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PSEUDO-DIFFERENTIAL OPERATORS AND EXISTENCE OF GABOR FRAMES

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Abstract. We study from a pseudo-differential point of view the frame operator associated with a Gabor system. In particular we show how an application of the classical boundedness theorem of Calderón -Vaillancourt yields sufficient conditions for a Gabor system to form a frame in $L^2(\mathbb{R}^d)$.

0. introduction

The central problem of time-frequency analysis is the extraction of information about the frequency content of a signal in dependence on time. In this context a signal is a complex function, or distribution, $f(t)$ of the time variable t, which for generality is usually supposed to be in $\mathbb{R}^d$. To this aim, *Gabor frames*, also known as *Weyl-Heisenberg frames*, have proved to be a particularly useful tool. Their theory has grown in the last decades to a vast subject of research in applied harmonic analysis, with connections to various aspects of abstract harmonic analysis.

At the core of Gabor frames is the idea of representing signals as expansions in terms of translations and modulations of a fixed analysing window function. More precisely suppose $g \in L^2(\mathbb{R}^d)$ is a non identically zero function, then for $\alpha, \beta \in \mathbb{R}_+$, its translations and modulations

$$g_{h,k}(t) = e^{2\pi i\beta k\cdot t} g(t - \alpha h), \quad h, k \in \mathbb{Z}^d, \tag{0.1}$$

are called *time-frequency shifts* of g of parameter α, β. The *Gabor system*

$$\mathcal{G}(g, \alpha, \beta) = \{g_{h,k}\}_{h,k\in\mathbb{Z}^d} \tag{0.2}$$

is said a *Gabor frame* in $L^2(\mathbb{R}^d)$ if there exist $A, B > 0$ such that

$$A\|f\|_2^2 \leq \sum_{h,k} |(f, g_{h,k})_{L^2}|^2 \leq B\|f\|_2^2, \tag{0.3}$$

for every $f \in L^2(\mathbb{R}^d)$. In this case a classical result asserts the existence of frames $\{\widetilde{g}_{h,k}\}_{h,k\in\mathbb{Z}}$, called *dual frames* of $\{g_{h,k}\}_{h,k\in\mathbb{Z}}$, such that the following *reconstruction formula* holds:

$$f = \sum_{h,k} (f, \widetilde{g}_{h,k}) g_{h,k}, \tag{0.4}$$

for all $f \in L^2(\mathbb{R}^d)$, with unconditional convergence in $L^2(\mathbb{R}^d)$.

The literature about Gabor frames theory is so vast that we do not attempt to give an exhausting references list, but we just indicate for example the monographs [5], [8], [15], [20], and the references therein. A problem of great interest is to find conditions on the window $g(t)$ and on the parameters α, β in order that $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame. In the most part of the literature the conditions are only sufficient or necessary, characterizations are known only for few cases, see for instance [17] and [18] for some recent results of Gröchenig and Stöckler about totally positive

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functions. The general leading idea is that, for suitable fixed window g, if α and β are "sufficiently small" then the lattice $\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ is refined enough to yield a frame. See [16], [19] for general surveys and, among others, [2], [4], [7], [9], [24], [25], [27], [28], [29], [30], [35] for specific contributions.

For any Gabor system $\mathcal{G}(g, \alpha, \beta)$ a *Gabor operator* may be formally defined by

$$Sf = \sum_{h,k} (f, g_{h,k}) g_{h,k}, \quad f \in \mathcal{S}(\mathbb{R}^d). \tag{0.5}$$

It is well known that the Gabor system (0.2) is a frame in L^2 if and only if the related Gabor operator is invertible in L^2. In particular the invertibility of the Gabor operator may be proved by estimating its closeness to the identity. See for example [11] where such estimation is obtained with the aid of the spreading function, in the case of more general lattices $\Lambda = M\mathbb{Z}^{2d}$, generated by a $2^{2d} \times 2^{2d}$ non singular matrix.

The results we present in this paper go exactly in this direction, but the technique used is not quite typical of time-frequency analysis. The main idea is to estimate the closeness of the Gabor operator to the identity by means of the classical Calderón -Vaillancourt Theorem for L^2-boundedness of pseudo-differential operators. We work with lattices $\Lambda = M\mathbb{Z}^{2d}$, where M is a diagonal matrix, therefore slightly generalizing the case $\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$.

Namely our Gabor systems $\mathcal{G}(g, a, b)$ are defined, for g measurable function, $a = (a_1, \ldots, a_d)$, $b = (b_1, \ldots b_d)$ in $\mathbb{R}^d_+$, by

$$\mathcal{G}(g, a, b) = \left\{ g_{h,k}(t) = e^{2\pi i bk\cdot t} g(t - ah) \right\}_{h,k\in\mathbb{Z}^d}, \tag{0.6}$$

where

$$ah = \begin{pmatrix} a_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & a_d \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_d \end{pmatrix}; \; bk = \begin{pmatrix} b_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & b_d \end{pmatrix} \begin{pmatrix} k_1 \\ \vdots \\ k_d \end{pmatrix}. \tag{0.7}$$

The aim is to find sufficient conditions on the vectors a, b and the windows g such that the *Gabor operator* (0.5) is a well-defined, bounded and invertible operator on $L^2(\mathbb{R}^d)$.

From a different perspective we remark that frames theory, and in general time-frequency analysis, has already revealed deep and interesting connections in a number of topics essentially related to pseudo-differential operators which act, or admit symbols in modulation or Wiener amalgam spaces, see for reference [1], [3], [6], [21], [26], [32], [33], [34].

The paper is organized as follows. In Section 1 we state the main results and in Section 2 we fix the notations and give the necessary definitions and tools.
In Section 3 we introduce a suitable class of pseudo-differential operators with periodic symbol. By means of Calderón -Vaillancourt Theorem we prove their boundedness and invertibility in $L^2(\mathbb{R}^d)$.

Incidentally we remark that we have chosen the Kohn-Nirenberg quantization in order to apply a convenient version of the Calderón -Vaillancourt Theorem. In principle however other quantizations can also be used.

In Section 4 we write S as a pseudo-differential operator in the class described above and we prove that, for sufficiently small a, b, we have $\|Id - cS\|_{L^2} < 1$, which, by the von Neumann series, implies the invertibility of the operator S. This invertibility is equivalent to the fact that $\{g_{h,k}\}$ is a frame and furnishes furthermore the frame bounds. The conditions on a, b and c depend only on the regularity and the decay at infinity of the window g.

We remark that the sufficient conditions on the density of the frame lattice, as well as the frames bounds that we determine in this paper are by far not optimal.

The interest of the method relies instead on the fact that our results (Thm. 1.1, Cor. 1.2) are valid for a rather general class of windows in dimension d, whereas most of the more sharp results are obtained in one dimension and refer to specific windows. Moreover Theorem 3.7 about boundedness and invertibility of pseudo-differential operators with periodized symbols is interesting by itself in the framework of the pseudo-differential calculus.

We have chosen to remain in the L^2 setting since this is the natural context for the Calderón -Vaillancourt theorem and our methods are new even in this context. A reformulation in more general contexts, e.g. in the framework of Banach Gelfand triple setting, see [10], could be possible. In Section 5 we end the paper using the pseudo-differential form for the Gabor operator (0.5) to obtain a characterization for dual frames of $\{g_{h,k}\}$. Here we also point out the connections between our functional setting and the Feichtinger algebra M^1 (Lemma 5.1). To this regard see also [12] for results in the more general context of *weak duality*.

1. Statement of the main result

We give here the essential tools for understanding the main result. More notations and definitions will be detailed in the next section.

For any $a, b \in \mathbb{R}^d_+$ and $x, \xi \in \mathbb{R}^d$ we define:

- $\langle x\rangle = \sqrt{1+|x|^2}$;
- $x \cdot \xi = \sum_{j=1}^d x_j \xi_j$;
- $ax = (a_1x_1, \ldots, a_dx_d)$;
- $\Pi a_jb_j = \Pi_{j=1}^d a_jb_j$;
- $T_d = \{(\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d_+ \,;\, \text{such that}\, \alpha_j = 0 \text{ or } 1\}$.

For $f : \mathbb{R}^d \mapsto \mathbb{C}$ measurable function and $\varepsilon > 0$ we say that

$$f \in L^\infty_\varepsilon = L^\infty_\varepsilon(\mathbb{R}^d) \quad \text{if} \quad \|f\|_{L^\infty_\varepsilon} := \|f(\cdot)\langle\cdot\rangle^{d+\varepsilon}\|_{L^\infty} < \infty, \tag{1.1}$$

$$f \in \hat{L}^\infty_\varepsilon \quad \text{if} \quad \hat{f} \in L^\infty_\varepsilon, \tag{1.2}$$

where $\hat{f}$ is the Fourier transform of f. We write moreover $f \in \hat{C}^d$ if $\hat{f}$ belongs to the set of d times differentiable functions.

The main result (Theorem 1.1) will make use of the following constants, depending only on the dimension d and the positive parameter ε:

$$N_d = 3^{5d}(2\pi)^{\frac{5d}{2}}(d+1)^{2d}, \qquad N_{d,\varepsilon} = (2(d+1))^{2\varepsilon}\left(\int \langle x\rangle^{-d-\varepsilon}\,dx\right)^2; \tag{1.3}$$

$$M_d = 3^{5d+1}4^d(d+1)^{4d}(2\pi)^{\frac{5d+2}{2}}, \quad M_{d,\varepsilon} = (2(d+1))^{4\varepsilon}\left(\int \langle x\rangle^{-d-\varepsilon}\,dx\right)^2. \tag{1.4}$$

Theorem 1.1. *Consider $g \in C^{d+1} \cap \hat{C}^{d+1}$, such that, for any $\alpha, \beta \in T_d$, $j = 1, \ldots, d$, $x^\alpha\partial_x^\beta g$, $x^\alpha\partial_{x_j}\partial_x^\beta g$, $x_jx^\alpha\partial_x^\beta g$ belong to $L^\infty_\varepsilon(\mathbb{R}^d) \cap \hat{L}^\infty_\varepsilon(\mathbb{R}^d)$, for some $\varepsilon > 0$. Assume that $a, b \in (0,1]^d$ satisfy the condition*

$$\sum_{j=0}^d (a_j + b_j) < \frac{\|g\|^2_{L^2}}{M_dM_{d,\varepsilon}\mathcal{K}_g}, \tag{1.5}$$

with $\mathcal{K}_g$ suitable positive constant depending on the norm of g in L^∞_ε, $\hat{L}^\infty_\varepsilon$. Then the Gabor system $\mathcal{G}(g, a, b)$ is a Gabor Frame in $L^2(\mathbb{R}^d)$.

Moreover the frame bounds A, B in (0.3) *are*

$$A = \frac{\|g\|^2_{L^2} - M_d M_{d,\varepsilon} \mathcal{K}_g \sum_{j=1}^d (a_j + b_j)}{\Pi a_j b_j}; \tag{1.6}$$

$$B = N_d N_{d,\varepsilon} \max_{\alpha,\beta \in T_d} \{\|x^\alpha \partial_x^\beta g\|_{L^\infty_\varepsilon} \|\xi^\beta \partial_\xi^\alpha \hat{g}\|_{L^\infty_\varepsilon}\}. \tag{1.7}$$

Precisely the constant $\mathcal{K}_g$ is:

$$\mathcal{K}_g = \max_{\substack{\alpha,\beta\in T_d \\ j=1,\dots,d}} \left\{ \begin{array}{l} \|x^\alpha \partial_x^\beta g\|_{L^\infty_\varepsilon} \|\xi^\beta \partial_\xi^\alpha \hat{g}\|_{L^\infty_\varepsilon}; \\ \|x^\alpha \partial_x^\beta g\|_{L^\infty_\varepsilon} \|\xi_j \xi^\beta \partial_\xi^\alpha \hat{g}\|_{L^\infty_\varepsilon}; \|x^\alpha \partial_{x_j} \partial_x^\beta g\|_{L^\infty_\varepsilon} \|\xi^\beta \partial_\xi^\alpha \hat{g}\|_{L^\infty_\varepsilon}; \\ \|x_j x^\alpha \partial_x^\beta g\|_{L^\infty_\varepsilon} \|\xi^\beta \partial_\xi^\alpha \hat{g}\|_{L^\infty_\varepsilon}; \|x^\alpha \partial_x^\beta g\|_{L^\infty_\varepsilon} \|\xi^\beta \partial_{\xi_j} \partial_\xi^\alpha \hat{g}\|_{L^\infty_\varepsilon} \end{array} \right\}. \tag{1.8}$$

The proof is obtained combining properties of frame operators in Hilbert spaces and a result of invertibility of pseudo-differential operators with periodic symbols, obtained by means of a careful application of the Calderón -Vaillancourt Theorem. The details are given in the next Section 4, after the preparation material of Sections 2 and 3.

Corollary 1.2. *Consider $g \in C^{2d+2}$, such that for any $\alpha \in T_d$, $|\beta| \leq 2d+1$, $j = 1, \dots, d$, $x^\alpha \partial_x^\beta g$, $x^\alpha \partial_{x_j} \partial_x^\beta g$, $x_j x^\alpha \partial_x^\beta g$ belong to $L^\infty_\varepsilon(\mathbb{R}^d)$. Then, with the same assumption on $a, b \in [0,1)^d$ and the same involved constants, all the results in Theorem 1.1 are true.*

Example 1.3. Despite the apparently complicated hypotheses, we notice that the previous results apply to very simple types of window function, which do not belong to the Schwartz space. For example in one dimension the functions

$$g(x) = \frac{1}{p + qx^{4k}}, \qquad p, q > 0, \quad k = 1, 2, 3, ...$$

satisfy the hypothesis of Theorem 1.1.

Windows (essentially) of this type are of considerable interest; actually results of Janssen [23] furnish a complete characterization of the Gabor sampling set for windows of the type $(1 + ax^2)^{-1}$, $a > 0$.

Let us notice that in the particular case $a_j = \alpha > 0$, $b_j = \beta > 0$, $j = 1, .., d$, we have the bounded inclusion $L^\infty_\varepsilon(\mathbb{R}^d) \hookrightarrow W(\mathbb{R}^d)$, where $W(\mathbb{R}^d)$ is the Wiener function space (see [15], Def. 6.1.1). In this case our results overlap with results of Walnut [35], where however conditions on the sampling parameters α, β, and as consequence the frame bounds A, B, are given in different way.

Finally, due to the inequality $\|.\|_2 \leq C\|.\|_{L^\infty_\varepsilon}$, $C = (\int \langle x \rangle^{-2(d+\varepsilon)}\, dx)^{1/2}$, we can notice that the sufficient condition (1.5) implies the estimate $\sum_{j=0}^\infty (a_j + b_j) < C/M_d M_{d,\varepsilon}$, independently of g. We conclude that, as noticed in the Introduction, the bound in the right-hand side of (1.5) is far from being optimal.

2. Notations and Background

Given $f(x)$, $u(x,t)$ in the Schwartz classes of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^{2d})$ respectively, we define the Fourier transforms $\mathcal{F}f(\xi) = \hat{f}(\xi) = \int e^{-2\pi i x\omega} f(x)\, dx$, $\mathcal{F}_2 u(x,\xi) = \int e^{-2\pi i t\omega} u(x,t)\, dt$, togheter with the inverse transforms: $\mathcal{F}^{-1} f(\xi) = \check{f}(\xi) = \int e^{2\pi i x\omega} f(x)\, dx$, $\mathcal{F}_2^{-1} u(x,\xi) = \int e^{2\pi i t\omega} u(x,t)\, dt$. In the same way we denote their extension to $\mathcal{S}'(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^{2d})$.

Let us recall the following basic properties which will be useful in the sequel.

i) If $x^\alpha f(x) \in L^1(\mathbb{R}^d)$, $|\alpha| \leq N$, then $\hat{f} \in C^N(\mathbb{R}^d)$ and

$$\mathcal{F}\left((-2\pi i x)^\alpha f\right)(\xi) = \partial^\alpha (\mathcal{F}f)(\xi), \quad |\alpha| \leq N. \tag{2.1}$$

ii) If $f \in C^N(\mathbb{R}^d)$ and $\partial_x^\alpha f \in L^1(\mathbb{R}^d)$, $|\alpha| \le N$, then

$$\mathcal{F}(\partial^\alpha f)(\xi) = (2\pi i \xi)^\alpha \mathcal{F}f(\xi). \tag{2.2}$$

Time frequency shifts. For $y, \omega \in \mathbb{R}^d$ we define the operators:

$$T_y f(t) = f(t-y), \qquad \ddot{T}_y u(x,\xi) = u(x,\xi-y) \qquad \text{(translation)}; \tag{2.3}$$

$$M_\omega f(t) = e^{2\pi i \omega\cdot t} f(t), \qquad \ddot{M}_\omega u(x,\xi) = e^{2\pi i \omega\cdot \xi} u(x,\xi) \qquad \text{(modulation)}. \tag{2.4}$$

The next properties follow:

$$\mathcal{F}(T_y f) = M_{-y}\mathcal{F}f, \qquad \mathcal{F}^{-1}(T_y f) = M_y \mathcal{F}^{-1} f, \tag{2.5}$$

$$\mathcal{F}(M_\omega f) = T_\omega \mathcal{F}f, \qquad \mathcal{F}^{-1}(M_\omega f) = T_{-\omega}\mathcal{F}^{-1} f, \tag{2.6}$$

$$\mathcal{F}(M_\omega T_y f) = T_\omega M_{-y}\mathcal{F}f; \qquad \mathcal{F}^{-1}(T_y M_\omega f) = M_{-y} T_\omega \mathcal{F}^{-1} f. \tag{2.7}$$

Function spaces. Together with the definition of L^∞_ε and $\hat{L}^\infty_\varepsilon$, given in (1.1), (1.2), we define $L^\infty_{\varepsilon,\varepsilon}(\mathbb{R}^{2d})$ as the set of the measurable functions $p(x,\xi)$ on $\mathbb{R}^d_x \times \mathbb{R}^d_\xi$ such that

$$\|p\|_{L^\infty_{\varepsilon,\varepsilon}} := \|p(x,\xi)\langle x\rangle^{d+\varepsilon}\langle \xi\rangle^{d+\varepsilon}\|_{L^\infty(\mathbb{R}^{2d})} < \infty. \tag{2.8}$$

Notice that for any $\varepsilon > 0$, $L^\infty_\varepsilon(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$ and $L^\infty_{\varepsilon,\varepsilon}(\mathbb{R}^{2d}) \subset L^1(\mathbb{R}^{2d})$.
Since for some $C > 0$: $\langle x\rangle^{d+\varepsilon/2}\langle \xi\rangle^{d+\varepsilon/2} \le C\langle (x,\xi)\rangle^{2d+\varepsilon}$, we obtain for any $\varepsilon > 0$

$$L^\infty_\varepsilon(\mathbb{R}^{2d}) \subset L^\infty_{\frac{\varepsilon}{2},\frac{\varepsilon}{2}}(\mathbb{R}^{2d}). \tag{2.9}$$

We say moreover that the function $p(x,\xi)$ is (a,b)-periodic, $a, b \in \mathbb{R}^d$, if for any $m, n \in \mathbb{Z}^d$ we have $p(x,\xi) = p(x+ma, \xi+nb)$

Invertibility in Banach algebras. We will make use of the properties of the von Neumann series in Banach algebras of operators in the following version

Proposition 2.1. *Consider $x \in \mathcal{A}$, where $(\mathcal{A}, \|\cdot\|)$ is a Banach algebra on the field of complex numbers, with multiplicative identity e. If there exist $c \in \mathbb{C}\setminus\{0\}$ such $\|e - cx\| < 1$ then x is invertible in $\mathcal{A}$ and*

$$x^{-1} = c\sum_{n=0}^{\infty}(e-cx)^n. \tag{2.10}$$

Frames in Hilbert Spaces. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a separable Hilbert space H is a *frame* if there exist $A, B > 0$ such that $A\|x\|^2 \le \sum_n |(x,x_n)|^2 \le B\|x\|^2$ for $x \in H$.

Notice now that for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset H$ we can formally define the operator

$$S : x \to Sx = \sum_{n\in\mathbb{N}}(x,x_n)x_n, \quad x \in H. \tag{2.11}$$

The main result in Theorem 1.1 is based on the fact that $\{x_n\}_{n\in\mathbb{N}} \subset H$ is a frame if and only if S is a bijection of H in itself.

Although this result is well-known it is not easy to find it explicitly mentioned and proved in the literature, therefore we give next the details.

Lemma 2.2 (Heil [20])**.** *$\{x_n\}_{n\in\mathbb{N}}$ is a frame for H if and only if there exist $A, B > 0$ such that $AI \le S \le BI$.*

Proposition 2.3. *$\{x_n\}_{n\in\mathbb{N}}$ is a frame for H if and only of $S : H \to H$ is a bijection. Furthermore in this case S is bi-continuous.*

Proof. See [20] for the proof that if $\{x_n\}_{n\in\mathbb{N}}$ is a frame then S is a bijection.
On the converse, suppose $S : H \to H$ is a bijection. Notice that for every $N \in \mathbb{N}$ the operators $S_N : x \in H \to S_N x = \sum_{n=1}^{N} (x, x_n)x_n \in H$ are bounded, namely $\|S_N x\| \leq \sum_{n=1}^{N} |(x, x_n)|\|x_n\| \leq \|x\| \sum_{n=1}^{N} \|x_n\|^2$.

Clearly $\lim_{N\to\infty} S_N x = Sx$ for every $x \in H$. Then from the Banach-Steinhaus theorem follows that S is bounded. From the Banach inverse operator theorem also S^{-1} is bounded.

For every $x \in H$, $(Sx, x) = \sum_{n\in\mathbb{N}} |(x, x_n)|^2 \geq 0$, therefore $S \geq 0$.
As $S \geq 0$ and is invertible, the same holds for S^{-1}, namely for $y = Sx$ we have $(S^{-1}y, y) = (x, Sx) \geq 0$.
From the boundedness of S we have $(Sx, x) \leq \|Sx\|\|x\| \leq \|S\|\|x\|^2 = \|S\|(x, x)$, i.e. $S \leq \|S\|I$.
Analogously, as S^{-1} continuous, $(S^{-1}y, y) \leq \|S^{-1}y\|\|y\| \leq \|S\|\|y\|^2 = \|S\|(y, y)$, i.e. $S^{-1} \leq \|S^{-1}\|I$.
Multiplying this inequality by S we get $\frac{1}{\|S^{-1}\|} I \leq S$. It follows $\frac{1}{\|S^{-1}\|} I \leq S \leq \|S\|I$, and this means that $\{x_n\}_{n\in\mathbb{N}}$ is a frame by the previous Lemma. □

Multiple expansions. We prove here a technical result about multiple expansions.

For fixed $j \leq d$, consider a set of indices $J \subset \{1, \dots, d\}$, containing exactly j elements and set:

$$K_J =: \left\{ h \in \mathbb{Z}^d; \quad \begin{array}{ll} h_i \neq 0, & i \in J \\ h_i = 0, & i \notin J \end{array} \right\}; \tag{2.12}$$

$$\mathbb{Z}_0^m =: \{h \in \mathbb{Z}^m;\ h_j \neq 0, \text{ for any } j = 1, \dots, m\} \tag{2.13}$$

$$\mathbb{Z}_{0,+}^m =: \{h \in \mathbb{Z}^m;\ h_j > 0, \text{ for any } j = 1, \dots, m\} \tag{2.14}$$

Lemma 2.4. *For any $f : \mathbb{R}^d \mapsto \mathbb{C}$, $a \in \mathbb{R}_+^d$, $N > 0$, we have:*

$$\sum_{h\in\mathbb{Z}^d} f(h) = \sum_{j=0}^{d} \sum_{|J|=j} \sum_{h\in K_J} f(h); \tag{2.15}$$

$$\sum_{h\in\mathbb{Z}_0^m} \langle ah\rangle^{-N} = 2^m \sum_{h\in\mathbb{Z}_{0,+}^m} \langle ah\rangle^{-N}, \quad m = 1, ..., d; \tag{2.16}$$

$$\sum_{h\in\mathbb{Z}_{0,+}^m} \langle ah\rangle^{-N} \leq \frac{1}{\prod_i a_i} \int_{[0,\infty)^m} \langle x\rangle^{-N}\, dx, \quad m = 1, ..., d; \tag{2.17}$$

$$\sum_{a\in\mathbb{Z}^d} \langle ah\rangle^{-N} \leq 1 + M_{a,d} \int_{[0,\infty)^d} \langle x\rangle^{-N}\, dx, \quad M_{a,d} = \sum_{j=1}^{d} 2^j \sum_{|J|=j} \frac{1}{\prod_{i\in J} a_i}. \tag{2.18}$$

Proof. Notice that $|\{J : J \subset \{1, \dots d\}\}| = \binom{d}{j}$. We can prove (2.15) by induction on the dimension d. Namely:
For $d = 1$, $\sum_{h\in\mathbb{Z}} f(h) = \sum_{j=0}^{1} \sum_{n=1}^{\binom{1}{j}} \sum_{h\in K_J} f(h) = f(0) + \sum_{h\neq 0} f(h)$ is verified.

Assume now that (2.15) holds for $d-1$, then:

$$\sum_{h\in\mathbb{Z}^d} f(h) = \sum_{\tilde h\in\mathbb{Z}^{d-1}} f(\tilde h,0) + \sum_{h_d\neq 0}\sum_{\tilde h\in\mathbb{Z}^{d-1}} f(\tilde h,h_d) =$$

$$=\sum_{j=0}^{d-1}\sum_{\substack{J\subseteq\{1,\dots,d-1\}\\|J|=j}}\sum_{\tilde h\in K_J} f(\tilde h,0) + \sum_{h_d\neq 0}\sum_{j=0}^{d-1}\sum_{\substack{J=\{1,\dots,d-1\}\\|J|=j}}\sum_{\tilde h\in K_J} f(\tilde h,h_d) =$$

$$=\sum_{j=0}^{d}\sum_{\substack{J\subseteq\{1,\dots,d\}\\|J|=j,d\notin J}}\sum_{h\in K_J} f(h) + \sum_{j=1}^{d}\sum_{\substack{J=\{1,\dots,d\}\\|J|=j,d\in J}}\sum_{h\in K_J} f(h) =$$

$$=\sum_{j=0}^{d}\sum_{\substack{J\subseteq\{1,\dots,d\}\\|J|=j}}\sum_{h\in K_J} f(h)$$

(2.16) directly follows by induction on the index m, since it is trivial for $m=1$ and

$$\sum_{h\in\mathbb{Z}_0^m}\langle ah\rangle^{-N} = \sum_{h_m\neq 0} 2^{m-1}\sum_{h\in\mathbb{Z}_{0,+}^{m-1}}\langle ah, a_m h_m\rangle = 2^m\sum_{h\in\mathbb{Z}_{0,+}^m}\langle ah\rangle^{-N} \tag{2.19}$$

Thanks to a classic result about multiple expansions, see e.g. [14], we obtain that the expansion in (2.17) converges if and only if the multiple integral $\int_{[0,+\infty)^m}\langle ax\rangle^{-N}\,dx$ is convergent, moreover

$$\sum_{h\in\mathbb{Z}_{0,+}^m}\langle ah\rangle^{-N} \leq \int_{[0,+\infty)^m}\langle ax\rangle^{-N} \leq \sum_{h\in\mathbb{Z}_{+}^m}\langle ax\rangle^{-N}. \tag{2.20}$$

Using both (2.16) and (2.17), we obtain

$$\begin{aligned}\textstyle\sum_{h\in\mathbb{Z}^d}\langle ah\rangle^{-N} &= \textstyle\sum_{j=0}^d\sum_{|J|=j}\sum_{h\in K_J}\langle ah\rangle^{-N}\\ &\leq 1+\textstyle\sum_{j=1}^d 2^j\sum_{|J|=j}\frac{1}{\prod_{i\in J}a_i}\int_{\tilde x\in[0,\infty)^j}\langle\tilde x\rangle^{-N}\,d\tilde x\\ &\leq 1+\textstyle\int_{x\in[0,\infty)^d}\langle x\rangle^{-N}\,dx\sum_{j=1}^d 2^j\sum_{|J|=j}\frac{1}{\prod_{i\in J}a_i}.\end{aligned} \tag{2.21}$$

Setting now $M_{a,d}=\sum_{j=1}^d 2^j\sum_{|J|=j}\frac{1}{\prod_{i\in J}ai}$ the proof is concluded. □

Remark 2.5. *If $a_j\leq 1$, $j=1,\dots,d$, we can notice that $M_{a,d}\leq\frac{1}{\Pi a_i}\sum_{j=1}^d 2^j\binom{d}{j}=\frac{1}{\Pi a_j}(3^d-1)$, then $1+M_{a,d}\int_{[0,+\infty)^d}\langle x\rangle^{-d-\varepsilon}\,dx\leq\frac{1}{\Pi a_j}\left(\frac{3}{2}\right)^d\int\langle x\rangle^{-d-\varepsilon}\,dx$. We can then conclude that*

$$\sum_{h\in\mathbb{Z}^d}\langle ah\rangle^{-d-\varepsilon}\leq\frac{1}{\Pi a_j}\left(\frac{3}{2}\right)^d\int\langle x\rangle^{-d-\varepsilon}\,dx,\quad \textit{when}\quad a\in(0,1]^d,\ \varepsilon>0. \tag{2.22}$$

3. Pseudo-differential Operators

Recall that, for $a(x,\xi)\in\mathcal{S}'(\mathbb{R}^{2d})$, the Kohn-Nirenberg quantization of $a(x,\xi)$ is the pseudo-differential operator which defines the bounded map

$$\varphi\in\mathcal{S}(\mathbb{R}^d)\quad\longrightarrow\quad a(x,D)\varphi=\mathcal{F}_2^{-1}(a(x,\xi)\hat\varphi(\xi))\in\mathcal{S}'(\mathbb{R}^d). \tag{3.1}$$

We notice that

$$
\begin{aligned}
a(x,D)\varphi(x) &= \iint e^{2\pi i\xi\cdot(x-t)}a(x,\xi)\varphi(t)\,dt\,d\xi \\
&= \int \varphi(t)\int e^{-2\pi i(t-x)\cdot\xi}a(x,\xi)\,d\xi\,dt \\
&= \langle(\mathcal{F}_2(x,\cdot-x),\varphi\rangle = \langle\ddot{T}_x(\mathcal{F}_2a)(x,\cdot),\varphi\rangle,
\end{aligned} \tag{3.2}
$$

and the following kernel theorem holds.

Proposition 3.1. *For any $a\in\mathcal{S}'(\mathbb{R}^{2d})$ the pseudo-differential operator $a(x,D)$ can be expressed as kernel operator, precisely for any $\varphi\in\mathcal{S}$ we have:*

$$a(x,D)\varphi=\langle K(x,\cdot),\varphi\rangle, \quad \textit{where} \tag{3.3}$$

$$K(x,t)=\ddot{T}_x(\mathcal{F}_2a)(x,t) \quad \textit{or equivalently} \quad a(x,\xi)=\ddot{M}_{-x}(\mathcal{F}_2^{-1}K)(x,\xi). \tag{3.4}$$

Concerning the L^2 boundedness of pseudo-differential operators let us recall the Calderón -Vaillancourt Theorem in the version of Hwang [22, Theorem 2].

Theorem 3.2. *Let $a:\mathbb{R}^d\times\mathbb{R}^d\mapsto\mathbb{C}$ be a continuous function whose derivatives $\partial_x^\alpha\partial_x^\beta a$ satisfy the following condition:*

$$
\begin{aligned}
&\textit{there is a constant}\, C>0 \textit{ such that } \|\partial_x^\alpha\partial_\xi^\beta a\|_{L^\infty(\mathbb{R}^{2d})}\le C,\\
&\textit{where}\, \alpha=(\alpha_1,\dots,\alpha_d), \beta=(\beta_1,\dots,\beta_d)\in\mathbb{Z}_+^d\\
&\textit{with}\, \alpha_j=0 \textit{ or } 1,\ \ \beta_j=0 \textit{ or } 1.
\end{aligned} \tag{3.5}
$$

Then $a(x,D)$ is continuous from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$, with norm estimate $\|a(x,D)\|\le C_d|||a|||$, where C_d is a constant depending only on d and $|||a|||$ is the smallest C such that (3.5) *holds.*

Remark 3.3. Arguing carefully on the proof of Theorem 2 in [22], assuming that the conditions (3.5) are satisfied, we can write for any $u,v\in C_0^\infty(\mathbb{R}^d)$,

$$\langle a(x,D)u,v\rangle=\sum_{\alpha\in T_d}\sum_{\beta\le\alpha}(-1)^{|\alpha|}\left(\frac{1}{2\pi}\right)^{2d}\int\partial_x^{\alpha-\beta}b(x,\xi)g_\beta(x,\xi)h(x,\xi)\,dxd\xi, \tag{3.6}$$

where $\|g_\beta\|_{L^2}\le(2\pi)^{d/2}\pi^{d/2}\|u\|_{L^2}$; $\|h\|_{L^2}=(2\pi)^d\pi^{d/2}\|v\|_{L^2}$; $\|\partial_x^{\alpha-\beta}b\|_{L^\infty}\le 2^d|||a|||$. Thus

$$|\langle a(x,D)u,v\rangle|\le(2\pi)^{d/2}K_d|||a|||\|u\|_{L^2}\|v\|_{L^2}, \quad K_d=\sum_{\alpha\in T_d}\sum_{\beta\le\alpha}1. \tag{3.7}$$

It may be easily proved by induction on the dimension d, that $K_d=3^d$. Namely $K_1=\sum_{\alpha_1=0}^{1}\sum_{\beta_1=0}^{\alpha_1}1=3$; assuming now that $K_d=3^d$ we have $K_{d+1}=\sum_{\alpha\in T_{d+1}}\sum_{\beta\le\alpha}1=\sum_{\alpha_{d+1}=0}^{1}\sum_{\beta_{d+1}=0}^{\alpha_{d+1}}K_d=3^{d+1}$.

We can conclude that the constant in Theorem 3.2 is

$$C_d=3^d(2\pi)^{d/2}. \tag{3.8}$$

Lemma 3.4. *Consider the following periodization of $p(x,\xi)\in L^\infty_{\varepsilon,\varepsilon}(\mathbb{R}^{2d})$:*

$$\sum_{h,k\in\mathbb{Z}^d}p(x-ah;\xi-bk). \tag{3.9}$$

Then for any $a,b\in(0,1]^d$ we have:

i) *the sum in* (3.9) *is convergent in $L^\infty(\mathbb{R}^{2d})$, and it satisfies the estimate:*

$$\|\sum_{h,k\in\mathbb{Z}^d}p(x-ah,\xi-bk))\|_{L^\infty(\mathbb{R}^{2d})}\le v^2_{\varepsilon,d}\frac{1}{\Pi a_jb_j}\|p\|_{L^\infty_{\varepsilon,\varepsilon}} \tag{3.10}$$

with

$$\upsilon_{\varepsilon,d} = 3^d 2^\varepsilon (d+1)^{d+\varepsilon} \int \langle y\rangle^{-d-\varepsilon}\, dy. \tag{3.11}$$

ii) *If moreover $p(x,\xi)$ is a continuous function, then the sum in (3.9) is a continuous function of $x,\xi \in \mathbb{R}^{2d}$.*

Remark that i) and ii) are still valid for general $a,b \in \mathbb{R}^d_+$, but the constants $\upsilon_{\varepsilon,d}$ also depends on $\langle a\rangle^{d+\varepsilon}$, $\langle b\rangle^{d+\varepsilon}$ and on $M_{a,d}$ and $M_{b,d}$ defined in (2.18).

Proof. Since the expression in (3.9) is (a,b) periodic, for any $(x,\xi) \in \mathbb{R}^{2d}$ we can find $(\bar{x},\bar{\xi})$ in the thorus $\mathbb{T}^{2d}_{a,b} := \Pi_{j=1}^d ([0,a_j]\times[0,b_j])$, such that $\sum_{h,k\in\mathbb{Z}^d} p(x-ah,\xi-bk) = \sum_{h,k\in\mathbb{Z}^d} p(\bar{x}-ah,\bar{\xi}-bk)$. Considering moreover that $p \in L^\infty_{\varepsilon,\varepsilon}$, $a,b\in(0,1]^d$, using Peetre's inequality and estimate (2.22) we have for almost any $(x,\xi)\in\mathbb{R}^{2d}$:

$$\begin{aligned}
&\sum_{h,k\in\mathbb{R}^{2d}} |p(x-ah,\xi-bk)| = \sum_{h,k\in\mathbb{Z}^d} |p(\bar{x}-ah,\bar{\xi}-bk)| \le \\
&\le \sum_{h,k\in\mathbb{Z}^d} \operatorname{ess\,sup}_{x,\xi\in\mathbb{T}^{2d}_{a,b}} \big[|p(\bar{x}-ah,\bar{\xi}-bk)|\langle\bar{x}-ah\rangle^{d+\varepsilon}\langle\bar{\xi}-bk\rangle^{d+\varepsilon}\times \\
&\quad\times \langle\bar{x}-ah\rangle^{-d-\varepsilon}\langle\bar{\xi}-bk\rangle^{-d-\varepsilon}\big] \le \\
&\le 2^{2(d+\varepsilon)}\langle\bar{x}\rangle^{d+\varepsilon}\langle\bar{\xi}\rangle^{d+\varepsilon}\|p\|_{L^\infty_{\varepsilon,\varepsilon}} \sum_{h,k\in\mathbb{Z}^d}\langle ah\rangle^{-d-\varepsilon}\langle bk\rangle^{-d-\varepsilon} \le \\
&\le 2^{2(d+\varepsilon)}3^{2d}\langle a\rangle^{d+\varepsilon}\langle b\rangle^{d+\varepsilon}\frac{1}{\Pi a_j b_j}\frac{1}{2^{2d}}\left(\int\langle y\rangle^{-d-\varepsilon}\,dy\right)^2\|p\|_{L^\infty_{\varepsilon,\varepsilon}} \le \\
&\le 3^{2d}2^{2\varepsilon}(d+1)^{2(d+\varepsilon)}\left(\int\langle y\rangle^{-d-\varepsilon}\,dy\right)^2\frac{1}{\Pi a_j b_j}\|p\|_{L^\infty_{\varepsilon,\varepsilon}}.
\end{aligned} \tag{3.12}$$

When $p(x,\xi)$ is continuous, the L^∞ convergence implies the uniform convergence, thus the sum is continuous. □

In the following we consider, for $p\in L^\infty_{\varepsilon,\varepsilon}(\mathbb{R}^{2d})$, the symbol

$$\varsigma(x,\xi) = \sum_{h,k\in\mathbb{Z}^d} p(x-ah,\xi-bk), \quad a,b\in[0,1)^d. \tag{3.13}$$

Since the expansion in (3.9) is convergent in $\mathcal{S}'(\mathbb{R}^{2d})$, thanks to (3.10) we have, for any $\alpha,\beta\in\mathbb{Z}^d_+$:

$$\|\partial^\alpha_\xi\partial^\beta_x\varsigma(x,\xi)\|_{L^\infty} = \|\sum_{h,k\in\mathbb{Z}^d}\partial^\alpha_\xi\partial^\beta_x p(x-ah,\xi-bk)\|_{L^\infty} \le \frac{\upsilon_{\varepsilon,d}}{\Pi_{a_j b_j}}\|\partial^\alpha_\xi\partial^\beta_x p\|_{L^\infty_{\varepsilon,\varepsilon}}. \tag{3.14}$$

Applying now the Calderón -Vaillancourt Theorem, we can state the following:

Proposition 3.5. *Consider a continuous function $p(x,\xi)$ such that, for any $\alpha,\beta\in T_d$, we have $\partial^\alpha_\xi\partial^\beta_x p\in L^\infty_{\varepsilon,\varepsilon}(\mathbb{R}^{2d})$. Then the operator $\varsigma(x,D)$ is bounded on $L^2(\mathbb{R}^d)$ with norm estimate:*

$$\|\varsigma(x,D)\|_{\mathcal{L}(L^2)} \le \frac{3^d(2\pi)^{d/2}\upsilon^2_{\varepsilon,d}}{\Pi a_j b_j}\max_{\alpha,\beta\in T_d}\left\{\|\partial^\alpha_\xi\partial^\beta_x p\|_{L^\infty_{\varepsilon,\varepsilon}}\right\}. \tag{3.15}$$

Lemma 3.6. *Consider $p\in C^{N+1}(\mathbb{R}^{2d})$ and assume that $\partial^\alpha_\xi\partial^\beta_x p\in L^\infty_{\varepsilon,\varepsilon}$, for $|\alpha+\beta|\le N+1$. Then for any $a,b\in(0,1]^d$ we have*

i) *If* $N = 0$

$$\begin{aligned} &\| \int p(y,\eta)dy\,d\eta - \Pi a_j b_j \sum_{h,h\in\mathbb{Z}^d} p(x-ah,\xi-b_k)\|_{L^\infty(\mathbb{R}^{2d})} \leq \\ &\leq \kappa_{\varepsilon,d}^2 \sum_{j=1}^d a_j \|\partial_{x_j} p\|_{L^\infty_{\varepsilon,\varepsilon}} + b_j \|\partial_{\xi_j} p\|_{L^\infty_{\varepsilon,\varepsilon}}, \end{aligned} \tag{3.16}$$

where

$$\kappa_{\varepsilon,d} = 3^d 2^d 2^{2\varepsilon}(d+1)^{2(d+\varepsilon)} \int \langle x\rangle^{-d-\varepsilon}\,dx. \tag{3.17}$$

ii) *If* $N \geq 1$ *and* $0 < |\alpha+\beta| \leq N$

$$\begin{aligned} &\|\Pi a_j b_j \sum_{h,k\in\mathbb{Z}^d} \partial_\xi^\alpha \partial_x^\beta p(x-ah,\xi-bk)\|_{L^\infty(\mathbb{R}^{2d})} \leq \\ &\leq \kappa_{\varepsilon,d}^2 \sum_{j=1}^d a_j \|\partial_{x_k} \partial_\xi^\alpha \partial_x^\beta p\|_{L^\infty_{\varepsilon,\varepsilon}} + b_j \|\partial_{\xi_k} \partial_\xi^\alpha \partial_x^\beta p\|_{L^\infty_{\varepsilon,\varepsilon}}, \end{aligned} \tag{3.18}$$

with $\kappa_{\varepsilon,d}$ *as before.*

Proof. Since $\int p(y,\eta)dy\,d\eta$ is a constant and $\sum_{h,k\in\mathbb{Z}^d} p(x-ah,\xi-bk)$ is (a,b)-periodic, we can reduce the norm in the left-hand side of (3.16) to the essential supremum on the Thorus $\mathbb{T}^{2d}_{a,b} = \Pi_{j=1}^d [0,a_j]\times[0,b_j]$.

Consider now the intervals $I_{h,k} = \Pi_{j=1}^d [a_j h_j; a_j(h_j+1)) \times [b_j k_j; b_j(k_j+1))$. Notice that $\bigcup_{h,k\in\mathbb{Z}^d} I_{h,k} = \mathbb{R}^{2d}$, $I_{h,k}\cap I_{m,n} = \emptyset$ if $(h,k)\neq(m,n)$. By setting $\tilde{p}(y,\eta) = p(-y,-\eta)$ and observing that the measure of any interval $I_{h,k}$ is $|I_{h,k}| = \Pi a_j b_j$, by means of a suitable change of variable we obtain, for any $(x,\xi)\in\mathbb{T}^{2d}_{a,b}$:

$$\begin{aligned} &\left|\int p(y,\eta)dy\,d\eta - \Pi a_j b_j \sum_{h,h\in\mathbb{Z}^d} p(x-ah,\xi-bk)\right| = \\ &= \left|\int T_{x,\xi}\tilde{p}(y,\eta)dy\,d\eta - \sum_{h,h\in\mathbb{Z}^d} |I_{h,k}|\, T_{x,\xi}\tilde{p}(ah,bk)\right| = \\ &= \left|\sum_{h,h\in\mathbb{Z}^d} \int_{I_{h,k}} T_{x,\xi}\tilde{p}(y,\eta)dy\,d\eta - \sum_{h,h\in\mathbb{Z}^d} \int_{I_{h,k}} T_{x,\xi}\tilde{p}(ah,bk)\,dy\,d\eta\right| \leq \\ &\leq \sum_{h,k\in\mathbb{Z}^d} \int_{I_{h,k}} |T_{x,\xi}\tilde{p}(y,\eta) - T_{x,\xi}\tilde{p}(ah,bk)|\,dy\,d\eta \leq \\ &\leq \sum_{h,k\in\mathbb{Z}^d} |I_{h,k}| \sup_{(y,\eta)\in I_{h,k}} |T_{x,\xi}\tilde{p}(y,\eta) - T_{x,\xi}\tilde{p}(ah,bk)| \leq \\ &\leq \Pi a_j b_j \sum_{h,k\in\mathbb{Z}^d} \sup_{(y,\eta)\in I_{h,k}} |T_{x,\xi}\tilde{p}(y,\eta) - T_{x,\xi}\tilde{p}(ah,bk)|. \end{aligned} \tag{3.19}$$

Assuming that $p\in C^1(\mathbb{R}^{2d})$ and $\partial_\xi^\alpha \partial_x^\beta p \in L^\infty_{\varepsilon,\varepsilon}$ when $|\alpha+\beta|\leq 1$, the Taylor expansion with integral remainder and the Peetre's inequality give the following estimate for any $(y,\eta)\in I_{h,k}$ and $(h,k)\in\mathbb{Z}^d$:

$$\begin{aligned}
&|T_{x,\xi}\tilde p(y,\eta)-T_{x,\xi}\tilde p(ah,bk))| =\\
=\ & \textstyle\sum_{j=1}^d |y_j-a_jh_j|\int_0^1 |\partial_{y_j}T_{x,\xi}\tilde p(ah+t(y-ah);bk+t(\eta-bk))|\,dt+\\
+\ & |\eta_j-b_jk_j|\textstyle\int_0^1 |\partial_{\eta_j}T_{x,\xi}\tilde p(ah+t(y-ah);bk+t(\eta-bk))|\,dt\le\\
\le\ & \textstyle\sum_{j=1}^d a_j\int_0^1 |\partial_{y_j}p(x-ah-t(y-ah);\xi-bk-t(\eta-bk))|\,dt+\\
+\ & b_j\textstyle\int_0^1 |\partial_{\eta_j}p(x-ah-t(y-ah);\xi-bk-t(\eta-bk))|\,dt\le\\
\le\ & \textstyle\int_0^1\langle x-ah-t(y-ah)\rangle^{-d-\varepsilon}\langle\xi-bk-t(\eta-bk)\rangle^{-d-\varepsilon}\,dt\times\\
& \times\textstyle\sum_{j=1}^d a_j\|\partial_{y_j}p\|_{L^\infty_{\varepsilon,\varepsilon}}+b_j\|\partial_{\xi_j}p\|_{L^\infty_{\varepsilon,\varepsilon}}\le\\
\le\ & 4^{d+\varepsilon}\langle x-ah\rangle^{-d-\varepsilon}\langle\xi-bk\rangle^{-d-\varepsilon}\textstyle\int_0^1\langle t(y-ah)\rangle^{d+\varepsilon}\langle t(\eta-bk)\rangle^{d+\varepsilon}\,dt\times\\
& \times\textstyle\sum_{j=1} a_j\|\partial_{x_j}p\|_{L^\infty_{\varepsilon,\varepsilon}}+b_j\|\partial_{\xi_j}p\|_{L^\infty_{\varepsilon,\varepsilon}}\le\\
\le\ & (16\langle x\rangle\langle\xi\rangle\langle a\rangle\langle b\rangle)^{d+\varepsilon}\textstyle\sum_{j=1}^d a_j\|\partial_{x_j}p\|_{L^\infty_{\varepsilon,\varepsilon}}\|b_j\partial_{\xi_j}p\|_{L^\infty_{\varepsilon,\varepsilon}}\langle ah\rangle^{-d-\varepsilon}\langle bk\rangle^{-d-\varepsilon}.
\end{aligned}$$

Thus recalling that $a,b\in(0,1]^d$ and using (2.22), we obtain

$$\|\int p(y,\eta)dy\,d\eta-\Pi a_jb_j\sum_{h,h\in\mathbb{Z}^d}p(x-ah,\xi-b_k)\|_{L^\infty(\mathbb{R}^{2d})}$$

$$\le 2^{4(d+\varepsilon)}(d+1)^{4(d+\varepsilon)}\Pi a_jb_j\sum_{j=1}^d a_j\|\partial_{x_j}p\|_{L^\infty_{\varepsilon,\varepsilon}}+b_j\|\partial_{\xi_j}p\|_{L^\infty_{\varepsilon,\varepsilon}}\sum_{h,k\in\mathbb{Z}^d}\langle ah\rangle^{-d-\varepsilon}\langle bk\rangle^{-d-\varepsilon}$$

$$\le 2^{2d}2^{4\varepsilon}3^{2d}(d+1)^{4(d+\varepsilon)}\left(\int\langle x\rangle^{-d-\varepsilon}\,dx\right)^2\Pi a_jb_j\sum_{j=1}^d a_j\|\partial_{x_j}p\|_{L^\infty_{\varepsilon,\varepsilon}}+b_j\|\partial_{\xi_j}p\|_{L^\infty_{\varepsilon,\varepsilon}}.$$

By setting $\kappa_{\varepsilon,d}=3^d2^d2^\varepsilon(d+1)^{2(d+\varepsilon)}\int\langle x\rangle^{-d-\varepsilon}\,dx$, the proof of $i)$ is concluded.

In order to prove $ii)$ notice that there exists $j=1,\dots,d$ such that at least one among α_j or β_j is different from 0 (assume $a_j\neq 0$). Moreover $\partial_\xi^{\alpha_1\dots(\alpha_j-1)\dots\alpha_d}\partial_x^\beta p\in L^\infty_{\varepsilon,\varepsilon}$, that is it vanishes at infinity. Since $\partial_\xi^\alpha\partial_x^\beta p\in L^1(\mathbb{R}^{2d})$, applying the Fubini's Theorem we obtain

$$\begin{aligned}
&\textstyle\int\partial_\xi^\alpha\partial_x^\beta p(x,\xi)\,d\xi dx=\\
&\textstyle\int_{\mathbb{R}^d_x}dx\int_{\mathbb{R}^{d-1}_{\xi_{i\neq j}}}d\xi_{i\neq j}\int_{\mathbb{R}_{\xi_j}}\partial_{\xi_j}\partial^{\alpha_1\dots(\alpha_j-1)\dots\alpha_d}\partial_x^\beta p(x,\xi)\,d\xi_j=0.
\end{aligned}\tag{3.20}$$

The proof then follows observing that $\partial_\xi^\alpha\partial_x^\beta p$ satisfies the assumptions in $i)$. □

Theorem 3.7. *Consider $p(x,\xi)\in C^{2d+1}(\mathbb{R}^{2d})$ such that, $\int p(x,\xi)\,dx\,d\xi\neq 0$ and for any $\alpha,\beta\in T_d$, $j=1,\dots,d$, $\partial_\xi^\alpha\partial_x^\beta p$, $\partial_{x_j}\partial_\xi^\alpha\partial_x^\beta p$ and $\partial_{\xi_j}\partial_\xi^\alpha\partial_x^\beta p$ belong to $L^\infty_{\varepsilon,\varepsilon}(\mathbb{R}^{2d})$. Set*

$$\mathcal{C}=\max_{\substack{\alpha,\beta\in T^d\\ j=1,\dots,d}}\left\{\|\partial_{x_j}\partial_\xi^\alpha\partial_x^\beta p\|_{L^\infty_{\varepsilon,\varepsilon}};\|\partial_{\xi_j}\partial_\xi^\alpha\partial_x^\beta p\|_{L^\infty_{\varepsilon,\varepsilon}}\right\}.\tag{3.21}$$

Then, for any $a,b\in(0,1]^d$:

i) *The operator $I-\frac{\Pi a_jb_j}{\int p(x,\xi)dxd\xi}\varsigma(x,D)$ is continuous from $L^2(\mathbb{R}^d)$ to itself and its operator norm satisfies:*

$$\left\|I-\frac{\Pi a_jb_j}{\int p(x,\xi)dxd\xi}\varsigma(x,D)\right\|_{\mathcal{L}(L^2)}\le\frac{3^d(2\pi)^{d/2}\mathcal{C}\kappa_{\varepsilon,d}^2}{|\int p(x,\xi)dxd\xi|}\sum_{j=1}^d(a_j+b_j),\tag{3.22}$$

where $\kappa_{\varepsilon,d}$ defined in (3.17);

ii) *If moreover*

$$\sum_{j=1}^{d} a_j + b_j < \frac{\left|\int p(x,\xi)\,dxd\xi\right|}{3^d(2\pi)^{d/2}\mathcal{C}\kappa_{\varepsilon,d}^2}, \tag{3.23}$$

then the operator $\varsigma(x,D)$ *is invertible in* $\mathcal{L}(L^2(\mathbb{R}^d))$ *and we have:*

$$\varsigma(x,D)^{-1} = \frac{\Pi a_j b_j}{\int p(x,\xi)\,dxd\xi}\sum_{n=0}^{\infty}\left(I - \frac{\Pi a_j b_j}{\int p(x,\xi)\,dxd\xi}\varsigma(x,D)\right)^n; \tag{3.24}$$

$$\|\varsigma(x,D)^{-1}\|_{\mathcal{L}(L^2)} \leq \frac{\Pi a_j b_j}{\left|\int p(x,\xi)\,dxd\xi\right| - 3^{3d}(2\pi)^{d/2}\mathcal{C}\kappa_{\varepsilon,d}^2\sum_{j=1}^{d}(a_j+b_j)} \tag{3.25}$$

Proof. *i*) is a straightforward application of Lemma 3.6, Theorem 3.2, jointly with Remark 3.3.

Working now in the Banach algebra $\mathcal{L}(L^2)$, *ii*) easily follows from (3.22), (3.23), by setting $x = I - \frac{\Pi a_j b_j}{\int p(x,\xi)dxd\xi}\varsigma(x,D)$ in Proposition 2.1. The estimate (3.25) directly follows from the well known identity $\sum_{n=0}^{\infty} t^n = \frac{1}{1-t}$ when $|t| < 1$. □

4. Gabor Operators

For $a, b \in \mathbb{R}_+^d$, g measurable function, consider the Gabor systems $\mathcal{G}(g,a,b)$ and $G(\gamma,a,b)$.

Assume $g \in L_\varepsilon^\infty$, $\gamma \in \hat{L}_\varepsilon^\infty$ and $a, b \in \mathbb{R}_+^d$, then for any $\phi, \varphi \in \mathcal{S}(\mathbb{R}^d)$:

$$\begin{aligned} &|\langle g_{h,k},\varphi\rangle\langle\gamma_{h,k},\phi\rangle| = |\langle g_{h,k},\varphi\rangle\langle\widehat{\gamma_{h,k}},\hat{\phi}\rangle| = |\langle M_{bk}T_{ah}g,\varphi\rangle\langle T_{bk}M_{-ah}\hat{\gamma},\hat{\phi}\rangle| \leq \\ &\leq \|T_{ah}g\|_{L^\infty}\|\varphi\|_{L^1}\|T_{bk}\hat{\gamma}\|_{L^\infty}\|\phi\|_{L^1}. \end{aligned} \tag{4.1}$$

Directly from Lemma 3.4 we obtain that the expansion $\sum_{h,k\in\mathbb{Z}^d}(\varphi, g_{h,k})\gamma_{h,k}$ is convergent in $\mathcal{S}'(\mathbb{R}^d)$, for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

For any $g \in L_\varepsilon^\infty$, $\gamma \in \hat{L}_\varepsilon^\infty$ we can then define the operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$:

$$S_{g,\gamma}\varphi = \sum_{h,k\in\mathbb{Z}^d}(\varphi, g_{h,k})\gamma_{h,k}; \qquad \varphi \in \mathcal{S}(\mathbb{R}^d), \tag{4.2}$$

which is said *(generalized) Gabor operator*. For $g \in L_\varepsilon^\infty \cap \hat{L}_\varepsilon^\infty$, we write $S_g = S_{g,g}$.

The explicit form of the kernel of $S_{g,\gamma}$ is given, for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$, by

$$S_{g,\gamma}\varphi(x) = \langle K(x,\cdot),\varphi\rangle, \quad \text{with} \quad K(x,t) = \sum_{h,k\in\mathbb{Z}^d}\bar{g}_{hk}(t)\gamma_{hk}(x), \tag{4.3}$$

where the sum converges in $\mathcal{S}'(\mathbb{R}^{2d})$.

The next result shows how a Gabor operator may be expressed as pseudo-differential operator with suitable periodic symbol.

Theorem 4.1. *For* $\gamma \in L_\varepsilon^\infty(\mathbb{R}^d)$, $g \in \hat{L}_\varepsilon^\infty(\mathbb{R}^d)$ *we obtain* $S_{g,\gamma} = \sigma(x,D)$, *where:*

$$\sigma(x,\xi) = \sum_{h,k\in\mathbb{Z}^d} e^{-2\pi i(x-ah)\cdot(\xi-bk)}\gamma(x-ah)\bar{\hat{g}}(\xi-bk), \tag{4.4}$$

with convergence in $L^\infty(\mathbb{R}^{2d})$. *Moreover, when* $a, b \in (0,1]^d$, *its norm satisfies the following estimate:*

$$\|\sigma(x,\xi)\|_\infty \leq \frac{\upsilon_{\varepsilon,d}^2}{\Pi a_j b_j}\|\hat{g}\|_{L_\varepsilon^\infty}\|\gamma\|_{L_\varepsilon^\infty} \tag{4.5}$$

with $\upsilon_{\varepsilon,d}$ *defined in* (3.11).

Proof. Using (4.3) and (3.4) we have:

$$\begin{aligned}
&\sigma(x,\xi) = \ddot{M}_{-x}(\mathcal{F}_2{}^{-1}K)(x,\xi) = \ddot{M}_{-x}\mathcal{F}_2{}^{-1}\left(\sum_{h,k\in\mathbb{Z}^d}\gamma_{hk}(x)\bar{g}_{hk}(t)\right)(\xi) = \\
&\ddot{M}_{-x}\sum_{h,k\in\mathbb{Z}^d}\gamma_{hk}(x)(\mathcal{F}_2{}^{-1}\bar{g}_{hk})(\xi) = \ddot{M}_{-x}\sum_{h,k\in\mathbb{Z}^d}\gamma_{hk}\mathcal{F}_2{}^{-1}\left(M_{-bk}T_{ah}\bar{g}\right)(\xi) = \\
&\ddot{M}_{-x}\sum_{h,k\in\mathbb{Z}^d}\gamma_{hk}(x)T_{bk}M_{ah}\check{\bar{g}}(\xi) = \\
&e^{-2\pi ix\cdot\xi}\sum_{h,k\in\mathbb{Z}^d}e^{2\pi ikb\cdot x}\gamma(x-ah)T_{bk}M_{ah}\bar{\hat{g}}(\xi) = \\
&e^{-2\pi ix\cdot\xi}\sum_{h,k\in\mathbb{Z}^d}e^{2\pi ibk\cdot x}\gamma(x-ah)e^{2\pi iah\cdot(\xi-bk)}\bar{\hat{g}}(\xi-bk) = \\
&\sum_{h,k\in\mathbb{Z}^d}e^{-2\pi i(x-ah)\cdot(\xi-bk)}\gamma(x-ah)\bar{\hat{g}}(\xi-bk).
\end{aligned}$$

The convergence of the expansion, estimate (4.5) and the continuity of $\sigma(x,\xi)$, directly follow from the conditions on the functions g,γ and Lemma 3.4. □

Lemma 4.2. *Consider $u,v\in\mathcal{S}'(\mathbb{R}^d)$, set $a(x,\xi)=e^{-2\pi ix\cdot\xi}u(x)v(\xi)$, then for any $\alpha,\beta\in\mathbb{Z}^d_+$ we have*

$$\partial^\alpha_\xi\partial^\beta_x a(x,\xi) = \sum_{\substack{\lambda\le\alpha\\ \nu\le\beta}}\sum_{\substack{\mu\le\lambda\\ \mu\le\nu}} C^{\alpha,\beta}_{\lambda,\nu,\mu}(-2\pi i)^{|\lambda+\nu-\mu|}e^{-2\pi ix\cdot\xi}U_{\lambda,\nu,\mu}(x)V_{\lambda,\nu,\mu}(\xi) \tag{4.6}$$

$$\text{with}\quad U^\beta_{\lambda,\nu,\mu}(x) = x^{\lambda-\mu}\partial^{\beta-\nu}_x u(x),\quad V^\alpha_{\lambda,\nu,\mu}(\xi) = \xi^{\nu-\mu}\partial^{\alpha-\lambda}_\xi v(\xi) \tag{4.7}$$

$$\text{and}\quad C^{\alpha,\beta}_{\lambda,\nu,\mu} = \binom{\alpha}{\lambda}\binom{\beta}{\nu}\binom{\nu}{\mu}\binom{\lambda}{\mu}\mu!. \tag{4.8}$$

If moreover $\alpha,\beta\in T_d$, $j=1,\dots,d$ we obtain the estimate:

$$\|\partial^\alpha_\xi\partial^\beta_x a(x,\xi)\|_{L^\infty} \le 3^{|\alpha+\beta|}(2\pi)^{|\alpha+\beta|}\max_{\substack{\lambda,\mu\le\alpha\\ \nu,\eta\le\beta}}\left\{\|x^\lambda\partial^\nu_x u\|_{L^\infty}\|\xi^\eta\partial^\mu_\xi v\|_{L^\infty}\right\}. \tag{4.9}$$

$$\|\partial_{x_j}\partial^\alpha_\xi\partial^\beta_x a(x,\xi)\|_{L^\infty} \le 3^{|\alpha+\beta|+1}(2\pi)^{|\alpha+\beta|+1}\mathcal{M}^\beta_\alpha(u,v), \tag{4.10}$$

$$\|\partial_{\xi_j}\partial^\alpha_\xi\partial^\beta_x a(x,\xi)\|_{L^\infty} \le 3^{|\alpha+\beta|+1}(2\pi)^{|\alpha+\beta|+1}\mathcal{N}^\beta_\alpha(u,v), \tag{4.11}$$

with

$$\mathcal{M}^\beta_\alpha(u,v) = \max_{\substack{\lambda,\mu\le\alpha\\ \nu,\eta\le\beta\\ j=1,\dots,d}}\left\{\begin{array}{c}\|x^\lambda\partial^\nu_x u\|_{L^\infty}\|\xi^\eta\partial^\mu_\xi v\|_{L^\infty},\\ \|x^\lambda\partial^\nu_x u\|_{L^\infty}\|\xi_j\xi^\nu\partial^\mu_\xi v\|_{L^\infty},\\ \|x^\lambda\partial_{x_j}\partial^\nu_x u\|_{L^\infty}\|\xi^\eta\partial^\mu_\xi v\|_{L^\infty}\end{array}\right\}, \tag{4.12}$$

$$\mathcal{N}^\beta_\alpha(u,v) = \max_{\substack{\lambda,\mu\le\alpha\\ \nu,\eta\le\beta\\ j=1,\dots,d}}\left\{\begin{array}{c}\|x^\lambda\partial^\nu_x u\|_{L^\infty}\|\xi^\eta\partial^\mu_\xi v\|_{L^\infty},\\ \|x_jx^\lambda\partial^\nu_x u\|_{L^\infty}\|\xi^\nu\partial^\mu_\xi v\|_{L^\infty},\\ \|x^\lambda\partial^\nu_x u\|_{L^\infty}\|\xi^\eta\partial_{\xi_j}\partial^\mu_\xi v\|_{L^\infty}\end{array}\right\}. \tag{4.13}$$

Proof. By a straighforward application of the Leibnitz rule we have:

$$\begin{aligned}
\partial^\beta_x\partial^\alpha_\xi a(x,\xi) &= \partial^\beta_x u(x)\sum_{\lambda\le\alpha}\binom{\alpha}{\lambda}\partial^\alpha_\xi(-2\pi i)^{|\lambda|}x^\lambda e^{-2\pi ix\cdot\xi}\partial^{\alpha-\lambda}_\xi v(\xi) = \\
&= \sum_{\lambda\le\alpha}\binom{\alpha}{\lambda}(-2\pi i)^{|\lambda|}\partial^{\alpha-\lambda}v(\xi)\sum_{\nu\le\beta}\binom{\beta}{\nu}\partial^\nu_x\left(x^\lambda e^{-2\pi ix\cdot\xi}\right)\partial^{\beta-\nu}u(x) = \\
&= \sum_{\substack{\lambda\le\alpha\\ \nu\le\beta}}\sum_{\substack{\mu\le\lambda\\ \mu\le\nu}}\binom{\alpha}{\lambda}\binom{\beta}{\nu}\binom{\nu}{\mu}(-2\pi i)^{|\lambda+\nu-\mu|}\frac{\lambda!}{(\lambda-\mu)!}\times \\
&\qquad\times e^{-2\pi ix\cdot\xi}x^{\lambda-\mu}\xi^{\nu-\mu}\partial^{\alpha-\lambda}v(\xi)\partial^{\beta-\nu}u(x).
\end{aligned} \tag{4.14}$$

By noticing that $\frac{\lambda!}{(\lambda-\mu)!} = \binom{\lambda}{\mu}\mu!$, setting $U^{\alpha}_{\lambda,\nu,\mu}(x)$, $V^{\beta}_{\lambda,\nu,\mu}(\xi)$ and $C^{\alpha,\beta}_{\lambda,\nu,\mu}$ as in (4.7), (4.8), we obtain (4.6). Estimate (4.9) directly follows by observing that $\mu! = 1$, when $\alpha, \beta \in T^d$ and by means of a suitable application of Newton's binomial identity:

$$(x+y)^{\alpha} = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} x^{\alpha-\beta} y^{\beta}, \quad x, y \in \mathbb{R}^d, \quad \alpha \in \mathbb{Z}^d_+. \tag{4.15}$$

Setting $e_j = (0, \ldots, 0, \alpha_j = 1, 0, \ldots 0)$ we have:

$$\partial_{x_j} U_{\lambda,\nu,\mu}(x) = (\lambda_j - \mu_j) x^{\lambda-\mu-e_j} \partial_x^{\beta-\nu} u(x) + x^{\lambda-\mu} \partial_x^{\beta-\nu+e_j} u(x), \tag{4.16}$$

thus

$$\partial_{x_j} \partial_{\xi}^{\alpha} \partial_x^{\beta} a(x,\xi) = \sum_{\substack{\lambda\leq\alpha \\ \nu\leq\beta}} \sum_{\substack{\mu\leq\lambda \\ \mu\leq\nu}} C^{\alpha.\beta}_{\lambda,\nu,\mu} (2\pi i)^{|\lambda-\mu-\nu|} e^{-2\pi i x\cdot\xi} \triangle(x,\xi) \tag{4.17}$$

with

$$\begin{aligned} \triangle(x,\xi) = \quad & (2\pi i)\xi_j \xi^{\nu-\mu} \partial_{\xi}^{\alpha-\lambda} v(\xi) x^{\lambda-\mu} \partial_x^{\beta-\nu} u(x) + \\ & (\lambda_j - \mu_j) \xi^{\nu-\mu} \partial_{\xi}^{\alpha-\lambda} v(\xi) x^{\lambda-\mu-e_j} \partial_x^{\beta-\nu} u(x) + \\ & \xi^{\nu-\mu} \partial_{\xi}^{\alpha-\lambda} v(\xi) x^{\lambda-\mu} \partial_{x_j} \partial_x^{\beta-\nu} u(x). \end{aligned} \tag{4.18}$$

Assuming that $\alpha, \beta \in T^d$, observing that $\mu_j \leq \lambda_j \leq \alpha_j \leq 1$ and arguing as in proof of (4.9), we obtain (4.10). Simmetrically we can prove (4.11). □

Thanks to Theorem 4.1 the Gabor operator $S_{g,\gamma}$ is a pseudo-differential operator with symbol obtained as periodization of $p(x,\xi) = e^{-2\pi i x\xi} \gamma(x) \bar{\hat{g}}(\xi)$. Using then Theorem 3.2, Theorem 3.7, Lemma 4.2 and observing that $\int p(x,\xi)\, dx\, d\xi = (\gamma, g)$, we obtain the following invertibility result.

Theorem 4.3. *Consider the functions* $\gamma \in C^{d+1}$, $g \in \hat{C}^{d+1}$ *such that, for any* $\alpha, \beta \in T_d$ *and* $j = 1, \ldots, d$, $x^{\alpha}\partial_x^{\beta}\gamma$, $x^{\alpha}\partial_{x_j}\partial_x^{\beta}\gamma$, $x_j x^{\alpha}\partial_x^{\beta}\gamma$ *belong to* $L^{\infty}_{\varepsilon}(\mathbb{R}^d)$ *and* $x^{\alpha}\partial_x^{\beta} g$, $x^{\alpha}\partial_{x_j}\partial_x^{\beta} g$, $x_j x^{\alpha}\partial_x^{\beta} g$ *belong to* $\hat{L}^{\infty}_{\varepsilon}(\mathbb{R}^d)$.
set:

$$\mathcal{K}_{g,\gamma} = \max_{\substack{\alpha,\beta\in T_d \\ j=1,\ldots,d}} \left\{ \begin{array}{lcl} \|x^{\alpha}\partial_x^{\beta}\gamma\|_{L^{\infty}_{\varepsilon}} \|\xi^{\beta}\partial_{\xi}^{\alpha}\hat{g}\|_{L^{\infty}_{\varepsilon}} & ; & \|x^{\alpha}\partial_x^{\beta}\gamma\|_{L^{\infty}_{\varepsilon}} \|\xi_j\xi^{\beta}\partial_{\xi}^{\alpha}\hat{g}\|_{L^{\infty}_{\varepsilon}}; \\ \|x^{\alpha}\partial_{x_j}\partial_x^{\beta}\gamma\|_{L^{\infty}_{\varepsilon}} \|\xi^{\beta}\partial_{\xi}^{\alpha}\hat{g}\|_{L^{\infty}_{\varepsilon}} & ; & \|x_j x^{\alpha}\partial_x^{\beta}\gamma\|_{L^{\infty}_{\varepsilon}} \|\xi^{\beta}\partial_{\xi}^{\alpha}\hat{g}\|_{L^{\infty}_{\varepsilon}}; \\ \|x^{\alpha}\partial_x^{\beta}\gamma\|_{L^{\infty}_{\varepsilon}} \|\xi^{\beta}\partial_{\xi_j}\partial_{\xi}^{\alpha}\hat{g}\|_{L^{\infty}_{\varepsilon}} & & \end{array} \right\},$$

then for some positive constants M_d, $M_{d,\varepsilon}$, N_d, $N_{d,\varepsilon}$ *we have*

i) *For any* $a, b \in (0,1]^d$ *the operator* $S_{g,\gamma}$ *is* L^2 *bounded and moreover*

$$\|S_{g,\gamma}\|_{\mathcal{L}(L^2)} \leq N_d N_{d,\varepsilon} \max_{\alpha,\beta\in T_d} \{\|x^{\alpha}\partial_x^{\beta} g\|_{L^{\infty}_{\varepsilon}} \|\xi^{\beta}\partial_{\xi}^{\alpha}\hat{g}\|_{L^{\infty}_{\varepsilon}}\}, \tag{4.19}$$

$$\|I - \frac{\Pi a_j b_j}{(\gamma,g)} S_{g,\gamma}\|_{\mathcal{L}(L^2)} \leq M_d M_{d,\varepsilon} \frac{\mathcal{K}_{g,\gamma}}{|(\gamma,g)|} \sum_{j=1}^{d} (a_j + b_j)\,; \tag{4.20}$$

ii) *If moreover* $\sum_{j=1}^{d}(a_j+b_j) < \frac{|(\gamma,g)|}{C_d C_{d,\varepsilon}\mathcal{K}_{g,\gamma}}$, *then the operator* $S_{g,\gamma}$ *is invertible in* $\mathcal{L}(L^2(\mathbb{R}^d))$, *more precisely*

$$S^{-1}_{g,\gamma} = \frac{\Pi a_j b_j}{|(\gamma,g)|} \sum_{n=0}^{\infty} \left(I - \frac{\Pi a_j b_j}{|(\gamma,g)|} S_{g,\gamma} \right)^n, \tag{4.21}$$

and

$$\|S^{-1}_{g,\gamma}\|_{\mathcal{L}(L^2)} \leq \frac{\Pi a_j b_j}{|(\gamma,g)| - M_d M_{d,\varepsilon}\mathcal{K}_{g,\gamma} \sum_{j=1}^{d}(a_j+b_j)}. \tag{4.22}$$

The constants M_d, $M_{d,\varepsilon}$, N_d, $N_{d,\varepsilon}$ depend only on d and ε, precisely:

$$N_d = 3^{5d}(2\pi)^{\frac{5d}{2}}(d+1)^{2d}, \qquad N_{d,\varepsilon} = (2(d+1))^{2\varepsilon}\left(\int \langle x\rangle^{-d-\varepsilon}\,dx\right)^2; \tag{4.23}$$

$$M_d = 3^{5d+1}4^d(d+1)^{4d}(2\pi)^{\frac{5d+2}{2}}, \quad M_{d,\varepsilon} = (2(d+1))^{4\varepsilon}\left(\int \langle x\rangle^{-d-\varepsilon}\,dx\right)^2. \tag{4.24}$$

Now applying Proposition 2.3 we obtain the main result of the paper: Theorem 1.1.

Proof of Corollary 1.2

Proof. In order to prove Corollary 1.2, let us notice that in general $\xi^\beta\partial_\xi^\alpha\hat{g}(\xi) \in L^\infty_\varepsilon$ if $(\langle\xi\rangle^{d+1+|\beta|})\widehat{x^\alpha g}(\xi) \in L^\infty$. Thanks to (2.1), (2.2), the last statement is verified if $x^\alpha g \in C^{d+1+|\beta|}$ and, for any $|\gamma| \le d+1+|\beta|$, $\partial_x^\gamma(x^\alpha g(x)) = \sum_{\eta\le\gamma} c_\eta x^{\alpha-\eta}\partial^{\gamma-\eta}g(x)$ belongs to $L^\infty_\varepsilon \subset L^1$. Remembering we need that $\xi^\beta\partial_\xi^\alpha\hat{g}$, $\xi_j\xi^\beta\partial_\xi^\alpha\hat{g}$ and $\xi^\beta\partial_{\xi_j}\partial_\xi^\alpha\hat{g}$ belong to L^∞_ε, when $\alpha, \beta \in T_d$, with straightforward calculation we obtain the results in Corollary 1.2 with the assumptions: $g \in C^{2d+2}$, $x^\alpha\partial^\beta g$, $x^\alpha\partial_{x_j}\partial_x^\beta g$, $x_j x^\alpha\partial_x^\beta g$ belong to L^∞_ε, for any $\alpha \in T_d$, $|\beta| \le 2d+1$, $j = 1,\ldots,d$. □

5. A characterization of dual Gabor windows

By means of the pseudo-differential calculus, we characterize in this section when two windows in the modulation space $g, \gamma \in M^1$ generate dual Gabor frames in terms of their τ-Wigner transform (for the definition of M^1 and more general modulation spaces see for example [15], Ch. 11). As Lemma 5.1 will show, the condition $g, \gamma \in M^1$ is a generalization, for suitably large ε, of the condition $\gamma, g \in L^\infty_\varepsilon(\mathbb{R}^d) \cap \hat{L}^\infty_\varepsilon(\mathbb{R}^d)$ which was considered in the paragraphs before.

We start by recalling some facts about the τ-Wigner peudo-differential calculus, see [31]

For $\tau \in [0,1]$ define the *torsion* operator $T_\tau : \Phi(x,y) \longrightarrow \Phi(x+\tau y, x-(1-\tau)y)$. The τ-Wigner transform of $f, g \in L^2(\mathbb{R}^d)$ is

$$Wig_\tau(f,g)(x,\omega) = \int_{\mathbb{R}^d} e^{-2\pi i t\cdot\omega} f(x+\tau t)\overline{g(x-(1-\tau)t)}\,dt = \mathcal{F}_2[T_\tau(f\otimes\overline{g})], \tag{5.1}$$

whereas the τ-Weyl operator with symbol $b_\tau(x,\omega)$ is

$$W_\tau^{b_\tau} : f \longrightarrow W_\tau^{b_\tau} f(x) = \int_{\mathbb{R}^{2d}} e^{-2\pi i(x-y)\cdot\xi} b_\tau((1-\tau)x+\tau y, \xi) f(y)\,dyd\xi.$$

In our result will be crucial the fact that the map $b_\tau \longrightarrow W_\tau^{b_\tau}$ is injective.

The τ-Wigner transforms and τ-Weyl operators are related by the formula

$$(W_\tau^{b_\tau} f, g) = (b_\tau, Wig_\tau(g,f))$$

The connection between the τ-symbol b_τ of the operator $W_\tau^{b_\tau}$ and its Schwartz kernel K_τ is given by:

$$\begin{aligned} K_\tau(x,y) &= \mathcal{F}_{\xi\to x-y}[b_\tau((1-\tau)x+\tau y, \xi)] \\ b_\tau(v,\xi) &= \mathcal{F}_{w\to\xi}[K_\tau(v+\tau w, v-(1-\tau)w)] = \mathcal{F}_2[T_\tau(K_\tau)] \end{aligned} \tag{5.2}$$

We remark that when $K_\tau = f\otimes\overline{g}$, we have

$$b_\tau = Wig_\tau(f,g).$$

Let $a, b \in \mathbb{R}^d_+$, for a function $f(x,\xi)$ defined on $\mathbb{R}^{2d}$ we define the periodization

$$P_{a,b}f(x,\xi) = \sum_{h,k\in\mathbb{Z}} T_{(ah,bk)}f(x,\xi) = \sum_{h,k\in\mathbb{Z}} f(x-ah, \xi-bk).$$

As proved in section 0.6, given two window functions g, γ, the Gabor operator formally defined as

$$S_{g,\gamma} : \phi(x) \longrightarrow S_{g,\gamma}\phi(x) = \sum_{h,k} (\phi, g_{h,k})\gamma_{h,k}(x),$$

has Schwartz kernel

$$K(x,y) = \sum_{h,k} \gamma_{h,k}(x)\overline{g_{h,k}(y)}. \tag{5.3}$$

where the sum converges in $\mathcal{S}'(\mathbb{R}^{2d})$.

Lemma 5.1.
a) If $\gamma, g \in L^\infty_\varepsilon(\mathbb{R}^d) \cap \hat{L}^\infty_\varepsilon(\mathbb{R}^d)$, with $\varepsilon > d/2$, then $\gamma, g \in M^1$.
b) If $\gamma, g \in M^1$, then $P_{a,b}Wig_\tau(\gamma, g) \in L^\infty(\mathbb{R}^{2d})$, for any $a, b > 0$.

Proof. a) For $\varepsilon' > 0$ consider the spaces $L^2_{\varepsilon'}(\mathbb{R}^d)$ which consists of the measurable functions f on $\mathbb{R}^d$ for which the norm $\|f\|_{L^2_{\varepsilon'}} = (\int_{\mathbb{R}^d} |f(x)|^2 \langle x\rangle^{2(d+\varepsilon')}\, dx)^{1/2}$ is finite. Choosing $0 < \varepsilon' < \varepsilon - d/2$, by an easy calculation we obtain $\|.\|_{L^2_{\varepsilon'}} \leq \|.\|_{L^\infty_\varepsilon} \left(\int \langle x\rangle^{2(\varepsilon'-\varepsilon)}\, dx\right)^{1/2}$ and therefore $L^\infty_\varepsilon(\mathbb{R}^d) \hookrightarrow L^2_{\varepsilon'}(\mathbb{R}^d)$. The conclusion follows now from the well-known fact that $L^2_{\varepsilon'}(\mathbb{R}^d) \cap \hat{L}^2_{\varepsilon'}(\mathbb{R}^d) \hookrightarrow M^1$ (see e.g. [15], Prop. 12.1.6).

b) If $\gamma, g \in M^1$, then $Wig_\tau(\gamma, g) \in W(\mathbb{R}^{2d})$, ([15], Prop. 12.1.11), (which is actually a characterization of M^1), and this in turn implies that the periodization $P_{a,b}Wig_\tau(\gamma, g)$ is in $L^\infty(\mathbb{R}^{2d})$ ([15], Lemma 6.1.2). □

Proposition 5.2. *Consider $\gamma, g \in M^1$, then the τ-symbol of the operator $S_{g,\gamma}$ belong to $L^\infty(\mathbb{R}^d)$ and has the following expression*

$$b_\tau(x,\xi) = P_{a,b}Wig_\tau(\gamma, g)(x,\xi).$$

Proof. We notice at first that, from Lemma 5.1 b), the periodization $P_{a,b}Wig_\tau(\gamma, g)$ is in $L^\infty(\mathbb{R}^{2d})$. We show now the *covariance property* of the $\tau-$Wigner transform. For $g, \gamma \in L^2(\mathbb{R}^d)$, we have

$$\begin{aligned} Wig_\tau(\gamma_{h,k}, g_{h,k})(x,\xi) &= Wig_\tau(M_k T_h\gamma, M_k T_h g)(x,\xi) \\ &= Wig_\tau(T_h\gamma, T_h g)(x, \xi - \tau bk - (1-\tau)bk)) \\ &= Wig_\tau(\gamma, g)(x - (1-\tau)ah - \tau ah, \xi - bk) \\ &= Wig_\tau(\gamma, g)(x - ah, \xi - bk). \end{aligned}$$

Therefore $Wig_\tau(\gamma_{h,k}, g_{h,k})(x,\xi) = Wig_\tau(\gamma, g)(x - ah, \xi - bk)$ for every $\tau \in [0,1]$.

By (5.3) the operator $S_{\gamma,g}$ has Schwartz kernel $K(x,y) = \sum_{h,k} \gamma_{h,k}(x)\overline{g_{h,k}(y)}$ and by (5.2) its τ-symbol is

$$\begin{aligned} b_\tau(x,\xi) &= \mathcal{F}_\in[T_\tau \textstyle\sum_{h,k} \gamma_{h,k}(x)\overline{g_{h,k}(y)}] \\ &= \textstyle\sum_{h,k} \mathcal{F}_\in[T_\tau(\gamma_{h,k}(x)\overline{g_{h,k}(y)})] \\ &= \textstyle\sum_{h,k} Wig_\tau(\gamma_{h,k}, g_{h,k})(x,\xi) \\ &= \textstyle\sum_{h,k} Wig_\tau(\gamma, g)(x - xh, \xi - b\xi) \\ &= P_{a,b}Wig_\tau(\gamma, g)(x,\xi). \end{aligned}$$

□

Proposition 5.3. *Let $\gamma \in M^1$ and suppose that $\{\gamma_{h,k}\}$ is a Gabor frame in $L^2(\mathbb{R}^d)$, then for a Gabor system $\{g_{h,k}\}$ with $g \in M^1$ the following are equivalent:*

(*a*) *$\{g_{h,k}\}$ is a dual frame of $\{\gamma_{h,k}\}$,*
(*b*) *$P_{a,b}Wig_\tau(\gamma, g) \equiv 1$ idendically on $\mathbb{R}^{2d}$.*

Proof. $\{g_{h,k}\}$ is a dual frame of $\{\gamma_{h,k}\}$ if and only if the operator $S_{g,\gamma} = Id$. From the injectivity of the correspondence between τ-symbols and operators and by Proposition 5.2, this happens if and only if $P_{a,b}Wig_\tau(\gamma, g) \equiv 1$ idendically on $\mathbb{R}^{2d}$. □

The property above can be reformulated using more general representations in the *Cohen class*, which is the set of time-frequency representations defined as quadratic forms of the type

$$f(x) \quad \longrightarrow \quad Q_\sigma(f)(x,\xi) = (\sigma * Wig(f))(x,\omega). \tag{5.4}$$

Here $\sigma(x,\xi)$ is the so-colled *Cohen kernel* (note that $Wig(f)$ is obtained for $\sigma = \delta$, the Dirac delta), see e.g. [3], [5], [15].

Clearly $f(x)$ and $\sigma(x,\xi)$ must be chosen in functional (or distributional) spaces such that the convolution in the *time-frequency space* $\mathbb{R}^d_x \times \mathbb{R}^d_\xi$ appearing in (5.4) makes sense.

Proposition 5.4. *In the same hypothesis of Proposition 5.3, suppose* $\sigma \in \mathcal{E}'(\mathbb{R}^{2d}) \cup L^1(\mathbb{R}^{2d})$ *and* $\mathcal{F}^{-1}(1/\widehat{\sigma}) \in \mathcal{E}'(\mathbb{R}^{2d}) \cup L^1(\mathbb{R}^{2d})$*, where* $\mathcal{E}'(\mathbb{R}^{2d})$ *is the space of compactly supported distributions on* $\mathbb{R}^{2d}$*. Then the following are equivalent:*

(*a*) $\{g_{h,k}\}$ *is a dual frame of* $\{\gamma_{h,k}\}$ *modulo a multiplicative constant,*
(*b*) $P_{a,b}Q_\sigma(\gamma, g)$ *is constant on* $\mathbb{R}^{2d}$.

Proof. Let $g_{h,k}$ be multiple of a dual frame of $\gamma_{h,k}$, then by Proposition 5.3 with $\tau = 1/2$ we have that $P_{a,b}Wig(\gamma, g)$ is constant. As $Q_\sigma = \sigma * Wig(\gamma, g)$ we have

$$\begin{aligned} P_{a,b}Q_\sigma(\gamma,g) &= \textstyle\sum_{h,k} T_{(ah,bk)}\big(\sigma * Wig(\gamma,g)\big) \\ &= \textstyle\sum_{h,k} \sigma * \big(T_{(ah,bk)} Wig(\gamma,g)\big) \\ &= \sigma * \textstyle\sum_{h,k} T_{(ah,bk)} Wig(\gamma,g) \\ &= \sigma * P_{a,b}Wig(\gamma,g) \\ &= c \end{aligned} \tag{5.5}$$

with $c = \int_{\mathbb{R}^{2d}} \sigma(x,\omega)\, dx d\omega$. We remark that the convolutions above make sense either considering $\sigma \in \mathcal{E}'(\mathbb{R}^{2d})$ and the constant $P_{a,b}Wig(\gamma, g) \in \mathcal{E}(\mathbb{R}^{2d})$ or $\sigma \in L^1(\mathbb{R}^{2d})$ and $P_{a,b}Wig(\gamma, g) \in L^\infty(\mathbb{R}^{2d})$.

Viceversa suppose that (b) holds and consider the Weyl symbol $b_{1/2}(x,\xi) = P_{a,b}Wig(\gamma, g)(x,\xi)$ of the Gabor operator. We have from (5.5)

$$c = \sigma * b_{1/2}$$

i.e. $b_{1/2} = \mathcal{F}^{-1}(\frac{c\delta}{\widehat{\sigma}}) = c\mathcal{F}^{-1}(\delta) * \mathcal{F}^{-1}(1/\widehat{\sigma}) = c * \mathcal{F}^{-1}(1/\widehat{\sigma}) = c'$, which makes sense as $\mathcal{F}^{-1}(1/\widehat{\sigma}) \in \mathcal{E}'(\mathbb{R}^{2d}) \cup L^1(\mathbb{R}^{2d})$. This means that the Gabor operator is a (multiple) of the identity and therefore (a) holds.

□

A simple example of Cohen kernel satisfying the hypothesis of the previous proposition is the function $\sigma(x,\xi) = e^{-\alpha|x| - \beta|\xi|}$ with $\alpha, \beta > 0$. Actually

$$\widehat{\sigma}(\omega, y) = \frac{4\alpha\beta}{(\alpha^2 + 4\pi^2\omega^2)(\beta^2 + 4\pi^2 y^2)}$$

and

$$\begin{aligned} \mathcal{F}^{-1}\big(1/\widehat{\sigma}\big)(x,\xi) &= \mathcal{F}^{-1}\left(\left(\tfrac{\alpha}{2} + \tfrac{4\pi^2\omega^2}{2a}\right)\left(\tfrac{\beta}{2} + \tfrac{2\pi^2 y^2}{a}\right)\right)(x,\xi) \\ &= \left(\tfrac{\alpha}{2}\delta_x - \tfrac{1}{2\alpha}\delta''_x\right)\left(\tfrac{\beta}{2}\delta_\xi - \tfrac{1}{2\beta}\delta''_\xi\right). \end{aligned}$$

Therefore $\mathcal{F}^{-1}\big(1/\widehat{\sigma}\big)$ belongs to $\mathcal{E}'(\mathbb{R}^2)$ and we have

$$c * \mathcal{F}^{-1}\big(1/\widehat{\sigma}\big) = \text{constant}.$$

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