Supplementary Material of "Stochastic Approximations to the Pitman–Yor Process"

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Appendix A: Random generation of $T_{\alpha,\theta}$

Let Y be a standard Exp(1) random variable. Note that

$$\begin{aligned}\mathrm{P}((Y/T_\alpha)^\alpha > x) &= \int_0^\infty \mathrm{P}(Y > x^{1/\alpha} t) f_\alpha(t)\mathrm{d}t = \int_0^\infty \exp[-x^{1/\alpha} t] f_\alpha(t)\mathrm{d}t \\ &= \mathrm{E}\big[\mathrm{e}^{-x^{1/\alpha} T_\alpha}\big] = \mathrm{e}^{-x} = \mathrm{P}(Y > x)\end{aligned}$$

so we have $Y =_d (Y/T_\alpha)^\alpha$. For $r < \alpha$, $\mathrm{E}(Y^{-r/\alpha}) < \infty$, so we find that $\mathrm{E}(Y^{-r/\alpha}) = \mathrm{E}(T_\alpha^r)\mathrm{E}(Y^{-r})$ and

$$\mathrm{E}(T_\alpha^r) = \frac{\mathrm{E}(Y^{-r/\alpha})}{\mathrm{E}(Y^{-r})} = \frac{\Gamma(1 - r/\alpha)}{\Gamma(1 - r)}. \quad (1)$$

The normalizing constant in $f_{\alpha,\theta}(t)$ is $\int_0^\infty t^{-\theta} f_\alpha(t)\mathrm{d}t = \mathrm{E}(T_\alpha^{-\theta})$, so set $r = -\theta$ and note that $-\theta < \alpha$. Let G_a be a gamma random variable with shape $a > 0$ and unit rate. Simple moment comparisons using (1) yield the distributional equality $G_{1+\theta/\alpha} \stackrel{d}{=} (G_{1+\theta}/T_{\alpha,\theta})^\alpha$, which, however, does not provide a way to generate from $T_{\alpha,\theta}$. For this we resort to Devroye (2009). First we recall how to generate a Zolotarev random variable $Z_{\alpha,b}$ for $\alpha \in (0, 1)$ and $b = \theta/\alpha > -1$. Let

$$C = \frac{\Gamma(1 + b\alpha)\Gamma(1 + b(1 - \alpha))}{\pi\Gamma(1 + b)}$$

and

$$B(u) = A(u)^{-(1-\alpha)} = \frac{\sin(u)}{\sin(\alpha u)^\alpha \sin((1 - \alpha)u)^{1-\alpha}}.$$

A simple asymptotic argument yields the value $B(0) = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}$. Then $f(x) = C\,B(x)^b, \quad 0 \le x \le \pi$. The following bound holds

$$f(x) \le CB(0)^b \mathrm{e}^{-\frac{x^2}{2\sigma^2}}, \quad \text{with } \sigma^2 = \frac{1}{b\alpha(1 - \alpha)}.$$

This Gaussian upper bound suggests a simple rejection sampler for sampling Zolotarev random variates. Following Devroye (2009), it is most efficient to adapt the sampler to

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the value of σ. If $\sigma \geq \sqrt{2\pi}$, rejection from a uniform random variate is best. Otherwise, use a normal dominating curve as suggested in the bound above. The details are given below.

Algorithm 3 (Sampler of $T_{\alpha,\theta}$)

1. `set` $b = \theta/\alpha$ `and` $\sigma = \sqrt{b\alpha(1-\alpha)}$
2. `if` $\sigma \geq \sqrt{2\pi}$:
   `then repeat:` `generate` $U \sim \mathrm{Unif}(0,\pi)$ `and` $V \sim \mathrm{Unif}(0,1)$.
   `set` $X \leftarrow U$, $W \leftarrow B(X)$.
   `until` $V \leq (W/B(0))^b$
   `else repeat:` `generate` $N \sim \mathrm{N}(0,1)$ `and` $V \sim \mathrm{Unif}(,1)$.
   `set` $X \leftarrow \sigma|N|$, $W \leftarrow B(X)$.
   `until` $X \leq \pi$ `and` $V\mathrm{e}^{-N^2/2} \leq (W/B(0))^b$
3. `generate` $G \stackrel{d}{=} G_{1+b(1-\alpha)/\alpha}$
4. `set` $T \leftarrow 1/(WG^{1-\alpha})^{1/\alpha}$
5. `return` T

References

Devroye, L. (2009). “Random variate generation for exponentially and polynomially tilted stable distributions.” *ACM Transactions on Modeling and Computer Simulation (TOMACS)*, 19(4): Article No. 18. 1