on ∂K, thus it is the $W^{1,p}$-limit (and also, up to extracting a subsequence, the pointwise limit) of some sequence $\{\varphi_j\} \subset C^\infty(\overline{\Omega \backslash K})$ where $\varphi_j \equiv 0$ in a neighbourhood of ∂K. Extending φ_j to be zero on K we have that $g + \varphi_j$ is Cauchy in $W^{1,p}$ and pointwise convergent to u, thus $g + \varphi_j \to u \in W^{1,p}(\Omega)$.

Proof. Let $\varphi \in \mathcal{D}(K, \Omega, g)$. First, we claim that $\hat{\varphi} = \min\{\varphi, g\} \in \mathcal{D}(K, \Omega, g)$ solves $Q_V(\hat{\varphi}) \leq Q_V(\varphi)$, whence we can assume, without loss of generality, that $\varphi \leq g$ on Ω (and hence $\varphi = g$ on a neighbourhood of K).

Consider the open set $U = \{\varphi > g\} \Subset \Omega$. We test $Q'_V(g) \geq 0$ with the non-negative function $(\varphi^p - g^p)_+/g^{p-1} \in W_0^{1,p}(\Omega)$ and use the non-negativity of the Lagrangian in (3.5) to deduce

$$
\begin{aligned}
0 \;\leq\; & \int_U |\nabla g|^{p-2}\langle\nabla g, \nabla\left(\frac{\varphi^p}{g^{p-1}}\right)\rangle \mathrm{d}\mu_f - \int_U V\varphi^p \mathrm{d}\mu_f - pQ_V\big(g_{|U}\big) \\
=\; & p\int_U \left(\frac{\varphi}{g}\right)^{p-1} |\nabla g|^{p-2}\langle\nabla g, \nabla\varphi\rangle \mathrm{d}\mu_f - (p-1)\int_U \left(\frac{\varphi}{g}\right)^p |\nabla g|^p \mathrm{d}\mu_f \\
& - \int_U V\varphi^p \mathrm{d}\mu_f - pQ_V\big(g_{|U}\big) \\
=\; & \int_U |\nabla\varphi|^p \mathrm{d}\mu_f - \int_U \mathcal{L}(\varphi, g)\mathrm{d}\mu_f - \int_U V\varphi^p \mathrm{d}\mu_f - pQ_V\big(g_{|U}\big) \\
\leq\; & pQ_V\big(\varphi_{|U}\big) - pQ_V\big(g_{|U}\big),
\end{aligned}
$$

(4.4)

hence $Q_V(\hat{\varphi}_{|U}) = Q_V(g_{|U}) \leq Q_V(\varphi_{|U})$. Since $\varphi \equiv \hat{\varphi}$ on $\Omega\backslash U$, the claim follows. Let now $\varphi \in \mathcal{D}(K, \Omega, g)$ be such that $\varphi = g$ on K. By density, we can assume that $\varphi \in \mathrm{Lip}_0(\Omega)$. We therefore have $u - \varphi \in \mathrm{Lip}_0(\Omega\backslash K)$. Again by density, we can further assume that $\varphi = 0$ in a neighbourhood of $\partial\Omega$. Thus, testing $Q'_V(u) = 0$ with $(\varphi^p - u^p)/u^{p-1} \in \mathrm{Lip}_0(\Omega\backslash K)$ and proceeding as above we obtain

$$
\begin{aligned}
0 \;=\; & \int_{\Omega\backslash K} |\nabla u|^{p-2}\langle\nabla u, \nabla\left(\frac{\varphi^p}{u^{p-1}}\right)\rangle \mathrm{d}\mu_f - \int_{\Omega\backslash K} V\varphi^p \mathrm{d}\mu_f - pQ_V\big(u_{|\Omega\backslash K}\big) \\
=\; & p\int_{\Omega\backslash K} \left(\frac{\varphi}{u}\right)^{p-1} |\nabla u|^{p-2}\langle\nabla u, \nabla\varphi\rangle \mathrm{d}\mu_f - (p-1)\int_{\Omega\backslash K} \left(\frac{\varphi}{u}\right)^p |\nabla u|^p \mathrm{d}\mu_f \\
& - \int_{\Omega\backslash K} V\varphi^p \mathrm{d}\mu_f - pQ_V\big(u_{|\Omega\backslash K}\big) \\
=\; & \int_{\Omega\backslash K} |\nabla\varphi|^p \mathrm{d}\mu_f - \int_{\Omega\backslash K} \mathcal{L}(\varphi, u)\mathrm{d}\mu_f - \int_{\Omega\backslash K} V\varphi^p \mathrm{d}\mu_f - pQ_V\big(u_{|\Omega\backslash K}\big) \\
\leq\; & pQ_V\big(\varphi_{|\Omega\backslash K}\big) - pQ_V\big(u_{|\Omega\backslash K}\big).
\end{aligned}
$$

(4.5)

As $u = \varphi = g$ on K, we conclude $Q_V(u) \leq Q_V(\varphi)$ and whence $Q_V(u) \leq \mathrm{cap}(K, \Omega, g)$. Since u lies in the $W^{1,p}$ closure of $\mathcal{D}(K, \Omega, g)$, equality (4.2) follows. $\qquad\square$

Remark 4.2. By the pasting Lemma 3.1, note that u solving (4.3) is a supersolution on the whole Ω, that is, $Q'_V(u) \geq 0$ on Ω.

Proposition 4.2. *In the assumptions of the previous theorem, suppose that $Q'_V(g) = 0$ on a neighbourhood of K, and that ∂K is smooth. Then,*

$$
Q_V(u) = \frac{1}{p}\int_{\partial K} g\left[|\nabla g|^{p-2}\frac{\partial g}{\partial\nu} - |\nabla u|^{p-2}\frac{\partial u}{\partial\nu}\right]\mathrm{d}\sigma_f,
\tag{4.6}
$$

where ν is the unit normal to ∂K pointing outward of K.

Proof. Let $T \approx \partial K \times (-\varepsilon_0, \varepsilon_0) \Subset \Omega$ be a tubular neighbourhood of ∂K where Fermi coordinates are defined, and let $\rho(x)$ be the smooth signed distance from ∂K, that is, $\rho(x) = \mathrm{dist}(x, \partial K)$ if $x \notin K$, and $\rho(x) = -\mathrm{dist}(x, \partial K)$ if $x \in K$. Let $h \in \mathrm{Lip}(\mathbb{R}_0^+)$

be such that $h(t) = 0$ if $t \leq 0$, $h(t) = t$ for $t \in [0, 1]$ and $h(t) = 1$ for $t \geq 1$ and, for small $\varepsilon \in (0, \varepsilon_0)$, set $h_\varepsilon(t) \doteq h(t/\varepsilon)$. Applying (4.3) on $\Omega \backslash K$ to the test function $h_\varepsilon(\rho)u \in \mathrm{Lip}_0(\Omega \backslash K)$, using the coarea's formula and letting $\varepsilon \to 0$, since g is C^1 we deduce

$$(4.7) \qquad 0 = \lim_{\varepsilon \to 0} Q'_V(u)\big[h_\varepsilon(\rho)u\big] = pQ_V\big(u_{\Omega \backslash K}\big) + \int_{\partial K} u|\nabla u|^{p-2}\frac{\partial u}{\partial \nu}\mathrm{d}\sigma_f.$$

In a similar way, applying $Q'_V(g) = 0$ on K to the non-negative test function $h_\varepsilon(-\rho)g \in \mathrm{Lip}_0(K)$ and letting $\varepsilon \to 0$ we deduce that

$$(4.8) \qquad 0 = \lim_{\varepsilon \to 0} Q'_V(u)\big[h_\varepsilon(-\rho)g\big] = pQ_V\big(u_K\big) - \int_{\partial K} g|\nabla g|^{p-2}\frac{\partial g}{\partial \nu}\mathrm{d}\sigma_f.$$

Subtracting the two identities and using $u = g$ on ∂K yields (4.6). $\qquad \square$

Next, we consider the Q_V-capacity of K in the whole M:

$$(4.9) \qquad \mathcal{D}(K, g) = \Big\{\varphi \in C_c^0(M) \cap W_{\mathrm{loc}}^{1,p}(M) \; : \; \varphi \geq g \text{ in a neighbourhood of } K\Big\}$$

$$\mathrm{cap}_{Q_V}(K, g) \doteq \inf_{\varphi \in \mathcal{D}(K, g)} Q_V(\varphi).$$

Let $\{\Omega_j\}$ be an exhaustion of M with $K \Subset \Omega_1$. Then, from the definitions it readily follows that

$$\mathrm{cap}_{Q_V}(K, g) = \inf_j \mathrm{cap}_{Q_V}(K, \Omega_j, g) = \lim_{j \to +\infty} \mathrm{cap}_{Q_V}(K, \Omega_j, g).$$

If K is the closure of a open set and ∂K is of class $C^{1,\alpha}$ for some $\alpha \in (0, 1)$, let u_j be the Q_V-capacitor of (K, Ω_j, g). By Proposition 4.1,

$$(4.10) \qquad \mathrm{cap}_{Q_V}(K, g) = \lim_{j \to +\infty} Q_V(u_j).$$

Proposition 3.3 implies that $0 \leq u_j \leq u_{j+1} \leq g$ for each j, whence, by Dini theorem and elliptic estimates, u_j converges locally uniformly on M, in $W_{\mathrm{loc}}^{1,p}(M)$ and in the C^1 topology on $M \backslash K$ to a weak solution $u \in C^0(M) \cap W_{\mathrm{loc}}^{1,p}(M) \cap C_{\mathrm{loc}}^{1,\mu}(M \backslash K)$ of

$$(4.11) \qquad \begin{cases} Q'_V(u) = 0 & \text{on } M \backslash K, \\ u = g \quad \text{on } K, & 0 < u \leq g \quad \text{on } M \backslash K. \end{cases}$$

The pasting Lemma 3.1 guarantees that $Q'_V(u) \geq 0$ on the whole M. We call such a u the Q_V-capacitor of (K, g).

Proposition 4.3. *In the assumptions of Proposition 4.2, if $Q'_V(g) = 0$ on a neighbourhood of K,*

$$(4.12) \qquad \mathrm{cap}_{Q_V}(K, g) = \frac{1}{p}\int_{\partial K} g\left[|\nabla g|^{p-2}\frac{\partial g}{\partial \nu} - |\nabla u|^{p-2}\frac{\partial u}{\partial \nu}\right]\mathrm{d}\sigma_f,$$

where ν is the unit normal to ∂K pointing outward of K.

Proof. By Proposition 4.1, $\mathrm{cap}_{Q_V}(K, g) = \lim_j Q_V(u_j)$, where u_j is the Q_V-capacitor of (K, Ω_j, g). Now, since $u_j \to u$ in $C^1(\partial K)$, it is enough to pass to the limit in 4.6. $\qquad \square$

Next Theorem, the core of this section, relates the subcriticality of Q_V and the Q_V-capacity with other basic properties, which we will define below. It is due to Y. Pinchover and K. Tintarev (see [67]), and it is known in the literature as the ground state alternative. The authors state it for f constant and $M = \mathbb{R}^m$. Our contribution here is to include the Q_V-capacity properties to the above picture. However, since at

some point of [67] the authors use inequalities for which we found no counterpart in a manifold setting, we prefer to provide a full proof which sometimes uses arguments that differ from those in [67, 68], though keeping the same guidelines.

Definition 4.1. *For $V \in L^\infty_{\mathrm{loc}}(M)$, define Q_V as in (2.7) and let $\Omega \subseteq M$ be an open set.*

   iii) *Q_V has a **weighted spectral gap** on Ω if there exists $W \in C^0(\Omega)$, $W > 0$ on Ω such that*

(4.13)
$$\int_\Omega W(x)|\varphi|^p \mathrm{d}\mu_f \leq Q_V(\varphi) \qquad \forall \, \varphi \in \mathrm{Lip}_c(\Omega).$$

   iv) *A sequence $\{\eta_j\} \in L^\infty_c(\Omega) \cap W^{1,p}(\Omega)$ is said to be a **null sequence** if $\eta_j \geq 0$ a.e. for each j, $Q_V(\eta_j) \to 0$ as $j \to +\infty$ and there exists a relatively compact open set $B \Subset M$ and $C > 1$ such that $C^{-1} \leq \|\eta_j\|_{L^p(B)} \leq C$ for each j.*

   v) *A function $0 \leq \eta \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $\eta \geq 0$, $\eta \not\equiv 0$ is a **ground state** for Q_V on Ω if it is the $L^p_{\mathrm{loc}}(\Omega)$ limit of a null sequence.*

Theorem 4.1. *Let $(M, \langle \, , \, \rangle)$ be connected and non-compact, and for $V \in L^\infty_{\mathrm{loc}}(M)$ consider an operator $Q_V \geq 0$ on M. Then, either Q_V has a weighted spectral gap or a ground state on M, and the two possibilities mutually exclude. Moreover, the following properties are equivalent:*

   $(i)_\mathrm{S}$ *Q_V has a weighted spectral gap.*

   $(ii)_\mathrm{S}$ *Q_V is subcritical on M.*

   $(iii)_\mathrm{S}$ *There exist two positive solutions $u_1, u_2 \in C^0(M) \cap W^{1,p}_{\mathrm{loc}}(M)$ of $Q'_V(u) \geq 0$ which are not proportional.*

   $(iv)_\mathrm{S}$ *For some (any) $K \Subset M$ compact with non-empty interior, and for some (any) $0 < g \in C^0(M) \cap W^{1,p}_{\mathrm{loc}}(M)$ solving $Q'_V(g) \geq 0$, $\mathrm{cap}(K, g) > 0$.*

When Q_V has a ground state η, η solves $Q'_V(\eta) = 0$, and in particular $\eta \in C^{1,\mu}_{\mathrm{loc}}(M)$, $\eta > 0$ on M. Furthermore, the next properties are equivalent:

   $(i)_\mathrm{GS}$ *Q_V has a ground state.*

   $(ii)_\mathrm{GS}$ *All positive solutions $g \in C^0(M) \cap W^{1,p}_{\mathrm{loc}}(M)$ of $Q'_V(g) \geq 0$ are proportional; in particular, each positive supersolution is indeed a solution (hence, a ground state).*

   $(iii)_\mathrm{GS}$ *For some (any) $K \Subset M$ compact with non-empty interior, and for some (any) $0 < g \in C^0(M) \cap W^{1,p}_{\mathrm{loc}}(M)$ solving $Q'_V(g) \geq 0$, $\mathrm{cap}(K, g) = 0$.*

Proof. Hereafter, L^p and $W^{1,p}$ spaces will be considered with respect to the measure $\mathrm{d}\mu_f$. We begin with the following fact.

Claim 1: Fix an open set $U \Subset M$. If $\{\phi_j\} \subset L^\infty_c(M) \cap W^{1,p}(M)$ is such that $\|\phi_j\|_{L^p(U)} + Q_V(\phi_j) \leq C$ for some $C > 0$ independent of j, then $\{\phi_j\}$ is locally bounded in $W^{1,p}(M)$.

Proof of Claim 1. Up to replacing ϕ_j with $|\phi_j|$, we can assume that $\phi_j \geq 0$ a.e. on M. Using that Q_V is non-negative on M, choose a positive solution $g \in C^{1,\mu}_{\mathrm{loc}}(M)$ of $Q'_V(g) = 0$ and consider the Lagrangian representation of $Q_V(\phi_j)$:

$$Q_V(\phi_j) = \int_M \mathcal{L}(\phi_j, g) \mathrm{d}\mu_f.$$

Fix $\Omega \Subset M$ containing $\overline{U}$. Since $Q_V(\phi_j) \le C$ and $\mathcal{L}(\phi_j, g) \ge 0$ on M, it holds

$$(4.14) \qquad 0 \le \limsup_{j \to +\infty} \int_\Omega \mathcal{L}(\phi_j, g) \mathrm{d}\mu_f \le C.$$

By using Cauchy-Schwarz and Young inequalities on the third addendum of the expression of $\mathcal{L}(\phi_j, g)$ we deduce that, for each $\varepsilon > 0$,

$$(4.15) \qquad \mathcal{L}(\phi_j, g) \ge (1 - \varepsilon^p)|\nabla \phi_j|^p + (p-1)\left(1 - \varepsilon^{-\frac{p}{p-1}}\right)\left(\frac{\phi_j}{g}\right)^p |\nabla g|^p.$$

In our assumptions, $|\nabla \log g| \in L^\infty(\Omega)$. Setting $\varepsilon = 1/2$ in (4.15), integrating and using (4.14) we get
(4.16)
$$\limsup_{j \to +\infty}\left[(1 - 2^{-p})\int_\Omega |\nabla \phi_j|^p \mathrm{d}\mu_f + (p-1)\left(1 - 2^{\frac{p}{p-1}}\right)\|\nabla \log g\|_{L^\infty(\Omega)}\int_\Omega |\phi_j|^p \mathrm{d}\mu_f\right] \le C.$$

From this inequality we argue the existence of constants $C_1, C_2 > 0$ independent of j such that

$$(4.17) \qquad \|\nabla \phi_j\|_{L^p(\Omega)} \le C_1 \|\phi_j\|_{L^p(\Omega)} + C_2.$$

Suppose now, by contradiction, that $\{\phi_j\}$ is not bounded in $W^{1,p}(\Omega)$. By (4.17), $\|\phi_j\|_{L^p(\Omega)}$ diverge. Set $\bar{\phi}_j = \phi_j/\|\phi_j\|_{L^p(\Omega)}$. Then, by (4.17) $\{\bar{\phi}_j\}$ is bounded in $W^{1,p}(\Omega)$, thus it has a subsequence (still called $\{\bar{\phi}_j\}$) converging weakly in $W^{1,p}$, strongly in L^p and pointwise almost everywhere to some non-negative function Ψ satisfying $\|\Psi\|_{L^p(\Omega)} = \lim_j \|\bar{\phi}_j\|_{L^p(\Omega)} = 1$. Since $|\nabla \log g| \in L^\infty(\Omega)$ we straightforwardly have that

$$(4.18) \qquad \begin{array}{ll} (i) & \bar{\phi}_j^p |\nabla \log g|^p \to \Psi^p |\nabla \log g|^p \qquad \text{in } L^1(\Omega), \\[2mm] (ii) & \bar{\phi}_j^{p-1} |\nabla \log g|^{p-1} \to \Psi^{p-1} |\nabla \log g|^{p-1} \qquad \text{in } L^{\frac{p}{p-1}}(\Omega). \end{array}$$

We just prove (ii), the other being a consequence of the proof. By the elementary inequalities

$$(4.19) \qquad (x-y)^\beta \le x^\beta - y^\beta \le C x^{\beta-1}(x-y) \qquad \text{for } y \in [0,x] \text{ and } \beta > 1,$$

for some C depending on β, with a final application of Hölder inequality we deduce that
(4.20)
$$\|(\bar{\phi}_j^{p-1} - \Psi^{p-1})|\nabla \log g|^{p-1}\|_{L^{p/(p-1)}(\Omega)}^{p/(p-1)} \le \|\nabla \log g\|_{L^\infty(\Omega)}^p \int_\Omega \left|\bar{\phi}_j^{p-1} - \Psi^{p-1}\right|^{\frac{p}{p-1}} \mathrm{d}\mu_f$$

$$\le \|\nabla \log g\|_{L^\infty(\Omega)}^p \int_\Omega |\bar{\phi}_j^p - \Psi^p| \mathrm{d}\mu_f \le C\|\nabla \log g\|_{L^\infty(\Omega)}^p \int_\Omega \left(\max\{\bar{\phi}_j, \Psi\}\right)^{p-1} |\bar{\phi}_j - \Psi| \mathrm{d}\mu_f,$$

$$\le C\|\nabla \log g\|_{L^\infty(\Omega)}^p \|\bar{\phi}_j + \Psi\|_{L^p(\Omega)}^{(p-1)/p} \|\bar{\phi}_j - \Psi\|_{L^p(\Omega)},$$

and this latter goes to zero $\bar{\phi}_j \to \Psi$ in $L^p(\Omega)$. Now, coupling (ii) with the weak convergence of $\bar{\phi}_j$ to Ψ in $W^{1,p}(\Omega)$, we get

$$\int_\Omega \left(\frac{\bar{\phi}_j}{g}\right)^{p-1} |\nabla g|^{p-2}\langle \nabla g, \nabla \bar{\phi}_j\rangle \mathrm{d}\mu_f \longrightarrow \int_\Omega \left(\frac{\Psi}{g}\right)^{p-1} |\nabla g|^{p-2}\langle \nabla g, \nabla \Psi\rangle \mathrm{d}\mu_f,$$

so that, combining $(i), (ii)$ and the weak lower semicontinuity of $\|\cdot\|_{W^{1,p}(\Omega)}$,

$$0 \le \int_\Omega \mathcal{L}(\Psi, g)\mathrm{d}\mu_f \le \liminf_{j \to +\infty}\int_\Omega \mathcal{L}(\bar{\phi}_j, g)\mathrm{d}\mu_f = \|\phi_j\|_{L^p(\Omega)}^{-p}\int_\Omega \mathcal{L}(\phi_j, g)\mathrm{d}\mu_f \to 0$$

as $j \to +\infty$. Hence $\mathcal{L}(\Psi, g) = 0$ on Ω, so by Proposition 3.1 $\Psi = cg$ for some constant $c \geq 0$. However, since $\|\phi_j\|_{L^p(U)} \leq C$ for each j,

$$0 \leq \|\Psi\|_{L^p(U)} = \lim_{j \to +\infty} \|\bar\phi_j\|_{L^p(U)} = \lim_{j \to +\infty} \frac{\|\phi_j\|_{L^p(U)}}{\|\phi_j\|_{L^p(\Omega)}} \leq \limsup_{j \to +\infty} \frac{C}{\|\phi_j\|_{L^p(\Omega)}} = 0,$$

hence $c = 0$ and $\Psi \equiv 0$ on Ω, contradicting the fact that $\|\Psi\|_{L^p(\Omega)} = 1$. This concludes the proof of the claim.

Claim 2: Either Q_V has a weighted spectral gap or a ground state, but not both.

Proof of Claim 2. For a relatively compact open set U, define the L^p-capacity c as follows:

$$\mathcal{D}(U) = \left\{ \varphi \in L_c^\infty(M) \cap W^{1,p}(M) : \|\varphi\|_{L^p(U)} = 1 \right\}, \qquad c(U) = \inf_{\varphi \in \mathcal{D}(U)} Q_V(\varphi),$$

where, as usual, the L^p-norm is computed with respect to $\mathrm{d}\mu_f$. Then, two mutually exclusive cases may occur: ether $c(U) > 0$ for each U, or $c(U) = 0$ for some U. In the first case, it is easy to see that Q_V has a weighted spectral gap. Indeed, let $\{U_j\}$ be a locally finite covering of M via relatively compact, open sets, set $c_j = c(U_j) > 0$ and let $\{t_j\}$ be a sequence of positive numbers such that $\sum_j t_j = 1$. For each $\varphi \in \mathrm{Lip}_c(M)$, by the definition of c_j we get $Q_V(\varphi) \geq c_j \|\varphi\|_{L^p(U_j)}^p$, and summing up we get

$$Q_V(\varphi) = \left[\sum_{j=1}^{+\infty} t_j \right] Q_V(\varphi) \geq \sum_{j=1}^{+\infty} t_j c_j \int_{U_j} |\varphi|^p \mathrm{d}\mu_f = \int_M \hat{W} |\varphi|^p \mathrm{d}\mu_f,$$

where

$$\hat{W}(x) = \sum_{j=1}^{+\infty} t_j c_j 1_{U_j}(x) > 0 \qquad \text{on } M.$$

We can thus choose a weighted spectral gap W by taking a positive, continuous function W not exceeding $\hat{W}$.

Now, suppose that $c(U) = 0$ for some U. We show that there exists a ground state. Indeed, by the definition of $c(U)$ there exists $\{\eta_j\} \subset L_c^\infty(M) \cap W^{1,p}(M)$ such that $\|\eta_j\|_{L^p(U)} = 1$ and $Q_V(\eta_j) \to 0$. Up to replacing η_j with $|\eta_j|$, we can suppose that $\eta_j \geq 0$ a.e. on M. By Claim 1, η_j is locally bounded in $W^{1,p}$, and a Cantor type argument on an increasing exhaustion of M ensures the existence of a subsequence, still called $\{\eta_j\}$, converging weakly in $W_{\mathrm{loc}}^{1,p}(M)$ and strongly in $L_{\mathrm{loc}}^p(M)$ to some function $\eta \in W_{\mathrm{loc}}^{1,p}(M)$. By definition, η is a ground state.

We now show our desired equivalences. To establish those involving $(iv)_{\mathrm{S}}$ and $(iii)_{\mathrm{GS}}$, where the "some/all" alternative appears, then we will always assume the weakest alternative and prove the strongest one.

$(\mathbf{i})_{\mathrm{GS}} \Rightarrow (\mathbf{ii})_{\mathrm{GS}}$. Let $\eta \geq 0$ be a ground state, and let $\{\eta_j\} \subset L_c^\infty(M) \cap W^{1,p}(M)$ be a null sequence converging in L_{loc}^p to η. Then, by Claim 1 $\{\eta_j\}$ is locally bounded in $W^{1,p}$, thus up to passing to a subsequence we can assume that also $\eta_j \to \eta$ weakly in $W_{\mathrm{loc}}^{1,p}$. Consider a positive solution $g \in C^0(M) \cap W_{\mathrm{loc}}^{1,p}(M)$ of $Q'_V(g) \geq 0$. Fix $\Omega \Subset M$. Up to multiply g by a large positive constant, we can suppose that $\{x \in \Omega : \eta(x) < g(x)\}$ has positive measure. Let $\bar\eta_j = \min\{\eta_j, g\}$, and note that $\{\bar\eta_j\}$ is still a null sequence,

converging weakly in $W^{1,p}_{\mathrm{loc}}$ to $\bar{\eta} = \min\{\eta, g\}$. Consider the Lagrangian representation

$$(4.21) \qquad Q_V(\bar{\eta}_j) \geq \int_M \mathcal{L}(\bar{\eta}_j, g) \mathrm{d}\mu_f$$

guaranteed by Proposition 3.1. We claim that

$$(4.22) \qquad \frac{\bar{\eta}_j^p}{g^{p-1}} \to \frac{\bar{\eta}^p}{g^{p-1}} \qquad \text{weakly in } W^{1,p}(\Omega).$$

To see this, we follows arguments analogous to those yielding (4.18). Choose a constant $c_\Omega > 0$ large enough to satisfy $g \geq c_\Omega^{-1}$ on Ω and $\bar{\eta}_j \leq c_\Omega$. By a direct computation, $\|\bar{\eta}_j^p/g^{p-1}\|_{W^{1,p}(\Omega)}$ is uniformly bounded so that, by density, it is enough to check the weak convergence with test function $\varphi \in \mathrm{Lip}_0(\Omega)$. From

$$\left| \int_\Omega \frac{\bar{\eta}_j^p - \bar{\eta}^p}{g^{p-1}} \varphi \mathrm{d}\mu_f \right| \leq c_\Omega^{p-1} \|\varphi\|_{L^\infty(\Omega)} \int_\Omega \left| \bar{\eta}_j^p - \bar{\eta}^p \right| \mathrm{d}\mu_f,$$

applying the inequalities in (4.20) from the second line to the end we deduce that $\bar{\eta}_j^p/g^{p-1} \to \bar{\eta}^p/g^{p-1}$ weakly in $L^p(\Omega)$. Regarding the gradient part,

$$\left| \int_\Omega \langle \nabla \left(\frac{\bar{\eta}_j^p - \bar{\eta}^p}{g^{p-1}} \right), \nabla \varphi \rangle \mathrm{d}\mu_f \right| \leq (\mathrm{I}) + (\mathrm{II}),$$

where

$$
\begin{aligned}
(\mathrm{I}) &= (p-1) \int_\Omega |\bar{\eta}_j^p - \bar{\eta}^p| |\nabla g| \frac{|\nabla \varphi|}{g^p} \\
(\mathrm{II}) &= p \left| \int_\Omega \langle \left(\frac{\bar{\eta}_j}{g} \right)^{p-1} \nabla \varphi, \nabla \bar{\eta}_j \rangle \mathrm{d}\mu_f - \int_\Omega \langle \left(\frac{\bar{\eta}}{g} \right)^{p-1} \nabla \varphi, \nabla \bar{\eta} \rangle \mathrm{d}\mu_f \right|.
\end{aligned}
$$

As for (I), by Hölder and both the inequalities in (4.19) we deduce

$$
\begin{aligned}
\frac{1}{p-1}(\mathrm{I}) &\leq c_\Omega^p \|\nabla \varphi\|_{L^\infty(\Omega)} \|\nabla g\|_{L^p(\Omega)} \left(\int_\Omega |\bar{\eta}_j^p - \bar{\eta}^p|^{\frac{p}{p-1}} \mathrm{d}\mu_f \right)^{\frac{p-1}{p}} \\
&\leq c_\Omega^p \|\nabla \varphi\|_{L^\infty(\Omega)} \|\nabla g\|_{L^p(\Omega)} \left(\int_\Omega \left| \bar{\eta}_j^{\frac{p^2}{p-1}} - \bar{\eta}^{\frac{p^2}{p-1}} \right| \mathrm{d}\mu_f \right)^{\frac{p-1}{p}} \\
&\leq c_\Omega^p \|\nabla \varphi\|_{L^\infty(\Omega)} \|\nabla g\|_{L^p(\Omega)} C^{\frac{p-1}{p}} \left(\int_\Omega \max\{\bar{\eta}_j, \bar{\eta}\}^{\frac{p^2}{p-1}-1} |\bar{\eta}_j - \bar{\eta}| \mathrm{d}\mu_f \right)^{\frac{p-1}{p}} \\
&\leq c_\Omega^{p+\frac{p}{p-1}-\frac{p-1}{p}} \|\nabla \varphi\|_{L^\infty(\Omega)} \|\nabla g\|_{L^p(\Omega)} C^{\frac{p-1}{p}} \left(\int_\Omega |\bar{\eta}_j - \bar{\eta}| \mathrm{d}\mu_f \right)^{\frac{p-1}{p}}.
\end{aligned}
$$

Again by Hölder inequality, the last integral goes to zero since $\bar{\eta}_j \to \bar{\eta}$ in $L^p(\Omega)$, which shows that $(\mathrm{I}) \to 0$ as $j \to +\infty$. Finally, we consider (II). To show that $(\mathrm{II}) \to 0$, using the weak convergence if $\bar{\eta}_j$ to $\bar{\eta}$ in $W^{1,p}(\Omega)$ and standard estimates it is enough to prove that

$$\left(\frac{\bar{\eta}_j}{g} \right)^{p-1} \nabla \varphi \to \left(\frac{\bar{\eta}}{g} \right)^{p-1} \nabla \varphi \qquad \text{strongly in } L^{\frac{p}{p-1}}(\Omega).$$

This follows from

$$\int_\Omega |\nabla \varphi|^{\frac{p}{p-1}} \left| \left(\frac{\bar{\eta}_j}{g} \right)^{p-1} - \left(\frac{\bar{\eta}}{g} \right)^{p-1} \right|^{\frac{p}{p-1}} \mathrm{d}\mu_f \leq c_\Omega^p \|\nabla \varphi\|_{L^\infty(\Omega)}^{\frac{p}{p-1}} \int_\Omega \left| \bar{\eta}_j^{p-1} - \bar{\eta}^{p-1} \right|^{\frac{p}{p-1}} \mathrm{d}\mu_f$$

and inequalities analogous to those in the second and third lines of (4.20). This concludes the proof of (4.22).

Now, (4.22) implies that

$$\int_\Omega |\nabla g|^{p-2} \langle \nabla g, \nabla \left(\frac{\bar\eta_j^p}{g^{p-1}} \right) \rangle \mathrm{d}\mu_f \longrightarrow \int_\Omega |\nabla g|^{p-2} \langle \nabla g, \nabla \left(\frac{\bar\eta^p}{g^{p-1}} \right) \rangle \mathrm{d}\mu_f,$$

so that integrating on Ω the Lagrangian identity

$$\mathcal{L}(\bar\eta_j, g) = |\nabla \bar\eta_j|^p - |\nabla g|^{p-2} \langle \nabla g, \nabla \left(\frac{\bar\eta_j^p}{g^{p-1}} \right) \rangle$$

and using the weak lower semicontinuity of the $W^{1,p}$ norm we deduce

(4.23) $$0 \le \int_\Omega \mathcal{L}(\bar\eta, g) \mathrm{d}\mu_f \le \liminf_{j \to +\infty} \int_\Omega \mathcal{L}(\bar\eta_j, g) \mathrm{d}\mu_f.$$

Inequalities (4.21) and $\mathcal{L}(\bar\eta_j, g) \ge 0$ on M then imply

$$0 \le \int_\Omega \mathcal{L}(\bar\eta_j, g) \mathrm{d}\mu_f \le \int_M \mathcal{L}(\bar\eta_j, g) \mathrm{d}\mu_f \le Q_V(\bar\eta_j) \longrightarrow 0$$

as $j \to +\infty$, so we conclude by (4.23) that $\mathcal{L}(\bar\eta, g) \equiv 0$ on Ω. By Proposition 3.1, $\bar\eta = cg$ on Ω for some $c \ge 0$. In fact, $c > 0$ since $\|\eta\|_{L^p(U)} = \lim_j \|\eta_j\|_{L^p(U)} \ne 0$, and $c < 1$ since $\{x \in \Omega : \eta(x) < g(x)\}$ has positive measure. Therefore, $\bar\eta \equiv \eta$ on Ω, showing that g is a positive multiple of η on Ω, that is, $(ii)_{\mathrm{GS}}$ holds.

Claim 3.   When Q_V has a ground state η, $\eta \in C^{1,\mu}_{\mathrm{loc}}(M)$, is positive and solves $Q'_V(\eta) = 0$.

Proof of Claim 3.   By $(i)_{\mathrm{GS}} \Rightarrow (ii)_{\mathrm{GS}}$, all solutions $g \in C^0(M) \cap W^{1,p}_{\mathrm{loc}}(M)$ of $Q'_V(g) \ge 0$ are proportional. Choosing g to be a positive solution of $Q'_V(g) = 0$ (which exists by Proposition 3.4), we get that $\eta > 0$ solves $Q'_V(\eta) = 0$ and $\eta \in C^{1,\mu}_{\mathrm{loc}}(M)$, proving the claim.

$(iii)_{\mathrm{GS}} \Rightarrow (i)_{\mathrm{GS}}$. Since $\mathrm{Int}(K) \ne \emptyset$ we select a closed smooth geodesic ball B contained in $\mathrm{Int}(K)$. By the monotonicity of capacity, $\mathrm{cap}_{Q_V}(B, g) = 0$. Fix an exhaustion $\{\Omega\}_j$ of M with $B \Subset \Omega_1$, let η_j be the Q_V-capacitor of (B, Ω_j, g) extended with zero outside Ω_j, and let η be the capacitor of (B, g). Then, (4.10) ensures that $Q_V(\eta_j) \to \mathrm{cap}_{Q_V}(B, g) = 0$, and since $\eta_j \to \eta$ in $L^p_{\mathrm{loc}}(M)$ we deduce that η is the desired ground state.

$(iv)_{\mathrm{S}} \Rightarrow (iii)_{\mathrm{S}}$. Up to enlarging K (capacity increases), we can assume that K is the closure of a relatively compact open set with smooth boundary. Now, consider a solution $\bar g \in C^{1,\mu}_{\mathrm{loc}}(M)$ of $Q'_V(\bar g) = 0$ on M. Up to multiplying $\bar g$ by a small positive constant, we can suppose that $\bar g \le g$ on K. By the very definition of Q_V-capacity,

$$\mathrm{cap}_{Q_V}(K, g) \ge \mathrm{cap}_{Q_V}(K, \bar g).$$

We claim that $\mathrm{cap}_{Q_V}(K, \bar g) > 0$. Indeed, otherwise, by $(iii)_{\mathrm{GS}} \Rightarrow (i)_{\mathrm{GS}}$ just proved, Q_V has a ground state and so all solutions $u \in C^0(M) \cap W^{1,p}_{\mathrm{loc}}(M)$ of $Q'_V(u) \ge 0$ are proportional. In particular, $\bar g = cg$ for some constant $c > 0$, which implies $\mathrm{cap}_{Q_V}(K, g) = c^{-p} \mathrm{cap}_{Q_V}(K, \bar g) = 0$ contradicting our assumptions. Now, since $\bar g$ solves $Q'_V(\bar g) = 0$ and $\mathrm{cap}_{Q_V}(K, \bar g) > 0$, applying formula (4.12) in Proposition 4.3 with $\bar g$ replacing g we deduce that necessarily the Q_V-capacitor u of $(K, \bar g)$ is different from $\bar g$. As u solves $Q'_V(u) \ge 0$ on M and $u = g$ on K, u and $\bar g$ are two non-proportional

solutions of $Q'_V(v) \geq 0$, proving $(iii)_{\mathrm{S}}$.

$(\mathbf{ii})_{\mathrm{GS}} \Rightarrow (\mathbf{iii})_{\mathrm{GS}}$. If, by contradiction, $\mathrm{cap}_{Q_V}(K, g) > 0$ for some (K, g), then by $(iv)_{\mathrm{S}} \Rightarrow (iii)_{\mathrm{S}}$ above there would exists two non-proportional solutions of $Q'_V(g) \geq 0$, contradicting $(ii)_{\mathrm{GS}}$.

Now, we have concluded the equivalence $(i)_{\mathrm{GS}} \Leftrightarrow (ii)_{\mathrm{GS}} \Leftrightarrow (iii)_{\mathrm{GS}}$ in the "ground state" part of the theorem. Combining with Claim 2, we automatically have the validity of
$(\mathbf{i})_{\mathrm{S}} \Leftrightarrow (\mathbf{iii})_{\mathrm{S}} \Leftrightarrow (\mathbf{iv})_{\mathrm{S}}$.

We are thus left to show $(i)_{\mathrm{S}} \Leftrightarrow (ii)_{\mathrm{S}}$. Having observed that $(i)_{\mathrm{S}} \Rightarrow (ii)_{\mathrm{S}}$ is obvious (set $w = W$), to conclude we prove that

$(\mathbf{ii})_{\mathrm{S}} \Rightarrow (\mathbf{i})_{\mathrm{S}}$. Suppose by contradiction that $(i)_{\mathrm{S}}$ is not true. Then, by Claim 2, Q_V has a ground state η, which is positive on M and solves $Q'_V(\eta) = 0$. Fix a smooth open set U such that $w \not\equiv 0$ on U, let $\{\Omega_j\} \uparrow M$ be an exhaustion of M with $U \Subset \Omega_1$, and let η_j be the Q_V-capacitor of (U, Ω_j, η). Then, by the equivalence $(i)_{\mathrm{GS}} \Leftrightarrow (iii)_{\mathrm{GS}}$ (in particular, by the proof of $(iii)_{\mathrm{GS}} \Rightarrow (i)_{\mathrm{GS}}$), $\{\eta_j\}$ is a null sequence. By the subcriticality assumption and since $\eta_j = \eta$ on U,

$$\int_U w|\eta|^p \mathrm{d}\mu_f \leq \int_M w|\eta_j|^p \mathrm{d}\mu_f \leq Q_V(\eta_j) \to 0$$

as $j \to +\infty$, contradicting the fact that $\eta > 0$ on M and $w \not\equiv 0$ on U. $\qquad\square$

Remark 4.3. We can now give a simple proof of the fact that the positivity of the Yamabe invariant $Y(M)$ in Theorem 1.1 implies the subcriticality of the conformal Laplacian $L_{\langle,\rangle}$. Indeed, suppose the contrary. Then, by Theorem 4.1, there exist a ground state $\eta > 0$ and a null sequence $\{\eta_j\}$ locally L^2-converging to η. From the chain of inequalities

$$Y(M)\left(\int_M |\eta_j|^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} \leq \int_M \left[|\nabla \eta_j|^2 + \frac{s(x)}{c_m}\eta_j^2\right] \to 0 \quad \text{as } j \to +\infty,$$

we deduce that $\eta_j \to 0$ in $L^{\frac{2m}{m-2}}(M)$, hence locally in $L^2(M)$, a contradiction.

It is now immediate to prove the equivalence partly mentioned in the Introduction (Proposition 2.1). We suggest the interested reader to consult the very recent [23, 24] for an investigation on the optimality of the Hardy weights given in items 5) and 6) below.

Proposition 4.4. *Let M^m be a Riemannian manifold of dimension $m \geq 2$, $p \in (1, +\infty)$ and $f \in C^\infty(M)$. The following properties are equivalent:*

1) *There exists a positive, non-constant solution $g \in C^0(M) \cap W^{1,p}_{\mathrm{loc}}(M)$ of $\Delta_{p,f} g \leq 0$.*

2) *For some (any) compact $K \subset M$ with non-empty interior, and for some (any) solution g of $\Delta_{p,f} g \leq 0$, $\mathrm{cap}(K, g) > 0$.*

3) *Q_0 is subcritical on M: there exists $w \in L^1_{\mathrm{loc}}(M)$, $w \geq 0$, $w \not\equiv 0$ on M such that*

$$(4.24) \qquad \int_M w(x)|\varphi|^p \mathrm{d}\mu_f \leq \int_M |\nabla \varphi|^p \mathrm{d}\mu_f \qquad \forall \varphi \in \mathrm{Lip}_c(M).$$

4) Q_0 has a weighted spectral gap on M: there exists $W \in C^0(M)$, $W > 0$ on M such that

$$(4.25) \qquad \int_M W(x)|\varphi|^p \mathrm{d}\mu_f \leq \int_M |\nabla\varphi|^p \mathrm{d}\mu_f \qquad \forall\, \varphi \in \mathrm{Lip}_c(M).$$

5) For each non-constant, positive weak solution $u \in W^{1,p}_{\mathrm{loc}}(M)$ of $\Delta_{p,f} u \leq 0$ the following Hardy type inequality holds:

$$(4.26) \qquad \left(\frac{p-1}{p}\right)^p \int_M \frac{|\nabla u|^p}{u^p}|\varphi|^p \mathrm{d}\mu_f \leq \int_M |\nabla\varphi|^p \mathrm{d}\mu_f \qquad \forall\, \varphi \in \mathrm{Lip}_c(M).$$

6) For each $y \in M$, $-\Delta_{p,f}$ has a positive Green kernel $\mathcal{G}(x,y)$ and the following Hardy inequality holds:

$$(4.27) \qquad \left(\frac{p-1}{p}\right)^p \int_M \frac{|\nabla_x \mathcal{G}|^p}{\mathcal{G}^p}|\varphi|^p \mathrm{d}\mu_f \leq \int_M |\nabla\varphi|^p \mathrm{d}\mu_f \qquad \forall\, \varphi \in \mathrm{Lip}_c(M).$$

Remark 4.4. Here, a Green kernel $\mathcal{G}(x,y)$ means a distributional solution of

$$\Delta_{p,f}\mathcal{G}(\cdot,y) = -\delta_y,$$

where δ_y is the Dirac delta function at y.

Proof. The equivalence 1) $\Leftrightarrow$ 2) $\Leftrightarrow$ 3) $\Leftrightarrow$ 4) is the particular case $V \equiv 0$ of Theorem 4.1. Indeed, as any positive constant is a solution of $\Delta_{p,f} u \leq 0$, 1) can be rephrased as the existence of two non-proportional solutions of $\Delta_{p,f} u \leq 0$. Clearly 5) $\Rightarrow$ 3), thus 5) yields that Q_0 is subcritical on M; therefore, we just need to show that 1) $\Rightarrow$ 5). Towards this aim observe that if $g \in C^0(M) \cap \in W^{1,p}_{\mathrm{loc}}(M)$, $g > 0$ is a positive, non-constant solution of $\Delta_{p,f} g \leq 0$, then by (3.14) $z = g^{\frac{p-1}{p}}$ is a positive weak solution of

$$\Delta_{p,f} z + \left(\frac{p-1}{p}\right)^p \frac{|\nabla u|^p}{u^p} z^{p-1} \leq 0 \qquad \text{on } M.$$

Proposition 3.1 and the non-negativity of $\mathcal{L}$ then imply that Q_V is non-negative for

$$V = \left(\frac{p-1}{p}\right)^p \frac{|\nabla u|^p}{u^p},$$

which gives (4.26).

To prove 6) $\Rightarrow$ 1), take a positive Green kernel $\mathcal{G}(x,y)$ for $-\Delta_{p,f}$ at y. For large $c > 0$, the function $G_c(x) = \min\{c, \mathcal{G}(x,y)\}$ is a non-constant, positive weak solution of $\Delta_{p,f} G_c \leq 0$ by the pasting Lemma 3.1, showing 1). Vice versa, if 1) holds, then by the equivalence 1) $\Leftrightarrow$ 2) we deduce that each compact set has positive capacity with respect to the "standard" supersolution $g \equiv 1$. This implies, by results in [38] and [39], that for each $y \in M$ there exists a positive Green kernel $\mathcal{G}(x,y)$ for $-\Delta_{p,f}$. Defining again G_c as above, $\Delta_{p,f} G_c \leq 0$ and the equivalence 1) $\Leftrightarrow$ 5) ensures that the Hardy inequality
(4.28)
$$\left(\frac{p-1}{p}\right)^p \int_M \frac{|\nabla G_c|^p}{G_c^p}|\varphi|^p \mathrm{d}\mu_f \leq \int_M |\nabla\varphi|^p \mathrm{d}\mu_f \qquad \text{holds for each } \varphi \in \mathrm{Lip}_c(M),$$

and (4.27) follows by letting $c \to +\infty$ and using the monotone convergence theorem. $\qquad\square$

Remark 4.5. As a consequence of [75, 76] and Theorem 1.1 in [44] (see also [45]), if $M = \mathbb{R}^m$ each positive Green kernel $\mathcal{G}$ satisfies

$$(4.29) \qquad \frac{|\nabla_x \mathcal{G}|^p}{\mathcal{G}^p}(x, y) \sim \begin{cases} C \operatorname{dist}(x, y)^{-m} \log^{-m} \operatorname{dist}(x, y) & \text{if } p = m, \\[2mm] C \operatorname{dist}(x, y)^{-p} & \text{if } p < m \end{cases}$$

as $\operatorname{dist}(x, y) \to 0$, for some explicit $C > 0$. In the linear case $p = 2$, the first order expansions in [75, 76] guarantee the validity of (4.29) on each Riemannian manifold. On the contrary, when $p \neq 2$, the scaling arguments used in [44] are typical of the Euclidean setting and, although we believe (4.29) to be true in general, to the best of our knowledge there is still no proof of (4.29) in a manifold setting.

The above proposition gives a useful, simple criterion to check the subcriticality of some Q_V.

Proposition 4.5. Let $(M, \langle\,,\,\rangle)$ be a Riemannian manifold, $f \in C^0(M)$ and $p > 1$. Let $V \in L^\infty_{\mathrm{loc}}(M)$. Suppose that Q_0 is subcritical on M. If, for some Hardy weight $\hat{w}$, it holds $V \leq \hat{w}$ and $V \not\equiv \hat{w}$, then Q_V is subcritical.

Proof. Indeed, using (4.24),

$$Q_V(\varphi) \doteq \int_M |\nabla \varphi|^p \mathrm{d}\mu_f - \int_M V |\varphi|^p \mathrm{d}\mu_f \geq \int_M (\hat{w} - V)|\varphi|^p \mathrm{d}\mu_f;$$

consequently, w in (2.10) can be chosen to be $\hat{w} - V$, proving the subcriticality of Q_V. $\qquad\square$

Remark 4.6. The alternative in Theorem 4.1 is also related to the existence of a global positive Green kernel for Q'_V. It has been shown in [67] (Theorems 5.4 and 5.5) that, on $\mathbb{R}^m$, Q_V has a weighted spectral gap if and only if Q'_V admits a global positive Green kernel. The result depends on Lemma 5.1 therein, which has been proved via a rescaling argument typical of the Euclidean space, and calls for a different strategy in a manifold setting. However, in the linear case we can easily obtain the above equivalence on general manifolds by relating Q'_V to a weighted Laplacian via a standard transformation. In fact, suppose that $p = 2$ and that Q_V is non-negative. Let g be a positive solution of $Q'_V(g) = 0$. Then, setting $h = f - 2 \log g$, by a simple computation the following formula holds weakly for $\varphi \in C^2(M)$:

$$(4.30) \qquad -\Delta_h \left(\frac{\varphi}{g}\right) = g^{-1} Q'_V(\varphi).$$

Note that, according to our notation, for smooth ϕ

$$\Delta_h \phi = g^{-2} e^f \operatorname{div}(e^{-f} g^2 \nabla \phi)$$

Integrating by parts, we infer

$$\frac{1}{2} \int_M \left|\nabla\left(\frac{\varphi}{g}\right)\right|^2 g^2 \mathrm{d}\mu_f = Q_V(\varphi) \qquad \forall \varphi \in C_c^2(M).$$

Therefore, it is readily seen that Q'_V is subcritical if and only if so is $-\Delta_h$ with respect to the measure $g^2 \mathrm{d}\mu_f$. Now, Proposition 4.4 guarantees that this happens if and only if $-\Delta_h$ has a positive Green kernel $\mathcal{G}(x, y)$. Coming back with the aid of (4.30), it is easy to see that $\mathcal{G}(x, y) g(x) g(y)$ is a global positive Green kernel for Q'_V.

5. HARDY WEIGHTS AND COMPARISON GEOMETRY

On a manifold M for which Q_0 is subcritical, the criterion in Proposition 4.5 shifts the problem of the subcriticality of Q_V to the one of finding explicit Hardy weights. As we will see in a moment, the construction of weights given in 4) and 5) of Proposition 4.4 is compatible with the usual geometric comparison theorems. Therefore, it gives a useful way to produce simple Hardy weights on manifolds satisfying suitable curvature assumptions. In this section, we describe some examples to illustrate the method.

First, we underline the following simple fact. By its very definition, the set of Hardy weights $w \in L^1_{\mathrm{loc}}(M)$, $w \geq 0$, $w \not\equiv 0$ is convex in $L^1_{\mathrm{loc}}(M)$. More generally, given a family $\{w_\alpha\}_{\alpha \in A}$ of Hardy weights on M whose index α lies in a measurable space $(A, \mathscr{F})$ ($\mathscr{F}$ a σ-algebra), and such that the map

$$w \quad : \quad (x, \alpha) \in M \times A \longrightarrow w_\alpha(x) \in [0, +\infty]$$

is measurable, for each measure λ on A such that $0 < \lambda(A) \leq 1$ the function

$$\chi(x) = \int_A w(x, \alpha) \mathrm{d}\lambda(\alpha) \tag{5.1}$$

is still a Hardy weight. Indeed, it is enough to apply Tonelli's theorem: for each $\varphi \in \mathrm{Lip}_c(M)$,

$$
\begin{aligned}
\int_M \chi |\varphi|^p \mathrm{d}\mu_f &= \int_A \left[\int_M w_\alpha(x) |\varphi(x)|^p \mathrm{d}\mu_f \right] \mathrm{d}\lambda(\alpha) \\
&\leq \int_A \left[\int_M |\nabla \varphi|^p \mathrm{d}\mu_f \right] \mathrm{d}\lambda(\alpha) \leq \int_M |\nabla \varphi|^p \mathrm{d}\mu_f.
\end{aligned}
\tag{5.2}
$$

Clearly, this construction makes sense also if for λ-almost all $\alpha \in A$, w_α is a Hardy weight.

Remark 5.1. By item 5) of Proposition 4.4, each Green kernel $\mathcal{G}$ generates a family w indexed by $A = M$:

$$w(x, y) \doteq \left(\frac{p-1}{p} \right)^p \frac{|\nabla_x \mathcal{G}(x, y)|^p}{\mathcal{G}(x, y)^p} \quad : \quad M \times M \longrightarrow [0, +\infty], \tag{5.3}$$

provided that w is measurable. If $p = 2$, the standard construction of a Green kernel in [34] produces a symmetric kernel, and measurability is obvious. However, measurability seems to be a subtle issue if $p \neq 2$, since $\mathcal{G}(x, y)$ is constructed in [38] and [39] by fixing y and finding a solution of $\Delta_{p,f} \mathcal{G}(x, y) = -\delta_y$. The dependence of $\mathcal{G}(x, y)$ from y could be, a priori, wild.

We now describe two simple measures λ that have been considered in the literature when $M = \mathbb{R}^m$, and the corresponding Hardy inequalities.

Example 5.1. *[Multipole Hardy weights]*
Fix a possibly infinite sequence $\{y_j\} \subset M$, $j \in I \subset \mathbb{N}$, let $\{t_j\}_{j \in I} \subset (0, 1]$ be such that $\sum_j t_j \leq 1$ and define

$$\lambda = \sum_j t_j \delta_{y_j},$$

where δ_{y_j} is the Dirac delta function at y_j. Being λ discrete, measurability of w follows automatically and thus, by (5.2),

$$(5.4) \qquad \left(\frac{p-1}{p}\right)^p \int_M \left[\sum_{j\in I} \frac{t_j|\nabla_x\mathcal{G}(x,y_j)|^p}{\mathcal{G}(x,y_j)^p}\right] |\varphi(x)|^p\mathrm{d}\mu_f \leq \int_M |\nabla\varphi(x)|^p\mathrm{d}\mu_f$$

holds for $\varphi \in \mathrm{Lip}_c(M)$.

Example 5.2. *[Hardy weights that blow-up along a submanifold]*
If $w(x,y)$ in (5.3) is measurable, for each rectifiable subset $\Sigma \hookrightarrow M^m$ of finite non-zero k-dimensional Hausdorff measure $\mathcal{H}^k$ we can set

$$\lambda = \frac{\mathcal{H}^k\llcorner\Sigma}{\mathcal{H}^k(\Sigma)},$$

Then, we have the Hardy inequality

$$(5.5) \quad \left(\frac{p-1}{p}\right)^p \int_M \left[\frac{1}{\mathcal{H}^k(\Sigma)}\int_\Sigma \frac{|\nabla_x\mathcal{G}(x,y)|^p}{\mathcal{G}(x,y)^p}\mathrm{d}\mathcal{H}^k(y)\right] |\varphi(x)|^p\mathrm{d}x \leq \int_M |\nabla\varphi(x)|^p\mathrm{d}x$$

for each $\varphi \in \mathrm{Lip}_c(M)$.
Hardy weights of this type have been considered, for instance, in [31]: in Theorem 1.1 therein, $M = \mathbb{R}^m = \mathbb{R}^2 \times \mathbb{R}^{m-2}$, $p = 2$, $f \equiv 0$ and $\Sigma \subset \mathbb{R}^m$ is a round circle $S_\rho \doteq \mathbb{S}^1(\rho) \times \{0\} \subset \mathbb{R}^2 \times \mathbb{R}^{m-2}$ centered at the origin and of radius ρ. We remark that Hardy weights of different type but still depending on the distance from a submanifold have been investigated, among others, in [10]. The reader is also suggested to see the references therein for deepening.

To introduce the results below, we first recall comparison geometry, starting with the definition of a model manifold. Briefly, fix a point $o \in \mathbb{R}^m$. Given $g \in C^2(\mathbb{R}_0^+)$ such that $g > 0$ on some open interval $(0, R) \subset \mathbb{R}^+$, $g(0) = 0$, $g'(0) = 1$, a model $(M_g^m, \mathrm{d}s_g^2)$ is the manifold $B_R(o) \subset \mathbb{R}^m$ equipped with a radially symmetric C^2 metric $\mathrm{d}s_g^2$ whose expression, in polar geodesic coordinates (ρ, θ) centered at o (where $\theta \in \mathbb{S}^{m-1}$), is given by

$$\mathrm{d}s_g^2 = \mathrm{d}\rho^2 + g(\rho)^2\mathrm{d}\theta^2,$$

$\mathrm{d}\theta^2$ being the standard metric on the unit sphere $\mathbb{S}^{m-1}$. Clearly, ρ is the distance function from o, and M_g is complete if and only if $R = +\infty$. At a point $x = (\rho, \theta)$, the radial sectional curvature K_{rad} of M_g (that is, the sectional curvature restricted to planes containing $\nabla\rho(x)$), the Hessian and the Laplacian of ρ are given by

$$(5.6) \quad K_{\mathrm{rad}}(x) = -\frac{g''(\rho)}{g(\rho)}, \quad \nabla\mathrm{d}\rho(x) = \frac{g'(\rho)}{g(\rho)}\left(\mathrm{d}s_g^2 - \mathrm{d}\rho\otimes\mathrm{d}\rho\right), \quad \Delta\rho(x) = (m-1)\frac{g'(\rho)}{g(\rho)}.$$

By the first formula, in (5.6), a model can also be given by prescribing the radial sectional curvature $-G \in C^0(\mathbb{R}_0^+)$ and recovering $g \in C^2(\mathbb{R}_0^+)$ as the solution of

$$(5.7) \qquad \begin{cases} g'' - Gg = 0 & \text{on } \mathbb{R}^+, \\ g(0) = 0, \quad g'(0) = 1, \end{cases}$$

on the maximal interval where $g > 0$. A sharp condition on G that ensures the positivity of g on the whole $\mathbb{R}^+$ is given by

$$G_- \in L^1(\mathbb{R}^+), \qquad t\int_t^{+\infty} G_-(s)\mathrm{d}s \leq \frac{1}{4} \qquad \text{on } \mathbb{R}^+,$$

see Proposition 1.21 in [12]. In particular, if $G \equiv \kappa^2$ for some constant $\kappa \geq 0$, we will denote with g_κ the solution g of (5.7):

$$(5.8) \qquad g_\kappa(\rho) = \begin{cases} \rho & \text{if } \kappa = 0, \\ \kappa^{-1} \sinh(\kappa\rho) & \text{if } \kappa > 0. \end{cases}$$

When $\kappa = 0$, M_g is the Euclidean space $\mathbb{R}^m$, while if $\kappa > 0$ our model is the hyperbolic space $\mathbb{H}^m_\kappa$ of sectional curvature $-\kappa^2$. Clearly, the two examples are radially symmetric with respect to any chosen origin o. Hereafter, we will always consider geodesically complete models.

Given $p \in (1, +\infty)$, $-\Delta_p$ is subcritical on M_g if and only if

$$(5.9) \qquad g(\rho)^{-\frac{m-1}{p-1}} \in L^1(+\infty)$$

(the case $p = 2$ can be found, for instance, in [34], and for $p \neq 2$ the argument of the proof goes along the same lines). In fact, under the validity of (5.9), up to an unessential constant, the function

$$G(x, o) = G\big((\rho, \theta), o\big) = \int_\rho^{+\infty} g(s)^{-\frac{m-1}{p-1}} \, \mathrm{d}s$$

is the minimal positive Green kernel for $-\Delta_p$ with singularity at o.

A simplified version of the Hessian and Laplacian comparison theorems from below (see [58, 62, 12]) can be stated as follows: suppose that $(M, \langle , \rangle)$ has a pole o and, denoting with $r(x) = \mathrm{dist}(x, o)$, that the radial sectional curvature K_{rad} satisfies

$$(5.10) \qquad K_{\mathrm{rad}}(x) \leq -G\big(r(x)\big)$$

(i.e., for each plane $\pi \leq T_x M$ containing $\nabla r(x)$, $K(\pi) \leq -G\big(r(x)\big)$). Denote with g the solution of (5.7), and let $(0, R)$ be the maximal interval where $g > 0$. Then, in the sense of quadratic forms,

$$(5.11) \qquad \nabla \mathrm{d}r(x) \geq \frac{g'(r(x))}{g(r(x))}\Big(\langle , \rangle - \mathrm{d}r \otimes \mathrm{d}r\Big)$$

pointwise on $B_R(o) \setminus \{o\}$, and tracing

$$(5.12) \qquad \Delta r(x) \geq (m-1)\frac{g'(r(x))}{g(r(x))},$$

whose validity holds weakly on the geodesic ball $B_R(o) \subset M$. As it is apparent from (5.6), M is compared with a model M_g of radial sectional curvature $-G$. Indeed, via Sturm comparison, for (5.12) to hold it is enough that $g \in C^2(\mathbb{R}_0^+)$ solves the inequality

$$(5.13) \qquad \begin{cases} g'' - Gg \leq 0 & \text{on } \mathbb{R}^+, \\ g(0) = 0, \quad g'(0) = 1, \end{cases}$$

so that the model to which M is compared has radial sectional curvature greater than or equal to $-G$. In [13], we have collected a number of examples of G for which explicit positive solutions g of (5.13) can be found. These also include manifolds whose sectional curvature can be positive but in a controlled way, loosely speaking manifolds whose compared model is some sort of paraboloid.

With this preparation, we are now ready to discuss the next cases.

5.1. **Hardy weights on manifolds with a pole.**

Theorem 5.1. *Let $(M, \langle \, , \, \rangle)$ be a Riemannian manifold with a pole o. Denoting with $r(x) = \mathrm{dist}(x, o)$, assume that the radial sectional curvature satisfies*

$$K_{\mathrm{rad}} \leq -G\big(r(x)\big),$$

for some $G \in C^2(\mathbb{R}_0^+)$. Suppose that g solving (5.13) is positive on $\mathbb{R}^+$, and, for $p \in (1, +\infty)$, assume that

$$(5.14) \qquad\qquad g(r)^{-\frac{m-1}{p-1}} \in L^1(+\infty).$$

Then, the Hardy inequality

$$(5.15) \qquad\qquad \int_M (\chi \circ r) |\varphi|^p \mathrm{d}x \leq \int_M |\nabla \varphi|^p \mathrm{d}x$$

holds for each $\varphi \in \mathrm{Lip}_c(M)$, where

$$(5.16) \qquad\qquad \chi(t) = \left(\frac{p-1}{p}\right)^p \left[g(t)^{\frac{m-1}{p-1}} \int_t^{+\infty} g(s)^{-\frac{m-1}{p-1}} \mathrm{d}s \right]^{-p}$$

Proof. Consider the transplanted Green kernel of the model M_g with pole at o:

$$(5.17) \qquad\qquad G(x) = \int_{r(x)}^{+\infty} g(s)^{-\frac{m-1}{p-1}}.$$

Then, a computation using (5.11) shows that $\Delta_p G \leq 0$ on $M \backslash \{o\}$.

Furthermore, G satisfies the asymptotic behaviour

$$(5.18) \qquad G(x) \sim \begin{cases} C > 0 & \text{if } p > m, \\ |\log r| & \text{if } p = m, \\ \frac{p-1}{m-p} r^{-\frac{m-p}{p-1}} & \text{if } p < m, \end{cases} \qquad \text{as } r \to 0^+,$$

for some constant $C > 0$. It thus follows that, for each $c \in (0, C)$ if $p > m$ and $c \in \mathbb{R}^+$ if $p \leq m$, by the pasting Lemma 3.1 the function $G_c = \min\{G, c\}$ is a solution of $\Delta_p G_c \leq 0$, hence by (4.4)

$$(5.19) \quad \left(\frac{p-1}{p}\right)^p \int_M \frac{|\nabla G_c|^p}{G_c^p} |\varphi|^p \mathrm{d}x \leq \int_M |\nabla \varphi|^p \mathrm{d}x \qquad \text{holds for each } \varphi \in \mathrm{Lip}_c(M).$$

Letting $c \to C^-$ when $p > m$, $C \to +\infty$ when $p \leq m$, and since

$$\left(\frac{p-1}{p}\right)^p \frac{|\nabla G|^p}{G^p}(x) = \chi\big(r(x)\big),$$

we conclude the validity of (5.15). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

The function $\chi \circ r$ has the following asymptotics as $r = r(x) \to 0^+$:

$$(5.20) \qquad\qquad \chi(r) \sim \begin{cases} \left(\frac{p-1}{Cp}\right)^p r^{-\frac{p(m-1)}{p-1}} & \text{if } p > m, \\ \left(\frac{m-1}{m}\right)^m \frac{1}{r^m \log^m r} & \text{if } p = m, \\ \left(\frac{m-p}{p}\right)^p \frac{1}{r^p} & \text{if } p < m, \end{cases}$$

where, for $p > m$, the constant C is the same as in (5.18). Note that, on each case, $\chi \circ r \in L^1_{\mathrm{loc}}(M)$, in particular the singularity at o is integrable.

Remark 5.2. Under the same assumptions on M, this example can be extended to deal with the weighted operator $\Delta_{p,f}$ provided that $f = f(r(x))$ is a radial function. In this case, it can be seen that (5.14) must be replaced by

$$\frac{e^{f(r)}}{g(r)^{\frac{m-1}{p-1}}} \in L^1(+\infty),$$

and that

$$G(x) = \int_{r(x)}^{+\infty} \frac{e^{f(s)}\mathrm{d}s}{g(s)^{\frac{m-1}{p-1}}}$$

is a solution of $\Delta_{p,f} G \leq 0$ on $M \backslash \{o\}$ that gives rise to a Hardy weight analogous to (5.15).

The Hardy weight in (5.15) is explicit once we have an explicit g solving (5.13), for example those related to the families of G described in the Appendix of [13]. We refer the reader to the above mentioned paper also for a detailed study of the corresponding Hardy weight (called the "critical curve" therein) for $p = 2$. The modifications needed to deal with general p are straightforward. Here, we just focus on the Euclidean and hyperbolic settings.

Example 5.3. *[Hardy weights for Euclidean and hyperbolic spaces]*
Consider the Euclidean space, where $g(r) = g_0(r) = r$. Condition (5.14) is met if and only if $p < m$, and the Hardy weight (5.15) has the simple expression

$$(5.21) \qquad\qquad \chi\big(r(x)\big) = \left(\frac{m-p}{p}\right)^p \frac{1}{r(x)^p}.$$

Consequently, the Hardy inequality

$$(5.22) \qquad \left(\frac{m-p}{p}\right)^p \int_M \frac{|\varphi|^p}{r^p}\mathrm{d}x \leq \int_M |\nabla\varphi|^p\mathrm{d}x \qquad \forall\,\varphi \in \mathrm{Lip}_c(M)$$

holds on each manifold with a pole and radial sectional curvature $K_{\mathrm{rad}} \leq 0$. Inequality (5.22) is classical and well-known in $\mathbb{R}^m$, see [10].
On the hyperbolic space $\mathbb{H}^m_\kappa$ of sectional curvature $-\kappa^2$, where $g_\kappa(r) = \kappa^{-1}\sinh(\kappa r)$, condition (5.14) is met for each $m \geq 2$ independently of p, but the expression of $\chi(r(x))$ is not so neat. However, an iterative argument allows us to explicitly compute the integral in the expression of χ in some relevant cases. For $\alpha > 0$, set

$$I_\alpha(r) \doteq \int_r^{+\infty} \frac{\mathrm{d}s}{g_\kappa(s)^\alpha} \qquad \text{and} \qquad \chi_\alpha(r) \doteq \left(\frac{p-1}{p}\right)^p [g_\kappa(r)^\alpha I_\alpha(r)]^{-p}.$$

In view of (5.16), our case of interest is $\alpha = \frac{m-1}{p-1}$. Writing

$$\frac{I_{\alpha+2}}{\kappa^{\alpha+2}} = \int_r^{+\infty} \frac{\cosh(\kappa s)\cosh(\kappa s)}{\sinh^{\alpha+2}(\kappa s)}\mathrm{d}s - \frac{I_\alpha}{\kappa^\alpha},$$

and integrating by parts, we obtain the recursion formula

$$\alpha I_\alpha(r) = \frac{\cosh(\kappa r)}{\kappa^2 g_\kappa(r)^{\alpha+1}} - \frac{\alpha+1}{\kappa^2}I_{\alpha+2}(r)$$

that yields

$$\alpha \chi_\alpha^{-1/p} = \frac{p\coth(\kappa r)}{(p-1)\kappa} - \frac{\alpha+1}{\kappa^2 g_\kappa(r)^2}\chi_{\alpha+2}^{-1/p}.$$

Therefore, one may inductively recover χ_α. In particular, if $\alpha = 1$ (i.e., in our case of interest, $p = m$), by explicit integration of $I_1(r)$ we get

$$(5.23) \qquad \chi_1(r) = \left(\frac{(m-1)\kappa}{m}\right)^m \left[\sinh(\kappa r) \log\left(\frac{e^{\kappa r}+1}{e^{\kappa r}-1}\right)\right]^{-m},$$

while if $\alpha = 2$ (i.e. $m = 2p - 1$), again by explicit integration of $I_2(r)$ we deduce

$$(5.24) \qquad \chi_2(r) = \left(\frac{2(m-1)\kappa}{m+1}\right)^{\frac{m+1}{2}} \left(1 - e^{-2\kappa r}\right)^{-\frac{m+1}{2}},$$

see Example 3.15 in [12]. An important feature of $\chi_\alpha(r)$ is the following:

$$(5.25) \qquad \chi_\alpha(r) \geq \left(\frac{p-1}{p}\right)^p \alpha^p \kappa^p, \qquad \chi_\alpha(r) \to \left(\frac{p-1}{p}\right)^p \alpha^p \kappa^p \quad \text{as } r \to +\infty.$$

Indeed, the limit is straightforwardly computable. As for the first relation, it follows from the following property. To state it, for fixed $\alpha > 0$ and for g satisfying $1/g^\alpha \in L^1(+\infty)$, write

$$\chi_g(r) \doteq \left(\frac{p-1}{p}\right)^p \left(g(r)^\alpha \int_r^{+\infty} \frac{\mathrm{d}s}{g(s)^\alpha}\right)^{-p}.$$

Then, the next comparison result holds:

if g_1/g_2 is non-decreasing on $\mathbb{R}^+$ (respectively, non-increasing),

then $\chi_{g_1} \geq \chi_{g_2}$ on $\mathbb{R}^+$ (resp, $\leq$).

The proof of this fact goes along the same lines as in Proposition 3.12 in [12], and is left to the interested reader. Using this with $g_1(r) \doteq g_\kappa(r)$ and $g_2(r) \doteq \exp\{\kappa r\}$ we get

$$\chi_{g_1}(r) \geq \chi_{g_2}(r) \equiv \left(\frac{p-1}{p}\right)^p \alpha^p \kappa^p,$$

as claimed.

When each point of the manifold M is a pole, we can construct multipole Hardy weights by the standard procedure described at the beginning of Section 5. This is the case if, for example, M is a Cartan-Hadamard manifold, that is, a simply-connected, complete manifold with non-positive sectional curvature.

Theorem 5.2. *For $m \geq 2$, let M^m be a Cartan-Hadamard manifold satisfying $K \leq -\kappa^2$, for some constant $H \geq 0$. Given the solution g_κ of (5.8), let*

$$(5.26) \qquad \begin{cases} p \in (1, m) & \text{if } \kappa = 0, \\ p \in (1, +\infty) & \text{if } \kappa > 0. \end{cases}$$

Then, for each unit measure λ on M, the Hardy inequality

$$(5.27) \qquad \int_M \left[\int_M \left(\chi\big(\mathrm{dist}(x,y)\big)\right)\mathrm{d}\lambda(y)\right] |\varphi(x)|^p \mathrm{d}x \leq \int_M |\nabla\varphi(x)|^p \mathrm{d}x$$

holds for each $\varphi \in \mathrm{Lip}_c(M)$, where

$$(5.28) \qquad \chi(t) = \left(\frac{p-1}{p}\right)^p \left[g_\kappa(t)^{\frac{m-1}{p-1}} \int_t^{+\infty} g_\kappa(s)^{-\frac{m-1}{p-1}} \mathrm{d}s\right]^{-p}.$$

In particular, when $\kappa = 0$, for each $\varphi \in \mathrm{Lip}_c(M)$

$$(5.29) \qquad \left(\frac{m-p}{p}\right)^p \int_M \left[\int_M \frac{\mathrm{d}\lambda(y)}{\mathrm{dist}(x,y)^p}\right] |\varphi(x)|^p \mathrm{d}x \leq \int_M |\nabla\varphi(x)|^p \mathrm{d}x.$$

Proof. By (5.26), (5.14) is met for $g = g_\kappa$. Since each $y \in M$ is a pole of M, setting $r_y(\cdot) = \mathrm{dist}(\cdot, y)$ one can apply Theorem 5.1 to deduce that (5.15) holds for each fixed y, namely,

$$(5.30) \qquad \int_M (\chi \circ r_y)|\varphi|^p \mathrm{d}x \leq \int_M |\nabla \varphi|^p \mathrm{d}x \qquad \forall \, \varphi \in \mathrm{Lip}_c(M),$$

where χ is as in (5.28). The generalization in (5.27) follows from the argument at the beginning of Section 5. □

5.2. **Hardy weights on minimally immersed submanifolds.**

Theorem 5.3. *Let $F : (M^m, \langle \, , \, \rangle) \to (N^n, (\, , \,))$ be an immersed, minimal submanifold of a Cartan-Hadamard ambient space N^n, and suppose that the sectional curvature $\bar{K}$ of N satisfies $\bar{K} \leq -\kappa^2$, for some constant $H \geq 0$. If $\kappa = 0$, we assume that $m \geq 3$. Then, given the solution g_κ of (5.8), and denoting with $\bar{r}_q$ the extrinsic distance from a point $q \in N$ evaluated along the immersion F, the following Hardy inequality holds:*

$$(5.31) \qquad \int_M (\chi \circ \bar{r}_q) \varphi^2 \mathrm{d}x \leq \int_M |\nabla \varphi|^2 \mathrm{d}x \qquad \forall \, \varphi \in \mathrm{Lip}_c(M),$$

where

$$(5.32) \qquad \chi(t) = \frac{1}{4}\left[g_\kappa(t)^{m-1} \int_t^{+\infty} g_\kappa(s)^{1-m} \mathrm{d}s \right]^{-2}.$$

In particular, if $\kappa = 0$,

$$(5.33) \qquad \left(\frac{m-2}{2}\right)^2 \int_M \frac{\varphi^2}{\bar{r}_q^2} \mathrm{d}x \leq \int_M |\nabla \varphi|^2 \mathrm{d}x, \qquad \forall \, \varphi \in \mathrm{Lip}_c(M),$$

while, if $\kappa > 0$,

- *when $m = 2$, that is, M is a surface, for each $\varphi \in \mathrm{Lip}_c(M)$*

$$(5.34) \qquad \left(\frac{\kappa}{2}\right)^2 \int_M \left[\sinh(\kappa \bar{r}_q) \log\left(\frac{e^{\kappa \bar{r}_q} + 1}{e^{\kappa \bar{r}_q} - 1}\right) \right]^{-2} \varphi^2 \mathrm{d}x \leq \int_M |\nabla \varphi|^2 \mathrm{d}x;$$

- *when $m = 3$, for each $\varphi \in \mathrm{Lip}_c(M)$*

$$(5.35) \qquad \kappa^2 \int_M \frac{\varphi^2}{(1 - e^{-2\kappa \bar{r}_q})^2} \mathrm{d}x \leq \int_M |\nabla \varphi|^2 \mathrm{d}x.$$

Remark 5.3. The inequality in (5.33) has been proved in [19, 49] by combining the comparison for the Hessian of the extrinsic distance function with an integration by parts argument. The case $\kappa > 0$ has been considered in [49, Example 1.8]. However, the Hardy weight found there is skew with (5.34) and (5.35), in particular it is quite smaller if $\bar{r}_q$ is close to zero. As before, other Hardy weights can be constructed in the way described in Examples (5.1) and (5.2).

Proof. We mark with a bar each quantity when referred to N, so that, for example, $\bar{\nabla}, \overline{\mathrm{dist}}$ are the Riemannian connection and the distance function of N. For simplicity, we denote with $\bar{r}_q$ the distance function from q in the manifold N, i.e. $\bar{r}_q(\cdot) = \overline{\mathrm{dist}}(\cdot, q)$, so that the function $\bar{r}_q$ in the statement of the theorem is, indeed, $\bar{r}_q \circ F$. By the Hessian comparison theorem (5.11), for each $q \in N$ it holds

$$\bar{\nabla} \mathrm{d}\bar{r}_q \geq \frac{g'_\kappa(\bar{r}_q)}{g_\kappa(\bar{r}_q)}\Big((\, ,\,) - \mathrm{d}\bar{r}_q \otimes \mathrm{d}\bar{r}_q\Big) \qquad \text{on } N \backslash \{q\}.$$

For $x \in M$, $q \in N$ define

(5.36) $$G_q(x) = h(\bar{r}_q(F(x))), \qquad \text{where } h(t) = \int_t^{+\infty} \frac{\mathrm{d}s}{g_\kappa(s)^{m-1}}.$$

Observe that the integral in $h(t)$ converges for each $m \geq 2$ when $\kappa > 0$, and for each $m \geq 3$ if $\kappa = 0$, which accounts for our dimensional restrictions. Denote with II the second fundamental form of F. By the chain rule and using $h' < 0$, for each vector field X on M (identified with $F_*(X)$) we get

$$\begin{aligned}
\nabla \mathrm{d}G_q(X, X) &= h''(\bar{\nabla}\bar{r}_q, X)^2 + h'\bar{\nabla}\mathrm{d}\bar{r}_q(X, X) + h'(\bar{\nabla}\bar{r}_q, \mathrm{II}(X, X)) \\
&\leq \left(h'' - \frac{g'_\kappa}{g_\kappa}h'\right)(\bar{\nabla}\bar{r}_q, X)^2 + h'\frac{g'_\kappa}{g_\kappa}|X|^2 + h'(\bar{\nabla}\bar{r}_q, \mathrm{II}(X, X)) \\
&= -mh'\frac{g'_\kappa}{g_\kappa}(\bar{\nabla}\bar{r}_q, X)^2 + h'\frac{g'_\kappa}{g_\kappa}|X|^2 + h'(\bar{\nabla}\bar{r}_q, \mathrm{II}(X, X)) \\
&= h'\frac{g'_\kappa}{g_\kappa}\left[|X|^2 - m(\bar{\nabla}\bar{r}_q, X)^2\right] + h'(\bar{\nabla}\bar{r}_q, \mathrm{II}(X, X))
\end{aligned}$$

Tracing with respect to an orthonormal frame $\{e_i\}$ of M, using minimality and again $h' < 0$, we obtain

(5.37) $$\Delta G_q \leq mh'\frac{g'_\kappa}{g_\kappa}\left(1 - |\bar{\nabla}^T\bar{r}_q|^2\right) \leq 0$$

where $\bar{\nabla}^T$ is the component of the gradient in N which is tangent to M. By Proposition 4.4,

(5.38) $$\frac{|\nabla G_q(x)|^2}{4G_q(x)^2} = \left(\frac{h'(\bar{r}_q)}{2h(\bar{r}_q)}\right)^2 |\bar{\nabla}^T\bar{r}_q|^2 = \chi(\bar{r}_q(F(x)))|\bar{\nabla}^T\bar{r}_q|^2.$$

is a Hardy weight. Unfortunately, such a weight is not effective, since we cannot control the size of $\bar{\nabla}^T\bar{r}_q$. However, we can improve (5.38) to the effective Hardy weight $\chi(\bar{r}_q)$ by using the full information coming from (5.37). In fact, since $h' < 0$, by (5.37) the function $u_1 \doteq \sqrt{G_q}$ solves, on $M\backslash f^{-1}\{q\}$,

(5.39)
$$\begin{aligned}
\Delta u_1 &= \left[\frac{\Delta G_q}{2G_q} - \chi(\bar{r}_q)|\bar{\nabla}^T\bar{r}_q|^2\right]u_1 \leq \left[m\frac{h'}{2h}\frac{g'_\kappa}{g_\kappa}\left(1 - |\bar{\nabla}^T\bar{r}_q|^2\right) - \chi(\bar{r}_q)|\bar{\nabla}^T\bar{r}_q|^2\right]u_1 \\
&\leq \left[-m\sqrt{\chi(\bar{r}_q)}\frac{g'_\kappa}{g_\kappa} + \sqrt{\chi(\bar{r}_q)}|\bar{\nabla}^T\bar{r}_q|^2\left(m\frac{g'_\kappa}{g_\kappa} - \sqrt{\chi(\bar{r}_q)}\right)\right]u_1.
\end{aligned}$$

Now, we claim that

(5.40) $$\zeta(t) \doteq m\frac{g'_\kappa(t)}{g_\kappa(t)} - \sqrt{\chi(t)} \geq 0 \qquad \text{for } t \in \mathbb{R}^+.$$

Indeed, in the Euclidean case $\kappa = 0$, $g_\kappa(t) = t$ and explicit computation gives

$$\zeta(t) = \frac{m+2}{2t} > 0 \qquad \text{on } \mathbb{R}^+.$$

When $\kappa > 0$, $g_\kappa(t) = \kappa^{-1}\sinh(\kappa t)$. A computation gives

(5.41) $$\zeta(t) \sim \frac{m+2}{2t} \quad \text{as } t \to 0, \qquad \zeta(t) \sim \frac{(m+1)\kappa}{2} \quad \text{as } t \to +\infty.$$

Now, $y(t) = \sqrt{\chi(t)}$ solves

$$(5.42) \qquad\qquad y' = 2y^2 - y(m-1)\frac{g'_\kappa}{g_\kappa} \qquad \text{on } \mathbb{R}^+,$$

Suppose that $\zeta(\bar{t}) \leq 0$ for some $\bar{t} > 0$. Then, an inspection of (5.42) and the fact that g'_κ/g_κ is decreasing show that $y' > 0$ on $[\bar{t}, +\infty)$, whence there exists $c > 0$ such that

$$y > \frac{m-1}{2}\frac{g'_\kappa}{g_\kappa} + c \qquad \text{on } [\bar{t}, +\infty).$$

But then

$$\lim_{t \to +\infty} \zeta(t) = m\kappa - \lim_{t \to +\infty} y(t) \leq m\kappa - c - \frac{m-1}{2}\kappa = \frac{m+1}{2}\kappa - c,$$

contradicting (5.41) and proving the claim. Next, by (5.40) we can use the estimate $|\bar{\nabla}^T \bar{r}_q| \leq 1$ to conclude

$$(5.43) \qquad \Delta u_1 \leq \left[-m\sqrt{\chi(\bar{r}_q)}\frac{g'_\kappa}{g_\kappa} + \sqrt{\chi(\bar{r}_q)}\left(m\frac{g'_\kappa}{g_\kappa} - \sqrt{\chi(\bar{r}_q)} \right) \right] u_1 = -\chi(\bar{r}_q)u_1.$$

For $\varepsilon > 0$, consider the truncated function $\chi_\varepsilon(t)$ given by $\chi_\varepsilon(t) = \chi(t)$ if $t \geq 2\varepsilon$, and 0 otherwise. Then, $\chi_\varepsilon(\bar{r}_q \circ F) \in L^\infty_{\mathrm{loc}}(M)$ and by (5.43)

$$(5.44) \qquad\qquad \Delta u_1 + \chi_\varepsilon(\bar{r}_q)u_1 \leq 0 \qquad \text{on } M \backslash F^{-1}\{q\}.$$

If $F^{-1}\{q\} \neq \emptyset$ observe that the constant function $u_2 \doteq \sqrt{G_q(\varepsilon)}$ solves (5.44) on $F^{-1}\{B_{2\varepsilon}(q)\}$. By the pasting Lemma 3.1 with the choices $\Omega_1 \doteq F^{-1}\{B_\varepsilon(q)\}\backslash F^{-1}\{q\}$, $\Omega_2 \doteq F^{-1}\{B_{2\varepsilon}(q)\}\backslash F^{-1}\{q\}$, and since $h' < 0$, we deduce that

$$u \doteq \begin{cases} \sqrt{G_q(\varepsilon)} & \text{on } F^{-1}\{B_\varepsilon(q)\}, \\ \sqrt{G_q} & \text{on } M\backslash F^{-1}\{B_\varepsilon(q)\} \end{cases}$$

solves $\Delta u + \chi_\varepsilon(\bar{r}_q)u \leq 0$ on $\Omega_2 = F^{-1}\{B_{2\varepsilon}(q)\}\backslash F^{-1}\{q\}$. Since the pasting region $F^{-1}\{\partial B_\varepsilon(q)\}$ is internal to Ω_2 and u is smooth in a neighbourhood of $F^{-1}\{q\}$, then clearly u solves

$$(5.45) \qquad\qquad \Delta u + \chi_\varepsilon(\bar{r}_q)u \leq 0 \qquad \text{on the whole } M.$$

By Proposition 3.4 with $p = 2$, $f = 1$ and $V = \chi_\varepsilon(\bar{r}_q)$ we deduce that $\lambda_V(M) \geq 0$, that is

$$\int_M \chi_\varepsilon(\bar{r}_q \circ F)\varphi^2 \mathrm{d}x \leq \int_M |\nabla\varphi|^2 \mathrm{d}x \qquad \text{for each } \varphi \in \mathrm{Lip}_c(M),$$

whence, letting $\varepsilon \to 0$ and using monotone convergence we deduce (5.31). The cases (5.33), (5.34), (5.35) follow by computing $\chi(r)$ according to Example 5.3, in particular see (5.21), (5.23), (5.24). □

5.3. **Specializing our main theorems: an example.** To illustrate the results of the last two sections, by way of an example we specialize our Theorem 2.1 to the case of quasilinear Yamabe type equations on Cartan-Hadamard manifolds. We underline that all the assumptions in the next Corollary are explicit and easy to check. Clearly, analogous results can be stated using any of Theorems 5.1, 5.2, 5.3, as well as Theorem 2.2 instead of Theorem 2.1.

Corollary 5.1. *Let M be a Cartan-Hadamard manifold of dimension $m \geq 3$, and let $p \in (1, m)$. Let $a, b \in L^\infty_{\text{loc}}(M)$, and suppose that, for some countable set of points $\{y_j\}_{j \in I} \subset M$ and $\{t_j\}_{j \in I} \subset [0, 1]$ with $\sum_j t_j \leq 1$,*

$$(5.46) \qquad a(x) \leq \left(\frac{m-p}{p}\right)^p \sum_{j \in I} \frac{t_j}{\text{dist}(x, y_j)^p}.$$

Furthermore, suppose that

- *i) $b_-(x)$ has compact support;*
- *ii) $a(x) = O\big(b(x)\big)$ as x diverges*
- *iii) for some $\theta > 0$, $\big(a(x) - \theta b_+(x)\big)_- \in L^1(M)$.*

Fix a nonlinearity $F(t)$ satisfying (2.6). Then, there exists $\delta > 0$ such that if

$$b(x) \geq -\delta \qquad on \ M,$$

there exists a weak solution $u \in C^{1,\mu}_{\text{loc}}(M)$ of

$$(5.47) \qquad \begin{cases} \Delta_p u + a(x)u^{p-1} - b(x)F(u) = 0 & on \ M, \\ 0 < u \leq \|u\|_{L^\infty(M)} < +\infty. \end{cases}$$

Proof. In view of Theorem 2.1, it is enough to prove that $-\Delta_p$ and Q_a are subcritical on M. By Theorem 5.2, the Hardy inequality

$$\left(\frac{m-p}{p}\right)^p \int_M \sum_{j \in I} \frac{t_j}{\text{dist}(x, y_j)^p} |\varphi(x)|^p \mathrm{d}x \leq \int_M |\nabla \varphi(x)|^p \mathrm{d}x$$

holds for $\varphi \in \text{Lip}_c(M)$. Therefore, by Proposition 4.4, $-\Delta_p$ is subcritical. Next, the fact that the inequality in (5.46) is strict on a set of positive measure (the right-hand side is essentially unbounded at each y_j) assures that Q_a be subcritical by Proposition 4.5. $\qquad \square$

6. Proofs of Theorems 2.1 and 2.2

We first address the local solvability of the Dirichlet problem for

$$\Delta_{p,f} u + A(x)u^{p-1} - B(x)F(u) = 0$$

when $B \geq 0$.

Lemma 6.1. *Let M be a Riemannian manifold, $p \in (1, +\infty)$ and $f \in C^\infty(M)$. Let $\Omega \Subset M$ be a smooth relatively compact open set, and let $A(x), B(x) \in L^\infty(\Omega)$ satisfy $\lambda_A(\Omega) > 0$ and $B \geq 0$ a.e. on Ω. Then, for each nonlinearity $F(t)$ that satisfies (2.6), and for each $\varphi \in C^{1,\alpha}(\partial\Omega)$, $\alpha \in (0, 1)$ such that $\varphi \geq 0$, $\varphi \not\equiv 0$, there exist $\mu \in (0, 1)$ and a unique $0 < z \in C^{1,\mu}(\overline{\Omega})$ solving*

$$(6.1) \qquad \begin{cases} \Delta_{p,f} z + A(x)z^{p-1} - B(x)F(z) = 0 & on \ \Omega, \\ z = \varphi & on \ \partial\Omega. \end{cases}$$

Proof. Take the positive solution $z_0 \in C^{1,\mu}(\overline{\Omega})$ of

$$(P_0) \qquad \begin{cases} \Delta_{p,f} z_0 + A|z_0|^{p-2} z_0 = 0 & on \ \Omega, \\ z_0 = \varphi & on \ \partial\Omega, \end{cases}$$

which exists by Proposition 3.4. Since $B \geq 0$, z_0 gives rise to a supersolution for (6.1). To construct a subsolution with boundary value φ, we solve

$$(P_n) \qquad \begin{cases} \Delta_{p,f} z_1 + (A - B_1)|z_1|^{p-2} z_1 = 0 & \text{on } \Omega, \qquad B_1 \doteq B \dfrac{F(z_0)}{z_0^{p-1}} \\ z_1 = \varphi & \text{on } \partial\Omega. \end{cases}$$

Since $A - B_1 \leq A$, $\lambda_{A-B_1}(\Omega) > 0$, thus z_1 exists and, by comparison, $z_1 \leq z_0$. Since $F(t)/t^{p-1}$ is increasing,

$$\Delta_{p,f} z_1 + \left(A - B \frac{F(z_1)}{z_1^{p-1}} \right) z_1^{p-1} \geq \Delta_{p,f} z_1 + \left(A - B \frac{F(z_0)}{z_0^{p-1}} \right) z_1^{p-1} = 0.$$

Hence z_1 is a subsolution for (6.1). Applying the subsolution-supersolution method (see [25], Theorem 4.14 page 272) we obtain the existence of z satisfying (6.1). The local Harnack inequality in Theorem 3.1 gives $z > 0$ on $\overline{\Omega}$, and uniqueness follows from the comparison Proposition 3.3. $\qquad\square$

Next, we investigate the existence of local uniform lower bounds for solutions of the Dirichlet problem when Ω varies.

Lemma 6.2. *Let $(M, \langle\,,\,\rangle)$ be a Riemannian manifold, $f \in C^\infty(M)$, $p \in (1, +\infty)$, and assume that Q_0 is subcritical on M. Let $A, B \in L^\infty_{\mathrm{loc}}(M)$ such that*

$$(6.2) \qquad \begin{cases} Q_A \text{ is non-negative,} \\ (A(x) - \theta B(x))_- \in L^1(M, \mathrm{d}\mu_f), & \text{for some constant } \theta > 0. \end{cases}$$

Fix $\varepsilon > 0$. Consider a triple of relatively compact, open sets $\Lambda \Subset \Lambda' \Subset \Omega \Subset M$, with $\partial\Omega$ smooth, and let $z \in C^{1,\mu}(\overline{\Omega})$ be a positive solution of

$$(6.3) \qquad \begin{cases} \Delta_{p,f} z + A(x) z^{p-1} - B(x) F(z) \leq 0 & \text{on } \Omega; \\ z = \varepsilon & \text{on } \partial\Omega. \end{cases}$$

Then, there exists $C > 0$ depending on ε, but independent of Ω, such that

$$(6.4) \qquad \inf_\Lambda z \geq C.$$

Proof. Observe that, by the half-Harnack inequality in Theorem 3.1, $z > 0$ on Ω. By comparison, we can also suppose that z solves (6.3) with the equality sign, whence $z \in C^{1,\mu}(\overline{\Omega})$. Fix $\delta \in (0, \varepsilon)$ small enough that

$$(6.5) \qquad \frac{F(t)}{t^{p-1}} < \theta \qquad \text{if } t \in (0, \delta).$$

This is possible by (2.6). Let $\eta \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$, $\eta(\log \varepsilon) = 0$ to be specified later, and define $u = \log z$ on Ω, so that, weakly,

$$(6.6) \qquad \begin{cases} \Delta_{p,f} u = -A + B \dfrac{F(z)}{z^{p-1}} - (p-1)|\nabla u|^p & \text{on } \Omega \\ u = \log \varepsilon & \text{on } \partial\Omega. \end{cases}$$

The function $\eta(u) \in \mathrm{Lip}_0(\Omega)$ can be used as a test function for (6.6) to obtain

$$(6.7) \qquad \int_\Omega \left[(p-1)\eta(u) - \eta'(u) \right] |\nabla u|^p \mathrm{d}\mu_f = \int_\Omega \left[-A + B \frac{F(z)}{z^{p-1}} \right] \eta(u) \mathrm{d}\mu_f.$$

Let $L(\delta) \doteq \{x : u(x) < \log \delta\}$. Choose

$$\eta(t) = \left[1 - \frac{e^{(p-1)t}}{\delta^{p-1}} \right] 1_{(-\infty, \log \delta)}(t) \in \mathrm{Lip}(\mathbb{R}).$$

Then, from (6.7) and (6.2), and since $\delta \in (0, \varepsilon)$, $\eta(\log \varepsilon) = 0$. Plugging in (6.7) and using (6.5) we deduce

(6.8)
$$\int_{L(\delta)} |\nabla u|^p \mathrm{d}\mu_f \leq \frac{1}{p-1} \int_{L(\delta)} \left[-A + B \frac{F(z)}{z^{p-1}} \right] \eta(u) \mathrm{d}\mu_f \leq \frac{1}{p-1} \|(A - \theta B)_-\|_{L^1(M, \mathrm{d}\mu_f)}$$

Since Q_0 is subcritical on M, by Proposition 4.4 there exists $W \in C^0(M)$, $W > 0$ on M such that

(6.9)
$$\int_M W|\varphi|^p \mathrm{d}\mu_f \leq \int_M |\nabla \varphi|^p \mathrm{d}\mu_f \qquad \forall \varphi \in \mathrm{Lip}_c(M).$$

Now the function $\varphi(x) = \big(u(x) - \log \delta\big) 1_{L(\delta)}(x)$, extended with 0 outside Ω, is an admissible test function for (6.9) and with this choice of φ we obtain

(6.10)
$$\int_{L(\delta)} |\nabla u|^p \mathrm{d}\mu_f \geq \int_{L(\delta)} W|u - \log \delta|^p \mathrm{d}\mu_f.$$

By contradiction assume that there exists a sequence of relatively compact open sets $\{\Omega_j\}$ with smooth boundary such that $\Lambda' \Subset \Omega_j$ and a sequence of associated solutions φ_j of (6.3), such that $\inf_\Lambda \varphi_j \to 0^+$ as $j \to +\infty$. Using Harnack inequality of Theorem 3.1 3), $\phi_j \to 0$ uniformly on Λ as $j \to +\infty$ (note that, to infer $\phi_j \to 0$, we need that each Ω_j contains a fixed domain larger than Λ, which accounts for the presence of Λ'). Having fixed $N > 0$, we choose j_0 large enough that $u_j = \log \varphi_j < -N + \log \delta$ on Λ when $j \geq j_0$. Consequently, $\Lambda \subset \{u_j < \log \delta\} \doteq L_j(\delta)$, and from (6.8) and (6.10) we deduce

$$
\begin{aligned}
N^p \int_\Lambda W \mathrm{d}\mu_f &\leq \int_{L_j(\delta)} W|u_j - \log \delta|^p \mathrm{d}\mu_f \leq \int_{L_j(\delta)} |\nabla u_j|^p \mathrm{d}\mu_f \\
&\leq \frac{1}{p-1} \|(A - \theta B)_-\|_{L^1(M, \mathrm{d}\mu_f)}.
\end{aligned}
$$

This gives a contradiction provided N is large enough, proving the validity of (6.4). $\square$

Remark 6.1. To guarantee (6.4), the second condition in (6.2) cannot be relaxed too much. To see this, let us suppose that $B \equiv 0$, for which the second in (6.2) reads $A_- \in L^1(M, \mathrm{d}\mu_f)$. We first observe that the validity of (6.4) is granted provided that there exists a positive, bounded solution of $Q'_A(u) = 0$ on M. Indeed, if such a u exists, comparing a solution z of (6.3) with $\varepsilon u/\|u\|_{L^\infty(M)}$ with the aid of Proposition 3.3 yields (6.4). If, on the other hand, there exists a positive solution of $Q'_A(u) = 0$ with $u(x) \to +\infty$ as x diverges, (6.4) fails: indeed, if we consider a smooth exhaustion $\{\Omega_j\}$ with $\partial \Omega_j \subset \{u \in [j, j+1)\}$, we compare the solution of (6.3) on Ω_j (with the equality sign) and the function $\varepsilon u/j$, and we let $j \to 0$, it is easy to see that the left-hand side of (6.4) is zero.

Now, consider the case $p = 2$, $f \equiv 0$. In Theorem 3 of [13] we have shown that, if M is a manifold with a pole o, dimension $m \geq 3$ and radial sectional curvature $K_{\mathrm{rad}} \leq 0$,

and for $A \in L^\infty_{\text{loc}}(M)$, conditions

(6.11)
$$\begin{cases} |A(x)| \leq \bar{A}(r(x)) \leq \dfrac{(m-2)^2}{4} \dfrac{1}{r(x)^2} & \text{on } M, \\ r\bar{A}(r) \in L^1(+\infty), \end{cases}$$

imply the existence of a positive bounded solution $u \in C^{1,\mu}_{\text{loc}}(M)$ of $Q'_A(u) = 0$. Thus, in this case (6.4) holds true. On the other hand, by Theorem 11 of [13] if, for $A \in L^\infty_{\text{loc}}(\mathbb{R}^m)$

(6.12)
$$\begin{cases} \bar{A}_1(r(x)) \leq A(x) \leq \bar{A}_2(r(x)) \leq \dfrac{(m-2)^2}{4} \dfrac{1}{r(x)^2} & \text{on } \mathbb{R}^m, \\ r\left[\bar{A}_j(r) - k\dfrac{(m-2)^2}{4r^2}\right] \in L^1(+\infty), \end{cases}$$

for some $k < 0$ and $j \in \{1,2\}$, then there exists a positive solution u of $Q'_A(u) = 0$ on $\mathbb{R}^m$ such that $u(x) \to +\infty$ as $x \to \infty$ and so (6.4) cannot hold. Note that (6.11) is weaker than $A_- \in L^1(M)$. However, we stress that it seems hard to find conditions analogous of (6.11) and (6.12) on more general manifolds and for nonlinear operators.

We now investigate the existence of uniform upper bounds, i.e. independent of Ω, for the solutions of (6.1) with boundary data $\varphi = 1$. The next lemma, Lemma 2.1 of the Introduction, ensures *global* L^∞-estimates. In view of a subtle asymmetry between bounds from below and above, to reach our goal we had to find a new strategy.

Lemma 6.3. *Let M be a Riemannian manifold, $f \in C^\infty(M)$, $p \in (1, +\infty)$ and fix $F(t)$ satisfying (2.6). Let $A, B \in L^\infty_{\text{loc}}(M)$ with $B \geq 0$ a.e. on M. Assume that either*

(i) $B \equiv 0$ *and Q_A is subcritical, or*
(ii) $B \not\equiv 0$ *and Q_A is non-negative.*

Suppose that there exist a smooth, relatively compact open set $\Lambda \Subset M$ and a constant $c > 0$ such that

(6.13)
$$A \leq cB \qquad \text{a.e. on } M\backslash\Lambda,$$

and fix a smooth, relatively compact open set Λ' such that $\Lambda \Subset \Lambda'$, and a constant $\varepsilon > 0$.
Then, there exists a constant $C_\Lambda > 0$ depending on $\varepsilon, p, f, F, c, A, B, \Lambda, \Lambda'$ but not on Ω such that for each smooth, relatively compact open set Ω with $\Lambda' \Subset \Omega$, the solution $0 < z \in C^{1,\mu}(\overline{\Omega})$ of

(6.14)
$$\begin{cases} \Delta_{p,f} z + A(x) z^{p-1} - B(x) F(z) = 0 & \text{on } \Omega, \\ z = \varepsilon & \text{on } \partial\Omega. \end{cases}$$

satisfies

(6.15)
$$z \leq C_\Lambda \qquad \text{on } \Omega.$$

Proof. Without loss of generality, we can suppose that $c > 1$. Using that, by (2.6), $F(t)/t^{p-1} \to +\infty$ as $t \to +\infty$, we can fix $\alpha > \varepsilon$ such that

(6.16)
$$\frac{F(t)}{t^{p-1}} \geq c \qquad \text{for } t \geq \alpha.$$

Consider the open set
$$U = \{x \in \Omega \ : \ z(x) > \alpha\}.$$

We note that $U \Subset \Omega$ and that, by (6.16), z solves

$$(6.17) \qquad \begin{cases} \Delta_{p,f}z + (A - cB)z^{p-1} \geq 0 & \text{on } U \\ z = \alpha & \text{on } \partial U, \end{cases}$$

If $U = \emptyset$, for each Ω, z, then (6.15) trivially holds with $C_\Lambda = \alpha$. Therefore, suppose that $U \neq \emptyset$ for some Ω, z. By (6.17) and (6.13), $\Delta_{p,f}z \geq 0$ on $U \backslash \Lambda$. Thus, in the case $U \cap \Lambda = \emptyset$, $\Delta_{p,f}z \geq 0$ on U. For $\varepsilon > 0$ small, choose a smooth increasing sequence $\{U_\varepsilon\}$ exhausting U as $\varepsilon \to 0$ and such that $z \leq \alpha + \varepsilon$ on ∂U_ε. Applying Proposition 3.3 we infer that $z \leq \alpha + \varepsilon$ on U_ε, thus letting $\varepsilon \to 0$ we get $z \leq \alpha$ on U, contradicting its very definition. Hence, we conclude that $U \cap \Lambda \neq \emptyset$. Note that

$$\sup_{U \cap \Lambda} z \geq \alpha = z_{|\partial U},$$

and since $\Delta_{p,f}z \geq 0$ on $U \backslash \Lambda$, again via Proposition 3.3 we obtain

$$\sup_{U \backslash \Lambda} z = \sup_{\partial(U \backslash \Lambda)} z \leq \max\left\{ \sup_{\partial U} z, \sup_{\partial \Lambda} z \right\} = \max\left\{ \alpha, \sup_{\partial \Lambda \cap U} z \right\} \leq \sup_{U \cap \Lambda} z.$$

It follows that

$$(6.18) \qquad \text{if } U = \{z > \alpha\} \neq \emptyset, \quad \text{then} \qquad \sup_U z \equiv \sup_\Lambda z.$$

To prove (6.15) we proceed by contradiction: if it does not hold, and in view of (6.18) there exists a sequence of triples (Ω_j, z_j, U_j), where Ω_j is a smooth, relatively compact open set such that $\Lambda' \Subset \Omega_j$, z_j is a solution of

$$(6.19) \qquad \begin{cases} \Delta_{p,f}z_j + \left(A - B\dfrac{F(z_j)}{z_j^{p-1}} \right) z_j^{p-1} = 0 & \text{on } \Omega_j, \\ z_j = \varepsilon & \text{on } \partial\Omega_j, \end{cases}$$

$U_j = \{x \in \Omega_j \ : \ z_j(x) > \alpha\}$, and

$$(6.20) \qquad \|z_j\| \doteq \|z_j\|_{L^\infty(\Lambda)} \to +\infty \qquad \text{as } j \to +\infty.$$

Up to taking j large enough, we can suppose that $\|z_j\| > 2\alpha$. Consider the rescaled functions

$$(6.21) \qquad h_j \doteq \frac{z_j}{\|z_j\|} \qquad \text{on } \Omega_j$$

and note that they all solve

$$(6.22) \qquad \begin{cases} \Delta_{p,f}h_j + Ah_j^{p-1} \geq 0 & \text{on } \Lambda', \\ \sup_\Lambda h_j = 1, \qquad h_j \leq \dfrac{\alpha}{\|z_j\|} < \dfrac{1}{2} & \text{on } \Omega_j \backslash U_j. \end{cases}$$

By (6.22) and the half-Harnack inequality of Theorem 3.1 $(3a)$ there exists $C_H > 0$ such that

$$(6.23) \qquad 1 = \|h_j\|_{L^\infty(\Lambda)} \leq C_H \|h_j\|_{L^p(\Lambda', d\mu_f)} \qquad \forall j.$$

Next we define

$$(6.24) \qquad \eta_j(x) = \begin{cases} \dfrac{z_j(x) - \alpha}{\|z_j\|} & \text{on } U_j, \\ 0 & \text{on } M \backslash U_j. \end{cases}$$

Then, $0 \leq \eta_j \in \mathrm{Lip}_c(M)$. Let $V = A - cB$. Thus, $V \leq 0$ on $M \backslash \Lambda$ and, by (6.17), $Q'_V(z_j) \leq 0$ on U_j, therefore

$$
\begin{aligned}
0 \leq Q_V(\eta_j) \quad &= \quad \frac{1}{p} Q'_V(\eta_j)[\eta_j] \\
&= \quad \frac{1}{p\|z_j\|^{p-1}} \int_{U_j} \left[|\nabla z_j|^{p-2} \langle \nabla z_j, \nabla \eta_j \rangle - V(z_j - \alpha)^{p-1} \eta_j \right] \mathrm{d}\mu_f \\
&\leq \quad \frac{1}{p\|z_j\|^p} \int_{U_j} V \left\{ z_j^{p-1} - (z_j - \alpha)^{p-1} \right\} (z_j - \alpha) \mathrm{d}\mu_f \\
&\leq \quad \frac{1}{p\|z_j\|^p} \int_{U_j \cap \Lambda} V \left[z_j^{p-1} - (z_j - \alpha)^{p-1} \right] (z_j - \alpha) \mathrm{d}\mu_f \\
&\leq \quad \frac{1}{p\|z_j\|^p} \int_{U_j \cap \Lambda} V_+(x) \left[z_j^p - (z_j - \alpha)^p \right] \mathrm{d}\mu_f .
\end{aligned}
$$

(6.25)

We observe that on $[\alpha, +\infty)$, the function

$$
\rho(t) = t^p - (t - \alpha)^p
$$

is increasing and

$$
\rho(t) \sim p\alpha t^{p-1} \qquad \text{as } t \to +\infty.
$$

We thus infer the existence of constants $c_1, c_2 > 0$ just depending on α and such that

$$
\rho(t) \leq c_1 t^{p-1} + c_2 \qquad \text{on } [\alpha, +\infty).
$$

Hence

$$
\frac{1}{p\|z_j\|^p} \int_{U_j \cap \Lambda} V_+(x) \left[z_j^p - (z_j - \alpha)^p \right] \mathrm{d}\mu_f \leq \frac{\|V_+\|_{L^\infty(\Lambda)}}{p\|z_j\|^p} \int_{U_j \cap \Lambda} \left(c_1 z_j^{p-1} + c_2 \right) \mathrm{d}\mu_f
$$

$$
\leq \frac{\|V_+\|_{L^\infty(\Lambda)}}{p\|z_j\|} \mathrm{vol}_f(\Lambda) c_1 + \frac{\|V_+\|_{L^\infty(\Lambda)}}{p\|z_j\|^p} \mathrm{vol}_f(\Lambda) c_2 \longrightarrow 0 \qquad \text{as } j \to +\infty.
$$

This fact, together with inequality (6.25), implies that $Q_V(\eta_j) \to 0$ as $j \to +\infty$. Now, in both cases (i) and (ii) in the statement of the lemma, Q_V is subcritical on M. In fact, if (i) holds, $V \equiv A$ and Q_A is assumed to be subcritical, while under the validity of (ii) the subcriticality property (2.10) follows from

$$
Q_V(\varphi) = Q_A(\varphi) + \int_M (cB)|\varphi|^p \mathrm{d}\mu_f \geq \int_M (cB)|\varphi|^p \mathrm{d}\mu_f .
$$

By Theorem 4.1, Q_V has thus a weighted spectral gap, and in particular $\eta_j \to 0$ in $L^p_{\mathrm{loc}}(M)$. Since, by definition (6.21),

$$
h_j = \eta_j + \frac{\alpha}{\|z_j\|} \qquad \text{on } U_j,
$$

we deduce that

$$
\|h_j\|_{L^p(\Lambda' \cap U_j, \mathrm{d}\mu_f)} \to 0 \qquad \text{as } j \to +\infty
$$

and since $h_j \leq \dfrac{\alpha}{\|z_j\|}$ on $\Lambda' \backslash U_j$ we conclude that

$$
h_j \to 0 \quad \text{in } L^p(\Lambda', \mathrm{d}\mu_f) \quad \text{as } j \to +\infty.
$$

This contradicts (6.23) and proves the claimed (6.15). □

Lemmas 6.2 and 6.3 enable us to prove

Proposition 6.1. *Let $(M, \langle\,,\,\rangle)$ be a Riemannian manifold, $f \in C^\infty(M)$, $p \in (1, +\infty)$ and suppose that Q_0 is subcritical on M. Let $V \in L^\infty(M)$ have compact support and assume that Q_V is subcritical on M. Then, there exists a positive solution $g \in C^{1,\mu}_{loc}(M)$ of*

$$(6.26) \qquad -Q'_V(g) = \Delta_{p,f}\,g + V|g|^{p-2}g = 0 \qquad \text{on } M$$

satisfying

$$(6.27) \qquad C^{-1} \le g(x) \le C \qquad \text{on } M$$

for some constant $C > 1$.

Proof. Let Λ be a smooth, relatively compact open set such that $V \equiv 0$ on $M\backslash\Lambda$, and consider an exhaustion $\{\Omega_j\}$ of M by smooth, relatively compact open sets with $\overline{\Lambda} \subset \Omega_1$. We let $\varphi_j \in C^{1,\mu_j}(\overline{\Omega}_j)$ be the positive solution of

$$(6.28) \qquad \begin{cases} Q'_V(\varphi_j) = 0 & \text{on } \Omega_j, \\ \varphi_j = 1 & \text{on } \partial\Omega_j, \end{cases}$$

whose existence is granted by $iv)$ of Proposition 3.4. Our assumptions enable us to apply Lemma 6.2 and 6.3 and to conclude the existence of a constant $C > 1$ independent of j, for which

$$(6.29) \qquad C^{-1} \le \varphi_j(x) \le C \qquad \text{on } \Lambda.$$

On the other hand, because of (6.28) and $V \equiv 0$ on $M\backslash\Lambda$, we have

$$\begin{cases} \Delta_{p,f}\varphi_j = 0 & \text{on } \Omega_j\backslash\overline{\Lambda} \\ \varphi_j = 1 \quad \text{on } \partial\Omega_j, \quad C^{-1} \le \varphi_j \le C \quad \text{on } \partial\Lambda, \end{cases}$$

and thus, by the comparison Proposition 3.3,

$$C^{-1} \le \varphi_j(x) \le C \qquad \text{on } \Omega_j.$$

Hence, the φ_j's are uniformly bounded from above and below. By elliptic estimates $\varphi_j \to g$ for some $g \in C^{1,\mu}_{loc}(M)$ solving $Q'_V(g) = 0$ and satisfying (6.27). $\square$

We are ready to prove Theorem 2.1. We rewrite its statement for the convenience of the reader.

Theorem 6.1. *Let M^m be a Riemannian manifold, $f \in C^\infty(M)$, $p \in (1, +\infty)$, and suppose that Q_0 is subcritical on M. Let $a, b \in L^\infty_{loc}(M)$. Assume that Q_a is subcritical and that*

  i) b_- has compact support;

  ii) for some $\theta > 0$, $(a - \theta b_+)_- \in L^1(M, \mathrm{d}\mu_f)$;

  iii) $a(x) = O\big(b_+(x)\big)$ as x diverges.

Then, there exists $\delta > 0$ such that, if

$$(6.30) \qquad b(x) \ge -\delta \qquad \text{on } M,$$

we can find a solution $u \in C^{1,\mu}_{loc}(M)$ of

$$(6.31) \qquad \begin{cases} \Delta_{p,f}\,u + a(x)u^{p-1} - b(x)F(u) = 0 & \text{on } M, \\ 0 < u \le \|u\|_{L^\infty(M)} < +\infty & \text{on } M. \end{cases}$$

Moreover, if ii) and iii) are replaced by the stronger condition

$$(6.32) \qquad\qquad a(x) \asymp b_+(x) \qquad as\ x\ diverges,$$

then u can be constructed with the further property that $\inf_M u > 0$.

Proof. By assumption $iii)$, there exists a relatively compact, open subset Λ containing $\mathrm{supp}(b_-)$, and a constant $C \geq \theta$ such that

$$(6.33) \qquad\qquad a(x) \leq Cb_+(x) \qquad \text{on } M\backslash\Lambda.$$

The subcriticality of Q_a on M implies, via Theorem 4.1, that Q_a has a weighted spectral gap, that is, there exists $W \in C^0(M)$, $W > 0$ on M such that

$$(6.34) \qquad\qquad \int_M W(x)|\varphi|^p \mathrm{d}\mu_f \leq Q_a(\varphi) \qquad \forall\,\varphi \in \mathrm{Lip}_c(M).$$

In particular, if $A(x) = a(x) + W(x)1_\Lambda(x)$ then Q_A is still subcritical on M. Fix $\varepsilon > 0$ and a relatively compact subset Λ' with $\Lambda \Subset \Lambda'$. Next, consider a smooth, relatively compact open set Ω with $\Lambda' \Subset \Omega$. Applying Lemma 6.1 with the choices $A = a$ (respectively, $A = a + W1_\Lambda$) and $B = b_+$, we produce solutions φ_∞, φ_0 of

$$(6.35) \qquad\qquad \begin{cases} \Delta_{p,f}\varphi_\infty + a\varphi_\infty^{p-1} - b_+F(\varphi_\infty) = 0 & \text{on } \Omega, \\[2mm] \varphi_\infty = \varepsilon & \text{on } \partial\Omega, \end{cases}$$

$$(6.36) \qquad\qquad \begin{cases} \Delta_{p,f}\varphi_0 + (a + W1_\Lambda)\varphi_0^{p-1} - b_+F(\varphi_0) = 0 & \text{on } \Omega, \\[2mm] \varphi_0 = \varepsilon & \text{on } \partial\Omega. \end{cases}$$

By comparison, $\varphi_\infty \leq \varphi_0$. By Lemma 6.2, because of assumption $ii)$ we can guarantee the existence of a constant $\bar{C}_\Lambda(\varepsilon) > 0$ (we emphasize its dependence on ε), independent of Ω, such that

$$(6.37) \qquad\qquad \varphi_\infty \geq \bar{C}_\Lambda(\varepsilon) \qquad \text{on } \Lambda.$$

On the other hand, (6.33) implies that $a + W1_\Lambda \leq Cb_+$ outside of Λ. Hence, Lemma 6.3 ensures the existence of $C_\Lambda(\varepsilon)$ independent of Ω and such that

$$(6.38) \qquad\qquad \big(\varphi_\infty \leq\big)\,\varphi_0 \leq C_\Lambda(\varepsilon) \qquad \text{on } \Omega,$$

and we can also consider the sharpest one, that is,

$$(6.39) \qquad\qquad C_\Lambda(\varepsilon) \doteq \sup_{\substack{\Omega\,:\,\Omega\ \text{open smooth},\\ \Lambda' \Subset \Omega \Subset M}} \|\varphi_0\|_{L^\infty(\Omega)},$$

where φ_0 solves (6.36). As a consequence of the comparison Proposition 3.3, $C_\Lambda(\varepsilon)$ is non-increasing as a function of ε.

Define

$$(6.40) \qquad\qquad \delta = \big(\min_{\bar\Lambda} W\big)\left[\frac{C_\Lambda(\varepsilon)^{p-1}}{F(C_\Lambda(\varepsilon))}\right],$$

and observe that, by assumption (6.30), our definition (6.40) of δ and the fact that $F(t)/t^{p-1}$ is increasing,

$$(6.41) \qquad b_-\left[\frac{F(\varphi_0)}{\varphi_0^{p-1}}\right] \leq \delta 1_\Lambda\left[\frac{F(C_\Lambda(\varepsilon))}{C_\Lambda(\varepsilon)^{p-1}}\right] = \big(\min_{\bar\Lambda} W\big)1_\Lambda \leq W1_\Lambda,$$

on Λ, and the same relation clearly holds on $\Omega \backslash \Lambda$ since there $b_- \equiv 0$. Inserting into (6.36) we obtain

$$\Delta_{p,f}\varphi_0 + a\varphi_0^{p-1} - bF(\varphi_0) \leq 0 \qquad \text{on } \Omega,$$

that is, φ_0 is a supersolution for the problem

(6.42) $$\begin{cases} \Delta_{p,f}u_\Omega + au_\Omega^{p-1} - bF(u_\Omega) = 0 & \text{on } \Omega, \\ u_\Omega = \varepsilon & \text{on } \partial\Omega. \end{cases}$$

On the other hand, φ_∞ is a subsolution for (6.42), and the subsolution-supersolution method (see [25], Theorem 4.14 page 272) gives the existence of u_Ω solving (6.42), satisfying

(6.43) $$\varphi_\infty \leq u_\Omega \leq \varphi_0 \leq C_\Lambda(\varepsilon),$$

and, by (6.37),

(6.44) $$u_\Omega \geq \bar{C}_\Lambda(\varepsilon) \qquad \text{on } \Lambda.$$

Indeed, in our setting we can describe a simple iteration scheme to produce u_Ω: set $V_0 = a + W1_\Lambda$. For $n \geq 1$, we inductively define $\varphi_n \in C^{1,\mu}(\overline{\Omega})$ as the positive solution of

(P_n) $$\begin{cases} \Delta_{p,f}\varphi_n + V_n\varphi_n^{p-1} - b_+F(\varphi_n) = 0 & \text{on } \Omega, \\ \varphi_n = \varepsilon & \text{on } \partial\Omega \end{cases}$$

where

(6.45) $$V_n \doteq a + b_- \left(\frac{F(\varphi_{n-1})}{\varphi_{n-1}^{p-1}} \right).$$

We claim that each φ_n exists and that $\{\varphi_n\}$ is a non-increasing sequence bounded below by φ_∞. Indeed, by (6.41), $V_1 \leq V_0$ and so $\lambda_{V_1}(\Omega) > 0$, which ensures the existence of φ_1 by Proposition 6.1. Moreover, φ_1 solves

(6.46) $$\begin{cases} \Delta_{p,f}\varphi_1 + V_0\varphi_1^{p-1} - b_+F(\varphi_1) \geq 0 & \text{on } \Omega; \\ \varphi_1 = \varepsilon & \text{on } \partial\Omega, \end{cases}$$

whence by comparison $\varphi_1 \leq \varphi_0$ on Ω. Now, this last inequality (and the monotonicity of $F(t)/t^{p-1}$) gives $V_2 \leq V_1$, so that $\lambda_{V_2}(\Omega) > 0$ and (P_2) admits a unique positive solution φ_2. Again from $V_2 \leq V_1$, φ_2 turns out to be a subsolution of (P_1), hence $\varphi_2 \leq \varphi_1$ by comparison. Repeating the argument above shows the monotonicity of $\{\varphi_n\}$. The positivity of φ_{n-1} ensures that $V_n \geq V_\infty$, so by comparison $\varphi_n \geq \varphi_\infty$ for each n. The inequalities $a \leq V_n \leq V$ and $\varphi_\infty \leq \varphi_n \leq \varphi_0$ for each n then guarantee, via Theorem 3.1 (2), that there exists $\mu \in (0,1)$ such that the $C^{1,\mu}$-norm of φ_n on Ω is uniformly bounded. Therefore, a subsequence of $\{\varphi_{n_k}\}_k \subset \{\varphi_n\}_n$ converges in the C^1-norm, as $k \to +\infty$, to some non-negative $v_\Omega \in C^1(\overline{\Omega})$, and since $\{\varphi_n\}$ is a non-increasing sequence the whole $\{\varphi_n\}$ converges to v_Ω uniformly. By letting $k \to +\infty$ in the weak definition of (P_n) along the subsequence $\{n_k\}$, we deduce that v_Ω is a weak solution of (6.42). Now, we choose an exhaustion Ω_j, and let $u_j = u_{\Omega_j}$ be the solution of (6.42) on Ω_j constructed above. Note that

$$\varphi_{\infty,j} \leq u_j \leq C_\Lambda(\varepsilon) \text{ on } \Omega_j, \qquad u_j \geq \bar{C}_\Lambda(\varepsilon) \text{ on } \Lambda.$$

where $\varphi_{\infty,j}$ solves (6.35) on Ω_j. Hence, by local elliptic estimates the sequence $\{u_j\}$ subconverges to some global solution $u \geq 0$ of (6.31) on the whole M, satisfying

$$u \leq C_\Lambda(\varepsilon) \quad \text{on } M, \qquad u \geq \bar{C}_\Lambda(\varepsilon) \quad \text{on } \Lambda.$$

By Harnack inequality, $u > 0$ on M, and we have proved the first part of our theorem. The final step is to guarantee that, when $a \asymp b_+$, $\inf_M u > 0$.

This will be accomplished by proving corresponding lower bounds for $\varphi_{\infty,j}$. Up to reducing θ, we can suppose that

$$a - \theta b_+ \geq 0$$

outside some compact set. Using the assumption that $F(t)/t^{p-1} \to 0$ as $t \to 0$, fix $\alpha \in (0, \varepsilon)$ small enough that

$$\frac{F(t)}{t^{p-1}} \leq \theta \qquad \text{for } t \in [0, \alpha].$$

Inspecting problem (6.35) and noting that $\alpha < \varepsilon$, we deduce that on the subset $U_j \doteq \{\varphi_{\infty,j} < \alpha\} \Subset \Omega_j$ the function $\varphi_{\infty,j}$ solves $\Delta_{p,f}\varphi_{\infty,j} + (a - \theta b_+)\varphi_{\infty,j}^{p-1} \leq 0$, hence in particular

$$(6.47) \qquad \begin{cases} \Delta_{p,f}\varphi_{\infty,j} - (a - \theta b_+)_-\varphi_{\infty,j}^{p-1} \leq 0 & \text{on } U_j, \\ \varphi_{\infty,j} = \alpha & \text{on } \partial U_j, \end{cases}$$

In view of the boundary regularity requirements to apply Proposition 3.3, we fix a smooth open set S_j satisfying

$$\{x : \varphi_{\infty,j} \leq \alpha/2\} \Subset S_j \Subset U_j,$$

so that

$$(6.48) \qquad \begin{cases} \Delta_{p,f}\varphi_{\infty,j} - (a - \theta b_+)_-\varphi_{\infty,j}^{p-1} \leq 0 & \text{on } S_j, \\ \varphi_{\infty,j} \geq \frac{\alpha}{2} & \text{on } \partial S_j, \end{cases}$$

Next, we use the fact that $V \doteq -(a(x) - \theta b_+(x))_-$ is compactly supported and Q_V is subcritical on M (being Q_0 subcritical by assumption, and $V \leq 0$): by Proposition 6.1, there exists $g \in C^{1,\mu}_{\text{loc}}(M)$ solution of

$$(6.49) \qquad \begin{cases} \Delta_{p,f}g - (a - \theta b_+)_-g^{p-1} = 0 & \text{on } M, \\ 0 < \inf_M g \leq \|g\|_{L^\infty(M)} < +\infty. \end{cases}$$

Rescaling g by multiplying by a positive constant we can assume that

$$(6.50) \qquad 0 < \inf_M g \leq \sup_M g \leq \frac{\alpha}{2}.$$

Comparing (6.48) and (6.49) on S_j, by Proposition 3.3 we infer that $\varphi_{\infty,j} \geq g$ on S_j, and thus by (6.50)

$$u_j \geq \varphi_{\infty,j} \geq \min\left\{\frac{\alpha}{2}, g\right\} = g \qquad \text{on } \Omega_j$$

Passing to the limit, we finally get

$$u(x) \geq g(x) \quad \text{on } M,$$

and $\inf_M u > 0$ follows since $\inf_M g > 0$. $\qquad\qquad\qquad\qquad\qquad\qquad \square$

In the proof of the above result, the parameter ε plays no role. However, a judicious choice of ε is crucial in the following proof of Theorem 2.2.

Theorem 6.2. *Let M^m be a Riemannian manifold, $f \in C^\infty(M)$ and $p \in (1, +\infty)$. Suppose that Q_0 is subcritical on M and let $a \in L^\infty_{\mathrm{loc}}(M)$ be such that Q_a is subcritical on M. Consider $b \in L^\infty_{\mathrm{loc}}(M)$, and assume*

    i) $b_-(x)$ *has compact support;*

    ii') $a(x) \leq 0$ *outside a compact set;*

    iii') $a(x), b(x) \in L^1(M, \mathrm{d}\mu_f)$.

Fix a nonlinearity $F(t)$ satisfying (2.6). Then, there exists a sequence $\{u_k\} \subset C^{1,\mu}_{\mathrm{loc}}(M)$ of distinct weak solutions of

(6.51)
$$\begin{cases} \Delta_{p,f} u_k + a(x)u_k^{p-1} - b(x)F(u_k) = 0 & \text{on } M \\ 0 < u_k \leq \|u_k\|_{L^\infty(M)} < +\infty, \end{cases}$$

such that $\|u_k\|_{L^\infty(M)} \to 0$ as $k \to +\infty$. If we replace ii') and iii') by the stronger condition

    iv') $a(x), b(x)$ *have compact support,*

then each u_k also satisfies $\inf_M u_k > 0$.

Proof. We first observe that our set of assumptions on a and b is a special case of the one in Theorem 6.1, namely we just include the requirement that $a \leq 0$ outside some compact set. Hence, the constructions in the previous theorem hold as well in our setting, and we will refer to the proof of Theorem 6.1 also for relevant definitions.

Up to enlarging Λ we can assume that $a \leq 0$ on $M \backslash \Lambda$. Let Ω satisfy $\Lambda' \Subset \Omega$. By Lemma 6.3 applied with $A = a + W1_\Lambda$, $B \equiv 0$ there exists a uniform constant $\hat{C}$, independent of Ω, such that the solution ψ of

(6.52)
$$\begin{cases} \Delta_{p,f}\psi + (a + W1_\Lambda)\psi^{p-1} = 0 & \text{on } \Omega, \\ \psi = 1 & \text{on } \partial\Omega \end{cases}$$

satisfies $\psi \leq \hat{C}$ on Ω. By comparison and recalling our definition of φ_0 in (6.36), we get $\varphi_0 \leq \varepsilon\psi$. Therefore, by our definition of $C_\Lambda(\varepsilon)$,

(6.53)
$$C_\Lambda(\varepsilon) \leq \varepsilon\hat{C} \to 0 \qquad \text{as } \varepsilon \to 0.$$

Combining (6.53), the monotonicity of $C_\Lambda(\varepsilon)$ and property $F(t)/t^{p-1} \to 0$ as $t \to 0$ in (2.6), we deduce that, given any b with b_- compactly supported, there exists ε_0 sufficiently small such that

(6.54)
$$b_- \leq \left(\min_{\overline{\Lambda}} W\right)\left[\frac{C_\Lambda(\varepsilon)^{p-1}}{F(C_\Lambda(\varepsilon))}\right] \qquad \text{for each } \varepsilon \leq \varepsilon_0.$$

Fix such ε_0 and follow the arguments in Theorem 6.1 with $\varepsilon = \varepsilon_0$. Note that, as in (6.41),

$$b_-(x)\left(\frac{F(\varphi_0(x))}{\varphi_0(x)^{p-1}}\right) \leq b_-(x)\left[\frac{F(C_\Lambda(\varepsilon))}{C_\Lambda(\varepsilon)^{p-1}}\right] \leq 1_\Lambda(x)\left(\min_{\overline{\Lambda}} W\right) \leq W(x)1_\Lambda(x),$$

which is the key step to produce the local solutions of (6.42), and now it does not require (6.30). Proceeding with the construction, we get a solution u_0 of

(6.55)
$$\begin{cases} \Delta_{p,f} u_0 + a(x)u_0^{p-1} - b(x)F(u_0) = 0 & \text{on } M, \\ 0 < u_0 \leq C_\Lambda(\varepsilon_0) & \text{on } M, \end{cases}$$

and $\inf_M u_0 > 0$ when (iv') holds. Next, choose $\varepsilon_1 < \varepsilon_0$ small enough that

$$C_\Lambda(\varepsilon_1) < \min_{\overline{\Lambda}} u_0.$$

Proceeding as above with ε_1 replacing ε_0 we get a solution u_1 of

(6.56)
$$\begin{cases} \Delta_{p,f} u_1 + a(x)u_1^{p-1} - b(x)F(u_1) = 0 & \text{on } M, \\ 0 < u_1 \le C_\Lambda(\varepsilon_1) & \text{on } M, \end{cases}$$

and $\inf_M u_1 > 0$ when (iv') is in force. By our choice of ε_1, $u_1 < u$ on Λ, thus in particular the two solutions are different. We can now repeat the procedure inductively by choosing, at each step, $\varepsilon_k < \varepsilon_{k-1}$ such that

$$C_\Lambda(\varepsilon_k) < \min_{\overline{\Lambda}} u_{k-1},$$

obtaining a solution u_k of

$$\Delta_{p,f} u_k + a(x)u_k^{p-1} - b(x)F(u_k) = 0 \qquad \text{on } M$$

satisfying

(6.57) $0 < u_k \le C_\Lambda(\varepsilon_k)$ on M, $u_k < u_{k-1}$ on Λ, $\inf_M u_k > 0$ when (iv') holds.

By construction, $C_\Lambda(\varepsilon_k) \to 0$ as $k \to +\infty$, hence $\{u_k\}$ is the desired sequence. □

Remark 6.2. The key point that allows, in Theorem 6.2, to get rid of (6.30) is the validity of the asymptotic relation

(6.58) $$C_\Lambda(\varepsilon) \to 0^+ \qquad \text{as } \varepsilon \to 0^+,$$

which is granted via the presence of an uniform L^∞-bound for solutions ψ of (6.52). For general a, b, just satisfying the assumptions of Proposition 6.1, (6.58) may not hold. As an example, consider the hyperbolic space $\mathbb{H}^m$ of sectional curvature -1. For each $\tau \ge 1$, the radial functions

$$u_\tau(x) = \left(2\cosh^2\left(\frac{r(x)}{2}\right)\right)^{-\frac{m-2}{2}} \beta_\tau\left(\tanh\left(\frac{r(x)}{2}\right)\right),$$

where

$$\beta_\tau(t) = \frac{(\tau^2 - t^2)^{-\frac{m-2}{2}}}{m(m-2)\tau^2}$$

are all solutions of

$$\Delta u + \frac{m(m-2)}{4}u - u^{\frac{m+2}{m-2}} = 0 \qquad \text{on } \mathbb{H}^m.$$

Moreover, they are decreasing functions of $r(x)$, and the sequence $\{u_\tau\}$ is monotone decreasing. For each $\varepsilon > 0$, consider $\Omega = u_1^{-1}\{(\varepsilon, +\infty)\}$. Then, for ε small enough in such a way that $\Omega \ne \emptyset$, by the definition of $C_\Lambda(\varepsilon)$ in (6.39) we deduce

$$C_\Lambda(\varepsilon) \ge \|u_1\|_{L^\infty(\Omega)} = u_1(0) = \frac{2^{-\frac{m-2}{2}}}{m(m-2)},$$

preventing from the validity of (6.58).

We briefly comment on the sharpness of the subcriticality assumption for Q_a in Theorem 2.1, and for this reason we now state a result that improves on a theorem in [48]. First of all, we extend definition (3.8) to arbitrary subsets $\Lambda \subset M$, that is, we define the fundamental tone $\lambda_V(\Lambda)$ by setting

$$(6.59) \qquad \lambda_V(\Lambda) = \sup \lambda_V(\Omega)$$

where the supremum is taken over all open subsets $\Omega \subset M$ with smooth boundary such that $\overline{\Lambda} \subset \Omega$.

Proposition 6.2. *Let $(M, \langle\,,\,\rangle)$ be a Riemannian manifold $f \in C^\infty(M)$, $p \in (1, \infty)$ and let $a(x) \in L^\infty_{\mathrm{loc}}(M)$, $b(x) \in C^0(M)$. Define*

$$B_0 \doteq \{x \in M \ : \ b(x) \le 0\},$$

Let Ω be an open domain containing $\overline{B}_0$ and such that there exists a positive, bounded solution $u \in C^0(\overline{\Omega}) \cap W^{1,p}_{\mathrm{loc}}(\Omega)$ of

$$(6.60) \qquad \Delta_{p,f} u + a(x) u^{p-1} - b(x) F(u) \le 0 \qquad on \ \Omega,$$

for some nonlinearity $F(t)$ that satisfies (2.6). Then

$$(6.61) \qquad \lambda_a(B_0) \ge 0.$$

Remark 6.3. Let us consider the case $p = 2$, $f \equiv 0$. The first Dirichlet eigenvalue of the Laplacian on a geodesic ball B_r grows like r^{-2} as $r \to 0^+$, thus $\lambda_a(B_r) > 0$ provided r is sufficiently small and one may think that condition (6.61) expresses the fact that b_- is, loosely speaking, small in a spectral sense.

Proof. Let u be as above and by contradiction assume that $\lambda_a(B_0) \doteq \lambda < 0$. Then, by definition (6.59) we can find a sequence $\{U_i\}$ of open sets with smooth boundaries such that

$$\overline{B}_0 \subset U_i \subset \overline{U}_i \subset \Omega, \qquad \lambda_a(U_i) \to \lambda \quad \text{as } i \to +\infty.$$

Take a nested sequence $\{V_i\}$ of smooth open sets shrinking to B_0 such that:

$$V_i \subset \left\{ x : b(x) < \frac{1}{i} \right\}, \qquad \overline{B}_0 \subset V_{i+1} \subset \overline{V}_{i+1} \subset V_i \subset \Omega, \qquad \bigcap_{i=1}^{+\infty} V_i = \overline{B}_0.$$

Then, up to replacing U_i with $V_i \cap U_i$ and smoothing corners, by the monotonicity of eigenvalues we can further suppose that $\{U_i\}$ satisfies

$$(6.62) \qquad \bigcap_{i=1}^{+\infty} U_i = \overline{B}_0, \qquad b < \frac{1}{i} \ \text{on } U_i.$$

By the definition of $\lambda_a(U_i)$, there exists a smooth, relatively compact open set $\Omega_i \Subset U_i$ for which $\lambda_i \doteq \lambda_a(\Omega_i) \le \lambda_a(U_i) + 1/i$, so that clearly $\lambda_i \to \lambda$ as i diverges. Corresponding to λ_i there exists a positive eigenfunction $\varphi_i \in C^{1,\mu}(\overline{\Omega}_i)$ satisfying

$$\begin{cases} Q'_a(\varphi_i) = \lambda_i |\varphi_i|^{p-2} \varphi_i & \text{on } \Omega_i \\ \varphi_i = 0 & \text{on } \partial\Omega_i. \end{cases}$$

Setting $h = \log u$ using (3.14) and (6.60) we see that h solves

$$\Delta_{p,f} h \le -a(x) + b(x) \frac{F(u)}{u^{p-1}} - (p-1) |\nabla h|^p.$$

Integrating on Ω_i against φ_i^p and proceeding as in the proof of $i) \Rightarrow iii)$ in Proposition 3.4 we obtain

$$\int_{\Omega_i} a(x)\varphi_i^p \mathrm{d}\mu_f - \int_{\Omega_i} b(x)\frac{F(u)}{u^{p-1}}\varphi_i^p \mathrm{d}\mu_f \leq \int_{\Omega_i} |\nabla\varphi_i|^p \mathrm{d}\mu_f.$$

Now, since the φ_i's are eigenfunctions, $pQ_a(\varphi) = \lambda_i \|\varphi_i\|_{L^p(\Omega_i, \mathrm{d}\mu_f)}^p$. Therefore, inserting into the above, observing that $b < 1/i$ on Ω_i and using the monotonicity of $f(t)/t^{p-1}$ we deduce

$$(6.63) \qquad 0 \geq \int_{\Omega_i} \left[-\lambda_i - b(x)\frac{F(u)}{u^{p-1}} \right] \varphi_i^p \mathrm{d}\mu_f \geq \int_{\Omega_i} \left[-\lambda_i - \frac{F(\|u\|_{L^\infty(\Omega)})}{i\|u\|_{L^\infty(\Omega)}^{p-1}} \right] \varphi_i^p \mathrm{d}\mu_f.$$

Concluding, as $\lambda_i \to \lambda < 0$, taking i sufficiently large the previous inequality yields the desired contradiction. $\qquad\square$

In view of Proposition 6.2, and since we made no assumptions on the size of the set $\{x : b(x) = 0\}$ in Theorem 2.1, the existence of a bounded solution of (2.12) on M when b changes sign requires at least that $Q_a \geq 0$ on the whole M. We feel interesting to investigate the validity of Theorem 2.1 when assumption $i)$ is replaced with the requirement that Q_a be non-negative and with a ground state.

Remark 6.4. We conclude this section with a remark on the role of (2.11) in Theorem 2.1. that is, $\|b_-\|_{L^\infty(M)} \leq \delta$. Denote with B_0 the relatively compact set $\{x : b(x) < 0\}$, and for each $\varepsilon \in (0, 1)$ choose a smooth relatively compact set $\Omega_\varepsilon \subset \{x : b_-(x) \geq (1 - \varepsilon)\|b_-\|_{L^\infty(M)}\}$. Then we let φ_ε be the positive eigenfunction of

$$\begin{cases} Q_a'(\varphi_\varepsilon) = \lambda_a(\Omega_\varepsilon)\varphi_\epsilon^{p-1} & \text{on } \Omega_\varepsilon \\ \varphi_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon, \qquad \varphi_\varepsilon > 0 \quad \text{on } \Omega_\varepsilon. \end{cases}$$

Note that $\lambda_a(\Omega_\varepsilon) > 0$ by assumption. Reasoning as in Proposition 6.2 to get (6.63) we obtain

$$\lambda_a(\Omega_\varepsilon)\|\varphi_\varepsilon\|_{L^p(\Omega_\varepsilon)}^p \geq \int_{\Omega_\varepsilon} b_-\frac{F(u)}{u^{p-1}}\varphi_\varepsilon^p \mathrm{d}\mu_f,$$

hence using the definition of Ω_ε

$$\lambda_a(\Omega_\varepsilon)\|\varphi_\varepsilon\|_{L^p(\Omega_\varepsilon)}^p \geq (1 - \varepsilon)\|b_-\|_{L^\infty(M)} \left[\inf_{B_0} \frac{F(u)}{u^{p-1}}\right] \|\varphi_\varepsilon\|_{L^p(\Omega_\varepsilon)}^p$$

in other words

$$(6.64) \qquad \|b_-\|_{L^\infty(M)} \left[\inf_{B_0} \frac{F(u)}{u^{p-1}}\right] \leq \inf_{\varepsilon \in (0,1)} \frac{\lambda_a(\Omega_\varepsilon)}{1 - \varepsilon}.$$

The above inequality helps to understand the relationship between the L^∞-norm of the negative part of b and a lower bound for u on B_0. In particular, in view of the relation $F(t)/t^{p-1} \to 0$ as $t \to 0$ in (2.6), a larger size of b_- forces u to squeeze towards zero on B_0.

7. Proofs of our geometric corollaries, and concluding comments

In this section, we prove Theorems 2.3, 1.5, 1.6 and their Corollaries 1.1 and 1.2.

Proof of Theorem 2.3. Taking into account that the conformal factor u in the deformation

$$\widetilde{\langle\,,\,\rangle} = u^{\frac{4}{m-2}}\langle\,,\,\rangle$$

shall satisfy (1.2), the theorem follows immediately from Theorems 2.1 and 2.2. When the two-sided bound

(7.1) $$C^{-1}\langle\,,\,\rangle \leq \widetilde{\langle\,,\,\rangle} \leq C\langle\,,\,\rangle$$

holds, $(M, \widetilde{\langle\,,\,\rangle})$ is complete if and only if $(M, \langle\,,\,\rangle)$ is so. Furthermore, because of (7.1) since $\langle\,,\,\rangle$ is non-parabolic the same holds for $\widetilde{\langle\,,\,\rangle}$. Indeed, it is easy to see that (7.1) induces a similar two-sided bound between the capacities cap and $\widetilde{\text{cap}}$ of the Laplace-Beltrami operators of the two metrics (with, say, supersolution $g = 1$), whence the preservation of parabolicity follows from Theorem 4.1. This concludes the proof. $\quad\square$

Proof of Theorem 1.5. It follows directly from case (II) of Theorem 2.3: it is enough to observe that, if $s(x) \geq 0$ on the whole M and M is non-parabolic (i.e., $-\Delta$ is subcritical), then the conformal Laplacian $L_{\langle\,,\,\rangle}$ is subcritical. $\quad\square$

Proof of Corollary 1.1. We begin with performing a "reduction" argument that goes back to the original works of Schoen-Yau on the positive mass theorem, see [47], pp. 82-83. Via a cut-off procedure and a careful analysis of Schrödinger operators on weighted spaces, they showed that there exists a conformal deformation

(7.2) $$\langle\,,\,\rangle_1 = u_1^{\frac{4}{m-2}}\langle\,,\,\rangle$$

of the original asymptotically flat metric in such a way that $(M, \langle\,,\,\rangle_1)$ is still asymptotically flat and has zero scalar curvature outside a compact set. Moreover, $\langle\,,\,\rangle_1$ is uniformly equivalent to $\langle\,,\,\rangle$ (actually much more is true, but this is enough for our purposes). Next, by the very definition of asymptotic flatness, the metric $\langle\,,\,\rangle_1$ on each end U_j with respect to the compact set K is bi-Lipschitz equivalent to the Euclidean one on $\mathbb{R}^m \backslash B_r(0)$, hence proceeding as in the proof of Theorem 2.3 we deduce that $(M, \langle\,,\,\rangle_1)$ is non-parabolic. Consequently, since $\langle\,,\,\rangle_1$ has non-negative scalar curvature, the conformal Laplacian $L_{\langle\,,\,\rangle_1}$ is subcritical. Applying previous Theorem 1.5 to the background manifold $(M, \langle\,,\,\rangle_1)$, we get the existence of a family of conformal deformations to scalar curvature $\widetilde{s}(x)$ which, after composing with the deformation (7.2), concludes the proof of the corollary. $\quad\square$

Proof of Theorem 1.6. It follows directly from (I) of Theorem 2.3. Note that, according to Remark 4.6, $L_{\langle\,,\,\rangle}$ is subcritical if and only if it admits a positive Green kernel. $\quad\square$

Proof of Corollary 1.2. Condition $K \leq -\kappa^2$ implies, via Theorem 5.1, that the Hardy inequality

$$\int_M (\chi \circ r)\varphi^2 \mathrm{d}x \leq \int_M |\nabla\varphi|^2 \mathrm{d}x,$$

holds for each $\varphi \in \text{Lip}_c(M)$, where

$$\chi(r) = \frac{1}{4}\left(g_\kappa(r)^{m-1}\int_r^{+\infty}\frac{\mathrm{d}s}{g_\kappa(s)^{m-1}}\right)^{-2}.$$

Now, since by (5.25) the Hardy weight satisfies

$$\chi(r) \geq \frac{(m-1)^2\kappa^2}{4} \qquad \text{on } \mathbb{R}^+,$$

using assumption (1.24) we deduce that

$$-\frac{s(x)}{c_m} = -\frac{m-2}{4(m-1)}s(x) \leq \frac{(m-1)^2\kappa^2}{4} \leq \chi\big(r(x)\big),$$

thus the conformal Laplacian $L_{\langle,\rangle} = -\Delta + s/c_m$ is subcritical by Proposition 4.5 (clearly, $-s/c_m \not\equiv (\chi \circ r)$ since this latter tends to infinity at o). Now, if (1.25) holds outside of a compact set, assumption i) of Theorem 2.3 is met. Tracing $K \leq -\kappa^2$ we deduce

$$s(x) \leq -m(m-1)\kappa^2,$$

which coupled with (1.24) and (1.25) implies $s(x) \asymp \widetilde{s}(x)$ as x diverges. Applying Theorem 1.6 we eventually have the desired conformal deformation to a uniformly equivalent metric $\widetilde{\langle,\rangle}$. $\qquad\square$

We conclude with a couple of remarks. In the Introduction, the prototype cases of Euclidean and hyperbolic space helped us to have a picture of the variety of phenomena concerning the prescribed curvature problem. We have seen that the uniqueness of the conformal deformation in Theorem 2.1 fails to hold for sign-changing $\widetilde{s}(x)$, and that fastly decaying solutions coexist with solutions bounded from below and above by positive constants. In particular, assumptions like (1.25) do not imply a control of the decay of the conformal factor from both sides by two comparable quantities. However, when $\widetilde{s} < 0$ on the whole M, something more precise can be said about uniqueness and asymptotic behaviour of solutions u of the Yamabe equation (1.2). As above, consider the prototype case of $\mathbb{H}^m_\kappa$, and suppose that

$$(7.3) \qquad\qquad -C_1 \leq \widetilde{s}(x) \leq -C_2 < 0 \qquad \text{on } \mathbb{H}^m_\kappa.$$

By Theorem 3.4 in [74] with the choice $\beta = 0$ (or even by Theorem 4 in [8]), assumption (7.3) guarantees that the conformal deformation given in Aviles-McOwen's Theorem 1.3 and in Corollary 1.2 is the *unique* conformal deformation realizing $\widetilde{s}(x)$ and such that the conformal factor satisfies $\inf_{\mathbb{H}^m_\kappa} u > 0$. Moreover, by [71, 61] (a simpler form can also be found in Theorem 2.3 of [74]), in the same assumptions each solution of (1.2) satisfies $\sup_{\mathbb{H}^m_\kappa} u < +\infty$. On the other hand, since $\widetilde{s} < 0$ on $\mathbb{H}^m_\kappa$, estimates from below for positive solutions of the Yamabe type equation (2.1) have been provided in [74]. Applying Theorem 2.4 of [74] with the choices

$$m > 4, \quad \delta = 0, \quad \beta = -1 - \epsilon, \quad \alpha < 0 \ \text{ arbitrary}, \quad \sigma = \frac{m+2}{m-2}, \quad \gamma = \frac{m-2}{2}H > H$$

with $\epsilon > 0$, we deduce that any solution u of (1.2) satisfies

$$(7.4) \qquad\qquad u(x) \geq Ce^{-\frac{m-2}{2}\kappa r(x)} \qquad \text{on } \mathbb{H}^m_\kappa,$$

for some $C > 0$. Indeed, the right-hand side in (7.4) is exactly the asymptotic decay of the solutions that create the conformally deformed metrics in Theorem 1.4, and in fact it is also the decay of a radial solution of $L_{\langle,\rangle}u = 0$ on $\mathbb{H}^m_\kappa$. In summary, when (7.3) holds, the decay of the solutions $\{u_j\}$ in Theorem 1.4 is the minimal one that a solution of the Yamabe equation with (7.3) in force can have (if $m > 4$), while the function u produced in Corollary 1.2 (we call it $\hat{u}$) is the unique solution which is bounded below by a positive constant, and indeed it also has the maximal possible order at infinity, as each solution shall be bounded above by a constant. This intriguing scenario is

enriched by the fact that, by Theorem 1.1 in [71], the Yamabe equation on $\mathbb{H}_\kappa^m$ also admits a solution u_c giving rise to a complete metric $\widetilde{\langle,\rangle}$ whenever

$$(7.5) \qquad \widetilde{s} < 0 \;\; \text{on } M, \text{ and} \quad \widetilde{s}(x) \geq -Cr(x)^2 \quad \text{as } r \to +\infty,$$

for some $C > 0$. It is still not clear whether u_c coincides with $\hat{u}$ or not, or even if (7.5) ensures the existence of a whole infinite family of solutions, distinct from $\{u_j\}$ and $\hat{u}$, giving rise to complete metrics.

Remark 7.1. Based on the hints in [72], we conjecture that there exists a conformal deformation of the hyperbolic metric that gives rise to a complete metric of scalar curvature $\widetilde{s}(x)$ whenever $|\widetilde{s}(x)| \leq Cr(x)^2$, where $r(x)$ is the distance from a fixed origin of $\mathbb{H}_\kappa^m$ and $C > 0$. See also [13] for some comments.

APPENDIX: THE OBSTACLE PROBLEM AND THE PASTING LEMMA

The aim of this section is to present a proof of the pasting Lemma 3.1. The argument is divided in three steps. First, observe that our assumptions in Lemma 3.1 imply $\lambda_V(\Omega_2) \geq 0$. Therefore, the obstacle problem that we shall consider below is solvable on relatively compact open subsets of Ω_2. Secondly, the minimizing properties of its solutions yield a quick proof of the fact that the minimum of two positive supersolutions is still a supersolution. Finally we obtain Lemma 3.1 by refining the argument used in the second step. The idea of the proof is close to that in Section 3 of [53]. Hereafter, each $W^{1,p}$-norm is intended to be with respect to the measure $\mathrm{d}\mu_f$

Let $\Omega \Subset M$ be a relatively compact, open subset and $V \in L^\infty(\Omega)$. Given ψ measurable and $\theta \in W^{1,p}(\Omega)$ such that $\psi \leq \theta$ a.e. on Ω, we define the non-empty, closed, convex set

$$(7.6) \qquad \mathcal{K}_{\psi,\theta} \doteq \left\{ \varphi \in W^{1,p}(\Omega) \mid \varphi \geq \psi \text{ a.e. and } \varphi - \theta \in W_0^{1,p}(\Omega) \right\}.$$

We say that $u \in \mathcal{K}_{\psi,\theta}$ solves the obstacle problem if

$$(7.7) \qquad Q_V'(u)[\varphi - u] \geq 0 \qquad \text{for each } \varphi \in \mathcal{K}_{\psi,\theta},$$

that is, weakly,

$$(7.8) \qquad \int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla(\varphi - u)\rangle \mathrm{d}\mu_f - \int_\Omega V|u|^{p-2}u(\varphi - u)\mathrm{d}\mu_f \geq 0$$

Note that, for each non-negative $\widehat{\varphi} \in C_c^1(\Omega)$, the function $\varphi = u + \widehat{\varphi} \in \mathcal{K}_{\psi,\theta}$, and putting into (7.7) we get that u solving (7.7) satisfies $Q_V'(u) \geq 0$, that is, u is a supersolution. We address the solvability of the obstacle problem in the next

Theorem 7.1. *Let M be a Riemannian manifold, $f \in C^\infty(M)$, $p \in (1,+\infty)$ and $V \in L_{\mathrm{loc}}^\infty(M)$. Let $\Omega \Subset M$ be a relatively compact open set with Lipschitz boundary for which $\lambda_V(\Omega) > 0$. Suppose that the obstacle ψ satisfies $0 \leq \psi \leq \theta$ a.e. on Ω, for some $\theta \in W^{1,p}(\Omega)$. Then, there exists a solution $u \in \mathcal{K}_{\psi,\theta}$ of (7.7).*

Proof. We consider the translated set

$$\bar{\mathcal{K}}_{\psi,\theta} \doteq \mathcal{K}_{\psi,\theta} - \theta = \left\{ \bar{g} \; : \; \bar{g} + \theta \in \mathcal{K}_{\psi,\theta} \right\} \subset W_0^{1,p}(\Omega).$$

Note that, since $\psi \geq 0$, $\forall \, \bar{g} \in \bar{\mathcal{K}}_{\psi,\theta}$ we have $\bar{g} + \theta \geq 0$.

We now define the functional $\mathcal{F} : \bar{\mathcal{K}}_{\psi,\theta} \to W_0^{1,p}(\Omega)^*$ by setting: $\forall \bar{u} \in \bar{\mathcal{K}}_{\psi,\theta}$ and $\varphi \in W_0^{1,p}(\Omega)$,

$$(7.9) \qquad \mathcal{F}(\bar{u})[\varphi] \doteq Q_V'(\bar{u} + \theta)[\varphi] = \mathcal{A}(\bar{u})[\varphi] - \mathcal{B}(\bar{u})[\varphi]$$

with

$$\mathcal{A}(\bar{u})[\varphi] = \int_\Omega |\nabla(\bar{u}+\theta)|^{p-2} \langle \nabla(\bar{u}+\theta), \nabla\varphi\rangle \mathrm{d}\mu_f$$

and

$$\mathcal{B}(\bar{u})[\varphi] = \int_\Omega V\,|\bar{u}+\theta|^{p-2}\,(\bar{u}+\theta)\,\varphi\mathrm{d}\mu_f.$$

Clearly, $\bar{u} \in \bar{\mathcal{K}}_{\psi,\theta}$ solves the obstacle problem

$$(7.10) \qquad\qquad \mathcal{F}(\bar{u})[\bar{\varphi} - \bar{u}] \geq 0 \qquad \forall\,\bar{\varphi} \in \bar{\mathcal{K}}_{\psi,\theta}$$

if and only if $u = \bar{u} + \theta \in \mathcal{K}_{\psi,\theta}$ is a solution of the obstacle problem (7.7).

According to Theorem 8.2, p. 247 in [51], to solve the obstacle problem it is enough to verify that:

1. $\mathcal{F}$ is pseudo-monotone on $\bar{\mathcal{K}}_{\psi,\theta}$, that is,
   i)
   $$\mathcal{F} \;:\; \left(W_0^{1,p}(\Omega), \|\cdot\|_{W_0^{1,p}(\Omega)}\right) \to \left(W_0^{1,p}(\Omega)^*, \|\cdot\|_{W_0^{1,p}(\Omega)^*}\right)$$

   is bounded.
   ii) if $u_i, u \in \bar{\mathcal{K}}_{\psi,\theta}$ and $u_i \rightharpoonup u$ in $\left(W_0^{1,p}(\Omega), \text{weak}\right)$ as $i \to +\infty$ and

   $$(7.11) \qquad\qquad \limsup_{i\to+\infty} \mathcal{F}(u_i)[u_i - u] \leq 0$$

   then

   $$(7.12) \qquad \liminf_{i\to+\infty} \mathcal{F}(u_i)[u_i - \varphi] \geq \mathcal{F}(u)[u - \varphi] \qquad \forall\,\varphi \in W_0^{1,p}(\Omega).$$

2. $\mathcal{F}$ is coercive on $\bar{\mathcal{K}}_{\psi,\theta}$, that is,
   iii) there exists $\bar{\varphi} \in \bar{\mathcal{K}}_{\psi,\theta}$ for which

   $$\frac{\mathcal{F}(\bar{u})[\bar{u} - \bar{\varphi}]}{\|\bar{u}\|_{W_0^{1,p}(\Omega)}} \to +\infty$$

   if $\|\bar{u}\|_{W_0^{1,p}(\Omega)} \to +\infty$ with $\bar{u} \in \bar{\mathcal{K}}_{\psi,\theta}$.

We first address the pseudo-monotonicity of $\mathcal{F}$.

i) Boundedness follows since, for each $\varphi \in W_0^{1,p}(\Omega)$, by Hölder inequality we have

$$|\mathcal{F}(\bar{u})[\varphi]| \leq C\left(1 + \|\bar{u}\|_{W^{1,p}(\Omega)}^{p-1}\right) \|\varphi\|_{W^{1,p}(\Omega)}$$

for some constant $C > 0$.

ii) Let $u_i \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$ with $u_i, u \in \bar{\mathcal{K}}_{\psi,\theta}$ and suppose (7.11). Since $u_i \rightharpoonup u$, $\{u_i\}$ is bounded in $W_0^{1,p}(\Omega)$; since $\partial\Omega$ is Lipschitz, by Rellich-Kondrachov compactness theorem $\|u_i - u\|_{L^p(\Omega)} \to 0$ and almost everywhere. Then, it is easy to see that $\mathcal{B}(u_i)[u_i - u] \to 0$. Therefore, by (7.11)

$$(7.13) \qquad\qquad \limsup_i \mathcal{A}(u_i)[u_i - u] \leq 0.$$

Since $\mathcal{A}$ is a monotone operator and $u_i \rightharpoonup u$, from (7.13) we have

$$0 \leq \big(\mathcal{A}(u_i) - \mathcal{A}(u)\big)[u_i - u] = \mathcal{A}(u_i)[u_i - u] - \mathcal{A}(u)[u_i - u]$$

$$= \mathcal{A}(u_i)[u_i - u] + o(1)$$

as $i \to +\infty$, and we therefore deduce that $\lim_i \left(\mathcal{A}(u_i) - \mathcal{A}(u)\right)[u_i - u] = 0$. By Browder lemma, see Lemma 3 in [18], since $\mathcal{A}$ is strictly monotone we obtain that $u_i \to u$ strongly in $W_0^{1,p}(\Omega)$. Now we show that $\mathcal{F}$ is sequentially continuous from $(W_0^{1,p}(\Omega), \|\cdot\|_{W_0^{1,p}(\Omega)})$ to $(W_0^{1,p}(\Omega)^*, \text{weak})$. Let u_i be a sequence in $W_0^{1,p}(\Omega)$ which converges strongly to u. Up a subsequence we have $(u_k(x), \nabla u_k(x)) \to (u(x), \nabla u(x))$ for a.e. $x \in \Omega$. Set for convenience

$$X_k(x) = \nabla\left(u_k(x) + \theta(x)\right), \qquad X(x) = \nabla\left(u(x) + \theta(x)\right).$$

If $k \to \infty$, $(X_k(x), |X_k(x)|) \to (X(x), |X(x)|)$ for a.e. $x \in \Omega$. If we set $\nabla\varphi = Y$, we have

$$\left(\mathcal{A}(u_k) - \mathcal{A}(u)\right)[\varphi] = \int_\Omega \langle Y, X_k |X_k|^{p-2} - X |X|^{p-2}\rangle \mathrm{d}\mu_f = (I) + (II),$$

where

$$
\begin{aligned}
(I) &= \int_\Omega \frac{|X_k|^{p-2}\langle Y, X_k\rangle}{1 + |X_k|^{p-1}}\left[|X_k|^{p-1} - |X|^{p-1}\right]\mathrm{d}\mu_f; \\
(II) &= \int_\Omega \langle Y, \frac{|X_k|^{p-2} X_k}{1 + |X_k|^{p-1}} - \frac{|X|^{p-2} X}{1 + |X|^{p-1}}\rangle \left(1 + |X|^{p-1}\right)\mathrm{d}\mu_f.
\end{aligned}
$$

Since the integrand in (II) is bounded by $2|Y|\left(1 + |X|^{p-1}\right) \in L^1(\Omega)$, by Lebesgue theorem

(7.14)
$$(II) \to 0 \qquad \text{as } k \to +\infty.$$

We now consider (I). Fix $\epsilon > 0$. Then, by Egoroff theorem there exists $\Omega_\epsilon \subset \Omega$ such that $\mu_f(\Omega\backslash\Omega_\epsilon) < \epsilon$ and $|X_k| \to |X|$ uniformly on Ω_ϵ as $k \to +\infty$. Since

$$\left|\frac{|X_k|^{p-2}\langle Y, X_k\rangle}{1 + |X_k|^{p-1}}\right| \leq |Y| \qquad \text{on } \Omega$$

we therefore obtain

$$\int_{\Omega_\epsilon}\left|\frac{|X_k|^{p-2}\langle Y, X_k\rangle}{1 + |X_k|^{p-1}}\left(|X_k|^{p-1} - |X|^{p-1}\right)\right|\mathrm{d}\mu_f \to 0 \qquad \text{as } k \to +\infty.$$

On the other hand, using Hölder inequality we deduce

$$\left|\int_{\Omega\backslash\Omega_\epsilon} \frac{|X_k|^{p-2}\langle Y, X_k\rangle}{1 + |X_k|^{p-1}}\left(|X_k|^{p-1} - |X|^{p-1}\right)\right| \leq \|Y\|_{L^p(\Omega)}\left\||X_k|^{p-1} - |X|^{p-1}\right\|_{L^{\frac{p}{p-1}}(\Omega\backslash\Omega_\epsilon)}$$

$$\leq C\|Y\|_{L^p(\Omega)}\left\||X_k|^p + |X|^p\right\|_{L^1(\Omega\backslash\Omega_\epsilon)}^{\frac{p-1}{p}} \leq C\|Y\|_{L^p(\Omega)}\left\|\,|X_k - X| + |X||^p + |X|^p\right\|_{L^1(\Omega\backslash\Omega_\epsilon)}^{\frac{p-1}{p}}$$

$$\leq C\|Y\|_{L^p(\Omega)}\left(2^p\left\|\,|X_k - X|^p\right\|_{L^1(\Omega)} + (2^p + 1)\left\|\,|X|^p\right\|_{L^1(\Omega\backslash\Omega_\varepsilon)}\right)^{\frac{p-1}{p}}.$$

For some constant $C > 0$ only depending on p. The first integral converges to 0 because $\|X_k - X\|_p \to 0$ as $k \to \infty$, while the second is infinitesimal, if $\epsilon \to 0$, by the absolute continuity of the integral. Thus

$$0 \leq \limsup_{k\to+\infty}|(I)| \leq C(\epsilon),$$

where $C(\epsilon) \to 0^+$ as $\epsilon \to 0$. By the arbitrariness of ε, $(I) \to 0$ as $k \to +\infty$ and, combining with (7.14), $(\mathcal{A}(u_k) - \mathcal{A}(u))\,[\varphi] \to 0$ as $k \to +\infty$. Next, in an analogous way, it can be shown that

$$\big(\mathcal{B}(u_k) - \mathcal{B}(u)\big)[\varphi] \to 0 \qquad \text{as } k \to +\infty.$$

We have thus proved that

$$\mathcal{F}(u_k) \rightharpoonup \mathcal{F}(u) \qquad \text{in } W_0^{1,p}(\Omega)^* \quad \text{as } k \to \infty,$$

for some subsequence $\{u_k\}$ of the original $\{u_i\}$. A simple reasoning by contradiction thus shows that the whole $\mathcal{F}(u_i) \rightharpoonup \mathcal{F}(u)$ weakly on $W_0^{1,p}(\Omega)^*$, proving the sequential continuity of $\mathcal{F}$. Therefore, since $\|u_i - u\|_{W_0^{1,p}(\Omega)} \to 0$ as $i \to +\infty$, for each $\varphi \in W_0^{1,p}(\Omega)$, we have:

$$\mathcal{F}(u_i)[u_i - \varphi] = \mathcal{F}(u_i)[u_i - u] + \mathcal{F}(u_i)[u - \varphi] \to 0 + \mathcal{F}(u)[u - \varphi]$$

as $i \to +\infty$, which because of (7.11) proves the pseudo-monotonicity of $\mathcal{F}$.

iii) We are left to prove the coercivity of $\mathcal{F}$. This is a more or less predictable consequence of the assumption $\lambda_V(\Omega) > 0$, but some technical details suggest to provide a full proof. We shall prove the validity of $iii)$ with the choice $\bar{\varphi} \equiv 0 \in \bar{\mathcal{K}}_{\psi,\theta}$.

First we observe that, by Cauchy-Schwarz inequality,

$$(7.15) \qquad \mathcal{A}(\bar{u})[\bar{u}] \geq \int_\Omega |\nabla(\bar{u} + \theta)|^p \mathrm{d}\mu_f - \int_\Omega |\nabla(\bar{u} + \theta)|^{p-1} |\nabla\theta| \mathrm{d}\mu_f.$$

Next, using

$$|X + Y|^p \geq |X|^p - p|X|^{p-1}|Y|, \qquad |X + Y|^{p-1} \leq 2^{p-1}\big(|X|^{p-1} + |Y|^{p-1}\big)$$

and Hölder inequality into (7.15) we obtain

$$(7.16) \qquad \mathcal{A}(\bar{u})[\bar{u}] \geq \|\nabla\bar{u}\|_{L^p(\Omega)}^p - C_3\|\nabla\bar{u}\|_{L^p(\Omega)}^{p-1} - C_2$$

for some constants $C_2, C_3 > 0$ independent of $\bar{u}$.

To deal with $\mathcal{B}(\bar{u})[\bar{u}]$ we first observe that, since $\theta \geq \psi \geq 0$ and $\bar{u} + \theta \geq 0$, we have

$$(7.17)$$

$$
\begin{aligned}
-\mathcal{B}(\bar{u})[\bar{u}] \;=\; & -\int_\Omega V|\bar{u}|^p \mathrm{d}\mu_f - \int_\Omega V\left[|\bar{u} + \theta|^{p-2}(\bar{u} + \theta) - |\bar{u}|^{p-2}\bar{u}\right]\bar{u}\,\mathrm{d}\mu_f \\
\geq\; & -\int_\Omega V|\bar{u}|^p \mathrm{d}\mu_f - \|V\|_{L^\infty(\Omega)}\int_{\{\bar{u}\geq\theta\}}\left[|\bar{u}+\theta|^{p-1} - |\bar{u}|^{p-1}\right]\bar{u}\,\mathrm{d}\mu_f \\
& -\|V\|_{L^\infty(\Omega)}\int_{\{\bar{u}<\theta\}}\left[|\bar{u}+\theta|^{p-1} - |\bar{u}|^{p-1}\right]\bar{u}\,\mathrm{d}\mu_f.
\end{aligned}
$$

Since $\bar{u} \geq -\theta$, on the set $\{x \in \Omega : \bar{u}(x) \leq \theta(x)\}$ we have $\bar{u}(x) \in [-\theta, \theta]$; hence the third integral in the right-hand side of the above is bounded above by $C_5 = 3^{p-1}2\|V\|_{L^\infty(\Omega)}\|\theta\|_{L^p(\Omega)}^p$. As for the second integral, on $\{\bar{u} \geq \theta\}$, the following elementary inequality holds:

$$(\bar{u} + \theta)^{p-1} - \bar{u}^{p-1} \leq \begin{cases} \bar{u}^{p-1} + \theta^{p-1} - \bar{u}^{p-1} = \theta^{p-1} & \text{if } p - 1 \leq 1, \\[2ex] \bar{u}^{p-1}\left[\left(1 + \dfrac{\theta}{\bar{u}}\right)^{p-1} - 1\right] \leq \bar{u}^{p-1}\left[p2^{p-1}\dfrac{\theta}{\bar{u}}\right] & \text{if } p - 1 > 1. \end{cases}$$

Therefore,

$$\int_{\{\bar{u} \geq \theta\}} \left[(\bar{u} + \theta)^{p-1} - \bar{u}^{p-1} \right] \bar{u} \mathrm{d}\mu_f$$

$$\leq \begin{cases} \displaystyle\int_{\{\bar{u} \geq \theta\}} \theta^{p-1} \bar{u} \mathrm{d}\mu_f \leq \|\theta\|_{L^p(\Omega)}^{p-1} \|\bar{u}\|_{L^p(\Omega)} & \text{if } p - 1 \leq 1, \\[3ex] p 2^{p-1} \displaystyle\int_{\{\bar{u} \geq \theta\}} \theta \bar{u}^{p-1} \mathrm{d}\mu_f \leq p 2^{p-1} \|\theta\|_{L^p(\Omega)} \|\bar{u}\|_{L^p(\Omega)}^{p-1} & \text{if } p - 1 > 1. \end{cases}$$

Inserting the obtained estimates into (7.17) we finally get

$$(7.18) \qquad -\mathcal{B}(\bar{u})[\bar{u}] \geq -\int_\Omega V|\bar{u}|^p \mathrm{d}\mu_f - C_4 \|\bar{u}\|_{L^p(\Omega)}^{\max\{p-1,1\}} - C_5$$

for some constants $C_4, C_5 > 0$ independent of $\bar{u}$.

Combining (7.16) and (7.18) we obtain

$$(7.19) \quad \mathcal{F}(\bar{u})[\bar{u}] \geq \|\nabla \bar{u}\|_{L^p(\Omega)}^p - C_3 \|\nabla \bar{u}\|_{L^p(\Omega)}^{p-1} - C_6 - C_4 \|\bar{u}\|_{L^p(\Omega)}^{\max\{p-1,1\}} - \int_\Omega V|\bar{u}|^p \mathrm{d}\mu_f.$$

On the other hand, using Rayleigh characterization of $\lambda_V(\Omega)$, since $\bar{u} \in W_0^{1,p}(\Omega)$ we get

$$\|\nabla \bar{u}\|_{L^p(\Omega)}^p - \int_\Omega V|\bar{u}|^p \mathrm{d}\mu_f \geq \lambda_V(\Omega)\|\bar{u}\|_{L^p(\Omega)}^p,$$

thus,

$$(7.20) \qquad \mathcal{F}(\bar{u})[\bar{u}] \geq \lambda_V(\Omega)\|\bar{u}\|_{L^p(\Omega)}^p - C_4 \|\bar{u}\|_{L^p(\Omega)}^{\max\{p-1,1\}} - C_3\|\nabla \bar{u}\|_{L^p(\Omega)}^{p-1} - C_6$$

for some constants $C_4, C_3, C_6 > 0$ and independent of $\bar{u}$.

Since $\bar{u} \in W_0^{1,p}(\Omega)$, by Poincaré inequality on Ω, there exists a constant $C_P > 0$ independent of $\bar{u}$ such that

$$\|\bar{u}\|_{L^p(\Omega)} \leq C_P \|\nabla \bar{u}\|_{L^p(\Omega)}.$$

Now, let $\bar{u}_k \in \bar{\mathcal{K}}_{\psi,\theta}$ be any sequence such that $\|\bar{u}_k\|_{W^{1,p}(\Omega)} \to +\infty$ as $k \to +\infty$. Again by Poincaré inequality, $\|\nabla \bar{u}_k\|_{L^p(\Omega)} \to +\infty$ as $k \to +\infty$, and two cases may occur: either

$$(a) \quad \frac{\|\bar{u}_k\|_{L^p(\Omega)}}{\|\nabla \bar{u}_k\|_{L^p(\Omega)}} \to 0, \qquad \text{or} \quad (b) \quad \limsup_{k \to +\infty} \frac{\|\bar{u}_k\|_{L^p(\Omega)}}{\|\nabla \bar{u}_k\|_{L^p(\Omega)}} = c > 0.$$

In the case (a), using (7.19), we deduce

$$\frac{\mathcal{F}(\bar{u}_k)[\bar{u}_k]}{\|\bar{u}_k\|_{W_0^{1,p}(\Omega)}} \geq$$

$$\geq \frac{\|\nabla \bar{u}_k\|_{L^p(\Omega)}^p - \|V\|_{L^\infty(\Omega)}\|\bar{u}_k\|_{L^p(\Omega)}^p - C_4\|\bar{u}_k\|_{L^p(\Omega)}^{\max\{p-1,1\}} - C_3\|\nabla \bar{u}_k\|_{L^p(\Omega)}^{p-1} - C_6}{(1 + C_P)\|\nabla \bar{u}_k\|_{L^p(\Omega)}} \to +\infty$$

as $\|\nabla \bar{u}_k\|_{L^p(\Omega)} \to +\infty$.

In case (b), for each subsequence (still denoted with $\{\bar{u}_k\}$) satisfying

$$\frac{\|\bar{u}_k\|_{L^p(\Omega)}}{\|\nabla \bar{u}_k\|_{L^p(\Omega)}} \to \bar{c} \in (0, c] \qquad \text{for } k \to +\infty$$

using (7.20) and $\lambda_V(\Omega) > 0$ we get

$$\frac{\mathcal{F}(\bar{u}_k)[\bar{u}_k]}{\|\bar{u}_k\|_{W_0^{1,p}(\Omega)}} \geq \frac{\lambda_V(\Omega)\|\bar{u}_k\|_{L^p(\Omega)}^p - C_4\|\bar{u}_k\|_{L^p(\Omega)}^{\max\{p-1,1\}} - C_3\|\nabla\bar{u}_k\|_{L^p(\Omega)}^{p-1} - C_6}{(1+C_P)\|\nabla\bar{u}_k\|_{L^p(\Omega)}} \to +\infty$$

as $\|\nabla\bar{u}_k\|_{L^p(\Omega)} \to +\infty$. This concludes the proof of the coercivity of $\mathcal{F}$.

$\square$

For $\theta = \psi$ we set $\mathcal{K}_\psi$ for $\mathcal{K}_{\psi,\psi}$. Next, we prove a minimizing properties for solutions of the obstacle problem.

Proposition 7.1. *Let $u \in \mathcal{K}_{\psi,\theta}$, $u \not\equiv 0$ be a solution of the obstacle problem with $\psi \geq 0$. Suppose that $w \in W^{1,p}(\Omega)$, $w \not\equiv 0$ solves $Q'_V(w) \geq 0$ on Ω, such that $\min\{u,w\} \in \mathcal{K}_{\psi,\theta}$. If*

$$(7.21) \qquad \frac{u}{w}, \ \frac{w}{u} \in L^\infty(\Omega),$$

then $u \leq w$ on Ω.

Proof. Since w, u are non-negative, nonzero supersolutions (u being a solution of the obstacle problem), by the half-Harnack inequality we have $u > 0$ and $w > 0$ on Ω. Combining with (7.21) and since Ω is relatively compact, we deduce that the following set is non-empty.

$$A \doteq \{c > 1 : cw(x) \geq u(x) \quad \text{for a.e. } x \in \Omega\}.$$

Let $c_0 = \inf A$, so that $c_0 w \geq u$ a.e. on Ω. We shall show that $c_0 = 1$. By contradiction suppose $c_0 > 1$, and choose $1 < c < c_0$ close enough to c_0 to have

$$(7.22) \qquad \frac{(cw)^p}{u^{p-1}} \geq \psi \quad \text{on } \Omega.$$

This is possible since, for $c \geq c_0^{\frac{p-1}{p}}$ we have

$$\frac{(cw)^p}{u^{p-1}} \geq \frac{(cw)^p}{(c_0 w)^{p-1}} = \frac{c^p}{c_0^{p-1}}w \geq w \geq \psi, \qquad \text{on} \qquad \overline{\Omega}.$$

Consider the non-empty set $U = \{x \in \Omega \ : \ u(x) > cw(x)\}$, and define

$$\varphi = u + \min\left\{\frac{(cw)^p - u^p}{u^{p-1}}, 0\right\} = \min\left\{u, \frac{(cw)^p}{u^{p-1}}\right\}.$$

Because of (7.22), $\varphi \geq \psi$ and $\varphi - \theta = u - \theta = 0$ on $\partial\Omega$; in other words, $\varphi \in \mathcal{K}_{\psi,\theta}$.

Since u solves the obstacle problem (7.7), using the above φ we deduce

$$(7.23) \qquad 0 \leq Q'_V(u)[\varphi - u] = \int_U |\nabla u|^{p-2}\langle \nabla u, \nabla\left(\frac{(cw)^p - u^p}{u^{p-1}}\right)\rangle \mathrm{d}\mu_f$$

$$- \int_U V\big((cw)^p - u^p\big)\mathrm{d}\mu_f.$$

On the other hand, applying the definition of supersolution to cw with the non-negative test function

$$\widetilde{\varphi} = \frac{\big(u^p - (cw)^p\big)_+}{(cw)^{p-1}} \in W_0^{1,p}(\Omega),$$

we get

$$
\begin{aligned}
(7.24) \quad 0 \leq Q_V'(cw)[\widetilde{\varphi}] \ &= \ \int_U |\nabla(cw)|^{p-2}\langle\nabla(cw),\nabla\left(\frac{u^p-(cw)^p}{(cw)^{p-1}}\right)\rangle \mathrm{d}\mu_f \\
&\quad - \int_U V\left(u^p-(cw)^p\right)\mathrm{d}\mu_f.
\end{aligned}
$$

Summing up (7.23) and (7.24) we get
$$
(7.25) \quad \int_U |\nabla(cw)|^{p-2}\langle\nabla(cw),\nabla\left(\frac{u^p-(cw)^p}{(cw)^{p-1}}\right)\rangle \mathrm{d}\mu_f - \int_U |\nabla u|^{p-2}\langle\nabla u,\nabla\left(\frac{u^p-(cw)^p}{u^{p-1}}\right)\rangle \mathrm{d}\mu_f \geq 0.
$$

Set $z = \max\{u, cw\}$, and note that (7.25) is equivalent to $I(cw, z) \leq 0$. Indeed, in the definition (3.7) of I, the part of the integral outside U is zero since $z \equiv cw$. To conclude, applying Proposition 3.2 we deduce $I(cw, z) = 0$, so that cw and z (hence u) are proportional. Since $cw = u$ on ∂U we conclude that $u \equiv cw$ on U, contradicting the definition of this latter. $\qquad\square$

Remark 7.2. A typical case when (7.21) is automatically met is when the data ψ, θ satisfy $\psi, \theta \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$ and $\theta > 0$ on $\partial\Omega$, which we will frequently use. In fact, when $\psi, \theta \in C^0(\overline{\Omega})$, the solution $u \in \mathcal{K}_{\psi,\theta}$ of the obstacle problem is continuous on $\overline{\Omega}$. In this respect, see Theorem 5.4, page 235 of [52].

Remark 7.3. Although it is not explicitly stated, condition $\lambda_V(\Omega) \geq 0$ is automatic by assuming the existence of a u solving the obstacle problem and $\psi \geq 0$, $\psi \not\equiv 0$. Indeed, u is a positive solution of $Q_V'(u) \geq 0$ on M, and $\lambda_V(\Omega) \geq 0$ follows from Proposition 3.4.

In particular from the above proposition we deduce the following characterization.

Corollary 7.1. *Let $\Omega \Subset M$ be a relatively compact open set, and let $0 \leq \theta \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$ be such that $\theta > 0$ on $\partial\Omega$. Then, a solution u of the obstacle problem on $\mathcal{K}_{\psi,\theta}$ with obstacle $0 \leq \psi \leq \theta$, $\psi \not\equiv 0$ is the minimal $w \in \mathcal{K}_{\psi,\theta}$ satisfying $Q_V'(w) \geq 0$ on Ω. Consequently, such a solution is unique.*

A second important consequence of Proposition 7.1 is

Proposition 7.2. *Let $w_1, w_2 \in W^{1,p}_{\mathrm{loc}}(M) \cap C^0(M)$ be positive solutions of $Q_V'(w_j) \geq 0$ on M, $j = 1, 2$. Then,*
$$
w \doteq \min\{w_1, w_2\}
$$
solves
$$
(7.26) \quad Q_V'(w) \geq 0 \qquad \text{on} \quad M.
$$

Proof. Fix any relatively compact open set Ω with smooth boundary and consider a solution s of the obstacle problem $\mathcal{K}_w$ on Ω. This is possible since $w \in W^{1,p}(\Omega)$. From $s \in \mathcal{K}_w$ we have $s \geq w$, and, being a solution of the obstacle problem, $Q_V'(s) \geq 0$. Next, since $w_j > 0$ on $\overline{\Omega}$ and $Q_V'(w_j) \geq 0$, we can apply Proposition 7.1 to obtain $s \leq w_j$ for each j, whence $s \leq w$. This shows that $s \equiv w$, so that w solves (7.26) as claimed. $\qquad\square$

We are now ready for the

Proof of Lemma 3.1. First we show that $u \in W^{1,p}_{\text{loc}}(\overline{\Omega}_2)$. Let $U \subset \overline{\Omega}_2$ be a relatively compact open set. Without loss of generality we can assume that $\Omega_1 \cap U \neq \emptyset$, for otherwise $u \equiv u_2$ on U and the sought is immediate. Since $z = \min\{u_1 - u_2, 0\} \in W^{1,p}(\Omega_1)$ is zero on $\partial\Omega_1 \cap \Omega_2$, there exists a sequence $\{\varphi_j\} \subset C^0(\overline{U \cap \Omega_1}) \cap W^{1,p}(U \cap \Omega_1)$, $\varphi_j \equiv 0$ on some neighbourhood of $\partial\Omega_1 \cap \Omega_2$, converging in the $W^{1,p}(U \cap \Omega_1)$ norm to z. Using that $z \leq 0$ on Ω_1, one can take, for instance,

$$(7.27) \qquad \varphi_j = \left(z + \frac{1}{j}\right)_-.$$

We extend each φ_j to a continuous function on $U \backslash \Omega_1$ by setting $\varphi_j \equiv 0$ on $U \backslash \Omega_1$, so that $\varphi_j \in W^{1,p}(U)$ and clearly $\varphi_j \to z 1_{\Omega_1}$ in $W^{1,p}(U)$, which shows that $z 1_{\Omega_1} \in W^{1,p}(U)$. It follows that $u = u_2 + \varphi_j \in W^{1,p}(U)$ converges to $u = u_2 + z 1_{\Omega_1}$, whence $u \in W^{1,p}(U)$.

To prove that $Q'_V(u) \geq 0$ on Ω_2, we shall reduce ourselves to Proposition 7.1. We take a smooth open set $U \Subset \Omega_2$ which, without loss of generality, intersects Ω_1. Since $\lambda_V(U) > 0$ by monotonicity of eigenvalues, Theorem 7.1 guarantees the existence of a solution s of the obstacle problem in $\mathcal{K}_u$ on U which, applying Theorem 5.4, page 235 in [52] is continuous on $\overline{U}$. We want to show that $s \equiv u$ on U. First, since $u > 0$ on $\overline{U}$, using Proposition 7.2 we get

$$(7.28) \qquad s \leq u_2 \qquad \text{on } U.$$

Hence, $s = u_2 = u$ on $U \backslash \Omega_1$. Consequently, since also $s \in \mathcal{K}_u$, $s - u = 0$ on $\partial U \cup (\partial\Omega_1 \cap U) = \partial(\Omega_1 \cap U)$, and so by standard theory

$$(7.29) \qquad s - u \in W^{1,p}_0(\Omega_1 \cap U).$$

(one can construct an approximating sequence as in (7.27) above). Therefore, s is also a solution of the obstacle problem on the closed convex set

$$\hat{\mathcal{K}}_u = \left\{ \varphi \in W^{1,p}(\Omega_1 \cap U) \mid \varphi \geq u \text{ a.e. and } \varphi - u \in W^{1,p}_0(\Omega_1 \cap U) \right\}.$$

As u_1 is a positive supersolution on $\Omega_1 \cap U$ and $\min\{s, u_1\} \in \hat{\mathcal{K}}_u$, by Proposition 7.1

$$(7.30) \qquad s \leq u_1 \qquad \text{on } \Omega_1 \cap U.$$

Coupling with (7.28) we get $s \leq u$ on $\Omega_1 \cap U$, and combining with $s \geq u$ on U ($s \in \mathcal{K}_u$), $s = u = u_1$ on $U \backslash \Omega_1$, we conclude $s \equiv u$ on U. This shows that u solves $Q'_V(u) \geq 0$ on U. As $U \Subset \Omega_2$ is arbitrary, $Q'_V(u) \geq 0$ on Ω_2, concluding the proof. $\square$

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