



“Generalized” algebraic Bethe ansatz, Gaudin-type models and Z_p -graded classical r -matrices

T. Skrypnyk ^{a,b,*}

^a *Universita degli Studi di Milano-Bicocca, via Roberto Cozzi, 53, 20125, Milano, Italy*

^b *Bogoliubov Institute for Theoretical Physics, Metrologichna st. 14-b, 03143, Kiev, Ukraine*

Received 29 June 2016; received in revised form 12 September 2016; accepted 12 September 2016

Available online 19 September 2016

Editor: Hubert Saleur

Abstract

We consider quantum integrable systems associated with reductive Lie algebra $gl(n)$ and Cartan-invariant non-skew-symmetric classical r -matrices. We show that under certain restrictions on the form of classical r -matrices “nested” or “hierarchical” Bethe ansatz usually based on a chain of subalgebras $gl(n) \supset gl(n-1) \supset \dots \supset gl(1)$ is generalized onto the other chains or “hierarchies” of subalgebras. We show that among the r -matrices satisfying such the restrictions there are “twisted” or Z_p -graded non-skew-symmetric classical r -matrices. We consider in detail example of the generalized Gaudin models with and without external magnetic field associated with Z_p -graded non-skew-symmetric classical r -matrices and find the spectrum of the corresponding Gaudin-type hamiltonians using nested Bethe ansatz scheme and a chain of subalgebras $gl(n) \supset gl(n-n_1) \supset gl(n-n_1-n_2) \supset gl(n-(n_1+\dots+n_{p-1}))$, where $n_1+n_2+\dots+n_p=n$.

© 2016 The Author. Published by Elsevier B.V. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>). Funded by SCOAP³.

1. Introduction

Quantum integrable models with long-range interaction play important role in the non-perturbative physics. They are applied in the theory of small metallic grains [1–4], where the

* Correspondence to: Universita degli Studi di Milano-Bicocca, via Roberto Cozzi, 53, 20125, Milano, Italy.
E-mail addresses: taras.skrypnyk@unimib.it, tskrypnyk@imath.kiev.ua.

famous Richardson’s model [5] is used, nuclear physics [6,7], theory of colored Fermi gases [8], in the so-called “central spin model” theory [9,10] etc.

The main example of integrable models with long-range interaction are the famous Gaudin spin chains [11] associated with simple (reductive) Lie algebras \mathfrak{g} and skew-symmetric $\mathfrak{g} \otimes \mathfrak{g}$ -valued classical r -matrices with spectral parameters. In the papers [12,13] we have proposed a generalization of classical and quantum Gaudin models with [13] and without [12] external magnetic field associated with arbitrary non-skew-symmetric $\mathfrak{g} \otimes \mathfrak{g}$ -valued non-dynamical classical r -matrices with spectral parameters that satisfy the so-called generalized or “permuted” classical Yang–Baxter equation [29]. Moreover, we have shown that these models are applied in order to construct new integrable fermion models of reduced BCS-type [14–16] and integrable “ s ”-type and “ $p + ip$ ”-type proton–neutron models of nuclear physics [17,18,20]. This makes a study of the generalized Gaudin models associated with non-skew-symmetric classical r -matrices important both from the mathematical and physical points of view.

The present paper is devoted to study of quantum integrable models based on classical non-skew-symmetric r -matrices and reductive Lie algebras $gl(n)$. The important problem of the theory of integrable systems is not only the construction of quantum integrable models themselves (i.e. not only construction of the mutually commuting hamiltonian and integrals of motion) but also the development of the methods of their exact solution. For the case of quantum integrable models based on higher rank Lie algebras the main method of the exact solution is the so-called “nested” or “hierarchical” algebraic Bethe ansatz. It was invented in [21] for the quantum group case and repeated in [22] for the Lie-algebraic case in the particular case of skew-symmetric rational and trigonometric r -matrices.

In our previous paper [23] we have considered quantum integrable systems associated with the Lie algebra $gl(n)$ and Cartan-invariant non-skew-symmetric classical r -matrices for which exists the standard procedure of the nested Bethe ansatz associated with the chain of embeddings $gl(n) \supset gl(n-1) \supset gl(n-2) \supset \dots \supset gl(1)$. It is necessary to observe that, contrary to Cartan-invariant skew-symmetric classical r -matrices, that are exhausted by *two* r -matrices, namely, rational and trigonometric ones, there are *many* Cartan-invariant non-skew-symmetric classical r -matrices. Moreover, contrary to Cartan-invariant skew-symmetric classical r -matrices [22], majority of Cartan-invariant non-skew-symmetric r -matrices do not support standard nested Bethe ansatz scheme associated with the chain of embeddings $gl(n) \supset gl(n-1) \supset gl(n-2) \supset \dots \supset gl(1)$. In fact, only three classes of non-skew-symmetric Cartan-invariant classical r -matrices support the standard nested Bethe ansatz scheme [23]. That is why, in order to make it compatible with the structure of other r -matrices, it is necessary to generalize the scheme of the nested Bethe ansatz onto other chains of embedded subalgebras.

In the present paper we generalize the “nesting” procedure of the algebraic Bethe ansatz onto the wide class of the chains or “hierarchies” of the embedded subalgebras of $gl(n)$. The standard idea of the nested Bethe ansatz in the $gl(n)$ case is to reduce the problem of the diagonalization of the generating function of the commutative integrals defined for the $gl(n)$ -valued Lax matrix to the same problem for the $gl(n-1)$ -valued Lax matrix and then apply this method recursively [21,22]. In the present paper we propose to reduce the problem of the diagonalization of the generating function of the commutative integrals of second order for $gl(n)$ -valued Lax matrix to the same problem for $gl(n-n_1) \oplus gl(n_1)$ -valued Lax matrix, where $n_1 \in \overline{1, n-1}$. It occurred to be possible under certain requirements imposed onto the considered classical r -matrix. In the cases $n_1 = 1$ or $n_1 = n-1$ we recover our previous results [23]. Other cases are new. In such a way, proceeding recursively, i.e. applying the similar steps to the Lie algebras $gl(n-n_1)$ and $gl(n_1)$ we obtain a lot of possible “nestings” or “hierarchies of embeddings” for

the nested Bethe ansatz. We realize this scheme to the final end using the chain of subalgebras $gl(n) \supset gl(n - n_1) \supset gl(n - n_1 - n_2) \supset gl(n - (n_1 + \dots + n_{p-1}))$, where $n_1 + n_2 + \dots + n_p = n$, complemented by the sub-chains $gl(n_k) \supset gl(n_k - 1) \supset gl(n_k - 2) \supset \dots \supset gl(1)$, $k \in \overline{1, p}$. We obtain the formula for the spectrum of the quadratic generating function of the quantum integrals for the integrable models, whose representation space possess the highest weight vector and whose r -matrix supports the above chain of the embedded subalgebras. The spectrum is given in terms of “rapidities” — solutions of the corresponding Bethe-type equations. In particular, we obtain the answer for the spectrum of the Gaudin-type hamiltonians (their representation spaces do possess the highest weight vectors) and the corresponding Bethe-type equations.

All the obtained answers are quite general and are given in terms of the components of the r -matrices that satisfy the above mentioned restrictions. Nevertheless, we also consider a concrete example of the r -matrix and Gaudin-type models with and without external magnetic field that are solved by our method. The r -matrices we concentrate on are the so-called “twisted” or Z_p -graded classical r -matrices [32]. The corresponding generalized Gaudin hamiltonians have been studied in detail in [24]. The considered Z_p -gradings are labeled by the decomposition of n into the sum of p integers: $n = n_1 + n_2 + \dots + n_p$. The corresponding Z_p -graded $gl(n) \otimes gl(n)$ -valued classical r -matrices after the appropriate equivalence transformation satisfy the conditions sufficient for the applicability of the nested Bethe ansatz based on the chain $gl(n) \supset gl(n - n_1) \supset gl(n - n_1 - n_2) \supset gl(n - (n_1 + \dots + n_{p-1}))$ and complemented by sub-chains $gl(n_k) \supset gl(n_k - 1) \supset gl(n_k - 2) \supset \dots \supset gl(1)$, $k \in \overline{1, p}$. Using this fact along with the developed in this paper general theory we find the spectrum and Bethe equations for the corresponding generalized Gaudin hamiltonians with and without external magnetic field. The case $p = 2$ is considered in some detail.

Observe, that example of Z_p -graded r -matrices is not the only example of the r -matrices that fit in the proposed scheme — some more general r -matrices from [24] also fit into it, but Z_p -graded example is the simplest one for which the nesting scheme based on the chain $gl(n) \supset gl(n - n_1) \supset gl(n - n_1 - n_2) \supset gl(n - (n_1 + \dots + n_{p-1}))$ is applicable.

At the end of the Introduction let us emphasize, that the main motivation for the proposed generalization of the nested Bethe ansatz scheme consists in the fact that the standard nested Bethe ansatz based on the chain $gl(n) \supset gl(n - 1) \supset \dots \supset gl(1)$ does not work for many physically interesting models, i.e. is not compatible with the structure of the corresponding r -matrices. Among such the models are Gaudin-type models in magnetic field associated with $gl(n)$ and Z_2 -graded classical r -matrices, where Z_2 -gradings are defined by decompositions $n = n_1 + n_2$, such that $n_1 > 1$ and $n_2 > 1$. In particular, in the case when $n = 4$, $n_1 = n_2 = 2$ such Gaudin-type model produce integrable “ $p + ip$ ” proton–neutron model [18], whose exact solution is not provided by standard nested Bethe ansatz scheme based on the chain $gl(4) \supset gl(3) \supset gl(2) \supset gl(1)$.

We would like also to outline that general non-skew-symmetric r -matrices do not satisfy usual classical Yang–Baxter equation, lie out of the Belavin–Driinfeld classification [28] and are not, generally speaking, connected with the quantum groups or related structures. That is why it is methodologically important to develop a theory of the corresponding quantum integrable systems independently of much better elaborated quantum-group formalism [33,34].

The structure of the present paper is the following: in the second section we remind the construction of the quantum integrable systems based on the classical r -matrices. In the third (main) section we develop a theory of the nested Bethe ansatz based on non-standard chains of subalgebras $gl(n) \supset gl(n - n_1) \supset gl(n - n_1 - n_2) \supset gl(n - (n_1 + \dots + n_{p-1}))$. In the fourth section we consider example of Z_p -graded classical r -matrix and the corresponding Gaudin models with and without external magnetic field.

2. Quantum integrable systems and r -matrices

2.1. Definitions and notations

Let $\mathfrak{g} = gl(n)$ be the Lie algebra of the general linear group over the field of complex numbers. Let X_{ij} , $i, j = \overline{1, n}$ be a standard basis in $gl(n)$ with the commutation relations:

$$[X_{ij}, X_{kl}] = \delta_{kj}X_{il} - \delta_{il}X_{kj}. \tag{1}$$

Definition 1. A function of two complex variables $r(u_1, u_2) = \sum_{i,j,k,l=1}^n r_{ij,kl}(u_1, u_2)X_{ij} \otimes X_{kl}$ with values in the tensor square of the algebra $\mathfrak{g} = gl(n)$ is called a classical r -matrix if it satisfies the following generalized classical Yang–Baxter equation [29–32]:

$$[r^{12}(u_1, u_2), r^{13}(u_1, u_3)] = [r^{23}(u_2, u_3), r^{12}(u_1, u_2)] - [r^{32}(u_3, u_2), r^{13}(u_1, u_3)], \tag{2}$$

where $r^{12}(u_1, u_2) \equiv \sum_{i,j,k,l=1}^n r_{ij,kl}(u_1, u_2)X_{ij} \otimes X_{kl} \otimes 1$, $r^{13}(u_1, u_3) \equiv \sum_{i,j,k,l=1}^n r_{ij,kl}(u_1, u_3)X_{ij} \otimes 1 \otimes X_{kl}$, etc. and $r_{ij,kl}(u, v)$ are matrix elements of the r -matrix $r(u, v)$.

It is easy to show that there are three classes of equivalences in the space of solutions of the equation (2). They are:

1. “gauge transformations”: $r(u_1, u_2) \rightarrow Ad_{g(u_1)} \otimes Ad_{g(u_2)}r(u_1, u_2)$.
2. “re-parametrization”: $r(u_1, u_2) \rightarrow r(v_1, v_2)$, where $u_i = u_i(v_i)$, $i = 1, 2$.
3. “rescaling”: $r(u_1, u_2) \rightarrow f(u_2)r(u_1, u_2)$, where function $f(u_2)$ is arbitrary.

Remark 1. In the case of skew-symmetric r -matrices, i.e. when $r^{12}(u_1, u_2) = -r^{21}(u_2, u_1)$, where $r^{21}(u_2, u_1) = P^{12}r^{12}(u_1, u_2)P^{12}$ and P^{12} interchanges the first and second spaces in tensor product, the generalized classical Yang–Baxter equation reduces to the usual classical Yang–Baxter equation [35,28,36]:

$$[r^{12}(u_1, u_2), r^{13}(u_1, u_3)] = [r^{23}(u_2, u_3), r^{12}(u_1, u_2) + r^{13}(u_1, u_3)]. \tag{3}$$

Observe, that gauge transformations and re-parametrizations are equivalences in the spaces of solutions of the equation (3), while “rescaling” is not, because it does not preserve skew-symmetry property of the r -matrix.

In the present paper we are interested only in the r -matrices that by re-parametrization and rescaling may be brought to the form possessing the following decomposition:

$$r(u_1, u_2) = \frac{\Omega}{u_1 - u_2} + r^0(u_1, u_2), \tag{4}$$

where $\Omega = \sum_{i,j=1}^n X_{ij} \otimes X_{ji}$ is the tensor Casimir and $r^0(u_1, u_2)$ is a regular on the “diagonal” $u = v$ function with values in $gl(n) \otimes gl(n)$.

For the subsequent we will also need the following three definitions:

Definition 2. We will call classical r -matrix to be $\mathfrak{g}_0 \subset \mathfrak{gl}(n)$ -invariant if

$$[r(u_1, u_2), X \otimes 1 + 1 \otimes X] = 0, \forall X \in \mathfrak{g}_0. \tag{5}$$

Observe, that on the level of Lie groups this definition means exactly the G_0 -invariance:

$$(Ad_g \otimes Ad_g) \cdot r(u_1, u_2) = r(u_1, u_2),$$

where $g \in G_0$ and G_0 is a Lie group of the algebra \mathfrak{g}_0 .

Definition 3. We will call classical $\mathfrak{gl}(n)$ -valued r -matrix to be diagonal in the root basis if it has the following form:

$$r(u_1, u_2) \equiv \sum_{i,j=1}^n r_{ji}(u_1, u_2) X_{ij} \otimes X_{ji}. \tag{6}$$

Diagonal in the root basis classical r -matrices are automatically Cartan-invariant. Inverse is not true: not all Cartan-invariant r -matrices are diagonal in the root basis.

Remark 2. Observe that in the case of the diagonal r -matrices the generalized classical Yang–Baxter equation (2), is re-written in the component form as follows:

$$r_{ij}(u_1, u_2)r_{jl}(u_1, u_3) - r_{il}(u_1, u_2)r_{jl}(u_2, u_3) - r_{il}(u_1, u_3)r_{ij}(u_3, u_2) = 0, \tag{7}$$

where $i, j, l \in \overline{1, n}$ and all three indices cannot coincide simultaneously.

Definition 4. A $\mathfrak{gl}(n)$ -valued function $c(u) = \sum_{i,j=1}^n c_{ij}(u) X_{ij}$ of one complex variable is called “generalized shift element” if it satisfies the following equation:

$$[r^{12}(u_1, u_2), c(u_1) \otimes 1] - [r^{21}(u_2, u_1), 1 \otimes c(u_2)] = 0. \tag{8}$$

In the subsequent we will be interested only in the diagonal shift elements, i.e. shift elements of the following form:

$$c(u) = \sum_{i=1}^n c_{ii}(u) X_{ii}.$$

Remark 3. Observe, that for skew-symmetric classical r -matrices any element of its symmetry algebra is a shift element. For non-skew-symmetric r -matrices it is not true.

2.2. Algebra of Lax operators and classical r -matrices

Using the classical r -matrix $r(u_1, u_2)$ it is possible to define the following “tensor” Lie bracket:

$$[\hat{L}(u_1) \otimes 1, 1 \otimes \hat{L}(u_2)] = [r^{12}(u_1, u_2), \hat{L}(u_1) \otimes 1] - [r^{21}(u_2, u_1), 1 \otimes \hat{L}(u_2)], \tag{9}$$

where $\hat{L}(u) = \sum_{i,j=1}^n \hat{L}_{ij}(u) X_{ij}, r^{21}(u_2, u_1) = P^{12}r^{12}(u_1, u_2)P^{12}$.

The tensor bracket (9) between the quantum Lax matrices $\hat{L}(u_1)$ and $\hat{L}(u_2)$ is a symbolical expression of the Poisson brackets between their matrix elements. In the case of the diagonal r -matrices (6) which are the main object of interest in the present article the corresponding commutation relations (9) have the following simple form:

$$[\hat{L}_{ij}(u_1), \hat{L}_{kl}(u_2)] = \delta_{il}(r_{kl}(u_1, u_2)\hat{L}_{kj}(u_1) + r_{ij}(u_2, u_1)\hat{L}_{kj}(u_2)) - \delta_{kj}(r_{kl}(u_1, u_2)\hat{L}_{il}(u_1) + r_{ij}(u_2, u_1)\hat{L}_{il}(u_2)). \tag{10}$$

The explicit form of the operators $\hat{L}_{ij}(u)$ as functions of the auxiliary spectral parameter u is not arbitrary. It agrees with the structure of the r -matrix $r(u_1, u_2)$ and depends on the concrete quantum system under consideration. In the present paper we will consider the simplest but the most important examples of such the systems.

Meantime, let us clarify the role of the shift elements in the algebra of Lax operators. The following proposition holds true:

Proposition 2.1. *Let $\hat{L}(u)$ be the Lax operator satisfying the commutation relations (9) and $c(u)$ be the shift element satisfying equation (8). Then the operator*

$$\hat{L}^c(u) \equiv \hat{L}(u) + c(u)$$

also satisfies the commutation relations (9).

(Proof of this proposition follows from the explicit form of commutation relations (9), definition (8) and the fact that $c^{ij}(u)$ are c -numbers.)

2.3. Quantum integrals

In this subsection we will explain the connection of classical non-skew-symmetric $gl(n) \otimes gl(n)$ -valued r -matrices with quantum integrability. It will be shown that, just like in the case of classical r -matrix Lie–Poisson brackets [29,31], the Lie bracket (9) leads to an algebra of mutually commuting quantum integrals.

Let us consider the following functions in generators of the Lax algebra:

$$\hat{t}_n(u) = \text{tr}_n \hat{L}(u) = \sum_{i=1}^n \hat{L}_{ii}(u) \quad \text{and} \quad \hat{\tau}_n(u) = \frac{1}{2} \text{tr}_n \hat{L}^2(u) = \frac{1}{2} \sum_{i,j=1}^n \hat{L}_{ij}(u) \hat{L}_{ji}(u). \tag{11}$$

The following important theorem holds true [26]:

Theorem 2.1. *Let $\hat{L}(u)$ be the Lax operator satisfying the commutation relations (9) with the diagonal in the root basis classical r -matrix. Assume that in some open region $U \times U \subset \mathbb{C}^2$ the function $r(u, v)$ possesses the decomposition (4). Then the operator-valued functions $\hat{\tau}_n(u)$ and $\hat{t}_n(u)$ are a generators of a commutative algebra, i.e.:*

$$[\hat{t}_n(u), \hat{t}_n(v)] = 0, \quad [\hat{\tau}_n(u), \hat{t}_n(v)] = 0, \quad [\hat{\tau}_n(u), \hat{\tau}_n(v)] = 0.$$

(The proof of the theorem involves the Leibnitz rule and Jacobi identity for the commutator, the $gl(n)$ -invariance of the corresponding quadratic form and some consequences of the generalized classical Yang–Baxter equations.)

Remark 4. It is necessary to emphasize that, generally speaking, operator

$$\hat{t}_n(u) = \text{tr}_n \hat{L}(u) = \sum_{i=1}^n \hat{L}_{ii}(u)$$

does not belong to a center of the algebra of Lax operators. Indeed, even in the simplest case of the diagonal r -matrices from the commutation relations (10) we obtain:

$$[\hat{t}_n(u), \hat{L}_{kl}(v)] = (r_{ll}(v, u) - r_{kk}(v, u)) \hat{L}_{kl}(v). \tag{12}$$

This expression is not zero if $r_{kk}(u, v) \neq r_{ll}(u, v)$.

From Theorem 2.1 follows the next corollary:

Corollary 2.1. *Let the points v_l belong to the open region U . Then all the operators of the form $\hat{H}_{v_l} = -\text{res}_{u=v_l} \hat{t}_n(u)$ and $\hat{C}_{v_m} = -\text{res}_{u=v_m} \hat{t}_n(u)$, $l, m \in \overline{1, N}$ mutually commute.*

Remark 5. Observe that the operators \hat{H}_{v_l} are exact analogs of the generalized Gaudin hamiltonians [12] and coincide with them for the special choice of the Lax operator. From the equality (12) it is easy to deduce that the operators \hat{C}_{v_l} belong to the center of the algebra of Lax operators.

To summarize: in this section we have constructed an algebra of quantum commutative operators that coincide with a quantization of linear and quadratic subalgebra of the algebra of Lie–Poisson commuting integrals of a classical integrable system admitting Lax representation. The problem of quantization of the other “higher” integrals is complicated. It is solved only in the partial case of the classical rational r -matrices (i.e. r -matrices for which $r^0(u, v) \equiv 0$) in [37]. Fortunately, for the physically applications the most important are quadratic integrals. Moreover, as we will show below, their diagonalization does not depend on the higher integrals. That is why we do not consider the problem of the quantization of the higher integrals here.

2.4. Example: generalized Gaudin systems

Let us now consider the most important for the applications (and the simplest at the same time) examples of quantum integrable systems associated with classical r -matrices.

Let $\hat{S}_{ij}^{(m)}$, $i, j = \overline{1, n}$, $m = \overline{1, N}$ be linear operators in some Hilbert space that span Lie algebra isomorphic to $gl(n)^{\oplus N}$ with the commutation relations:

$$[\hat{S}_{ij}^{(m)}, \hat{S}_{kl}^{(p)}] = \delta^{pm} (\delta_{kj} \hat{S}_{il}^{(m)} - \delta_{il} \hat{S}_{kj}^{(m)}).$$

Let us fix N distinct points of the complex plain v_m , $m = 1, 2, \dots, N$. It is possible to introduce the following quantum Lax operator [12]:

$$\hat{L}(u) = \sum_{i,j=1}^n \hat{L}_{ij}(u) X_{ij} \equiv \sum_{m=1}^N \sum_{i,j=1}^n r_{ij,kl}(v_m, u) \hat{S}_{ij}^{(m)} X_{kl}. \tag{13}$$

Using generalized classical Yang–Baxter equation it is possible to show that it satisfies a linear r -matrix algebra (9). This quantum Lax operator is a Lax operator of the generalized $gl(n)$ -valued Gaudin spin chains.

For the applications important are also the shifted Lax operators [13]:

$$\hat{L}^c(u) = \sum_{i,j=1}^n \hat{L}_{ij}^c(u) X_{ij} \equiv \sum_{m=1}^N \sum_{i,j=1}^n r_{ij,kl}(v_m, u) \hat{S}_{ij}^{(m)} X_{kl} + \sum_{i,j=1}^n c_{ij}(u) X_{ij}. \tag{14}$$

This quantum Lax operator is a Lax operator of the generalized $gl(n)$ -valued Gaudin spin chains in an external magnetic field and a generalized shift element $c(u) = \sum_{i,j=1}^n c_{ij}(u) X_{ij}$ plays a role of an external nonhomogeneous magnetic field.

Hereafter we will consider only diagonal r -matrices and diagonal shift elements.

As it was noted above, residues of generating function $\hat{t}_n(u)$ produces only trivial integrals proportional to the Casimir functions $\hat{C}_{v_k} = \sum_{i=1}^n S_{ii}^{(k)}$. Residues of the second order generating function $\hat{\tau}(u)$ produces nontrivial integrals \hat{H}_{v_l} :

$$\hat{H}_{v_l} = \sum_{k=1, k \neq l}^N \sum_{i,j=1}^n r_{ij}(v_k, v_l) \hat{S}_{ji}^{(k)} \hat{S}_{ij}^{(l)} + \frac{1}{2} \sum_{i,j=1}^n (r_{ji}^0(v_l, v_l) + r_{ij}^0(v_l, v_l)) \hat{S}_{ij}^{(l)} \hat{S}_{ji}^{(l)} \tag{15}$$

in the case of the “unshifted” Lax operators and

$$\begin{aligned} \hat{H}_{v_l}^c &= \sum_{k=1, k \neq l}^N \sum_{i,j=1}^n r_{ij}(v_k, v_l) \hat{S}_{ji}^{(k)} \hat{S}_{ij}^{(l)} + \frac{1}{2} \sum_{i,j=1}^n (r_{ji}^0(v_l, v_l) + r_{ij}^0(v_l, v_l)) \hat{S}_{ij}^{(l)} \hat{S}_{ji}^{(l)} \\ &+ \sum_{i=1}^n c_{ii}(v_l) \hat{S}_{ii}^{(l)} \end{aligned} \tag{16}$$

in the case of “shifted” Lax operators.

The hamiltonians (15) are the generalized Gaudin hamiltonians corresponding to the diagonal $gl(n) \otimes gl(n)$ -valued r -matrix and hamiltonians (16) are the generalized Gaudin hamiltonians in external magnetic field corresponding to the same diagonal $gl(n) \otimes gl(n)$ -valued classical r -matrix.

3. Hierarchical Bethe ansatz

The standard idea of the nested or hierarchical algebraic Bethe ansatz [21,22] in the $gl(n)$ case is to reduce the problem of the diagonalization of the generating function of the commutative integrals of $gl(n)$ to the same problem for the Lie algebra $gl(n - 1)$ and then apply this method recursively, using the hierarchy of subalgebras $gl(n) \supset gl(n - 1) \supset \dots \supset gl(1)$.

In this section we will show how to generalize the nested Bethe ansatz onto the wide class of hierarchies of subalgebras starting from the reduction of the problem of diagonalization of the generating function of the commutative integrals on $gl(n)$ to the same problem on $gl(n_1) \oplus gl(n - n_1)$. This procedure allows to solve by this method quantum integrable models associated with the r -matrices not compatible with the usual sequence of embeddings $gl(n) \supset gl(n - 1) \supset \dots \supset gl(1)$.

3.1. Reduction to subalgebra $gl(n_1) \oplus gl(n - n_1)$

Let V be a space of an irreducible representation of the algebra of Lax operators. Let us assume that there exist a vacuum vector $\Omega \in V$ such that:

$$\hat{L}_{ii}(u)\mathbf{\Omega} = \Lambda_{ii}(u)\mathbf{\Omega}, \hat{L}_{kl}(u)\mathbf{\Omega} = 0, i, k, l \in \overline{1, n}, k > l \tag{17}$$

and the whole space V is generated by the action of $\hat{L}_{kl}(u), k < l$ on the vector $\mathbf{\Omega}$.

In order to diagonalize the generating function $\hat{\tau}(u)$ and construct the corresponding set of eigen-vectors we will reduce this problem to the same problem on the subalgebra $gl(n_1) \oplus gl(n - n_1)$. We will do this in several steps.

3.1.1. Reduction of the problem to subalgebra on “vacuum” subspace

Let us at first consider a subspace $V_0 \subset V$ consisting of the vectors \mathbf{v} such that:

$$\hat{L}_{lk}(u)\mathbf{v} = 0, k \leq n_1, l > n_1,$$

where $n_1 \in \overline{1, n}$ is some fixed integer.

Using the commutation relations in the algebra of the Lax operators it is easy to see that this subspace is invariant with respect to the action of the subalgebra of Lax operators taking values in subalgebra $gl(n_1) \oplus gl(n - n_1)$.

Let us return to the generating function $\hat{\tau}_n(u)$. We have:

$$\begin{aligned} \hat{\tau}_n(u) &= \frac{1}{2} \sum_{i,j=1}^n \hat{L}_{ij}(u)\hat{L}_{ji}(u) \\ &= \frac{1}{2} \sum_{i,j=1}^{n_1} \hat{L}_{ij}(u)\hat{L}_{ji}(u) + \frac{1}{2} \sum_{i=1}^{n_1} \sum_{j=n_1+1}^n (\hat{L}_{ij}(u)\hat{L}_{ji}(u) + \hat{L}_{ji}(u)\hat{L}_{ij}(u)) \\ &\quad + \frac{1}{2} \sum_{i,j=n_1+1}^n \hat{L}_{ij}(u)\hat{L}_{ji}(u). \end{aligned}$$

Let us take into account that $\hat{\tau}_{n_1}(u) = \frac{1}{2} \sum_{i,j=1}^{n_1} \hat{L}_{ij}(u)\hat{L}_{ji}(u), \hat{\tau}_{n-n_1}(u) = \frac{1}{2} \sum_{i,j=n_1+1}^n \hat{L}_{ij}(u)\hat{L}_{ji}(u)$ are the generating function of commutative integrals of a $gl(n_1)$ -valued and $gl(n - n_1)$ -valued subalgebras of the algebra of Lax operators and:

$$[\hat{L}_{ij}(u)\hat{L}_{ji}(u)] = (\partial_u + (r_{ji}^0(u, u) + r_{ij}^0(u, u)))(L_{jj}(u) - L_{ii}(u)).$$

(This equality is obtained by taking the limit $u_2 \rightarrow u_1 = u$ in the corresponding commutation relations of the Lax algebra.)

Let us assume that the following conditions on the r -matrix are satisfied:

$$r_{ji}(u, v) = r_{-,n_1}(u, v), r_{ij}(u, v) = r_{+,n_1}(u, v), \forall i \in \overline{1, n_1}, j \in \overline{n_1 + 1, n}. \tag{18}$$

In this case we will have:

$$\begin{aligned} &\sum_{i=1}^{n_1} \sum_{j=n_1+1}^n [\hat{L}_{ji}(u)\hat{L}_{ij}(u)] \\ &= (\partial_u + (r_{-,n_1}^0(u, u) + r_{+,n_1}^0(u, u))((n - n_1)\hat{t}_{n_1}(u) - n_1\hat{t}_{n-n_1}(u)), \end{aligned}$$

where $\hat{t}_{n_1}(u) \equiv \sum_{i=1}^{n_1} \hat{L}_{ii}(u), \hat{t}_{n-n_1}(u) \equiv \sum_{i=n_1+1}^n \hat{L}_{ii}(u)$.

In the result we obtain:

$$\hat{\tau}_n(u)\mathbf{v} = \left(\hat{\tau}_{n_1}(u) + \hat{\tau}_{n-n_1}(u) + \frac{1}{2}(\partial_u + (r_{-,n_1}^0(u, u) + r_{+,n_1}^0(u, u))) \right. \\ \left. \times ((n - n_1)\hat{t}_{n_1}(u) - n_1\hat{t}_{n-n_1}(u)) \right)\mathbf{v}, \tag{19}$$

where $\mathbf{v} \in V_0$. That is the problem of diagonalization of $\hat{\tau}_n(u)$ on this subspace is reduced to the problem of the simultaneous diagonalization of the linear and quadratic generating functions of the subalgebras $gl(n_1)$ and $gl(n - n_1)$.

So we have performed the needed reduction on the space V_0 . Using this result, in the next sections we will perform it on the whole space V .

3.1.2. Preparation for the reduction of the problem to the subalgebra

In order to construct the needed Bethe vectors and perform the required reduction to subalgebra we need to introduce some notations and perform many auxiliary calculations.

By the direct calculations, using the commutation relations in the Lax algebra (9) and definition of the function $\hat{\tau}_n(u)$ one can prove the following formula:

$$[2\hat{\tau}_n(u), \hat{L}_{kl}(v)] = \sum_{j=1}^n \left(r_{lj}(v, u)(\hat{L}_{kj}(v)\hat{L}_{jl}(u) + \hat{L}_{jl}(u)\hat{L}_{kj}(v)) \right. \\ \left. - r_{jk}(v, u)(\hat{L}_{jl}(v)\hat{L}_{kj}(u) + \hat{L}_{kj}(u)\hat{L}_{jl}(v)) \right), \tag{20}$$

which will be used by us in the subsequent calculations.

Let us introduce the following operator-valued matrix:

$$\hat{T}_n(u) = \hat{\tau}_n(u)1_n.$$

We will also use the following operator-valued matrices (parts of the Lax matrix):

$$\hat{A}(u) = \sum_{i,j=1}^{n_1} \hat{L}_{ij}(u)X_{ij}^{(0)}, \quad \hat{B}(u) = \sum_{i=1}^{n_1} \sum_{j=n_1+1}^n \hat{L}_{ij}(u)X_{ij}^{(0)}, \quad \hat{D}(u) = \sum_{i,j=n_1+1}^n \hat{L}_{ij}(u)X_{ij}^{(0)},$$

where $X_{ij}^{(0)}$ is a standard basis element of $gl(n)$ acting in some auxiliary linear space \mathbb{C}^n carrying additional index 0. Hereafter it will be convenient for us to label auxiliary linear subspace by the indices 0 and 0' (instead of the indices 1 and 2 that were used for this purpose before).

Observe that the matrix $\hat{A}(u)$ belonging to $gl(n_1)$ may act (as a matrix) on $\hat{B}(u)$ only from the left, while the matrix $\hat{D}(u)$ belonging to $gl(n - n_1)$ may act (as a matrix) on the matrix $\hat{B}(u)$ only from the right. On the other hand in order to find the spectrum we will need to write the components of $\hat{A}(u)$ to the right of operators $\hat{B}(u)$. That is why we need to introduce the following symbolical notation for the multiplication of two operator-valued matrices $\hat{Y}(u) =$

$$\sum_{i,j=1}^n \hat{y}_{ij}(u)X_{ij}^{(0)} \text{ and } \hat{X}(u) = \sum_{k,l=1}^n \hat{x}_{kl}(u)X_{kl}^{(0)}:$$

$$\hat{Y}(u)\hat{X}^L(u) = \sum_{i,j,k,l=1}^n \hat{y}_{ij}(u)\hat{x}_{kl}(u)X_{kl}^{(0)}X_{ij}^{(0)} = \sum_{i,j,k=1}^n \hat{y}_{ij}(u)\hat{x}_{ki}(u)X_{kj}^{(0)}$$

along with the standard matrix multiplication:

$$\hat{Y}(u)\hat{X}^R(u) = \sum_{i,j,k,l=1}^n \hat{y}_{ij}(u)\hat{x}_{kl}(u)X_{ij}^{(0)}X_{kl}^{(0)} = \sum_{i,j,l=1}^n \hat{y}_{ij}(u)\hat{x}_{jl}(u)X_{il}^{(0)}.$$

Now we can re-write relation (20) in the matrix form. The following proposition holds:

Proposition 3.1. *Let the r -matrix $r(u, v)$ satisfy the conditions (18) then the commutation relations (20) are written as follows:*

$$\begin{aligned}
 [\hat{T}_n(u), \hat{B}(v)] = & r_{-,n_1}(v, u) (\hat{B}(u) \hat{A}^L(v) - \hat{B}(u) \hat{D}^R(v) + \hat{B}(u) \rho^{L,R}(v)) + \\
 & + (\hat{B}(v) \langle r_{00'}(v, u), \hat{D}_{0'}(u) \rangle_{0'}^R - \hat{B}(v) \langle r_{00'}(v, u), \hat{A}_{0'}(u) \rangle_{0'}^L + \\
 & + \hat{B}(v) \rho^{L,R}(v, u)),
 \end{aligned} \tag{21}$$

where $\langle , \rangle_{0'}$ is a pairing in a second tensor space labeled by the index $0'$ and:

$$\begin{aligned}
 2\rho^{L,R}(v, u) = & \sum_{i,j=n_1+1}^n r_{ji}(v, u) r_{ij}(v, u) X_{jj}^{(0)R} + \sum_{i,j=1}^{n_1} r_{ji}(v, u) r_{ij}(v, u) X_{jj}^{(0)L} \\
 & - nr_{-,n_1}(v, u) r_{+,n_1}(v, u) Id, \\
 2\rho^{L,R}(v) = & n(r_{-,n_1}^0(v, v) + r_{+,n_1}^0(v, v)) Id - \sum_{i,j=n_1+1}^n (r_{ji}^0(v, v) + r_{ij}^0(v, v)) X_{jj}^{(0)R} \\
 & - \sum_{i,j=1}^{n_1} (r_{ji}^0(v, u) + r_{ij}^0(v, u)) X_{jj}^{(0)L}
 \end{aligned}$$

Proof. In order to prove this proposition it is necessary to re-write the formula (20) in the following form:

$$\begin{aligned}
 [2\hat{\tau}_n(u), \hat{L}_{kl}(v)] = & 2 \sum_{j=1}^{n_1} (r_{lj}(v, u) \hat{L}_{jl}(u) \hat{L}_{kj}(v) - r_{jk}(v, u) \hat{L}_{jl}(v) \hat{L}_{kj}(u)) + \\
 & + 2 \sum_{j=n_1+1}^n (r_{lj}(v, u) \hat{L}_{kj}(v) \hat{L}_{jl}(u) - r_{jk}(v, u) \hat{L}_{kj}(u) \hat{L}_{jl}(v)) + \\
 & + \sum_{j=1}^{n_1} r_{lj}(v, u) [\hat{L}_{kj}(v), \hat{L}_{jl}(u)] + \sum_{j=n_1+1}^n r_{lj}(v, u) [\hat{L}_{jl}(u), \hat{L}_{kj}(v)] - \\
 & - \sum_{j=1}^{n_1} r_{jk}(v, u) [\hat{L}_{kj}(u), \hat{L}_{jl}(v)] - \sum_{j=n_1+1}^n r_{jk}(v, u) [\hat{L}_{jl}(v), \hat{L}_{kj}(u)].
 \end{aligned} \tag{22}$$

Now, using the very definitions of the matrices $\hat{A}(u)$ and $\hat{D}(u)$, multiplying the both sides of (22) by $X_{kl}^{(0)}$, summing the both sides over $k \in \overline{1, n_1}$, $l \in \overline{n_1 + 1, n}$ and taking into account that by our assumption on the form of the r -matrix: $r_{ji}(u, v) = r_{-,n_1}(u, v)$, $\forall i \in \overline{1, n_1}$, $j \in \overline{n_1 + 1, n}$ we obtain from (22) that:

$$\begin{aligned}
 [\hat{T}_n(u), \hat{B}(v)] = & r_{-,n_1}(v, u) (\hat{B}(u) \hat{A}^L(v) - \hat{B}(u) \hat{D}^R(v)) + \hat{B}(v) \langle r_{00'}(v, u), \hat{D}_{0'}(u) \rangle_{0'}^R - \\
 & - \hat{B}(v) \langle r_{00'}(v, u), \hat{A}_{0'}(u) \rangle_{0'}^L + \frac{1}{2} [[\hat{T}_n(u), \hat{B}(v)]],
 \end{aligned} \tag{23}$$

where we have introduced the following notation:

$$\begin{aligned}
 & [[\hat{T}_n(u), \hat{B}(v)]] \\
 & \equiv \sum_{k=1}^{n_1} \sum_{l=n_1+1}^n \left(\sum_{j=1}^{n_1} r_{lj}(v, u) [\hat{L}_{kj}(v), \hat{L}_{jl}(u)] X_{kl} + \sum_{j=n_1+1}^n r_{lj}(v, u) [\hat{L}_{jl}(u), \hat{L}_{kj}(v)] X_{kl} \right. \\
 & \quad \left. - \sum_{j=1}^{n_1} r_{jk}(v, u) [\hat{L}_{kj}(u), \hat{L}_{jl}(v)] X_{kl} - \sum_{j=n_1+1}^n r_{jk}(v, u) [\hat{L}_{jl}(v), \hat{L}_{kj}(u)] X_{kl} \right).
 \end{aligned}$$

Let us now calculate this expression. Using the commutation relation of the Lax algebra (10) and re-grouping the summands we obtain:

$$\begin{aligned}
 & [[\hat{T}_n(u), \hat{B}(v)]] = \\
 & = \sum_{k=1}^{n_1} \sum_{l=n_1+1}^n \left(\sum_{j=1}^{n_1} (r_{jk}(v, u)r_{kj}(v, u) - r_{lj}(v, u)r_{jl}(v, u)) \right. \\
 & \quad \left. - \sum_{j=n_1+1}^n (r_{jk}(v, u)r_{kj}(v, u) - r_{lj}(v, u)r_{jl}(v, u)) \right) L_{kl}(v) X_{kl} \\
 & + \sum_{k=1}^{n_1} \sum_{l=n_1+1}^n \left(\sum_{j=1}^{n_1} (r_{jl}(u, v)r_{jk}(v, u) - r_{lj}(v, u)r_{kj}(u, v)) \right. \\
 & \quad \left. - \sum_{j=n_1+1}^n (r_{jk}(v, u)r_{jl}(u, v) - r_{lj}(v, u)r_{kj}(u, v)) \right) L_{kl}(u) X_{kl}. \tag{24}
 \end{aligned}$$

Now, using the fact that by our assumption on the form of the r -matrix: $r_{ji}(u, v) = r_{-,n_1}(u, v)$, $r_{ij}(u, v) = r_{+,n_1}(u, v) \forall i \in \overline{1, n_1}, j \in \overline{n_1 + 1, n}$ we obtain that the first two sums in the right-hand-side of (24) is equal to the expression $2\hat{B}(v)\rho^{LR}(v, u)$. That is why it is left to transform the last two sums of the expression (24).

For this purpose it is necessary to use the following identities:

$$\begin{aligned}
 r_{jk}(v, u)r_{jl}(u, v) & = r_{lk}(v, u)(r_{jl}^0(v, v) - r_{jk}^0(v, v)) + \partial_v r_{lk}(v, u), \\
 r_{kj}(u, v)r_{lj}(v, u) & = r_{lk}(v, u)(r_{kj}^0(v, v) - r_{lj}^0(v, v)) + \partial_v r_{lk}(v, u),
 \end{aligned}$$

which are the differential consequences of the generalized classical Yang–Baxter equation written in the component form (7).

Using them we obtain:

$$\begin{aligned}
 & \sum_{j=1}^{n_1} (r_{jl}(u, v)r_{jk}(v, u) - r_{lj}(v, u)r_{kj}(u, v)) \\
 & = \sum_{j=1}^{n_1} r_{lk}(v, u) ((r_{jl}^0(v, v) + r_{lj}^0(v, v)) - (r_{jk}^0(v, v) + r_{kj}^0(v, v))), \\
 & \sum_{j=n_1+1}^n (r_{lj}(v, u)r_{kj}(u, v) - r_{jk}(v, u)r_{jl}(u, v)) \\
 & = \sum_{j=n_1+1}^n r_{lk}(v, u) ((r_{jk}^0(v, v) + r_{kj}^0(v, v)) - (r_{jl}^0(v, v) + r_{lj}^0(v, v))).
 \end{aligned}$$

Using this and the condition $r_{ji}(u, v) = r_{-,n_1}(u, v)$, $r_{ij}(u, v) = r_{+,n_1}(u, v) \forall i \in \overline{1, n_1}, j \in \overline{n_1 + 1, n}$ we finally obtain that the last sums in the left-hand-side of the expression (24) are equal to the expression $r_{-,n_1}(u, v)\hat{B}(u)\rho^{L,R}(v)$.

Proposition is proven. \square

Let us re-write the obtained in the above proposition formula in the different form, which will be important for the subsequent. For this purpose we need to introduce some new operators:

$$\hat{D}^{(1)}(u) = \sum_{i,j=n_1+1}^n \hat{L}_{ij}(u)X_{ij}^{(0')} + \sum_{j=n_1+1}^n r_{ij}(v, u)X_{ji}^{(0)R}X_{ij}^{(0')}$$

$$\hat{A}^{(1)}(u) = \sum_{i,j=1}^{n_1} \hat{L}_{ij}(u)X_{ij}^{(0')} - \sum_{i,j=1}^{n_1} r_{ij}(v, u)X_{ji}^{(0)L}X_{ij}^{(0')}$$

It is easy to see that the operators $\hat{D}^{(1)}(u)$ and $\hat{A}^{(1)}(u)$ are the $gl(n - n_1)$ and $gl(n_1)$ -valued Lax operators, satisfying the Lax algebra (9) were the r -matrix is the corresponding $gl(n - n_1) \otimes gl(n - n_1)$ and $gl(n_1) \otimes gl(n_1)$ -valued sub r -matrices of the initial $gl(n) \otimes gl(n)$ -valued r -matrix $r(u, v)$. Let, furthermore,

$$\hat{\tau}_{n-n_1}^{(1)}(u) = \frac{1}{2}tr_{n-n_1}(\hat{D}^{(1)}(u))^2, \hat{\tau}_{n_1}^{(1)}(u) = \frac{1}{2}tr_{n_1}(\hat{A}^{(1)}(u))^2.$$

The following corollary of Proposition 3.1 holds true:

Corollary 3.1. *The equality (21) are written in the new notations as follows:*

$$\begin{aligned} [\hat{T}_n(u), \hat{B}(v)] &= r_{-,n_1}(v, u)\hat{B}(u)res_{u=v}(\hat{\tau}_{n_1}^{(1)}(u) + \hat{\tau}_{n-n_1}^{(1)}(u) - \frac{n}{2}r_{-,n_1}(v, u)r_{+,n_1}(v, u)Id) \\ &\quad + \hat{B}(v)(\hat{\tau}_{n_1}^{(1)}(u) + \hat{\tau}_{n-n_1}^{(1)}(u) - \hat{\tau}_{n_1}(u) - \hat{\tau}_{n-n_1}(u) \\ &\quad - \frac{n}{2}r_{-,n_1}(v, u)r_{+,n_1}(v, u)Id). \end{aligned} \tag{25}$$

Proof. To prove this corollary it is enough to observe that the expressions for $\rho^{LR}(v, u)$ and $\rho_0^{LR}(v)$ are re-written in the following form:

$$\begin{aligned} 2\rho^{LR}(v, u) &= tr_{0'}(\sum_{j=n_1+1}^n r_{ij}(v, u)X_{ji}^{(0)R}X_{ij}^{(0')})^2 + tr_{0'}(\sum_{i,j=1}^{n_1} r_{ij}(v, u)X_{ji}^{(0)L}X_{ij}^{(0')})^2 \\ &\quad - nr_{-,n_1}(v, u)r_{+,n_1}(v, u)Id, \\ \rho^{LR}(v) &= res_{u=v}\rho^{LR}(v, u). \end{aligned}$$

Then calculating explicitly $\frac{1}{2}tr_{n-n_1}(\hat{D}^{(1)}(u))^2$ and $\frac{1}{2}tr_{n_1}(\hat{A}^{(1)}(u))^2$ and their residues and comparing them with the right-hand-side of (21) one obtains the equality (25). \square

For the subsequent we will need to prove similar proposition and corollary for the tensor products of many $B_i(v_i)$. For this purpose we will need to “lift” all the above matrices with the non-commuting entries to the tensor products of the M copies of $gl(n)$ and we will use the following “tensorial” notations: $\hat{B}_k(v_k) \equiv 1_n \otimes \dots \otimes \hat{B}(v_k) \otimes \dots \otimes 1_n$, $\hat{A}_k(v_k) \equiv 1_n \otimes \dots \otimes \hat{A}(v_k) \otimes \dots \otimes 1_n$, $\hat{D}_k(v_k) \equiv 1_n \otimes \dots \otimes \hat{D}(v_k) \otimes \dots \otimes 1_n$, where $\hat{A}(v_k)$, $\hat{B}(v_k)$, $\hat{D}(v_k)$ stand in

the k -th component of the tensor product and 1_n is the unit matrix in the space \mathbb{C}^n . We will also “lift” the matrix $\hat{\tau}_n(u)$ to this tensor product in the following way:

$$\hat{T}_{n,M}(u) \equiv \hat{\tau}_n(u)1_n \otimes \cdots \otimes 1_n \otimes \cdots \otimes 1_n. \tag{26}$$

The most important technical step we need to accomplish is to calculate the commutator $[\hat{T}_{n,M}(u), \hat{B}_1(v_1)\hat{B}_2(v_2)\cdots\hat{B}_M(v_M)]$. The following important lemma holds:

Lemma 3.1. *The following commutation relations are valid:*

$$\begin{aligned} & [\hat{T}_{n,M}(u), \hat{B}_1(v_1)\dots\hat{B}_M(v_M)] \\ &= \sum_{i=1}^M r_{-,n_1}(v_i, u) \hat{B}_1(v_1)\dots\overset{\check{}}{\hat{B}}_i(v_i)\hat{B}_i(u)\dots\hat{B}_M(v_M) \\ &\quad \times res_{u=v_i}(\hat{\tau}_{n_1}^{(1)}(u) + \hat{\tau}_{n-n_1}^{(1)}(u) - \frac{n}{2} \sum_{j=1}^M r_{-,n_1}(v_j, u)r_{+,n_1}(v_j, u)Id) \\ &\quad + \hat{B}_1(v_1)\dots\hat{B}_M(v_M)(\hat{\tau}_{n_1}^{(1)}(u) + \hat{\tau}_{n-n_1}^{(1)}(u) - \hat{\tau}_{n_1}(u) - \hat{\tau}_{n-n_1}(u) \\ &\quad - \frac{n}{2} \sum_{j=1}^M r_{-,n_1}(v_j, u)r_{+,n_1}(v_j, u)Id), \end{aligned} \tag{27}$$

where “check” over the operator $\hat{B}_i(v_i)$ means that it is absent in the product, $\hat{\tau}_{n_1}^{(1)}(u) = \frac{1}{2}tr_{n_1}(\hat{A}^{(1)}(u))^2$, $\hat{\tau}_{n-n_1}^{(1)}(u) = \frac{1}{2}tr_{n-n_1}(\hat{D}^{(1)}(u))^2$ where $\hat{A}^{(1)}(u)$ and $\hat{D}^{(1)}(u)$ are defined as follows:

$$\begin{aligned} \hat{D}^{(1)}(u) &= \sum_{i,j=n_1+1}^n \hat{L}_{ij}(u)X_{ij}^{(0')} + \sum_{k=1}^M \sum_{j=n_1+1}^n r_{ij}(v_k, u)X_{ji}^{(k)R}X_{ij}^{(0')}, \\ \hat{A}^{(1)}(u) &= \sum_{i,j=1}^{n_1} \hat{L}_{ij}(u)X_{ij}^{(0')} - \sum_{k=1}^M \sum_{i,j=1}^{n_1} r_{ij}(v_k, u)X_{ji}^{(k)L}X_{ij}^{(0')}. \end{aligned}$$

Idea of the proof. The lemma is proven in the direct way, using the Leibnitz rule for the commutator, Proposition 3.1, Corollary 3.1 and the classical Yang–Baxter equation written in the component form. We will not expose the detailed proof here, due to its long and technical character. □

3.1.3. Reduction of the problem to subalgebra on the Bethe vectors

Now we are ready to reduce a problem of the diagonalization of the generating function of quadratic commutative quantum integrals on $gl(n)$ to the same problem on the subalgebra $gl(n_1) \oplus gl(n - n_1)$.

Using the action formula (19) and Lemma 3.1 we obtain the following theorem:

Theorem 3.1. *Let the r -matrix $r(u, v)$ be such that the conditions (18) are satisfied. Let, moreover, the r -matrix satisfy the following conditions:*

$$r_{aa}(u, v) = r_{bb}(u, v), \forall a, b \in \overline{1, n_1} \quad \text{and} \quad r_{kk}(u, v) = r_{ll}(u, v), \forall k, l \in \overline{n_1 + 1, n} \tag{28}$$

or

$$\begin{aligned} \partial_v r_{aa}(u, v) &= \partial_v r_{bb}(u, v), \forall a, b \in \overline{1, n_1}, \quad \partial_v r_{kk}(u, v) = \partial_v r_{ll}(u, v), \forall k, l \in \overline{n_1 + 1, n} \\ \text{and } (r_{-,n_1}^0(u, u) + r_{+,n_1}^0(u, u)) &= 0. \end{aligned} \tag{29}$$

Let $v_{i_1 \dots i_M, j_1 \dots j_M} \in V_0$ and let the subspace V_0 be a vacuum subspace defined in the previous subsections. Then the vectors of the form:

$$\mathbf{v}(v_1, \dots, v_M) = \sum_{i_1, \dots, i_M=1}^{n_1} \sum_{j_1, \dots, j_M=n_1+1}^n \hat{L}_{i_1 j_1}(v_1) \dots \hat{L}_{i_M j_M}(v_M) v_{i_1 \dots i_M, j_1 \dots j_M}$$

are the eigenvectors of the operator $\hat{t}_n(u)$ if the vectors $\mathbf{V} = \sum_{i_1, \dots, i_M=1}^{n_1} v_{i_1, \dots, i_M, j_1, \dots, j_M} e_{i_1}^* \otimes \dots \otimes e_{i_M}^* \otimes e_{j_1} \otimes \dots \otimes e_{j_M}$ are the common eigen-vectors of the operators $\hat{t}_{n_1}^{(1)}(u)$ and $\hat{t}_{n-n_1}^{(1)}(u)$ and the following conditions (Bethe-type equations) are satisfied:

$$\text{res}_{u=v_i}(\hat{t}_{n_1}^{(1)}(u) + \hat{t}_{n-n_1}^{(1)}(u) - \frac{n}{2} \sum_{j=1}^M r_{-,n_1}(v_j, u) r_{+,n_1}(v_j, u) Id) = 0, \forall i \in \overline{1, M}. \tag{30}$$

(Here e_i are basis vectors in the space \mathbb{C}^n — vector-columns with unit on the place i and zeros elsewhere and e_i^* are dual basis vectors in the space \mathbb{C}^n — vector-rows with unit on the place i and zeros elsewhere.)

Then the eigen-values $\Lambda_n(u)$ and $\Lambda_{n_1}^{(1)}(u), \Lambda_{n-n_1}^{(1)}(u)$ of the operators $\hat{t}_n(u)$ and $\hat{t}_{n_1}^{(1)}(u), \hat{t}_{n-n_1}^{(1)}(u)$ are connected as follows:

$$\begin{aligned} \Lambda_n(u) &= \Lambda_{n_1}^{(1)}(u) + \Lambda_{n-n_1}^{(1)}(u) - \frac{n}{2} \sum_{j=1}^M r_{-,n_1}(v_j, u) r_{+,n_1}(v_j, u) + \\ &+ \frac{1}{2} (\partial_u + (r_{-,n_1}^0(u, u) + r_{+,n_1}^0(u, u))) ((n - n_1) c_{n_1}(u) - n_1 c_{n-n_1}(u)), \end{aligned}$$

where $c_{n_1}(u)$ and $c_{n-n_1}(u)$ is a spectrum of the linear Casimir operators $\hat{t}_{n_1}(u)$ and $\hat{t}_{n-n_1}(u)$ of Lax subalgebras with the values in subalgebras $gl(n_1)$ and $gl(n - n_1)$ in the representations of these Lax algebras with the highest weights $(\Lambda_{11}(u), \dots, \Lambda_{n_1 n_1}(u))$ and $(\Lambda_{n_1+1 n_1+1}(u), \dots, \Lambda_{nn}(u))$ correspondingly.

Proof. In order to prove the theorem, let us calculate the action $\hat{T}_{n,M}(u) \cdot \mathbf{V}(v_1, \dots, v_M)$, where the Bethe vector $\mathbf{V}(v_1, \dots, v_M)$ is defined as follows:

$$\mathbf{V}(v_1, \dots, v_M) = \hat{B}_1(v_1) \dots \hat{B}_M(v_M) \mathbf{V}.$$

We will have:

$$\hat{T}_{n,M}(u) \mathbf{V}(v_1, \dots, v_M) = \hat{B}_1(v_1) \dots \hat{B}_M(v_M) \hat{T}_{n,M}(u) \mathbf{V} + [\hat{T}_{n,M}(u), \hat{B}_1(v_1) \dots \hat{B}_M(v_M)] \mathbf{V}. \tag{31}$$

Using the action formula (19) and the Lemma 3.1 we obtain:

$$\begin{aligned}
 & \hat{T}_{n,M}(u)\mathbf{V}(v_1, \dots, v_M) \\
 &= \sum_{i=1}^M r_{-,n_1}(v_i, u) \hat{B}_1(v_1) \dots \hat{B}_i(v_i) \hat{B}_i(u) \dots \hat{B}_M(v_M) \\
 & \quad \times \text{res}_{u=v_i} \left(\hat{\tau}_{n_1}^{(1)}(u) + \hat{\tau}_{n-n_1}^{(1)}(u) - \frac{n}{2} \sum_{j=1}^M r_{-,n_1}(v_j, u) r_{+,n_1}(v_j, u) Id \right) \mathbf{V} \\
 & \quad + \hat{B}_1(v_1) \dots \hat{B}_M(v_M) \left(\hat{\tau}_{n_1}^{(1)}(u) + \hat{\tau}_{n-n_1}^{(1)}(u) - \frac{n}{2} \sum_{j=1}^M r_{-,n_1}(v_j, u) r_{+,n_1}(v_j, u) Id \right) \\
 & \quad + \frac{1}{2} (\partial_u + (r_{-,n_1}^0(u, u) + r_{+,n_1}^0(u, u))) ((n - n_1) \hat{t}_{n_1}(u) - n_1 \hat{t}_{n-n_1}(u)) \mathbf{V}. \tag{32}
 \end{aligned}$$

Now, in order to prove this theorem it is left to identify the action of the operators $\hat{\tau}_n(u)$ with the action of the operator $\hat{T}_{n,M}(u)$ in the corresponding spaces.

Let us consider the spaces $V \otimes (\mathbb{C}^{n^*})^{\otimes M} \otimes (\mathbb{C}^n)^{\otimes M}$. There is a canonical projection of this space onto V : $V \otimes (\mathbb{C}^{n^*})^{\otimes M} \otimes (\mathbb{C}^n)^{\otimes M} \rightarrow V$ made with the help of the pairing $\langle \cdot, \cdot \rangle: (\mathbb{C}^{n^*})^{\otimes M} \otimes (\mathbb{C}^n)^{\otimes M} \rightarrow \mathbb{C}$ which is obtained as a prolongation of the pairing $\mathbb{C}^{n^*} \times \mathbb{C}^n \rightarrow \mathbb{C}$ onto the tensor product of M copies of these spaces. It is easy to show that eigen-vectors of $\hat{T}_{n,M}(u)$ pass into the eigen-vectors of $\hat{\tau}_{n,M}(u)$ under this projection. The Bethe vectors

$$\begin{aligned}
 & \hat{B}_1(v_1) \dots \hat{B}_M(v_M) \mathbf{V} \\
 &= \sum_{k_1, \dots, k_M=1}^{n_1} \sum_{l_1, \dots, l_M=n_1+1}^n \sum_{i_1, \dots, i_M=1}^{n_1} \hat{L}_{k_1 l_1}(v_1) \dots \hat{L}_{k_M l_M}(v_M) v_{i_1, \dots, i_M, j_1, \dots, j_M} e_{i_1}^* \otimes \dots \otimes e_{i_M}^* \\
 & \quad \otimes X_{k_1 l_1} e_{j_1} \otimes \dots \otimes X_{k_M l_M} e_{j_M}
 \end{aligned}$$

under this map pass into the vectors

$$\mathbf{v}(v_1, \dots, v_M) = \sum_{i_1, \dots, i_M=1}^{n_1} \sum_{j_1, \dots, j_M=n_1+1}^n \hat{L}_{i_1 j_1}(v_1) \dots \hat{L}_{i_M j_M}(v_M) v_{i_1, \dots, i_M, j_1, \dots, j_M}.$$

Taking into account all the above and the fact that under the condition (28) the operators $\hat{t}_{n_1}(u)$ and $\hat{t}_{n-n_1}(u)$ are the Casimir operators of Lax subalgebras with the values in subalgebras $gl(n_1)$ and $gl(n - n_1)$ and are proportional to unit operators, or, due to the condition (29) the operators $\partial_u \hat{t}_{n_1}(u)$ and $\partial_u \hat{t}_{n-n_1}(u)$ are the Casimir operators of Lax subalgebras with the values in subalgebras $gl(n_1)$ and $gl(n - n_1)$ and are proportional to unit operators, we obtain the statement of the theorem as a direct consequence of the formula (32).

The theorem is proven. \square

3.2. Diagonalization of $\hat{\tau}(u)$

In order to diagonalize quadratic generating function of the commutative quantum integrals it is necessary to apply in a recursive manner Theorem 3.1. For this purpose it is necessary to fix the chain of embedded subalgebras which is used in the recursion. We will use the following chain of the subalgebras: $gl(n) \supset gl(n - n_1) \supset gl(n - n_1 - n_2) \supset gl(n - (n_1 + \dots + n_{p-1}))$, where $n_1 + n_2 + \dots + n_p = n$, complemented by the following sub-chains: $gl(n_k) \supset gl(n_k - 1) \supset \dots \supset$

$gl(1)$, $k \in \overline{1, p}$. The choice of the chain of the embedded subalgebras for the recursion in the nested Bethe ansatz will impose certain conditions on the corresponding classical r -matrices.

The following theorem holds true:

Theorem 3.2. *Let the r -matrix $r(u, v)$ satisfy the condition (4) and the following two conditions¹:*

$$\begin{aligned}
 r_{ji}(u, v) &= r_{-,n_1+\dots+n_s}(u, v), \quad r_{ij}(u, v) \\
 &= r_{+,n_1+\dots+n_s}(u, v), \quad \forall i \in \overline{1+n_1+\dots+n_{s-1}, n_1+\dots+n_s}, \\
 & \quad j \in \overline{n_1+\dots+n_s+1, n}, \quad s \in \overline{1, p-1}
 \end{aligned}
 \tag{33}$$

$$\begin{aligned}
 r_{ji}(u, v) &= r_{-,i}(u, v), \quad r_{ij}(u, v) \\
 &= r_{+,i}(u, v), \quad \forall i \in \overline{1+n_1+\dots+n_{s-1}, n_1+\dots+n_s-1}, \\
 & \quad \forall j \in \overline{i+1, n_1+\dots+n_s}, \quad s \in \overline{1, p}
 \end{aligned}
 \tag{34}$$

and the following condition²:

$$r_{aa}(u, v) = r_{bb}(u, v), \quad \forall a, b \in \overline{1, n_1} \text{ and } r_{kk}(u, v) = r_{ll}(u, v), \quad \forall k, l \in \overline{n_1+1, n}
 \tag{35}$$

Then the spectrum of the generating function of the second order integrals is given by the following formula:

$$\begin{aligned}
 2\Lambda_n(u) &= \sum_{k=1}^n (\Lambda_{kk}(u) + \sum_{j=1}^{M_{k-1}} r_{kk}(v_j^{(k-1)}, u) - \sum_{j=1}^{M_k} r_{kk}(v_j^{(k)}, u))^2 \\
 &+ \sum_{k=1}^p \sum_{l=1}^{n_k} (\partial_u + r_{-,l+n_1+\dots+n_{k-1}}^0(u, u) + r_{+,l+n_1+\dots+n_{k-1}}^0(u, u)) \\
 &\times \left((n_k - l)\Lambda_{l+n_1+\dots+n_{k-1}, l+n_1+\dots+n_{k-1}}(u) - \sum_{i=l+1}^{n_k} \Lambda_{i+n_1+\dots+n_{k-1}, i+n_1+\dots+n_{k-1}}(u) \right. \\
 &\left. + (n_k - l) \sum_{j=1}^{M_{l+n_1+\dots+n_{k-1}-1}} r_{l+n_1+\dots+n_{k-1}, l+n_1+\dots+n_{k-1}}(v_j^{(l+n_1+\dots+n_{k-1}-1)}, u) \right) \\
 &- \sum_{k=1}^p \sum_{l=1}^{n_k-1} (n_k - l + 1) \sum_{j=1}^{M_{l+n_1+\dots+n_{k-1}}} r_{-,l+n_1+\dots+n_{k-1}}(v_j^{(l+n_1+\dots+n_{k-1})}, u) r_{+,l+n_1+\dots+n_{k-1}}(v_j^{(l+n_1+\dots+n_{k-1})}, u) \\
 &+ \sum_{k=1}^{p-1} (n_k - 1) (\partial_u + r_{-,n_1+\dots+n_k+1}^0(u, u) + r_{+,n_1+\dots+n_k+1}^0(u, u)) \sum_{j=1}^{M_{n_1+\dots+n_k}} r_{n_1+\dots+n_k+1, n_1+\dots+n_k+1}(v_j^{(n_1+\dots+n_k)}, u) \\
 &+ \sum_{k=1}^{p-1} \sum_{l=1}^{n_k-1} (\partial_u + r_{-,n_1+\dots+n_{k-1}+l}^0(u, u) + r_{+,n_1+\dots+n_{k-1}+l}^0(u, u)) \sum_{j=1}^{M_{n_1+\dots+n_k}} r_{n_1+\dots+n_k, n_1+\dots+n_k}(v_j^{(n_1+\dots+n_k)}, u) \\
 &- \sum_{k=1}^{p-1} (n - (n_1 + \dots + n_{k-1})) \sum_{j=1}^{M_{n_1+\dots+n_k}} r_{-,n_1+\dots+n_k}(v_j^{(n_1+\dots+n_k)}, u) r_{+,n_1+\dots+n_k}(v_j^{(n_1+\dots+n_k)}, u) \\
 &+ \sum_{k=2}^p (n - (n_1 + \dots + n_k)) (\partial_u + r_{-,n_1+\dots+n_k}^0(u, u) + r_{+,n_1+\dots+n_k}^0(u, u))
 \end{aligned}$$

¹ Hereafter it is naturally implied that $1 + n_1 + \dots + n_{s-1} \equiv 1$ if $s = 1$.

² Theorem 3.1 and the chosen set of the embedded subalgebras permit one to consider more complicated (branching) set of conditions instead of one condition (35). For simplicity we will restrict ourselves by simple condition (35).

$$\begin{aligned} & \times \sum_{j=1}^{M_{n_1+\dots+n_{k-1}}} r_{n_1+\dots+n_{k-1}+1, n_1+\dots+n_{k-1}+1} (v_j^{(n_1+\dots+n_{k-1})}, u) + \sum_{k=1}^p (\partial_u + r_{-,n_1+\dots+n_{k-1}}^0(u, u) + r_{+,n_1+\dots+n_{k-1}}^0(u, u)) \\ & \times ((n - (n_1 + \dots + n_k) \sum_{l=1}^{n_k} \Lambda_{n_1+\dots+n_k+l, n_1+\dots+n_k+l}(u) - n_k \sum_{l=1}^{n-(n_1+\dots+n_k)} \Lambda_{n_1+\dots+n_k+l, n_1+\dots+n_k+l}(u)), \end{aligned} \tag{36}$$

where $M_0 = M_n = 0$, $M_k, k \in \overline{1, n-1}$ are non-negative integers, $r_{-,n}(u, v) \equiv r_{+,n}(u, v) \equiv 0$ and “rapidities” $v_i^{(k)}, k \in \overline{1, n-1}, i \in \overline{1, M_k}$ satisfy the following Bethe-type equations:

$$\begin{aligned} & \Lambda_{n_1+\dots+n_k, n_1+\dots+n_k}(v_i^{(n_1+\dots+n_k)}) - \Lambda_{n_1+\dots+n_k+1, n_1+\dots+n_k+1}(v_i^{(n_1+\dots+n_k)}) \\ & - (r_{n_1+\dots+n_k, n_1+\dots+n_k}^0(v_i^{(n_1+\dots+n_k)}, v_i^{(n_1+\dots+n_k)}) + r_{n_1+\dots+n_k+1, n_1+\dots+n_k+1}^0(v_i^{(n_1+\dots+n_k)}, v_i^{(n_1+\dots+n_k)})) \\ & - \sum_{j=1; j \neq i}^{M_{n_1+\dots+n_k}} (r_{n_1+\dots+n_k, n_1+\dots+n_k}(v_j^{(n_1+\dots+n_k)}, v_i^{(n_1+\dots+n_k)}) + r_{n_1+\dots+n_k+1, n_1+\dots+n_k+1}(v_j^{(n_1+\dots+n_k)}, v_i^{(n_1+\dots+n_k)})) \\ & + \sum_{j=1}^{M_{n_1+\dots+n_{k-1}}} r_{n_1+\dots+n_k, n_1+\dots+n_k}(v_j^{(n_1+\dots+n_{k-1})}, v_i^{(n_1+\dots+n_k)}) \\ & + \sum_{j=1}^{M_{n_1+\dots+n_{k+1}}} r_{n_1+\dots+n_k+1, n_1+\dots+n_k+1}(v_j^{(n_1+\dots+n_{k+1})}, v_i^{(n_1+\dots+n_k)}) \\ & + \frac{(n - (n_1 + \dots + n_{k-1}))}{2} (r_{-,n_1+\dots+n_k}^0(v_i^{(n_1+\dots+n_k)}, v_i^{(n_1+\dots+n_k)}) + r_{+,n_1+\dots+n_k}^0(v_i^{(n_1+\dots+n_k)}, v_i^{(n_1+\dots+n_k)})) \\ & - \frac{(n - (n_1 + \dots + n_{k+1}))}{2} (r_{-,n_1+\dots+n_{k+1}}^0(v_i^{(n_1+\dots+n_k)}, v_i^{(n_1+\dots+n_k)}) + r_{+,n_1+\dots+n_{k+1}}^0(v_i^{(n_1+\dots+n_k)}, v_i^{(n_1+\dots+n_k)})) \\ & - (n_k - 1)(r_{-,n_1+\dots+n_{k+1}}^0(v_i^{(n_1+\dots+n_k)}, v_i^{(n_1+\dots+n_k)}) + r_{+,n_1+\dots+n_{k+1}}^0(v_i^{(n_1+\dots+n_k)}, v_i^{(n_1+\dots+n_k)})) \\ & - \frac{1}{2} \sum_{l=1}^{n_k-1} (r_{-,n_1+\dots+n_k+l}^0(v_i^{(n_1+\dots+n_k)}, v_i^{(n_1+\dots+n_k)}) + r_{+,n_1+\dots+n_k+l}^0(v_i^{(n_1+\dots+n_k)}, v_i^{(n_1+\dots+n_k)})) = 0, \end{aligned} \tag{37}$$

corresponding to the chain $gl(n) \supset gl(n - n_1) \supset gl(n - n_1 - n_2) \supset gl(n - (n_1 + \dots + n_{p-1}))$, $n_1 + n_2 + \dots + n_p = n$, and the following set of Bethe-type equations:

$$\begin{aligned} & \Lambda_{kk}(v_i^{(k)}) - \Lambda_{k+1k+1}(v_i^{(k)}) - (r_{kk}^0(v_i^{(k)}, v_i^{(k)}) + r_{k+1k+1}^0(v_i^{(k)}, v_i^{(k)})) \\ & + \frac{n_1 + \dots + n_s - k + 1}{2} (r_{-,k}^0(v_i^{(k)}, v_i^{(k)}) + r_{+,k}(v_i^{(k)}, v_i^{(k)})) \\ & - \frac{n_1 + \dots + n_s - k - 1}{2} (r_{-,k+1}^0(v_i^{(k)}, v_i^{(k)}) + r_{+,k+1}^0(v_i^{(k)}, v_i^{(k)})) \\ & = \sum_{j=1; j \neq i}^{M_k} (r_{kk}(v_j^{(k)}, v_i^{(k)}) + r_{k+1k+1}(v_j^{(k)}, v_i^{(k)})) - \sum_{j=1}^{M_{k-1}} r_{kk}(v_j^{(k-1)}, v_i^{(k)}) \\ & - \sum_{j=1}^{M_{k+1}} r_{k+1k+1}(v_j^{(k+1)}, v_i^{(k)}), \end{aligned} \tag{38}$$

$k \in \overline{n_1 + \dots + n_{s-1} + 1, n_1 + \dots + n_s}$ corresponding to the sub-chains

$$gl(n_s) \supset gl(n_s - 1) \supset \dots \supset gl(1), s \in \overline{1, p}.$$

Remark 6. Observe, that we have chosen the numeration of the upper index of the rapidities $v_j^{(k)}$ in the same way as in the upper index of the rapidities in Bethe ansatz corresponding to the simplest possible chain of embedded subalgebras $gl(n) \supset gl(n - 1) \supset gl(n - 2) \supset \dots \supset gl(1)$.

Proof. The proof of the theorem is based on the recursive usage of the formula (30) and the following chain of subalgebras: $gl(n) \supset gl(n - n_1) \supset gl(n - n_1 - n_2) \supset gl(n - (n_1 + \dots + n_{p-1}))$, where $n_1 + n_2 + \dots + n_p = n$, complemented by the sub-chains: $gl(n_k) \supset gl(n_k - 1) \supset \dots \supset gl(1)$, $k \in \overline{1, p}$.

Let us at first apply the theorem (30) $(p - 1)$ -times using the chain $gl(n) \supset gl(n - n_1) \supset gl(n - n_1 - n_2) \supset gl(n - (n_1 + \dots + n_{p-1}))$, where $n_1 + n_2 + \dots + n_p = n$.

We will obtain the following formula for the spectrum of the generating function:

$$\begin{aligned} 2\Lambda_n(u) &= 2 \sum_{k=1}^p \Lambda_{n_k}^{(k)}(u) - \sum_{k=1}^{p-1} (n - (n_1 + \dots + n_{k-1})) \\ &\quad \times \sum_{j=1}^{M_{n_1+\dots+n_k}} r_{-,n_1+\dots+n_k}(v_j^{(n_1+\dots+n_k)}, u) r_{+,n_1+\dots+n_k}(v_j^{(n_1+\dots+n_k)}, u) \\ &\quad + \sum_{k=1}^{p-1} (\partial_u + r_{-,n_1+\dots+n_k}^0(u, u) + r_{+,n_1+\dots+n_k}^0(u, u)) \\ &\quad \times ((n - (n_1 + \dots + n_k)) c_{n_k}^{(k-1)}(u) - n_k c_{n-(n_1+\dots+n_k)}^{(k-1)}(u)). \end{aligned}$$

Here $\Lambda_{n_k}^{(k)}(u)$ is an eigen-value of the quadratic generating functions of quantum integrals of the subalgebra $gl(n_k)$ after k -times application to it the procedure described in Lemma 3.1, $c_{n_k}^{(k-1)}(u)$ and $c_{n-(n_1+\dots+n_k)}^{(k-1)}(u)$ are eigen-values of the linear generating functions of quantum integrals of the subalgebras $gl(n_k)$ and $gl(n - (n_1 + \dots + n_k))$ after $(k - 1)$ -times application to it the procedure described in Lemma 3.1.

Let us specify at first the values of $c_{n_k}^{(k-1)}(u)$ and $c_{n-(n_1+\dots+n_k)}^{(k-1)}(u)$. By the very definition these are eigen-values of the linear Casimir functions of the Lie subalgebras $gl(n_k)$ and $gl(n - (n_1 + \dots + n_k))$ correspondingly. They have the following form:

$$c_{n_k}^{(k-1)}(u) = \sum_{i=n_1+\dots+n_{k-1}+1}^{n_1+\dots+n_k} \Lambda_{ii}^{(k-1)}(u), \quad c_{n-(n_1+\dots+n_k)}^{(k-1)}(u) = \sum_{i=n_1+\dots+n_k+1}^n \Lambda_{ii}^{(k-1)}(u).$$

It is necessary only to find $\Lambda_{ii}^{(k-1)}(u)$, keeping in mind the procedure described in Lemma 3.1 that had been applied to the Lax operator $k - 1$ times. We will have:

$$\begin{aligned} c_{n_k}^{(k-1)}(u) &= \sum_{i=n_1+\dots+n_{k-1}+1}^{n_1+\dots+n_k} \Lambda_{ii}(u) + \sum_{j=1}^{M_{n_1+\dots+n_{k-1}}} r_{n_1+\dots+n_{k-1}+1, n_1+\dots+n_{k-1}+1}(v_j^{(n_1+\dots+n_{k-1})}, u), \\ c_{n-(n_1+\dots+n_k)}^{(k-1)}(u) &= \sum_{i=n_1+\dots+n_k+1}^n \Lambda_{ii}(u). \end{aligned}$$

In order to see this it is necessary to take into account that, by the very procedure described in Lemma 3.1 and by the $k - 1$ steps based on the chain of subalgebras $gl(n) \supset gl(n - n_1) \supset gl(n - n_1 - n_2) \supset \dots \supset gl(n - (n_1 + \dots + n_{k-1}))$, we will have:

$$\begin{aligned} \Lambda_{ii}^{(k-1)}(u) &= \Lambda_{ii}^{(k-2)}(u) + \delta_{i,n_1+\dots+n_{k-1}+1} \sum_{j=1}^{M_{n_1+\dots+n_{k-1}}} r_{n_1+\dots+n_{k-1}+1,n_1+\dots+n_{k-1}+1}(v_j^{(n_1+\dots+n_{k-1})}, u) \\ &\quad - \delta_{i,n_1+\dots+n_{k-1}} \sum_{j=1}^{M_{n_1+\dots+n_{k-1}}} r_{n_1+\dots+n_{k-1},n_1+\dots+n_{k-1}}(v_j^{(n_1+\dots+n_{k-1})}, u). \end{aligned} \tag{39}$$

The formula (39) gives the change of the highest weight vector $\Lambda_{ii}^{(k-2)}(u)$ after the application of the $(k - 1)$ -th procedure described in Lemma 3.1 to the Lax submatrices. Applying this formula recursively we come to the above expressions for $c_{n_k}^{(k-1)}(u)$ and $c_{n-(n_1+\dots+n_k)}^{(k-1)}(u)$.

It is left to describe $\Lambda_{n_k}^{(k)}(u)$. In order to do this we have to apply additionally the chain of the embedded subalgebras $gl(n_k) \supset gl(n_k - 1) \supset gl(n_{k-1} - 2) \supset \dots \supset gl(1)$, keeping in mind that the initial highest weight vector of the representation of the corresponding sub-Lax matrix is changed by the previous steps of the nested Bethe ansatz.

Let us consider the case $\Lambda_{n_1}^{(1)}(u)$. We have that the corresponding spectrum is described by the following formula [23]:

$$\begin{aligned} 2\Lambda_{n_1}^{(1)}(u) &= (\Lambda_{11}^{(1)}(u) - \sum_{j=1}^{M_1} r_{11}(v_j^{(1)}, u))^2 + \sum_{k=2}^{n_1-1} (\Lambda_{kk}^{(1)}(u) + \sum_{j=1}^{M_{k-1}} r_{kk}(v_j^{(k-1)}, u) - \sum_{j=1}^{M_k} r_{kk}(v_j^{(k)}, u))^2 \\ &\quad + (\Lambda_{n_1 n_1}^{(1)}(u) + \sum_{j=1}^{M_{k-1}} r_{n_1 n_1}(v_j^{(n_1-1)}, u))^2 + \sum_{k=1}^{n_1} (\partial_u + (r_{-,k}^0(u, u) + r_{+,k}^0(u, u))) \\ &\quad \times ((n_1 - k)\Lambda_{kk}^{(1)}(u) - \sum_{i=k+1}^{n_1} \Lambda_{ii}^{(1)}(u) + (n_1 - k) \sum_{j=1}^{M_{k-1}} r_{kk}(v_j^{(k-1)}, u)) \\ &\quad - \sum_{k=1}^{n_1-1} (n_1 - k + 1) \sum_{j=1}^{M_k} r_{-,k}(v_j^{(k)}, u) r_{+,k}(v_j^{(k)}, u), \end{aligned} \tag{40}$$

where $M_k, k \in \overline{1, n_1 - 1}$ are non-negative integers.

Using the fact (see above) that

$$\Lambda_{ii}^{(1)}(u) = \Lambda_{ii}(u) + \delta_{i,n_1+1} \sum_{j=1}^{M_{n_1}} r_{n_1+1,n_1+1}(v_j^{(n_1)}, u) - \delta_{i,n_1} \sum_{j=1}^{M_{n_1}} r_{n_1,n_1}(v_j^{(n_1)}, u), \quad i \in \overline{1, n_1},$$

we obtain the following answer:

$$\begin{aligned} 2\Lambda_{n_1}^{(1)}(u) &= \sum_{k=1}^{n_1} (\Lambda_{kk}(u) + \sum_{j=1}^{M_{k-1}} r_{kk}(v_j^{(k-1)}, u) - \sum_{j=1}^{M_k} r_{kk}(v_j^{(k)}, u))^2 \\ &\quad + \sum_{k=1}^{n_1} (\partial_u + (r_{-,k}^0(u, u) + r_{+,k}^0(u, u))) ((n_1 - k)\Lambda_{kk}(u) - \sum_{i=k+1}^{n_1} \Lambda_{ii}(u) + (n_1 - k) \sum_{j=1}^{M_{k-1}} r_{kk}(v_j^{(k-1)}, u)) \\ &\quad - \sum_{k=1}^{n_1-1} (n_1 - k + 1) \sum_{j=1}^{M_k} r_{-,k}(v_j^{(k)}, u) r_{+,k}(v_j^{(k)}, u) + \sum_{k=1}^{n_1-1} (\partial_u + (r_{-,k}^0(u, u) + r_{+,k}^0(u, u))) \sum_{j=1}^{M_{n_1}} r_{n_1,n_1}(v_j^{(n_1)}, u), \end{aligned} \tag{41}$$

where $M_0 \equiv 0$.

In the similar way we have:

$$\begin{aligned}
 2\Lambda_{n_2}^{(2)}(u) &= \sum_{k=n_1+1}^{n_1+n_2} (\Lambda_{kk}(u) + \sum_{j=1}^{M_{k-1}} r_{kk}(v_j^{(k-1)}, u) - \sum_{j=1}^{M_k} r_{kk}(v_j^{(k)}, u))^2 \\
 &+ \sum_{k=n_1+1}^{n_1+n_2} (\partial_u + (r_{-,k}^0(u, u) + r_{+,k}^0(u, u))) \\
 &\times ((n_1 + n_2 - k)\Lambda_{kk}(u) - \sum_{i=k+1}^{n_1+n_2} \Lambda_{ii}(u) + (n_1 + n_2 - k) \sum_{j=1}^{M_{k-1}} r_{kk}(v_j^{(k-1)}, u)) \\
 &- \sum_{k=n_1+1}^{n_1+n_2-1} (n_1 + n_2 - k + 1) \sum_{j=1}^{M_k} r_{-,k}(v_j^{(k)}, u)r_{+,k}(v_j^{(k)}, u) \\
 &+ (n_2 - 1)(\partial_u + (r_{-,n_1+1}^0(u, u) + r_{+,n_1+1}^0(u, u))) \sum_{j=1}^{M_{n_1}} r_{n_1+1, n_1+1}(v_j^{(n_1)}, u) \\
 &+ \sum_{k=n_1+1}^{n_1+n_2-1} (\partial_u + (r_{-,k}^0(u, u) + r_{+,k}^0(u, u))) \sum_{j=1}^{M_{n_1+n_2}} r_{n_1+n_2, n_1+n_2}(v_j^{(n_1+n_2)}, u), \tag{42}
 \end{aligned}$$

where we have used (in a recursive manner) two times the formula (39), i.e. we have used that:

$$\begin{aligned}
 \Lambda_{ii}^{(2)}(u) &= \Lambda_{ii}(u) + \delta_{i, n_1+1} \sum_{j=1}^{M_{n_1}} r_{n_1+1, n_1+1}(v_j^{(n_1)}, u) - \delta_{i, n_1} \sum_{j=1}^{M_{n_1}} r_{n_1, n_1}(v_j^{(n_1)}, u) \\
 &+ \delta_{i, n_1+n_2+1} \sum_{j=1}^{M_{n_1+n_2}} r_{n_1+n_2+1, n_1+n_2+1}(v_j^{(n_1+n_2)}, u) - \delta_{i, n_1+n_2} \sum_{j=1}^{M_{n_1+n_2}} r_{n_1+n_2, n_1+n_2}(v_j^{(n_1+n_2)}, u).
 \end{aligned}$$

In the analogous way we obtain the following formula:

$$\begin{aligned}
 2\Lambda_{n_s}^{(s)}(u) &= \sum_{k=n_1+\dots+n_{s-1}+1}^{n_1+\dots+n_s} (\Lambda_{kk}(u) + \sum_{j=1}^{M_{k-1}} r_{kk}(v_j^{(k-1)}, u) - \sum_{j=1}^{M_k} r_{kk}(v_j^{(k)}, u))^2 \\
 &+ \sum_{k=n_1+\dots+n_{s-1}+1}^{n_1+\dots+n_s} (\partial_u + (r_{-,k}^0(u, u) + r_{+,k}^0(u, u))) \\
 &\times ((n_1 + \dots + n_s - k)\Lambda_{kk}(u) - \sum_{i=k+1}^{n_1+\dots+n_s} \Lambda_{ii}(u) + (n_1 + \dots + n_s - k) \sum_{j=1}^{M_{k-1}} r_{kk}(v_j^{(k-1)}, u)) \\
 &- \sum_{k=n_1+\dots+n_{s-1}+1}^{n_1+\dots+n_s-1} (n_1 + \dots + n_s - k + 1) \sum_{j=1}^{M_k} r_{-,k}(v_j^{(k)}, u)r_{+,k}(v_j^{(k)}, u) + (n_s - 1) \\
 &\times (\partial_u + (r_{-,n_1+\dots+n_{s-1}+1}^0(u, u) + r_{+,n_1+\dots+n_{s-1}+1}^0(u, u))) \sum_{j=1}^{M_{n_1+\dots+n_{s-1}}} r_{n_1+\dots+n_{s-1}+1, n_1+\dots+n_{s-1}+1}(v_j^{(n_1+\dots+n_{s-1})}, u) \\
 &+ \sum_{k=n_1+\dots+n_{s-1}+1}^{n_1+\dots+n_s-1} (\partial_u + (r_{-,k}^0(u, u) + r_{+,k}^0(u, u))) \sum_{j=1}^{M_{n_1+\dots+n_s}} r_{n_1+\dots+n_s, n_1+\dots+n_s}(v_j^{(n_1+\dots+n_s)}, u), \tag{43}
 \end{aligned}$$

for $1 < s < p$, and the last “closing” expression for $s = p$:

$$\begin{aligned}
 2\Lambda_{n_p}^{(p)}(u) &= \sum_{k=n_1+\dots+n_{p-1}+1}^{n_1+\dots+n_p} (\Lambda_{kk}(u) + \sum_{j=1}^{M_{k-1}} r_{kk}(v_j^{(k-1)}, u) - \sum_{j=1}^{M_k} r_{kk}(v_j^{(k)}, u))^2 \\
 &+ \sum_{k=n_1+\dots+n_{p-1}+1}^{n_1+\dots+n_p} (\partial_u + (r_{-,k}^0(u, u) + r_{+,k}^0(u, u))) \\
 &\times ((n_1 + \dots + n_p - k)\Lambda_{kk}(u) - \sum_{i=k+1}^{n_1+\dots+n_p} \Lambda_{ii}(u) + (n_1 + \dots + n_p - k) \sum_{j=1}^{M_{k-1}} r_{kk}(v_j^{(k-1)}, u)) \\
 &- \sum_{k=n_1+\dots+n_{p-1}+1}^{n_1+\dots+n_{p-1}} (n_1 + \dots + n_p - k + 1) \sum_{j=1}^{M_k} r_{-,k}(v_j^{(k)}, u)r_{+,k}(v_j^{(k)}, u) + (n_p - 1) \\
 &\times (\partial_u + r_{-,n_1+\dots+n_{p-1}+1}^0(u, u) + r_{+,n_1+\dots+n_{p-1}+1}^0(u, u)) \\
 &\times \sum_{j=1}^{M_{n_1+\dots+n_{p-1}}} r_{n_1+\dots+n_{p-1}+1, n_1+\dots+n_{p-1}+1}(v_j^{(n_1+\dots+n_{p-1})}, u),
 \end{aligned}$$

where we have used that $n_1 + n_2 + \dots + n_p = n$ and $M_n = 0$.

Taking into account all the above we obtain the formula (36) for the spectrum.

Finally the Bethe-type equations (37)–(38) are obtained as the condition of the absence of the poles of the functions (36) in the points $v_j^{(k)}$.

Theorem is proven. \square

3.3. Case of the Gaudin models

Let us specify the obtained answers for spectrum and Bethe equations for the cases of the generalized Gaudin models with and without an external magnetic field. By other words, let us calculate the spectrum of the corresponding Gaudin-type hamiltonians. In order to do this it will be necessary to specify the explicit form of the eigen-values $\Lambda_{ii}(u)$.

In more detail: in the cases of Gaudin-type models Lax algebra coincides with the direct sum of N -copies of the Lie algebra $gl(n)$. A space of an irreducible representation of the algebra $gl(n)^{\oplus N}$ has the form: $V = (\otimes_{l=1}^N V^{\lambda^{(l)}})$, where $V^{\lambda^{(l)}}$ is the space of an irreducible representation of the l -th copy of $gl(n)$ labeled by the highest weights $\lambda^{(l)} = (\lambda_1^{(l)}, \dots, \lambda_n^{(l)})$, $l \in \overline{1, N}$. In the representation space V there exists the highest weight vector Ω such that:

$$\hat{S}_{ii}^{(l)} \Omega = \lambda_i^{(l)} \Omega, \hat{S}_{ij}^{(l)} \Omega = 0, i, j \in \overline{1, n}, i < j; l \in \overline{1, N}.$$

Using the definition of the Lax operators (13) and (14) it is easy to see that Ω coincides with the vacuum vector for the corresponding representation of the Lax algebra (we assume that the corresponding “shift element” is diagonal).

The following proposition (corollary of Theorem 3.2) holds true:

Proposition 3.2. *Let the r -matrix $r(u, v)$ satisfy the conditions (4), (33), (34), (35). Then the spectrum of the generalized Gaudin hamiltonians (15) and the generalized Gaudin hamiltonians in an external magnetic field (16) has the form:*

$$h_m = h_m^0 + \sum_{k=1}^n \lambda_k^{(m)} \left(\sum_{j=1}^{M_{k-1}} r_{kk}(v_j^{(k-1)}, v_m) - \sum_{j=1}^{M_k} r_{kk}(v_j^{(k)}, v_m) \right), \tag{44}$$

where $M_0 = M_n = 0$ and h_l^0 is an eigen-value of these hamiltonians on the vacuum vector Ω :

$$\begin{aligned}
 h_m^0 = & \sum_{k=1}^n \left(\sum_{s=1, s \neq l}^N r_{kk}(v_s, v_m) \lambda_k^{(s)} \lambda_k^{(m)} + r_{kk}^0(v_l, v_m) (\lambda_k^{(l)})^2 \right) + \sum_{k=1}^n c_{kk}(v_m) \lambda_k^{(m)} \\
 & + \sum_{k=1}^p \sum_{l=1}^{n_k} \left(r_{-,l+n_1+\dots+n_{k-1}}^0(v_m, v_m) + r_{+,l+n_1+\dots+n_{k-1}}^0(v_m, v_m) \right) \\
 & \times \left((n_k - l) \lambda_{l+n_1+\dots+n_{k-1}}^{(m)} - \sum_{i=l+1}^{n_k} \lambda_{i+n_1+\dots+n_{k-1}}^{(m)} \right) \\
 & + \sum_{k=1}^p \left(r_{-,n_1+\dots+n_{k-1}}^0(v_m, v_m) + r_{+,n_1+\dots+n_{k-1}}^0(v_m, v_m) \right) \\
 & \times \left((n - (n_1 + \dots + n_k)) \sum_{l=1}^{n_k} \lambda_{n_1+\dots+n_k+l}^{(m)} - n_k \sum_{l=1}^{n-(n_1+\dots+n_k)} \lambda_{n_1+\dots+n_k+l}^{(m)} \right), \tag{45}
 \end{aligned}$$

$c_{kk}(v_m)$ are the components of the shift element — external magnetic field and the “rapidities” $v_k^{(i)}$, $k \in \overline{1, M_i}$, $i \in \overline{1, n-1}$ satisfy Bethe equations (37), (38) with $\Lambda_{kk}(u) = \sum_{m=1}^N \lambda_k^{(m)} r_{kk}(v_m, u) + c_{kk}(u)$.

Proof. The proof is achieved by the direct calculation, using the formulas (36), taking into account that in the case of the generalized Gaudin models in an external magnetic field $\Lambda_{kk}(u) = \sum_{m=1}^N \lambda_k^{(m)} r_{kk}(v_m, u) + c_{kk}(u)$ and taking the residue in the point $u = v_m$ in the formula (36). \square

4. Example: case of “ Z_p -graded” classical r -matrices

4.1. Z_p -graded classical r -matrices

4.1.1. General case

Let \mathfrak{g} be semisimple (reductive) Lie algebra. Let $\mathfrak{g} = \sum_{j=0}^{p-1} \mathfrak{g}_{\bar{j}}$ be $Z_p = \mathbb{Z}/p\mathbb{Z}$ grading of \mathfrak{g} , i.e.:

$[\mathfrak{g}_{\bar{i}}, \mathfrak{g}_{\bar{j}}] \subset \mathfrak{g}_{\overline{i+j}}$ where \bar{j} denotes the class of equivalence of the elements $j \in \mathbb{Z} \bmod p \mathbb{Z}$. It is known, that \mathbb{Z}_p -grading of \mathfrak{g} may be defined with the help of some automorphism σ of the order p , such that $\sigma(\mathfrak{g}_{\bar{i}}) = e^{2\pi i k/p} \mathfrak{g}_{\bar{i}}$ and $\mathfrak{g}_{\bar{0}}$ is the algebra of σ -invariants: $\sigma(\mathfrak{g}_{\bar{0}}) = \mathfrak{g}_{\bar{0}}$. It is also known [32] (see also [24]), that using this data it is possible to define the following classical r -matrix:

$$r^\sigma(u_1, u_2) = \frac{\sum_{j=0}^{p-1} u_1^j u_2^{p-j} \Omega_{12}^{(\bar{j})}}{(u_1^p - u_2^p)}, \tag{46}$$

where $X_{\alpha, \bar{j}}$ is a basis in the space $\mathfrak{g}_{\bar{j}}$, $X^{\alpha, -\bar{j}}$ is a dual basis in the space $\mathfrak{g}_{-\bar{j}}$ and $\Omega_{12}^{(\bar{j})} = \sum_{\alpha=1}^{\dim \mathfrak{g}_{\bar{j}}} g_{\bar{j}}^{\alpha\beta} X_{\alpha, \bar{j}} \otimes X_{\beta, -\bar{j}}$ is a projection operator onto the subspace $\mathfrak{g}_{\bar{j}}$, $g_{\bar{j}}^{\alpha\beta} \equiv (X^{\alpha, -\bar{j}}, X^{\beta, \bar{j}})$. In particular, $\Omega^{(\bar{0})}$ is the tensor Casimir of the subalgebra $\mathfrak{g}_{\bar{0}}$.

4.1.2. Case of the $gl(n)$

Let us return to the case $\mathfrak{g} = gl(n)$ and consider a Z_p -grading of $gl(n)$ corresponding to the decomposition $n = n_1 + n_2 + \dots + n_p$ such that the graded subspaces consist of the block matrices. In more detail:

$$\mathfrak{g}_{\bar{0}} = \left\{ \begin{pmatrix} G_{11} & 0 & \dots & 0 \\ 0 & G_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & G_{pp} \end{pmatrix} \right\}, \quad \mathfrak{g}_{-\bar{1}} = \left\{ \begin{pmatrix} 0 & 0 & \dots & G_{1p} \\ G_{21} & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots \\ 0 & \dots & G_{pp-1} & 0 \end{pmatrix} \right\},$$

$$\mathfrak{g}_{\bar{1}} = \left\{ \begin{pmatrix} 0 & G_{12} & \dots & 0 \\ 0 & 0 & \dots & \dots \\ \dots & \dots & \dots & G_{p-1p} \\ G_{p1} & \dots & 0 & 0 \end{pmatrix} \right\}$$

etc., where $G_{ii} \in gl(n_i)$, $G_{ij} \in Mat(n_i, n_j)$. This grading corresponds to the internal automorphism $\sigma = Ad_g$, where g is a diagonal matrix $g = \text{diag}(g_1, g_2, \dots, g_n)$ with $g_1 = g_2 \dots = g_{n_1} = 1$, $g_{n_1+1} = \dots = g_{n_1+n_2} = \epsilon$, $g_{n_1+n_2+1} = \dots = g_{n_1+n_2+n_3} = \epsilon^2, \dots, g_{n_1+\dots+n_{p-1}+1} = \dots = g_n = \epsilon^{p-1}$, $\epsilon = e^{\frac{2\pi i}{p}}$.

The corresponding r -matrix written in the form (46) for $p > 2$ does not satisfy Bethe-ansatz conditions (33), (34) for any possible chain of the embedded subalgebras. In order to make it satisfy the conditions (33), (34) it is necessary to apply to it certain equivalence transformations. The resulting r -matrix is given in the following proposition:

Proposition 4.1. *Let $\mathfrak{g} = gl(n)$ and its Z_p -grading be defined as above. Then by the equivalence transformations the corresponding r -matrix (46) is brought to the following form:*

$$r^{12}(u, v) = \frac{\Omega^{12}}{u - v} + \frac{C^{12}}{v}, \tag{47}$$

where $\Omega^{12} = \sum_{i,j=1}^n X_{ij} \otimes X_{ji}$ and tensor C^{12} is defined as follows:

$$C^{12} = \sum_{s=1}^{p-1} \sum_{i=1+n_1+\dots+n_{s-1}}^{n_1+n_2+\dots+n_s} \sum_{j=n_1+n_2+\dots+n_s+1}^n X_{ij} \otimes X_{ji}. \tag{48}$$

Sketch of the proof. In order to prove the proposition it is necessary to apply to the r -matrix (46), specified for the given Lie algebra and the given grading, the gauge transformation $r(u_1, u_2) \rightarrow g^{-1}(u_1) \otimes g^{-1}(u_2)r(u_1, u_2)g(u_1) \otimes g(u_2)$ with the following matrix $g(u_i)$:

$$g(u_i) = \text{diag}(g_1(u_i), g_2(u_i), \dots, g_n(u_i)),$$

where $g_1(u_i) = g_2(u_i) \dots = g_{n_1}(u_i) = 1$, $g_{n_1+1}(u_i) = g_{n_1+2}(u_i) = \dots = g_{n_1+n_2}(u_i) = u_i$, $g_{n_1+n_2+1}(u_i) = g_{n_1+n_2+2}(u_i) = \dots = g_{n_1+n_2+n_3}(u_i) = u_i^2, \dots, g_{n_1+\dots+n_{p-1}+1}(u_i) =$

$g_{n_1+\dots+n_{p-1}+2}(u_i) = \dots = g_n(u_i) = u_i^{p-1}$, $i \in 1, 2$, divide the obtained r -matrix by u_2^p and make the substitution of variables $u = u_1^p$, $v = u_2^p$. \square

Remark 7. Observe, that the considered r -matrix satisfies the regularity property (4) with

$$r^0(u, v) = \frac{C^{12}}{v}$$

Observe also that this r -matrix is diagonal in the root basis.

Example 1. In the case $p = 2$ the r -matrix (46) has the form:

$$r^{12}(u_1, u_2) = \frac{u_2^2}{u_1^2 - u_2^2} \left(\sum_{i,j=1}^{n_1} X_{ij} \otimes X_{ji} + \sum_{i,j=n_1+1}^n X_{ij} \otimes X_{ji} \right) + \frac{u_1 u_2}{u_1^2 - u_2^2} \sum_{i=1}^{n_1} \sum_{j=n_1+1}^n (X_{ij} \otimes X_{ji} + X_{ji} \otimes X_{ij}). \tag{49}$$

After the gauge transformation and re-parametrization described in Proposition 4.1 the corresponding r -matrix acquires the following form:

$$r^{12}(u, v) = \frac{\sum_{i,j=1}^n X_{ij} \otimes X_{ji}}{u - v} + \frac{\sum_{i=1}^{n_1} \sum_{j=n_1+1}^n X_{ij} \otimes X_{ji}}{v}. \tag{50}$$

Let us now specify the diagonal shift elements for the r -matrix (47):

Proposition 4.2. *The general diagonal shift element for the r -matrix (47) has the form:*

$$c(v) = \frac{1}{v} \text{diag}(c_1, \dots, c_n), \text{ where} \\ c_1 = c_2 = \dots = c_{n_1}, c_{n_1+1} = c_{n_1+2} \dots = c_{n_1+n_2}, \dots, c_{n_1+\dots+n_{p-1}+1} = c_{n_1+\dots+n_{p-1}+2} = \dots \\ = c_{n_1+n_2+\dots+n_p}.$$

Proof. As was shown in our previous paper [25], the “diagonal” shift elements for the r -matrices (46) are constant and belong to the center of the reductive subalgebra \mathfrak{g}_0 , i.e. in the case at hand have the form: $c = \text{diag}(c_1, \dots, c_n)$ where $c_1 = c_2 = \dots = c_{n_1}, c_{n_1+1} = c_{n_1+2} \dots = c_{n_1+n_2}, \dots, c_{n_1+\dots+n_{p-1}+1} = c_{n_1+\dots+n_{p-1}+2} = \dots = c_{n_1+n_2+\dots+n_p}$. Now, in order to obtain the shift element for the r -matrix (47) it is enough to apply the equivalence transformation from the previous proposition to this shift element, i.e. in our case simply to divide it by v .

Proposition is proven. \square

Example 2. In the simplest non-banal $p = 2$ case the diagonal shift element has the form:

$$c(u) = \frac{1}{u} (c_{n_1} \sum_{i=1}^{n_1} X_{ii} + c_{n_1+n_2} \sum_{i=n_1+1}^n X_{ii}).$$

4.2. Z_p -graded classical r -matrices and generalized Gaudin models

For Z_p -graded r -matrix (46) corresponding to $\mathfrak{g} = \mathfrak{gl}(n)$, the specified Z_p -grading and the gauge (47) the Gaudin-type Lax operator is written as follows:

$$\hat{L}(u) = \sum_{m=1}^N \left(\sum_{i,j=1}^n \frac{\hat{S}_{ij}^{(m)} X_{ji}}{v_m - u} + \frac{1}{u} \sum_{s=1}^{p-1} \sum_{i=1+n_1+\dots+n_{s-1}}^{n_1+n_2+\dots+n_s} \sum_{j=1+n_1+\dots+n_s}^n \hat{S}_{ij}^{(m)} X_{ji} \right). \tag{51}$$

The Gaudin-type Lax operator with an external “diagonal” magnetic field has the form:

$$\begin{aligned} \hat{L}^c(u) = & \sum_{m=1}^N \left(\sum_{i,j=1}^n \frac{\hat{S}_{ij}^{(m)} X_{ji}}{v_m - u} + \frac{1}{u} \sum_{s=1}^{p-1} \sum_{i=1+n_1+\dots+n_{s-1}}^{n_1+n_2+\dots+n_s} \sum_{j=1+n_1+\dots+n_s}^n \hat{S}_{ij}^{(m)} X_{ji} \right) + \\ & + \frac{1}{u} \sum_{k=1}^p c_{n_1+\dots+n_k} \sum_{i=1+n_1+\dots+n_{k-1}}^{n_1+\dots+n_k} X_{ii}. \end{aligned} \tag{52}$$

The corresponding Gaudin-type hamiltonians (15) and Gaudin-type hamiltonians in an external magnetic field (16) are written as follows:

$$\begin{aligned} \hat{H}_{v_l} = & \sum_{m=1, m \neq l}^N \left(\sum_{i,j=1}^n \frac{\hat{S}_{ij}^{(m)} \hat{S}_{ji}^{(l)}}{v_m - v_l} + \frac{1}{v_l} \sum_{s=1}^{p-1} \sum_{i=1+n_1+\dots+n_{s-1}}^{n_1+n_2+\dots+n_s} \sum_{j=1+n_1+\dots+n_s}^n \hat{S}_{ij}^{(m)} \hat{S}_{ji}^{(l)} \right) + \\ & + \frac{1}{2v_l} \sum_{s=1}^{p-1} \sum_{i=1+n_1+\dots+n_{s-1}}^{n_1+n_2+\dots+n_s} \sum_{j=1+n_1+\dots+n_s}^n (\hat{S}_{ij}^{(l)} \hat{S}_{ji}^{(l)} + \hat{S}_{ji}^{(l)} \hat{S}_{ij}^{(l)}), \end{aligned} \tag{53}$$

$$\begin{aligned} \hat{H}_{v_l}^c = & \sum_{m=1, m \neq l}^N \left(\sum_{i,j=1}^n \frac{\hat{S}_{ij}^{(m)} \hat{S}_{ji}^{(l)}}{v_m - v_l} + \frac{1}{v_l} \sum_{s=1}^{p-1} \sum_{i=1+n_1+\dots+n_{s-1}}^{n_1+n_2+\dots+n_s} \sum_{j=1+n_1+\dots+n_s}^n \hat{S}_{ij}^{(m)} \hat{S}_{ji}^{(l)} \right) + \\ & + \frac{1}{2v_l} \sum_{s=1}^{p-1} \sum_{i=1+n_1+\dots+n_{s-1}}^{n_1+n_2+\dots+n_s} \sum_{j=1+n_1+\dots+n_s}^n (\hat{S}_{ij}^{(l)} \hat{S}_{ji}^{(l)} + \hat{S}_{ji}^{(l)} \hat{S}_{ij}^{(l)}) + \\ & + \frac{1}{v_l} \sum_{k=1}^p c_{n_1+\dots+n_k} \sum_{i=1+n_1+\dots+n_{k-1}}^{n_1+\dots+n_k} \hat{S}_{ii}^{(l)}. \end{aligned} \tag{54}$$

In the next subsection we will consider the diagonalization of these hamiltonians by means of the nested Bethe ansatz, according to new “nesting” scheme described in the previous sections. But at first let us consider in some detail the simplest non-banal (i.e. different from the standard rational r -matrix case) example of the described in this subsection Lax operators and hamiltonians.

Example 3. In the $p = 2$ case the Gaudin-type Lax operator (52) with an external “diagonal” magnetic field has the form:

$$\begin{aligned} \hat{L}^c(u) = & \sum_{m=1}^N \left(\sum_{i,j=1}^n \frac{\hat{S}_{ij}^{(m)} X_{ji}}{v_m - u} + \frac{1}{u} \sum_{i=1}^{n_1} \sum_{j=n_1+1}^n \hat{S}_{ij}^{(m)} X_{ji} \right) + \\ & + \frac{1}{u} (c_{n_1} \sum_{i=1}^{n_1} X_{ii} + c_{n_1+n_2} \sum_{i=n_1+1}^n X_{ii}). \end{aligned}$$

The corresponding Gaudin-type hamiltonian in the external magnetic field is written as follows:

$$\begin{aligned} \hat{H}_{v_l}^c = & \sum_{m=1, m \neq l}^N \left(\sum_{i,j=1}^n \frac{\hat{S}_{ij}^{(m)} \hat{S}_{ji}^{(l)}}{v_m - v_l} + \frac{1}{v_l} \sum_{i=1}^{n_1} \sum_{j=n_1+1}^n \hat{S}_{ij}^{(m)} \hat{S}_{ji}^{(l)} \right) + \\ & + \frac{1}{2v_l} \sum_{i=1}^{n_1} \sum_{j=n_1+1}^n (\hat{S}_{ij}^{(l)} \hat{S}_{ji}^{(l)} + \hat{S}_{ji}^{(l)} \hat{S}_{ij}^{(l)}) + \frac{1}{v_l} (c_{n_1} \sum_{i=1}^{n_1} \hat{S}_{ii}^{(l)} + c_{n_1+n_2} \sum_{i=n_1+1}^n \hat{S}_{ii}^{(l)}). \end{aligned}$$

The Lax operator and Gaudin-type hamiltonians without external magnetic field are obtained simply by putting $c_{n_1} = c_{n_1+n_2} = 0$ in the above formulas.

4.2.1. Nested Bethe ansatz and Z_p -graded classical r -matrices

Let us now pass to the nested Bethe ansatz. First of all we note, that the considered r -matrix in the gauge (47) satisfies the conditions (33), (34) for the chain of the embedded subalgebras of the form $gl(n) \supset gl(n - n_1) \supset gl(n - n_1 - n_2) \supset gl(n - (n_1 + \dots + n_{p-1}))$, where $n_1 + n_2 + \dots + n_p = n$. Indeed, due to the fact that

$$r_{ji}(u, v) = \frac{1}{u - v} + \frac{1}{v},$$

if $i \in \overline{1 + \dots + n_{s-1}, n_1 + \dots + n_s}, j \in \overline{n_1 + \dots + n_s + 1, n}, s \in \overline{1, p - 1}$, (55)

$$r_{ji}(u, v) = \frac{1}{u - v}, \text{ for all other indices } i, j, \tag{56}$$

the conditions (33), (34) are satisfied. In more detail we have:

$$\begin{aligned} r_{-,n_1+\dots+n_s}(u, v) &= \frac{1}{u - v} + \frac{1}{v}, \\ r_{+,n_1+\dots+n_s}(u, v) &= \frac{1}{u - v}, \forall s \in \overline{1, p - 1}, r_{-,i}(u, v) = r_{+,i}(u, v) = \frac{1}{u - v}. \end{aligned}$$

Moreover, instead of the condition (35) the following stronger condition holds true:

$$r_{ii}(u, v) = \frac{1}{u - v}, i \in \overline{1, n}. \tag{57}$$

Now, for the applicability of the nested Bethe ansatz it is necessary to guaranty the existence of the “vacuum vector” in the representation space. But, as it was argued above, for the Gaudin-type models it is automatic. That is why we have only to specify the results of Proposition 3.2 for the r -matrix (47). The following corollary of Proposition 3.2 holds true:

Corollary 4.1. *The spectrum of the generalized Gaudin hamiltonians (53) and the generalized Gaudin hamiltonians in an external magnetic field (54) in the space of an irreducible representation of the algebra $gl(n)^{\oplus N}$ labeled by the highest weights $\lambda^{(l)} = (\lambda_1^{(l)}, \dots, \lambda_n^{(l)})$, $l \in \overline{1, N}$ has the form:*

$$h_l = h_l^0 + \sum_{k=1}^n \lambda_k^{(l)} \left(\sum_{j=1}^{M_{k-1}} \frac{1}{(v_j^{(k-1)} - v_l)} - \sum_{j=1}^{M_k} \frac{1}{(v_j^{(k)} - v_l)} \right), \tag{58}$$

where h_l^0 is an eigen-value of these hamiltonians on the vacuum vector Ω :

$$\begin{aligned}
 h_l^0 = & \sum_{k=1}^n \sum_{s=1, s \neq l}^N \frac{\lambda_k^{(s)} \lambda_k^{(l)}}{v_s - v_l} + \frac{1}{v_l} \sum_{k=1}^p c_{n_1+\dots+n_k} \sum_{i=n_1+\dots+n_{k-1}+1}^{n_1+\dots+n_k} \lambda_i^{(l)} + \\
 & + \sum_{k=1}^p \frac{1}{v_l} \left((n - (n_1 + \dots + n_k)) \sum_{m=1}^{n_k} \lambda_{n_1+\dots+n_k+m}^{(l)} - n_k \sum_{m=1}^{n-(n_1+\dots+n_k)} \lambda_{n_1+\dots+n_k+m}^{(l)} \right),
 \end{aligned}
 \tag{59}$$

$c_{n_1+\dots+n_k}$ are the components of the shift element — external magnetic field and the “rapidities” $v_k^{(i)}$, $k \in \overline{1, M_i}$, $i \in \overline{1, n-1}$ satisfy the following Bethe-type equations:

$$\begin{aligned}
 \sum_{m=1}^N \frac{\lambda_{n_1+\dots+n_k}^{(m)} - \lambda_{n_1+\dots+n_{k+1}}^{(m)}}{(v_m - v_i^{(n_1+\dots+n_k)})} + \frac{c_{n_1+\dots+n_k} - c_{n_1+\dots+n_{k+1}}}{v_i^{(n_1+\dots+n_k)}} + \frac{(n_k + n_{k+1})}{2} \frac{1}{v_i^{(n_1+\dots+n_k)}} = \\
 = \sum_{j=1; j \neq i}^{M_{n_1+\dots+n_k}} \frac{2}{(v_j^{(n_1+\dots+n_k)} - v_i^{(n_1+\dots+n_k)})} - \sum_{j=1}^{M_{n_1+\dots+n_k}-1} \frac{1}{(v_j^{(n_1+\dots+n_k-1)} - v_i^{(n_1+\dots+n_k)})} - \\
 - \sum_{j=1}^{M_{n_1+\dots+n_k}+1} \frac{1}{(v_j^{(n_1+\dots+n_k+1)} - v_i^{(n_1+\dots+n_k)})},
 \end{aligned}
 \tag{60a}$$

$k \in \overline{1, p-1}$, corresponding to the chain $gl(n) \supset gl(n - n_1) \supset gl(n - n_1 - n_2) \supset gl(n - (n_1 + \dots + n_{p-1}))$, $n_1 + n_2 + \dots + n_p = n$, and the following set of Bethe-type equations:

$$\sum_{m=1}^N \frac{\lambda_k^{(m)} - \lambda_{k+1}^{(m)}}{(v_m - v_i^{(k)})} = \sum_{j=1; j \neq i}^{M_k} \frac{2}{v_j^{(k)} - v_i^{(k)}} - \sum_{j=1}^{M_{k-1}} \frac{1}{(v_j^{(k-1)} - v_i^{(k)})} - \sum_{j=1}^{M_{k+1}} \frac{1}{(v_j^{(k+1)} - v_i^{(k)})},
 \tag{61}$$

$k \in \overline{n_1 + \dots + n_{s-1} + 1, n_1 + \dots + n_s}$ corresponding to the sub-chains

$$gl(n_s) \supset gl(n_s - 1) \supset \dots \supset gl(1), \quad s \in \overline{1, p}.$$

Remark 8. Observe, that in the case of the Gaudin systems without external magnetic field one has simply to put in the above formulas $c_{n_1+\dots+n_k} = 0$, $k \in \overline{1, p}$.

Remark 9. Observe, that the spectrum (59) of the Gaudin-type hamiltonians for the considered Z_p -graded classical r -matrices has (formally) the same form as for the case of the rational r -matrix, but the “rapidities” $v_i^{(k)}$ satisfy different system of Bethe equations. Observe also that among the system of the Bethe equations only the equations that correspond to the subchain $gl(n) \supset gl(n - n_1) \supset gl(n - n_1 - n_2) \supset gl(n - (n_1 + \dots + n_{p-1}))$ differ from those in the case of the rational r -matrices.

Example 4. In the simplest “non-banal” $p = 2$ case the only group of Bethe equations different from the corresponding equations in the case of the rational r -matrix has the following form:

$$\begin{aligned} & \sum_{m=1}^N \frac{\lambda_{n_1}^{(m)} - \lambda_{n_1+1}^{(m)}}{(v_m - v_i^{(n_1)})} + \frac{c_{n_1} - c_{n_1+n_2}}{v_i^{(n_1)}} + \frac{n}{2} \frac{1}{v_i^{(n_1)}} \\ &= \sum_{j=1; j \neq i}^{M_{n_1}} \frac{2}{(v_j^{(n_1)} - v_i^{(n_1)})} - \sum_{j=1}^{M_{n_1-1}} \frac{1}{(v_j^{(n_1-1)} - v_i^{(n_1)})} - \sum_{j=1}^{M_{n_1+1}} \frac{1}{(v_j^{(n_1+1)} - v_i^{(n_1)})}. \end{aligned}$$

Observe also, that the case $p = 2$ can be treated in the framework of our scheme directly, without the application of the gauge transform to the r -matrix (49). The Bethe equations obtained using this direct treatment are equivalent to those described above (see [27] where integrable spin, boson and spin–boson models corresponding to the case $p = 2$ have been considered). They are also equivalent to the Bethe equations obtained in the case $p = 2$ using the analytical Bethe ansatz [19].

5. Conclusion and discussion

In the present paper we have generalized nested Bethe ansatz onto a wide class of chains or “hierarchies” of embedded subalgebras of Lie algebra $gl(n)$ and certain sub-classes of the Cartan-invariant non-skew-symmetric classical r -matrices. We have shown that among such the r -matrices there are “twisted” or Z_p -graded non-skew-symmetric classical r -matrices with spectral parameters. We have considered an example of the corresponding generalized Gaudin models with and without external magnetic field and found the spectrum of their hamiltonians using nested Bethe ansatz.

The important on-going problem is a solution (at least a numerical one) of the obtained nested Bethe equations, which is necessary for concrete applications of the corresponding integrable models in quantum physics.

Finally we would like to suggest that some of the results of the present paper on the generalized nested algebraic Bethe ansatz for the case of linear Lax algebras can be prolonged onto the quadratic “quantum-group” cases. In particular, we suppose that for the case of the Reflection Equation Algebras corresponding to the considered “ Z_2 -graded” classical r -matrices [19], there should be a generalization of the nested algebraic Bethe ansatz scheme onto a chain of subalgebras $gl(n) \supset gl(n_1)$ or $gl(n) \supset gl(n_2)$, $n_1 + n_2 = n$, $n_1 \geq 1$, $n_2 > 1$, complimented by sub-chains $gl(n_k) \supset gl(n_k - 1) \supset gl(n_k - 2) \supset \dots \supset gl(1)$, $k \in \{1, 2\}$.

References

- [1] J. von Delft, D.C. Ralph, Phys. Rep. 345 (2001) 61.
- [2] L. Amico, A. Di Lorenzo, A. Osterloh, Phys. Rev. Lett. 86 (2001) 5759.
- [3] J. Links, H.-Q. Zhou, R.H. McKenzie, M. Gould, J. Phys. A 36 (19) (2003) R.63.
- [4] J. Dukelsky, S. Pittel, G. Sierra, Rev. Mod. Phys. 76 (2004) 643.
- [5] R.W. Richardson, N. Sherman, Nucl. Phys. 52 (1964) 221;
R.W. Richardson, N. Sherman, Nucl. Phys. 52 (1964) 253.
- [6] R.W. Richardson, Phys. Rev. 144 (1966) 874;
R.W. Richardson, Phys. Rev. 154 (1967) 1007.
- [7] J. Links, H.-Q. Zhou, M.D. Gould, R.H. McKenzie, J. Phys. A 35 (2002) 6459–6470.
- [8] B. Errea, J. Dukelsky, G. Ortiz, Phys. Rev. A 79 (2009) 051603.
- [9] E. Yuzbashyan, B. Altshuler, V. Kuznetsov, V. Enolskii, J. Phys. A 38 (2005) 7831.
- [10] M. Bortz, J. Stolze, Phys. Rev. B 76 (2007) 014304.
- [11] M. Gaudin, La Fonction d’onde de Bethe, Masson, Paris, 1983.
- [12] T. Skrypynyk, J. Geom. Phys. 56 (12) (2006) 53.

- [13] T. Skrypnyk, *J. Phys. A* 40 (2007) 13357.
- [14] T. Skrypnyk, *Nucl. Phys. B* 806 (3) (2009) 504–528.
- [15] T. Skrypnyk, *J. Math. Phys.* 50 (2009) 033504.
- [16] T. Skrypnyk, *J. Phys. A* 42 (47) (2009) 472004.
- [17] T. Skrypnyk, *Nucl. Phys. B* 864 (3) (2012) 770.
- [18] T. Skrypnyk, *Nucl. Phys. B* 864 (3) (2012) 770.
- [19] T. Skrypnyk, *Nucl. Phys. B* 870 (3) (2013) 495.
- [20] T. Skrypnyk, *J. Phys. A* 45 (41) (2012) 415203.
- [21] P. Kulish, N. Reshetikhin, *J. Phys. A* 16 (1983) L591.
- [22] B. Jurco, *J. Math. Phys.* 30 (1989) 1289.
- [23] T. Skrypnyk, *Nucl. Phys. B* 891 (2015) 200–229.
- [24] T. Skrypnyk, *J. Geom. Phys.* 60 (3) (2010) 491.
- [25] T. Skrypnyk, *J. Geom. Phys.* 80 (2014) 71–87.
- [26] T. Skrypnyk, *J. Geom. Phys.* 97 (2015) 133–155.
- [27] T. Skrypnyk, *J. Phys. A* 49 (36) (2016) 365201.
- [28] A. Belavin, V. Drinfeld, *Funct. Anal. Appl.* 16 (3) (1982) 1.
- [29] J.M. Maillet, *Phys. Lett. B* 167 (1986) 401.
- [30] L. Fredel, J.M. Maillet, *Phys. Lett. B* 262 (1991) 278.
- [31] O. Babelon, C. Viallet, *Phys. Lett. B* 237 (1990) 411.
- [32] J. Avan, M. Talon, *Phys. Lett. B* 241 (1990) 77.
- [33] L. Fadeev, N. Reshetikhin, L. Tahtadjan, *Algebra Anal.* 1 (1) (1986) 178.
- [34] V. Chari, A. Prisley, *A Guide to Quantum Groups*, Cambridge Univ. Press, Cambridge, 1994.
- [35] E. Sklyanin, On integrability of Landau–Lifschitz equation, LOMI Preprint E-3-79, 1979.
- [36] L. Fadeev, L. Takhtadgian, *Hamiltonian Approach in Theory of Solitons*, Nauka, Moscow, 1986.
- [37] D. Talalaev, *Funct. Anal. Appl.* 40 (1) (2006) 73–77.