

AperTO - Archivio Istituzionale Open Access dell'Università di Torino

## On the spectrum of bounded immersions

### **This is the author's manuscript**

*Original Citation:*

*Availability:*

This version is available <http://hdl.handle.net/2318/1693216> since 2019-02-18T18:34:10Z

*Terms of use:*

Open Access

Anyone can freely access the full text of works made available as "Open Access". Works made available under a Creative Commons license can be used according to the terms and conditions of said license. Use of all other works requires consent of the right holder (author or publisher) if not exempted from copyright protection by the applicable law.

(Article begins on next page)

# ON THE SPECTRUM OF BOUNDED IMMERSIONS

G. PACELLI BESSA, LUQUÉSIO P. JORGE, AND LUCIANO MARI

ABSTRACT. In this paper, we investigate some of the relations between the spectrum of a non-compact, extrinsically bounded submanifold  $\varphi: M^m \rightarrow N^n$  and the Hausdorff dimension of its limit set  $\lim \varphi$ . In particular, we prove that if  $\varphi: M^2 \rightarrow \Omega \subset \mathbb{R}^3$  is a minimal immersion into an open, bounded, strictly convex subset  $\Omega$  with  $C^3$ -boundary, then  $M$  has discrete spectrum provided that  $\mathcal{H}_\Psi(\lim \varphi \cap \Omega) = 0$ , where  $\mathcal{H}_\Psi$  is the generalized Hausdorff measure of order  $\Psi(t) = t^2 |\log t|$ . Our theorem applies to a number of examples recently constructed by various authors in the light of N. Nadirashvili's discovery of complete, bounded minimal disks in  $\mathbb{R}^3$ , as well as to solutions of Plateau's problems for non-rectifiable Jordan curves, giving a fairly complete answer to a question posed by S.T. Yau in his Millenium Lectures [63], [64]. On the other hand, we present a criterion, called *the ball property*, whose fulfilment guarantee the existence of elements in the essential spectrum. As an application of the ball property, we show that some of the examples of Jorge-Xavier [36] and Rosenberg-Toubiana [59] of complete minimal surfaces between two planes have essential spectrum  $\sigma_{\text{ess}}(-\Delta) = [0, \infty)$ .

## 1. INTRODUCTION

An interesting problem in geometry is the study of the spectrum of the Laplacian  $\Delta$  of a Riemannian manifold in terms of Riemannian invariants. There is a huge literature studying the spectrum of complete Riemannian manifolds under various curvature restrictions. To have a glimpse of them, we point out few references with geometric restrictions implying that the spectrum is purely continuous, see [22], [23], [27], [37], [55], [61] or implying that the spectrum is discrete see [5], [24], [33], [38], [39]. However, in the study of the spectrum of submanifolds, the relevant geometric restrictions are related to extrinsic bounds, ambient manifold curvature bounds and the mean curvature of the submanifold, see [8], [9], [10], [15]. A particularly interesting problem is the part of the *Calabi-Yau conjectures on minimal hypersurfaces* related to the spectrum of the Laplacian. S. T. Yau, in his Millennium Lectures [63], [64], revisiting E. Calabi conjectures on the existence of bounded minimal surfaces, [13], [16], in the light of Jorge-Xavier and Nadirashvili's counter-examples, [36], [48], proposed a new set of questions about bounded minimal surfaces of  $\mathbb{R}^3$ .

He wrote: “*It is known [48] that there are complete minimal surfaces properly immersed into the [open] ball. What is the geometry of these surfaces? Can they be embedded? Since the curvature must tend to minus infinity, it is important to find the precise asymptotic behavior of these surfaces near their ends. Are their [Laplacian] spectra discrete?*”

This set of questions is known in the literature as the Calabi-Yau conjectures on minimal surfaces. They motivated the construction of a large number of exotic examples of minimal surfaces in  $\mathbb{R}^3$ , see [1], [2], [3], [28], [40], [41], [43], [44], [45], [46], [62].

The purpose of this article is to study the essential spectrum of bounded submanifolds, in particular, the spectrum of those examples constructed after the Calabi-Yau conjectures.

---

*Date:* April 26, 2016.

*1991 Mathematics Subject Classification.* Primary 53C40, 53C42, Secondary 58J50.

*Key words and phrases.* Essential spectrum, bounded submanifolds, limit set,  $\psi$ -Hausdorff measure.

The new ingredient we introduce in the study of the spectrum of bounded submanifolds is the size of their limit sets. Before we announce our main results with precision, we will present some of the examples, concerning the Calabi-Yau conjectures, that motivate our work. The problem about the existence of bounded, complete, *embedded* minimal surfaces in  $\mathbb{R}^3$  was negatively answered by T. Colding-W. Minicozzi in the finite topology case, see [17]. Although Yau's question suggests that Nadirashvili's example [48] is *properly* immersed into an open ball  $B_r(0) \subset \mathbb{R}^3$ , this is not clear from his construction. However, the question: “Does there exist a complete minimal surface properly immersed into a ball of  $\mathbb{R}^3$ ?” may be considered as the first problem of the Calabi-Yau conjectures. This question was answered by F. Martin and S. Morales in [43], see below. Recall that the limit set of an isometric immersion  $\varphi: M \rightarrow \Omega \subset N$  is the set

$$\lim \varphi := \{y \in \bar{\Omega}; \exists \{x_n\} \subseteq M \text{ divergent in } M, \text{ such that } \varphi(x_n) \rightarrow y \text{ in } N\},$$

and that  $\varphi$  is proper in  $\Omega$  if and only if  $\lim \varphi \subset \partial\Omega$ . The question “What is the geometry of these surfaces?” motivated the construction of bounded complete minimal surfaces of arbitrary topology in  $\mathbb{R}^3$  and the understanding their shape and the size of their limit sets stimulated intense research in the last fifteen years, see [2], [10], [17], [18], [28], [36], [40], [41], [43], [44], [45], [46], [48], [62]. We briefly recall the main achievements:

- (i.) Martin and Morales constructed for each convex domain  $\Omega \subseteq \mathbb{R}^3$ , not necessarily bounded or smooth, a complete minimal disk  $\varphi: \mathbb{D} \hookrightarrow \Omega$  properly immersed into  $\Omega$ , see [43].
- (ii.) M. Tokuomaru constructed a complete minimal annulus  $\varphi: A \hookrightarrow B_1^{\mathbb{R}^3}(0)$  properly immersed into the unit ball of  $\mathbb{R}^3$ , see [62].
- (iii.) Martin and Morales improved the results of [43], showing that, if  $\Omega$  is a bounded, strictly convex domain of  $\mathbb{R}^3$ , with  $\partial\Omega$  of class  $C^{2,\alpha}$ , then there exists a complete, minimal disk properly immersed into  $\Omega$  whose limit set is close to a prescribed Jordan curve<sup>1</sup> on  $\partial\Omega$ , see [44].
- (iv.) A. Alarcon, L. Ferrer and F. Martin extended the results of [43] and [62]. They showed that for any convex domain  $\Omega \subset \mathbb{R}^3$ , not necessarily bounded or smooth, there exists a proper minimal immersion  $\varphi: M \rightarrow \Omega$  of a complete non-compact surface  $M$  with arbitrary finite topology into  $\Omega$ , see [2, Thm B.].
- (v.) Ferrer, Martin and W. Meeks in [28], improving the main results on [44], proved that given a bounded smooth domain  $\Omega \subset \mathbb{R}^3$  and given any open surface  $M$ , there exists a complete, proper, minimal immersion  $\varphi: M \rightarrow \Omega$  with the property that the limit sets of different ends are disjoint, compact, connected subsets of  $\partial\Omega$ . It should be remarked that the Ferrer-Martin-Meeks' surfaces [28] immersed into a bounded smooth domain  $\Omega$  can have either finite or infinite topology. They can have uncountably many ends and be either orientable or non-orientable. Moreover, the convexity of  $\Omega$  is not a necessary hypothesis, although they need its smoothness. In fact, one can not drop the convexity and the smoothness condition of  $\Omega$  altogether, see [42] for a counter-example. They also prove that for every convex open set  $\Omega$  and every non-compact, orientable surface  $M$ , there exists a complete, proper minimal immersion  $\varphi: M \rightarrow \Omega$  such that  $\lim \varphi \equiv \partial\Omega$ , see [28, Prop.1].
- (vi.) Martin and Nadirashvili constructed complete minimal immersions  $\varphi: \mathbb{D} \rightarrow \mathbb{R}^3$  of the unit disk  $\mathbb{D} \subseteq \mathbb{C}$  admitting continuous extensions to the closed disk  $\bar{\varphi}: \bar{\mathbb{D}} \rightarrow \mathbb{R}^3$  such that  $\bar{\varphi}|_{\partial\mathbb{D}}: \partial\mathbb{D} = \mathbb{S}^1 \rightarrow \bar{\varphi}(\mathbb{S}^1)$  is a homeomorphism and  $\bar{\varphi}(\mathbb{S}^1)$  is a non-rectifiable Jordan curve of Hausdorff dimension  $\dim_{\mathcal{H}}(\bar{\varphi}(\mathbb{S}^1)) = 1$ . They also showed that the set of Jordan curves  $\bar{\varphi}(\mathbb{S}^1)$  constructed via this procedure is dense in the space of Jordan curves of  $\mathbb{R}^3$  with respect to the Hausdorff metric, see [46].

---

<sup>1</sup>A continuous embedding  $\gamma: \mathbb{S}^1 \rightarrow \mathbb{R}^3$ .

- (vii.) Alarcon proved that for any arbitrary finite topological type there exists a compact Riemann surface  $\mathcal{M}$ , an open domain  $M \subset \mathcal{M}$  with this fixed topological type and a conformal complete minimal immersion  $\varphi: M \rightarrow \mathbb{R}^3$  which can be extended to a continuous map  $\bar{\varphi}: \bar{M} \rightarrow \mathbb{R}^3$  such that  $\bar{\varphi}|_{\partial M}$  is an embedding and the Hausdorff dimension of  $\bar{\varphi}(\partial M)$  is 1, see [1].

In this paper, we address Yau's question whether the spectrum of bounded minimal surfaces of  $\mathbb{R}^3$  is discrete or not. We provide a sharp, general criterion that applies to each of the examples in (i.), ..., (vii.). A preliminary answer was given by Bessa-Jorge-Montenegro in [10], where they proved that the spectrum of a complete minimal surface properly immersed into a ball of  $\mathbb{R}^3$  is discrete. Despite the generality of this result, there is the technical detail that the "bounding" convex domains  $\Omega \subset \mathbb{R}^3$  are restricted to balls. Moreover, their proof uses, in a fundamental way, the properness condition that cannot be generalized to deal with non-proper immersions. On the other hand, if the limit set resembles a curve in the sense that it has Hausdorff dimension 1, as the examples of Martin-Nadirashvili in (vi.) or the examples of Alarcon [1], we could think of that the minimal surfaces is not too far from a compact set with boundary and thus it has discrete spectrum. We will show that is the case. In Theorem 2.4, we show that the spectrum of a bounded minimal surface is discrete provided its limit set has zero Hausdorff measure of order  $\Psi(t) = t^2 |\log t|$ . Moreover, we consider bounded immersions where the "bounding" set satisfies a weaker notion of convexity. On the other hand, we will set a simple geometric criterion that implies that the essential spectrum is not empty. In particular, we show that the essential spectrum of non-proper isometric immersions with locally bounded geometry is non-empty. We will also study the spectrum of the examples of Jorge-Xavier as well as of Rosenberg-Toubiana complete minimal surfaces between two planes.

The structure of this paper is as follows. In section 2 we state the main result and its corollaries. The first result says, roughly, that the zero  $\Psi$ -Hausdorff measure  $\mathcal{H}_\Psi(\lim \varphi) = 0$  of the limit set  $\lim \varphi$  implies  $\sigma_{\text{ess}}(-\Delta) = \emptyset$  and whereas the second says that  $\sigma_{\text{ess}}(-\Delta) \neq \emptyset$  in the presence of the ball property. We also show, via some examples, that the criterion  $\mathcal{H}_\Psi(\lim \varphi) = 0$  implying  $\sigma_{\text{ess}}(-\Delta) = \emptyset$  is sharp in dimension 2. In section 3 we introduce the notation and the necessary material to prove all results, that is done in section 4.

## 2. MAIN RESULTS

We start with the definition of  $j$ -convex open subsets.

**Definition 2.1.** An open subset  $\Omega \subset N^n$  with smooth  $C^2$ -boundary is strictly  $j$ -convex, for some  $j \in \{1, \dots, n-1\}$ , if for every  $q \in \partial\Omega$ , the ordered eigenvalues  $\xi_1(q) \leq \dots \leq \xi_{n-1}(q)$  of the second fundamental form  $\alpha$  of the boundary  $\partial\Omega$  at  $q$  with respect to the unit normal vector field  $\nu$  pointing towards  $\Omega$  satisfies  $\xi_1(q) + \dots + \xi_j(q) > 0$ . If for some constant  $c > 0$   $\xi_1(q) + \dots + \xi_j(q) \geq c$ , then we say that  $\Omega$  is strictly  $j$ -convex with constant  $c$ .

A well known result, due to Hadamard [32], states that if the second fundamental form of a compact immersed hypersurface  $M$  of  $\mathbb{R}^n$  is positive definite then  $M$  is embedded as the boundary  $M = \partial\Omega$  of a strictly convex body  $\Omega$ . In other words, a compact 1-convex subset  $\Omega \subset \mathbb{R}^n$  is a convex body, this is, any two points in  $\Omega$  can be joined by a segment contained in  $\Omega$ . The classical notions of convexity and mean convexity are respectively 1-convexity and  $(n-1)$ -convexity. The following example due to Jorge-Tomi [35] shows that a set can be 2-convex without being 1-convex. Let

$$(1) \quad \mathbb{T}^n(r_1, r_2) = \{(z, w) \in \mathbb{R}^2 \times \mathbb{R}^{n-2} : (|z| - r_2)^2 + |w|^2 \leq r_1^2\}, \quad 0 < r_1 < r_2$$

be the solid torus homeomorphic to  $\mathbb{S}^1 \times \mathbb{B}^{n-1}$  where  $\mathbb{B}^{n-1}$  is the unit ball of  $\mathbb{R}^{n-1}$ . It was shown in [35] that  $\mathbb{T}^n$  is 2-convex whenever  $r_1 \leq r_2/2$ . Finally, we will show that strictly

$j$ -convexity of an open set  $\Omega$  with constant  $c > 0$  and  $C^3$ -smooth boundary  $\partial\Omega$  is equivalent to the existence of suitable  $j$ -subharmonic  $C^2$ -function  $f: \Omega \rightarrow \mathbb{R}$ , see Lemma 3.5 for details.

**2.1. Discrete Spectrum.** Let  $\Omega \subset N$  be a bounded open set in a Riemannian manifold. For given  $r > 0$  let  $T_r(\Omega) = \{y \in N: \text{dist}_N(y, \Omega) \leq r\}$  be the closed tube around  $\Omega$  and let

$$(2) \quad b = \sup\{K_N(z), z \in T_{\text{diam}(\Omega)}(\Omega)\}.$$

For each  $y \in \Omega$  define  $r(y) = \min\{\text{inj}_N(y), \pi/2\sqrt{b}\}$ , where  $\pi/2\sqrt{b}$  is replaced by  $+\infty$  if  $b \leq 0$ . Set  $r_\Omega = \inf_{y \in \Omega} r(y)$ .

**Definition 2.2.** A bounded domain  $\Omega \subset N$  is totally regular if  $\text{diam}_N(\Omega) < r_\Omega$ .

**Example 2.3.** If  $N$  is a Hadamard manifold then any bounded domain  $\Omega$  is totally regular. On the other hand,  $\Omega \subset \mathbb{S}^n(1)$  is totally regular if and only if  $\text{diam}_{\mathbb{S}^n(1)}(\Omega) < \pi/2$ .

For  $b \in \mathbb{R}$  define the function  $\mu_b: [0, \infty) \rightarrow \mathbb{R}$  given by

$$(3) \quad \mu_b(t) = \begin{cases} \frac{1}{\sqrt{b}} \tan(\sqrt{b}t) & \text{if } b > 0 \\ t & \text{if } b = 0 \\ \frac{1}{\sqrt{-b}} \tanh(\sqrt{-b}t) & \text{if } b < 0 \end{cases}$$

The notion of the generalized Hausdorff measures or  $\Psi$ -Hausdorff measures  $\mathcal{H}_\Psi$  given by the Caratheodory construction, where  $\Psi: [0, \infty) \rightarrow [0, \infty)$  is a non-negative continuous function, can be found (with every detail) in the beautiful book of P. Mattila [47, Chapter 4], see the Definition 3.1 later on in this article. Our first main result on this paper is the following theorem.

**Theorem 2.4.** *Let  $\varphi: M \rightarrow N$  be an isometric immersion of a Riemannian  $m$ -manifold  $M$  into a Riemannian  $n$ -manifold  $N$  with mean curvature vector  $H$ . Suppose that  $\varphi(M) \subset \Omega$ , a bounded, totally regular, open subset of  $N$  and let  $b$  be as in (2) and  $\mu_b$  as defined in (3). Assume that*

$$(4) \quad \|H\|_{L^\infty(M)} < \frac{m-1}{m \cdot \mu_b(\text{diam}(\Omega))}.$$

Define  $\theta = [m-1 - m \cdot \mu_b(\text{diam}(\Omega)) \cdot \|H\|_{L^\infty(M)}] > 0$  and let  $\Psi: [0, \infty) \rightarrow [0, \infty)$  be given by

$$(5) \quad \Psi(t) = \begin{cases} t^2 & \text{if } \theta > 1 \\ t^2 |\log t| & \text{if } \theta = 1 \\ t^{\theta+1} & \text{if } \theta \in (0, 1), \end{cases}$$

If one of the following conditions holds

$$(1) \quad \lim \varphi \cap \partial\Omega = \emptyset \text{ and } \mathcal{H}_\Psi(\lim \varphi) = 0,$$

$$(2) \quad \lim \varphi \cap \partial\Omega \neq \emptyset, \mathcal{H}_\Psi(\lim \varphi \cap \Omega) = 0, \Omega \text{ is strictly } m\text{-convex with constant } c > 0, \partial\Omega \text{ is of class } C^3, \text{ and the mean curvature vector } H \text{ satisfies the further restriction}$$

$$(6) \quad \|H\|_{L^\infty(M)} < \frac{c}{m},$$

then the spectrum of  $-\Delta$  is discrete.

We shall make few comments about Theorem 2.4.

- We remark that in item 2, the Hausdorff measure of  $\lim \varphi \cap \partial\Omega$  does not need to be zero. In particular, the examples of Ferrer-Martin-Meeks [28] of complete, proper minimal immersions  $\varphi: M \rightarrow \Omega$  such that  $\lim \varphi \equiv \partial\Omega \subset \mathbb{R}^3$  have discrete spectrum, provided  $\Omega$  is strictly 2-convex. One illustrative example is the 2-convex solid torus  $\mathbb{T}^2(r_1, r_2)$ ,  $r_1 \leq r_2/2$  described in [35], see (1). By the result [28, Prop. 1] there exists, for any open surface  $M$ , a complete, proper minimal immersion  $\varphi: M \rightarrow \mathbb{T}^2(r_1, r_2)$  such that  $\lim \varphi \equiv \partial\mathbb{T}^2(r_1, r_2)$ , hence by Theorem 2.4, item 2, its spectrum is discrete.
- A more technical observation is: our definition of  $\Omega$  being totally regular implies that  $\mu_b(\text{diam}(\Omega)) > 0$  thus (4) is meaningful, where  $b = \sup\{K_N(z), z \in T_{\text{diam}(\Omega)}(\Omega)\}$ . However, if one knows only an upper bound for the sectional curvatures  $b_0 > b$  instead, then Theorem 2.4 is still valid, provided  $\mu_{b_0}(\text{diam}(\Omega)) > 0$ .
- The case that  $\lim \varphi \cap \Omega = \emptyset$  is equivalent to the properness of  $\varphi$  in  $\Omega$ , therefore the statement of Theorem 2.4 extends in many aspects the main result of [10].
- Theorem 2.4 also applies to non-orientable manifolds  $M$ . In fact, its proof can be applied to the two-sheeted oriented covering of  $M$  yielding the same conclusions<sup>2</sup>.
- The Riemannian manifold  $M$  may be geodesically incomplete and the statement regards the spectrum of the Friedrichs extension of  $\Delta: C_c^\infty(M) \rightarrow C_c^\infty(M)$ .

The minimal surfaces in the set of examples (i.), (ii.), (iii.) and (iv.) are properly immersed in 1-convex domains  $\Omega$  of  $\mathbb{R}^3$ , whereas the minimal surfaces in (v.) are properly immersed in smooth domains  $\Omega$ . In those examples  $\lim \varphi \cap \Omega = \emptyset$  thus  $\mathcal{H}_\Psi(\lim \varphi \cap \Omega) = 0$ . The examples in (vi.) and (vii.) are bounded and  $\lim \varphi$  is a non-rectifiable Jordan curve of Hausdorff dimension 1. Thus  $\mathcal{H}_\Psi(\lim \varphi \cap \Omega) = 0$  for  $\Psi(t) = t^2 |\log(t)|$ . By Theorem 2.4, all of those examples of (i.), (ii.), (iii.), (iv.), (v.), (vi.) and (vii.) have discrete spectrum, provided  $\Omega$  is bounded strictly 2-convex with  $C^3$ -boundary. That can be summarized in the following corollary as follows.

**Corollary 2.5.** *Let  $\varphi: M^m \rightarrow N^n$  be a minimal submanifold, possibly incomplete, immersed into a bounded open  $m$ -convex subset  $\Omega$  of a Hadamard manifold with constant  $c > 0$ . Suppose that  $\partial\Omega$  is  $C^3$ -smooth and  $\Psi(t) = t^2$  if  $m \geq 3$  and  $\Psi(t) = t^2 |\log(t)|$  if  $m = 2$ . If  $\mathcal{H}_\Psi(\lim \varphi \cap \Omega) = 0$  then the spectrum of  $-\Delta$  is discrete. In particular, those minimal surfaces constructed in (i.), (ii.), (iii.), (iv.), (v.), (vi.) and (vii.) have discrete spectrum provided  $\Omega$  is bounded, strictly 2-convex with  $C^3$ -boundary.*

Let  $\gamma$  be a Jordan curve and let  $a_\gamma$  be the infimum of all the areas of the disks spanning  $\gamma$ . It is well known that J. Douglas [25] and T. Radó [53] proved the existence of minimal disks  $\varphi: \mathbb{D} \rightarrow \mathbb{R}^3$  spanning  $\gamma^3$  if  $\text{area } a_\gamma < +\infty$ , therefore their spectrum are discrete (provided  $\mathcal{H}_\Psi(\gamma) = 0$ ). When  $a_\gamma = +\infty$ , there is no sense to speak about the least area surface spanning  $\gamma$ , however, Douglas [26] proved that there exists a globally stable minimal disk  $\varphi: \mathbb{D} \rightarrow \mathbb{R}^3$  with infinite area spanning  $\gamma$ . On the other hand, the set  $\mathbb{J} = \{\gamma: \mathbb{S}^1 \rightarrow \mathbb{R}^3\}$  of all non-rectifiable Jordan curves of  $\mathbb{R}^3$  coming from the Martin-Nadirashvili's procedure is dense in the set of Jordan curves of  $\mathbb{R}^3$  with respect to the Hausdorff metric, see [46]. Hence, the globally stable minimal disks  $D_\gamma$  of Douglas spanning a non-rectifiable Jordan curves  $\gamma \in \mathbb{J}$  can not be complete since complete stable minimal surfaces (either orientable or nonorientable) of  $\mathbb{R}^3$  are planar by Do Carmo-Peng, Fischer-Colbrie-Schoen, Pogorelov, Ros' Theorem<sup>4</sup> [20], [29], [52], [57], [60]. For the same  $\gamma \in \mathbb{J}$  there exists a complete minimal disk  $M_\gamma$  spanning  $\gamma$  by Martin-Nadirashvili's result [46]. Hence, every non-rectifiable curve  $\gamma \in \mathbb{J}$  considered by Martin-Nadirashvili are spanned by, at least, two minimal disks. This together with our main result yields the following corollary that has interest on its own.

<sup>2</sup>We thank an anonymous referee for pointing this out.

<sup>3</sup>Douglas proved the existence of minimal disks spanning Jordan curves in  $\mathbb{R}^n$ .

<sup>4</sup>A. Ros proved this characterization of the plane in the orientable case.

**Corollary 2.6.** *Let  $\gamma \in \mathbb{J}$  be a non-rectifiable Jordan curve spanning a Martin-Nadirashvili minimal surface  $M_\gamma$  as in [46]. Then*

- (1)  $\gamma$  is spanned by at least two minimal disks. A geodesically complete minimal surface  $M_\gamma$  given by Martin-Nadirashvili result and a geodesically incomplete but globally stable minimal surface  $D_\gamma$  given by Douglas' result [26].
- (2) Any Douglas' solution  $D_\gamma$  for the classical Plateau problem for  $\gamma \in \mathbb{J}$ , as well as  $M_\gamma$ , has discrete spectrum.
- (3) Any minimal surface spanning a Jordan curve  $\gamma$  with  $\mathcal{H}_\Psi(\gamma) = 0$ ,  $\Psi(t) = t^2 |\log t|$ , has discrete spectrum.

Notice that there are examples of embedded continuous curves  $\gamma: [0, 1] \rightarrow \mathbb{R}^2$  with  $\dim_{\mathcal{H}}(\gamma([0, 1])) = 2$ , see [26], [49]. It would be interesting to know the spectrum of the minimal solutions of the Plateau problem spanning such curve  $\gamma$  with Hausdorff dimension  $\dim_{\mathcal{H}}(\gamma) \geq 2$ .

*Remark 2.7.* The hypothesis concerning the measure of the limit set  $\lim \varphi$  in Theorem 2.4 is sharp. Consider a bounded, complete proper minimal annulus  $\varphi: M \rightarrow B_1^{\mathbb{R}^3}(0)$  as in [62] with  $\lim \varphi \cap \Omega = \emptyset$ , thus with discrete spectrum by Theorem 2.4 or [10, Thm.1]. Considering the universal cover  $\pi: \widetilde{M} \rightarrow M$  and setting  $\phi = \varphi \circ \pi: \widetilde{M} \rightarrow \mathbb{R}^3$  one has a bounded, complete minimal surface with non-empty essential spectrum. In fact, if  $\pi: (\widetilde{M}, \pi^* ds^2) \rightarrow (M, ds^2)$  is an infinite sheeted covering then the induced metric  $\pi^* ds^2$  satisfies the “ball property”, see Definition 2.8, therefore the essential spectrum of  $(\widetilde{M}, \pi^* ds^2)$  is non-empty, regardless the spectrum of  $(M, ds^2)$ . Observe that the immersed submanifold  $\varphi(M) = \phi(\widetilde{M})$  but the limit sets are different,  $\lim \varphi \neq \lim \phi = \overline{\phi(\widetilde{M})}$  and Theorem 2.4 could not be applied since the Hausdorff dimension  $\dim_{\mathcal{H}}(\lim \phi \cap B_1^{\mathbb{R}^3}(0)) \geq 2$ .

## 2.2. Essential spectrum.

**2.2.1. Ball property.** As a counterpart to Theorem 2.4, it would be interesting to find a set of geometric conditions for a Riemannian manifold to have non-empty essential spectrum. In this regard, we will establish a criterion that does not involve curvatures and thus it can be used to study the spectrum of the complete minimal surfaces, for instance, those immersed into a slab of  $\mathbb{R}^3$  constructed by Jorge-Xavier [36] and Rosenberg-Toubiana [59]. We begin with the following definition.

**Definition 2.8.** A Riemannian manifold  $M$  is said to have the **ball property** if there exists  $R > 0$  and a collection of disjoint balls  $\{B_R^M(x_j)\}_{j=1}^{+\infty}$  of radius  $R$  centered at  $x_j$  such that for some constants  $C > 0$ ,  $\delta \in (0, 1)$ , possibly depending on  $R$ ,

$$(7) \quad \text{vol}(B_{\delta R}^M(x_j)) \geq C^{-1} \text{vol}(B_R^M(x_j)) \quad \forall j \in \mathbb{N}$$

Observe that (7) is *not* a doubling condition since it needs to hold only along the sequence  $\{x_j\}$  and the constant  $C$  may depend on  $R$ . The importance of the ball property is that its validity implies that the essential spectrum is nonempty.

**Theorem 2.9.** *If a Riemannian manifold  $M$  has the ball property (with parameters  $R, \delta, C$ ), then*

$$(8) \quad \inf \sigma_{\text{ess}}(-\Delta) \leq \frac{C}{R^2(1-\delta)^2}.$$

The well-known Bishop-Gromov volume comparison theorem, see [12], [31], shows that any complete non-compact Riemannian  $m$ -manifold  $M$  with Ricci curvature bounded from below has the ball property, therefore it has non-empty essential spectrum. This was known to H. Donnelly, that proved sharp results in the class of Riemannian manifolds with Ricci

curvature bounded below. He showed that the essential spectrum of a complete non-compact Riemannian  $m$ -manifold  $M$  with Ricci curvature  $\text{Ric}_M \geq -(m-1)c^2 > -\infty$  intersects the interval  $[0, (m-1)^2c^2/4]$ , see [21, Thm. 3.1]. However, there are examples of complete non-compact Riemannian manifolds with the ball property and  $\inf \text{Ric} = -\infty$ . For instance, the examples of Jorge-Xavier of minimal surfaces between two planes that have Ricci curvature satisfying  $\inf \text{Ric} = -\infty$ , see [7], [58] and some of them have the ball property and therefore have non-empty essential spectrum. H. Rosenberg and E. Toubiana, in [59], constructed a complete minimal annulus between two parallel planes of  $\mathbb{R}^3$  such that the immersion is proper in the slab. Jorge-Xavier's and Rosenberg-Toubiana's examples are constructed with a flexible method depending on a chosen set of parameters and we will show that, depending on this choice of parameters, the spectrum of the complete minimal surfaces immersed in the slab can be the half-line  $[0, \infty)$ .

There are other examples of manifolds with the ball property, for instance, the non-proper submanifolds with locally bounded geometry. An isometric immersion  $\varphi: M \rightarrow N$  is said to have *locally bounded geometry* if for each compact set  $W \subset N$  there is a constant  $\Lambda = \Lambda(W)$  such that

$$\|\alpha_\varphi\|_{L^\infty(\varphi^{-1}(W))} \leq \Lambda$$

Here  $\alpha_\varphi$  is the second fundamental form of the immersion  $\varphi$ . To complete this section about the ball property we will prove the following result.

**Theorem 2.10.** *Let  $\varphi: M \rightarrow N$  be an isometric immersion with locally bounded geometry of a complete non-compact Riemannian  $m$ -manifold  $M$  into a complete Riemannian  $n$ -manifold  $N$ . If the immersion is non-proper then  $M$  has the ball property. Thus it has non empty essential spectrum.*

2.2.2. *Spectrum of complete minimal surfaces in the slab.* We will need to give a brief description of the examples of complete minimal surfaces between two parallel planes. Jorge and Xavier in [36] constructed a complete minimal immersion of the disk  $\varphi: \mathbb{D} \rightarrow \mathbb{R}^3$  with  $\mathbb{R}^3$ ,  $\varphi(M) \subset \{(x, y, z) \in \mathbb{R}^3 : |z| < 1\}$ . Let  $\{D_n \subset \mathbb{D}\}$  be a sequence of closed disks centered at the origin such that  $D_n \subset \text{int}(D_{n+1})$ ,  $\cup D_n = \mathbb{D}$ . Let  $K_n \subset D_n$  be a compact set so that  $K_n \cap D_{n-1} = \emptyset$  and  $D_n \setminus K_n$  is connected as in the figure 1. below.

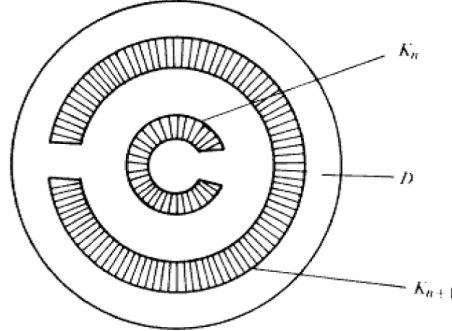


Fig. 1. The compact sets  $K_n$ .

By Runge's Theorem, [34, p. 96], there exists a holomorphic function  $h: \mathbb{D} \rightarrow \mathbb{C}$  such that  $|h - c_n| < 1$  on  $K_n$ , for each  $n$ . Letting  $g = e^h$  and  $f = e^{-h}$  and setting  $\phi = (f(1-g^2)/2, i \cdot f(1+g^2)/2, fg)$ , by the Weierstrass representation, one has that  $\varphi = \text{Re} \int \phi: \mathbb{D} \rightarrow \mathbb{R}^3$  is a minimal surface with bounded third coordinate. Let  $r_n$  denote the Euclidean distance between the inner and the outer circle of  $K_n$  and for each  $n$  choose a constant  $c_n$  such that

$$(9) \quad \sum_{n \text{ even}}^{+\infty} r_n e^{c_n - 1} = +\infty, \quad \sum_{n \text{ odd}}^{+\infty} r_n e^{c_n - 1} = +\infty.$$



Condition (9) implies that this minimal surface is complete. The induced metric  $ds^2$  by this minimal immersion is conformal to the Euclidean metric  $|dz|^2$  given by  $ds^2 = \lambda^2|dz|^2$ , where

$$(10) \quad \lambda(z) = \frac{1}{2} \left( |e^{h(z)}| + |e^{-h(z)}| \right).$$

The choice of the compact subsets  $K_n \subset D_n$  with width  $r_n$  and the set of constants  $c_n$  satisfying (9) and yielding a complete minimal surface of  $\mathbb{R}^3$  between two parallel planes is what we are calling a choice of parameters,  $(\{(r_n, c_n)\})$ , in Jorge-Xavier's construction. We should give a brief description of Rosenberg-Toubiana construction of a complete minimal annulus properly immersed into a slab of  $\mathbb{R}^3$ , see details in [59]. They start considering a labyrinth in the annulus  $\mathbb{A}(1/c, c) = \{z \in \mathbb{C} : 1/c < |z| < c\}$ ,  $c > 1$  composed by compact sets  $K_n$  contained in the annulus  $\mathbb{A}(1, c)$  and compact sets  $L_n = \{1/\bar{z} : -z \in K_n\}$  contained in the annulus  $\mathbb{A}(1/c, 1)$  as in the figure below. The compact sets  $L_n$  are converging to the boundary  $|z| = 1/c$  and the compact sets  $K_n$  are converging to the boundary  $|z| = c$ .

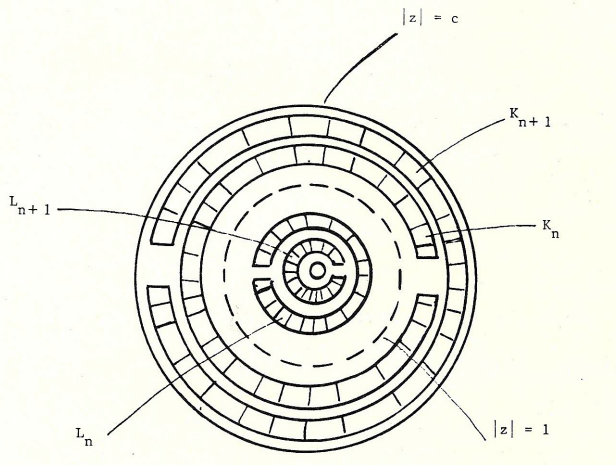


Fig. 2.

They need two non-vanishing holomorphic functions  $f, g: \mathbb{A}(1/c, c) \rightarrow \mathbb{C}$ , in order to construct a minimal surface via Weierstrass representation formula, such that the resulting minimal surface is geodesically complete and properly immersed into a slab. They construct  $f$  and  $g$  satisfying  $f(z) \cdot g(z) = 1/z$  where  $|g(z) - e^{2c_n}| < 1$  on  $K_n$  and  $|g(z) - e^{-2c_n}| < 1$  on  $L_n$ , where  $\{c_n\}$  is a sequence of positive numbers such that

$$\sum_n r_n e^{2c_n} = \infty, \quad \sum_n s_n e^{2c_n} = \infty$$

and  $r_n$  and  $s_n$  are the width of  $K_n$  and  $L_n$  respectively. The induced metric by the immersion on the annulus  $\mathbb{A}(1/c, c)$  is given by  $ds^2 = \lambda^2|dz|^2$  where

$$\lambda = \frac{1}{2|z|} \left( \frac{1}{|g(z)|} + |g(z)| \right).$$

On  $K_n$  we have

$$(11) \quad e^{2c_n} \geq \left( 1 + \frac{e^{2c_n}}{2} \right) \geq \lambda \geq \frac{1}{2|c|} (e^{2c_n} - 1)$$

The choice of the parameters  $\{(r_n, c_n)\}$  in Jorge-Xavier's construction or  $\{(r_n, s_n, c_n)\}$  in Rosenberg-Toubiana's construction gives information about the essential spectrum of the resulting surfaces. Set  $\lambda_n := \sup_{z \in K_n} \lambda(z)$ .

**Theorem 2.11.** *Let  $\varphi: \mathbb{D}, \mathbb{A}(1/c, c) \rightarrow \mathbb{R}^3$  be either Jorge-Xavier's or Rosenberg-Toubiana's complete minimal surface immersed into the slab with parameters  $\{(r_n, c_n)\}$  or  $\{(r_n, s_n, c_n)\}$ . If  $\limsup \lambda_n r_n = \infty$  then  $\sigma_{\text{ess}}(-\Delta) = [0, \infty)$ . And if  $\limsup \lambda_n r_n > 0$  then  $\varphi(\mathbb{D})$  or  $\varphi(\mathbb{A}(1/c, c))$  has the ball property and  $\sigma_{\text{ess}}(-\Delta) \neq \emptyset$ .*

At points  $z \in K_n$  we have  $e^{1+c_n} \geq \lambda(z) \geq \frac{1}{2}e^{c_n-1}$ , therefore  $e^{c_n+1} \geq \lambda_n \geq e^{c_n}/2e$ . If  $c_n = -\log(r_n^2)$  we have that the parameters  $\{(r_n, c_n)\}$  satisfies (9) and  $\lambda_n r_n = 1/(2er_n)$ . Thus  $\limsup \lambda_n r_n = \infty$  yielding a complete minimal surface between two parallel planes with spectrum  $\sigma(-\Delta) = [0, \infty)$ . In the original construction in [36], Jorge-Xavier choose  $c_n = -\log r_n$  that yields  $e \geq r_n \lambda_n \geq 1/2e$  and the resulting minimal surfaces has nonempty essential spectrum.

### 3. PRELIMINARIES

In this section we set the basic notation and definitions used in the rest of this paper. For instance, we will denote by  $\varphi: M \rightarrow N$  an isometric immersion of a complete Riemannian  $m$ -manifold  $M$  into a Riemannian  $n$ -manifold  $N$ . The Riemannian connections of  $N$  and  $M$  are denoted by  $\bar{\nabla}$  and  $\nabla$  respectively. The second fundamental form  $\alpha = \bar{\nabla}d\varphi^\perp$  and mean curvature vector  $H = \text{tr}\alpha/m$ . The gradient of a function  $g: N \rightarrow \mathbb{R}$ , is denoted by  $\bar{\nabla}g$  whereas  $\nabla(g \circ \varphi) = (\bar{\nabla}g)^\top$  is the gradient of  $g \circ \varphi$ , the restriction of  $g$  to  $M$ . The hessian of  $g$  is denoted by  $\bar{\nabla}dg$  and the hessian  $\nabla d(g \circ \varphi)$  of  $g \circ \varphi$  are related by

$$(12) \quad \nabla d(g \circ \varphi) = \bar{\nabla}dg + \langle \bar{\nabla}d\varphi^\perp, \bar{\nabla}g \rangle$$

The symbol  $B_r^N(x)$  denotes the geodesic ball of  $N$  centered at  $x \in N$  with radius  $r$ . However the unit ball  $B_1^{\mathbb{R}^2}(0)$  of  $\mathbb{R}^2$ , will be denoted by  $\mathbb{D}$ . Similarly, for  $X \subset N$  the symbol  $T_r^N(X)$ , called the tube of radius  $r$  around  $X$ , denotes the open set of points (in  $N$ ) whose distance from  $X$  is less than  $r$ . Finally, denote by  $\mathbb{R}^+ = (0, +\infty)$  and  $\mathbb{R}_0^+ = [0, +\infty)$ .

**3.1. Carathéodory's Construction.** In this section we shall review the notion of generalized  $\Psi$ -Hausdorff measures. We do follow the elegant exposition of P. Mattila, in [47, Chap.4].

**Definition 3.1** (Carathéodory's Construction). Let  $X$  be a metric space,  $\mathcal{J}$  a family of subsets of  $X$  and  $\zeta \geq 0$  a non-negative function on  $\mathcal{J}$ . Make the following assumptions.

1. For every  $\delta > 0$  there are  $E_1, E_2, \dots \in \mathcal{J}$  such that  $X = \bigcup_{i=1}^{\infty} E_i$  and  $\text{diam}(E_i) \leq \delta$ .
2. For every  $\delta > 0$  there is  $E \in \mathcal{J}$  such that  $\zeta(E) \leq \delta$  and  $\text{diam}(E) \leq \delta$ .

For  $0 < \delta \leq \infty$  and  $A \subset X$  we define

$$\zeta_\delta(A) = \inf \left\{ \sum_{i=1}^{\infty} \zeta(E_i) : A \subset \bigcup_{i=1}^{\infty} E_i, \text{diam}(E_i) \leq \delta, E_i \in \mathcal{J} \right\}.$$

It is easy to see that  $\zeta_\delta(A) \leq \zeta_\epsilon(A)$  whenever  $0 < \epsilon < \delta \leq \infty$ . Therefore,

$$\mathcal{H}_\zeta(A) = \lim_{\delta \rightarrow 0} \zeta_\delta(A) = \sup_{\delta > 0} \zeta_\delta(A)$$

defines the generalized  $\zeta$ -Hausdorff measure  $\mathcal{H}_\zeta$ .

In this construction above, let  $X = M$ , be a complete Riemannian manifold  $M$  and let  $\mathcal{J}$  be the family of Borel subsets of  $M$ . Let  $\Psi: [0, \infty) \rightarrow [0, \infty)$  a continuous function such that  $\Psi(0) = 0$ . The  $\Psi$ -Hausdorff measure is defined by  $\mathcal{H}_\Psi(A) = \mathcal{H}_\zeta(A)$  where  $\zeta(A) = \Psi(\text{diam}(A))$  and it is Borel regular, see [47, Thm. 4.2]. Taking  $\mathcal{J} = \{\text{open subsets of } M\}$  instead of the Borel sets and the same  $\Psi$ , the generalized Hausdorff measures obtained by the Carathéodory construction coincides, i.e they are the same  $\Psi$ -Hausdorff measure  $\mathcal{H}_\Psi$ , see [47, Thm. 4.4]. The choice  $\Psi(t) = t^\beta$ , for some fixed  $\beta > 0$ , gives the standard  $\beta$ -dimensional Hausdorff measure  $\mathcal{H}_{t^\beta} = \mathcal{H}^\beta$ .

*Remark 3.2.* If  $\mathcal{J}$  is the family of geodesic balls of  $M$ , the resulting measure  $\overline{\mathcal{H}}_\Psi$  does not coincide, in general, with generalized Hausdorff measure  $\mathcal{H}_\Psi$ , see [47, Chap. 5]. However, if for some constant  $c > 0$  the following inequality holds  $\Psi(2t) \leq c \cdot \Psi(t)$  then  $\mathcal{H}_\Psi \leq \overline{\mathcal{H}}_\Psi \leq c\mathcal{H}_\Psi$ .

The first inequality  $\mathcal{H}_\Psi \leq \overline{\mathcal{H}}_\Psi$  is obvious from the definition. To prove  $\overline{\mathcal{H}}_\Psi \leq c\mathcal{H}_\Psi$  we proceed as follows. Since every open set  $E_j$  is contained in a ball  $B_{r_j}^M(x_j)$  of radius  $r_j = \text{diam}(E_j)$ , we have that for every covering  $\{E_j\}$  of  $A \subseteq M$  with  $\text{diam}(E_j) < \delta$  that

$$\sum_{i=1}^{+\infty} \Psi(\text{diam}(E_j)) \geq \frac{1}{c} \cdot \sum_{i=1}^{+\infty} \Psi(2\text{diam}(E_j)) = \frac{1}{c} \cdot \sum_{i=1}^{+\infty} \Psi(\text{diam}(B_{r_j}^M(x_j))).$$

Taking the infimum, in the right hand-side, with respect to all covering  $\{B_{r_j}^M(x_j)\}$  by balls of diameter less than  $2\delta$  and taking the infimum in the left hand side with respect of  $E_i$  we have  $\overline{\zeta}_\delta \leq c \cdot \zeta_\delta$ , ( $\zeta = \Psi(\text{diam})$ ). Letting  $\delta \downarrow 0$  we obtain the desired  $\overline{\mathcal{H}}_\Psi \leq c\mathcal{H}_\Psi$ .

**3.2. Strategy of proof of Theorem 2.4.** In this section we give a brief description of the strategy for the proof of Theorem 2.4. Let  $M$  be a Riemannian manifold. The Laplace operator  $\Delta = \text{div} \circ \text{grad}$  acting on  $C_0^\infty$ , the space of smooth functions with compact support, is symmetric with respect to the  $L^2$ -scalar product. If  $M$  is complete, it is known that  $\Delta$  is essentially self-adjoint, thus it has a unique (unbounded) self-adjoint extension to an operator on  $L^2(M)$ , also denoted by  $\Delta$  whose domain  $D(\Delta) = \{f \in L^2(M) : \Delta f \in L^2(M)\}$ . If  $M$  is not geodesically complete then  $\Delta$  may fail to be essentially self-adjoint in  $C_c^\infty(M)$  and in this case we will consider the Friedrichs extension of  $\Delta$  (that is, the unique self-adjoint extension of  $(\Delta, C_c^\infty(M))$  whose domain lies in that of the closure of the associated quadratic form). Moreover,  $-\Delta$  is positive semi-definite so that the spectrum of  $-\Delta$  is contained in  $[0, \infty)$ . The spectrum of a self-adjoint operator  $-\Delta$ , denoted by  $\sigma(-\Delta)$ , is formed by all  $\lambda \in \mathbb{R}$  for which  $-(\Delta + \lambda I)$  is not injective or the inverse operator  $-(\Delta + \lambda I)^{-1}$  is unbounded, see [19]. The set of all eigenvalues of  $\sigma(M)$  is the *point spectrum*  $\sigma_p(M)$ , while the *discrete spectrum*  $\sigma_d(M)$  is the set of all isolated eigenvalues of finite multiplicity. The complement of the discrete spectrum is the *essential spectrum*,  $\sigma_{\text{ess}}(M) = \sigma(M) \setminus \sigma_d(M)$ .

To show that  $-\Delta$  has discrete spectrum we rely on the characterization (13) of the essential spectrum, see [24], [50, Thm. 2.1], and Barta's eigenvalue lower bound, see [6], [9]. This characterization relates the infimum  $\inf \sigma_{\text{ess}}(-\Delta)$  of the essential spectrum of  $-\Delta$  to the fundamental tone of the complements of compact sets. This is,

$$(13) \quad \inf \sigma_{\text{ess}}(-\Delta) = \sup_{K \subset M} \lambda^*(M \setminus K)$$

where  $K$  is compact and  $\lambda^*(M \setminus K)$  is the bottom of the spectrum of the Friedrichs extension of  $(-\Delta, C_c^\infty(M \setminus K))$ , given by

$$\lambda^*(M \setminus K) = \inf \left\{ \frac{\int_{M \setminus K} |\nabla u|^2}{\int_{M \setminus K} u^2}, 0 \neq u \in C_0^\infty(M \setminus K) \right\}.$$

On the other hand, Barta inequality gives a lower bound for  $\lambda^*(M \setminus K)$  via positive functions, this is

$$(14) \quad \lambda^*(M \setminus K) \geq \inf_{M \setminus K} \frac{-\Delta w}{w} \quad \text{for every } 0 < w \in C^2(M \setminus K).$$

To prove that  $-\Delta$  has discrete spectrum or equivalently, by the min-max theorem, to prove that  $\inf \sigma_{\text{ess}}(-\Delta) = +\infty$ , it is enough to find, for each small  $\epsilon > 0$ , a compact set  $K_\epsilon \subset M$  and a function  $0 < w_\epsilon \in C^2(M \setminus K_\epsilon)$  such that

$$(15) \quad \frac{-\Delta w_\epsilon}{w_\epsilon} \geq c(\epsilon) \quad \text{on } M \setminus K_\epsilon,$$

Where  $c(\epsilon) \rightarrow +\infty$  as  $\epsilon \rightarrow 0$ . Each  $w_\epsilon$  will be constructed as a sum of suitable strictly positive superharmonic functions, depending on a good covering of  $\lim \varphi$  by balls.

### 3.3. Technical lemmas.

3.3.1. *Main Lemma.* Let  $\varphi: M \rightarrow N$  be an isometric immersion of a complete Riemannian  $m$ -manifold  $M$  into a Riemannian  $n$ -manifold  $N$ , with mean curvature vector  $H$ . Suppose that  $\varphi(M) \subset \Omega$ , a bounded, totally regular subset and let  $b = \sup\{K_N(z), z \in T_{\text{diam}(\Omega)}(\Omega)\}$ . Fix  $\bar{a} > 0$  such that  $(\log(\bar{a}))^2 > \log(\text{diam}(\Omega))$  and if  $b > 0$ , suppose in addition that  $\bar{a} \leq \min\{\pi/3\sqrt{b}, \pi/2(1+\theta)\sqrt{b}\}$ . Recall that  $\theta = [m-1-m \cdot \mu_b(\text{diam}(\Omega)) \cdot \|H\|_{L^\infty(M)}]$ .

**Lemma 3.3** (Main Lemma). *Suppose that  $\theta > 0$ . For each  $a \in (0, \bar{a}/3]$  and  $x \in \Omega$  such that  $\varphi(M) \subset B_{\text{diam}(\Omega)}^N(x)$  there exists  $u \in C^\infty(M)$  satisfying these three conditions.*

- i.  $u \geq 0$  and  $u(p) = 0$  if and only if  $\varphi(p) = x$ .
- ii.  $\Delta u \geq \theta/3$  on  $\varphi^{-1}(B_a^N(x))$  if  $\varphi^{-1}(B_a^N(x)) \neq \emptyset$ .
- iii.  $\Delta u \geq 0$  on  $M$ .
- iv.

$$\|u\|_{L^\infty(M)} \leq \begin{cases} Ca^2 & \text{if } \theta > 1 \\ Ca^2 |\log a| & \text{if } \theta = 1 \\ Ca^{\theta+1} & \text{if } 0 < \theta < 1 \end{cases}$$

Where  $C$  is positive constant depending on  $m$ ,  $\text{diam}(\Omega)$ ,  $\|H\|_{L^\infty(M)}$ .

*Proof.* Fix  $x \in \Omega$  such that  $\varphi(M) \subset B_{\text{diam}(\Omega)}^N(x) \subset B_{r_\Omega}^N(x)$ . Thus, the distance function  $\rho(y) = \text{dist}_N(x, y)$  is smooth (except at  $y = x$ ) and the geodesic ball  $B_{\text{diam}(\Omega)}^N(x)$  is 1-convex. In fact, by the Hessian comparison theorem, [11, Theorem 1.15],

$$(16) \quad \bar{\nabla} d\rho \geq \frac{h'(\rho)}{h(\rho)} (\langle \cdot, \cdot \rangle - d\rho \otimes d\rho).$$

where  $h: [0, \infty) \rightarrow [0, \infty)$  given by

$$h(t) = \begin{cases} \frac{1}{\sqrt{b}} \sin(\sqrt{bt}) & \text{if } b > 0 \\ t & \text{if } b = 0 \\ \frac{1}{\sqrt{-b}} \sinh(\sqrt{-bt}) & \text{if } b < 0. \end{cases}$$

Let  $f \in C^2(N)$  be defined by  $f(y) = g(\rho(y))$  for some  $g \in C^2(\mathbb{R}_0^+)$  that will be chosen later. The chain rule applied to the composition  $f \circ \varphi \in C^2(M)$  implies that

$$\nabla d(f \circ \varphi) = \bar{\nabla} df(d\varphi, d\varphi) + df(\bar{\nabla} d\varphi^\perp)$$

where  $\nabla, \bar{\nabla}$  are the connections of  $M$  and  $N$  respectively and  $\bar{\nabla} d\varphi^\perp$  is the second fundamental form of the immersion. Let  $\{e_i, e_\alpha\}$  be a local Darboux frame along  $\varphi$ , with  $\{e_i\}$  tangent to  $M$ . Tracing the above equality, it yields

$$(17) \quad \Delta(f \circ \varphi) = \sum_{j=1}^m \bar{\nabla} df(e_j, e_j) + mdf(H).$$

On the other hand

$$\bar{\nabla} df = g''(\rho)d\rho \otimes d\rho + g'(\rho)\bar{\nabla} d\rho.$$

If  $g' \geq 0$  and by (16)

$$(18) \quad \bar{\nabla} df \geq \frac{g'(\rho)h'(\rho)}{h(\rho)} (\langle \cdot, \cdot \rangle - d\rho \otimes d\rho) + g''(\rho)d\rho \otimes d\rho.$$

Using  $|d\rho| = 1$  and by (18)

$$(19) \quad \begin{aligned} \sum_{j=1}^m \bar{\nabla} df(e_j, e_j) + mdf(H) &= \frac{g'h'}{h} \left( m - \sum_{j=1}^m d\rho(e_j)^2 \right) + g'' \sum_{j=1}^m d\rho(e_j)^2 + mg'd\rho(H) \\ &\geq \frac{g'h'}{h} \left( m - \sum_{j=1}^m d\rho(e_j)^2 - m \frac{h}{h'} \|H\| \right) + g'' \sum_{j=1}^m d\rho(e_j)^2 \\ &\geq \frac{g'h'}{h} \left( \overbrace{m - 1 - m\mu_b(\text{diam}(\Omega)) \|H\|_{L^\infty(M)}}^{=\theta} \right) \\ &\quad + g'' \sum_{j=1}^m d\rho(e_j)^2. \\ &= \frac{g'h'}{h} \theta + g'' \sum_{j=1}^m d\rho(e_j)^2. \end{aligned}$$

In other words,

$$(20) \quad \Delta(f \circ \varphi) \geq \frac{g'h'}{h} \theta + g'' \sum_{j=1}^m d\rho(e_j)^2.$$

Define  $\omega: [0, \infty) \rightarrow \mathbb{R}$  by

$$\omega(t) = \begin{cases} \left(1 - \frac{t}{3a(1+\theta)}\right)(\theta+1)h'(t) & \text{if } t \leq 3a(1+\theta) \\ 0 & \text{if } t \geq 3a(1+\theta). \end{cases}$$

where  $3a \leq \bar{a}$ . Setting

$$(21) \quad g(t) = \int_0^t \frac{1}{h(s)^\theta} \left[ \int_0^s h(\sigma)^\theta \omega(\sigma) d\sigma \right] ds.$$

We have that  $g$  is solution of

$$(22) \quad g'(t) \frac{h'(t)}{h(t)} \theta + g''(t) = \omega(t).$$

It is easy to show that  $g \in C^2([0, \infty))$ . From (22) we have that if  $t \leq 3a(1+\theta)$  then

$$(23) \quad \begin{aligned} g''(t) &= \omega(t) - \frac{\theta h'(t)}{h(t)^{1+\theta}} \int_0^t \left(1 - \frac{s}{3a(1+\theta)}\right) \frac{d}{ds} (h^{1+\theta}(s)) ds, \\ &= \omega(t) - \theta h'(t) + \frac{\theta h'(t)}{h^{(\theta+1)}(t)} \int_0^t \frac{s}{3a} h^\theta(s) h'(s) ds \\ &= \left(1 - \frac{t}{3a}\right) h'(t) + \frac{\theta h'(t)}{h^{(\theta+1)}(t)} \int_0^t \frac{s}{3a} h^\theta(s) h'(s) ds. \end{aligned}$$

From (23) we have that  $g''(t) \geq 0$  if  $t \leq 3a$ . Moreover,  $h'(t) \geq 1/2$  if  $t \leq 3a$ . Then at any  $x' \in \varphi^{-1}(B_a^N(x))$  we have from (20)

$$\begin{aligned}
\Delta f \circ \varphi(x') &\geq \frac{g'h'}{h}\theta + g'' \sum_{j=1}^m d\rho(e_j)^2. \\
&\geq g'(\rho(\varphi(x))) \frac{h'}{h} \rho(\varphi(x))\theta, \\
(24) \quad &\geq \left(1 - \frac{\rho(\varphi(x))}{3a(1+\theta)}\right) \theta h'(\rho(\varphi(x))) \\
&\geq \frac{1}{2} \left(1 - \frac{\rho(\varphi(x))}{3a(1+\theta)}\right) \theta \\
&\geq \frac{\theta}{3}
\end{aligned}$$

Let  $M = \{y \in M : g''(\rho(\varphi(y))) \geq 0\} \cup \{y \in M : g''(\rho(\varphi(y))) < 0\} = A \cup B$ . Clearly, the inequalities (24) also shows that if  $x' \in A$  then  $\Delta f \circ \varphi(x) \geq 0$ .

On the other hand, at any point  $x' \in B$  we have by (20), using

$$|\bar{\nabla} \rho|^2 = 1 = \sum_{j=1}^m d\rho(e_j)^2 + \sum_{\alpha=m+1}^n d\rho(e_\alpha)^2 \geq \sum_{j=1}^m d\rho(e_j)^2,$$

that

$$\begin{aligned}
\Delta f \circ \varphi(x) &\geq \left[ g' \frac{h'}{h} \theta + g'' \sum_{j=1}^m d\rho(e_j)^2 \right], \\
(25) \quad &\geq g' \frac{h'}{h} \theta + g'' \\
&\geq \omega \\
&\geq 0.
\end{aligned}$$

Observe that

$$(26) \quad \int_0^t h(s)^\theta \omega(s) ds \leq \begin{cases} h(t)^{1+\theta} & \text{if } 0 \leq t \leq 3a(1+\theta) \\ h(t_1)^{1+\theta} & \text{if } t > t_1 = 3a(1+\theta). \end{cases}$$

Taking in account that  $c_1 \cdot t \leq h(t) \leq c_2 \cdot t$ ,  $t \in [0, \text{diam}(\Omega)]$  for some positive constants  $c_1, c_2$ , we have the following upper bounds for  $g$ .

If  $0 \leq t \leq t_1 = 3a(1+\theta)$ ,

$$\begin{aligned}
(27) \quad g(t) &= \int_0^t \frac{1}{h(s)^\theta} \left[ \int_0^s h(\sigma)^\theta \omega(\sigma) d\sigma \right] ds \\
&\leq \int_0^t h(s) ds. \\
&\leq c_2 \frac{(t_1)^2}{2} = 9 \cdot c_2 \cdot \frac{(1+\theta)^2}{2} \cdot a^2
\end{aligned}$$

If  $t \geq t_1 = 3a(1 + \theta)$ ,

$$\begin{aligned}
g(t) &= \int_0^a \frac{1}{h(s)^\theta} \left[ \int_0^s h(\sigma)^\theta \omega(\sigma) d\sigma \right] ds + \int_a^t \frac{1}{h(s)^\theta} \left[ \int_0^{t_1} h(\sigma)^\theta \omega(\sigma) d\sigma \right] ds \\
&\leq \int_0^a h(s) ds + h^{1+\theta}(t_1) \int_a^t \frac{1}{h(s)^\theta} ds \\
&\leq \frac{c_2}{2} \cdot a^2 + \frac{c_2^{(1+\theta)} (3a(1+\theta))^{(1+\theta)}}{c_1} \int_a^t \frac{1}{s^\theta} ds \\
&= c_3 \cdot a^2 + c_4 \cdot a^{(\theta+1)} \int_a^t \frac{1}{s^\theta} ds \\
(28) \quad &\leq c_3 \cdot a^2 + c_4 \cdot a^{(\theta+1)} \begin{cases} \frac{a^{1-\theta}}{\theta-1} & \text{if } \theta > 1 \\ c_5 \cdot |\ln a| & \text{if } \theta = 1 \\ \frac{t^{1-\theta}}{1-\theta} \leq \frac{\text{diam}(\Omega)^{1-\theta}}{1-\theta} & \text{if } 0 < \theta < 1 \end{cases}
\end{aligned}$$

We can deduce from (27) and (28) that there exists a positive constant  $C$  depending on  $m, \text{diam}(\Omega), b$  and  $\|H\|_{L^\infty(M)}$  such that

$$(29) \quad \|g\|_{L^\infty([0, \text{diam}(\Omega)])} \leq \begin{cases} Ca^2 & \text{if } \theta > 1 \\ Ca^2 |\log a| & \text{if } \theta = 1 \\ Ca^{\theta+1} & \text{if } \theta \in (0, 1). \end{cases}$$

Taking  $u = f \circ \varphi: M^m \rightarrow \mathbb{R}$  we have:

- By construction  $u(p) = 0$  if and only if  $\varphi(p) = x$
- By (24) and (26) we have  $\Delta u \geq \theta/3$  on  $\varphi^{-1}(B_a^N(x))$  and  $\Delta u \geq 0$  on  $M$ , respectively.
- By (29) we have  $\|u\|_{L^\infty(M)} \leq \|f\|_{L^\infty(\varphi^{-1}(B_{\text{diam}(\Omega)}^N(x)))} = \|g\|_{L^\infty([0, \text{diam}(\Omega)])}$ .

This proves the Lemma 3.3.  $\square$

**3.3.2. Strictly  $m$ -convex domains.** A strictly  $m$ -convex domain  $\Omega \subset N$  with constant  $c > 0$  is related to the existence of strictly  $m$ -subharmonic functions on  $\Omega$ .

**Definition 3.4.** A  $C^2$ -function  $\phi: \Omega \rightarrow \mathbb{R}$  is said to be **strictly  $m$ -subharmonic** with constant  $c > 0$  if  $\lambda_1(p) \leq \lambda_2(p) \leq \dots \leq \lambda_n(p)$  are the ordered eigenvalues of the hessian  $\nabla^2 \phi(p)$  then there exists an  $\epsilon > 0$  such that

$$\begin{cases} \lambda_1(p) + \dots + \lambda_m(p) \geq c, & \forall p \in T_\epsilon^N(\partial\Omega) = \{y \in N : \text{dist}_N(y, \partial\Omega) \leq \epsilon\} \\ \lambda_1(p) + \dots + \lambda_m(p) \geq 0, & \forall p \in \Omega. \end{cases}$$

Let  $\Omega \subset N$  be a strictly  $m$ -convex domain of  $N$  with constant  $c > 0$  and  $\Gamma = \partial\Omega$  of class  $C^3$ . Let  $t: N \rightarrow \mathbb{R}$  be the oriented distance function to  $\Gamma$  with orientation outward  $\Omega$ . This is,

$$(30) \quad t(y) = \begin{cases} -\text{dist}_N(y, \partial\Omega) & \text{if } y \in \Omega \\ \text{dist}_N(y, \partial\Omega) & \text{if } y \in N \setminus \Omega \end{cases}$$

The oriented distance  $t(y)$  is Lipschitz in  $N$  and of class  $C^2$  in a tubular neighborhood  $T_{\epsilon_0}^N(\partial\Omega)$  for some  $\epsilon_0$ . Let  $\alpha_s$  be the shape operator of the parallel hypersurface  $\Gamma_s = t^{-1}(s)$ ,

$|s| \leq \epsilon_0$  with respect to the normal vector field  $-\bar{\nabla}t$ . At each point of  $\Gamma_s$  there is an orthonormal bases of  $T\Gamma_s$  such that  $\alpha_s$  is diagonalized

$$\alpha_s = \text{diag}(\xi_1^s, \xi_2^s, \dots, \xi_{n-1}^s),$$

where  $\xi_1^s \leq \xi_2^s \leq \dots \leq \xi_{n-1}^s$ . By the uniform continuity of each  $\xi_j^s$  and the compactness of  $\overline{T_{\epsilon_0}^N(\partial\Omega)}$ , for each  $\delta \in (0, 1)$  one can choose  $\epsilon_0$  small enough to have

$$\xi_1^s(y) + \dots + \xi_m^s(y) \geq \delta c$$

$\forall y \in T_{\epsilon_0}^N(\partial\Omega)$ . Let  $\epsilon_1$  be a positive number so that

$$\epsilon_1 < \min \left\{ 1, \epsilon_0, \|\alpha_s\|_{L^\infty(T_{\epsilon_0}^N(\partial\Omega))}^{-1} \right\}.$$

Define  $\Phi_\epsilon: N \rightarrow \mathbb{R}$ ,  $0 < \epsilon < \epsilon_1/2$ , by

$$(31) \quad \Phi_\epsilon(y) = \begin{cases} -2\epsilon & \text{if } t(y) \leq -2\epsilon \\ 2\epsilon \left[ \left( \frac{t(y)}{2\epsilon} + 1 \right)^3 - 1 \right] & \text{if } t(y) \geq -2\epsilon \end{cases}$$

The function  $\Phi_\epsilon$  is Lipschitz on  $N$  and of class  $C^2$  in  $\overline{T_{\epsilon_0}^N(\Omega)} = t^{-1}((-\infty, \epsilon_0])$ .

For  $t(y) \leq \epsilon_0$ , we can compute the gradient and the hessian of  $\Phi_\epsilon$  as follows.

$$\bar{\nabla}\Phi_\epsilon(y) = \begin{cases} 0 & \text{if } t(y) \leq -2\epsilon \\ 3 \left( \frac{t(y)}{2\epsilon} + 1 \right)^2 \bar{\nabla}t(y) & \text{if } -2\epsilon \leq t(y) \leq \epsilon_0 \end{cases}$$

$$\bar{\nabla}d\Phi_\epsilon(y)(X, Y) = \begin{cases} 0 & \text{if } t(y) \leq -2\epsilon \\ 3 \left( \frac{t(y)}{2\epsilon} + 1 \right)^2 \bar{\nabla}dt(y)(X, Y) + \frac{3}{\epsilon} \left( \frac{t(y)}{2\epsilon} + 1 \right) X(t)Y(t) & \text{if } -2\epsilon \leq t(y) \leq \epsilon_0 \end{cases}$$

Write  $\bar{\nabla}d\Phi_\epsilon(y)(X, Y) = \langle S(X), Y \rangle$ , for an endomorphism  $S: TN \rightarrow TN$ . We have that for  $-2\epsilon \leq t(y) \leq 2\epsilon$ ,  $S(y)$  can be represented by a diagonal matrix,

$$S(y) = \text{diag} \left( 3 \left( \frac{t(y)}{2\epsilon} + 1 \right)^2 \xi_1^t(y), \dots, 3 \left( \frac{t(y)}{2\epsilon} + 1 \right)^2 \xi_{n-1}^t(y), \frac{3}{\epsilon} \left( \frac{t(y)}{2\epsilon} + 1 \right) \right)$$

Since

$$(32) \quad \begin{aligned} 3 \left( \frac{t(y)}{2\epsilon} + 1 \right)^2 \xi_j^t(y) - \frac{3}{\epsilon} \left( \frac{t(y)}{2\epsilon} + 1 \right) &= 3 \left( \frac{t(y)}{2\epsilon} + 1 \right) \left[ \left( \frac{t(y)}{2\epsilon} + 1 \right) \xi_j^t(y) - \frac{1}{\epsilon} \right] \\ &\leq 6 \left[ 2\xi_j^t - \frac{1}{\epsilon_1} \right] \\ &\leq 12 \left( \xi_j^t(y) - 2\|\alpha_t\|_{L^\infty(T_{\epsilon_0}^N(\partial\Omega))} \right) \\ &\leq 0 \end{aligned}$$

We get  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ ,  $\lambda_j = 3 \left( \frac{t}{2\epsilon} + 1 \right) \xi_j^t$ ,  $j = 1, \dots, n-1$ ,  $\lambda_n(y) = \frac{3}{\epsilon} \left( \frac{t(y)}{2\epsilon} + 1 \right)$  with  $S = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ . By Lemma 2.3 of [35], we have that for any subspace  $V \subset T_y N$ ,



$y \in T_{2\epsilon}^N(\partial\Omega)$  and  $1 \leq \dim V = m \leq n-1$  that

$$\begin{aligned}
(33) \quad \text{Trace}(\overline{\nabla}d\Phi_\epsilon(y)|V) &\geq \lambda_1(y) + \cdots + \lambda_m(y) \\
&\geq 3 \left( \frac{t(y)}{2\epsilon} + 1 \right)^2 \left[ \xi_1^{t(y)} + \cdots + \xi_m^{t(y)} \right] \\
&\geq 3 \left( \frac{t(y)}{2\epsilon} + 1 \right)^2 \delta c,
\end{aligned}$$

Then

- If  $t(y) \leq 2\epsilon$ , we obtain that,  $\text{Trace}(\overline{\nabla}d\Phi_\epsilon(y)|V) \geq 0$  and
- for  $|t(y)| \leq \epsilon(1 - \sqrt{\delta})$ , we obtain,  $\text{Trace}(\overline{\nabla}d\Phi_\epsilon(y)|V) \geq 3(1 + \sqrt{\delta})^2 \delta c/4$ .

This proves the following lemma.

**Lemma 3.5.** *Let  $\Omega$  be a strictly  $m$ -convex,  $1 \leq m \leq n-1$ , with constant  $c > 0$ . There exists a Lipschitz function  $\Phi_\epsilon: N \rightarrow \mathbb{R}$ ,  $C^2$  in  $T_{2\epsilon}(\Omega)$ ,  $2\epsilon < \epsilon_1$ ,  $\epsilon_1$  is a positive number depending on the geometry of  $\partial\Omega$ , and such that*

1.  $\Phi_\epsilon^{-1}((-\infty, 0)) = \Omega$ ,  $\Phi_\epsilon^{-1}(0) = \partial\Omega$
2.  $|\Phi_\epsilon| \leq 2\epsilon$  in  $\overline{\Omega}$ .
3.  $\text{Trace}(\overline{\nabla}d\Phi_\epsilon(y)|V) \geq 3(1 + \sqrt{\delta})^2 \delta c/4$ , for  $|t(y)| \leq \epsilon(1 - \sqrt{\delta})$  and any subspace  $V \subset T_y N$  of dimension  $m$ .
4.  $\text{Trace}(\overline{\nabla}d\Phi_\epsilon(y)|V) \geq 0$ , in  $\Omega$  for any subspace  $V \subset T_y N$  of dimension  $m$ .

In other words,  $\Phi_\epsilon$  is strictly  $m$ -subharmonic function with constant  $3(1 + \sqrt{\delta})^2 \delta c/4$ .

We will need the following lemma for the proof of Theorem 2.4.

**Lemma 3.6.** *Let  $\varphi: M^m \rightarrow N^n$  be an isometric immersion such that there exists a bounded, totally regular, strictly  $m$ -convex domain  $\Omega \subset N$  with constant  $c > 0$  and  $C^3$ -boundary  $\partial\Omega$  such that  $\varphi(M) \subset \Omega$ ,  $\mathcal{H}_\Psi(\lim \varphi \cap \Omega) = 0$  and*

$$(34) \quad \|H\|_{L^\infty(M)} < \min\left\{ \frac{m-1}{m \cdot \mu_b(\text{diam}(\Omega))}, \frac{c}{m} \right\}.$$

Take  $\delta \in (0, 1)$  such that

$$\|H\|_{L^\infty(M)} < \frac{\delta^2 c}{m}$$

and let  $\epsilon < \epsilon_1/2$  as above in Lemma 3.5. Then the function  $u: M^m \rightarrow \mathbb{R}$  given by  $u = \Phi_\epsilon \circ \varphi$ , where  $\Phi_\epsilon$  is also given in Lemma 3.5, satisfies

- (1)  $|u(x)| \leq 2\epsilon$  for all  $x \in M$
- (2)  $\Delta u(x) \geq 0$  for all  $x \in M$
- (3)  $\Delta u(x) \geq C_\delta$ , if  $|t(\varphi(x))| \leq \epsilon(1 - \sqrt{\delta})$ , where  $C_\delta = 3c \cdot \delta \cdot (1 - \delta) \cdot (1 + \sqrt{\delta})^2/4$ .
- (4)  $\varphi(M) \cap \partial\Omega = \emptyset$

*Proof.* Taking  $u = \Phi_\epsilon \circ \varphi$  the item 1. holds by the item 2. of Lemma 3.5 and the fact that  $\varphi(M) \subset \Omega$ . On the other hand, we have by (33)

$$\begin{aligned}
(35) \quad \Delta u(x) &= \text{Trace}(\overline{\nabla}d\Phi_\epsilon|_{T_{\varphi(x)}M}) + \langle \nabla\Phi_\epsilon, mH \rangle \\
&\geq 3 \left( \frac{t(\varphi(x))}{2\epsilon} + 1 \right)^2 \delta c - 3 \left( \frac{t(\varphi(x))}{2\epsilon} + 1 \right)^2 \delta^2 c \\
&= 3 \left( \frac{t(\varphi(x))}{2\epsilon} + 1 \right)^2 \delta c(1 - \delta) \\
&\geq 0
\end{aligned}$$

This proves item 2. If  $|t(\varphi(x))| \leq \epsilon(1 - \sqrt{\delta})$  we get

$$(36) \quad \Delta u(x) \geq \frac{3}{4}(1 + \sqrt{\delta})^2(1 - \delta)\delta c$$

and that proves item 3. If there exists a  $x \in \varphi^{-1}(\varphi(M) \cap \partial\Omega)$  then  $\Delta u(x) > 0$  by (35). On the other hand  $u$  has a maximum at  $x$  therefore  $\Delta u(x) \leq 0$  a contradiction. This proves item 4 and finishes the proof of Lemma 3.6.  $\square$

#### 4. PROOF OF THE RESULTS

**4.1. Proof of Theorem 2.4.** Let  $\varphi: M \rightarrow N$  be an isometric immersion of a Riemannian  $m$ -manifold  $M$  into a Riemannian  $n$ -manifold  $N$  with mean curvature vector  $H$ . Suppose that  $\varphi(M) \subset \Omega \subset B_{\text{diam}(\Omega)}^N(x_0)$ , for a bounded totally regular subset  $\Omega$  and a point  $x_0 \in \Omega$ ,  $b = \sup\{K_N(z), z \in T_{\text{diam}(\Omega)}(\Omega)\}$  and  $\|H\|_{L^\infty(M)} < (m-1)/m \cdot \mu_b(\text{diam}(\Omega))$ .

We are going to prove the Theorem 2.4 under the assumption of item 1. Suppose that  $\mathcal{H}_\Psi(\lim \varphi) = 0$ . Choose a positive number  $\bar{a} > 0$  such that  $(\log(\bar{a}))^2 > \log(\text{diam}(\Omega))$  and if  $b > 0$  take  $\bar{a} \leq \min\{\pi/3\sqrt{b}, \pi/2(1 + \theta)\sqrt{b}\}$ , where  $\theta = m-1 - m\mu(\text{diam}(\Omega))\|H\|_{L^\infty(M)}$ . Choose  $r_1 \ll \text{diam}(\Omega)$  such that the  $2r_1$ -tubular neighborhood  $T_{2r_1}(\lim \varphi) \subset B_{\text{diam}(\Omega)}^N(x_0)$ . Fix  $\epsilon \in (0, r_1)$ . Since  $\mathcal{H}_\Psi(\lim \varphi) = 0$  and Remark 3.2, there is  $a > 0$  and a countable covering of  $\lim \varphi$  by balls  $B_j = B_{a_j}^N(y_j) \subset N$  of radius  $2a_j \leq a \leq \min\{r_1, \bar{a}/3\}$  such that

$$(37) \quad \lim \varphi \subset \bigcup_j B_j \quad \text{and} \quad \left| \sum_j \Psi(2a_j) \right| < \epsilon.$$

Since  $\lim \varphi$  is compact we can extract a finite sub-covering  $\{B_j\}_{j=1}^k$  of  $\lim \varphi$  such that (37) holds, and each  $B_j \subset T_{2r_1}(\lim \varphi)$  for all  $j = 1, \dots, k$ . Applying Lemma 3.3, we construct, for every  $j = 1, \dots, k$ , a function  $u_j: M \rightarrow \mathbb{R}$  such that

$$(38) \quad \begin{cases} u_j \geq 0, & u_j(p) = 0 \text{ if and only if } \varphi(p) = y_j, \\ \|u_j\|_{L^\infty(M)} \leq C\Psi(2a_j) \\ \Delta u_j \geq 0 \text{ on } M, & \Delta u_j \geq \theta/3 \text{ on } \varphi^{-1}(B_j), \end{cases}$$

where  $C$  is positive constant depending on  $m, \text{diam}(\Omega), \|H\|_{L^\infty(M)}$ .

Let  $w_1 = \sum_{j=1}^{k_1} (2\|u_j\|_{L^\infty} - u_j) > 0$ . By the boundedness of  $\varphi(M)$  the set

$$K_\epsilon = M \setminus \varphi^{-1}\left(\bigcup_{j=1}^{k_1} B_j\right)$$

is compact in  $M$ . Now, by (14) the fundamental tone  $\lambda^*(M \setminus K_\epsilon) \geq \inf_{M \setminus K_\epsilon} \left(-\frac{\Delta_M w_1}{w_1}\right)$ .

Let  $q \in M \setminus K_\epsilon$  then  $\varphi(q) \in \bigcup_{j=1}^{k_1} B_j$ . Let  $j'$  be so that  $\varphi(q) \in B_{j'}$ . Then  $\Delta_M u_{j'}(q) \geq \theta/3$  and  $\Delta_M u_j(q) \geq 0$  for all other  $j$ 's. Therefore,

$$(39) \quad \begin{aligned} -\frac{\Delta w_1}{w_1}(q) &\geq \frac{\sum_j \Delta_M u_j(q)}{2 \sum_j \|u_j\|_{L^\infty}} \\ &\geq \frac{\Delta_M u_{j'}(q)}{2C \sum_j \Psi(2a_j)} \\ &\geq \frac{\theta}{6C\epsilon} \end{aligned}$$

Here  $C = C(m, R_1, \|H\|_{L^\infty(M)})$ . This shows that  $\lambda^*(M \setminus K_\epsilon) \geq \frac{\theta}{6C\epsilon}$  for each  $\epsilon \in (0, r_1)$ . Therefore  $\lambda^*(M \setminus K_\epsilon) \rightarrow +\infty$  if  $\epsilon \rightarrow 0$  and proves item 1.

To prove item 2. we recall that we have an isometric immersion  $\varphi: M^m \rightarrow N^n$  of a Riemannian  $m$ -manifold  $M$  into a Riemannian  $n$ -manifold  $N$  with mean curvature vector  $H$  such that  $\varphi(M) \subset \Omega$ ,  $\Omega \subset B_{\text{diam}(\Omega)}^N(y_0)$  a totally regular, strictly  $m$ -convex domain with constant  $c > 0$  and  $C^3$ -boundary  $\partial\Omega$  and  $\Psi$ -Hausdorff measure  $\mathcal{H}_\Psi(\lim \varphi \cap \Omega) = 0$ . The mean curvature vector is assumed to satisfy  $\|H\|_{L^\infty(M)} < \min\{(m-1)/m \cdot \mu_b(\text{diam}(\Omega)), c/m\}$ . We may assume that  $\lim \varphi \cap \partial\Omega \neq \emptyset$ , otherwise we can apply item 1. By Lemma 3.6, there exist positive numbers  $\delta = \delta(\varphi)$ ,  $C_\delta > 0$  and  $\epsilon_1 = \epsilon_1(\Omega)$  such that for any  $\epsilon < \epsilon_1/2$ , there exists a  $C^2$  function  $u: M \rightarrow \mathbb{R}$ , such that

1.  $u^{-1}(-\infty, 0) = M$ .
2.  $|u(x)| \leq 2\epsilon$  in  $M$ .
3.  $\Delta u(x) \geq 0$  for all  $x \in M$ .
4.  $\Delta u(x) \geq C_\delta$ , if  $\varphi(x) \in T_{\epsilon(1-\sqrt{\delta})}(\partial\Omega)$ .

Fix one  $\epsilon$ ,  $0 < \epsilon < \epsilon_1/2$  and set  $K = \lim \varphi \setminus T_{\epsilon(1-\sqrt{\delta})}(\partial\Omega)$ . We have  $K \subset \lim \varphi \cap \Omega$  compact  $\mathcal{H}_\Psi(K) = 0$ . By the first part of this proof we have finite functions  $u_j: M \rightarrow \mathbb{R}$  and balls  $B_j \subset \Omega$  (covering  $K$ ) such that (37) and (38) holds. Take  $w_1 = \sum_{j=1}^{k_1} (2\|u_j\|_{L^\infty} - u_j) > 0$  (related to  $K$ ) and  $u: M \rightarrow \mathbb{R}$  given by Lemma 3.6. Define  $\omega: M \rightarrow \mathbb{R}$  by

$$\omega(x) = \omega_1(x) + \epsilon - u(x), \quad x \in M$$

and

$$K_\epsilon = M \setminus \varphi^{-1} \left( (\cup_{j=1}^{k_1} B_j) \cup T_{\epsilon(1-\sqrt{\delta})}(\partial\Omega) \right)$$

The set  $K_\epsilon$  is compact and for  $x \in M \setminus K_\epsilon$  we get

$$-\Delta\omega \geq c_0 = \min\left\{\frac{\theta}{3}, C_\delta\right\} > 0.$$

Since  $0 < \omega(x) < (2C + 3)\epsilon$ ,  $x \in M$ , we get

$$-\frac{\Delta\omega}{\omega} \geq \frac{c_0}{(2C + 3)\epsilon}.$$

Then  $\lambda^*(M \setminus K_\epsilon) \rightarrow \infty$  if  $\epsilon \rightarrow 0$  what proves item 2.

**4.2. Proof of Theorem 2.9.** In this section we show that the ball property, introduced in Definition 2.8, implies the existence of elements in the essential spectrum of  $-\Delta$ . Let  $M$  be a Riemannian manifold with the ball property, this is, there exists  $R > 0$  and a collection of disjoint balls  $\{B_R^M(x_j)\}_{j=1}^\infty$  such that for some constants  $C > 0$  and  $\delta \in (0, 1)$  the inequalities

$$\text{vol}(B_{\delta R}^M(x_j)) \geq C^{-1} \text{vol}(B_R^M(x_j)), \quad j = 1, 2, \dots$$

hold. For each  $j$ , define the compactly supported, Lipschitz function  $\phi_j(x) = \zeta(\rho_j(x))$ , where  $\rho_j(x) = \text{dist}(x, x_j)$  and

$$(40) \quad \zeta(t) = \begin{cases} 1 & \text{if } t \leq \delta R \\ \frac{R-t}{R(1-\delta)} & \text{if } t \in [\delta R, R] \\ 0 & \text{if } t \geq R \end{cases}$$

Observe that  $|\zeta'| \leq \frac{1}{R(1-\delta)}$ . By the ball property (7),

$$\begin{aligned}
(41) \quad I_\lambda(\phi_j, \phi_j) &= \int_{B_R^M(x_j)} |\nabla \phi_j|^2 - \lambda \int_{B_R^M(x_j)} \phi_j^2 \\
&\leq \frac{\text{vol}(B_R^M(x_j))}{R^2(1-\delta)^2} - \lambda \text{vol}(B_{\delta R}^M(x_j)) \\
&\leq \text{vol}(B_R^M(x_j)) \left( \frac{1}{R^2(1-\delta)^2} - \lambda C^{-1} \right) \\
&< 0
\end{aligned}$$

provided that  $\lambda > C/(R^2(1-\delta)^2)$ . Since  $\{\phi_j\}$  span an infinite-dimensional subspace of the domain of  $-\Delta$ , the Friedrichs extension of the operator  $-(\Delta + \lambda)$  has infinite index, or equivalently,  $-\Delta$  has infinite eigenvalues below  $\lambda$ , for each  $\lambda > C/(R(1-\delta))^2$ . By the Min-Max Theorem, see [21, Prop. 2.1 & 2.2], [51, Section 3] or [54], the inequality  $\inf \sigma_{\text{ess}}(-\Delta) \leq C/(R(1-\delta))^2$  follows.

*Remark 4.1.* In virtue of Bishop-Gromov volume comparison theorem, [12], [31], all Riemannian  $n$ -manifolds  $M$  with Ricci curvature bounded below  $\text{Ric}_M \geq -(n-1)k^2$  has the ball property. In fact, if we denote by  $\text{vol}_k(r)$  the volume of a geodesic ball of radius  $r$  in the hyperbolic space  $\mathbb{H}^n(-k^2)$  of constant sectional curvature  $-k^2$ . By the Bishop-Gromov volume comparison theorem, the ratio  $\text{vol}(B_r(x_j))/\text{vol}_k(r)$  is non-increasing on  $[0, R]$ . Hence, for each  $\delta > 0$

$$\text{vol}(B_{\delta R}^M(x_j)) \geq \frac{\text{vol}_k(\delta R)}{\text{vol}_k(R)} \text{vol}(B_R^M(x_j)) = C(\delta, R)^{-1} \text{vol}(B_R^M(x_j)).$$

4.2.1. *Application of the ball property.* Now, we will show that, for a suitable choice of their parameters, the Jorge-Xavier and Rosenberg-Toubiana complete minimal surfaces immersed into slabs of  $\mathbb{R}^3$  have the ball property. Denoting by  $\varphi: \mathbb{D} \rightarrow \{(x_1, x_2, x_3): |x_3| < 1\}$  and  $\varphi: \mathbb{A}(1/c/c) \rightarrow \{(x_1, x_2, x_3): 1/c < x_3 < c\}$  with parameters  $\{(r_n, c_n)\}$ ,  $\{(r_n, s_n, c_n)\}$  respectively, the examples of Jorge-Xavier and Rosenberg-Toubiana, we shall show that with the choice  $c_n = -\log(r_n^2)$ , we have that  $0 = \inf \sigma_{\text{ess}}(-\Delta)$  in both surfaces. The induced metric  $ds^2$  in Jorge-Xavier minimal immersion is conformal to the Euclidean metric  $|dz|^2$ . More precisely,  $ds^2 = \lambda^2 |dz|^2$ , where

$$\lambda = \frac{1}{2} (|e^h| + |e^{-h}|).$$

At points of  $K_n$ ,

$$e^{1+c_n} \geq \lambda \geq \frac{1}{2} e^{c_n-1}$$

thus,

$$e^{2+2c_n} |dz|^2 \geq ds^2 = \lambda^2 |dz|^2 \geq \frac{1}{4} e^{2c_n-2} |dz|^2$$

Choosing  $c_n = -\log(r_n^2)$  and letting  $I_n$  be the segment of the real axis that crosses  $K_n$  one has that the length  $\ell(I_n)$  of this segment in the metric  $ds^2$  has the following lower and upper bound

$$\frac{e^2}{r_n^4} \geq \ell(I_n) \geq r_n e^{c_n-1} \geq \frac{e^{-1}}{r_n}$$

Let  $p_n$  be the center of the  $I_n$  and denote by  $B_R^{\text{ds}^2}(p_n)$  and  $B_R^{|dz|^2}(p_n)$  the geodesic balls of radius  $R$  and center  $p_n$  with respect to the metric  $ds^2$  and the metric  $|dz|^2$  respectively. Giving  $R > 0$ , there exists  $n_R$  such that for all  $n \geq n_R$  the geodesic ball  $B_R^{\text{ds}^2}(p_n) \subset K_n$

for all  $n \geq n_R$ . Indeed, since  $r_n \rightarrow 0$  as  $n \rightarrow \infty$ , just choose  $n_R$  be such that  $r_{n_R} \leq \frac{e^{-1}}{3R}$ . Moreover, these inclusions

$$B_{R/(e^{1+c_n})}^{|dz|^2}(p_n) \subset B_R^{\text{ds}^2}(p_n) \subset B_{2R/(e^{c_n-1})}^{|dz|^2}(p_n)$$

holds. Therefore, for  $\delta \in (0, 1)$ , we have

$$\begin{aligned} \text{vol}_{\text{ds}^2}(B_{\delta R}^{\text{ds}^2}(p_n)) &\geq \text{vol}_{\text{ds}^2}(B_{\delta R/(e^{1+c_n})}^{|dz|^2}(p_n)) \\ (42) \qquad \qquad \qquad &\geq \frac{1}{4}e^{2c_n-2} \text{vol}_{|dz|^2}(B_{\delta R/(e^{1+c_n})}^{|dz|^2}(p_n)) \\ &= \frac{1}{4e^4} \text{vol}_{|dz|^2}(B_{\delta R}^{|dz|^2}(p_n)) \end{aligned}$$

$$\begin{aligned} \text{vol}_{\text{ds}^2}(B_R^{\text{ds}^2}(p_n)) &\leq \text{vol}_{\text{ds}^2}(B_{2R/(e^{c_n-1})}^{|dz|^2}(p_n)) \\ (43) \qquad \qquad \qquad &\leq e^{2c_n+2} \text{vol}_{|dz|^2}(B_{2R/(e^{c_n-1})}^{|dz|^2}(p_n)) \\ &= 4e^4 \text{vol}_{|dz|^2}(B_R^{|dz|^2}(p_n)) \end{aligned}$$

From (42) and (43) we have

$$(44) \qquad \qquad \qquad \text{vol}_{\text{ds}^2}(B_{\delta R}^{\text{ds}^2}(p_n)) \geq \frac{\delta^2}{e^{10}} \cdot \text{vol}_{\text{ds}^2}(B_R^{\text{ds}^2}(p_n))$$

This shows that Jorge-Xavier minimal surfaces with those choices of  $c_n$  above has the ball property, (along the sequence  $p_n$ , for  $n \geq n_R$ ), with parameters  $R$ ,  $\delta$  and  $C = e^{10}/\delta^2$ . By Theorem 2.9,

$$\inf \sigma_{\text{ess}}(-\Delta) \leq \frac{C}{R^2(1-\delta)^2}.$$

Letting  $R \rightarrow \infty$ , we conclude that  $0 \in \sigma_{\text{ess}}(-\Delta)$ . Likewise, in the construction of Rosenberg-Toubiana's complete minimal annulus properly immersed into a slab of  $\mathbb{R}^3$  the induced metric is given by  $\text{ds}^2 = \lambda^2|dz|^2$ ,  $\lambda = \frac{1}{2|z|} \left( \frac{1}{|g(z)|} + |g(z)| \right)$ . On  $K_n$  we have

$$e^{2c_n} \geq \left( 1 + \frac{e^{2c_n}}{2} \right) \geq \lambda \geq \frac{1}{2|c|} (e^{2c_n} - 1)$$

Letting  $I_n$  be the segment of the real axis crossing  $K_n$  and  $p_n$  the middle point of  $I_n$  we have that the geodesic ball (in the metric  $\text{ds}^2$ ) with radius  $R > 0$  and center  $p_n$  is contained in  $K_n$ , for sufficiently large  $n$ ,

$$B_R^{\text{ds}^2}(p_n) \subset K_n$$

Moreover,

$$B_{\frac{1}{e^{2c_n}}R}^{|dz|^2}(p_n) \subset B_R^{\text{ds}^2}(p_n) \subset B_{\frac{2|c|}{e^{2c_n-1}}R}^{|dz|^2}(p_n)$$

Thus

$$\text{vol}_{\text{ds}^2}(B_{\delta R}^{\text{ds}^2}(p_n)) \geq \text{vol}_{\text{ds}^2}(B_{\frac{1}{e^{2c_n}}\delta R}^{|dz|^2}(p_n)) \geq \frac{(e^{2c_n} - 1)^2}{4|c|^2 e^{4c_n}} \text{vol}_{|dz|^2}(B_{\delta R}^{|dz|^2}(p_n))$$

and

$$\text{vol}_{\text{ds}^2}(B_R^{\text{ds}^2}(p_n)) \leq \text{vol}_{\text{ds}^2}(B_{\frac{2|c|}{e^{2c_n-1}}R}^{|dz|^2}(p_n)) \leq \frac{4|c|^2 e^{4c_n}}{(e^{2c_n} - 1)^2} \text{vol}_{|dz|^2}(B_R^{|dz|^2}(p_n)).$$

Therefore, for  $n$  so that  $1 - r_n \geq 2/3$  we have  $\text{vol}_{\text{ds}^2}(B_{\delta R}^{\text{ds}^2}(p_n)) \geq \frac{\delta^2}{81|c|^4} \text{vol}_{\text{ds}^2}(B_R^{\text{ds}^2}(p_n))$ .

This shows that Rosenberg-Toubiana minimal surfaces with those choices of  $c_n$  above has

the ball property, (along the sequence  $p_n$ ), with parameters  $R$ ,  $\delta$  and  $C = 81|c|^4/\delta^2$ . By Theorem 2.9,

$$\inf \sigma_{ess}(-\Delta) \leq \frac{C}{R^2(1-\delta)^2}.$$

Again, letting  $R \rightarrow \infty$ , we conclude that  $0 \in \sigma_{ess}(-\Delta)$ . This finishes the proof.

We conclude this section calling the attention to an example of a bounded minimal surface  $\varphi: M \rightarrow \mathbb{R}^3$  with  $\dim_{\mathcal{H}}(\varphi(M)) = 3$ , which is not a covering and  $\sigma_{ess}(-\Delta) \neq \emptyset$ . In [4] P. Andrade constructed a complete minimal immersion  $\varphi: \mathbb{C} \rightarrow \mathbb{R}^3$  with bounded curvature with the property that  $\varphi(\mathbb{C})$  was an unbounded subset of the Euclidean space  $\mathbb{R}^3$  with  $\text{vol}_3(\varphi(\mathbb{C})) = \infty$ , see also [56]. In other words, he constructed a dense complete minimal surface with bounded curvature thus, with the ball property. However, the restriction of the parametrization of Andrade's surface to a strip  $U = \{u+iv \in \mathbb{C} : |u| < 1\}$ , yields a bounded, simply-connected minimal immersion with the ball property and dense in a bounded subset of  $\mathbb{R}^3$ . To give more details, we will keep Andrade's notation, thus, here and only here,  $H$  will be a holomorphic function.

**Example 4.2.** Choose  $r_1, r_2 > 0$  such that  $r_1/r_2$  is irrational and strictly less than 1, and set  $d = r_2 - r_1$ . Define the map  $\chi: \mathbb{C} \rightarrow \mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ ,  $\chi(z) = (L(z) - \overline{H(z)}, h(z))$ , for the following choice of holomorphic functions  $L, H$  and harmonic function  $h$ :

$$L(z) = (r_1 - r_2)e^z, \quad H(z) = -de^{\left(\frac{r_1}{r_2}-1\right)z}, \quad h(z) = 4\left(\frac{d}{r_2}\right)^{1/2} \left| \frac{r_2}{r_1} \right| |r_2 - r_1| \Re \left( ie^{\frac{r_1}{2r_2}z} \right),$$

where  $\Re$  means the real part. Then, a computation gives that

$$|L'(z)| + |H'(z)| > 0, \quad L'H' = \left( \frac{\partial h}{\partial z} \right)^2 \quad \text{on } \mathbb{C},$$

which is a necessary and sufficient set of condition on  $\chi$  to be a conformal minimal immersion of  $\mathbb{C}$  in  $\mathbb{R}^3$ . Restricting  $\chi$  to the region  $U = \{z = u + iv \in \mathbb{C} : |u| < 1\}$ , we get a bounded, simply-connected minimal immersion  $\varphi = \chi|_U$ . For each fixed  $u \in (-1, 1)$ ,  $\varphi(u + iv)$  is a dense immersed trochoid in the cylinder  $\Gamma_u = [B_{s_1(u)} \setminus B_{s_2(u)}] \times (-l(u), l(u))$ , where  $s_1, s_2, l$  are explicit functions of  $u$  depending on  $r_1$  and  $r_2$ . Therefore,  $\lim \varphi$  is dense in the open subset  $\bigcup_{u \in (-1, 1)} \Gamma_u$  of  $\mathbb{R}^3$ , which gives  $\dim_{\mathcal{H}}(\lim \varphi) = 3$ . Moreover, the induced metric  $ds^2$  satisfies

$$(45) \quad ds^2 = (|L'| + |H'|)^2 |dz|^2 = \left( |r_2 - r_1|e^u + de^{\left(\frac{r_1}{r_2}-1\right)u} \right)^2 |dz|^2 \geq 4(r_2 - r_1)^2 |dz|^2.$$

Considering  $z_k = 2ik \in U$ , each of the unit balls  $B_1^{|dz|^2}(z_k) \subseteq U$  in the metric  $|dz|^2$  contains a ball  $B_R(z_k)$  in the metric  $ds^2$  of radius at least  $R = 2|r_2 - r_1|$ . Since the sectional curvature of  $\chi$  satisfies

$$K = -c_1 \left( e^{\left(1 - \frac{r_1}{4r_2}\right)u} + c_2 e^{\left(\frac{3r_1}{4r_2}-1\right)u} \right)^{-4},$$

for some positive constants  $c_1, c_2$ , and  $1 - \frac{r_1}{4r_2}$  and  $\frac{3r_1}{4r_2} - 1$  have opposite signs,  $\chi$  has globally bounded curvature. In particular,  $\{B_R(z_k)\}$  is a collection of disjoint balls in  $(U, ds^2)$  with uniformly bounded sectional curvature, therefore,  $\sigma_{ess}(-\Delta) \neq \emptyset$  on  $(U, ds^2)$ , by Theorem 2.9 and Remark 4.1.

**4.3. Proof of Theorem 2.10.** Let  $\varphi: M \rightarrow N$  be a non-proper isometric immersion with locally bounded geometry of a complete Riemannian manifold into a complete Riemannian manifold  $N$ . We are going to show that there exists a sequence  $\{x_j\} \subset M$  a radius  $R$ , a constant  $C > 0$  and  $\delta \in (0, 1)$  such that

$$\text{vol}_M(B_{\delta R}^M(x_j)) \geq C^{-1} \text{vol}_M(B_R^M(x_j)).$$

In other words,  $M$  has the ball property. Let  $y_0 \in \lim \varphi$  and let  $W \subset N$  be a compact subset with  $y_0 \in \text{int}(W)$ . Let  $\Lambda_0 = \Lambda_0(W)$  be such that  $\|\alpha_\varphi\|_{L^\infty(\varphi^{-1}(W))} \leq \Lambda_0$ . The Gauss equation and the upper bound  $\sup_W |K_N| < \infty$  of the sectional curvatures of  $N$  on  $W$  gives a positive number  $b_0 > 0$  such that

$$\sup_{x \in \varphi^{-1}(W)} |K_M(x)| \leq 2\Lambda_0^2 + \sup_W |K_N| \leq b_0$$

where  $K_M$  are the sectional curvatures of  $M$ . In particular, each connected component  $U \subset \varphi^{-1}(W)$  has sectional curvatures uniformly bounded  $|K_U| \leq b_0$ . Set

$$(46) \quad 2r_0 = \min\{i_W, (2\Lambda_0)^{-1}, b_0^{-1/2} \cdot \cot^{-1}(1/(2\sqrt{b_0})), \text{dist}_N(y_0, N \setminus W)\}$$

where  $i_W = \inf\{\text{inj}_N(x), x \in W\}$ . Let  $B_0 = \overline{B_{r_0}^N(y_0)}$  be the closure of the geodesic ball of  $N$  with radius  $r_0$  and center  $y_0$ . There exists a sequence of points  $q_j \in M$ ,  $q_j \rightarrow \infty$  in  $M$  such that  $\varphi(q_j) \rightarrow y_0$  in  $N$ . Passing to a subsequence if necessary we may assume that  $q_j \in B_0$  and  $q_j \neq q_{j'}$  if  $j \neq j'$ . Define  $\rho_{y_0}: N \rightarrow \mathbb{R}$  by  $\rho_{y_0}(z) = \text{dist}_N(y_0, z)^2/2$ ,  $z \in N$ . Since  $r_0 < \text{inj}_N(y_0)$ , the function  $\rho_{y_0}(z)$  is at least  $C^2$  if  $\text{dist}_N(y_0, z) \leq r_0$ . If  $d_{b_0}(x) = \text{dist}_{\mathbb{N}^n(b_0)}(0, x)$  is the distance to a origin 0 in a simply connected  $n$ -space form  $\mathbb{N}^n(b_0)$  of constant sectional curvature  $b_0$  then by the hessian comparison theorem we obtain

$$(47) \quad \begin{aligned} \text{Hess} \rho_{y_0}(z)(Y, Y) &\geq \text{Hess} \frac{1}{2} d_{b_0}(p_0, p)^2(Y', Y') \\ &\geq \sqrt{b_0} \cot(\sqrt{b_0} r_0) |Y'|^2 \\ &\geq \frac{1}{2} |Y|^2, \end{aligned}$$

where  $d_N(y_0, z) = d_{b_0}(p_0, p) \leq r_0$ ,  $|Y| = |Y'|$ ,  $Y \perp \nabla \rho_y$  and  $Y' \perp \nabla d_{b_0}$ . We need part of the following result that might have interest in its own.

**Lemma 4.3.** *Let  $r \leq r_0/8$ . Then*

i. *For each  $x \in \varphi^{-1}(B_0)$  we have  $\text{inj}_M(x) > r_0$ .*

ii. *Let  $U_j$  be a connected component of  $\varphi^{-1}(\overline{B_{4r}^N(y_0)})$  containing  $q_j$ , then*

$$\text{dist}_N(\varphi(z_1), \varphi(z_2)) \leq \text{dist}_M(z_1, z_2) \leq 2\text{dist}_N(\varphi(z_1), \varphi(z_2)), \forall z_1, z_2 \in U$$

*Thus the map  $\varphi|_{U_j}: U_j \rightarrow N$  is an embedding.*

iii. *Take  $x_j \in U_j$  such that  $\text{dist}_N(y_0, \varphi(x_j)) = \text{dist}_N(y_0, \varphi(U_j))$ . If  $j$  is large enough then  $B_{3r}^M(x_j) \subset U_j \subset B_{10r}^M(x_j)$ .*

*Proof.* Let  $x \in \varphi^{-1}(B_0)$ . Suppose, by contradiction, that  $\text{dist}_M(x, \text{cut}_M(x)) < r_0$ . Let  $z \in \text{cut}_M(x)$  such that  $\text{dist}_M(x, z) = \text{dist}_M(x, \text{cut}_M(x))$ . By the restrictions (46),  $z$  is not conjugated to  $x$ , thus, there are two distinct minimal geodesics  $\gamma_1$  and  $\gamma_2$  joining  $x$  to  $z$ , making a geodesic loop  $\gamma = \gamma_1 \cup \gamma_2$  based at  $x$ , [14, Lemma 5.6]. Since  $r_0 > \text{dist}_M(x, z) \geq \text{dist}_N(\varphi(x), \varphi(z))$ , the closed curve  $\varphi(\gamma)$  is the region in  $N$  where  $\rho_{y_0}$  is  $C^2$ . The function  $h(s) = \rho_{y_0}(\varphi(\gamma(t)))$  has a maximum at  $s = \text{inj}_M(x)$ , however

$$(48) \quad \begin{aligned} h''(s) &= \nabla d \rho_{y_0}(d\varphi\gamma', d\varphi\gamma') + \langle \nabla \rho_{y_0}, \alpha(\gamma', \gamma') \rangle \\ &\geq 1/2 - r_0 \Lambda_0 \\ &\geq 1/4, \quad 0 \leq s \leq 2\text{inj}_M(x). \end{aligned}$$

This contradiction proves item (i). To prove (ii), let  $U_j \subset \varphi^{-1}(\overline{B_{4r}^N(y_0)})$  be a connected component containing  $q_j$ . Let  $z_1, z_2 \in U_j$  and  $y_1 = \varphi(z_1)$  and  $y_2 = \varphi(z_2)$ . Let  $\gamma(s)$ ,  $s \in [0, \text{dist}_M(z_1, z_2)]$  be a minimal geodesic in  $M$  joining  $z_1$  to  $z_2$ . We may assume without loss of generality that  $\text{dist}_N(y_0, y_1) \leq \text{dist}_N(y_0, y_2)$ . Observe that  $\rho_{y_0}(\varphi(\gamma(s))) \leq \rho_{y_0}(y_2)$  for all  $s$ . Otherwise,  $s \mapsto \rho_{y_0}(\varphi(\gamma(s)))$  has a maximum at some interior point  $s_0 \in (0, \text{dist}_M(z_1, z_2))$

and  $\text{dist}_N(y_0, \varphi_\iota(\gamma(s_0))) < r_0$ . Taking the second derivative at this point of maximum and we get a contradiction, as above, and that proves our assertion. Moreover,  $s \mapsto \rho_{y_1}(\varphi(\gamma(s)))$  is of class at least  $C^2$ . It is clear that  $(\rho_{y_1}(\varphi_\iota(\gamma(s))))'' \geq 1/4$  for all  $s \in [0, t = \text{dist}_M(z_1, z_2)]$ . Then

$$\begin{aligned}
\frac{\text{dist}_N^2(y_1, y_2)}{2} &= \rho_{y_1}(\varphi(\gamma(t))) \\
&= \rho_{y_1}(\varphi(\gamma(0))) + t\rho_{y_1}(\varphi(\gamma(s)))'|_{s=0} + \int_0^1 (1-s)(\rho_{y_1}(\varphi(\gamma(st))))'' ds \\
(49) \quad &\geq \frac{t^2}{4} \int_0^1 (1-s) ds \\
&= \frac{t^2}{8}
\end{aligned}$$

It follows that  $\text{dist}_M(z_1, z_2) \leq 2\text{dist}_N(\varphi(z_1), \varphi(z_2))$ .

To prove item iii. Pick  $x_j \in U_j$  such that  $\text{dist}_N(y_0, \varphi(x_j)) = \text{dist}_N(y_0, \varphi(U_j))$ . We may choose  $j$  large enough so that  $\text{dist}_N(y_0, \varphi(x_j)) < r$ . Let  $x \in B_{3r}^M(x_j)$ . Then

$$\begin{aligned}
\text{dist}_N(\varphi(x), y_0) &\leq \text{dist}_N(\varphi(x), \varphi(x_j)) + \text{dist}_N(\varphi(x_j), y) \\
&< \text{dist}_M(x, x_j) + r \\
&\leq 4r
\end{aligned}$$

Now let  $x \in U$  then

$$\begin{aligned}
\text{dist}_M(x_j, x) &\leq 2\text{dist}_N(\varphi(x_j), \varphi(x)) \\
&\leq 2[\text{dist}_N(\varphi(x_j), y_0) + \text{dist}_N(y_0, \varphi(x))] \\
&< 10r
\end{aligned}$$

By the Lemma 4.3, there exists a sequence  $x_j \in M$  such that  $B_{3r}^M(x_j) \subset U_j \subset B_{10r}^M(x_j)$  for all  $j$ . Observe that  $\text{dist}_N(q_j, y_0) \geq \text{dist}_N(\varphi(x_j), y_0) \rightarrow 0$  as  $j \rightarrow \infty$  and  $y_0 \in \lim \varphi$ . Therefore passing to a subsequence we have that  $x_j \neq x_{j+k}$  for all  $k \geq 1$ . Recall that the sectional curvatures of  $U_j$  are bounded below  $K_{U_j} \geq -b_0$ . Let  $\mathbb{N}^m(-b_0)$  the simply connected space form of constant sectional curvature  $-b_0$ . Choose any  $\delta \in (0, 1)$ . By the Bishop-Gromov volume comparison theorem we have

$$\text{vol}(B_{\delta 3r}^M(x_j)) \geq \frac{\text{vol}(B_{\delta 3r}^{\mathbb{N}^m(-b_0)})}{\text{vol}(B_{3r}^{\mathbb{N}^m(-b_0)})} \text{vol}(B_{3r}^M(x_j)) = C(b_0, m, \delta, 3r)^{-1} \text{vol}(B_{3r}^M(x_j)).$$

This shows that  $M$  has the ball property with respect to the parameters  $\{x_j\}$ ,  $R = 3r$ ,  $C^{-1} = \text{vol}(B_{\delta 3r}^{\mathbb{N}^m(-b_0)})/\text{vol}(B_{3r}^{\mathbb{N}^m(-b_0)})$  and any  $\delta \in (0, 1)$ . Since  $3r \in (0, 3r_0/8)$  and  $\delta \in (0, 1)$  we have by Theorem 2.9 (taking  $\delta = 1/2$ ) that

$$\inf \sigma_{\text{ess}}(-\Delta) \leq \frac{256}{9r_0^2} \cdot \frac{\text{vol}(B_{3r}^{\mathbb{N}^m(-b_0)})}{\text{vol}(B_{\delta 3r/2}^{\mathbb{N}^m(-b_0)})}.$$

□

**4.4. Proof of Theorem 2.11.** To prove Theorems 2.11 we will apply the following proposition derived from the Spectral Theorem, see details in [21, Prop.2], [30, pp. 13-15].

**Proposition 4.4.** *Let  $M$  be a Riemannian manifold. A necessary and sufficient condition for  $(\eta - \epsilon, \eta + \epsilon) \cap \sigma_{\text{ess}}(-\Delta) \neq \emptyset$  is that there exists an infinite dimensional subspace  $G_\epsilon$  of the domain  $D(-\Delta)$  of  $-\Delta$ , for which  $\|(\Delta + \eta I)\psi\|_{L^2(M)} < \epsilon\|\psi\|_{L^2(M)}$ ,  $\psi \in G_\epsilon$ .*



To show that  $\eta \in \sigma_{\text{ess}}(-\Delta)$ , using this proposition, we need to take a sequence  $v_n \rightarrow 0$  as  $n \rightarrow \infty$  and a sequence of functions  $\psi_n \in C_0^\infty(M)$  satisfying  $\|(\Delta + \eta I)\psi_n\|_{L^2(M)} < v_n \|\psi_n\|_{L^2(M)}$  with  $\text{supp } \psi_n \cap \text{supp } \psi_{n'} = \emptyset$  if  $n \neq n'$ . Consider a sequence of compact subsets  $K_n \subset D_n$  with Euclidean width  $r_n \rightarrow 0$  as  $n \rightarrow \infty$  and the set of constants  $c_n$  satisfying (9) in Jorge-Xavier's or Rosenberg-Toubiana's construction. The induced metric  $ds^2 = \varphi^*|dz|^2$  on the minimal surface is conformal to the Euclidean metric  $|dz|^2$  on the disk  $\mathbb{D}$ . More precisely,  $ds^2 = \lambda^2|dz|^2$ . Set  $\lambda_n = \sup_{K_n} \lambda(z)$  and  $\zeta_n = \lambda_n/(\inf_{K_n} \lambda(z))$  so that  $\lambda_n/\zeta_n \leq \lambda \leq \lambda_n$  in  $K_n$ . Let  $I_n$  be the segment of the real axis that crosses  $K_n$ . The length  $\ell_{ds^2}(I_n)$  of  $I_n$  in the metric  $ds^2$  has the following lower and upper bound

$$\frac{\lambda_n r_n}{\zeta_n} \leq \ell_{ds^2}(I_n) \leq \lambda_n r_n$$

Let  $p_n$  be the center of the  $I_n$  and denote by  $B_t^{ds^2}(p_n)$  and  $B_t^{|dz|^2}(p_n)$  the geodesic balls of radius  $t$  and center  $p_n$  with respect to the metrics  $ds^2$  and  $|dz|^2$  respectively. Denote by  $\Delta^{|dz|^2}$  and by  $dx$ , respectively the Laplace operator and the Lebesgue measure of  $\mathbb{R}^2$  with respect to the metric  $|dz|^2$  and denote by  $\Delta^{ds^2}$  and by  $\lambda^2 dx$  the Laplace operator and the Riemannian measure on  $M$  with respect to the metric  $ds^2$ . The Laplace operators  $\Delta^{|dz|^2}$  and  $\Delta^{ds^2}$  are related, on  $\mathbb{D}$ , by  $\Delta^{ds^2} = \frac{1}{\lambda^2} \Delta^{|dz|^2}$ . Given  $\eta > 0$  and  $f \in C_0^\infty(B_{r_n}^{|dz|^2}(p_n))$  be a smooth function with compact support in  $B_{r_n}^{|dz|^2}(p_n) \subset K_n$  to be chosen later. We have that

$$\begin{aligned} \|\Delta^{ds^2} f + \eta f\|_{L^2(M)}^2 &= \int_{B_{r_n}^{|dz|^2}(p_n)} \left( \frac{1}{\lambda^2} \Delta^{|dz|^2} f + \eta f \right)^2 \lambda^2 dx \\ &= \int_{B_{r_n}^{|dz|^2}(p_n)} \frac{1}{\lambda^2} (\Delta^{|dz|^2} f)^2 dx + \eta^2 \int_{B_{r_n}^{|dz|^2}(p_n)} f^2 \lambda^2 dx \\ (50) \quad &+ 2\eta \int_{B_{r_n}^{|dz|^2}(p_n)} f \Delta^{|dz|^2} f dx \\ &\leq \int_{B_{r_n}^{|dz|^2}(p_n)} \frac{\epsilon_n^2}{\lambda_n^2} (\Delta^{|dz|^2} f)^2 dx + \eta^2 \zeta_n^2 \int_{B_{r_n}^{|dz|^2}(p_n)} f^2 \lambda_n^2 dx \\ &+ 2\eta \zeta_n^2 \int_{B_{r_n}^{|dz|^2}(p_n)} f \Delta^{|dz|^2} f dx + 2\eta(\zeta_n^2 - 1) \int_{B_{r_n}^{|dz|^2}(p_n)} |\nabla^{|dz|^2} f|^2 dx \\ &= \zeta_n^2 \int_{B_{r_n}^{|dz|^2}(p_n)} \left( \frac{1}{\lambda_n^2} \Delta^{|dz|^2} f + \eta f \right)^2 \lambda_n^2 dx + 2\eta(\zeta_n^2 - 1) \int_{B_{r_n}^{|dz|^2}(p_n)} |\nabla^{|dz|^2} f|^2 dx \end{aligned}$$

Consider the ball  $B_{\lambda_n r_n}^{|dz|^2}(p_n) = p_n + B_{\lambda_n r_n}^{|dz|^2}(0) \subset \mathbb{R}^2$  of radius  $\lambda_n r_n$  and center  $p_n$  and the map  $\xi: B_{\lambda_n r_n}^{|dz|^2}(p_n) \rightarrow B_{r_n}^{|dz|^2}(p_n)$  given by  $\xi(p_n + x) = p_n + x/\lambda_n$  and define  $h: B_{\lambda_n r_n}^{|dz|^2}(p_n) \rightarrow \mathbb{R}$  by  $h = f \circ \xi$ . We have that  $\Delta^{|dz|^2} h = \Delta^{|dz|^2} f(\xi)/\lambda_n^2$  and the Jacobian  $J(\xi)(x) = 1/\lambda_n^2$ . Considering the change of variables  $x = \xi(y)$  we have that

$$\begin{aligned} &\bullet \int_{B_{r_n}^{|dz|^2}(p_n)} \left( \frac{1}{\lambda_n^2} \Delta^{|dz|^2} f + \eta f \right)^2 \lambda_n^2 dx = \int_{B_{\lambda_n r_n}^{|dz|^2}(p_n)} (\Delta^{|dz|^2} h + \eta h)^2 dx \\ &\bullet \int_{B_{r_n}^{|dz|^2}(p_n)} |\nabla^{|dz|^2} f|^2 dx = \int_{B_{\lambda_n r_n}^{|dz|^2}(p_n)} |\nabla^{|dz|^2} h|^2 dx \end{aligned}$$

Therefore from (50) and the change of variable above we have the following inequality (51)

$$\|\Delta^{ds^2} f + \eta f\|_{L^2(M)} \leq \zeta_n \|\Delta^{|dz|^2} h + \eta h\|_{L^2(B_{\lambda_n r_n}^{|dz|^2}(p_n))} + \sqrt{2\eta(\zeta_n^2 - 1)} \|\nabla^{|dz|^2} h\|_{L^2(B_{\lambda_n r_n}^{|dz|^2}(p_n))}$$

Where  $f: B_{r_n}^{|dz|^2}(p_n) \subset K_n \rightarrow \mathbb{R}$ ,  $h = f \circ \xi: B_{\lambda_n r_n}^{|dz|^2}(p_n) \rightarrow \mathbb{R}$  defined by  $h(p_n + x) = f(p_n + x/\lambda_n)$ . Observe that  $f = h \circ \xi^{-1}: B_{r_n}^{|dz|^2}(p_n) \rightarrow \mathbb{R}$  so that  $f(p_n + x) = h(p_n + \lambda_n x)$ ,

$x \in B_{r_n}^{|\mathrm{d}z|^2}(0)$ . In other words given  $h \in C_0^\infty(B_{\lambda_n r_n}^{|\mathrm{d}z|^2}(p_n))$  we obtain  $f \in C_0^\infty(B_{r_n}^{|\mathrm{d}z|^2}(p_n))$  and vice-versa, satisfying the inequality (51).

Given a positive real number  $\eta > 0$  and since  $\sigma(-\Delta^{|\mathrm{d}z|^2}) = \sigma_{\mathrm{ess}}(-\Delta^{|\mathrm{d}z|^2}) = [0, \infty)$  we have that  $\eta \in \sigma_{\mathrm{ess}}(-\Delta^{|\mathrm{d}z|^2})$ . Therefore for each  $\delta > 0$  there exists, (by Proposition 4.4),  $h \in C_0^\infty(\mathbb{R}^2)$  such that

$$(52) \quad \|\Delta^{|\mathrm{d}z|^2} h + \eta h\|_{L^2(\mathbb{R}^2)} < \delta \|h\|_{L^2(\mathbb{R}^2)}.$$

Suppose that  $\limsup_{n \rightarrow \infty} r_n \lambda_n = \infty$ . Then there exists  $n_0$  such that for all  $n \geq n_0$  the ball  $B_{\lambda_n r_n}^{|\mathrm{d}z|^2}(p_n)$  contains the support of  $h$  since for large  $n$  we have  $1 \leq e_n < 2$  and the length  $\ell_{ds^2}(I_n) \geq \lambda_n r_n / \zeta_n \rightarrow \infty$ . For this function  $h \in C_0^\infty(B_{\lambda_n r_n}^{|\mathrm{d}z|^2}(p_n))$  we have

- $\int_{B_{\lambda_n r_n}^{|\mathrm{d}z|^2}(p_n)} |\nabla^{|\mathrm{d}z|^2} h|^2 dx \leq \mu_1(n) \int_{B_{\lambda_n r_n}^{|\mathrm{d}z|^2}(p_n)} h^2 dy$ , where  $\mu_1(n) = \lambda_1(B_{\lambda_n r_n}^{|\mathrm{d}z|^2}(p_n))$  is the first Dirichlet eigenvalue of the ball  $B_{\lambda_n r_n}^{|\mathrm{d}z|^2}(p_n)$ .
- Letting  $f(p_n + x) = h(p_n + \lambda_n x) \in C_0^\infty(B_{r_n}^{|\mathrm{d}z|^2}(p_n))$  we have

$$(53) \quad \begin{aligned} \int_{B_{\lambda_n r_n}^{|\mathrm{d}z|^2}(p_n)} h^2 dy &= \int_{B_{r_n}^{|\mathrm{d}z|^2}(p_n)} \lambda_n^2 f^2 dx \\ &\leq 4 \int_{B_{r_n}^{|\mathrm{d}z|^2}(p_n)} f^2 \lambda^2 dx \\ &= 4 \|f\|_{L^2(M)}^2, \end{aligned}$$

since  $\lambda_n \leq 2\lambda$ .

- Putting together these information we have

$$\int_{B_{\lambda_n r_n}^{|\mathrm{d}z|^2}(p_n)} |\nabla^{|\mathrm{d}z|^2} h|^2 dx \leq 4 \|f\|_{L^2(M)}^2$$

From the inequality (51) we have then

$$\|\Delta^{ds^2} f + \eta f\|_{L^2(M)} \leq \left( 2\zeta_n \delta + 2\sqrt{2\eta(\zeta_n^2 - 1)\mu_1(n)} \right) \|f\|_{L^2(M)}.$$

We are ready to conclude that each  $\eta > 0$  belongs to  $\sigma_{\mathrm{ess}}(-\Delta^{ds^2})$ . Let us consider a sequence of positive numbers  $v_i \rightarrow 0$ . For each  $i$ , choose  $n$  such that  $2\sqrt{2\eta(\epsilon_n^2 - 1)\mu_1(n_i)} < v_i/2$ . This  $n$  exists since  $\mu_1(n) = \lambda_1(B_{\lambda_n r_n}^{|\mathrm{d}z|^2}(p_n)) = c/(\lambda_n r_n)^2 \rightarrow 0$  and  $\epsilon_n \rightarrow 1$  as  $n \rightarrow \infty$ . Take  $\delta < v_i/4$  and choose  $h_i \in C_0^\infty(\mathbb{R}^2)$  such that (51) holds and choosing  $n_i$  large enough so that  $\mathrm{supp} h_i \subset B_{\lambda_{n_i} r_{n_i}}^{|\mathrm{d}z|^2}(p_n)$ . Then the function  $f_i$  associated to  $h_i$  satisfies

$$\|\Delta^{ds^2} f_i + \eta f_i\|_{L^2(M)} < v_i \|f_i\|_{L^2(M)}.$$

It is clear that we can choose the family  $h_i$  with support in different balls. All that shows that  $\eta \in \sigma_{\mathrm{ess}}(-\Delta^{ds^2})$ . To finish the proof of Theorems 2.11 we need to address the case that  $\limsup r_n \lambda_n > 0$ . Observe that in  $K_n$  we have that

$$\frac{\lambda_n}{\zeta_n} \leq \lambda \leq \lambda_n.$$

This implies that, in  $K_n$ ,

$$\left( \frac{\lambda_n}{\zeta_n} \right)^2 |\mathrm{d}z|^2 \leq \mathrm{d}s^2 \leq \lambda_n^2 |\mathrm{d}z|^2.$$

From this point on, is easy to see that  $(\mathbb{D}, \mathrm{d}s^2)$  or  $(\mathbb{A}(1/c, c), \mathrm{d}s^2)$  has the ball property, see details in the application the subsection 4.2.1. Thus  $\sigma_{\mathrm{ess}}(\mathrm{d}s^2) \neq \emptyset$ . This finishes the proof of Theorem 2.11.

#### 4.5. Open problems.

- (1) We presented an example of a complete bounded surface with non-empty essential spectrum and limit set with positive 2-dimensional Hausdorff measure, see Remark 2.7. This shows that Theorem 2.4 is sharp. However, for submanifolds of dimension  $m \geq 3$ , it seems that requiring that the 2-dimensional Hausdorff measure of the limit set be zero is a technicality of our proof. A natural question arises.

**Question 4.5.** Let  $\varphi: M^m \rightarrow N^n$  be a bounded, minimally immersed submanifold of dimension  $m \geq 3$  of a Hadamard manifold  $N$ . Let  $\Omega \subset N$  be a bounded open subset so that  $\varphi(M) \subset \Omega$ . If  $\mathcal{H}^m(\lim \varphi \cap \Omega) = 0$ , does  $-\Delta$  has discrete spectrum?

- (2) Infinite sheeted coverings of complete bounded minimal surfaces always have non-empty essential spectrum. On the other hand, Example 4.2 establishes the existence of incomplete minimal surfaces with  $\sigma_{\text{ess}}(-\Delta) \neq \emptyset$  and whose immersion map  $\varphi$  is not a Riemannian covering. One could naturally ask the following: is it possible to find a complete, bounded minimal surface  $\varphi: M \rightarrow \mathbb{R}^3$  with non-empty essential spectrum and such that  $\varphi$  is not a Riemannian covering map?
- (3) Although Theorem 2.4 can be applied for each of the examples (i.), . . . , (vii.), it is still unapplicable to the original example of Nadirashvili [48]. Is it possible to find a choice of parameters in Nadirashvili's construction, such that the essential spectrum of the resulting minimal surface is not empty?
- (4) In the Jorge-Xavier's or in the Rosenberg-Toubiana's construction, what can be said about the essential spectrum if the choice of the parameters  $\{(r_n, c_n)\}$  is such that  $\limsup r_n \lambda_n = 0$ ?

**Acknowledgements:** This paper was completed while the third author was enjoying the hospitality of the Universidade Federal do Ceará-Brazil. The third author is indebted to Andrea Carlo Mennucci and Carlo Mantegazza for useful discussions. The first two authors were partially supported by FUNCAP-PRONEX and CNPq grants # 301041/2009-1 & # 305778/2012-9.

#### REFERENCES

- [1] A. Alarcón, *Compact complete minimal immersions in  $\mathbb{R}^3$* . Trans. Amer. Math. Soc. **362** (2010), 4063–4076.
- [2] A. Alarcón, L. Ferrer and F. Martin, *Density theorems for complete minimal surfaces in  $\mathbb{R}^3$* . Geom. Funct. Anal. **18** (2008), no. 1, 1–49.
- [3] A. Alarcón, F. Martin, *Null curves in  $\mathbb{C}^3$  and Calabi-Yau conjectures*. Math. Ann. **355**, no. 2, 429–455.
- [4] P. Andrade, *A wild minimal plane in  $\mathbb{R}^3$* . Proc. Amer. Math. Soc. **128** (2000), no. 5, 1451–1457.
- [5] A. Baidier, *Noncompact Riemannian manifolds with discrete spectra*, J. Diff. Geom. **14** (1979), 41–57.
- [6] J. Barta, *Sur la vibration fondamentale d'une membrane*. C. R. Acad. Sci. **204** (1937), 472–473.
- [7] G. P. Bessa, L. P. Jorge, G. Oliveira-Filho *Half-Space Theorems for Minimal Surfaces with Bounded Curvature*. J. Differential Geometry **57** (2001), 493–508.
- [8] G. P. Bessa, J. F. Montenegro, *Eigenvalue estimates for submanifolds with locally bounded mean curvature*. Ann. Global Anal. and Geom. **24**, (2003), 279–290.
- [9] G. P. Bessa, J. F. Montenegro, *An extension of Barta's Theorem and geometric applications*. Ann. Global Anal. and Geom. **31** (2007), 345–362.
- [10] G. Pacelli Bessa, L. Jorge, J. Fabio Montenegro, *The Spectrum of the Martin-Morales-Nadirashvili Minimal Surfaces is Discrete*. J. Geom. Anal. **20** (2010), 63–71.
- [11] B. Bianchini and L. Mari and M. Rigoli, *On some aspects of Oscillation Theory and Geometry*, Mem. Amer. Math. Soc. **225**, (2013) 195 pp.
- [12] R. L. Bishop, R. J. Crittenden, *Geometry of manifolds*. Pure and Applied Mathematics, Vol. XV Academic Press, New York-London (1964) ix+273 pp.
- [13] E. Calabi, *Problems in Differential Geometry* (S. Kobayashi and J. Eells, Jr., eds.) Proc. of the United States-Japan Seminar in Differential Geometry, Kyoto, Japan, 1965, Nippon Hyoronsha Co. Ltd., Tokyo (1966) 170.

- [14] J. Cheeger, D. Ebin, *Comparison theorems in Riemannian geometry*. North-Holland Mathematical Library, Vol. 9. North-Holland Publishing Co., Amsterdam-Oxford; American Elsevier Publishing Co., Inc., New York, 1975. viii+174 pp.
- [15] S. Y. Cheng, P. Li, and S. T. Yau, *Heat equations on minimal submanifolds and their applications*. Amer. J. Math. **106**, (1984), 1033–1065.
- [16] S. S. Chern, *The geometry of G-structures*. Bull. Amer. Math. Soc. **72** (1966), 167–219.
- [17] T. H. Colding, W. P. Minicozzi, *The Calabi-Yau conjectures for embedded surfaces*. Ann. of Math. (2) **167**, no. 1 (2008), 211–243.
- [18] P. Collin, H. Rosenberg, *Notes sur la démonstration de N. Nadirashvili des conjectures de Hadamard et Calabi-Yau*. Bull. Sci. Math. **123**, no. 7, (1999), 563–575.
- [19] E. B. Davies, *Spectral theory and differential operators*, Cambridge University Press, 1995. (regularity), Springer-Verlag, Berlin, 1992.
- [20] M. DoCarmo, C. K. Peng, *Stable complete minimal surfaces in  $\mathbb{R}^3$  are planes*. Bull. Amer. Math. Soc. (N.S.) **1**, no. 6 (1979), 903–906.
- [21] H. Donnelly *On the essential spectrum of a complete Riemannian manifold*. Topology **20** (1981), no. 1, 1–14.
- [22] H. Donnelly, *Negative curvature and embedded eigenvalues*. Math. Z. **203**, (1990), 301–308.
- [23] H. Donnelly, N. Garofalo, *Riemannian manifolds whose Laplacian have purely continuous spectrum*. Math. Ann. **293**, (1992), 143–161.
- [24] H. Donnelly and P. Li, *Pure point spectrum and negative curvature for noncompact manifolds*. Duke Math. J. **46** (1979), 497–503.
- [25] J. Douglas, *Solution of the problem of Plateau*. Trans. Amer. Math. Soc. **33** (1931), 263–321
- [26] J. Douglas, *The least area property of the minimal surface determined by an arbitrary Jordan contour*. Proc. Natl. Acad. Sci. U. S. A. **17**, (1931), no.4, 211–216.
- [27] J. F. Escobar, *On the spectrum of the Laplacian on complete Riemannian manifolds*. Comm. Partial Differ. Equations **11**, (1985), 63–85.
- [28] L. Ferrer, F. Martin, W. Meeks III, *Existence of proper minimal surfaces of arbitrary topological type*. Adv. Math. **231**, no. 1, (2012), 378–413.
- [29] D. Fischer-Colbrie R. Schoen, *The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature*. Comm. Pure Appl. Math. **33** (1980), 199–211.
- [30] I. M. Glazman, *Directed methods of qualitative spectral analysis of singular differential operators*. Daniel Davey, New York, (1965)
- [31] M. Gromov, *Structures mtriques pour les varits riemanniennes*. Textes Mathématiques 1. CEDIC, Paris, 1981. iv+152 pp. ISBN: 2-7124-0714-8. J. Lafontaine and P. Pansu, editors.
- [32] J. Hadamard, *Les surfaces à courbures opposées et leurs lignes géodésiques*. J. Math. Pures Appl. **4** (1898), 27–73. **23** (1970), 97–114.
- [33] M. Harmer, *Discreteness of the spectrum of the Laplaceian and Stochastic incompleteness*, J. Geom. Anal. **19** (2009), 358–372.
- [34] K. Hoffman, *Banach spaces of analytic functions*. Prentice Hall, N.J. 1962
- [35] L. P. Jorge and F. Tomi, *The barrier principle for minimal submanifolds of arbitrary codimension*. Ann. Global Anal. Geom. **24** (2003), no. 3, 261–267.
- [36] L. P. Jorge and F. Xavier, *A complete minimal surface in  $\mathbb{R}^3$  between two parallel planes*. Annals of Math. (2) **112** (1980), 203–206.
- [37] L. Karp, *Noncompact manifolds with purely continuous spectrum*, Mich. Math. J. **31**, (1984), 339–347.
- [38] R. Kleine, *Discreteness conditions for the Laplacian on complete noncompact Riemannian manifolds*, Math. Z. **198** (1988), 127–141.
- [39] R. Kleine, *Warped products with discrete spectra*, Results Math. **15** (1989), 81–103.
- [40] F. Lopez, F. Martin, and S. Morales, *Adding handles to Nadirashvili’s surfaces*. J. Diff. Geom. **60**, no. 1, (2002), 155–175.
- [41] F. Lopez, F. Martin, and S. Morales, *Complete nonorientable minimal surfaces in a ball of  $\mathbb{R}^3$* . Trans. Amer. Math. Soc. **358**, no. 9, (2006), 3807–3820.
- [42] F. Martín, W. Meeks III, N. Nadirashvili, *Bounded domains which are universal for minimal surfaces*. Amer. J. Math. **129** (2007), no. 2, 455–461.
- [43] F. Martín, S. Morales, *Complete proper minimal surfaces in convex bodies of  $\mathbb{R}^3$* . Duke Math. J. **128** (2005), 559–593.
- [44] F. Martín, S. Morales, *Complete proper minimal surfaces in convex bodies of  $\mathbb{R}^3$ . II. The behavior of the limit set*. Comment. Math. Helv. **81** (2006), 699–725.
- [45] F. Martín, S. Morales, *Construction of a complete bounded minimal annulus in  $\mathbb{R}^3$* . Proceedings of the Meeting of Andalusian Mathematicians, Vol. II (Spanish) (Sevilla, 2000), Colecc. Abierta **52** (2001), Univ. Sevilla Secr. Publ., Seville, 649–653.

- [46] F. Martín, N. Nadirashvili, *A Jordan curve spanned by a complete minimal surface*. Arch. Ration. Mech. Anal. **184** (2007), no. 2, 285–301.
- [47] P. Mattila, *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability*. Cambridge Studies in Advanced Mathematics **44**, Cambridge University Press, Cambridge, 1995.
- [48] N. Nadirashvili, *Hadamard's and Calabi-Yau's conjectures on negatively curved and minimal surfaces*. Invent. Math. **126** (1996), 457–465.
- [49] W. F. Osgood, *A Jordan curve of positive area*. Trans. Amer. Math. Soc. **4** (1903), no.1, 107–112.
- [50] A. Persson, *Bounds for the discrete part of the spectrum of a semibounded Schrödinger operator*. Math. Scand. **8** (1960), 143–153.
- [51] S. Pigola, M. Rigoli, A.G. Setti, *Vanishing and finiteness results in Geometric Analysis. A generalization of the Böchner technique*. Progress in Math. **266**, Birkhäuser, (2008).
- [52] A. V. Pogorelov, *On the stability of minimal surfaces*. Dokl. Akad. Nauk SSSR **260** (1981), 293-295.
- [53] T. Radó, *On the Problema of Plateau* Ann. of Math. (2) **31** (1930), no. 3, 457-469.
- [54] M. Reed, B. Simon, *Methods of Modern Mathematical Physics. IV. Analysis of Operators*. Academic Press, New York-London, (1978).
- [55] F. Rellich, *Über das asymptotische Verhalten der Lösungen von  $\Delta u + \lambda u = 0$  in unendlichen Gebieten*. Jahresber. Dtsch. Math.-Ver. **53**, (1943), 57–65.
- [56] Robson M. Mesquita, *O Conjunto Limite da Superfície de P. Andrade*. Master's Thesis, Universidade Federal do Ceará-UFC, (1999).
- [57] A. Ros, *One-sided complete stable minimal surfaces.*, J. Diff. Geom. **74** (2006), 69-92.
- [58] H. Rosenberg, *Intersection of minimal surfaces of bounded curvatures*, Bull. Sci. Math. **125** (2) (2001), 161-168.
- [59] H. Rosenberg, E. Toubiana, *A cylindrical type complete minimal surface in the slab of  $\mathbb{R}^3$* . Bull. Sci. Math. (2) **111**, no. 3, (1987), 241-245.
- [60] R. Schoen, *Estimates for stable minimal surfaces in three dimensional manifolds*. Seminar on minimal submanifolds, 111-126, Ann. of Math. Stud., **103**, Princeton Univ. Press, Princeton, NJ, (1983).
- [61] T. Tayoshi, *On the spectrum of the Laplace-Beltrami operator on noncompact surface*, Proc. Japan Acad. **47**, (1971), 579–585.
- [62] M. Tokuomaru, *Complete minimal cylinders properly immersed in the unit ball*. Kyushu J. math. **61** (2007), no. 2, 373–394.
- [63] S. T. Yau, *Review of Geometry and Analysis*. Kodaira's issue. Asian J. Math. **4** (2000), 235–278.
- [64] S. T. Yau, *Review of Geometry and Analysis*. Mathematics: frontier and perspectives. Amer. Math. Soc. Providence. RI. (2000), 353–401.

G. PACELLI BESSA: DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DO CEARÁ-UFC CAMPUS DO PICI, 60440-900 FORTALEZA-CE, (BRAZIL)

*E-mail address:* `bessa@mat.ufc.br`

LUQUÉSIO P. JORGE: DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DO CEARÁ-UFC CAMPUS DO PICI, 60440-900 FORTALEZA-CE, (BRAZIL)

*E-mail address:* `ljorge@mat.ufc.br`

LUCIANO MARI: DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DO CEARÁ-UFC CAMPUS DO PICI, 60440-900 FORTALEZA-CE, (BRAZIL)

*E-mail address:* `lucio.mari@libero.it`