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Abstract

In this paper we characterize global regularity in the sense of Shubin of twisted partial differential operators of second order in dimension 2. These operators form a class containing the twisted Laplacian, and in bi-unique correspondence with second order ordinary differential operators with polynomial coefficients and symbol of degree 2. This correspondence is established by a transformation of Wigner type. In this way the global regularity of twisted partial differential operators turns out to be equivalent to global regularity and injectivity of the corresponding ordinary differential operators, which can be completely characterized in terms of the asymptotic behavior of the Weyl symbol. In conclusion we observe that we have obtained a new class of globally regular partial differential operators which is disjoint from the class of hypo-elliptic operators in the sense of Shubin.

Keywords: Global regularity, twisted operators, non hypo-elliptic operators.

Mathematics subject classification: 35B40, 34E05, 42A38.

1 Introduction

In this paper we deal with the problem of global regularity for non hypo-elliptic partial differential operators with polynomial coefficients. An operator $A : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is globally regular if

$$u \in \mathcal{S}(\mathbb{R}^n) \quad \text{whenever} \quad Au \in \mathcal{S}(\mathbb{R}^n). \tag{1}$$

It is well known that hypo-elliptic partial differential operators in the sense of Definition 25.2 of [14] are globally regular. On the other hand, the problem of finding necessary and sufficient conditions for the global regularity of a differential operator with polynomial coefficients is still open. In the case of ordinary differential equations, in [13] necessary and sufficient conditions for global regularity are found under additional hypotheses. For partial differential equations the problem is much more complicated.

In this paper we study twisted differential operators of second order in $\mathbb{R}^2$, that is, partial differential operators of the kind

$$A = \sum_{j+k\leqslant 2} (-1)^{j+k} a_{kj} (\alpha D_y - \beta M_x)^j (\gamma D_x - \delta M_y)^k \tag{2}$$

with complex coefficients a_{kj}, where $D_x = -\mathrm{i}\partial_x$, $D_y = -\mathrm{i}\partial_y$, M_x and M_y are the multiplication operators by the corresponding variables x and y, and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ are such that

$$\alpha\delta - \beta\gamma = 1 \quad \text{and} \quad \beta\delta \neq 0. \tag{3}$$

An important example is the twisted Laplacian

$$L = \left(D_x + \frac{1}{2}M_y\right)^2 + \left(D_y - \frac{1}{2}M_x\right)^2, \tag{4}$$

that can be viewed as a Schrödinger operator with magnetic potential. It is well-known that L has a discrete spectrum, consisting of the set of positive odd numbers, and that each of the corresponding

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eigenspaces is infinite-dimensional. The literature on operators of the kind of (4) is wide. For general results on the twisted Laplacian and its relations with the sublaplacian on the Heisenberg group and the Harmonic Oscillator see for instance [15]. In [8] the eigenspaces of the twisted Laplacian are described and the spectral projections P_λ are studied, finding the optimal exponent $\rho(p)$ such that $\|P_\lambda u\|_{L^p} \leqslant \lambda^{\rho(p)}\|u\|_{L^2}$, for $p \in [2, \infty]$. Dispersive estimates of the wave flow for the twisted Laplacian (and the Harmonic Oscillator) are investigated in [5]. Moreover, problems related to regularity of the solution of the twisted Laplacian are studied in different frames. In particular, in [10] analytic and Gevrey regularity is analyzed, whereas in [17] the global regularity in the sense of (1) is proved, by explicit computation of the heat kernel and Green function. Here we follow a new approach, related to transformations of Wigner type, to characterize global regularity of second order twisted operators. The approach consists in applying a Wigner-like transform to a general differential equation. This idea is already present in some works related to engineering applications, see [4], [6]. In these papers some equations are analyzed, looking for the Wigner transform of the solution. Instead of finding first a solution u, and then computing its Wigner transform $\mathcal{T}u$, the equation itself is Wigner-transformed obtaining an equation in $\mathcal{T}u$. In this way it is possible to find, in some cases, the exact expression of $\mathcal{T}u$.

In this paper, by using the approach of [4], [6] (see also [3],) we establish a link between twisted operators (2) and general second order ordinary differential operators with polynomial coefficients of the form

$$B = \sum_{j+k\leqslant 2} a_{kj} M^j D^k. \tag{5}$$

We call B the *source* of A. We prove in Theorem 14 that (2) is globally regular in the sense of (1) if and only if (5) is globally regular and one-to-one as an operator from $\mathcal{S}'(\mathbb{R})$ into $\mathcal{S}'(\mathbb{R})$. In Proposition 18 we give a complete characterization of all operators (5) that are globally regular, in terms of the behavior of the complex roots of its Weyl symbol. In particular we avoid the additional hypotheses required in [13]. Among the operators (5) that are globally regular we then characterize those that are also one-to-one (see Theorem 27.) This is done through a careful analysis of the asymptotic behavior of the solutions of $Bu = 0$. As a consequence we characterize all the operators (2) that are globally regular. Then we recover as a particular case the global regularity of the twisted Laplacian, (already proved in [17],) since the source of the twisted Laplacian is the Harmonic Oscillator, that is globally regular and one-to-one.

As already observed, hypo-elliptic differential operators in the sense of Definition 25.2 of [14] are globally regular. Then, starting from an hypo-elliptic and one-to-one source, the corresponding twisted operator is globally regular. It is worthwhile to stress that twisted operators (2) are never hypo-elliptic, as shown in Proposition 6, so the class of twisted globally regular operators that we find is completely disjoint from the class of hypo-elliptic operators. Moreover, we observe that there are globally regular twisted operators that have an hypo-elliptic source, as the twisted Laplacian, but not all twisted globally regular operators have an hypo-elliptic source. For example the operator with constant coefficients

$$B_1 = a_{20} D_x^2 + a_{10} D_x + a_{00}$$

is globally regular and one-to-one if and only if the polynomial

$$a_{20}\xi^2 + a_{10}\xi + a_{00} \tag{6}$$

never vanishes. This is consequence of Theorem 27 below, but it can be easily proved directly since B_1, on the Fourier transform side, is the multiplication by (6). The corresponding twisted operator is

$$A_1 = a_{20}\gamma^2(D_x - \mu M_y)^2 - a_{10}\gamma(D_x - \mu M_y) + a_{00},$$

with $\gamma, \mu \in \mathbb{R}$, $\mu \neq 0$. If (6) never vanishes, A_1 is then globally regular but its source B_1 is never hypo-elliptic. We can find examples of this kind also in the case of sources with variable coefficients. Consider for example the twisted operator

$$A_2 = (\gamma D_x - \delta M_y)^2 - \mathrm{i}(\gamma D_x - \delta M_y) - (\alpha D_y - \beta M_x)^2$$

with source

$$B_2 = D_x^2 + \mathrm{i}D_x - M_x^2.$$

In view of the results of the present paper, for $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ satisfying (3), both A_2 and B_2 are globally regular, and B_2 is one-to-one, but both B_2 and A_2 are not hypo-elliptic.

In this paper we only treat the case of second order operators in dimension 2. Our results can be probably generalized to dimension greater than 2, but this depends on how to extend the Definition 5 to higher dimensions. On the other hand, the extension of Theorem 17 to operators of order greater than 2 looks very difficult because already a complete characterization of globally regular ordinary differential operators of order greater than 2 and with polynomial coefficients is an open problem.

Lastly, since the technique used in this paper to link a source to the corresponding twisted operator recaptures well-known connections between the Harmonic Oscillator and the twisted Laplacian, we think that it can be fruitfully used to prove that results holding for the twisted Laplacian (see for example [8], or [10]) hold in fact for larger classes of operators.

The paper is organized as follows. After some basic results in Section 2, we study properties of twisted operators and the relations with their sources in Section 3. The main results on global regularity are proved in Section 4. As already observed, we need a careful analysis of the asymptotic behavior of the solutions of second order ordinary differential equations. As a consequence we then need precise asymptotic expansions of special functions, as well as of their linear combinations, in suitable sectors of the complex plane. Since we have not found in the literature all the results in the form we need, for the sake of completeness we prove them in Sections 5 and 6.

We end this introduction with some notations and definitions.

Given a subset S of the complex numbers $\mathbb{C}$, we set $S^* = S \setminus \{0\}$. If $S \subset \mathbb{R}$, we set $S_+ = \{x \in S : x \geqslant 0\}$, and $S_- = \{x \in S : x \leqslant 0\}$. Thus in particular $\mathbb{Z}_+^* = \{1, 2, \ldots\}$.

To avoid ambiguity due to polar representation of complex numbers we define the *principal branch of the argument of* $z \in \mathbb{C}^*$ as

$$\operatorname{Arg} z = \begin{cases} 2\arctan \frac{\operatorname{Im} z}{\operatorname{Re} z + |z|}, & \text{if } \operatorname{Im} z \neq 0 \text{ or } \operatorname{Im} z = 0 \text{ and } \operatorname{Re} z > 0, \\ \pi, & \text{if } \operatorname{Im} z = 0 \text{ and } \operatorname{Re} z < 0. \end{cases} \tag{7}$$

Observe that (7) implies

$$\operatorname{Arg}(-z) = \operatorname{Arg} z + \sigma(z)\pi,$$

where

$$\sigma(z) = \begin{cases} 1, & \text{if } \operatorname{Arg} z \leqslant 0, \\ -1, & \text{if } 0 < \operatorname{Arg} z. \end{cases}$$

Given a complex number λ we *define*

$$z^\lambda = \mathrm{e}^{\lambda \log|z| + \mathrm{i}\lambda \operatorname{Arg} z}, \qquad \text{for } z \in \mathbb{C}^*.$$

With this definition we have

$$\operatorname{Arg}(z^\lambda) = \operatorname{Im}\lambda \log|z| + \operatorname{Re}\lambda \operatorname{Arg} z \iff -\pi < \operatorname{Im}\lambda \log|z| + (\operatorname{Re}\lambda) \operatorname{Arg} z \leqslant \pi.$$

In particular, given a *real* number p such that $|p| < 1$, we have $\operatorname{Arg}(z^p) = p \operatorname{Arg} z$, and therefore $(z^p)^\lambda = z^{\lambda p}$, for all $\lambda \in \mathbb{C}$.

2 Globally regular operators

Definition 1. A linear operator A on $\mathcal{S}'(\mathbb{R}^n)$ is *globally regular* if

$$Au \in \mathcal{S}(\mathbb{R}^n) \implies u \in \mathcal{S}(\mathbb{R}^n), \qquad \text{for all } u \in \mathcal{S}'(\mathbb{R}^n).$$

We employ standard multi-index notation. In particular, a linear differential operator A has *symbol*

$$a(x,\xi) = \sum_{|\alpha| \leqslant m} a_\alpha(x)\xi^\alpha, \tag{8}$$

if

$$A = \sum_{|\alpha| \leqslant m} a_\alpha(x) D^\alpha, \tag{9}$$

with

$$D_j = -\mathrm{i}\partial_j, \qquad \text{for } 1 \leqslant j \leqslant n$$

and $\mathrm{i}^2 = -1$.

Definition 2 (See [12, Definition 1.3.2])**.** A linear differential operator on $\mathcal{S}'(\mathbb{R}^n)$, *with polynomial symbol*:

$$a(x,\xi) = \sum_{|\alpha+\beta| \leqslant m} a_{\alpha,\beta} x^\alpha \xi^\beta$$

is *globally hypo-elliptic* if $a(x,\xi)$ does not vanish outside a compact set and

$$\lim_{|x|+|\xi|\to\infty} \frac{\partial_x^\alpha \partial_\xi^\beta a(x,\xi)}{a(x,\xi)} = 0, \qquad \text{for } |\alpha| + |\beta| = 1. \tag{10}$$

Theorem 3. Assumption (10) implies that

$$\lim_{|x|+|\xi|\to\infty} \frac{\partial_x^\alpha \partial_\xi^\beta a(x,\xi)}{a(x,\xi)} = 0, \qquad \text{for } |\alpha| + |\beta| \geqslant 1, \tag{11}$$

and that there exists $0 < m_0 \leqslant m$ such that

$$\inf_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n} \frac{1 + |a(x,\xi)|}{(1+|x|+|\xi|)^{m_0}} > 0. \tag{12}$$

Proof. Statement (11) follows from Propositions 2.4.1 and 2.4.4 of [12]. □

Theorem 4. A globally hypo-elliptic linear differential operator with polynomial symbol is globally regular.

Proof. Thanks to Theorem 3 the symbol satisfies the hypothesis of Theorem 25.3 of [14]. □

3 Twisted differential operators

Define the *multiplication operators*

$$M_1 u(x,y) = M_x u(x,y) = xu(x,y), \qquad M_2 u(x,y) = M_y u(x,y) = yu(x,y),$$

where $u \in \mathcal{S}'(\mathbb{R}^2)$.

The *twisted Laplacian*

$$\left(D_x + \frac{1}{2}M_y\right)^2 + \left(D_y - \frac{1}{2}M_x\right)^2 \tag{13}$$

is an important example of an operator which is globally regular but not globally hypo-elliptic (see [17].)

Definition 5. A *twisted differential operator of order* m is a linear differential operator on $\mathbb{R}^2$ of the kind

$$A = \sum_{j+k\leqslant m} (-1)^{j+k} a_{kj} (\alpha D_y - \beta M_x)^j (\gamma D_x - \delta M_y)^k, \tag{14}$$

where $\alpha, \beta, \gamma, \delta$ are real numbers such that

$$\alpha\delta - \beta\gamma = 1 \quad \text{and} \quad \beta\delta \neq 0 \tag{15}$$

and the coefficients $a_{k,j}$ are complex numbers such that $\sum_{j+k=m} |a_{kj}| \neq 0$.

For example, if we set

$$m = 2, \quad a_{20} = a_{02} = 1, \quad a_{jk} = 0, \quad \text{for } j, k \leqslant 1,$$

and

$$\alpha = -1, \quad \beta = -\frac{1}{2}, \quad \gamma = 1, \quad \delta = -\frac{1}{2},$$

the operator (14) becomes the twisted Laplacian (13).

The class of twisted differential operators is completely disjoint from the class of globally hypo-elliptic operators.

Proposition 6. Twisted differential operators are *never* globally hypo-elliptic.

Proof. By Theorem 3.4 of [14] we have that the symbol of the operator (14) is given by

$$\begin{aligned} a(x, y; \xi, \eta) &= \sum_{j+k\leqslant m} (-1)^{j+k} a_{kj} \sum_{n\in\mathbb{Z}_+} \frac{(-\mathrm{i})^n}{n!} \partial_\eta^n (\alpha\eta - \beta x)^j \partial_y^n (\gamma\xi - \delta y)^k \\ &= \sum_{j+k\leqslant m} (-1)^{j+k} a_{kj} \sum_{n\leqslant \min\{j,k\}} (\mathrm{i}\alpha\delta)^n \binom{j}{n} \binom{k}{n} n! (\alpha\eta - \beta x)^{j-n} (\gamma\xi - \delta y)^{k-n}. \end{aligned}$$

Since a is constant along the plane

$$\begin{cases} \alpha\eta - \beta x = 0, \\ \gamma\xi - \delta y = 0, \end{cases}$$

we have that the operator (14) cannot be globally hypo-elliptic. $\square$

Given four real numbers $\alpha, \beta\, \gamma, \delta$ satisfying (15), define the integral transform of a function $u \in \mathcal{S}(\mathbb{R}^2)$:

$$\mathcal{T}u(x, y) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}zy} u(\beta x + \alpha z, \beta x + \beta\gamma\delta^{-1} z)\, \mathrm{d}z.$$

A simple computation shows that $\mathcal{T}$ is an isomorphism on $\mathcal{S}(\mathbb{R}^2)$ with inverse given by

$$\mathcal{T}^{-1}v(x,y) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\delta(x-y)} v(\alpha\delta\beta^{-1}y - \gamma x, t)\,\mathrm{d}t.$$

Since $\mathcal{T}$ and its inverse extend to $\mathcal{S}'(\mathbb{R}^2)$, we may define the transform of an operator A on $\mathcal{S}'(\mathbb{R}^2)$ as

$$\mathcal{T}[A] = \mathcal{T}A\mathcal{T}^{-1}.$$

Of course this transformation is invertible, with inverse given by

$$\mathcal{T}^{-1}[B] = \mathcal{T}^{-1}B\mathcal{T}.$$

Since $\mathcal{T}$ is an isomorphism on $\mathcal{S}(\mathbb{R}^2)$ and on $\mathcal{S}'(\mathbb{R}^2)$, we have that

$$A \text{ is globally regular } \iff \mathcal{T}[A] \text{ is globally regular.} \tag{16}$$

Compute

$$\begin{aligned}
D_1\mathcal{T}u(x,y) &= (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}zy} D_x\left(u(\beta x + \alpha z, \beta x + \beta\gamma\delta^{-1}z)\right)\,\mathrm{d}z \\
&= \beta\mathcal{T}D_1u(x,y) + \beta\mathcal{T}D_2u(x,y),
\end{aligned} \tag{17}$$

$$\begin{aligned}
D_2\mathcal{T}u(x,y) &= (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} -z\mathrm{e}^{-\mathrm{i}zy} u(\beta x + \alpha z, \beta x + \beta\gamma\delta^{-1}z)\,\mathrm{d}z \\
&= -(2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}zy} \delta\left(\beta x + \alpha z\right) u(\beta x + \alpha z, \beta x + \beta\gamma\delta^{-1}z)\,\mathrm{d}z \\
&\quad + (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}zy} \delta\left(\beta x + \beta\gamma\delta^{-1}z\right) u(\beta x + \alpha z, \beta x + \beta\gamma\delta^{-1}z)\,\mathrm{d}z \\
&= -\delta\mathcal{T}M_1u(x,y) + \delta\mathcal{T}M_2u(x,y),
\end{aligned} \tag{18}$$

$$\begin{aligned}
M_1\mathcal{T}u(x,y) &= (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}zy} xu(\beta x + \alpha z, \beta x + \beta\gamma\delta^{-1}z)\,\mathrm{d}z \\
&= -(2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}zy} \gamma(\beta x + \alpha z) u(\beta x + \alpha z, \beta x + \beta\gamma\delta^{-1}z)\,\mathrm{d}z \\
&\quad + (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}zy} \alpha\delta\beta^{-1}(\beta x + \beta\gamma\delta^{-1}z) u(\beta x + \alpha z, \beta x + \beta\gamma\delta^{-1}z)\,\mathrm{d}z \\
&= -\gamma\mathcal{T}M_1u(x,y) + \alpha\delta\beta^{-1}\mathcal{T}M_2u(x,y),
\end{aligned} \tag{19}$$

$$\begin{aligned}
M_2\mathcal{T}u(x,y) &= (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \left(-D_z\mathrm{e}^{-\mathrm{i}zy}\right) u(\beta x + \alpha z, \beta x + \beta\gamma\delta^{-1}z)\,\mathrm{d}z \\
&= (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}zy} \alpha D_1u(\beta x + \alpha z, \beta x + \beta\gamma\delta^{-1}z)\,\mathrm{d}z \\
&\quad + (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}zy} \beta\gamma\delta^{-1} D_2u(\beta x + \alpha z, \beta x + \beta\gamma\delta^{-1}z)\,\mathrm{d}z \\
&= \alpha\mathcal{T}D_1u(x,y) + \beta\gamma\delta^{-1}\mathcal{T}D_2u(x,y).
\end{aligned} \tag{20}$$

It follows that

$$\begin{aligned}
\mathcal{T}[M_x] &= -\alpha D_y + \beta M_x, \\
\mathcal{T}[D_x] &= -\gamma D_x + \delta M_y,
\end{aligned}$$

and more generally the twisted differential operator (14) can be written as

$$A = \mathcal{T}[\check{A}] \tag{21}$$

where

$$\check{A} = \sum_{j+k\leqslant m} a_{kj} M_x^j D_x^k. \tag{22}$$

Observe that $\check{A}$ is an operator on $\mathbb{R}^2$, acting only on the first variable:

$$\check{A}u(x,y) = \sum_{j+k\leqslant m} a_{kj} x^j D_x^k u(x,y).$$

Recall now that $\mathcal{S}(\mathbb{R}^2)$ is the tensor product of $\mathcal{S}(\mathbb{R})$ by $\mathcal{S}(\mathbb{R})$. This means that $\mathcal{S}(\mathbb{R}^2)$ is the completion $\mathcal{S}(\mathbb{R})\widehat{\otimes}\mathcal{S}(\mathbb{R})$ of the space $\mathcal{S}(\mathbb{R})\otimes\mathcal{S}(\mathbb{R})$ of linear combinations of products

$$(f\otimes g)(x,y) = f(x)g(y).$$

The same is true for temperate distributions:

$$\mathcal{S}'(\mathbb{R}^2) = \big(\mathcal{S}(\mathbb{R})\widehat{\otimes}\mathcal{S}(\mathbb{R})\big)' = \mathcal{S}'(\mathbb{R})\widehat{\otimes}\mathcal{S}'(\mathbb{R}).$$

Given two continuous linear operators A_1 and A_2 on $\mathcal{S}'(\mathbb{R})$, there exists a unique continuous linear operator $A_1\widehat{\otimes}A_2$ on $\mathcal{S}'(\mathbb{R})\widehat{\otimes}\mathcal{S}'(\mathbb{R}) = \mathcal{S}'(\mathbb{R}^2)$ such that

$$(A_1\widehat{\otimes}A_2)(u_1\otimes u_2) = A_1u_1\otimes A_2u_2, \quad (u_1,u_2)\in\mathcal{S}'(\mathbb{R})\times\mathcal{S}'(\mathbb{R}).$$

If A_1 and A_2 are continuous on $\mathcal{S}(\mathbb{R})$, the tensor product $A_1\widehat{\otimes}A_2$ is continuous on $\mathcal{S}(\mathbb{R})\widehat{\otimes}\mathcal{S}(\mathbb{R}) = \mathcal{S}(\mathbb{R}^2)$.

Define the operators on $\mathcal{S}'(\mathbb{R})$:

$$Du(x) = -\mathrm{i}u'(x), \qquad Mu(x) = xu(x), \qquad Iu(x) = u(x).$$

then we have

$$M_x = M\widehat{\otimes}I, \qquad D_x = D\widehat{\otimes}I,$$

and more generally

$$\sum_{j+k\leqslant m} a_{kj} M_x^j D_x^k = \left(\sum_{j+k\leqslant m} a_{kj} M^j D^k\right)\widehat{\otimes}I.$$

In other words, if we keep into account (14), (21) and (22), we obtain the following identity:

$$A = \mathcal{T}[\tilde{A}\widehat{\otimes}I]$$

where A is the operator (14) and

$$\tilde{A} = \sum_{j+k\leqslant m} a_{kj} M^j D^k. \tag{23}$$

Definition 7. The ordinary differential operator $\tilde{A}$ defined in (23) is the *source* of the twisted differential operator A given by (14).

We always consider the kernel of the source $\tilde{A}$ in the sense of temperate distributions:

$$\ker \tilde{A} = \{u \in \mathcal{S}'(\mathbb{R}) \ : \ \tilde{A}u = 0\}.$$

Observe that $\ker \tilde{A} \subset \mathcal{S}(\mathbb{R})$, if $\tilde{A}$ is globally regular.

From (16), we obtain the following proposition.

Proposition 8. A twisted differential operator A is globally regular if and only if $\tilde{A}\widehat{\otimes}I$ is globally regular.

Proposition 9. The source of a globally regular twisted differential operator is globally regular and one-to-one.

In particular a globally regular twisted differential operator is one-to-one.

Proof. Let A be the twisted operator. We know from Proposition 8 that $\tilde{A}\widehat{\otimes}I$ is globally regular.

Consider $u \in \mathcal{S}'(\mathbb{R})$ such that $\tilde{A}u \in \mathcal{S}(\mathbb{R})$. Then $(\tilde{A}\widehat{\otimes}I)(u \otimes v) = (\tilde{A}u) \otimes v \in \mathcal{S}(\mathbb{R}^2)$ for all $v \in \mathcal{S}(\mathbb{R})$. Since $\tilde{A}\widehat{\otimes}I$ is globally regular, $u \otimes v$ must belong to $\mathcal{S}(\mathbb{R}^2)$ for all $v \in \mathcal{S}(\mathbb{R})$. But this is impossible, unless u belongs to $\mathcal{S}(\mathbb{R})$. In fact , given $v \in \mathcal{S}(\mathbb{R})$ such that $v(0) = 1$, let (ψ_n) be a sequence in $\mathcal{S}(\mathbb{R})$ converging to the Dirac distribution δ. Then for all $\phi \in \mathcal{S}(\mathbb{R})$, we have

$$\begin{aligned}
\int_{\mathbb{R}} (u \otimes v)(x, 0)\phi(x)\,\mathrm{d}x &= \lim_{n\to+\infty} \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} (u \otimes v)(x, y)\phi(x)\,\mathrm{d}x \right\} \psi_n(y)\,\mathrm{d}y \\
&= \lim_{n\to+\infty} \int_{\mathbb{R}^2} (u \otimes v)(x, y)\,(\phi \otimes \psi_n)(x, y)\,\mathrm{d}x\mathrm{d}y \\
&= \langle u \,|\, \phi\rangle \lim_{n\to+\infty} \int_{\mathbb{R}} v(y)\psi_n(y)\,\mathrm{d}y = \langle u \,|\, \phi\rangle.
\end{aligned}$$

But this means that $u(x) = (u \otimes v)(x, 0) \in \mathcal{S}(\mathbb{R})$.

Now we show that $\tilde{A}$ is one-to-one. Assume there exists $\phi \in \mathcal{S}(\mathbb{R}) \setminus \{0\}$ such that $\tilde{A}\phi = 0$. Then $\phi \otimes \delta$ belongs to the kernel of $\tilde{A}\widehat{\otimes}I$, but not to $\mathcal{S}(\mathbb{R}^2)$, in contradiction with the global regularity of $\tilde{A}\widehat{\otimes}I$.

If A is the globally regular twisted differential operator with source $\tilde{A}$, we have that $\ker \tilde{A} = 0$. Then $\ker A = \mathcal{T}\left((\ker \tilde{A})\widehat{\otimes}I\right) = 0$, that is A is one-to-one. □

Denote by $(\tilde{A})'$ the transpose of the source (23):

$$(\tilde{A})' = \sum_{j+k\leqslant m} (-1)^k a_{kj} D^k M^j.$$

Observe that $\tilde{A}$ and $(\tilde{A})'$ are dual to each other, that is $(\tilde{A})'' = \tilde{A}$. In other words, we have

$$\begin{cases} \langle(\tilde{A})'u \,|\, \phi\rangle = \langle u \,|\, \tilde{A}\phi\rangle, \\ \langle\tilde{A}u \,|\, \phi\rangle = \langle u \,|\, (\tilde{A})'\phi\rangle, \end{cases} \qquad \text{for all } u \in \mathcal{S}'(\mathbb{R}), \text{ and } \phi \in \mathcal{S}(\mathbb{R}).$$

Recall now the following Theorem of [11].

Theorem 10. An ordinary differential operator with polynomial coefficients, has closed range in $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$.

Thanks to Theorem 10, the images $\tilde{A}\,(\mathcal{S}(\mathbb{R}))$ and $\tilde{A}\,(\mathcal{S}'(\mathbb{R}))$ are closed subspaces of $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$, respectively. Then by Closed Range Theorem [2, Theorem 1.2], it follows that

$$\tilde{A}\,(\mathcal{S}(\mathbb{R})) = \left\{ f \in \mathcal{S}(\mathbb{R}) \ : \ \langle\phi \,|\, f\rangle = 0, \, \forall\phi \in \ker(\tilde{A})' \right\}$$

and

$$\tilde{A}\,(\mathcal{S}'(\mathbb{R})) = \Big\{ f \in \mathcal{S}'(\mathbb{R}) \ : \ \langle f \mid \phi \rangle = 0, \, \forall \phi \in \ker(\tilde{A})' \cap \mathcal{S}(\mathbb{R}) \Big\}.$$

Since $\ker(\tilde{A})'$ is finite-dimensional, both $\tilde{A}\,(\mathcal{S}(\mathbb{R}))$ and $\tilde{A}\,(\mathcal{S}'(\mathbb{R}))$ have a topological supplementary, we can choose as follows. Fix a basis $\phi_1, \dots, \phi_n$ of $\ker(\tilde{A})'$, and let $\psi_1, \dots, \psi_n$ be functions in $\mathcal{S}(\mathbb{R})$ such that $\langle \phi_j \mid \psi_k \rangle = \delta_{jk}$ for $j, k \in \{1, \dots, n\}$. Let $\mathcal{N}((\tilde{A})')$ be the subspace of $\mathcal{S}(\mathbb{R})$ generated by $\psi_1, \dots, \psi_n$. Then

$$\mathcal{S}(\mathbb{R}) = \tilde{A}\,(\mathcal{S}(\mathbb{R})) \oplus \mathcal{N}((\tilde{A})'). \tag{24}$$

Without loss of generality, we can assume that $\ker(\tilde{A})' \cap \mathcal{S}(\mathbb{R})$ either equals 0 or it is generated by $\phi_1, \dots, \phi_m$, with $m \leqslant n$. Then

$$\mathcal{S}'(\mathbb{R}) = \tilde{A}\,(\mathcal{S}'(\mathbb{R})) \oplus \mathcal{M}((\tilde{A})'), \tag{25}$$

where $\mathcal{M}((\tilde{A})')$ is either 0 or the subspace of $\mathcal{N}((\tilde{A})')$ generated by $\psi_1, \dots, \psi_m$.

Moreover, by Propositions 43.7 and 43.9 of [16], it follows from (24) and (25) that

$$\mathcal{S}(\mathbb{R}^2) = (\tilde{A} \widehat{\otimes} I)\mathcal{S}(\mathbb{R}^2) \oplus \mathcal{N}((\tilde{A})') \widehat{\otimes} \mathcal{S}(\mathbb{R}) \tag{26}$$

and

$$\mathcal{S}'(\mathbb{R}^2) = (\tilde{A} \widehat{\otimes} I)\mathcal{S}'(\mathbb{R}^2) \oplus \mathcal{M}((\tilde{A})') \widehat{\otimes} \mathcal{S}'(\mathbb{R}). \tag{27}$$

Proposition 11. Given a twisted differential operator A, the images $A\left(\mathcal{S}(\mathbb{R}^2)\right)$ and $A\left(\mathcal{S}'(\mathbb{R}^2)\right)$ are closed subspaces of $\mathcal{S}(\mathbb{R}^2)$ and $\mathcal{S}'(\mathbb{R}^2)$ respectively.

Proof. Let $\tilde{A}$ be the source of A. Then $A = \mathcal{T}[\tilde{A} \widehat{\otimes} I]$. Since $\mathcal{T}$ is an automorphism of $\mathcal{S}(\mathbb{R}^2)$ and of $\mathcal{S}'(\mathbb{R}^2)$, the closure of the images follows from (26) and (27). $\square$

Proposition 12. Given a twisted differential operator A the following conditions are equivalent.

(A) $\ker \tilde{A} \subset \mathcal{S}(\mathbb{R})$ and $\ker(\tilde{A})' \subset \mathcal{S}(\mathbb{R})$.

(B) $\tilde{A}$ and $(\tilde{A})'$ are globally regular.

Proof. It is clear that (B) $\implies$ (A).

Let us prove that (A) implies that $(\tilde{A})'$ is globally regular. Consider $u \in \mathcal{S}'(\mathbb{R})$ such that $f = (\tilde{A})'u \in \mathcal{S}(\mathbb{R})$. By the dual to (24), there exist $v \in \mathcal{S}(\mathbb{R})$ and $h \in \mathcal{N}(\tilde{A})$ such that $f = (\tilde{A})'v + h$. Since $\ker \tilde{A} \subset \mathcal{S}(\mathbb{R})$, we have $\mathcal{N}(\tilde{A}) = \mathcal{M}(\tilde{A})$. Then the dual to (25) implies that $h = 0$, that is that $u - v \in \ker(\tilde{A})' \subset \mathcal{S}(\mathbb{R})$. Since $v \in \mathcal{S}(\mathbb{R})$ also $u \in \mathcal{S}(\mathbb{R})$.

The proof that (A) implies that $\tilde{A}$ is globally regular is very similar and is left to the reader. $\square$

Theorem 13. Consider a twisted differential operator A. If $\ker \tilde{A} = 0$ and $\ker(\tilde{A})' \subset \mathcal{S}(\mathbb{R})$, the operator A is globally regular.

Proof. Thanks to Proposition 8 it is sufficient to prove that $\tilde{A} \widehat{\otimes} I$ is globally regular.

Consider $u \in \mathcal{S}'(\mathbb{R}^2)$ such that $f = (\tilde{A} \widehat{\otimes} I)u \in \mathcal{S}(\mathbb{R}^2)$. Thanks to Proposition 12, $(\tilde{A})'$ is globally regular. Since f belongs to $\mathcal{S}(\mathbb{R}^2)$, by (26) there exist $v \in \mathcal{S}(\mathbb{R}^2)$ and $h \in \mathcal{N}((\tilde{A})') \widehat{\otimes} \mathcal{S}(\mathbb{R})$ such that $(\tilde{A} \widehat{\otimes} I)u = (\tilde{A} \widehat{\otimes} I)v + h$. Since $\ker(\tilde{A})' \subset \mathcal{S}(\mathbb{R})$, we have $\mathcal{M}((\tilde{A})') = \mathcal{N}((\tilde{A})')$ and identity (27) implies that $h = 0$. Then $u = v \in \mathcal{S}(\mathbb{R}^2)$, because $\ker(\tilde{A} \widehat{\otimes} I) = (\ker \tilde{A}) \widehat{\otimes} \mathcal{S}(\mathbb{R}) = 0$. $\square$

4 Global regularity of second order twisted differential operators

4.1 Statement of the results

Global regularity of second order twisted differential operators can be characterized in a rather complete way. We state two theorems, which are the main results of the paper. We prove these theorems in Subsections 4.2, and 4.3.2.

Consider the second order twisted differential operator

$$A = \sum_{j+k\leqslant 2} (-1)^{j+k} a_{kj} (\alpha D_y - \beta M_x)^j (\gamma D_x - \delta M_y)^k,$$

with source

$$\tilde{A} = \sum_{j+k\leqslant 2} a_{kj} M^j D^k.$$

Theorem 14. The following statements are equivalent.

(A) A is globally regular.

(B) $\ker \tilde{A} = 0$, and $\tilde{A}$ is globally regular.

(C) $\ker \tilde{A} = 0$, and $(\tilde{A})'$ is globally regular.

(D) $\ker \tilde{A} = 0$, and $\ker(\tilde{A})' \subset \mathcal{S}(\mathbb{R})$.

Definition 15. Two polynomials $p(x,\xi)$ and $q(x,\xi)$ are symplectically equivalent if there exists a symplectic transformation [(3)] χ such that $q = p \circ \chi$.

Lemma 16. For any polynomial

$$p(x,\xi) = \sum_{j+k\leqslant 2} p_{kj} x^j \xi^k,$$

such that $|p_{20}| + |p_{11}| + |p_{02}| > 0$, there is an infinite number of polynomials

$$q(x,\xi) = \sum_{j+k\leqslant 2} q_{kj} x^j \xi^k,$$

symplectically equivalent to p and such that $q_{20} \neq 0$.

Proof. It is sufficient to consider $\chi(x,\xi) = (x + \theta\xi, \xi)$, where $\theta \in \mathbb{R}$ is such that $p_{20} + \theta p_{11} + \theta^2 p_{02} \neq 0$. $\square$

Recall that the *Weyl symbol* (see [14, Definition 23.5]) of a differential operator

$$P = p_{20} D^2 + p_{11} MD + p_{02} M^2 + p_{10} D + p_{01} M + p_{00} I$$

is given by

$$p(x,\xi) = p_{20}\xi^2 + p_{11} x\xi + p_{02} x^2 + p_{10}\xi + p_{01} x + p_{00} + \frac{\mathrm{i}}{2} p_{11}.$$

Denote by $\mathcal{B}$ the set of polynomials

$$b(x,\xi) = b_{20}\xi^2 + b_{11} x\xi + b_{02} x^2 + b_{10}\xi + b_{01} x + b_{00} + \frac{\mathrm{i}}{2} b_{11},$$

(3) In dimension 2 a symplectic transformation is a linear map with determinant equal to 1.

with $b_{20} \neq 0$, and symplectically equivalent to the Weyl symbol of $\tilde{A}$.

Since the order of A is 2, we have $|a_{20}| + |a_{11}| + |a_{02}| > 0$. Then Lemma 16 implies that $\mathcal{B} \neq \emptyset$.

For all $b \in \mathcal{B}$, set

$$\begin{cases} \Delta_2 = b_{11}^2 - 4b_{20}b_{02}, \\ \Delta_1 = 2b_{11}b_{10} - 4b_{20}b_{01}, \\ \Delta_0 = b_{10}^2 - 4b_{20}b_{00} - 2\mathrm{i}b_{20}b_{11}, \end{cases} \tag{28}$$

$$\lambda = \frac{1}{8}\left(-\frac{\Delta_2}{b_{20}^2}\right)^{-\frac{3}{2}} \frac{\Delta_1^2 - 4\Delta_2\Delta_0}{b_{20}^4},$$

and

$$\Xi_\pm(x) = \begin{cases} -\dfrac{1}{2}\left\{\dfrac{b_{11}}{b_{20}}x + \dfrac{b_{10}}{b_{20}} \pm \sigma\!\left(\dfrac{\Delta_2}{b_{20}^2}\right)\left(\dfrac{\Delta_2}{b_{20}^2}\right)^{\frac{1}{2}} x\left(1 + \dfrac{\Delta_1}{\Delta_2 x} + \dfrac{\Delta_0}{\Delta_2 x^2}\right)^{\frac{1}{2}}\right\}, & \text{if } \Delta_2 \neq 0, \\ -\dfrac{1}{2}\left\{\dfrac{b_{11}}{b_{20}}\,x + \dfrac{b_{10}}{b_{20}} \pm \sigma\!\left(\dfrac{\Delta_1}{b_{20}^2}\right)\left(\dfrac{\Delta_1}{b_{20}^2}\right)^{\frac{1}{2}} x^{\frac{1}{2}}\left(1 + \dfrac{\Delta_0}{\Delta_1 x}\right)^{\frac{1}{2}}\right\}, & \text{if } \Delta_2 = 0 \neq \Delta_1, \\ -\dfrac{1}{2}\left\{\dfrac{b_{11}}{b_{20}}\,x + \dfrac{b_{10}}{b_{20}} \pm \sigma\!\left(\dfrac{\Delta_0}{b_{20}^2}\right)\left(\dfrac{\Delta_0}{b_{20}^2}\right)^{\frac{1}{2}}\right\}, & \text{if } \Delta_2 = \Delta_1 = 0. \end{cases} \tag{29}$$

$\xi = \Xi_\pm$ are the complex roots of the Weyl symbol of B:

$$b(x,\xi) = b_{20}\xi^2 + (b_{11}x + b_{10})\xi + b_{02}x^2 + b_{01}x + b_{00} + \frac{\mathrm{i}}{2}\,b_{11}.$$

Theorem 17. The following conditions are equivalent.

(A) A is globally regular.

(B) There exists $b \in \mathcal{B}$ such that

$$\mathrm{e}^{\mathrm{i}x\Xi_\pm} \notin \mathcal{S}',$$

or

$$\mathrm{e}^{\mathrm{i}x\Xi_-} \notin \mathcal{S}', \quad \mathrm{e}^{\mathrm{i}x\Xi_+} \in \mathcal{S}, \quad \Delta_2 \neq 0, \quad \lambda \notin \{1 + 2n : n \in \mathbb{Z}_+\},$$

or

$$\mathrm{e}^{\mathrm{i}x\Xi_-} \notin \mathcal{S}', \quad \mathrm{e}^{\mathrm{i}x\Xi_+} \in \mathcal{S}, \quad \Delta_2 = 0.$$

(C) For all $b \in \mathcal{B}$ we have

$$\mathrm{e}^{\mathrm{i}x\Xi_\pm} \notin \mathcal{S}',$$

or

$$\mathrm{e}^{\mathrm{i}x\Xi_-} \notin \mathcal{S}', \quad \mathrm{e}^{\mathrm{i}x\Xi_+} \in \mathcal{S}, \quad \Delta_2 \neq 0, \quad \lambda \notin \{1 + 2n : n \in \mathbb{Z}_+\},$$

or

$$\mathrm{e}^{\mathrm{i}x\Xi_-} \notin \mathcal{S}', \quad \mathrm{e}^{\mathrm{i}x\Xi_+} \in \mathcal{S}, \quad \Delta_2 = 0.$$

4.2 Proof of Theorem 14

Let

$$B = b_{20}D^2 + b_{11}MD + b_{02}M^2 + b_{10}D + b_{01}M + b_{00}I \tag{30}$$

be a differential operator with Weyl symbol $b \in \mathcal{B}$.

As for the source of a twisted differential operator, also the kernel of B is considered in the sense of temperate distributions:

$$\ker B = \{u \in \mathcal{S}'(\mathbb{R}) \ : \ Bu = 0\}.$$

Proposition 18. The following conditions are equivalent.

(A) $\lim_{|x|\to\infty} |x \operatorname{Im} \Xi_\pm(x)| = \infty$.

(B) $\mathrm{e}^{\mathrm{i}x\Xi_\pm(x)} \in \mathcal{S} \cup (\mathcal{C}^\infty \setminus \mathcal{S}')$.

(C) B is globally regular.

Proof. It is obvious that (A) $\iff$ (B). Let us prove (A) $\iff$ (C).

Assume $\Delta_2 = \operatorname{Im} \frac{b_{11}}{b_{20}} = 0$. Then it is easy to verify that the following conditions are equivalent.

(a) There exists $\epsilon > 0$ such that

$$\max\left\{\left|\Xi_+(x) + \frac{b_{11}}{2b_{20}}x\right|, \left|\Xi_-(x) + \frac{b_{11}}{2b_{20}}x\right|, |x|^{\epsilon-1}\right\} = \mathcal{O}\left(|\Xi_+(x) - \Xi_-(x)|\right), \quad \text{for } |x| \to \infty.$$

(b) $\Delta_1 x + \Delta_0$ does not vanish identically.

If $\Delta_2 x^2 + \Delta_1 x + \Delta_0$ does not vanish identically, it follows that we can apply Theorem 1.2 of [13], obtaining that (A) is equivalent to (C).

If $\Delta_2 = \Delta_1 = \Delta_0 = 0$, the equation $Bu = f$ can be solved explicitly:

$$u(x) = -\mathrm{e}^{-\frac{\mathrm{i}}{4b_{20}}(b_{11}x^2+2b_{10}x)} \left\{\frac{1}{b_{20}} \int_0^x (x-t)\, \mathrm{e}^{\frac{\mathrm{i}}{4b_{20}}(b_{11}t^2+2b_{10}t)} f(t)\,\mathrm{d}t + c_0 x + c_1\right\}, \tag{31}$$

where c_0 and c_1 are arbitrary constants.

Since $x\Xi_\pm = -\frac{1}{2b_{20}}(b_{11}x^2 + b_{10}x)$, we have to show that

$$\left(\operatorname{Im} \frac{b_{11}}{b_{20}}\right)^2 + \left(\operatorname{Im} \frac{b_{10}}{b_{20}}\right)^2 > 0 \iff B \text{ is globally regular.}$$

Assume $\left(\operatorname{Im} \frac{b_{11}}{b_{20}}\right)^2 + \left(\operatorname{Im} \frac{b_{10}}{b_{20}}\right)^2 > 0$, and $f \in \mathcal{S}$. Then we have to prove that u belongs to $\mathcal{S} \cup (\mathcal{C}^\infty \setminus \mathcal{S}')$.

If $\operatorname{Im} \frac{b_{11}}{b_{20}} < 0$, set

$$v(x) = \mathrm{e}^{-h(x)} \int_0^x (x-t)\mathrm{e}^{h(t)} f(t)\,\mathrm{d}t,$$

with

$$h(x) = \frac{\mathrm{i}}{4b_{20}}(b_{11}x^2 + 2b_{10}x).$$

If we show that $v \in \mathcal{S}$, we have that $u \in \mathcal{S}$.

It is clear that for all $n \in \mathbb{Z}_+$ there exist polynomials $P_n(x)$ and $Q_n(x)$ of degree n such that [4]

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\,\mathrm{e}^{-h(x)} = P_n(x)\mathrm{e}^{-h(x)}, \quad \frac{\mathrm{d}^n}{\mathrm{d}x^n}\,\mathrm{e}^{h(x)} = Q_n(x)\mathrm{e}^{h(x)}, \qquad \text{for } n \geqslant 0. \tag{32}$$

Then we have

$$v'(x) = -h'(x)\mathrm{e}^{-h(x)}\int_0^x (x-t)\mathrm{e}^{h(t)}f(t)\,\mathrm{d}t + \mathrm{e}^{-h(x)}\int_0^x \mathrm{e}^{h(t)}f(t)\,\mathrm{d}t,$$

and

$$v^{(n)}(x) = P_n(x)\mathrm{e}^{-h(x)}\int_0^x (x-t)\mathrm{e}^{h(t)}f(t)\,\mathrm{d}t + nP_{n-1}(x)\mathrm{e}^{-h(x)}\int_0^x \mathrm{e}^{h(t)}f(t)\,\mathrm{d}t +$$
$$+\sum_{k=2}^{n}\binom{n}{k}P_{n-k}(x)\sum_{j=0}^{k-2}\binom{k-2}{j}Q_{k-2-j}(x)f^{(j)}(x), \qquad \text{for } n \geqslant 2.$$

Since $f \in \mathcal{S}$, we have

$$\lim_{|x|\to\infty} x^m \sum_{k=2}^{n}\binom{n}{k}P_{n-k}(x)\sum_{j=0}^{k-2}\binom{k-2}{j}Q_{k-2-j}(x)f^{(j)}(x) = 0, \qquad \forall m \in \mathbb{Z}_+.$$

On the other side, since $\operatorname{Re}\left(\frac{\mathrm{i}}{4b_{20}}b_{11}(t^2-x^2)\right) = \operatorname{Im}\frac{b_{11}}{4b_{20}}(x^2-t^2) < 0$, for $x > t$, we have

$$\lim_{|x|\to\infty} x^m\Big\{P_n(x)\mathrm{e}^{-h(x)}\int_0^x (x-t)\mathrm{e}^{h(t)}f(t)\,\mathrm{d}t + nP_{n-1}(x)\mathrm{e}^{-h(x)}\int_0^x \mathrm{e}^{h(t)}f(t)\,\mathrm{d}t\Big\} =$$
$$= \lim_{|x|\to\infty}\int_0^x x^m\Big\{xP_n(x) + nP_{n-1}(x) - P_n(x)t\Big\}\mathrm{e}^{\frac{\mathrm{i}}{4b_{20}}[b_{11}(t^2-x^2)+2b_{10}(t-x)]}f(t)\,\mathrm{d}t = 0,$$

by Dominated Convergence Theorem. Then we have shown that $\lim_{|x|\to 0} x^m v^{(n)}(x) = 0$ for all $m, n \in \mathbb{Z}_+$. It follows that $u \in \mathcal{S}$, that is that B is globally regular.

If $\operatorname{Im}\frac{b_{11}}{b_{20}} > 0$, $\mathrm{e}^{\frac{\mathrm{i}}{4b_{20}}(b_{11}t^2+2b_{10}t)}f(t)$ belongs to $\mathcal{S}$. Then

$$\int_0^{\pm\infty} \mathrm{e}^{\frac{\mathrm{i}}{4b_{20}}(b_{11}t^2+2b_{10}t)}f(t)\,\mathrm{d}t \quad \text{and} \quad \int_0^{\pm\infty} t\,\mathrm{e}^{\frac{\mathrm{i}}{4b_{20}}((b_{11}t^2+2b_{10}t)}f(t)\,\mathrm{d}t$$

are convergent, so u grows at infinity as $(1+|x|)\mathrm{e}^{\operatorname{Im}\frac{b_{11}}{4b_{20}}x^2}$ and cannot belong to $\mathcal{S}'$.

If $\operatorname{Im}\frac{b_{11}}{b_{20}} = 0$ and $\operatorname{Im}\frac{b_{10}}{b_{20}} > 0$,

$$\int_0^{+\infty} \mathrm{e}^{\frac{\mathrm{i}}{4b_{20}}(b_{11}t^2+2b_{10}t)}f(t)\,\mathrm{d}t \quad \text{and} \quad \int_0^{+\infty} t\,\mathrm{e}^{\frac{\mathrm{i}}{4b_{20}}(b_{11}t^2+2b_{10}t)}f(t)\,\mathrm{d}t$$

are convergent, so u grows as $(1+x)\mathrm{e}^{\operatorname{Im}\frac{b_{10}}{2b_{20}}x}$ for $x \to +\infty$ and cannot belong to $\mathcal{S}'$.

If $\operatorname{Im}\frac{b_{10}}{b_{20}} < 0$, u grows as $(1-x)\mathrm{e}^{\operatorname{Im}\frac{b_{10}}{2b_{20}}x}$ for $x \to -\infty$ and again cannot belong to $\mathcal{S}'$.

[4] Definition (32) is equivalent to define by induction

$$P_n = \begin{cases} 1, & \text{if } n = 0, \\ P'_{n-1} - P_{n-1}h', & \text{if } n \geqslant 1, \end{cases}, \qquad Q_n = \begin{cases} 1, & \text{if } n = 0, \\ Q'_{n-1} + Q_{n-1}h', & \text{if } n \geqslant 1. \end{cases}$$

On the contrary, if B is globally regular, from (31) with $f = 0$, $c_0 = 0$, and $c_1 = 1$, we get that

$$\mathrm{e}^{-\frac{\mathrm{i}}{b_{20}}(b_{11}x^2+2b_{10}x)} \in \mathcal{S} \cup (\mathcal{C}^\infty \setminus \mathcal{S}'),$$

which in turn implies $\left(\operatorname{Im}\frac{b_{11}}{b_{20}}\right)^2 + \left(\operatorname{Im}\frac{b_{10}}{b_{20}}\right)^2 > 0$. $\square$

Proposition 19. B is globally regular if and only if B' is globally regular.

Proof. Consider the *formal adjoint* $B^* = \overline{B'}$. Since $B' = f$ is equivalent to $B^* = \bar{f}$, B' is globally regular if and only if B^* is globally regular.

A simple computation shows that the Weyl symbol of B^* is the complex conjugate of the Weyl symbol of B. Then, since $|x \operatorname{Im} \Xi_\pm(x)| = \left|x \operatorname{Im} \overline{\Xi_\pm(x)}\right|$, the statement follows from Proposition 18. $\square$

Proposition 20. We have

$$\ker \tilde{A} = 0 \iff \ker B = 0, \tag{33}$$

$$\tilde{A} \text{ is globally regular if and only if } B \text{ is globally regular,} \tag{34}$$

and

$$(\tilde{A})' \text{ is globally regular if and only if } B' \text{ is globally regular.} \tag{35}$$

Proof. Thanks to [7, Theorem 18.5.9], there exists a unitary operator U on $L^2(\mathbb{R})$, which is an automorphism of $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$, such that $B = U^{-1}\tilde{A}U$. Since the dual is globally regular if and only if the formal adjoint is globally regular, this implies the result. $\square$

Proof of Theorem 14 (A) $\Longrightarrow$ (B): follows from Proposition 9.
(B) $\Longrightarrow$ (C): follows from Propositions 19, and 20.
(C) $\Longrightarrow$ (D): obvious.
(D) $\Longrightarrow$ (A): follows from Theorem 13. $\square$

4.3 Proof of Theorem 17

4.3.1 Asymptotic behavior of the general solution to equation $Bu = 0$

Consider the operator B given by (30) with $b_{20} \neq 0$.

Define

$$\Sigma_\pm(x) = \begin{cases} -\dfrac{1}{4}\left\{\dfrac{b_{11}}{b_{20}}x^2 + 2\dfrac{b_{10}}{b_{20}}x \pm \sigma\left(\dfrac{\Delta_2}{b_{20}^2}\right)\left(\dfrac{\Delta_2}{b_{20}^2}\right)^{\frac{1}{2}} x^2 \left(1 + \dfrac{\Delta_1}{2\Delta_2 x}\right)^2\right\}, & \text{if } \Delta_2 \neq 0, \\ -\dfrac{1}{4}\left\{\dfrac{b_{11}}{b_{20}}x^2 + 2\dfrac{b_{10}}{b_{20}}x \pm \dfrac{4}{3}\sigma\left(\dfrac{\Delta_1}{b_{20}^2}\right)\left(\dfrac{\Delta_1}{b_{20}^2}\right)^{\frac{1}{2}} x^{\frac{3}{2}} \left(1 + \dfrac{\Delta_0}{\Delta_1 x}\right)^{\frac{3}{2}}\right\}, & \text{if } \Delta_2 = 0 \neq \Delta_1, \\ -\dfrac{1}{4}\left\{\dfrac{b_{11}}{b_{20}}x^2 + 2\dfrac{b_{10}}{b_{20}}x \pm 2\sigma\left(\dfrac{\Delta_0}{b_{20}^2}\right)\left(\dfrac{\Delta_0}{b_{20}^2}\right)^{\frac{1}{2}} x\right\}, & \text{if } \Delta_2 = \Delta_1 = 0, \end{cases} \tag{36}$$

where Δ_0, Δ_1, and Δ_2 are given by (28).

Assume $\Delta_2 \neq 0$. The *confluent hypergeometric function of the first kind, of parameters $p \in \mathbb{C}$ and $q \in \mathbb{C} \setminus \mathbb{Z}_-$*, is the solution to the differential equation in the complex domain

$$zu'' + (q - z)u' - pu = 0,$$

given by the entire analytic function (see [9, (9.9.1)])

$$\Phi(p, q; z) = \sum_{k=0}^{\infty} \frac{(p)_k}{k!(q)_k} z^k, \tag{37}$$

where

$$(p)_k = \frac{\Gamma(p+k)}{\Gamma(p)} = \begin{cases} 1, & \text{if } k = 0, \\ p(p+1)\cdots(p+k-1), & \text{if } k \geqslant 1. \end{cases} \tag{38}$$

and Γ is the Euler Gamma Function.

Proposition 21. Consider a complex number λ. The *Hermite-Weber equation* (in the complex domain)

$$w''(z) - (z^2 - \lambda)w(z) = 0 \tag{39}$$

has two linearly independent solutions given by

$$w_1(z) = \mathrm{e}^{-\frac{1}{2}z^2} \Phi\left(\frac{1-\lambda}{4}, \frac{1}{2}; z^2\right), \qquad w_2(z) = \mathrm{e}^{-\frac{1}{2}z^2} z \Phi\left(\frac{3-\lambda}{4}, \frac{3}{2}; z^2\right). \tag{40}$$

Proof. A straightforward computation shows that w_1 and w_2 given by (40) solve (39).

Now we show that w_1 and w_2 are linearly independent. Since the Wronskian $\mathcal{W}$ of w_1 and w_2 is constant, it suffices to compute it at the origin, where we have

$$\mathcal{W}(0) = \begin{vmatrix} w_1(0) & w_2(0) \\ w_1'(0) & w_2'(0) \end{vmatrix} = \Phi\left(\frac{1-\lambda}{4}, \frac{1}{2}; 0\right) \Phi\left(\frac{3-\lambda}{4}, \frac{3}{2}; 0\right) = 1. \qquad \square$$

Proposition 22. The equation $Bu = 0$ has two linearly independent analytic solutions u_1 and u_2 given by

$$u_j(x) = \mathrm{e}^{-\frac{\mathrm{i}}{4b_{20}}\left(b_{11}x^2 + 2b_{10}x\right)} v_j(x), \tag{41}$$

where $j \in \{1, 2\}$,

$$v_j(x) = w_j\left(\left(-\frac{\Delta_2}{4b_{20}^2}\right)^{\frac{1}{4}} \left(x + \frac{\Delta_1}{2\Delta_2}\right)\right), \tag{42}$$

and w_1, and w_2 are given by (40), with

$$\lambda = \frac{1}{8}\left(-\frac{\Delta_2}{b_{20}^2}\right)^{-\frac{3}{2}} \frac{\Delta_1^2 - 4\Delta_2\Delta_0}{b_{20}^4}.$$

Proof. Set

$$v(x) = \mathrm{e}^{\frac{\mathrm{i}}{4b_{20}}\left(b_{11}x^2 + 2b_{10}x\right)} u(x).$$

A simple computation shows that $Bu = 0$ if and only if

$$v''(x) + \frac{1}{4b_{20}^2}\left(\Delta_2 x^2 + \Delta_1 x + \Delta_0\right) v(x) = 0. \tag{43}$$

Define

$$w(z) = v\left(\left(-\frac{\Delta_2}{4b_{20}^2}\right)^{-\frac{1}{4}} z - \frac{\Delta_1}{2\Delta_2}\right).$$

Then v satisfies equation (43) if and only if w is a solution to equation (39).

It follows that Proposition 22 is a consequence of Proposition 21. □

Proposition 23. Let u_1 and u_2 be as in Proposition 22 and assume $\operatorname{Arg} \frac{\Delta_2}{b_{20}^2} \neq 0$. For all $c_1, c_2 \in \mathbb{C}$ we have the following asymptotic expansions, with $\Sigma_\pm$ defined by (36).

(A) If $\frac{c_1}{\Gamma\left(\frac{1-\lambda}{4}\right)} \pm \frac{c_2}{2\Gamma\left(\frac{3-\lambda}{4}\right)} \neq 0$, we have

$$c_1 u_1(x) + c_2 u_2(x) = \sqrt{\pi}\left(-\frac{\Delta_2}{4b_{20}^2}\right)^{-\frac{1+\lambda}{8}} \left(\frac{c_1}{\Gamma\left(\frac{1-\lambda}{4}\right)} + \frac{c_2}{2\Gamma\left(\frac{3-\lambda}{4}\right)}\right) \mathrm{e}^{\mathrm{i}\Sigma_-(x)} |x|^{-\frac{1+\lambda}{2}} \left\{1 + \mathcal{O}\left(|x|^{-1}\right)\right\}, \quad \text{for } x \to +\infty,$$

$$c_1 u_1(x) + c_2 u_2(x) = \sqrt{\pi}\left(-\frac{\Delta_2}{4b_{20}^2}\right)^{-\frac{1+\lambda}{8}} \left(\frac{c_1}{\Gamma\left(\frac{1-\lambda}{4}\right)} - \frac{c_2}{2\Gamma\left(\frac{3-\lambda}{4}\right)}\right) \mathrm{e}^{\mathrm{i}\Sigma_-(x)} |x|^{-\frac{1+\lambda}{2}} \left\{1 + \mathcal{O}\left(|x|^{-1}\right)\right\}, \quad \text{for } x \to -\infty.$$

(B) If $c_1 = \frac{c}{\Gamma\left(\frac{3-\lambda}{4}\right)}$, $c_2 = -\frac{2c}{\Gamma\left(\frac{1-\lambda}{4}\right)}$, with $c \neq 0$, and $\lambda \notin \{1 + 2n : n \in \mathbb{Z}_+\}$, we have

$$c_1 u_1(x) + c_2 u_2(x) = \frac{c}{\sqrt{\pi}}\left(-\frac{\Delta_2}{4b_{20}^2}\right)^{-\frac{1-\lambda}{8}} \mathrm{e}^{\mathrm{i}\Sigma_+(x)} |x|^{-\frac{1-\lambda}{2}} \left\{1 + \mathcal{O}\left(|x|^{-1}\right)\right\}, \qquad \text{for } x \to +\infty,$$

$$c_1 u_1(x) + c_2 u_2(x) = \frac{2\sqrt{\pi}c}{\Gamma\left(\frac{1-\lambda}{4}\right)\Gamma\left(\frac{3-\lambda}{4}\right)}\left(-\frac{\Delta_2}{4b_{20}^2}\right)^{-\frac{1+\lambda}{8}} \mathrm{e}^{\mathrm{i}\Sigma_-(x)} |x|^{-\frac{1+\lambda}{2}} \left\{1 + \mathcal{O}\left(|x|^{-1}\right)\right\}, \quad \text{for } x \to -\infty,$$

(C) If $c_1 = \frac{c}{\Gamma\left(\frac{3-\lambda}{4}\right)}$, $c_2 = \frac{2c}{\Gamma\left(\frac{1-\lambda}{4}\right)}$, with $c \neq 0$, and $\lambda \notin \{1 + 2n : n \in \mathbb{Z}_+\}$, we have

$$c_1 u_1(x) + c_2 u_2(x) = \frac{2\sqrt{\pi}c}{\Gamma\left(\frac{1-\lambda}{4}\right)\Gamma\left(\frac{3-\lambda}{4}\right)}\left(-\frac{\Delta_2}{4b_{20}^2}\right)^{-\frac{1+\lambda}{8}} \mathrm{e}^{\mathrm{i}\Sigma_-(x)} |x|^{-\frac{1+\lambda}{2}} \left\{1 + \mathcal{O}\left(|x|^{-1}\right)\right\}, \quad \text{for } x \to +\infty,$$

$$c_1 u_1(x) + c_2 u_2(x) = \frac{c}{\sqrt{\pi}}\left(-\frac{\Delta_2}{4b_{20}^2}\right)^{-\frac{1-\lambda}{8}} \mathrm{e}^{\mathrm{i}\Sigma_+(x)} |x|^{-\frac{1-\lambda}{2}} \left\{1 + \mathcal{O}\left(|x|^{-1}\right)\right\}, \quad \text{for } x \to -\infty.$$

(D) If $c_1 = \frac{c}{\Gamma\left(\frac{3-\lambda}{4}\right)}$, $c_2 = \mp\frac{2c}{\Gamma\left(\frac{1-\lambda}{4}\right)}$, with $c \neq 0$, and $\lambda = 1 + 4n$, with $n \in \mathbb{Z}_+$, we have

$$c_1 u_1(x) + c_2 u_2(x) = \frac{c}{\sqrt{\pi}}\left(-\frac{\Delta_2}{4b_{20}^2}\right)^{\frac{n}{2}} \mathrm{e}^{\mathrm{i}\Sigma_+(x)} x^{2n} \left\{1 + \mathcal{O}\left(|x|^{-1}\right)\right\}, \qquad \text{for } |x| \to \infty.$$

(E) If $c_1 = \frac{c}{\Gamma\left(\frac{3-\lambda}{4}\right)}$, $c_2 = \mp\frac{2c}{\Gamma\left(\frac{1-\lambda}{4}\right)}$, with $c \neq 0$, and $\lambda = 3 + 4n$, with $n \in \mathbb{Z}_+$, we have

$$c_1 u_1(x) + c_2 u_2(x) = \pm\frac{c}{\sqrt{\pi}}\left(-\frac{\Delta_2}{4b_{20}^2}\right)^{\frac{1}{4}+\frac{n}{2}} \mathrm{e}^{\mathrm{i}\Sigma_+(x)} x^{2n+1} \left\{1 + \mathcal{O}\left(|x|^{-1}\right)\right\}, \qquad \text{for } |x| \to \infty.$$

Proof. Set

$$z=\left(-\frac{\Delta_2}{4b_{20}^2}\right)^{\frac14}\left(x+\frac{\Delta_1}{2\Delta_2}\right).$$

From (40), (41), and (42), it follows that

$$c_1u_1(x)+c_2u_2(x)=\mathrm{e}^{-\frac{\mathrm{i}}{4}\left(\frac{b_{11}}{b_{20}}x^2+2\frac{b_{10}}{b_{20}}x\right)}\Big(c_1w_1(z)+c_2w_2(z)\Big). \tag{44}$$

On the other side, since

$$\left(-\frac{\Delta_2}{b_{20}^2}\right)^{\frac12}=\left|\frac{\Delta_2}{b_{20}^2}\right|^{\frac12}\mathrm{e}^{\frac{\mathrm{i}}{2}\operatorname{Arg}\left(-\frac{\Delta_2}{b_{20}^2}\right)}=\left|\frac{\Delta_2}{b_{20}^2}\right|^{\frac12}\mathrm{e}^{\frac{\mathrm{i}}{2}\left(\operatorname{Arg}\frac{\Delta_2}{b_{20}^2}+\sigma\left(\frac{\Delta_2}{b_{20}^2}\right)\pi\right)}= \tag{45}$$
$$=\mathrm{e}^{\frac{\mathrm{i}}{2}\sigma\left(\frac{\Delta_2}{b_{20}^2}\right)\pi}\left(\frac{\Delta_2}{b_{20}^2}\right)^{\frac12}=\mathrm{i}\sigma\left(\frac{\Delta_2}{b_{20}^2}\right)\left(\frac{\Delta_2}{b_{20}^2}\right)^{\frac12},$$

we have

$$-\frac{\mathrm{i}}{4}\left(\frac{b_{11}}{b_{20}}x^2+2\frac{b_{10}}{b_{20}}x\right)\pm\frac12 z^2=-\frac{\mathrm{i}}{4}\left(\frac{b_{11}}{b_{20}}x^2+2\frac{b_{10}}{b_{20}}x\right)\pm\frac12\left(-\frac{\Delta_2}{4b_{20}^2}\right)^{\frac12}\left(x+\frac{\Delta_1}{2\Delta_2}\right)^2=\mathrm{i}\Sigma_{\mp}(x). \tag{46}$$

Moreover, since

$$\left|\operatorname{Arg}\left(\left(-\frac{\Delta_2}{b_{20}^2}\right)^{\frac14}\right)\right|<\frac{\pi}{4},$$

and

$$\lim_{|x|\to\infty}\operatorname{Arg}\left(1+\frac{\Delta_1}{2\Delta_2x}\right)=0,$$

there exists $0<\epsilon<\frac{\pi}{4}$, such that

$$|\operatorname{Arg}(\pm z)|\leqslant\frac{\pi}{4}-\epsilon,\qquad \text{for } x\to\pm\infty. \tag{47}$$

In particular

$$\pm z=\left(-\frac{\Delta_2}{4b_{20}^2}\right)^{\frac14}|x|\left(1+\mathcal{O}\left(|x|^{-1}\right)\right),\qquad \text{for } x\to\pm\infty. \tag{48}$$

In conclusion the statement follows from (44), (46), (47), (48), and Proposition 37. $\square$

Proposition 24. Let u_1 and u_2 be as in Proposition 22 and assume $\operatorname{Arg}\frac{\Delta_2}{b_{20}^2}=0$. For all $c_1,c_2\in\mathbb{C}$ we have the following asymptotic expansions.

(A) If $\left(\frac{\mathrm{i}c_1}{\Gamma\left(\frac{1+\lambda}{4}\right)}\mp\frac{c_2}{2\Gamma\left(\frac{3+\lambda}{4}\right)}\right)\left(\frac{c_1}{\Gamma\left(\frac{1-\lambda}{4}\right)}\pm\frac{c_2}{2\Gamma\left(\frac{3-\lambda}{4}\right)}\right)\neq0$, we have

$$c_1u_1(x)+c_2u_2(x)=\sqrt{\pi}\mathrm{e}^{-\mathrm{i}\frac{1+\lambda}{4}\pi}\left(-\frac{\Delta_2}{4b_{20}^2}\right)^{-\frac{1-\lambda}{8}}\left(\frac{\mathrm{i}c_1}{\Gamma\left(\frac{1+\lambda}{4}\right)}-\frac{c_2}{2\Gamma\left(\frac{3+\lambda}{4}\right)}\right)\mathrm{e}^{\mathrm{i}\Sigma_+(x)}|x|^{-\frac{1-\lambda}{2}}\left\{1+\mathcal{O}\left(|x|^{-1}\right)\right\}+$$
$$+\sqrt{\pi}\left(-\frac{\Delta_2}{4b_{20}^2}\right)^{-\frac{1+\lambda}{8}}\left(\frac{c_1}{\Gamma\left(\frac{1-\lambda}{4}\right)}+\frac{c_2}{2\Gamma\left(\frac{3-\lambda}{4}\right)}\right)\mathrm{e}^{\mathrm{i}\Sigma_-(x)}|x|^{-\frac{1+\lambda}{2}}\left\{1+\mathcal{O}\left(|x|^{-1}\right)\right\},$$

for $x \to +\infty$,

$$c_1 u_1(x) + c_2 u_2(x) = \sqrt{\pi} e^{-i \frac{1+\lambda}{4}\pi} \left(-\frac{\Delta_2}{4b_{20}^2}\right)^{-\frac{1-\lambda}{8}} \left(\frac{i c_1}{\Gamma\left(\frac{1+\lambda}{4}\right)} + \frac{c_2}{2\Gamma\left(\frac{3+\lambda}{4}\right)}\right) e^{i\Sigma_+(x)} |x|^{-\frac{1-\lambda}{2}} \left\{1 + \mathcal{O}\left(|x|^{-1}\right)\right\}$$

$$+ \sqrt{\pi} \left(-\frac{\Delta_2}{4b_{20}^2}\right)^{-\frac{1+\lambda}{8}} \left(\frac{c_1}{\Gamma\left(\frac{1-\lambda}{4}\right)} - \frac{c_2}{2\Gamma\left(\frac{3-\lambda}{4}\right)}\right) e^{i\Sigma_-(x)} |x|^{-\frac{1+\lambda}{2}} \left\{1 + \mathcal{O}\left(|x|^{-1}\right)\right\},$$

for $x \to -\infty$,

(B) If $c_1 = \frac{c}{\Gamma\left(\frac{3+\lambda}{4}\right)}$, $c_2 = \frac{2ic}{\Gamma\left(\frac{1+\lambda}{4}\right)}$, with $c \neq 0$, and $\lambda \notin \{-(1+2n) : n \in \mathbb{Z}_+\}$ we have

$$c_1 u_1(x) + c_2 u_2(x) = \frac{c}{\sqrt{\pi}} e^{i\Sigma_-(x)} e^{i \frac{1+\lambda}{4}\pi} \left(-\frac{\Delta_2}{4b_{20}^2}\right)^{-\frac{1+\lambda}{8}} |x|^{-\frac{1+\lambda}{2}} \left\{1 + \mathcal{O}\left(|x|^{-1}\right)\right\},$$

for $x \to +\infty$,

$$c_1 u_1(x) + c_2 u_2(x) = \frac{2i\sqrt{\pi} e^{-i \frac{1+\lambda}{4}\pi} c}{\Gamma\left(\frac{1+\lambda}{4}\right) \Gamma\left(\frac{3+\lambda}{4}\right)} \left(-\frac{\Delta_2}{4b_{20}^2}\right)^{-\frac{1-\lambda}{8}} e^{i\Sigma_+(x)} |x|^{-\frac{1-\lambda}{2}} \left\{1 + \mathcal{O}\left(|x|^{-1}\right)\right\} +$$

$$+ \frac{c\, e^{-i \frac{1+\lambda}{4}\pi}}{\sqrt{\pi}} \left(-\frac{\Delta_2}{4b_{20}^2}\right)^{-\frac{1+\lambda}{8}} e^{i\Sigma_-(x)} |x|^{-\frac{1+\lambda}{2}} \left\{1 + \mathcal{O}\left(|x|^{-1}\right)\right\}, \qquad \text{for } x \to -\infty.$$

(C) If $c_1 = \frac{c}{\Gamma\left(\frac{3+\lambda}{4}\right)}$, $c_2 = -\frac{2ic}{\Gamma\left(\frac{1+\lambda}{4}\right)}$, with $c \neq 0$, and $\lambda \notin \{-(1+2n) : n \in \mathbb{Z}_+\}$, we have

$$c_1 u_1(x) + c_2 u_2(x) = \frac{2i\sqrt{\pi} e^{-i \frac{1+\lambda}{4}\pi} c}{\Gamma\left(\frac{1+\lambda}{4}\right) \Gamma\left(\frac{3+\lambda}{4}\right)} \left(-\frac{\Delta_2}{4b_{20}^2}\right)^{-\frac{1-\lambda}{8}} e^{i\Sigma_+(x)} |x|^{-\frac{1-\lambda}{2}} \left\{1 + \mathcal{O}\left(|x|^{-1}\right)\right\} +$$

$$+ \frac{c\, e^{-i \frac{1+\lambda}{4}\pi}}{\sqrt{\pi}} \left(-\frac{\Delta_2}{4b_{20}^2}\right)^{-\frac{1+\lambda}{8}} e^{i\Sigma_-(x)} |x|^{-\frac{1+\lambda}{2}} \left\{1 + \mathcal{O}\left(|x|^{-1}\right)\right\}, \qquad \text{for } x \to +\infty,$$

$$c_1 u_1(x) + c_2 u_2(x) = \frac{c}{\sqrt{\pi}} e^{i \frac{1+\lambda}{4}\pi} \left(-\frac{\Delta_2}{4b_{20}^2}\right)^{-\frac{1+\lambda}{8}} e^{i\Sigma_-(x)} |x|^{-\frac{1+\lambda}{2}} \left\{1 + \mathcal{O}\left(|x|^{-1}\right)\right\}, \qquad \text{for } x \to -\infty.$$

(D) If $c_1 = \frac{c}{\Gamma\left(\frac{3+\lambda}{4}\right)}$, $c_2 = \pm\frac{2ic}{\Gamma\left(\frac{1+\lambda}{4}\right)}$, with $c \neq 0$, and $\lambda = -(1+4n)$, with $n \in \mathbb{Z}_+$, we have

$$c_1 u_1(x) + c_2 u_2(x) = \frac{(-1)^n c}{\sqrt{\pi}} \left(-\frac{\Delta_2}{4b_{20}^2}\right)^{\frac{n}{2}} e^{i\Sigma_-(x)} x^{2n} \left\{1 + \mathcal{O}\left(|x|^{-1}\right)\right\}, \qquad \text{for } |x| \to \infty.$$

(E) If $c_1 = \frac{c}{\Gamma\left(\frac{3+\lambda}{4}\right)}$, $c_2 = \pm\frac{2ic}{\Gamma\left(\frac{1+\lambda}{4}\right)}$, with $c \neq 0$, and $\lambda = -(3+4n)$, with $n \in \mathbb{Z}_+$, we have

$$c_1 u_1(x) + c_2 u_2(x) = \mp i \frac{(-1)^n c}{\sqrt{\pi}} \left(-\frac{\Delta_2}{4b_{20}^2}\right)^{\frac{1}{4}+\frac{n}{2}} e^{i\Sigma_-(x)} x^{2n+1} \left\{1 + \mathcal{O}\left(|x|^{-1}\right)\right\}, \qquad \text{for } |x| \to \infty.$$

(F) If $c_1 = \frac{c}{\Gamma\left(\frac{3-\lambda}{4}\right)}$, $c_2 = -\frac{2c}{\Gamma\left(\frac{1-\lambda}{4}\right)}$, with $c \neq 0$, and $\lambda \notin \{1+2n : n \in \mathbb{Z}_+\}$, we have

$$c_1 u_1(x) + c_2 u_2(x) = \frac{c}{\sqrt{\pi}} \left(-\frac{\Delta_2}{4b_{20}^2}\right)^{-\frac{1-\lambda}{8}} e^{i\Sigma_+(x)} |x|^{-\frac{1-\lambda}{2}} \left\{1 + \mathcal{O}\left(|x|^{-1}\right)\right\}, \qquad \text{for } x \to +\infty,$$

$$c_1 u_1(x) + c_2 u_2(x) = -\frac{c e^{-i \frac{1+\lambda}{2}\pi}}{\sqrt{\pi}} \left(-\frac{\Delta_2}{4b_{20}^2}\right)^{-\frac{1-\lambda}{8}} e^{i\Sigma_+(x)} |x|^{-\frac{1-\lambda}{2}} \left\{1 + \mathcal{O}\left(|x|^{-1}\right)\right\} +$$

$$+\frac{2\sqrt{\pi}c}{\Gamma\left(\frac{1-\lambda}{4}\right)\Gamma\left(\frac{3-\lambda}{4}\right)}\left(-\frac{\Delta_2}{4b_{20}^2}\right)^{-\frac{1+\lambda}{8}}\mathrm{e}^{\mathrm{i}\Sigma_-(x)}\,|x|^{-\frac{1+\lambda}{2}}\left\{1+\mathcal{O}\left(|x|^{-1}\right)\right\},\qquad \text{for } x\to-\infty,$$

(G) If $c_1=\frac{c}{\Gamma\left(\frac{3-\lambda}{4}\right)}$, $c_2=\frac{2c}{\Gamma\left(\frac{1-\lambda}{4}\right)}$, with $c\neq 0$, and $\lambda\notin\{1+2n : n\in\mathbb{Z}_+\}$, we have

$$c_1u_1(x)+c_2u_2(x)=-\frac{c\,\mathrm{e}^{-\mathrm{i}\frac{1+\lambda}{2}\pi}}{\sqrt{\pi}}\left(-\frac{\Delta_2}{4b_{20}^2}\right)^{-\frac{1-\lambda}{8}}\mathrm{e}^{\mathrm{i}\Sigma_+(x)}\,|x|^{-\frac{1-\lambda}{2}}\left\{1+\mathcal{O}\left(|x|^{-1}\right)\right\}+$$

$$+\frac{2\sqrt{\pi}c}{\Gamma\left(\frac{1-\lambda}{4}\right)\Gamma\left(\frac{3-\lambda}{4}\right)}\left(-\frac{\Delta_2}{4b_{20}^2}\right)^{-\frac{1+\lambda}{8}}\mathrm{e}^{\mathrm{i}\Sigma_-(x)}\,|x|^{-\frac{1+\lambda}{2}}\left\{1+\mathcal{O}\left(|x|^{-1}\right)\right\},\qquad \text{for } x\to+\infty,$$

$$c_1u_1(x)+c_2u_2(x)=\frac{c}{\sqrt{\pi}}\left(-\frac{\Delta_2}{4b_{20}^2}\right)^{-\frac{1-\lambda}{8}}\mathrm{e}^{\mathrm{i}\Sigma_+(x)}\,|x|^{-\frac{1-\lambda}{2}}\left\{1+\mathcal{O}\left(|x|^{-1}\right)\right\},\qquad \text{for } x\to-\infty,$$

(H) If $c_1=\frac{c}{\Gamma\left(\frac{3-\lambda}{4}\right)}$, $c_2=\mp\frac{2c}{\Gamma\left(\frac{1-\lambda}{4}\right)}$, with $c\neq 0$, and $\lambda=1+4n$, with $n\in\mathbb{Z}_+$, we have

$$c_1u_1(x)+c_2u_2(x)=\frac{c}{\sqrt{\pi}}\left(-\frac{\Delta_2}{4b_{20}^2}\right)^{\frac{n}{2}}\mathrm{e}^{\mathrm{i}\Sigma_+(x)}x^{2n}\left\{1+\mathcal{O}\left(|x|^{-1}\right)\right\},\qquad \text{for } |x|\to\infty.$$

(I) If $c_1=\frac{c}{\Gamma\left(\frac{3-\lambda}{4}\right)}$, $c_2=\mp\frac{2c}{\Gamma\left(\frac{1-\lambda}{4}\right)}$, with $c\neq 0$, and $\lambda=3+4n$, with $n\in\mathbb{Z}_+$, we have

$$c_1u_1(x)+c_2u_2(x)=\pm\frac{c}{\sqrt{\pi}}\left(-\frac{\Delta_2}{4b_{20}^2}\right)^{\frac{1}{4}+\frac{n}{2}}\mathrm{e}^{\mathrm{i}\Sigma_+(x)}x^{2n+1}\left\{1+\mathcal{O}\left(|x|^{-1}\right)\right\},\qquad \text{for } |x|\to\infty.$$

Proof. Set

$$z=\left(-\frac{\Delta_2}{4b_{20}^2}\right)^{\frac{1}{4}}\left(x+\frac{\Delta_1}{2\Delta_2}\right).$$

From (40), (41), and (42), it follows that

$$c_1u_1(x)+c_2u_2(x)=\mathrm{e}^{-\frac{\mathrm{i}}{4}\left(\frac{b_{11}}{b_{20}}x^2+2\frac{b_{10}}{b_{20}}x\right)}\Big(c_1w_1(z)+c_2w_2(z)\Big). \tag{49}$$

On the other side we have (see (46))

$$-\frac{\mathrm{i}}{4}\left(\frac{b_{11}}{b_{20}}x^2+2\frac{b_{10}}{b_{20}}x\right)\pm\frac{1}{2}\,z^2=\mathrm{i}\Sigma_\mp(x). \tag{50}$$

Moreover, since

$$\operatorname{Arg}\left(\left(-\frac{\Delta_2}{4b_{20}^2}\right)^{\frac{1}{4}}\right)=\frac{\pi}{4},$$

and

$$\lim_{|x|\to\infty}\operatorname{Arg}\left(1+\frac{\Delta_1}{2\Delta_2 x}\right)=0,$$

given $0<\epsilon<\frac{\pi}{4}$, we have

$$\left|\operatorname{Arg}(\pm z)-\frac{\pi}{4}\right|\leqslant\epsilon,\qquad \text{for } x\to\pm\infty. \tag{51}$$

In particular

$$\pm z = \left(-\frac{\Delta_2}{4b_{20}^2}\right)^{\frac{1}{4}} |x| \left(1 + \mathcal{O}\left(|x|^{-1}\right)\right), \qquad \text{for } x \to \pm\infty, \tag{52}$$

and

$$\mp \mathrm{i} z = \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}} \left(-\frac{\Delta_2}{4b_{20}^2}\right)^{\frac{1}{4}} |x| \left(1 + \mathcal{O}\left(|x|^{-1}\right)\right), \qquad \text{for } x \to \pm\infty. \tag{53}$$

In conclusion the statement follows from (49), (50), (51), (52), (53), and Proposition 38. □

Assume $\Delta_2 = 0$ and $\Delta_1 \neq 0$. The *Airy functions* are two linearly independent solutions to the differential equation in the complex domain

$$u''(z) - zu(z) = 0,$$

given by the entire analytic functions (see [9, (5.17.3)])

$$\mathrm{Ai}(z) = \sum_{k=0}^{\infty} \frac{z^{3k}}{3^{2k+\frac{2}{3}} k! \Gamma\left(k + \frac{2}{3}\right)} - \sum_{k=0}^{\infty} \frac{z^{3k+1}}{3^{2k+\frac{4}{3}} k! \Gamma\left(k + \frac{4}{3}\right)},$$

and

$$\mathrm{Bi}(z) = 3^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{z^{3k}}{3^{2k+\frac{2}{3}} k! \Gamma\left(k + \frac{2}{3}\right)} + 3^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{z^{3k+1}}{3^{2k+\frac{4}{3}} k! \Gamma\left(k + \frac{4}{3}\right)}.$$

Proposition 25. The equation $Bu = 0$ has two linearly independent analytic solutions u_1 and u_2 given by

$$u_j(x) = \mathrm{e}^{-\frac{\mathrm{i}}{4b_{20}}\left(b_{11}x^2 + 2b_{10}x\right)} v_j(x), \tag{54}$$

where $j \in \{1, 2\}$, and

$$v_1(x) = \mathrm{Ai}\left(\left(-\frac{\Delta_1}{4b_{20}^2}\right)^{\frac{1}{3}} \left(x + \frac{\Delta_0}{\Delta_1}\right)\right), \qquad v_2(x) = \mathrm{Bi}\left(\left(-\frac{\Delta_1}{4b_{20}^2}\right)^{\frac{1}{3}} \left(x + \frac{\Delta_0}{\Delta_1}\right)\right). \tag{55}$$

Proof. Set

$$v(x) = \mathrm{e}^{\frac{\mathrm{i}}{4b_{20}}\left(b_{11}x^2 + 2b_{10}x\right)} u(x),$$

and

$$w(z) = v\left(\left(-\frac{\Delta_1}{4b_{20}^2}\right)^{-\frac{1}{3}} z - \frac{\Delta_0}{\Delta_1}\right).$$

Then a simple computation shows that $Bu = 0$ if and only if w solves the Airy equation

$$w''(z) - zw(z) = 0.$$

□

Proposition 26. Let u_1 and u_2 be as in Proposition 25. Then we have the following asymptotic expansions.

$$\begin{aligned} c_1 u_1(x) + c_2 u_2(x) = {} & \frac{1}{2\sqrt{\pi}} \left(-\frac{\Delta_1}{4b_{20}^2}\right)^{-\frac{1}{12}} |x|^{-\frac{1}{4}} \cdot \\ & \cdot \left\{2c_2 \mathrm{e}^{\mathrm{i}\Sigma_-(x)} \left(1 + \mathcal{O}\left(|x|^{-1}\right)\right) - (c_1 + \mathrm{i}c_2)\, \mathrm{e}^{\mathrm{i}\Sigma_+(x)} \left(1 + \mathcal{O}\left(|x|^{-1}\right)\right)\right\}, \qquad \text{for } x \to +\infty, \end{aligned} \tag{56}$$

$$(57)\quad c_1u_1(x)+c_2u_2(x)=\frac{1}{2\sqrt{2\pi}}\left(-\frac{\Delta_1}{4b_{20}^2}\right)^{-\frac{1}{12}}|x|^{-\frac14}\cdot$$
$$\cdot\left\{\left((1-\mathrm{i})c_1+(1+\mathrm{i})c_2\right)\mathrm{e}^{\mathrm{i}\Sigma_+(x)}\left(1+\mathcal{O}\left(|x|^{-1}\right)\right)+\right.$$
$$\left.+\left((1+\mathrm{i})c_1+(1-\mathrm{i})c_2\right)\mathrm{e}^{\mathrm{i}\Sigma_-(x)}\left(1+\mathcal{O}\left(|x|^{-1}\right)\right)\right\},\qquad \text{for } x\to-\infty.$$

Proof. First we prove the following asymptotic expansions.

$$(58)\qquad v_1(x)=\frac{1}{2\sqrt{\pi}}\left(-\frac{\Delta_1}{4b_{20}^2}\right)^{-\frac{1}{12}}|x|^{-\frac14}\,\mathrm{e}^{-\frac23\left(-\frac{\Delta_1}{4b_{20}^2}\right)^{\frac12}|x|^{\frac32}\left(1+\frac{\Delta_0}{\Delta_1 x}\right)^{\frac32}}\left(1+\mathcal{O}\left(|x|^{-1}\right)\right),$$

$$(59)\qquad v_2(x)=\frac{1}{2\sqrt{\pi}}\left(-\frac{\Delta_1}{4b_{20}^2}\right)^{-\frac{1}{12}}|x|^{-\frac14}\left\{2\mathrm{e}^{\frac23\left(-\frac{\Delta_1}{4b_{20}^2}\right)^{\frac12}|x|^{\frac32}\left(1+\frac{\Delta_0}{\Delta_1 x}\right)^{\frac32}}\left(1+\mathcal{O}\left(|x|^{-1}\right)\right)+\right.$$
$$\left.+\mathrm{i}\mathrm{e}^{-\frac23\left(-\frac{\Delta_1}{4b_{20}^2}\right)^{\frac12}|x|^{\frac32}\left(1+\frac{\Delta_0}{\Delta_1 x}\right)^{\frac32}}\left(1+\mathcal{O}\left(|x|^{-1}\right)\right)\right\},$$

for $x\to+\infty$, and

$$(60)\qquad v_1(x)=\frac{1}{2\sqrt{2\pi}}\left(-\frac{\Delta_1}{4b_{20}^2}\right)^{-\frac{1}{12}}|x|^{-\frac14}\left\{(1-\mathrm{i})\,\mathrm{e}^{\mathrm{i}\frac23\left(-\frac{\Delta_1}{4b_{20}^2}\right)^{\frac12}|x|^{\frac32}\left(1+\frac{\Delta_0}{\Delta_1 x}\right)^{\frac32}}\left(1+\mathcal{O}\left(|x|^{-1}\right)\right)+\right.$$
$$\left.+(1+\mathrm{i})\,\mathrm{e}^{-\mathrm{i}\frac23\left(-\frac{\Delta_1}{4b_{20}^2}\right)^{\frac12}|x|^{\frac32}\left(1+\frac{\Delta_0}{\Delta_1 x}\right)^{\frac32}}\left(1+\mathcal{O}\left(|x|^{-1}\right)\right)\right\},$$

$$(61)\qquad v_2(x)=\frac{1}{2\sqrt{2\pi}}\left(-\frac{\Delta_1}{4b_{20}^2}\right)^{-\frac{1}{12}}|x|^{-\frac14}\left\{(1+\mathrm{i})\,\mathrm{e}^{\mathrm{i}\frac23\left(-\frac{\Delta_1}{4b_{20}^2}\right)^{\frac12}|x|^{\frac32}\left(1+\frac{\Delta_0}{\Delta_1 x}\right)^{\frac32}}\left(1+\mathcal{O}\left(|x|^{-1}\right)\right)+\right.$$
$$\left.+(1-\mathrm{i})\,\mathrm{e}^{-\mathrm{i}\frac23\left(-\frac{\Delta_1}{4b_{20}^2}\right)^{\frac12}|x|^{\frac32}\left(1+\frac{\Delta_0}{\Delta_1 x}\right)^{\frac32}}\left(1+\mathcal{O}\left(|x|^{-1}\right)\right)\right\},$$

for $x\to-\infty$.

Let $0<\epsilon<\pi/3$. Airy functions have the following asymptotic expansions for $|z|\to\infty$, see [1, 10.4.59, and 10.4.65]:

$$(62)\qquad \mathrm{Ai}(z)=\frac{z^{-\frac14}}{2\sqrt{\pi}}\,\mathrm{e}^{-\frac23 z^{\frac32}}\left(1+\mathcal{O}\left(|z|^{-\frac32}\right)\right),\qquad \text{for } |\operatorname{Arg} z|\leqslant\pi-\epsilon.$$

$$(63)\qquad \mathrm{Bi}(z)=\sqrt{\frac{2}{\pi}}\,\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\left(\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}z\right)^{-\frac14}\cdot\left\{\sin\left(\frac23\left(\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}z\right)^{\frac32}+\frac{\pi}{4}-\frac{\log 2}{2}\,\mathrm{i}\right)\left(1+\mathcal{O}\left(|z|^{-3}\right)\right)-\right.$$
$$\left.-\cos\left(\frac23\left(\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}z\right)^{\frac32}+\frac{\pi}{4}-\frac{\log 2}{2}\,\mathrm{i}\right)\cdot\mathcal{O}\left(|z|^{-\frac32}\right)\right\}$$
$$=\frac{\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}z^{-\frac14}}{\sqrt{2\pi}}\left\{(1-\mathrm{i})\,\mathrm{e}^{\frac23 z^{\frac32}}\left(1+\mathcal{O}\left(|z|^{-\frac32}\right)\right)+\frac{1+\mathrm{i}}{2}\,\mathrm{e}^{-\frac23 z^{\frac32}}\left(1+\mathcal{O}\left(|z|^{-\frac32}\right)\right)\right\},$$
$$\text{for } -\frac{\pi}{3}+\epsilon\leqslant\operatorname{Arg} z\leqslant\frac{\pi}{3}+\epsilon,$$

and, see [1, 10.4.60, and 10.4.64]:

$$\begin{aligned}
\text{(64)}\qquad \operatorname{Ai}(z) &= \frac{(-z)^{-\frac{1}{4}}}{\sqrt{\pi}}\left\{\sin\left(\frac{2}{3}(-z)^{\frac{3}{2}}+\frac{\pi}{4}\right)\left(1+\mathcal{O}\left(|z|^{-3}\right)\right)-\cos\left(\frac{2}{3}(-z)^{\frac{3}{2}}+\frac{\pi}{4}\right)\cdot\mathcal{O}\left(|z|^{-\frac{3}{2}}\right)\right\}\\
&= \frac{(-z)^{-\frac{1}{4}}}{2\sqrt{2\pi}}\left\{(1-\mathrm{i})\,\mathrm{e}^{\mathrm{i}\frac{2}{3}(-z)^{\frac{3}{2}}}\left(1+\mathcal{O}\left(|z|^{-\frac{3}{2}}\right)\right)+(1+\mathrm{i})\,\mathrm{e}^{-\mathrm{i}\frac{2}{3}(-z)^{\frac{3}{2}}}\left(1+\mathcal{O}\left(|z|^{-\frac{3}{2}}\right)\right)\right\},\\
&\qquad\qquad \text{for } |\operatorname{Arg}(-z)|\leqslant\frac{2\pi}{3}-\epsilon,\\
\text{(65)}\qquad \operatorname{Bi}(z) &= \frac{(-z)^{-\frac{1}{4}}}{\sqrt{\pi}}\left\{\cos\left(\frac{2}{3}(-z)^{\frac{3}{2}}+\frac{\pi}{4}\right)\left(1+\mathcal{O}\left(|z|^{-3}\right)\right)+\sin\left(\frac{2}{3}(-z)^{\frac{3}{2}}+\frac{\pi}{4}\right)\cdot\mathcal{O}\left(|z|^{-\frac{3}{2}}\right)\right\}\\
&= \frac{(-z)^{-\frac{1}{4}}}{2\sqrt{2\pi}}\left\{(1+\mathrm{i})\,\mathrm{e}^{\mathrm{i}\frac{2}{3}(-z)^{\frac{3}{2}}}\left(1+\mathcal{O}\left(|z|^{-\frac{3}{2}}\right)\right)+(1-\mathrm{i})\,\mathrm{e}^{-\mathrm{i}\frac{2}{3}(-z)^{\frac{3}{2}}}\left(1+\mathcal{O}\left(|z|^{-\frac{3}{2}}\right)\right)\right\},\\
&\qquad\qquad \text{for } |\operatorname{Arg}(-z)|\leqslant\frac{2\pi}{3}-\epsilon.
\end{aligned}$$

Let

$$z=\left(-\frac{\Delta_1}{4b_{20}^2}\right)^{\frac{1}{3}}\left(x+\frac{\Delta_0}{\Delta_1}\right), \tag{66}$$

and

$$0<\epsilon<\frac{\pi}{6}.$$

Since

$$\pm z=\left(-\frac{\Delta_1}{4b_{20}^2}\right)^{\frac{1}{3}}|x|\left(1+\frac{\Delta_0}{\Delta_1 x}\right),\qquad \text{for } x\to\pm\infty,$$

$$-\frac{\pi}{3}<\operatorname{Arg}\left(\left(-\frac{\Delta_1}{4b_{20}^2}\right)^{\frac{1}{3}}\right)\leqslant\frac{\pi}{3},$$

and

$$\lim_{|x|\to\infty}\left|\operatorname{Arg}\left(1+\frac{\Delta_0}{\Delta_1 x}\right)\right|=0,$$

we have for $x\to\pm\infty$:

$$\operatorname{Arg}(\pm z)=\operatorname{Arg}\left(\left(-\frac{\Delta_1}{4b_{20}^2}\right)^{\frac{1}{3}}\right)+\operatorname{Arg}\left(1+\frac{\Delta_0}{\Delta_1 x}\right)\leqslant\frac{\pi}{3}+\epsilon\leqslant\frac{2\pi}{3}-\epsilon,$$

for

$$\left|\operatorname{Arg}\left(1+\frac{\Delta_0}{\Delta_1 x}\right)\right|\leqslant\epsilon,$$

and

$$\operatorname{Arg}(\pm z)=\operatorname{Arg}\left(\left(-\frac{\Delta_1}{4b_{20}^2}\right)^{\frac{1}{3}}\right)+\operatorname{Arg}\left(1+\frac{\Delta_0}{\Delta_1 x}\right)\geqslant-\frac{\pi}{3}+\epsilon\geqslant-\frac{2\pi}{3}+\epsilon,$$

for

$$\left|\operatorname{Arg}\left(1+\frac{\Delta_0}{\Delta_1 x}\right)\right|\leqslant\frac{1}{2}\left[\frac{\pi}{3}+\operatorname{Arg}\left(\left(-\frac{\Delta_1}{4b_{20}^2}\right)^{\frac{1}{3}}\right)\right],$$

and

$$\epsilon \leqslant \frac{1}{2}\left[\frac{\pi}{3} + \operatorname{Arg}\left(\left(-\frac{\Delta_1}{4b_{20}^2}\right)^{\frac{1}{3}}\right)\right].$$

This shows that we can make the substitution (66) into expansions (62), (63), (64), and (65).

Since

$$(\pm z)^{-\frac{1}{4}} = \left(-\frac{\Delta_1}{4b_{20}^2}\right)^{-\frac{1}{12}} |x|^{-\frac{1}{4}} \left(1 + \mathcal{O}\left(|x|^{-1}\right)\right), \qquad \text{for } x \to \pm\infty,$$

thanks to (55), we obtain (58), (59), (60), and (61).

Now observe that

$$-\mathrm{i}(-x)^{\frac{3}{2}} = \mathrm{e}^{\mathrm{i}\frac{3\pi}{2}}(-x)^{\frac{3}{2}} = \mathrm{e}^{\frac{3}{2}(\log(-x)+\mathrm{i}\pi)} = x^{\frac{3}{2}}, \quad \text{for } x < 0,$$

and (see (45))

$$\left(-\frac{\Delta_1}{4b_{20}^2}\right)^{\frac{1}{2}} = \mathrm{i}\sigma\left(\frac{\Delta_1}{4b_{20}^2}\right)\left(\frac{\Delta_1}{4b_{20}^2}\right)^{\frac{1}{2}}.$$

It follows that

$$\begin{aligned}
&-\frac{\mathrm{i}}{4b_{20}}\left(b_{11}x^2 + 2b_{10}x\right) \pm \frac{2}{3}\left(-\frac{\Delta_1}{4b_{20}^2}\right)^{\frac{1}{2}} |x|^{\frac{3}{2}} \left(1 + \frac{\Delta_0}{\Delta_1 x}\right)^{\frac{3}{2}} = \\
&\qquad = -\frac{\mathrm{i}}{4}\left\{\frac{b_{11}}{b_{20}}\,x^2 + 2\frac{b_{10}}{b_{20}}\,x \mp \frac{4}{3}\sigma\left(\frac{\Delta_1}{b_{20}^2}\right)\left(\frac{\Delta_1}{b_{20}^2}\right)^{\frac{1}{2}} x^{\frac{3}{2}}\left(1 + \frac{\Delta_0}{\Delta_1 x}\right)^{\frac{3}{2}}\right\} = \mathrm{i}\Sigma_{\mp}(x), \qquad \text{for } x \to +\infty,
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{\mathrm{i}}{4b_{20}}\left(b_{11}x^2 + 2b_{10}x\right) \pm \mathrm{i}\frac{2}{3}\left(-\frac{\Delta_1}{4b_{20}^2}\right)^{\frac{1}{2}} |x|^{\frac{3}{2}} \left(1 + \frac{\Delta_0}{\Delta_1 x}\right)^{\frac{3}{2}} = \\
&\qquad = -\frac{\mathrm{i}}{4}\left\{\frac{b_{11}}{b_{20}}\,x^2 + 2\frac{b_{10}}{b_{20}}\,x \pm \frac{4}{3}\sigma\left(\frac{\Delta_1}{b_{20}^2}\right)\left(\frac{\Delta_1}{b_{20}^2}\right)^{\frac{1}{2}} x^{\frac{3}{2}}\left(1 + \frac{\Delta_0}{\Delta_1 x}\right)^{\frac{3}{2}}\right\} = \mathrm{i}\Sigma_{\pm}(x), \qquad \text{for } x \to -\infty.
\end{aligned}$$

It follows that (54), (58), (59), (60), and (61) imply (56), and (57). □

Assume $\Delta_2 = \Delta_1 = 0$. In this case it is sufficient to observe that the general solution is given by (see (45))

$$cu_1(x) + c_2u_2(x) = \begin{cases} \mathrm{e}^{-\frac{\mathrm{i}}{4}\left(\frac{b_{11}}{b_{20}}x^2 + 2\frac{b_{10}}{b_{20}}x\right)}\left\{c_1\mathrm{e}^{\frac{\mathrm{i}}{2}\sigma\left(\frac{\Delta_0}{b_{20}^2}\right)\left(\frac{\Delta_0}{b_{20}^2}\right)^{\frac{1}{2}}x} + c_2\mathrm{e}^{-\frac{\mathrm{i}}{2}\sigma\left(\frac{\Delta_0}{b_{20}^2}\right)\left(\frac{\Delta_0}{b_{20}^2}\right)^{\frac{1}{2}}x}\right\} = \\ \qquad = c_1\mathrm{e}^{\mathrm{i}\Sigma_-(x)} + c_2\mathrm{e}^{\mathrm{i}\Sigma_+(x)}, & \text{if } \Delta_0 \neq 0, \\ \mathrm{e}^{-\frac{\mathrm{i}}{4}\left(\frac{b_{11}}{b_{20}}x^2 + 2\frac{b_{10}}{b_{20}}x\right)}\left(c_1 + c_2x\right) = \left(c_1 + c_2x\right)\mathrm{e}^{\mathrm{i}\Sigma_{\pm}(x)}, & \text{if } \Delta_0 = 0. \end{cases} \tag{67}$$

4.3.2 Proof of Theorem 17

Theorem 27. *B is globally regular and one-to-one if and only if*

$$\mathrm{e}^{\mathrm{i}x\Xi_{\pm}} \notin \mathcal{S}', \tag{68}$$

or

$$\text{(69)} \qquad \mathrm{e}^{\mathrm{i}x\Xi_-} \notin \mathcal{S}', \quad \mathrm{e}^{\mathrm{i}x\Xi_+} \in \mathcal{S}, \quad \Delta_2 \neq 0, \quad \lambda \notin \{1+2n : n \in \mathbb{Z}_+\},$$

or

$$\text{(70)} \qquad \mathrm{e}^{\mathrm{i}x\Xi_-} \notin \mathcal{S}', \quad \mathrm{e}^{\mathrm{i}x\Xi_+} \in \mathcal{S}, \quad \Delta_2 = 0.$$

Proof. We have the following asymptotic expansions for $|x| \to \infty$.

If $\Delta_2 \neq 0$,

$$\begin{aligned} x\Xi_\pm(x) &= -\frac{1}{2}\left(\frac{b_{11}}{b_{20}} \pm \sigma\!\left(\frac{\Delta_2}{b_{20}^2}\right)\left(\frac{\Delta_2}{b_{20}^2}\right)^{\frac{1}{2}}\right)x^2 - \frac{1}{2}\left(\frac{b_{10}}{b_{20}} \pm \frac{1}{2}\,\sigma\!\left(\frac{\Delta_2}{b_{20}^2}\right)\left(\frac{\Delta_2}{b_{20}^2}\right)^{\frac{1}{2}}\frac{\Delta_1}{\Delta_2}\right)x + \mathcal{O}(1), \\ \Sigma_\pm(x) &= -\frac{1}{4}\left(\frac{b_{11}}{b_{20}} \pm \sigma\!\left(\frac{\Delta_2}{b_{20}^2}\right)\left(\frac{\Delta_2}{b_{20}^2}\right)^{\frac{1}{2}}\right)x^2 - \frac{1}{4}\left(2\,\frac{b_{10}}{b_{20}} \pm \sigma\!\left(\frac{\Delta_2}{b_{20}^2}\right)\left(\frac{\Delta_2}{b_{20}^2}\right)^{\frac{1}{2}}\frac{\Delta_1}{\Delta_2}\right)x + \mathcal{O}(1). \end{aligned} \tag{71}$$

If $\Delta_2 = 0$ and $\Delta_1 \neq 0$,[5]

$$\begin{aligned} x\Xi_\pm(x) &= -\frac{1}{2}\frac{b_{11}}{b_{20}}\,x^2 \mp \frac{1}{2}\sigma\!\left(\frac{\Delta_1}{b_{20}^2}\right)\left(\frac{\Delta_1}{b_{20}^2}\right)^{\frac{1}{2}}x^{\frac{3}{2}} - \frac{1}{2}\frac{b_{10}}{b_{20}}\,x \mp \frac{1}{4}\,\sigma\!\left(\frac{\Delta_1}{b_{20}^2}\right)\left(\frac{\Delta_1}{b_{20}^2}\right)^{\frac{1}{2}}\frac{\Delta_0}{\Delta_1}\,x^{\frac{1}{2}} + \mathcal{O}\left(|x|^{-\frac{1}{2}}\right), \\ \Sigma_\pm(x) &= -\frac{1}{4}\frac{b_{11}}{b_{20}}\,x^2 \mp \frac{1}{3}\,\sigma\!\left(\frac{\Delta_1}{b_{20}^2}\right)\left(\frac{\Delta_1}{b_{20}^2}\right)^{\frac{1}{2}}x^{\frac{3}{2}} - \frac{1}{2}\frac{b_{10}}{b_{20}}\,x \mp \frac{1}{2}\,\sigma\!\left(\frac{\Delta_1}{b_{20}^2}\right)\left(\frac{\Delta_1}{b_{20}^2}\right)^{\frac{1}{2}}\frac{\Delta_0}{\Delta_1}\,x^{\frac{1}{2}} + \mathcal{O}\left(|x|^{-\frac{1}{2}}\right). \end{aligned} \tag{72}$$

If $\Delta_2 = \Delta_1 = 0$,

$$\begin{aligned} x\Xi_\pm(x) &= -\frac{1}{2}\frac{b_{11}}{b_{20}}\,x^2 - \frac{1}{2}\left(\frac{b_{10}}{b_{20}} \pm \sigma\!\left(\frac{\Delta_0}{b_{20}^2}\right)\left(\frac{\Delta_0}{b_{20}^2}\right)^{\frac{1}{2}}\right)x, \\ \Sigma_\pm(x) &= -\frac{1}{4}\,\frac{b_{11}}{b_{20}}\,x^2 - \frac{1}{2}\left(\frac{b_{10}}{b_{20}} \pm \sigma\!\left(\frac{\Delta_0}{b_{20}^2}\right)\left(\frac{\Delta_0}{b_{20}^2}\right)^{\frac{1}{2}}\right)x. \end{aligned} \tag{73}$$

From these asymptotic expansions it follows that

(I) (68), (69), and (70) are equivalent to

$$\text{(74)} \qquad \mathrm{e}^{\mathrm{i}\Sigma_\pm} \notin \mathcal{S}',$$

or

$$\text{(75)} \qquad \mathrm{e}^{\mathrm{i}\Sigma_-} \notin \mathcal{S}', \quad \mathrm{e}^{\mathrm{i}\Sigma_+} \in \mathcal{S}, \quad \Delta_2 \neq 0, \quad \lambda \notin \{1+2n : n \in \mathbb{Z}_+\},$$

or

$$\text{(76)} \qquad \mathrm{e}^{\mathrm{i}\Sigma_-} \notin \mathcal{S}', \quad \mathrm{e}^{\mathrm{i}\Sigma_+} \in \mathcal{S}, \quad \Delta_2 = 0.$$

(II) Thanks to Proposition 18, global regularity is equivalent to

$$\text{(77)} \qquad \mathrm{e}^{\mathrm{i}\Sigma_\pm} \in \mathcal{S} \cup (\mathcal{C}^\infty \setminus \mathcal{S}').$$

[5] Observe that when $x < 0$ we have $x \cdot x^{\frac{1}{2}} = -\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}(-x)^{\frac{3}{2}} = \mathrm{e}^{\mathrm{i}\frac{3\pi}{2}}(-x)^{\frac{3}{2}} = x^{\frac{3}{2}}$.

(III) Then, if B is globally regular, there are only three possible behaviors of $\mathrm{e}^{\mathrm{i}\Sigma_\pm}$:

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}\Sigma_\pm} \notin \mathcal{S}', \\ &\mathrm{e}^{\mathrm{i}\Sigma_-} \notin \mathcal{S}', \quad \mathrm{e}^{\mathrm{i}\Sigma_+} \in \mathcal{S}, \\ &\mathrm{e}^{\mathrm{i}\Sigma_\pm} \in \mathcal{S}. \end{aligned}$$

Since (74), (75), and (76) imply (77), we have only to show that

(A) $\mathrm{e}^{\mathrm{i}\Sigma_\pm} \notin \mathcal{S}' \implies u \notin \mathcal{S}'$,

(B) $\mathrm{e}^{\mathrm{i}\Sigma_-} \notin \mathcal{S}'$, $\mathrm{e}^{\mathrm{i}\Sigma_+} \in \mathcal{S}$, $\Delta_2 \neq 0$, and $\lambda \notin \{1+2n : n \in \mathbb{Z}_+\} \implies u \notin \mathcal{S}'$,

(C) $\mathrm{e}^{\mathrm{i}\Sigma_-} \notin \mathcal{S}'$, $\mathrm{e}^{\mathrm{i}\Sigma_+} \in \mathcal{S}$, $\Delta_2 \neq 0$, and $\lambda \in \{1+2n : n \in \mathbb{Z}_+\} \implies u \in \mathcal{S}$,

(D) $\mathrm{e}^{\mathrm{i}\Sigma_-} \notin \mathcal{S}'$, $\mathrm{e}^{\mathrm{i}\Sigma_+} \in \mathcal{S}$, $\Delta_2 = 0$, $\implies u \notin \mathcal{S}'$,

(E) $\mathrm{e}^{\mathrm{i}\Sigma_\pm} \in \mathcal{S} \implies u \in \mathcal{S}$,

where

$$u = c_1 u_1 + c_2 u_2,$$

u_1, and u_2 are as in Propositions 23, 24, and 26, and formula (67), and $|c_1| + |c_2| > 0$.

Since all assumptions in (A)–(E) imply that B is globally regular, we have that

$$u \in \mathcal{S} \iff \lim_{|x|\to\infty} u(x) = 0.$$

At last implications (A)–(E) follow by computing the limit of u as $|x| \to \infty$ by making use of the asymptotic expansions (71), (72), and (73), and Propositions 23, 24, and 26, and formula (67).

We leave the details to the reader. □

Proof of Theorem 17 It follows from Theorem 14, Lemma 16, Proposition 20, and Theorem 27. □

5 Asymptotic expansions of functions Φ, and Θ

5.1 Lemmas on Gamma Function.

The *Euler Gamma Function* is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} \mathrm{e}^{-t}\, \mathrm{d}t, \qquad \text{for } \operatorname{Re} z > 0.$$

This function can be extended to a meromorphic function with simple pole at every $k \in \mathbb{Z}_-$, by the formula (see [9, 1.1]):

$$\Gamma(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{z+k} + \int_1^\infty t^{z-1} \mathrm{e}^{-t}\, \mathrm{d}t.$$

Lemma 28. Given two complex numbers p and q such that $\operatorname{Re} p > 0$, and $\operatorname{Re} q > 0$, we have

$$\mathrm{e}^{\mathrm{i}\theta p} \int_0^\infty \frac{t^{p-1}}{(1+\mathrm{e}^{\mathrm{i}\theta} t)^{p+q}}\, \mathrm{d}t = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \qquad \text{for all } \theta \in \mathbb{R}. \tag{78}$$

Proof. Since ([9, (1.5.3) and (1.5.6)])

$$\int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}}\,\mathrm{d}t = B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

where B is the *Euler Beta Function*, it suffices to show that the left-hand side of (78) is constant with respect to θ. But this follows from

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left\{\mathrm{e}^{\mathrm{i}\theta p}\int_0^\infty \frac{t^{p-1}}{(1+\mathrm{e}^{\mathrm{i}\theta}t)^{p+q}}\,\mathrm{d}t\right\} = \\ = \mathrm{i}p\mathrm{e}^{\mathrm{i}\theta p}\int_0^\infty \frac{t^{p-1}}{(1+\mathrm{e}^{\mathrm{i}\theta}t)^{p+q}}\,\mathrm{d}t - (p+q)\mathrm{i}\mathrm{e}^{\mathrm{i}\theta(p+1)}\int_0^\infty \frac{t^{p}}{(1+\mathrm{e}^{\mathrm{i}\theta}t)^{p+q+1}}\,\mathrm{d}t = 0,$$

because

$$p\int_0^\infty \frac{t^{p-1}}{(1+\mathrm{e}^{\mathrm{i}\theta}t)^{p+q}}\,\mathrm{d}t = \left[\frac{t^p}{(1+\mathrm{e}^{\mathrm{i}\theta}t)^{p+q}}\right]_{t=0}^{t=\infty} - \int_0^\infty t^p\frac{\mathrm{d}}{\mathrm{d}t}(1+\mathrm{e}^{\mathrm{i}\theta}t)^{-(p+q)}\,\mathrm{d}t \\ = (p+q)\mathrm{e}^{\mathrm{i}\theta}\int_0^\infty \frac{t^p}{(1+\mathrm{e}^{\mathrm{i}\theta}t)^{p+q+1}}\,\mathrm{d}t.$$

□

Lemma 29. If $\operatorname{Re} z > 0$, and $\operatorname{Re} p > 0$, we have

$$z^p\int_0^\infty t^{p-1}\mathrm{e}^{-tz}\,\mathrm{d}t = \Gamma(p). \tag{79}$$

Proof. Since $\operatorname{Re} z > 0$, the left-hand side of (79) is analytic. Let

Differentiate the left-end side of (79):

$$\frac{\mathrm{d}}{\mathrm{d}z}\left\{z^p\int_0^\infty t^{p-1}\mathrm{e}^{-zt}\,\mathrm{d}t\right\} = pz^{p-1}\int_0^\infty t^{p-1}\mathrm{e}^{-zt}\,\mathrm{d}t - z^p\int_0^\infty t^p\mathrm{e}^{-zt}\,\mathrm{d}t. \tag{80}$$

Since $\operatorname{Re} p > 0$, an integration by parts yields:

$$pz^{p-1}\int_0^\infty t^{p-1}\mathrm{e}^{-zt}\,\mathrm{d}t = z^p\int_0^\infty t^p\mathrm{e}^{-zt}\,\mathrm{d}t. \tag{81}$$

Then (80), and (81) imply

$$\frac{\mathrm{d}}{\mathrm{d}z}\left\{z^p\int_0^\infty t^{p-1}\mathrm{e}^{-zt}\,\mathrm{d}t\right\} = 0,$$

that is that the left-end side of (79) is constant with respect to z. It follows that

$$z^p\int_0^\infty t^{p-1}\mathrm{e}^{-zt}\,\mathrm{d}t = \int_0^\infty t^{p-1}\mathrm{e}^{-t}\,\mathrm{d}t = \Gamma(p).$$

□

Lemma 30. Let $\operatorname{Re} p > 0$ and $0 < \epsilon < \frac{\pi}{2}$. Then

$$\int_0^1 t^{p-1}\,\mathrm{e}^{-tz}\,\mathrm{d}t \;=\; z^{-p}\left\{\Gamma(p) + \mathcal{O}\left(|z|^{\operatorname{Re} p-1}\,\mathrm{e}^{-(\sin\epsilon)|z|}\right)\right\}, \qquad \text{for } |z|\to\infty, \text{ and } |\operatorname{Arg} z| \leqslant \frac{\pi}{2}-\epsilon. \tag{82}$$

Proof. From (79) it follows that

$$z^p\int_0^1 t^{p-1}\,\mathrm{e}^{-tz}\,\mathrm{d}t - \Gamma(p) = -z^p\int_1^\infty t^{p-1}\mathrm{e}^{-tz}\,\mathrm{d}t. \tag{83}$$

Let $N = \min\{k \in \mathbb{Z}_+ : \operatorname{Re} p - 1 - k \leqslant 0\}$. Integrating by parts we get

$$\int_1^\infty t^{p-1} \mathrm{e}^{-tz} \, \mathrm{d}t = \sum_{k=0}^{N} \frac{\gamma_k}{z^k} \, \mathrm{e}^{-z} + \frac{\gamma_{N+1}}{z^N} \int_1^\infty t^{p-1-N} \mathrm{e}^{-tz} \, \mathrm{d}t, \tag{84}$$

where

$$\gamma_k = \begin{cases} 0, & \text{if } k = 0, \\ 1, & \text{if } k = 1, \\ (p-1)(p-2)\cdots(p-k+1), & \text{if } k > 1. \end{cases} \tag{85}$$

Since $\operatorname{Re} p - 1 - N \leqslant 0$, and $|\operatorname{Arg} z| \leqslant \frac{\pi}{2} - \epsilon$, we have

$$\left| \int_1^\infty t^{p-1-N} \mathrm{e}^{-tz} \, \mathrm{d}t \right| \leqslant \int_1^\infty \mathrm{e}^{-t|z| \cos(\operatorname{Arg} z)} \, \mathrm{d}t = \frac{\mathrm{e}^{-|z| \cos(\operatorname{Arg} z)}}{|z| \cos(\operatorname{Arg} z)}. \tag{86}$$

Since $|\operatorname{Arg} z| \leqslant \frac{\pi}{2} - \epsilon$, from (84), (85), and (86), it follows that

$$\begin{aligned} \left| z^p \int_1^\infty t^{p-1} \mathrm{e}^{-tz} \, \mathrm{d}t \right| &\leqslant \sum_{k=0}^{N} |\gamma_k| \left| z^{p-k} \right| \left| \mathrm{e}^{-z} \right| + |\gamma_{N+1}| \left| z^{p-N} \right| \frac{\mathrm{e}^{-|z| \cos(\operatorname{Arg} z)}}{|z| \cos(\operatorname{Arg} z)}, \\ &\leqslant \left(\sum_{k=0}^{N+1} |\gamma_k| \right) \mathrm{e}^{-\operatorname{Im} p \operatorname{Arg} z} \, |z|^{\operatorname{Re} p - 1} \, \frac{\mathrm{e}^{-|z| \cos(\operatorname{Arg} z)}}{\cos(\operatorname{Arg} z)} \\ &\leqslant \left(\sum_{k=1}^{N+1} |\gamma_k| \right) \mathrm{e}^{\frac{\pi |\operatorname{Im} p|}{2}} \, |z|^{\operatorname{Re} p - 1} \, \frac{\mathrm{e}^{-(\sin \epsilon)|z|}}{\sin \epsilon}, \qquad \text{for } |z| \geqslant 1. \end{aligned}$$

This inequality together with (83) implies (82). $\square$

5.2 Asymptotic behavior of Φ.

Proposition 31. We have the following integral representation:

$$\Phi(p, q; z) = \frac{\Gamma(q)}{\Gamma(p)\Gamma(q-p)} \, \mathrm{e}^z \int_0^1 (1-t)^{p-1} t^{q-p-1} \mathrm{e}^{-tz} \, \mathrm{d}t, \qquad \text{for } \operatorname{Re} q > \operatorname{Re} p > 0. \tag{87}$$

Proof. We have (see [9, (1.5.2), and (1.5.6)])

$$\begin{aligned} \frac{(p)_n}{(q)_n} = \frac{\Gamma(p+n)\Gamma(q)}{\Gamma(p)\Gamma(q+n)} &= \frac{\Gamma(p+n)\Gamma(q-p)}{\Gamma(q+n)} \, \frac{\Gamma(q)}{\Gamma(p)\Gamma(q-p)} \\ &= \frac{\Gamma(q)}{\Gamma(p)\Gamma(q-p)} \, B(p+n, q-p) = \frac{\Gamma(q)}{\Gamma(p)\Gamma(q-p)} \int_0^1 s^{p+n-1} (1-s)^{q-p-1} \, \mathrm{d}s, \end{aligned}$$

Thus, from (37) we obtain

$$\begin{aligned} \Phi(p, q; z) &= \sum_{n=0}^{\infty} \frac{(p)_n}{n!(q)_n} \, z^n = \frac{\Gamma(q)}{\Gamma(p)\Gamma(q-p)} \sum_{n=0}^{\infty} \int_0^1 s^{p-1} (1-s)^{q-p-1} \, \frac{(sz)^n}{n!} \, \mathrm{d}s \\ &= \frac{\Gamma(q)}{\Gamma(p)\Gamma(q-p)} \int_0^1 s^{p-1} (1-s)^{q-p-1} \, \mathrm{e}^{sz} \, \mathrm{d}s = \frac{\Gamma(q)}{\Gamma(p)\Gamma(q-p)} \, \mathrm{e}^z \int_0^1 (1-t)^{p-1} t^{q-p-1} \mathrm{e}^{-tz} \, \mathrm{d}t. \square \end{aligned}$$

Proposition 32 (Kummer identity)**.** For all $q \notin \mathbb{Z}_-$ we have

$$\Phi(p, q; z) = \mathrm{e}^z \Phi(q-p, q; -z). \tag{88}$$

Proof. Assume $\operatorname{Re} q > \operatorname{Re} p > 0$, and put $t = 1 - s$ in the right hand side of (87). We get

$$\Phi(p, q; z) = \frac{\Gamma(q)}{\Gamma(p)\Gamma(q-p)} \int_0^1 s^{p-1}(1-s)^{q-p-1}\mathrm{e}^{sz}\,\mathrm{d}s.$$

Then using again (87) we have

$$\frac{\Gamma(q)}{\Gamma(p)\Gamma(q-p)} \int_0^1 s^{p-1}(1-s)^{q-p-1}\mathrm{e}^{sz}\,\mathrm{d}s = \mathrm{e}^z\Phi(q-p, q, -z).$$

This proves (88) under the additional hypothesis $\operatorname{Re} q > \operatorname{Re} p > 0$. However by analytic continuity with respect to p and q, (88) is true for all $p \in \mathbb{C}$, and $q \in \mathbb{C} \setminus \mathbb{Z}_-$. □

Theorem 33. Let $0 < \epsilon < \pi/2$, $p \in \mathbb{C}$, and $q \in \mathbb{C}\setminus\mathbb{Z}_-$. For all $N \in \mathbb{Z}_+$, we have the following asymptotic expansions for $|z| \to \infty$.

$$\Phi(p, q; z) = \mathrm{e}^z z^{p-q} \left\{ \frac{\Gamma(q)}{\Gamma(p)} \sum_{k=0}^{N} \frac{(q-p)_k(1-p)_k}{k!} z^{-k} + \mathcal{O}\left(|z|^{-N-1}\right) \right\}, \qquad \text{for } |\operatorname{Arg} z| \leqslant \frac{\pi}{2} - \epsilon. \tag{89}$$

Proof. Assume

$$\operatorname{Re} q > \operatorname{Re} p > 1. \tag{90}$$

Using the binomial expansion and the identity

$$(-1)^k \binom{p-1}{k} = \frac{(1-p)_k}{k!}, \qquad \text{for all } k \in \mathbb{Z}_+,$$

we obtain

$$\begin{aligned} \int_0^1 (1-t)^{p-1}t^{q-p-1}\mathrm{e}^{-tz}\,\mathrm{d}t &= \sum_{k=0}^{N} \frac{(1-p)_k}{k!} \int_0^1 t^{k+q-p-1}\mathrm{e}^{-tz}\,\mathrm{d}t + \\ &+ \frac{(1-p)_{N+1}}{N!} \int_0^1 \left(\int_0^1 (1-s)^N (1-st)^{p-N-2}\,\mathrm{d}s \right) t^{N+q-p}\mathrm{e}^{-tz}\,\mathrm{d}t. \end{aligned} \tag{91}$$

Now, if $N + 2 \geqslant \operatorname{Re} p > 1$ we have

$$\left| \int_0^1 (1-s)^N (1-st)^{p-N-2}\,\mathrm{d}s \right| \leqslant \int_0^1 (1-s)^{\operatorname{Re} p - 2}\,\mathrm{d}s = \frac{1}{\operatorname{Re} p - 1}.$$

Then we get

$$\begin{aligned} &\left| z^{q-p} \int_0^1 \left(\int_0^1 (1-s)^N (1-st)^{p-N-2}\,\mathrm{d}s \right) t^{N+q-p}\mathrm{e}^{-tz}\,\mathrm{d}t \right| \leqslant \\ &\leqslant \frac{|z|^{\operatorname{Re}(q-p)}\,\mathrm{e}^{-\operatorname{Im}(p-q)\operatorname{Arg} z}}{\operatorname{Re} p - 1} \int_0^1 t^{N+\operatorname{Re}(q-p)}\mathrm{e}^{-t|z|\cos(\operatorname{Arg} z)}\,\mathrm{d}t \leqslant \\ &\leqslant \frac{|z|^{\operatorname{Re}(q-p)}\,\mathrm{e}^{\frac{\pi}{2}|\operatorname{Im}(p-q)|}}{\operatorname{Re} p - 1} \int_0^1 t^{N+\operatorname{Re}(q-p)}\mathrm{e}^{-t(\sin\epsilon)|z|}\,\mathrm{d}t = \end{aligned} \tag{92}$$

$$= \frac{|z|^{-N-1}\, \mathrm{e}^{\frac{\pi}{2}|\mathrm{Im}(p-q)|}}{\mathrm{Re}\, p-1} \int_0^{|z|} s^{N+\mathrm{Re}(q-p)} \mathrm{e}^{-(\sin\epsilon)s}\, \mathrm{d}s \leqslant \frac{|z|^{-N-1}\, \mathrm{e}^{\frac{\pi}{2}|\mathrm{Im}(p-q)|}}{\mathrm{Re}\, p-1} \int_0^{+\infty} s^{N+\mathrm{Re}(q-p)} \mathrm{e}^{-(\sin\epsilon)s}\, \mathrm{d}s,$$

for $z \neq 0$.

In conclusion, when $N \geqslant \mathrm{Re}\, p - 2$, from (91), and (92) it follows that

$$\int_0^1 (1-t)^{p-1} t^{q-p-1} \mathrm{e}^{-tz}\, \mathrm{d}t = \sum_{k=0}^{N} \frac{(1-p)_k}{k!} \int_0^1 t^{k+q-p-1} \mathrm{e}^{-tz}\, \mathrm{d}t + z^{p-q} \mathcal{O}\left(|z|^{-N-1}\right), \tag{93}$$

$$\text{for } |z| \to \infty, \text{ and } |\mathrm{Arg}\, z| \leqslant \frac{\pi}{2} - \epsilon.$$

On the other hand, by Lemma 30, we have

$$\begin{aligned} \int_0^1 t^{k+q-p-1} \mathrm{e}^{-tz}\, \mathrm{d}t &= z^{p-q-k} \left\{ \Gamma(k+q-p) + \mathcal{O}\left(|z|^{\mathrm{Re}(q-p)+k-1}\, \mathrm{e}^{-(\sin\epsilon)|z|}\right) \right\} = \\ &= z^{p-q} \left\{ \Gamma(k+q-p)\, z^{-k} + \mathcal{O}\left(|z|^{-N-1}\right) \right\}, \end{aligned} \tag{94}$$

$$\text{for } |z| \to \infty, \text{ and } |\mathrm{Arg}\, z| \leqslant \frac{\pi}{2} - \epsilon.$$

At last (89) follows from (87), (93), and (94), when $N \geqslant \mathrm{Re}\, p - 2$. However this restriction can easily be eliminated, because, we have

$$\begin{aligned} \Phi(p,q;z) &= \mathrm{e}^z z^{p-q} \left\{ \frac{\Gamma(q)}{\Gamma(p)} \sum_{k=0}^{N+M} \frac{(q-p)_k (1-p)_k}{k!}\, z^{-k} + \mathcal{O}\left(|z|^{-N-M-1}\right) \right\} \\ &= \mathrm{e}^z z^{p-q} \left\{ \frac{\Gamma(q)}{\Gamma(p)} \sum_{k=0}^{N} \frac{(q-p)_k (1-p)_k}{k!}\, z^{-k} + \frac{\Gamma(q)}{\Gamma(p)} \sum_{k=N+1}^{N+M} \frac{(q-p)_k (1-p)_k}{k!}\, z^{-k} + \mathcal{O}\left(|z|^{-N-M-1}\right) \right\} \\ &= \mathrm{e}^z z^{p-q} \left\{ \frac{\Gamma(q)}{\Gamma(p)} \sum_{k=0}^{N} \frac{(q-p)_k (1-p)_k}{k!}\, z^{-k} + \mathcal{O}\left(|z|^{-N-1}\right) \right\}, \end{aligned}$$

where

$$M = \min\{m \in \mathbb{Z}_+^* \;:\; m \geqslant \mathrm{Re}\, p - N - 2\}.$$

It remains to eliminate the restriction $\mathrm{Re}\, q > \mathrm{Re}\, p > 1$ and prove (89) for all $p \in \mathbb{C}$, and $q \in \mathbb{C} \setminus \mathbb{Z}_-$.

Rewrite the recurrence relation [9, (9.9.11)] as

$$\Phi(p,q;z) = \frac{q+z}{q}\, \Phi(p,q+1;z) - \frac{(q+1-p)z}{q(q+1)}\, \Phi(p,q+2;z). \tag{95}$$

If $\operatorname{Re} q + 1 > \operatorname{Re} p > 1$, from (95) and (89) we obtain that

$$
\begin{aligned}
\Phi(p,q;z) &= \mathrm{e}^z z^{p-q-1} \left\{ \frac{q+z}{q} \frac{\Gamma(q+1)}{\Gamma(p)} \sum_{k=0}^{N} \frac{(q+1-p)_k(1-p)_k}{k!} z^{-k} + \mathcal{O}\left(|z|^{-N-1}\right) \right\} \\
&\quad - \mathrm{e}^z z^{p-q-2} \left\{ \frac{(q+1-p)z}{q(q+1)} \frac{\Gamma(q+2)}{\Gamma(p)} \sum_{k=0}^{N} \frac{(q+2-p)_k(1-p)_k}{k!} z^{-k} + \mathcal{O}\left(|z|^{-N-1}\right) \right\} \\
&= \left\{ \frac{\Gamma(q)}{\Gamma(p)} \sum_{k=0}^{N} \frac{(q+1-p)_k(1-p)_k}{k!} z^{-k} + \mathcal{O}\left(|z|^{-N-1}\right) \right\} z^{p-q}\mathrm{e}^z \\
&\quad + \left\{ \frac{\Gamma(q)}{\Gamma(p)} \sum_{k=0}^{N} \frac{C_k}{k!} z^{-k} + \mathcal{O}\left(|z|^{-N-1}\right) \right\} z^{p-q-1}\mathrm{e}^z, \\
&\qquad \text{for } |z| \to \infty, \text{ and } |\operatorname{Arg} z| \leqslant \frac{\pi}{2} - \epsilon,
\end{aligned} \tag{96}
$$

with

$$
\begin{aligned}
C_k &= q(q+1-p)_k(1-p)_k - (q+1-p)(q+2-p)_k(1-p)_k \\
&= (1-p)_k \left\{ q(q+1-p)_k - (q+1-p)_k(q+1+k-p) \right\} = -(q+1-p)_k(1-p)_{k+1}.
\end{aligned} \tag{97}
$$

Substituting (97) into (96) gives

$$
\begin{aligned}
\Phi(p,q;z) &= \mathrm{e}^z z^{p-q} \left\{ \frac{\Gamma(q)}{\Gamma(p)} \sum_{k=0}^{N} \frac{(q+1-p)_k(1-p)_k}{k!} z^{-k} + \mathcal{O}\left(|z|^{-N-1}\right) \right\} \\
&\quad - \mathrm{e}^z z^{p-q-1} \left\{ \frac{\Gamma(q)}{\Gamma(p)} \sum_{k=0}^{N} \frac{(q+1-p)_k(1-p)_{k+1}}{k!} z^{-k} + \mathcal{O}\left(|z|^{-N-1}\right) \right\} \\
&= \mathrm{e}^z z^{p-q} \left\{ \frac{\Gamma(q)}{\Gamma(p)} + \frac{\Gamma(q)}{\Gamma(p)} \sum_{k=1}^{N} \left[\frac{(q+1-p)_k(1-p)_k}{k!} - \frac{(q+1-p)_{k-1}(1-p)_k}{(k-1)!} \right] z^{-k} + \mathcal{O}\left(|z|^{-N-1}\right) \right\} \\
&= \mathrm{e}^z z^{p-q} \left\{ \frac{\Gamma(q)}{\Gamma(p)} \sum_{k=0}^{N} \frac{(q-p)_k(1-p)_k}{k!} z^{-k} + \mathcal{O}\left(|z|^{-N-1}\right) \right\}, \\
&\qquad \text{for } |z| \to \infty, \text{ and } |\operatorname{Arg} z| \leqslant \frac{\pi}{2} - \epsilon.
\end{aligned}
$$

This shows that (89) holds for $\operatorname{Re} q > \operatorname{Re} p - 1$ and $\operatorname{Re} p > 1$. Iterating we get that (89) holds for all $q \in \mathbb{C} \setminus \mathbb{Z}_-$ and $\operatorname{Re} p > 1$.

Now consider the recurrence relation [9, (9.9.12)]:

$$
\Phi(p,q;z) = \Phi(p+1,q;z) - \frac{z}{q} \Phi(p+1,q+1;z). \tag{98}
$$

Substituting (89) into (98) gives:

$$\begin{aligned}
\Phi(p,q;z) &= \mathrm{e}^z z^{p+1-q}\left\{\frac{\Gamma(q)}{\Gamma(p+1)}\sum_{k=0}^{N+1}\frac{(q-p-1)_k(-p)_k}{k!}\,z^{-k}\ +\mathcal{O}\left(|z|^{-N-2}\right)\right\}+\\
&\quad -\mathrm{e}^z z^{p+1-q}\left\{\frac{1}{q}\,\frac{\Gamma(q+1)}{\Gamma(p+1)}\sum_{k=0}^{N+1}\frac{(q-p)_k(-p)_k}{k!}\,z^{-k}\ +\mathcal{O}\left(|z|^{-N-2}\right)\right\}\\
&= \mathrm{e}^z z^{p-q}\left\{\frac{\Gamma(q)}{p\Gamma(p)}\sum_{k=1}^{N+1}\frac{(q-p-1)_k(-p)_k-(q-p)_k(-p)_k}{k!}\,z^{-k+1}\ +\mathcal{O}\left(|z|^{-N-1}\right)\right\}\\
&= \mathrm{e}^z z^{p-q}\left\{\frac{\Gamma(q)}{\Gamma(p)}\sum_{k=1}^{N+1}\frac{(q-p)_{k-1}(1-p)_{k-1}}{(k-1)!}\,z^{-k+1}\ +\mathcal{O}\left(|z|^{-N-1}\right)\right\}\\
&= \mathrm{e}^z z^{p-q}\left\{\frac{\Gamma(q)}{\Gamma(p)}\sum_{k=0}^{N}\frac{(q-p)_k(1-p)_k}{k!}\,z^{-k}\ +\mathcal{O}\left(|z|^{-N-1}\right)\right\},\\
&\qquad\qquad\text{for } |z|\to\infty, \text{ and } |\operatorname{Arg} z|\leqslant\frac{\pi}{2}-\epsilon.
\end{aligned}$$

This means that (89) holds for $\operatorname{Re} p>0$ and, by iteration, for all $p\in\mathbb{C}$. □

5.3 Asymptotic behavior of Θ.

For all $p\in\mathbb{C}$ set

$$\Theta(p;z)=\sqrt{\pi}\left\{\frac{1}{\Gamma\left(p+\frac{1}{2}\right)}\Phi\left(p,\frac{1}{2};z^2\right)-\frac{2z}{\Gamma(p)}\Phi\left(p+\frac{1}{2},\frac{3}{2};z^2\right)\right\}. \tag{99}$$

Observe that Θ is an entire analytic function of z. Moreover, since $\frac{1}{\Gamma(-n)}=0$ for all $n\in\mathbb{Z}_+$, Θ is also an entire analytic function of p.

Proposition 34. Consider $p\in\mathbb{C}$ such that $\operatorname{Re} p>0$. For all $\theta\in\left\{-\frac{\pi}{2},0,\frac{\pi}{2}\right\}$, we have the integral representation

$$\Theta(p;z)=\frac{\mathrm{e}^{\mathrm{i}p\theta}}{\Gamma(p)}\int_0^\infty t^{p-1}\left(1+\mathrm{e}^{\mathrm{i}\theta}t\right)^{-\left(p+\frac{1}{2}\right)}\mathrm{e}^{-\exp(\mathrm{i}\theta)z^2t}\,\mathrm{d}t,\qquad\text{for all } z\in\mathcal{S}_\theta, \tag{100}$$

where

$$\mathcal{S}_\theta=\left\{z\in\mathbb{C}^*\ :\ \left|\operatorname{Arg} z+\frac{\theta}{2}\right|<\frac{\pi}{4}\right\}.$$

Proof. We have

$$\left|\operatorname{Arg}(z^2)+\theta\right|<\frac{\pi}{2},\qquad\text{for all } z\in\mathcal{S}_\theta. \tag{101}$$

Then on $\mathcal{S}_\theta$ we have

$$\operatorname{Re}\left(\mathrm{e}^{\mathrm{i}\theta}z^2\right)=\cos\left(\operatorname{Arg}(z^2)+\theta\right)|z|^2>0$$

and the following integral is convergent:

$$w(z)=\mathrm{e}^{-\frac{1}{2}z^2}\int_0^\infty t^{p-1}\left(1+\mathrm{e}^{\mathrm{i}\theta}t\right)^{-\left(p+\frac{1}{2}\right)}\mathrm{e}^{-\exp(\mathrm{i}\theta)z^2t}\,\mathrm{d}t. \tag{102}$$

We have

$$
\begin{aligned}
&w'' - (z^2 + 4p - 1)w = \\
&\quad = \mathrm{e}^{-\frac{1}{2}z^2} \int_0^\infty t^{p-1}\left(1 + \mathrm{e}^{\mathrm{i}\theta} t\right)^{-\left(p+\frac{1}{2}\right)} \mathrm{e}^{-\exp(\mathrm{i}\theta) z^2 t} \Big\{ \left(1 + 2\mathrm{e}^{\mathrm{i}\theta} t\right)^2 z^2 - \left(1 + 2\mathrm{e}^{\mathrm{i}\theta} t\right) - (z^2 + 4p - 1) \Big\}\, \mathrm{d}t \\
&\quad = -4\mathrm{e}^{-\frac{1}{2}z^2} \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t} \Big\{ t^p \left(1 + \mathrm{e}^{\mathrm{i}\theta} t\right)^{-\left(p-\frac{1}{2}\right)} \mathrm{e}^{-\exp(\mathrm{i}\theta) z^2 t} \Big\}\, \mathrm{d}t \\
&\quad = -4\mathrm{e}^{-\frac{1}{2}z^2} \Big[t^p \left(1 + \mathrm{e}^{\mathrm{i}\theta} t\right)^{-\left(p-\frac{1}{2}\right)} \mathrm{e}^{-\exp(\mathrm{i}\theta) z^2 t} \Big]_{t=0}^{t=\infty} = 0.
\end{aligned}
$$

This means that (102) is a solution to (39) with $\lambda = 1 - 4p$. By Proposition 21 there exist $c_1, c_2 \in \mathbb{C}$ such that

$$
\int_0^\infty t^{p-1}\left(1 + \mathrm{e}^{\mathrm{i}\theta} t\right)^{-\left(p+\frac{1}{2}\right)} \mathrm{e}^{-\exp(\mathrm{i}\theta) z^2 t}\, \mathrm{d}t = c_1 \Phi\left(p, \frac{1}{2}; z^2\right) + c_2 z \Phi\left(p + \frac{1}{2}, \frac{3}{2}; z^2\right), \tag{103}
$$

for all $z \in \mathcal{S}_\theta$.

Set $z = \mathrm{e}^{-\mathrm{i}\frac{\theta}{2}} |s|$, with $s \in \mathbb{R}^*$, in (103), and take the limit for $s \to 0$. Since $\mathrm{e}^{-\mathrm{i}\frac{\theta}{2}} |s| \in \mathcal{S}_\theta$, thanks to Lemma 28 we get

$$
c_1 = \int_0^\infty t^{p-1}\left(1 + \mathrm{e}^{\mathrm{i}\theta} t\right)^{-\left(p+\frac{1}{2}\right)} \mathrm{d}t = \mathrm{e}^{-\mathrm{i}p\theta} \frac{\Gamma(p)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(p + \frac{1}{2}\right)} = \sqrt{\pi}\mathrm{e}^{-\mathrm{i}p\theta} \frac{\Gamma(p)}{\Gamma\left(p + \frac{1}{2}\right)}. \tag{104}
$$

Now we compute c_2. Differentiate (103) with respect to z, set $z = \mathrm{e}^{-\mathrm{i}\frac{\theta}{2}} |s|$, with $s \in \mathbb{R}^*$, and take the limit for $s \to 0$. We get

$$
\begin{aligned}
c_2 &= -2\mathrm{e}^{\mathrm{i}\frac{\theta}{2}} \lim_{s\to 0} |s| \int_0^\infty t^p \left(1 + \mathrm{e}^{\mathrm{i}\theta} t\right)^{-\left(p+\frac{1}{2}\right)} \mathrm{e}^{-s^2 t}\, \mathrm{d}t \\
&= -2\mathrm{e}^{\mathrm{i}\frac{\theta}{2}} \lim_{s\to 0} |s| \int_0^\infty \left(\frac{t}{s^2}\right)^p \left(1 + \mathrm{e}^{\mathrm{i}\theta} \frac{t}{s^2}\right)^{-\left(p+\frac{1}{2}\right)} \mathrm{e}^{-t} \frac{\mathrm{d}t}{s^2} \\
&= -2\mathrm{e}^{\mathrm{i}\frac{\theta}{2}} \int_0^\infty t^{-\frac{1}{2}} \mathrm{e}^{-\mathrm{i}\left(p+\frac{1}{2}\right)\theta} \mathrm{e}^{-t}\, \mathrm{d}t = -2\mathrm{e}^{-\mathrm{i}p\theta} \Gamma\left(\tfrac{1}{2}\right) = -2\sqrt{\pi}\mathrm{e}^{-\mathrm{i}p\theta}.
\end{aligned} \tag{105}
$$

From (103), (104), and (105) we obtain

$$
\int_0^\infty t^{p-1}\left(1 + \mathrm{e}^{\mathrm{i}\theta} t\right)^{-\left(p+\frac{1}{2}\right)} \mathrm{e}^{-\exp(\mathrm{i}\theta) z^2 t}\, \mathrm{d}t = \frac{\sqrt{\pi}}{\mathrm{e}^{\mathrm{i}p\theta}} \left\{ \frac{\Gamma(p)}{\Gamma\left(p + \frac{1}{2}\right)} \Phi\left(p, \frac{1}{2}; z^2\right) - 2z\Phi\left(p + \frac{1}{2}, \frac{3}{2}; z^2\right) \right\},
$$
$$
\text{for all } z \in \mathcal{S}_\theta,
$$

which is equivalent to (100). □

Theorem 35. Let $0 < \epsilon < \frac{\pi}{2}$. For all $N \in \mathbb{Z}_+$ we have

$$
\Theta(p; z) = z^{-2p} \left\{ \sum_{k=0}^N \frac{(-1)^k}{k!} (p)_k \left(p + \frac{1}{2}\right)_k z^{-2k} + \mathcal{O}\left(|z|^{-2(N+1)}\right) \right\}, \tag{106}
$$
$$
\text{for } |z| \to \infty, \text{ and } |\operatorname{Arg} z| \leqslant \frac{\pi}{2} - \epsilon.
$$

Remark. Observe that when either p or $p+\frac{1}{2}$ belong to $\mathbb{Z}_-$, $\Theta(p;z)$ becomes a polynomial. So (106) holds on the whole complex plane.

Let $n \in \mathbb{Z}_+$. Then from (99), (37), (38), and [9, (1.2.2)] we obtain

$$\Theta(-n;z) = \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2}-n\right)}\Phi\left(-n,\frac{1}{2};z^2\right) = \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2}-n\right)}\sum_{k=0}^{n}\frac{(-n)_k}{k!\left(\frac{1}{2}\right)_k}z^{2k} = z^{2n}\sum_{k=0}^{n}\frac{(-1)^k}{k!}(-n)_k\left(\frac{1}{2}-n\right)_k z^{-2k}, \tag{107}$$

and

$$\Theta\left(-\frac{1}{2}-n;z\right) = -\frac{2\sqrt{\pi}}{\Gamma\left(-\frac{1}{2}-n\right)}z\Phi\left(-n,\frac{3}{2};z^2\right) = -\frac{2\sqrt{\pi}}{\Gamma\left(-\frac{1}{2}-n\right)}z\sum_{k=0}^{n}\frac{(-n)_k}{k!\left(\frac{3}{2}\right)_k}z^{2k} = z^{2n+1}\sum_{k=0}^{n}\frac{(-1)^k}{k!}(-n)_k\left(-\frac{1}{2}-n\right)_k z^{-2k}. \tag{108}$$

Proof. (I) First we observe that it suffices to prove (106) for $\operatorname{Re} p > 0$.

Using (37), and (99), a long, but straightforward computation shows that

$$\Theta(p;z) = \left(2p+\frac{3}{2}+z^2\right)\Theta(p+1;z) - (p+1)\left(p+\frac{3}{2}\right)\Theta(p+2;z). \tag{109}$$

Assume now $\operatorname{Re} p > -1$, and (106) true for $\operatorname{Re} p > 0$. By (109) we obtain

$$\begin{aligned}
\Theta(p;z) &= \left(2p+\frac{3}{2}+z^2\right)z^{-2(p+1)}\left\{\sum_{k=0}^{N}\frac{(-1)^k(p+1)_k\left(p+\frac{3}{2}\right)_k}{k!}z^{-2k} + \mathcal{O}\left(|z|^{-2(N+1)}\right)\right\}\\
&\quad - (p+1)\left(p+\frac{3}{2}\right)z^{-2(p+2)}\left\{\sum_{k=0}^{N}\frac{(-1)^k(p+2)_k\left(p+\frac{5}{2}\right)_k}{k!}z^{-2k} + \mathcal{O}\left(|z|^{-2(N+1)}\right)\right\} =\\
&= z^{-2p}\left(2p+\frac{3}{2}\right)\left\{\sum_{k=1}^{N+1}\frac{(-1)^{k-1}(p+1)_{k-1}\left(p+\frac{3}{2}\right)_{k-1}}{(k-1)!}z^{-2k} + \mathcal{O}\left(|z|^{-2(N+2)}\right)\right\}\\
&\quad + z^{-2p}\left\{\sum_{k=0}^{N}\frac{(-1)^k(p+1)_k\left(p+\frac{3}{2}\right)_k}{k!}z^{-2k} + \mathcal{O}\left(|z|^{-2(N+1)}\right)\right\}\\
&\quad - z^{-2p}(p+1)\left(p+\frac{3}{2}\right)\left\{\sum_{k=2}^{N+2}\frac{(-1)^{k-2}(p+2)_{k-2}\left(p+\frac{5}{2}\right)_{k-2}}{(k-2)!}z^{-2k} + \mathcal{O}\left(|z|^{-2(N+3)}\right)\right\} =\\
&= z^{-2p} + z^{-2p}\left\{2p+\frac{3}{2}-(p+1)\left(p+\frac{3}{2}\right)\right\}z^{-2}\\
&\quad + z^{-2p}\left\{\left(-\frac{\left(2p+\frac{3}{2}\right)k}{p\left(p+\frac{1}{2}\right)} + \frac{(p+k)\left(p+\frac{1}{2}+k\right)}{p\left(p+\frac{1}{2}\right)} - \frac{k(k-1)}{p\left(p+\frac{1}{2}\right)}\right)\sum_{k=2}^{N}\frac{(-1)^k(p)_k\left(p+\frac{1}{2}\right)_k}{k!}z^{-2k}\right\}\\
&\quad + z^{-2p}\mathcal{O}\left(|z|^{-2(N+1)}\right) =\\
&= z^{-2p}\left\{\sum_{k=0}^{N}\frac{(-1)^k(p)_k\left(p+\frac{1}{2}\right)_k}{k!}z^{-2k} + \mathcal{O}\left(|z|^{-2(N+1)}\right)\right\}.
\end{aligned}$$

This shows that (106) is true for $\operatorname{Re} p > -1$. By iteration we get that (106) is true for all $p \in \mathbb{C}$.

(II) Since $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\subset\mathcal{S}_{-\frac{\pi}{2}}\cup\mathcal{S}_0\cup\mathcal{S}_{\frac{\pi}{2}}$ it suffices to prove (106) for

$$\left|\operatorname{Arg} z+\frac{\theta}{2}\right|\leqslant\frac{\pi}{4}-\frac{\epsilon}{2}, \tag{110}$$

for all $\theta\in\left\{-\frac{\pi}{4},0,\frac{\pi}{4}\right\}$.

According to (I), we may assume $\operatorname{Re} p>0$. Integrating term by term the binomial expansion

$$\left(1+\mathrm{e}^{\mathrm{i}\theta}t\right)^{-\left(p+\frac{1}{2}\right)}=\sum_{k=0}^{N}\frac{(-1)^k\left(p+\frac{1}{2}\right)_k}{k!}\mathrm{e}^{\mathrm{i}k\theta}t^k+\\+\frac{(-1)^{N+1}\left(p+\frac{1}{2}\right)_{N+1}}{N!}\mathrm{e}^{\mathrm{i}(N+1)\theta}t^{N+1}\int_0^1(1-s)^N\left(1+\mathrm{e}^{\mathrm{i}\theta}st\right)^{-\left(p+N+\frac{3}{2}\right)}\mathrm{d}s,$$

thanks to (100) we obtain

$$\Theta(p;z)=\sum_{k=0}^{N}\frac{(-1)^k\left(p+\frac{1}{2}\right)_k}{\Gamma(p)k!}\mathrm{e}^{\mathrm{i}(p+k)\theta}\int_0^\infty t^{p+k-1}\mathrm{e}^{-\exp(\mathrm{i}\theta)z^2t}\,\mathrm{d}t+\\+\frac{(-1)^{N+1}\left(p+\frac{1}{2}\right)_{N+1}}{\Gamma(p)N!}\mathrm{e}^{\mathrm{i}(p+N+1)\theta}\int_0^\infty\left(\int_0^1(1-s)^N\left(1+\mathrm{e}^{\mathrm{i}\theta}st\right)^{-\left(p+N+\frac{3}{2}\right)}\mathrm{d}s\right)t^{p+N}\mathrm{e}^{-\exp(\mathrm{i}\theta)z^2t}\,\mathrm{d}t. \tag{111}$$

Thanks to Lemma 29, we have

$$\frac{\mathrm{e}^{\mathrm{i}(p+k)\theta}}{\Gamma(p)}\int_0^\infty t^{p+k-1}\mathrm{e}^{-\exp(\mathrm{i}\theta)z^2t}\,\mathrm{d}t=\frac{\Gamma(p+k)}{\Gamma(p)}\left(z^2\right)^{-(p+k)}=(p)_kz^{-2(p+k)}=(p)_kz^{-2p}z^{-2k}. \tag{112}$$

Moreover

$$\left|1+\mathrm{e}^{\mathrm{i}\theta}st\right|^2=1+2(\cos\theta)st+s^2t^2\geqslant 1,\qquad\text{for } |\theta|\leqslant\frac{\pi}{2}.$$

Then we have

$$\left|\int_0^1(1-s)^N\left(1+\mathrm{e}^{\mathrm{i}\theta}st\right)^{-\left(p+N+\frac{3}{2}\right)}\mathrm{d}s\right|\leqslant\int_0^1(1-s)^N\mathrm{e}^{(\operatorname{Im}p)\operatorname{Arg}\left(1+\mathrm{e}^{\mathrm{i}\theta}st\right)}\mathrm{d}s\leqslant\frac{\mathrm{e}^{|\operatorname{Im}p|\pi}}{N+1}. \tag{113}$$

On the other hand from (110) we obtain

$$\operatorname{Re}\left(\mathrm{e}^{\mathrm{i}\theta}z^2\right)=\cos\left(\operatorname{Arg}(z^2)+\theta\right)\left|z^2\right|\geqslant\cos\left(\frac{\pi}{2}-\epsilon\right)\left|z^2\right|=(\sin\epsilon)\left|z\right|^2.$$

Then (113) implies that

$$\begin{aligned}&\left|\int_0^\infty\left(\int_0^1(1-s)^N\left(1+\mathrm{e}^{\mathrm{i}\theta}st\right)^{-\left(p+N+\frac{3}{2}\right)}\mathrm{d}s\right)t^{p+N}\mathrm{e}^{-\exp(\mathrm{i}\theta)z^2t}\,\mathrm{d}t\right|\leqslant\\&\quad\leqslant\frac{\mathrm{e}^{|\operatorname{Im}p|\pi}}{N+1}\left|\int_0^\infty t^{\operatorname{Re}p+N}\mathrm{e}^{-\operatorname{Re}\left(\exp(\mathrm{i}\theta z^2\right)t}\,\mathrm{d}t\right|\leqslant\frac{\mathrm{e}^{|\operatorname{Im}p|\pi}}{N+1}\left|\int_0^\infty t^{\operatorname{Re}p+N}\mathrm{e}^{-(\sin\epsilon)|z|^2t}\,\mathrm{d}t\right|\\&\quad=\frac{\mathrm{e}^{|\operatorname{Im}p|\pi}}{N+1}\left((\sin\epsilon)\left|z\right|^2\right)^{-(\operatorname{Re}p+N+1)}\left|\int_0^\infty s^{\operatorname{Re}p+N}\mathrm{e}^{-s}\,\mathrm{d}s\right|\\&\quad\leqslant\frac{\mathrm{e}^{|p|\pi}\Gamma(\operatorname{Re}p+N+1)}{(N+1)(\sin\epsilon)^{\operatorname{Re}p+N+1}}\left|z\right|^{-2(\operatorname{Re}p+N+1)},\\&\qquad\qquad\text{for }\left|\operatorname{Arg}z+\frac{\theta}{2}\right|\leqslant\frac{\pi}{4}-\frac{\epsilon}{2},\text{ and }\theta\in\left\{-\frac{\pi}{4},0,\frac{\pi}{4}\right\}.\end{aligned} \tag{114}$$

In conclusion, the expansion (106) follows from (111), (112), and (114). $\square$

6 Asymptotic expansions of the general solution to Hermite-Weber equation.

Let w_1, and w_2 be the solutions to equation (39) given by (40).

Proposition 36. We have the following identities (recall that $\frac{1}{\Gamma}$ extends to an entire function):

$$\frac{w_1(z)}{\Gamma\left(\frac{3-\lambda}{4}\right)} \mp \frac{2w_2(z)}{\Gamma\left(\frac{1-\lambda}{4}\right)} = \frac{\mathrm{e}^{-\frac{1}{2}z^2}}{\sqrt{\pi}}\Theta\left(\frac{1-\lambda}{4}; \pm z\right), \tag{115}$$

$$\frac{w_1(z)}{\Gamma\left(\frac{3+\lambda}{4}\right)} \pm \frac{2\mathrm{i}w_2(z)}{\Gamma\left(\frac{1+\lambda}{4}\right)} = \frac{\mathrm{e}^{\frac{1}{2}z^2}}{\sqrt{\pi}}\Theta\left(\frac{1+\lambda}{4}; \mp\mathrm{i}z\right). \tag{116}$$

and

$$w_1(z) = \sqrt{\pi}\,\mathrm{e}^{-\mathrm{i}\frac{1+\lambda}{4}\pi}\left\{\frac{\mathrm{i}\mathrm{e}^{-\frac{1}{2}z^2}}{\Gamma\left(\frac{1+\lambda}{4}\right)}\Theta\left(\frac{1-\lambda}{4}; \pm z\right) + \frac{\mathrm{e}^{\frac{1}{2}z^2}}{\Gamma\left(\frac{1-\lambda}{4}\right)}\Theta\left(\frac{1+\lambda}{4}; \mp\mathrm{i}z\right)\right\}, \tag{117}$$

$$w_2(z) = \mp\frac{\sqrt{\pi}}{2}\,\mathrm{e}^{-\mathrm{i}\frac{1+\lambda}{4}\pi}\left\{\frac{\mathrm{e}^{-\frac{1}{2}z^2}}{\Gamma\left(\frac{3+\lambda}{4}\right)}\Theta\left(\frac{1-\lambda}{4}; \pm z\right) - \frac{\mathrm{e}^{\frac{1}{2}z^2}}{\Gamma\left(\frac{3-\lambda}{4}\right)}\Theta\left(\frac{1+\lambda}{4}; \mp\mathrm{i}z\right)\right\}. \tag{118}$$

Proof. From (99), and Proposition 32 we have

$$\begin{aligned}\Theta(p; \mp\mathrm{i}z) &= \sqrt{\pi}\left\{\frac{1}{\Gamma\left(\frac{1}{2}+p\right)}\Phi\left(p, \frac{1}{2}; -z^2\right) \pm \frac{2\mathrm{i}z}{\Gamma(p)}\Phi\left(\frac{1}{2}+p, \frac{3}{2}; -z^2\right)\right\}\\ &= \sqrt{\pi}\left\{\frac{\mathrm{e}^{-z^2}}{\Gamma\left(\frac{1}{2}+p\right)}\Phi\left(\frac{1}{2}-p, \frac{1}{2}; z^2\right) \pm \frac{2\mathrm{i}\mathrm{e}^{-z^2}}{\Gamma(p)}z\Phi\left(1-p, \frac{3}{2}; z^2\right)\right\}.\end{aligned}$$

This identity can be rewritten as

$$\Theta\left(\frac{1}{2}-p; \mp\mathrm{i}z\right) = \sqrt{\pi}\left\{\frac{\mathrm{e}^{-z^2}}{\Gamma(1-p)}\Phi\left(p, \frac{1}{2}; z^2\right) \pm \frac{2\mathrm{i}z\mathrm{e}^{-z^2}}{\Gamma\left(\frac{1}{2}-p\right)}\Phi\left(\frac{1}{2}+p, \frac{3}{2}; z^2\right)\right\}. \tag{119}$$

Then from (99), and (119), we obtain

$$\frac{1}{\Gamma\left(\frac{1}{2}+p\right)}\Phi\left(p, \frac{1}{2}; z^2\right) \mp \frac{2}{\Gamma(p)}z\Phi\left(\frac{1}{2}+p, \frac{3}{2}; z^2\right) = \frac{1}{\sqrt{\pi}}\Theta(p; \pm z), \tag{120}$$

$$\frac{1}{\Gamma(1-p)}\Phi\left(p, \frac{1}{2}; z^2\right) \pm \frac{2\mathrm{i}}{\Gamma\left(\frac{1}{2}-p\right)}z\Phi\left(\frac{1}{2}+p, \frac{3}{2}; z^2\right) = \frac{\mathrm{e}^{z^2}}{\sqrt{\pi}}\Theta\left(\frac{1}{2}-p; \mp\mathrm{i}z\right). \tag{121}$$

Letting $p = \frac{1-\lambda}{4}$ in (120) and (121), and using (40), we obtain (115), and (116).

From [9, (1.2.2)] we get

$$\frac{1}{\Gamma\left(\frac{1-\lambda}{4}\right)\Gamma\left(\frac{3+\lambda}{4}\right)} \pm \frac{\mathrm{i}}{\Gamma\left(\frac{1+\lambda}{4}\right)\Gamma\left(\frac{3-\lambda}{4}\right)} = \frac{\mathrm{e}^{\pm\mathrm{i}\frac{1+\lambda}{4}\pi}}{\pi}. \tag{122}$$

Using this identity, we can solve the system given by (115), and (116), obtaining (117), and (118). $\square$

Proposition 37. Let $0 < \epsilon < \frac{\pi}{4}$. For all c_1, $c_2 \in \mathbb{C}$, and $N \in \mathbb{Z}_+$.

(A) If $\frac{c_1}{\Gamma\left(\frac{1-\lambda}{4}\right)} \pm \frac{c_2}{2\Gamma\left(\frac{3-\lambda}{4}\right)} \neq 0$, we have

$$c_1 w_1(z) + c_2 w_2(z) =$$

$$= \sqrt{\pi} \mathrm{e}^{\frac{1}{2} z^2} z^{-\frac{1+\lambda}{2}} \left\{ \left(\frac{c_1}{\Gamma\left(\frac{1-\lambda}{4}\right)} + \frac{c_2}{2\Gamma\left(\frac{3-\lambda}{4}\right)} \right) \sum_{k=0}^{N} \frac{1}{k!} \left(\frac{1+\lambda}{4} \right)_k \left(\frac{3+\lambda}{4} \right)_k z^{-2k} + \mathcal{O}\left(|z|^{-2(N+1)} \right) \right\},$$

$$\text{for } |z| \to \infty, \text{ and } |\operatorname{Arg} z| \leqslant \frac{\pi}{4} - \epsilon,$$

and

$$c_1 w_1(z) + c_2 w_2(z) =$$

$$= \sqrt{\pi} \mathrm{e}^{\frac{1}{2} z^2} (-z)^{-\frac{1+\lambda}{2}} \left\{ \left(\frac{c_1}{\Gamma\left(\frac{1-\lambda}{4}\right)} - \frac{c_2}{2\Gamma\left(\frac{3-\lambda}{4}\right)} \right) \sum_{k=0}^{N} \frac{1}{k!} \left(\frac{1+\lambda}{4} \right)_k \left(\frac{3+\lambda}{4} \right)_k z^{-2k} + \mathcal{O}\left(|z|^{-2(N+1)} \right) \right\},$$

$$\text{for } |z| \to \infty, \text{ and } |\operatorname{Arg}(-z)| \leqslant \frac{\pi}{4} - \epsilon.$$

(B) If

$$c_1 = \frac{c}{\Gamma\left(\frac{3-\lambda}{4}\right)}, \qquad c_2 = -\frac{2c}{\Gamma\left(\frac{1-\lambda}{4}\right)}, \tag{123}$$

with $c \neq 0$, and $\lambda \notin \{1 + 2n : n \in \mathbb{Z}_+\}$, we have

$$c_1 w_1(z) + c_2 w_2(z) =$$

$$= \frac{c}{\sqrt{\pi}} \mathrm{e}^{-\frac{1}{2} z^2} z^{-\frac{1-\lambda}{2}} \left\{ \sum_{k=0}^{N} \frac{(-1)^k}{k!} \left(\frac{1-\lambda}{4} \right)_k \left(\frac{3-\lambda}{4} \right)_k z^{-2k} + \mathcal{O}\left(|z|^{-2(N+1)} \right) \right\},$$

$$\text{for } |z| \to \infty, \text{ and } |\operatorname{Arg} z| \leqslant \frac{\pi}{4} - \epsilon,$$

$$c_1 w_1(z) + c_2 w_2(z) =$$

$$= \sqrt{\pi} \mathrm{e}^{\frac{1}{2} z^2} (-z)^{-\frac{1+\lambda}{2}} \left\{ \frac{2c}{\Gamma\left(\frac{1-\lambda}{4}\right) \Gamma\left(\frac{3-\lambda}{4}\right)} \sum_{k=0}^{N} \frac{1}{k!} \left(\frac{1+\lambda}{4} \right)_k \left(\frac{3+\lambda}{4} \right)_k z^{-2k} + \mathcal{O}\left(|z|^{-2(N+1)} \right) \right\},$$

$$\text{for } |z| \to \infty, \text{ and } |\operatorname{Arg}(-z)| \leqslant \frac{\pi}{4} - \epsilon,$$

(C) If

$$c_1 = \frac{c}{\Gamma\left(\frac{3-\lambda}{4}\right)}, \qquad c_2 = \frac{2c}{\Gamma\left(\frac{1-\lambda}{4}\right)}, \tag{124}$$

with $c \neq 0$, and $\lambda \notin \{1 + 2n : n \in \mathbb{Z}_+\}$, we have

$$c_1 w_1(z) + c_2 w_2(z) =$$

$$= \sqrt{\pi} \mathrm{e}^{\frac{1}{2} z^2} z^{-\frac{1+\lambda}{2}} \left\{ \frac{2c}{\Gamma\left(\frac{1-\lambda}{4}\right) \Gamma\left(\frac{3-\lambda}{4}\right)} \sum_{k=0}^{N} \frac{1}{k!} \left(\frac{1+\lambda}{4} \right)_k \left(\frac{3+\lambda}{4} \right)_k z^{-2k} + \mathcal{O}\left(|z|^{-2(N+1)} \right) \right\},$$

$$\text{for } |z| \to \infty, \text{ and } |\operatorname{Arg} z| \leqslant \frac{\pi}{4} - \epsilon,$$

$$c_1 w_1(z) + c_2 w_2(z) =$$

$$= \frac{c}{\sqrt{\pi}}\, \mathrm{e}^{-\frac{1}{2}z^2}(-z)^{-\frac{1-\lambda}{2}} \left\{ \sum_{k=0}^{N} \frac{(-1)^k}{k!} \left(\frac{1-\lambda}{4}\right)_k \left(\frac{3-\lambda}{4}\right)_k z^{-2k} + \mathcal{O}\left(|z|^{-2(N+1)}\right) \right\}.$$

for $|z| \to \infty$, and $|\mathrm{Arg}(-z)| \leqslant \frac{\pi}{4} - \epsilon$.

(D) If

$$c_1 = \frac{c}{\Gamma\left(\frac{3-\lambda}{4}\right)}, \qquad c_2 = \mp \frac{2c}{\Gamma\left(\frac{1-\lambda}{4}\right)},$$

with $c \neq 0$, and $\lambda = 1 + 4n$, with $n \in \mathbb{Z}_+$, we have

$$c_1 w_1(z) + c_2 w_2(z) = \frac{c}{\sqrt{\pi}}\, \mathrm{e}^{-\frac{1}{2}z^2} z^{2n} \sum_{k=0}^{n} \frac{(-1)^k}{k!} (-n)_k \left(\frac{1}{2} - n\right)_k z^{-2k}, \qquad \text{for all } z.$$

(E) If

$$c_1 = \frac{c}{\Gamma\left(\frac{3-\lambda}{4}\right)}, \qquad c_2 = \mp \frac{2c}{\Gamma\left(\frac{1-\lambda}{4}\right)},$$

with $c \neq 0$, and $\lambda = 3 + 4n$, with $n \in \mathbb{Z}_+$, we have

$$c_1 w_1(z) + c_2 w_2(z) = \pm \frac{c}{\sqrt{\pi}} \mathrm{e}^{-\frac{1}{2}z^2} z^{2n+1} \sum_{k=0}^{n} \frac{(-1)^k}{k!} (-n)_k \left(-\frac{1}{2} - n\right)_k z^{-2k} \qquad \text{for all } z.$$

Proof. (A) follows from (40), and Theorem 33, with $p = \frac{1-\lambda}{4}$. Observe that

$$|\mathrm{Arg}\, z| \leqslant \frac{\pi}{4} - \epsilon \implies \left|\mathrm{Arg}(z^2)\right| \leqslant \frac{\pi}{2} - 2\epsilon,$$

and that

$$(z^2)^p = \begin{cases} z^{2p}, & \text{if } -\frac{\pi}{4} < \mathrm{Arg}\, z \leqslant \frac{\pi}{4}, \\ (-z)^{2p}, & \text{if } -\frac{\pi}{4} < \mathrm{Arg}(-z) \leqslant \frac{\pi}{4}. \end{cases}$$

From (123), (124), and (115) we have

$$c_1 w_1(z) + c_2 w_2(z) = \frac{c}{\Gamma\left(\frac{3-\lambda}{4}\right)} w_1(z) \mp \frac{2c}{\Gamma\left(\frac{1-\lambda}{4}\right)} w_2(z) = \frac{c}{\sqrt{\pi}}\, \mathrm{e}^{-\frac{1}{2}z^2} \Theta\left(\frac{1-\lambda}{4}; \pm z\right).$$

Then (B) and (C) follow from Theorem 35 with $p = \frac{1-\lambda}{4}$; while (D) and (E) follow from (107), and (108). ☐

Proposition 38. Let $0 < \epsilon < \frac{\pi}{8}$. For all c_1, $c_2 \in \mathbb{C}$, and $N \in \mathbb{Z}_+$.

(A) If $\left(\frac{\mathrm{i}c_1}{\Gamma\left(\frac{1+\lambda}{4}\right)} \mp \frac{c_2}{2\Gamma\left(\frac{3+\lambda}{4}\right)}\right)\left(\frac{c_1}{\Gamma\left(\frac{1-\lambda}{4}\right)} \pm \frac{c_2}{2\Gamma\left(\frac{3-\lambda}{4}\right)}\right) \neq 0$, we have

$$c_1 w_1(z) + c_2 w_2(z) = \sqrt{\pi} \mathrm{e}^{-\mathrm{i}\frac{1+\lambda}{4}\pi} \cdot$$
$$\cdot \left\{ \mathrm{e}^{-\frac{1}{2}z^2} z^{-\frac{1-\lambda}{2}} \left[\left(\frac{\mathrm{i}c_1}{\Gamma\left(\frac{1+\lambda}{4}\right)} - \frac{c_2}{2\Gamma\left(\frac{3+\lambda}{4}\right)} \right) \sum_{k=0}^{N} \frac{(-1)^k}{k!} \left(\frac{1-\lambda}{4}\right)_k \left(\frac{3-\lambda}{4}\right)_k z^{-2k} + \mathcal{O}\left(|z|^{-2(N+1)}\right) \right] + \right.$$
$$\left. + \mathrm{e}^{\frac{1}{2}z^2} (-\mathrm{i}z)^{-\frac{1+\lambda}{2}} \left[\left(\frac{c_1}{\Gamma\left(\frac{1-\lambda}{4}\right)} + \frac{c_2}{2\Gamma\left(\frac{3-\lambda}{4}\right)} \right) \sum_{k=0}^{N} \frac{1}{k!} \left(\frac{1+\lambda}{4}\right)_k \left(\frac{3+\lambda}{4}\right)_k z^{-2k} + \mathcal{O}\left(|z|^{-2(N+1)}\right) \right] \right\},$$

$$\text{for } |z|\to\infty, \text{ and } \left|\operatorname{Arg}(z)-\frac{\pi}{4}\right|\leqslant\epsilon,$$

$$c_1w_1(z)+c_2w_2(z)=\sqrt{\pi}\mathrm{e}^{-\mathrm{i}\frac{1+\lambda}{4}\pi}\cdot$$
$$\cdot\left\{\mathrm{e}^{-\frac{1}{2}z^2}(-z)^{-\frac{1-\lambda}{2}}\left[\left(\frac{\mathrm{i}c_1}{\Gamma\left(\frac{1+\lambda}{4}\right)}+\frac{c_2}{2\Gamma\left(\frac{3+\lambda}{4}\right)}\right)\sum_{k=0}^{N}\frac{(-1)^k}{k!}\left(\frac{1-\lambda}{4}\right)_k\left(\frac{3-\lambda}{4}\right)_k z^{-2k}+\mathcal{O}\left(|z|^{-2(N+1)}\right)\right]+\right.$$
$$\left.+\mathrm{e}^{\frac{1}{2}z^2}(\mathrm{i}z)^{-\frac{1+\lambda}{2}}\left[\left(\frac{c_1}{\Gamma\left(\frac{1-\lambda}{4}\right)}-\frac{c_2}{2\Gamma\left(\frac{3-\lambda}{4}\right)}\right)\sum_{k=0}^{N}\frac{1}{k!}\left(\frac{1+\lambda}{4}\right)_k\left(\frac{3+\lambda}{4}\right)_k z^{-2k}+\mathcal{O}\left(|z|^{-2(N+1)}\right)\right]\right\},$$
$$\text{for } |z|\to\infty, \text{ and } \left|\operatorname{Arg}(-z)-\frac{\pi}{4}\right|\leqslant\epsilon,$$

(B) If

$$c_1=\frac{c}{\Gamma\left(\frac{3+\lambda}{4}\right)},\qquad c_2=\frac{2\mathrm{i}c}{\Gamma\left(\frac{1+\lambda}{4}\right)}, \tag{125}$$

with $c\neq 0$, and $\lambda\notin\{-(1+2n):n\in\mathbb{Z}_+\}$, we have

$$c_1w_1(z)+c_2w_2(z)=\frac{c}{\sqrt{\pi}}\,\mathrm{e}^{\frac{1}{2}z^2}(-\mathrm{i}z)^{-\frac{1+\lambda}{2}}\left\{\sum_{k=0}^{N}\frac{1}{k!}\left(\frac{1+\lambda}{4}\right)_k\left(\frac{3+\lambda}{4}\right)_k z^{-2k}+\mathcal{O}\left(|z|^{-2(N+1)}\right)\right\}.$$
$$\text{for } |z|\to\infty, \text{ and } \left|\operatorname{Arg}(z)-\frac{\pi}{4}\right|\leqslant\epsilon,$$

$$c_1w_1(z)+c_2w_2(z)=\sqrt{\pi}\mathrm{e}^{-\mathrm{i}\frac{1+\lambda}{4}\pi}\cdot$$
$$\cdot\left\{\mathrm{e}^{-\frac{1}{2}z^2}(-z)^{-\frac{1-\lambda}{2}}\left[\frac{2\mathrm{i}c}{\Gamma\left(\frac{1+\lambda}{4}\right)\Gamma\left(\frac{3+\lambda}{4}\right)}\sum_{k=0}^{N}\frac{(-1)^k}{k!}\left(\frac{1-\lambda}{4}\right)_k\left(\frac{3-\lambda}{4}\right)_k z^{-2k}+\mathcal{O}\left(|z|^{-2(N+1)}\right)\right]+\right.$$
$$\left.+\mathrm{e}^{\frac{1}{2}z^2}(\mathrm{i}z)^{-\frac{1+\lambda}{2}}\left[\frac{c\,\mathrm{e}^{-\mathrm{i}\frac{1+\lambda}{4}\pi}}{\pi}\sum_{k=0}^{N}\frac{1}{k!}\left(\frac{1+\lambda}{4}\right)_k\left(\frac{3+\lambda}{4}\right)_k z^{-2k}+\mathcal{O}\left(|z|^{-2(N+1)}\right)\right]\right\},$$
$$\text{for } |z|\to\infty, \text{ and } \left|\operatorname{Arg}(-z)-\frac{\pi}{4}\right|\leqslant\epsilon.$$

(C) If

$$c_1=\frac{c}{\Gamma\left(\frac{3+\lambda}{4}\right)},\qquad c_2=-\frac{2\mathrm{i}c}{\Gamma\left(\frac{1+\lambda}{4}\right)}, \tag{126}$$

with $c\neq 0$, and $\lambda\notin\{-(1+2n):n\in\mathbb{Z}_+\}$, we have

$$c_1w_1(z)+c_2w_2(z)=\sqrt{\pi}\mathrm{e}^{-\mathrm{i}\frac{1+\lambda}{4}\pi}\cdot$$
$$\cdot\left\{\mathrm{e}^{-\frac{1}{2}z^2}z^{-\frac{1-\lambda}{2}}\left[\frac{2\mathrm{i}c}{\Gamma\left(\frac{1+\lambda}{4}\right)\Gamma\left(\frac{3+\lambda}{4}\right)}\sum_{k=0}^{N}\frac{(-1)^k}{k!}\left(\frac{1-\lambda}{4}\right)_k\left(\frac{3-\lambda}{4}\right)_k z^{-2k}+\mathcal{O}\left(|z|^{-2(N+1)}\right)\right]+\right.$$
$$\left.+\mathrm{e}^{\frac{1}{2}z^2}(-\mathrm{i}z)^{-\frac{1+\lambda}{2}}\left[\frac{c\,\mathrm{e}^{-\mathrm{i}\frac{1+\lambda}{4}\pi}}{\pi}\sum_{k=0}^{N}\frac{1}{k!}\left(\frac{1+\lambda}{4}\right)_k\left(\frac{3+\lambda}{4}\right)_k z^{-2k}+\mathcal{O}\left(|z|^{-2(N+1)}\right)\right]\right\},$$
$$\text{for } |z|\to\infty, \text{ and } \left|\operatorname{Arg}(z)-\frac{\pi}{4}\right|\leqslant\epsilon,$$

$$c_1w_1(z)+c_2w_2(z)=\frac{c}{\sqrt{\pi}}\,\mathrm{e}^{\frac{1}{2}z^2}(\mathrm{i}z)^{-\frac{1+\lambda}{2}}\left\{\sum_{k=0}^{N}\frac{1}{k!}\left(\frac{1+\lambda}{4}\right)_k\left(\frac{3+\lambda}{4}\right)_k z^{-2k}+\mathcal{O}\left(|z|^{-2(N+1)}\right)\right\},$$

for $|z| \to \infty$, and $\left|\operatorname{Arg}(-z) - \frac{\pi}{4}\right| \leqslant \epsilon$.

(D) If

$$c_1 = \frac{c}{\Gamma\left(\frac{3+\lambda}{4}\right)}, \qquad c_2 = \pm\frac{2\mathrm{i}c}{\Gamma\left(\frac{1+\lambda}{4}\right)},$$

with $c \neq 0$, and $\lambda = -(1+4n)$, with $n \in \mathbb{Z}_+$, we have

$$c_1 w_1(z) + c_2 w_2(z) = \frac{(-1)^n c}{\sqrt{\pi}} \, \mathrm{e}^{\frac{1}{2}z^2} z^{2n} \sum_{k=0}^{n} \frac{1}{k!} \, (-n)_k \left(\frac{1}{2} - n\right)_k z^{-2k}, \qquad \text{for all } z.$$

(E) If

$$c_1 = \frac{c}{\Gamma\left(\frac{3+\lambda}{4}\right)}, \qquad c_2 = \pm\frac{2\mathrm{i}c}{\Gamma\left(\frac{1+\lambda}{4}\right)},$$

with $c \neq 0$, and $\lambda = -(3+4n)$, with $n \in \mathbb{Z}_+$, we have

$$c_1 w_1(z) + c_2 w_2(z) = \mp\mathrm{i} \, \frac{(-1)^n c}{\sqrt{\pi}} \, \mathrm{e}^{\frac{1}{2}z^2} z^{2n+1} \sum_{k=0}^{n} \frac{1}{k!} \, (-n)_k \left(-\frac{1}{2} - n\right)_k z^{-2k}, \qquad \text{for all } z.$$

(F) If

$$c_1 = \frac{c}{\Gamma\left(\frac{3-\lambda}{4}\right)}, \qquad c_2 = -\frac{2c}{\Gamma\left(\frac{1-\lambda}{4}\right)}, \tag{127}$$

with $c \neq 0$, and $\lambda \notin \{1 + 2n : n \in \mathbb{Z}_+\}$, we have

$$c_1 w_1(z) + c_2 w_2(z) = \frac{c}{\sqrt{\pi}} \, \mathrm{e}^{-\frac{1}{2}z^2} z^{-\frac{1-\lambda}{2}} \left\{ \sum_{k=0}^{N} \frac{(-1)^k}{k!} \left(\frac{1-\lambda}{4}\right)_k \left(\frac{3-\lambda}{4}\right)_k z^{-2k} + \mathcal{O}\left(|z|^{-2(N+1)}\right) \right\},$$

for $|z| \to \infty$, and $\left|\operatorname{Arg}(z) - \frac{\pi}{4}\right| \leqslant \epsilon$,

$$\begin{aligned} c_1 w_1(z) + c_2 w_2(z) = {} & \sqrt{\pi} \mathrm{e}^{-\mathrm{i}\frac{1+\lambda}{4}\pi} \cdot \\ & \cdot \Bigg\{ \mathrm{e}^{-\frac{1}{2}z^2} (-z)^{-\frac{1-\lambda}{2}} \left[-\frac{c \, \mathrm{e}^{-\mathrm{i}\frac{1+\lambda}{4}\pi}}{\pi} \sum_{k=0}^{N} \frac{(-1)^k}{k!} \left(\frac{1-\lambda}{4}\right)_k \left(\frac{3-\lambda}{4}\right)_k z^{-2k} + \mathcal{O}\left(|z|^{-2(N+1)}\right) \right] + \\ & + \mathrm{e}^{\frac{1}{2}z^2} (\mathrm{i}z)^{-\frac{1+\lambda}{2}} \left[\frac{2c}{\Gamma\left(\frac{1-\lambda}{4}\right) \Gamma\left(\frac{3-\lambda}{4}\right)} \sum_{k=0}^{N} \frac{1}{k!} \left(\frac{1+\lambda}{4}\right)_k \left(\frac{3+\lambda}{4}\right)_k z^{-2k} + \mathcal{O}\left(|z|^{-2(N+1)}\right) \right] \Bigg\}, \end{aligned}$$

for $|z| \to \infty$, and $\left|\operatorname{Arg}(-z) - \frac{\pi}{4}\right| \leqslant \epsilon$,

(G) If

$$c_1 = \frac{c}{\Gamma\left(\frac{3-\lambda}{4}\right)}, \qquad c_2 = \frac{2c}{\Gamma\left(\frac{1-\lambda}{4}\right)}, \tag{128}$$

with $c \neq 0$, and $\lambda \notin \{1 + 2n : n \in \mathbb{Z}_+\}$, we have

$$c_1 w_1(z) + c_2 w_2(z) = \sqrt{\pi} \mathrm{e}^{-\mathrm{i}\frac{1+\lambda}{4}\pi} \cdot$$

$$\cdot \left\{ \mathrm{e}^{-\frac{1}{2}z^2} z^{-\frac{1-\lambda}{2}} \left[-\frac{c\,\mathrm{e}^{-\mathrm{i}\frac{1+\lambda}{4}\pi}}{\pi} \sum_{k=0}^{N} \frac{(-1)^k}{k!} \left(\frac{1-\lambda}{4}\right)_k \left(\frac{3-\lambda}{4}\right)_k z^{-2k} + \mathcal{O}\left(|z|^{-2(N+1)}\right) \right] + \right.$$
$$\left. + \mathrm{e}^{\frac{1}{2}z^2} (-\mathrm{i}z)^{-\frac{1+\lambda}{2}} \left[\frac{2c}{\Gamma\left(\frac{1-\lambda}{4}\right)\Gamma\left(\frac{3-\lambda}{4}\right)} \sum_{k=0}^{N} \frac{1}{k!} \left(\frac{1+\lambda}{4}\right)_k \left(\frac{3+\lambda}{4}\right)_k z^{-2k} + \mathcal{O}\left(|z|^{-2(N+1)}\right) \right] \right\},$$
$$\text{for } |z| \to \infty, \text{ and } \left|\mathrm{Arg}(z) - \frac{\pi}{4}\right| \leqslant \epsilon,$$
$$c_1 w_1(z) + c_2 w_2(z) = \frac{c}{\sqrt{\pi}}\, \mathrm{e}^{-\frac{1}{2}z^2} (-z)^{-\frac{1-\lambda}{2}} \left\{ \sum_{k=0}^{N} \frac{(-1)^k}{k!} \left(\frac{1-\lambda}{4}\right)_k \left(\frac{3-\lambda}{4}\right)_k z^{-2k} + \mathcal{O}\left(|z|^{-2(N+1)}\right) \right\},$$
$$\text{for } |z| \to \infty, \text{ and } \left|\mathrm{Arg}(-z) - \frac{\pi}{4}\right| \leqslant \epsilon,$$

(H) If

$$c_1 = \frac{c}{\Gamma\left(\frac{3-\lambda}{4}\right)}, \qquad c_2 = \mp \frac{2c}{\Gamma\left(\frac{1-\lambda}{4}\right)},$$

with $c \neq 0$, and $\lambda = 1 + 4n$, with $n \in \mathbb{Z}_+$, we have

$$c_1 w_1(z) + c_2 w_2(z) = \frac{c}{\sqrt{\pi}}\, \mathrm{e}^{-\frac{1}{2}z^2} z^{2n} \sum_{k=0}^{n} \frac{(-1)^k}{k!} (-n)_k \left(\frac{1}{2} - n\right)_k z^{-2k}, \qquad \text{for all } z.$$

(I) If

$$c_1 = \frac{c}{\Gamma\left(\frac{3-\lambda}{4}\right)}, \qquad c_2 = \mp \frac{2c}{\Gamma\left(\frac{1-\lambda}{4}\right)},$$

with $c \neq 0$, and $\lambda = 3 + 4n$, with $n \in \mathbb{Z}_+$, we have

$$c_1 w_1(z) + c_2 w_2(z) = \pm \frac{c}{\sqrt{\pi}} \mathrm{e}^{-\frac{1}{2}z^2} z^{2n+1} \sum_{k=0}^{n} \frac{(-1)^k}{k!} (-n)_k \left(-\frac{1}{2} - n\right)_k z^{-2k} \qquad \text{for all } z.$$

Proof. In the computations we make use of identity (122).

(A) follows from (117), and (118), and Theorem 35, with $p = \frac{1 \mp \lambda}{4}$. Observe that

$$\left|\mathrm{Arg}(\pm z) - \frac{\pi}{4}\right| \leqslant \epsilon \iff \left|\mathrm{Arg}(\mp \mathrm{i}z) + \frac{\pi}{4}\right| \leqslant \epsilon,$$

and

$$\left|\mathrm{Arg}(\pm z) - \frac{\pi}{4}\right| \leqslant \epsilon \implies |\mathrm{Arg}(\pm z)| \leqslant \frac{\pi}{2} - \epsilon.$$

From (125), (126), and (116) we have

$$c_1 w_1(z) + c_2 w_2(z) = c \left\{ \frac{w_1(z)}{\Gamma\left(\frac{3+\lambda}{4}\right)} \pm \frac{2\mathrm{i}w_2(z)}{\Gamma\left(\frac{1+\lambda}{4}\right)} \right\} = \frac{c\mathrm{e}^{\frac{1}{2}z^2}}{\sqrt{\pi}} \Theta\left(\frac{1+\lambda}{4}; \mp \mathrm{i}z\right).$$

Then (B) and (C) follow from Theorem 35 with $p = \frac{1+\lambda}{4}$; while (D), and (E) follow from (107), and (108).

From (127), (128), and (115) we have

$$c_1 w_1(z) + c_2 w_2(z) = c \left\{ \frac{w_1(z)}{\Gamma\left(\frac{3-\lambda}{4}\right)} \mp \frac{2w_2(z)}{\Gamma\left(\frac{1-\lambda}{4}\right)} \right\} = \frac{c\mathrm{e}^{-\frac{1}{2}z^2}}{\sqrt{\pi}} \Theta\left(\frac{1-\lambda}{4}; \pm z\right).$$

Then (F) and (G) follow from Theorem 35 with $p = \frac{1-\lambda}{4}$; while (H), and (I) follow from (107), and (108). ∎

References

[1] Abramowitz, M. and Stegun, I. A., *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, National Bureau of Standards Applied Mathematics Series, vol. 55, For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.

[2] Browder, F.E., *Functional analysis and partial differential equations I*, Math. Ann. **138** (1959), 55–79.

[3] Cohen, L., *The Weyl operator and its generalization*, Pseudo-Differential Operators. Theory and Applications, vol. 9, Birkhäuser/Springer Basel AG, Basel, 2013.

[4] Cohen, L., Galleani, L., *Nonlinear transformation of differential equations into phase space*, EURASIP J. Appl. Signal Process **12** (2004), 1770–1777.

[5] D'ancona, P., Pierfelice, V., Ricci, F., *On the wave equation associated to the Hermite and the twisted Laplacian*, J. Fourier Anal. Appl., **16**, 2 (2010), 294–310.

[6] Galleani, L., Cohen, L., *The Wigner distribution for classical systems*, Phys. Lett. A **302**, 4 (2002), 149–155.

[7] Hörmander, L., *The Analysis of Linear Partial Differential Operators III*, Grundlehren der mathematischen Wissenschaften, vol. 274, Springer-Verlag, Berlin, 1985.

[8] Koch, H., Ricci, F., *Spectral projections for the twisted Laplacian*, Studia Math., **180**, 2 (2007), 103–110.

[9] Lebedev, N. N., *Special functions and their applications*, Revised English edition. Translated and edited by Richard A. Silverman, Prentice-Hall Inc., Englewood Cliffs, N.J., 1965.

[10] Li, W.-X., Parmeggiani, A., *Global Gevrey hypoellipticity for twisted Laplacians*, J. Pseudo-Differ. Oper. Appl., **4**, 3 (2013), 279–296.

[11] Nacinovich, M., *A remark on differential equations with polynomial coefficients*, Boll. Un. Mat. Ital. B (6) **1** (1982), no. 3, 979–989.

[12] Nicola, F. and Rodino, L., *Global pseudo-differential calculus on Euclidean spaces*, Pseudo-Differential Operators. Theory and Applications, vol. 4, Birkhäuser Verlag, Basel, 2010.

[13] Nicola, F. and Rodino, L., *Global regularity for ordinary differential operators with polynomial coefficients*, J. Differential Equations **255** (2013), no. 9, 2871–2890.

[14] Shubin, M.A., *Pseudodifferential Operators and Spectral Theory*, Springer-Verlag, Berlin, 1987.

[15] Thangavelu, S. *Harmonic analysis on the Heisenberg group*, Progress in Mathematics, 159. Birkhäuser Boston, 1998.

[16] Treves, F., *Topological vector spaces, distributions and kernels*, Academic Press, New York, 1967.

[17] Wong, M.-W., *Weyl Transforms, the Heat Kernel and Green Functions of a Degenerate Elliptic Operator*, Ann. Global Anal. Geom., **28**, 3 (2005), 271–283.