Large N limit of quiver matrix models and Sasaki-Einstein manifolds

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The large N limit of quiver matrix models and Sasaki-Einstein manifolds

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Abstract

We study the matrix models that result from localization of the partition functions of $\mathcal{N} = 2$ Chern-Simons-matter theories on the three-sphere. A large class of such theories are conjectured to be holographically dual to M-theory on Sasaki-Einstein seven-manifolds. We study the M-theory limit (large N and fixed Chern-Simons levels) of these matrix models for various examples, and show that in this limit the free energy reproduces the expected AdS/CFT result of $N^{3/2}/\mathrm{Vol}(Y)^{1/2}$, where $\mathrm{Vol}(Y)$ is the volume of the corresponding Sasaki-Einstein metric. More generally we conjecture a relation between the large N limit of the partition function, interpreted as a function of trial R-charges, and the volumes of Sasakian metrics on links of Calabi-Yau four-fold singularities. We verify this conjecture for a family of $U(N)^2$ Chern-Simons quivers based on M2 branes at hypersurface singularities, and for a $U(N)^3$ theory based on M2 branes at a toric singularity.

Contents

1 Introduction

For a long time the low energy theory on multiple M2 branes had remained rather mysterious, and consequently the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence poorly understood. A breakthrough occurred with the construction by Bagger-Lambert [1] and Gustavsson [2] of new maximally supersymmetric Chern-Simons-matter theories, which they proposed to describe the low energy limit of M2 branes. Inspired by these results, subsequently Aharony *et al* (ABJM) [3] conjectured the equivalence of a certain $\mathcal{N} = 6$ supersymmetric Chern-Simons-matter theory with gauge group $U(N) \times U(N)$ and Chern-Simons levels $(k, -k)$ with the M-theory backgrounds $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$. In particular, the ABJM theory is believed to describe N M2 branes at a $\mathbb{C}^4/\mathbb{Z}_k$ orbifold singularity. Despite this improved understanding of the microscopic theory of multiple M2 branes, a field theory derivation of the famous $N^{3/2}$ scaling of the number of degrees of freedom [4] on multiple M2 branes, in the large N limit, had remained elusive. Remarkably, in [5] this AdS/CFT prediction has been confirmed by a purely field theoretic calculation in the ABJM model. The large N limit of a BPS Wilson loop in this model was also computed in [6].[1] More recently in [7] and [8] similar results have been obtained for classes of $\mathcal{N} = 3$ Chern-Simons-quivers [9] with M-theory duals of the form $\mathrm{AdS}_4 \times Y$, where Y are tri-Sasakian manifolds [10]. The key to these results is the computation of [11], showing that the path integral of a (Euclidean) superconformal field theory on S^3 *localizes* to a matrix integral. As we will review momentarily, the results of [11] are effectively applicable to theories with $\mathcal{N} \geq 3$ supersymmetry.

$\mathcal{N} = 2$ supersymmetric field theories in three dimensions are expected to share certain properties with $\mathcal{N} = 1$ supersymmetric field theories in four dimensions, since the number of supercharges is the same. For the latter theories, the exact NSVZ beta functions and a-maximization [12] provide stringent constraints on the R-symmetry of superconformal theories, and indeed allow one to unambiguosly determine R-charges and related trace anomaly coefficients in most cases. Vanishing of beta functions and a-maximization may also be used as diagnostic tests of the existence of strongly coupled superconformal fixed points. In the context of the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence, a-maximization has a geometric counterpart in the volume minimization of Sasaki-Einstein manifolds[2] [15, 16, 17]. Intriguingly, it has been shown that the trial a function and the reciprocal of the trial volume function are equal, even before being extremized [18, 19]. The geometric results of [15, 16, 17] hold in any dimension, and hence crucially also in seven dimensions. Via the AdS/CFT correspondence, this suggests that a large class of $\mathcal{N} = 2$ Chern-Simons-matter theories, characterized by having AdS_4 gravity

[1] We would like to thank Takao Suyama for pointing out reference [6].

[2] See [13, 14] for extensions of these results to more general AdS_5 geometries of Type IIB supergravity.

duals, should in many ways behave similarly to $\mathcal{N}=1$ superconformal field theories in four dimensions. However, until now this has remained only wishful thinking.

Very recently Jafferis [20] (see also [21]) has extended the results of [11] by showing that the partition function Z of a general $\mathcal{N}=2$ supersymmetric field theory on S^3 reduces to a matrix integral. Putting the theory on the three-sphere, in a manner which preserves supersymmetry, requires the introduction of additional couplings between the matter fields and the curvature of S^3. These couplings are determined by a choice of R-symmetry. One may now regard these as *trial* R-charges for a putative superconformal fixed point. Furthermore, [20] conjectured that the R-charges of the matter fields at the superconformal fixed point are determined by extremizing (the modulus of) the partition function. If correct, this proposal gives support to the idea that Z plays a similar role, for $\mathcal{N}=2$ field theories in three dimensions, to the central charge function a in $\mathcal{N}=1$ field theories in four dimensions. In this paper we will initiate an investigation of this idea by evaluating the relevant matrix integrals, in the large N limit, in some $\mathcal{N}=2$ models. In particular, we will find supporting evidence for the following:

Conjecture: For $\mathcal{N}=2$ Chern-Simons-matter gauge theories with candidate Sasaki-Einstein gravity duals, in the large N limit the leading contribution to the free energy, as a function of trial R-charges R_a, is related to the Sasakian volume $\mathrm{Vol}(Y)[\xi]$ as a function of the Reeb vector field ξ via the formula

$$\lim_{N\to\infty} -\log|Z[R_a]| \quad = \quad N^{3/2}\sqrt{\frac{2\pi^6}{27\,\mathrm{Vol}(Y)[\xi]}}\ . \tag{1.1}$$

Here by "candidate Sasaki-Einstein dual" we mean that the field theory (at least) possesses a branch of the Abelian vacuum moduli space that is an isolated Calabi-Yau four-fold conical singularity [22, 23]. We refer the reader to [15, 16] for a detailed explanation of the volume function $\mathrm{Vol}(Y)[\xi]$. In (1.1) the trial R-charges should in general be understood to be functions of the Reeb vector field ξ, as in [18]. The method that we will use for computing the leading free energy in the large N limit follows very closely [7]; in the latter reference this method was applied to compute the same quantity for $\mathcal{N}=3$ matrix models, of the type studied in [11].

The rest of the paper is organized as follows. In section 2 we review relevant aspects of $\mathcal{N}=2$ Chern-Simons-matter theories, localization of the partition functions of such theories on S^3 to matrix integrals, and the expected AdS/CFT dual gravitational results in M-theory. In section 3 we discuss the M-theory large N limit of the partition functions of such theories, following similar methods to [6, 7]. In particular, we derive general formulas for the leading one-loop contributions for non-chiral theories. Section 4 contains an explicit verification of (1.1) for a family of examples describing M2 branes at certain hypersurface singularities [24]; these are in some sense a simple $\mathcal{N} = 2$ generalization of the ABJM theory. In section 5 we verify the matching of functions in (1.1) for a three node theory related to the suspended pinch point (SPP) singularity [25]. In section 6 we briefly outline directions for future work. In Appendix A we collect some series expansions used in the main text. Appendix B derives a technical result on the range of validity of a Fourier series used in section 5.

Note: As this paper was being completed we learned of the paper [26], which we understand has overlap with the results presented here.

2 Localization of $\mathcal{N} = 2$ Chern-Simons-matter theories on S^3

2.1 $\mathcal{N} = 2$ Chern-Simons-matter theories

We begin by reviewing the $\mathcal{N} = 2$ Chern-Simons-matter theories of interest, following [22]. Earlier foundational work on this topic includes [27, 3, 28].

A three-dimensional $\mathcal{N} = 2$ vector multiplet $\mathcal{V}$ consists of a gauge field $\mathscr{A}_\mu$, a scalar field σ, a two-component Dirac spinor χ, and another scalar field D, all transforming in the adjoint representation of the gauge group $\mathcal{G}$. This is simply the dimensional reduction of the usual four-dimensional $\mathcal{N} = 1$ vector multiplet. In particular, σ arises from the zero mode of the component of the vector field in the direction along which one reduces. The matter fields Φ_a are chiral multiplets, consisting of a complex scalar ϕ_a, a fermion ψ_a and an auxiliary scalar F_a, which may be in arbitrary representations $\mathcal{R}_a$ of $\mathcal{G}$. An $\mathcal{N} = 2$ Chern-Simons-matter Lagrangian then consists of three terms:

$$S = S_{\text{CS}} + S_{\text{matter}} + S_{\text{potential}} \ . \tag{2.1}$$

We will be interested in product gauge groups of the form $\mathcal{G} = \prod_{I=1}^{G} U(N_I)$, where the Chern-Simons level for the Ith factor $U(N_I)$ is $k_I \in \mathbb{Z}$. If $\mathcal{V}_I$ denotes the projection of $\mathcal{V}$ onto the Ith gauge group factor, then in component notation the Chern-Simons action, in Wess-Zumino gauge, takes the form

$$S_{\rm CS} = \sum_{I=1}^{G} \frac{k_I}{4\pi} \int \mathrm{Tr} \left(\mathscr{A}_I \wedge \mathrm{d}\mathscr{A}_I + \frac{2}{3} \mathscr{A}_I \wedge \mathscr{A}_I \wedge \mathscr{A}_I - \bar{\chi}_I \chi_I + 2 D_I \sigma_I \right) , \qquad (2.2)$$

where the trace in (2.2) is normalized in the fundamental representation.

The matter kinetic term takes a simple form in superspace, namely

$$\begin{aligned} S_{\rm matter} &= \int \mathrm{d}^3 x \, \mathrm{d}^4\theta \sum_a \mathrm{Tr} \, \bar{\Phi}_a \mathrm{e}^{\mathcal{V}} \Phi_a \\ &= \int \mathrm{d}^3 x \sum_a \mathscr{D}_\mu \bar{\phi}_a \mathscr{D}^\mu \phi_a - \bar{\phi}_a \sigma^2 \phi_a + \bar{\phi}_a D \phi_a + \text{fermions} \, . \qquad (2.3) \end{aligned}$$

In the second line we have expanded into component fields, and we have not written the terms involving the fermions ψ_a. $\mathscr{D}_\mu$ is the covariant derivative, and the auxiliary fields σ and D are understood to act on ϕ_a in the appropriate representation $\mathcal{R}_a$. In this paper we shall mainly focus on theories with matter in bifundamental or adjoint representations of a gauge group $\mathcal{G} = \prod_{I=1}^{G} U(N_I)$. In this case one can represent the gauge group and matter content by a quiver diagram with G nodes, with a directed arrow from node I to node J corresponding to a bifundamental field in the representation $\mathbf{N}_I \otimes \overline{\mathbf{N}}_J$; when $I = J$ this is understood to be the adjoint representation.

Finally, the superpotential term is

$$\begin{aligned} S_{\rm potential} &= \int \mathrm{d}^3 x \, \mathrm{d}^2\theta \, W(\Phi) + \text{c.c.} \\ &= - \int \mathrm{d}^3 x \sum_a \left| \frac{\partial W}{\partial \phi_a} \right|^2 + \text{fermions} \, . \qquad (2.4) \end{aligned}$$

One typically takes the superpotential to be a polynomial in the chiral fields Φ_a, and we have included the couplings (which in general will be renormalized) in the definition of W.

In [22, 23] the moduli spaces of vacua of such theories were studied. As usual this means that all scalar fields take constant vacuum expectation values, and one seeks absolute minima of the total potential. This is equivalent to imposing the F-terms $\mathrm{d}W = 0$, together with an appropriate Kähler quotient by the gauge group. The latter is more subtle than the corresponding quotient for four-dimensional $\mathcal{N} = 1$ gauge theories, due to the effects of the Chern-Simons interactions. The result is that for a large class of such theories, provided

$$\sum_{I=1}^{G} k_I = 0 \, , \qquad (2.5)$$

and one takes the ranks $N_I = N$ to be equal[3] for all $I = 1, \ldots, G$, then the moduli space of vacua contains the symmetric product $\mathrm{Sym}^N X$, where X is an affine Calabi-Yau four-fold variety.

The simplest construction of $\mathcal{N} = 2$ Chern-Simons-matter theories within this class is to begin with the gauge group and matter content of a "parent" four-dimensional $\mathcal{N} = 1$ gauge theory which has a Calabi-Yau *three-fold* as Abelian moduli space [22], and reinterpret this as a three-dimensional $\mathcal{N} = 2$ theory by adding the Chern-Simons interactions (2.2). This has an elegant string theory interpretation [30]: if the initial four-dimensional gauge theory is the effective theory on N D3-branes probing a Calabi-Yau three-fold singularity, then one can T-dualize along a worldvolume direction to obtain a corresponding theory on D2-branes in Type IIA string theory. The addition of Chern-Simons couplings may then be understood in terms of turning on Ramond-Ramond two-form (and more generally four-form) flux and lifting to M-theory. One can also add D6-branes in this set-up, which introduces new chiral matter fields in fundamental representations of the gauge group factors [31, 32]. The resulting theories are generally conjectured to be low energy effective field theories on N M2 branes probing a Calabi-Yau four-fold singularity X.

In this gauge theory construction the Calabi-Yau four-fold X is topologically a cone over a compact seven-manifold Y, and a Calabi-Yau metric on X of the conical form

$$g_X = \mathrm{d}r^2 + r^2 g_Y \ , \tag{2.6}$$

implies that Y is a Sasaki-Einstein manifold. The AdS/CFT correspondence conjectures that the IR limit of the Chern-Simons-matter theory, for fixed Chern-Simons levels and large N, is holographically dual to M-theory on the Freund-Rubin background $\mathrm{AdS}_4 \times Y$ with N units of $\star G_4$-flux through Y, where G_4 is the M-theory four-form and $\star$ denotes the eleven-dimensional Hodge dual.

We note that one can relax the condition (2.5), which leads to theories that are conjecturally dual to *massive* Type IIA string theory backgrounds [33]. This may be understood in the same context as [30], via the effects of turning on Ramond-Ramond fluxes in Type IIA on the Chern-Simons couplings on fractional branes. The results in this paper presumably extend to these gravity backgrounds also, although they are no longer described by Sasaki-Einstein geometry.

[3] In fact this is not really necessary. For a detailed discussion in a particular class of examples, see [29, 24].

2.2 Localization of the partition function to a matrix model

In [11] it was shown that the partition function of a *superconformal* Chern-Simons-matter theory on S^3 localizes to a matrix model. The derivation follows the usual method of localization: one adds an appropriately chosen Q-exact operator to the action of the theory, where Q is a supercharge with $Q^2 = 0$. One can formally argue that this does not affect the partition function, but at the same time it localizes fields to certain constant values, for which the one-loop approximation is exact. More precisely, in this case the adjoint scalar σ in the vector multiplet is localized to constant field configurations, with all other fields being zero.

For the theories described in section 2.1, the gauge group is of the product form $\mathcal{G} = \prod_{I=1}^{G} U(N_I)$ and correspondingly σ_I is a Hermitian $N_I \times N_I$ matrix. Up to gauge equivalence σ_I is described by its N_I real eigenvalues λ_i^I, where $i = 1, \ldots, N_I$, and the matrix model in question is then a multi-matrix model for these eigenvalues. Such matrix models were subsequently studied by a number of authors [5, 6, 7, 34, 35]. In particular, the matrix model for the ABJM theory was solved in [34, 5] in the limit in which N is large and k/N is held fixed. In [7] instead the authors studied what we will refer to as the *M-theory limit* of large N and k held fixed for the ABJM theory. In addition they studied some closely related theories with $\mathcal{N} = 3$ superconformal symmetry, to which the same techniques apply.

In the present paper we are interested in the more general situation in which one has some UV $\mathcal{N} = 2$ Chern-Simons-matter theory, with action (2.1), which one believes flows to a superconformal fixed point in the IR. In this case the results of [11] do not directly apply, since at the fixed point the chiral matter fields Φ_a will in general not have canonical scaling dimensions $\Delta = \frac{1}{2}$. Fortunately, this problem has recently been addressed in [20]. Here the author considered $\mathcal{N} = 2$ Chern-Simons-matter theories with a choice of R-symmetry, with appropriate supersymmetry-preserving R-charge dependent couplings to the curvature of S^3, and showed that this partition function on S^3 still localizes as in [11]. The resulting partition function then depends on this choice of R-symmetry. For a superconformal theory, one should of course choose this to be the superconformal R-symmetry of the IR theory.

It is straightforward to apply the localization results of [11, 20] to the theories described in section 2.1. As already mentioned, the adjoint scalars σ_I in the vector multiplet localize to constant field configurations, and using the gauge freedom we may parametrize these by their eigenvalues λ_i^I. All other fields, and in particular the chiral scalar fields ϕ_a, are localized to zero. The partition function then localizes to the finite matrix integral

$$Z = \frac{1}{(\prod_{I=1}^{G} N_I!)} \int \left(\prod_{I=1}^{G} \prod_{i=1}^{N_I} \frac{\mathrm{d}\lambda_i^I}{2\pi} \right) \exp\left[\mathrm{i} \sum_{I=1}^{G} \frac{k_I}{4\pi} \sum_{i=1}^{N_I} \left(\lambda_i^I\right)^2 \right] \exp\left[-F_{\mathrm{loop}}\right] , \tag{2.7}$$

where the one-loop term is

$$\exp\left[-F_{\mathrm{loop}}\right] = \prod_{I=1}^{G} \prod_{i \neq j} 2 \sinh\left(\frac{\lambda_i^I - \lambda_j^I}{2} \right) \cdot \exp[-F_{\mathrm{matter}}] . \tag{2.8}$$

Here the first exponential term in (2.7) is simply that of the Euclideanized Chern-Simons action (2.2), localized onto the constant field configuration of the σ_I. The localization renders the one-loop approximation to the path integral exact, and the first term in (2.8) is precisely this one-loop contribution for the gauge sector of the theory. As usual, this is a regularized determinant, derived from the action (2.2).

The matter sector contributes nothing at tree level, since the matter multiplets localize to zero, but there is a one-loop determinant factor given by [20]

$$\exp\left[-F_{\mathrm{matter}}\right] = \prod_a \det{}_{\mathcal{R}_a} \exp\left[\ell\left(1 - \Delta_a + \mathrm{i}\sigma\right)\right] . \tag{2.9}$$

Recall here that the index a labels chiral matter fields Φ_a in the representation $\mathcal{R}_a$ of $\mathcal{G}$. We have denoted the conformal dimension/R-charge of Φ_a by $\Delta_a = \Delta[\Phi_a]$. In (2.9), the determinant in the representation $\mathcal{R}_a$ is understood to be a product over weights ρ in the weight-space decomposition of this representation, and σ is then understood to mean $\rho(\sigma)$. For example, for a bifundamental field $\Phi_a = \Phi_{\mathrm{Bi}}$ in the representation $\mathbf{N}_I \otimes \overline{\mathbf{N}}_J$ we have

$$\exp\left[-F(\Phi_{\mathrm{Bi}})\right] = \prod_{i=1}^{N_I} \prod_{j=1}^{N_J} \exp\left[\ell\left(1 - \Delta_{\mathrm{Bi}} + \mathrm{i}\frac{\lambda_i^I - \lambda_j^J}{2\pi} \right) \right] , \tag{2.10}$$

while for an adjoint field Φ_{Ad} for the gauge group factor $U(N_I)$ we have

$$\exp\left[-F(\Phi_{\mathrm{Ad}})\right] = \prod_{i,j=1}^{N_I} \exp\left[\ell\left(1 - \Delta_{\mathrm{Ad}} + \mathrm{i}\frac{\lambda_i^I - \lambda_j^I}{2\pi} \right) \right] . \tag{2.11}$$

Finally, the function ℓ in (2.9) arises from the (zeta function) regularized one-loop determinant of the matter action (2.3), and is given explicitly by

$$\ell(z) = -z \log\left(1 - \mathrm{e}^{2\pi \mathrm{i} z}\right) + \frac{\mathrm{i}}{2}\left[\pi z^2 + \frac{1}{\pi} \mathrm{Li}_2\left(\mathrm{e}^{2\pi i z}\right) \right] - \frac{\mathrm{i}\pi}{12} . \tag{2.12}$$

Here $\mathrm{Li}_2(\zeta)$ denotes the dilogarithm function, defined as the analytic continuation of

$$\mathrm{Li}_2(\zeta) \;=\; \sum_{m=1}^{\infty} \frac{\zeta^m}{m^2} \;, \qquad |\zeta| < 1 \;, \tag{2.13}$$

to $\mathbb{C} \setminus [1, \infty)$. The function $\ell(z)$ satisfies the simple differential equation

$$\frac{\mathrm{d}\ell}{\mathrm{d}z} \;=\; -\pi z \cot \pi z \;, \tag{2.14}$$

together with the boundary condition that $\ell(0) = 0$. For canonical scaling dimensions $\Delta = \frac{1}{2}$, this reduces to the original result of [11].

2.3 The M-theory limit of the free energy and volumes

We are interested in computing the *free energy*, $F = -\log Z$, for these matrix models in the limit in which the Chern-Simons levels k_I, and in particular their greatest common divisor $k = \gcd\{k_I\}$, are held fixed and $N \to \infty$. This is the *M-theory limit*, as opposed to the *'t Hooft limit* in which N/k is held fixed while $N \to \infty$. For simplicity, here we assume that the ranks of the gauge group factors are all equal, so $N_I = N$ for all $I = 1, \ldots, G$.

Consider matrix models arising from Chern-Simons-matter theories which flow to superconformal fixed points, with holographic duals of the form $\mathrm{AdS}_4 \times Y$, as described at the end of section 2.1. If the Δ_a are taken to be the conformal dimensions/R-charges at this fixed point, then the AdS/CFT correspondence predicts that to leading order as $N \to \infty$ [36, 7]

$$F \;=\; -\log Z \;=\; N^{3/2} \sqrt{\frac{2\pi^6}{27 \mathrm{Vol}(Y)}} \;, \tag{2.15}$$

where $\mathrm{Vol}(Y)$ denotes the volume of the Sasaki-Einstein metric on Y. This formula follows from the saddle point approximation to the gravitational partition function on AdS_4, regularized using counterterm subtraction. In particular, we see the famous $N^{3/2}$ scaling of the free energy of N M2 branes in the large N limit in (2.15). This leads to a general concrete prediction: for a Chern-Simons-matter theory with Abelian moduli space X, which is a cone over Y, the free energy of the corresponding localized matrix model is related to the volume of a Sasaki-Einstein metric on Y via (2.15), in the M-theory large N limit.

This prediction has been confirmed for the ABJM theory in [34, 5, 7], and in the latter reference even more remarkably it has been confirmed for a simple class of $\mathcal{N}=3$ Chern-Simons-matter theories [10] that are holographically dual to certain *tri-Sasakian* seven-manifolds. The matrix models in these cases are somewhat simpler than the generic case described in the previous subsection, as originally noted in [11]. In particular, the combinations of ℓ-functions in (2.9) simplify to hyperbolic cosines for these $\mathcal{N}=3$ theories. Related to this, the fields all have canonical scaling dimensions of $\Delta=\frac{1}{2}$.

In this paper we wish to extend the correspondence further by checking that (2.15) holds for $\mathcal{N}=2$ theories which are only conjectured to be superconformal in the IR limit, with non-canonical scaling dimensions $\Delta\neq\frac{1}{2}$. In turn, this acts as an AdS/CFT check on the field theory results of [20]. In fact, we will provide non-trivial checks of our stronger conjecture (1.1).

3 The M-theory limit of Chern-Simons-matter matrix models

In this section we discuss the M-theory large N limit of the partition function (2.7). In sections 4 and 5 we shall compute this explicitly for families of Chern-Simons-matter theories, and verify that (2.7) leads to precisely (2.15), where $\mathrm{Vol}(Y)$ is the volume of the appropriate Sasakian manifold.

3.1 Massive adjoint fields

Before beginning our general analysis, we pause to comment on how *massive* adjoint fields should be treated in the partition function. As we shall see, this is relevant for the $\mathcal{A}_{n-1}$ theories discussed in section 4. Clearly, a theory with massive adjoint scalar fields cannot itself be superconformal. However, after integrating these out the theory may flow to a superconformal fixed point. How are we to treat such fields in the UV partition function (2.7)? As we now explain, the matrix model partition function effectively integrates out these fields for us.

Consider an adjoint scalar field Φ_{Ad} for a gauge group $U(N)$. From (2.11) this contributes

$$\prod_{i,j=1}^{N}\exp\left[\ell\left(1-\Delta_{\mathrm{Ad}}+\mathrm{i}\frac{\lambda_i-\lambda_j}{2\pi}\right)\right] \tag{3.1}$$

to the matrix model partition function. Since for a massive field $\Delta_{\rm Ad}=1$, this simplifies to

$$\exp[N\ell(0)]\cdot\prod_{i>j}\exp\left[\ell\left(\mathrm{i}\frac{\lambda_{ij}}{2\pi}\right)+\ell\left(-\mathrm{i}\frac{\lambda_{ij}}{2\pi}\right)\right]\ , \tag{3.2}$$

where we have defined $\lambda_{ij}=\lambda_i-\lambda_j$. However, we now recall that $\ell(0)=0$, and in fact it is easy to show more generally that

$$\ell(\mathrm{i}u)\ +\ \ell(-\mathrm{i}u)\ \ =\ \ 0\ ,\qquad u\in\mathbb{R}\ . \tag{3.3}$$

Thus the contribution (3.1) of a massive adjoint to the partition function is identically equal to 1.

3.2 The saddle point approximation

The strategy for evaluating (2.7) in the M-theory limit will be to use the saddle point, or stationary phase, approximation to the integral. Here N^2 plays the role of $1/\hbar$, so that the $N\to\infty$ limit is dominated by saddle point configurations that extremize the quantum effective action. This is a somewhat standard technique, and we refer the reader to the review [37] for further details.

The quantum effective action is

$$F\ \ =\ \ F_{\rm classical}+F_{\rm loop}\ , \tag{3.4}$$

where $F_{\rm loop}$ is defined by (2.8), (2.9) and

$$F_{\rm classical}\ \ =\ \ -\mathrm{i}\sum_{I=1}^{G}\frac{k_I}{4\pi}\sum_{i=1}^{N_I}\left(\lambda_i^I\right)^2 \tag{3.5}$$

is the localized Chern-Simons action. The saddle point approximation to the partition function (2.7) is dominated by solutions to the equations of motion $\partial F/\partial\lambda_i^I=0$. At this point it will be convenient to assume that the matter content is described by a quiver diagram with $N_I=N$, $I=1,\ldots,G$, so that in particular all matter fields are in bifundamental or adjoint representations of the gauge group $U(N)^G$. A straightforward computation then gives

$$
\begin{aligned}
-\frac{\partial F}{\partial \lambda_i^I} &= \frac{\mathrm{i}k_I}{2\pi}\lambda_i^I + \sum_{j\neq i} \coth \tfrac{\lambda_i^I-\lambda_j^I}{2} \\
&\quad -\frac{\mathrm{i}}{2}\sum_{\text{fixed } I\to J}\sum_{j=1}^{N}\left(1-\Delta_{IJ}+\mathrm{i}\tfrac{\lambda_i^I-\lambda_j^J}{2\pi}\right)\cot\left(\pi(1-\Delta_{IJ})+\mathrm{i}\tfrac{\lambda_i^I-\lambda_j^J}{2}\right) \\
&\quad +\frac{\mathrm{i}}{2}\sum_{\text{fixed } I\leftarrow J}\sum_{j=1}^{N}\left(1-\Delta_{JI}-\mathrm{i}\tfrac{\lambda_i^I-\lambda_j^J}{2\pi}\right)\cot\left(\pi(1-\Delta_{JI})-\mathrm{i}\tfrac{\lambda_i^I-\lambda_j^J}{2}\right) \ .
\end{aligned} \tag{3.6}
$$

Here the index I is fixed, and the sums are over outgoing arrows $a = (I \to J)$ or incoming arrows $a = (I \leftarrow J)$. The conformal dimension of a bifundamental field from node I to node J is denoted $\Delta_a = \Delta_{IJ}$. Notice that an adjoint field with $I = J$ contributes to both terms in (3.6).

Let us analyze these equations of motion assuming that the eigenvalues grow in the large N limit. More precisely, as we shall see the saddle point approximation is in fact dominated by a *complex* solution, in which the eigenvalues λ_i^I have an imaginary part that is small compared to the real part. This is a common phenomenon: one is effectively deforming the real integral in (2.7) into the complex plane in order to find the steepest descent. Then more precisely we are assuming that the real parts $\xi_i^I = \Re\lambda_i^I$ of the eigenvalues grow[4] with N, where we introduce the real and imaginary parts:

$$
\lambda_i^I = \xi_i^I + \mathrm{i}y_i^I \ . \tag{3.7}
$$

Using

$$
\cot(a+\mathrm{i}b) = \frac{1-\mathrm{i}\tan a \tanh b}{\tan a + \mathrm{i}\tanh b} \ , \tag{3.8}
$$

together with

$$
\tanh w = \mathrm{sgn}(\Re w) + \mathcal{O}\left(\mathrm{e}^{-2|\Re w|}\right) \ , \tag{3.9}
$$

we see that we may approximate $\cot(a+\mathrm{i}b) \simeq -\mathrm{i}\,\mathrm{sgn}(\Re b)$ for $|\Re b| \gg 0$. Using this, the

[4]We shall be more precise about this later.

equation of motion (3.6) simplifies to

$$
\begin{aligned}
-\frac{\partial F}{\partial \lambda_i^I} \quad &\simeq \quad \frac{\mathrm{i}k_I}{2\pi}\lambda_i^I + \sum_{j\neq i} \mathrm{sgn}\left(\xi_i^I - \xi_j^I\right) \\
&-\frac{1}{2}\sum_{\text{fixed } I\to J}\sum_{j=1}^{N}\left(1-\Delta_{IJ}+\mathrm{i}\tfrac{\lambda_i^I-\lambda_j^J}{2\pi}\right)\mathrm{sgn}\left(\xi_i^I-\xi_j^J\right) \\
&-\frac{1}{2}\sum_{\text{fixed } I\leftarrow J}\sum_{j=1}^{N}\left(1-\Delta_{JI}-\mathrm{i}\tfrac{\lambda_i^I-\lambda_j^J}{2\pi}\right)\mathrm{sgn}\left(\xi_i^I-\xi_j^J\right) \ .
\end{aligned}
\tag{3.10}
$$

We now introduce the *continuum limit*. Here the sums over N eigenvalues tend to Riemann integrals as $N \to \infty$. Again, this procedure is somewhat standard, and a review may be found in [37]. We begin by defining functions $\xi^I(s)$, $y^I(s)$, so $\xi^I, y^I : [0,1] \to \mathbb{R}$, via

$$
\xi^I\left[\tfrac{1}{N}\left(i-\tfrac{1}{2}\right)\right] \;=\; \xi_i^I \ , \qquad y^I\left[\tfrac{1}{N}\left(i-\tfrac{1}{2}\right)\right] \;=\; y_i^I \ , \qquad i = 1,\ldots,N \ . \tag{3.11}
$$

In the continuum limit where $N \to \infty$ the sums over N become Riemann integrals, so for example

$$
\frac{1}{N}\sum_{j=1}^{N} f(\xi_j) \longrightarrow \int_0^1 f(\xi(s))\,\mathrm{d}s \ . \tag{3.12}
$$

For each $I = 1,\ldots,G$ we may also introduce the *density* $\rho_I(\xi)$ via

$$
\rho_I(\xi^I(s))\mathrm{d}\xi^I \quad = \quad \mathrm{d}s \ . \tag{3.13}
$$

These are G functions of a single real variable. Assuming that the real parts of the eigenvalues dominate over the imaginary parts at large N, the leading order term in (3.10) in the continuum limit gives

$$
\sum_{\text{fixed } I\to J}\int\left(\xi-\xi'\right)\rho_J(\xi')\mathrm{sgn}(\xi-\xi')\mathrm{d}\xi' = \sum_{\text{fixed } I\leftarrow J}\int\left(\xi-\xi'\right)\rho_J(\xi')\mathrm{sgn}(\xi-\xi')\mathrm{d}\xi' \tag{3.14}
$$

Differentiating this twice with respect to ξ then implies

$$
\sum_{\text{fixed } I\to J}\rho_J(\xi) \quad = \quad \sum_{\text{fixed } I\leftarrow J}\rho_J(\xi) \ , \qquad I = 1,\ldots,G \ . \tag{3.15}
$$

Remarkably, these G linear equations for the G density functions $\rho_I(\xi)$ take the form of ABJ anomaly conditions, when thought of in the four-dimensional context. If one considers the parent $\mathcal{N} = 1$ four-dimensional quiver gauge theory, then it is a general result that there is a $(b_3(Y_5)+1)$-dimensional space of solutions to (3.15); that is, the skew part of the adjacency matrix of the quiver has a kernel of dimension $b_3(Y_5)+1$. Here $b_3(Y_5)$ is the third Betti number of Y_5, which is the link of the Calabi-Yau *three-fold* singularity of the parent theory vacuum moduli space. Given (3.15), the leading order part of the equation of motion (3.14) is then zero.

The next-to-leading order term in (3.6) then gives

$$\begin{aligned} 0 \; = \; & \int \rho_I(\xi')\,\mathrm{sgn}(\xi-\xi')\,\mathrm{d}\xi' \\ & -\frac{1}{2}\sum_{\text{fixed } I\to J}\int\left(1-\Delta_{IJ}-\tfrac{y^I(\xi)-y^J(\xi')}{2\pi}\right)\rho_J(\xi')\,\mathrm{sgn}(\xi-\xi')\,\mathrm{d}\xi' \\ & -\frac{1}{2}\sum_{\text{fixed } I\leftarrow J}\int\left(1-\Delta_{JI}+\tfrac{y^I(\xi)-y^J(\xi')}{2\pi}\right)\rho_J(\xi')\,\mathrm{sgn}(\xi-\xi')\,\mathrm{d}\xi'\;. \end{aligned} \tag{3.16}$$

Using (3.15) the $y^I(\xi)$ terms cancel, and differentiating gives

$$2\rho_I(\xi) \;=\; \sum_{\text{fixed } I\to J}\left(1-\Delta_{IJ}+\tfrac{y^J(\xi)}{2\pi}\right)\rho_J(\xi)+\sum_{\text{fixed } I\leftarrow J}\left(1-\Delta_{JI}-\tfrac{y^J(\xi)}{2\pi}\right)\rho_J(\xi)\;. \tag{3.17}$$

At this point we shall make the assumption that

$$\rho_I(\xi) \;=\; \rho(\xi) \tag{3.18}$$

holds for all $I=1,\ldots,G$. This condition is satisfied for all the $\mathcal{N}\geq 3$ examples discussed in [7], and as we shall see later also holds for the $\mathcal{N}=2$ $\mathcal{A}_{n-1}$ theories by a symmetry argument. As we shall also see, without the constraint (3.18) the behaviour of the matrix model is qualitatively different. Notice that for theories with parents for which $b_3(Y_5)=0$, the condition (3.18) *necessarily* holds as the skew adjacency matrix has a one-dimensional kernel. This kernel is due to gauge anomaly cancellation in four dimensions, which for equal ranks $N_I=N$ is equivalent to the number of incoming arrows equalling the number of outgoing arrows at each node. With (3.18), the last equation becomes

$$2 \;=\; \sum_{\text{fixed } I\to J}\left(1-\Delta_{IJ}+\tfrac{y^J(\xi)}{2\pi}\right)+\sum_{\text{fixed } I\leftarrow J}\left(1-\Delta_{JI}-\tfrac{y^J(\xi)}{2\pi}\right)\;. \tag{3.19}$$

One can now sum this equation over I. This gives

$$2G \;=\; 2\sum_{\text{all } I\to J}(1-\Delta_{IJ})+\sum_{\text{all } I\to J}\frac{y^J(\xi)-y^I(\xi)}{2\pi}\;. \tag{3.20}$$

Here the sums are over *all* matter content, or equivalently arrows $a = (I \to J)$ in the quiver. The first factor of 2 arises because we double count every arrow in the quiver (each arrow is incoming and outgoing to precisely one node each). The last term in (3.20) is then zero due to four-dimensional gauge anomaly cancellation of the parent theory: note that for a fixed node I an outgoing arrow contributes $-y^I(\xi)$, while an incoming arrow contributes $+y^I(\xi)$. We thus derive the following constraint on the conformal dimensions:

$$G = \sum_{\text{all } I\to J} (1 - \Delta_{IJ}) \ . \tag{3.21}$$

Given (3.18), it follows that $\Re\lambda_i^I = \Re\lambda_i^J \equiv \xi_i$ holds for all I and J. We may thus write

$$\lambda_i^I = \xi_i + \mathrm{i}y_i^I \ , \tag{3.22}$$

with ξ_i, y_i^I both real. At this point it is also convenient to order the eigenvalues in such a way that ξ_i is monotonically increasing with i. We may then simplify the expression (2.10) for $F(\Phi_{\mathrm{Bi}})$ for a generic bifundamental field Φ_{Bi}. We first rearrange the terms in the products, separating the terms with $i > j$ and $i < j$ from those with $i = j$:

$$F(\Phi_{\mathrm{Bi}}) = F_1(\Phi_{\mathrm{Bi}}) + F_2(\Phi_{\mathrm{Bi}}) \ . \tag{3.23}$$

Here

$$F_1(\Phi_{\mathrm{Bi}}) = \sum_{i=1}^{N} \ell\left(1 - \Delta_{\mathrm{Bi}} - \frac{y_i^I - y_i^J}{2\pi}\right) \ . \tag{3.24}$$

In fact this term will be subleading in the large N expansion, as we shall see momentarily. Using the expressions for $\ell_\pm(z)$ in Appendix A, we also compute

$$F_2(\Phi_{\mathrm{Bi}}) = \frac{1}{2}\sum_{i\neq j} \mathrm{sgn}(\xi_i - \xi_j)\left(1 - \Delta^{I,J} + \mathrm{i}\frac{\lambda_i^I - \lambda_j^J}{4\pi}\right)(\lambda_i^I - \lambda_j^J) + \text{sums of exponentials} \ . \tag{3.25}$$

Here the sums of exponentials are precisely the sums over m in equations (A.7). Of course, differentiating (3.25) with respect to λ_i^I leads to the corresponding term in the leading order equation of motion (3.10).

3.3 Quantum effective action: non-chiral theories

For a general Chern-Simons-matter theory, the terms written in (3.25) are at leading order in the large N expansion, and generically do not cancel on summing over all matter content. In contrast to this, for the $\mathcal{N} \geq 3$ examples studied in [7] these terms precisely cancelled. More generally, it is straightforward to see that this cancellation always occurs for a *non-chiral* Chern-Simons-matter theory. In terms of quivers, this means that for every arrow from node I to node J, there is a corresponding arrow from node J to node I. At this point it is expedient to therefore restrict to non-chiral theories. The matrix models in question will then closely resemble that for the ABJM theory, but will nevertheless still be general enough to allow for non-trivial $\mathcal{N} = 2$ theories with anomalous (and in fact irrational) dimensions Δ_a of the chiral matter fields.

Requiring that the Chern-Simons-matter theory be non-chiral has a number of interesting consequences. Firstly, the terms depending on y_i^J in (3.19) cancel, leaving us with the following set of constraints on conformal dimensions:

$$2 \;=\; 2 \sum_{\text{fixed } I \leftrightarrow J} (1 - \Delta_{IJ}) \ , \tag{3.26}$$

where we have assumed[5] that $\Delta_{IJ} = \Delta_{JI}$ for each bifundamental pair. These are precisely the conditions imposed by setting the G NSVZ beta functions in the four-dimensional parent theory to zero. Remarkably, the same conditions must hold for non-chiral Chern-Simons-matter theories, at least under the assumptions we have made thus far. It is then straightforward to check that the terms written in (3.25) cancel, leaving only the exponential sums. More precisely, the quadratic and constant terms cancel pairwise between the two arrows going between $I \leftrightarrow J$, while the linear terms only cancel on summing over the whole quiver, where one must also include the contribution from the gauge sector one-loop term in (2.8).

Thus the terms written explicitly in (3.25) cancel for a non-chiral quiver, and we are left with the sums of exponentials. For a fixed pair of arrows going between nodes

[5] For the explicit examples that we shall study in this paper this will follow from symmetry; more generally one might have to relax this condition.

$I \leftrightarrow J$, we compute the contribution

$$\begin{aligned} F_2(\Phi_{I\leftrightarrow J}) &= \sum_{i>j}\sum_{m=1}^{\infty} \mathrm{e}^{-m\xi_{ij}} \left\{ \frac{1}{\pi m}\left[\xi_{ij}+\frac{1}{m}\right] \sin 2\pi m(1-\Delta)\left[\mathrm{e}^{\mathrm{i}m(y_j^I-y_i^J)}+\mathrm{e}^{-\mathrm{i}m(y_i^I-y_j^J)}\right] \right. \\ & \quad -\frac{2}{m}(1-\Delta)\cos 2\pi m(1-\Delta)\left[\mathrm{e}^{\mathrm{i}m(y_j^I-y_i^J)}+\mathrm{e}^{-\mathrm{i}m(y_i^I-y_j^J)}\right] \\ & \quad \left. +\frac{\mathrm{i}}{\pi m}\sin 2\pi m(1-\Delta)\left[(y_i^I-y_j^J)\mathrm{e}^{-\mathrm{i}m(y_i^I-y_j^J)}-(y_j^I-y_i^J)\mathrm{e}^{\mathrm{i}m(y_j^I-y_i^J)}\right]\right\} . \end{aligned} \tag{3.27}$$

Here $\Delta \equiv \Delta_{IJ} = \Delta_{JI}$, and we have defined

$$\xi_{ij} = \xi_i - \xi_j \ . \tag{3.28}$$

In the continuum limit of the previous subsection, we have $y^I = y^I(\xi)$ with density function

$$\rho(\xi)\mathrm{d}\xi = \mathrm{d}s \ . \tag{3.29}$$

It is straightforward to apply this to the classical part of the action:

$$\begin{aligned} F_{\text{classical}} &= -\mathrm{i}\sum_{I=1}^{G}\frac{k_I}{4\pi}\sum_{i=1}^{N}\left(\lambda_i^I\right)^2 = \frac{1}{2\pi}\sum_{I=1}^{G}\sum_{i=1}^{N}k_I\xi_i y_i^I + \text{lower order} \\ &\longrightarrow \frac{N}{2\pi}\int \xi\rho(\xi)\sum_{I=1}^{G}k_I y^I(\xi)\,\mathrm{d}\xi \ . \end{aligned} \tag{3.30}$$

At this point we will be more precise about the growth of ξ_i. Following [7], we write the ansatz

$$\xi_i = N^\alpha x_i \tag{3.31}$$

for the leading order behaviour of the real parts of the eigenvalues, where $\alpha > 0$ is some real constant. Then more precisely the last equation (3.30) becomes

$$F_{\text{classical}} = \frac{N^{1+\alpha}}{2\pi}\int_{x_1}^{x_2} x\rho(x)\sum_{I=1}^{G}k_I y^I(x)\,\mathrm{d}x + o(N^{1+\alpha}) \ . \tag{3.32}$$

We then notice that (3.24) is $\mathcal{O}(N)$, and is thus *subleading* to the classical action, assuming $\alpha > 0$. Of course, this is intuitively clear, since this term came from $i = j$ in the original sum over both i and j, and thus should be measure zero compared to this latter term, in the continuum limit.

We turn next to the leading order contribution to (3.27) in the continuum limit. In order to evaluate this, notice that since the sum over $i > j$ leads to a double integral with $x - x' > 0$, the latter may be evaluated in the large N limit essentially using the representation of the delta function

$$\delta(x) = \lim_{c\to\infty} \frac{c}{2}\mathrm{e}^{-c|x|} \ . \tag{3.33}$$

More precisely, consider the general equality

$$\begin{aligned}\int_{x_1}^{x} \mathrm{d}x' \mathrm{e}^{-mN^\alpha(x-x')} f(x,x') &= \frac{1}{mN^\alpha}\left[\mathrm{e}^{-mN^\alpha(x-x')} f(x,x')\right]_{x_1}^{x} \\ &\quad -\frac{1}{mN^\alpha}\int_{x_1}^{x} \mathrm{d}x' \mathrm{e}^{-mN^\alpha(x-x')}\frac{\mathrm{d}}{\mathrm{d}x'} f(x,x') \ , \end{aligned} \tag{3.34}$$

where we have made a trivial integration by parts. The first term is simply $\frac{1}{mN^\alpha} f(x,x)$ plus a term which is *exponentially* suppressed in the large N limit. One has to use this identity twice on the first term of (3.27), proportional to $\xi_{ij} = N^\alpha x_{ij}$, and once for each of the remaining terms, to derive the leading order result

$$\begin{aligned} F_2(\Phi_{I\leftrightarrow J}) &= \frac{4N^{2-\alpha}}{\pi}\int_{x_1}^{x_2} (\rho(x))^2 \mathrm{d}x \sum_{m=1}^{\infty}\left[\frac{1}{m^3}\sin 2\pi m(1-\Delta)\cos m\left[y^I(x)-y^J(x)\right]\right. \\ &\quad -\frac{\pi}{m^2}(1-\Delta)\cos 2\pi m(1-\Delta)\cos m\left[y^I(x)-y^J(x)\right] \\ &\quad \left.+\frac{1}{2m^2}\left[y^I(x)-y^J(x)\right]\sin 2\pi m(1-\Delta)\sin m\left[y^I(x)-y^J(x)\right]\right] \ . \end{aligned} \tag{3.35}$$

The crucial point here is that the first term in the first line of (3.27) is naively of order N^2. However, the identity (3.34) implies that this leading contribution is itself identically zero. One should then worry about subleading corrections to this term coming from approximating the sum over N eigenvalues with an integral. However, one can use the general estimate

$$\left|\int_0^1 f(x(s))\mathrm{d}s - \frac{1}{N}\sum_{i=1}^{N} f(x_i)\right| \leq \frac{\sup_{s\in[0,1]}|f''(x(s))|}{24N^2} \ , \tag{3.36}$$

with x_i related to $x(s)$ as in (3.11), to show this correction is subleading to (3.35).

It is straightforward to similarly compute the leading order contribution to the partition function for an *adjoint* field of conformal dimension Δ:

$$\begin{aligned} F_2(\Phi_{\mathrm{Ad}}) &= \frac{2N^{2-\alpha}}{\pi}\int_{x_1}^{x_2} (\rho(x))^2 \mathrm{d}x \sum_{m=1}^{\infty}\left[\frac{1}{m^3}\sin 2\pi m(1-\Delta)\right. \\ &\quad \left. -\frac{\pi}{m^2}(1-\Delta)\cos 2\pi m(1-\Delta)\right] \ . \end{aligned} \tag{3.37}$$

The leading order contribution for the gauge sector one-loop term in (2.8) is

$$\begin{aligned} F_2(\text{gauge}) &= 2GN^{2-\alpha}\int_{x_1}^{x_2}(\rho(x))^2\mathrm{d}x\sum_{m=1}^{\infty}\frac{1}{m^2} \\ &= \frac{\pi^2 GN^{2-\alpha}}{3}\int_{x_1}^{x_2}(\rho(x))^2\mathrm{d}x\ . \end{aligned} \tag{3.38}$$

Here we have used the expansion (A.8); recall that the linear term in (A.8) has already been cancelled (see the paragraph after equation (3.26)). Notice that each $U(N)$ gauge group factor makes the same contribution, hence the overall factor of G in (3.38).

The sums over m in (3.35) and (3.37) may be evaluated in closed form using simple Fourier series results. We shall do this for our examples in the following sections.

4 The $U(N)^2$ $\mathcal{A}_{n-1}$ theories

In this section we study the M-theory limit of the partition function for a particular class of $\mathcal{N}=2$ non-chiral Chern-Simons-matter theories.

4.1 The quiver theories

In [24] a particular family of Chern-Simons-matter theories was studied in considerable detail. For these theories the gauge group is $\mathcal{G}=U(N)_k\times U(N)_{-k}$, so that the number of gauge group factors is $G=2$, and we have denoted the Chern-Simons levels as $k_1=k=-k_2$. The matter content consists of bifundamental fields A_α, B_α, $\alpha=1,2$, transforming in the $\mathbf{N}\otimes\overline{\mathbf{N}}$ and $\overline{\mathbf{N}}\otimes\mathbf{N}$ representations of the two gauge group factors, respectively, together with adjoint fields Ψ_1, Ψ_2 for each. The superpotential is

$$W = \mathrm{Tr}\left[s\left((-1)^n\Psi_2^{n+1}+\Psi_1^{n+1}\right)+\Psi_1(A_1B_1+A_2B_2)+\Psi_2(B_1A_1+B_2A_2)\right]\ , \tag{4.1}$$

where s is a coupling constant and $n\in\mathbb{N}$ is a positive integer. The superpotential is invariant under an $SU(2)$ flavour symmetry under which the adjoints Ψ_I are singlets and both pairs of bifundamentals A_α, B_α transform as doublets. There is also a $\mathbb{Z}_2^{\text{flip}}$ symmetry which exchanges $\Psi_1\leftrightarrow\Psi_2$, $A_\alpha\leftrightarrow B_\alpha$, $s\leftrightarrow(-1)^n s$. The quiver diagram is shown in Figure 1.

The case $n=1$ is special, since then the first two terms in (4.1) give a mass to the adjoint fields Ψ_1, Ψ_2. At energy scales below this mass, we may therefore integrate out these fields. On setting $s=k/8\pi$, one recovers the ABJM theory [3] with quartic

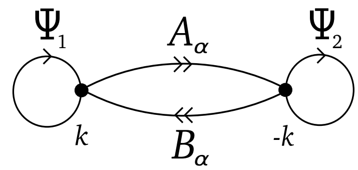

Figure 1: The $\mathcal{A}_{n-1}$ Chern-Simons quiver.

superpotential

$$W_{\mathrm{ABJM}} \;=\; \frac{4\pi}{k}(A_1B_2A_2B_1 - A_1B_1A_2B_2)\ . \tag{4.2}$$

This theory is superconformal with enhanced $\mathcal{N}=6$ supersymmetry. We may thus regard this family of theories, which we refer to as the $\mathcal{A}_{n-1}$ theories, as a generalization of the ABJM theory.

The Abelian ($N=1$) moduli space of vacua of this theory, for $k=1$, is the hypersurface singularity [24]

$$X_n \;=\; \left\{z_0^n + z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\right\} \subset \mathbb{C}^5\ . \tag{4.3}$$

For $n=1$ one easily sees that $X_1 \cong \mathbb{C}^4$, as one expects since the Abelian ABJM theory with $k=1$ describes a single M2 brane in flat spacetime. For $n>1$ there is an isolated singularity of X_n at the origin in $\mathbb{C}^5$. For $k>1$ one obtains instead the quotient $X_n/\mathbb{Z}_k$, where the $\mathbb{Z}_k$ action is free away from the origin. Then $X_n/\mathbb{Z}_k$ is a cone over a smooth compact Sasakian seven-manifold $Y_n/\mathbb{Z}_k$. Recall that a Sasakian metric on an odd-dimensional manifold Y is equivalent to the cone metric (2.6) being Kähler.

The $\mathcal{A}_{n-1}$ theories are not classically conformally invariant, as one immediately sees from the non-quartic superpotential (4.1). However, one might be tempted to conjecture that the quantum theories flow to a strongly coupled interacting superconformal fixed point in the IR, at which (4.1) is marginal. At such a superconformal fixed point the theory develops an R-symmetry, where the superpotential necessarily has R-charge 2. Since a chiral superfield Φ_a saturates the BPS bound that $R[\Phi_a]=\Delta[\Phi_a]$, where $\Delta[\Phi_a]$ denotes the conformal dimension, it follows that at such a superconformal fixed point we must have

$$\begin{aligned}\Delta_{\mathrm{Bi}} \;&\equiv\; \Delta[A_\alpha] \;=\; \Delta[B_\alpha] \;=\; \frac{n}{n+1}\ ,\\ \Delta_{\mathrm{Ad}} \;&\equiv\; \Delta[\Psi_1] \;=\; \Delta[\Psi_2] \;=\; \frac{2}{n+1}\ .\end{aligned} \tag{4.4}$$

As explained in [24], this is problematic for $n > 2$, since $\mathrm{Tr}\, \Psi_I$ would then be gauge-invariant chiral primary operators with conformal dimensions $\Delta[\mathrm{Tr}\, \Psi_I] = \frac{2}{n+1}$, and this violates the *unitarity bound* of $\Delta \geq \frac{1}{2}$, with equality only for a *free* field, for $n > 2$. Thus only for $n = 2$ could this conjecture be true. In [24] we conjectured instead that the $\mathcal{A}_{n-1}$ theories all flow to the *same* conformal fixed point for $n > 2$, in which the coupling constant $s = 0$ and thus the Ψ_I^{n+1} terms are absent in the superpotential (4.1). In this case the vacuum moduli space of the theory is $\mathbb{C} \times \mathrm{Con}/\mathbb{Z}_k$, where $\mathrm{Con} = \{xy = uv\} \subset \mathbb{C}^4$ is the usual conifold three-fold singularity [24], and the link Y certainly admits a (singular) Sasaki-Einstein metric. We shall present further field theory evidence for this conjecture in section 4.5.

There is a gravitational dual to this [17]. In general the existence of Sasaki-Einstein metrics, for example on links of hypersurface singularities, is a difficult unsolved problem. In [17] it was pointed out that there are some simple holomorphic obstructions, which moreover have AdS/CFT dual interpretations. In particular, the unitarity bound obstruction above is dual to the *Lichnerowicz* obstruction to the existence of Sasaki-Einstein metrics described in [17]. One can then show that no Sasaki-Einstein metric exists on the link Y_n of X_n in (4.3), for $n > 2$. On the other hand, for $n = 2$ the quadric hypersurface certainly admits a Ricci-flat Kähler cone metric, where the Sasaki-Einstein metric on Y_2 is the homogeneous $V_{5,2} = SO(5)/SO(3)$ metric. In spite of this, certainly there exist *Sasakian*, but *non*-Einstein, metrics on Y_n, and the volumes of these manifolds are then independent of the choice of such a metric [16]. One easily calculates this volume using the techniques in the latter reference, to obtain

$$\mathrm{Vol}(Y_n) \;=\; \frac{(n+1)^4}{16n^3}\mathrm{Vol}(S^7) \;=\; \frac{(n+1)^4\pi^4}{48n^3}\ . \tag{4.5}$$

We stress again that for $n = 2$ this is the volume of the homogeneous Sasaki-Einstein manifold $V_{5,2}$, while for $n > 2$ it is the volume of *any* Sasakian metric on the link of X_n (with canonical choice of Reeb vector field), although there is no Sasaki-Einstein metric.

4.2 The partition function

The matrix model for the $\mathcal{A}_{n-1}$ theories is easily obtained from the general formula (2.7). We have gauge group $\mathcal{G} = U(N)_k \times U(N)_{-k}$, so that $G = 2$, and the partition function is

$$Z[\mathcal{A}_{n-1}] \;=\; \frac{1}{(N!)^2}\int \left(\prod_{i=1}^{N} \frac{\mathrm{d}\lambda_i}{2\pi}\frac{\mathrm{d}\tilde{\lambda}_i}{2\pi}\right) \exp\left[\frac{\mathrm{i}k}{4\pi}\sum_{i=1}^{N}\left(\lambda_i^2 - \tilde{\lambda}_i^2\right)\right] \exp\left[-F_{\mathrm{loop}}\right]\ , \tag{4.6}$$

where

$$\exp\left[-F_{\text{loop}}\right] = \prod_{i\neq j} 2\sinh\frac{\lambda_{ij}}{2}\cdot 2\sinh\frac{\tilde{\lambda}_{ij}}{2}\cdot\prod_{i,j}\exp\left[2\ell\left(1-\Delta_{\text{Bi}}+\mathrm{i}\frac{\hat{\lambda}_{ij}}{2\pi}\right)\right.$$
$$\left.+2\ell\left(1-\Delta_{\text{Bi}}-\mathrm{i}\frac{\hat{\lambda}_{ij}}{2\pi}\right)+\ell\left(1-\Delta_{\text{Ad}}+\mathrm{i}\frac{\lambda_{ij}}{2\pi}\right)+\ell\left(1-\Delta_{\text{Ad}}+\mathrm{i}\frac{\tilde{\lambda}_{ij}}{2\pi}\right)\right]. \quad (4.7)$$

Here we have denoted the two sets of eigenvalues as $\lambda_i^1 = \lambda_i$, $\lambda_i^2 = \tilde{\lambda}_i$, and have introduced the notations

$$\lambda_{ij} = \lambda_i - \lambda_j\ , \qquad \tilde{\lambda}_{ij} = \tilde{\lambda}_i - \tilde{\lambda}_j\ , \qquad \hat{\lambda}_{ij} = \lambda_i - \tilde{\lambda}_j\ . \quad (4.8)$$

We also note from (4.4) that, provided $s\neq 0$, the marginality of the superpotential (4.1) implies

$$1-\Delta_{\text{Bi}} = \frac{1}{n+1}\ , \qquad 1-\Delta_{\text{Ad}} = \frac{n-1}{n+1}\ . \quad (4.9)$$

For the time being we shall assume that $s\neq 0$, and thus that (4.9) hold, and return to the case that $s=0$ in section 4.5. The two factors of 2 before the ℓ functions in (4.7) arise from the sum over $\alpha = 1,2$ on the bifundamental fields A_α, B_α.

4.3 Symmetries and saddle point equations

For the $\mathcal{A}_{n-1}$ theories the equation of motion (3.6) may be written as

$$-\frac{\partial F}{\partial\lambda_i} = \frac{\mathrm{i}k}{2\pi}\lambda_i + \sum_{j\neq i}\coth\frac{\lambda_{ij}}{2} + \sum_{j=1}^{N}\frac{\frac{\hat{\lambda}_{ij}}{2\pi}\sin\frac{2\pi}{n+1} - \frac{2}{n+1}\sinh\frac{\hat{\lambda}_{ij}}{2}\cosh\frac{\hat{\lambda}_{ij}}{2}}{\sin^2\frac{\pi}{n+1}+\sinh^2\frac{\hat{\lambda}_{ij}}{2}}$$
$$+\frac{1}{2}\sum_{j=1}^{N}\frac{\frac{\lambda_{ij}}{2\pi}\sin\frac{2\pi(n-1)}{n+1} - \frac{2(n-1)}{n+1}\sinh\frac{\lambda_{ij}}{2}\cosh\frac{\lambda_{ij}}{2}}{\sin^2\frac{\pi(n-1)}{n+1}+\sinh^2\frac{\lambda_{ij}}{2}}\ . \quad (4.10)$$

The corresponding equation of motion for $\lambda_i^2 = \tilde{\lambda}_i$ is obtained via the replacements $\lambda_i \leftrightarrow \tilde{\lambda}_i$, $k\to -k$. This is a remnant of the $\mathbb{Z}_2^{\text{flip}}$ symmetry which exchanges $\Psi_1\leftrightarrow\Psi_2$ and $A_\alpha\leftrightarrow B_\alpha$. In fact we note the following symmetries:

1. The equations of motion for λ_i and $\tilde{\lambda}_i$ are interchanged via $\lambda_i\leftrightarrow\tilde{\lambda}_i$, $k\leftrightarrow -k$, which as mentioned is the $\mathbb{Z}_2^{\text{flip}}$ symmetry of the $\mathcal{A}_{n-1}$ Chern-Simons-matter theory.

2. The equations of motion for λ_i and $\tilde{\lambda}_i$ are interchanged via $\tilde{\lambda}_i \leftrightarrow \bar{\lambda}_i$. In fact this follows from the previous comment, together with the fact that the equation of motion (4.10) is real up to the classical term involving the Chern-Simons level k.

3. The equations of motion are invariant under $\lambda_i \to -\lambda_i$, $\tilde{\lambda}_i \to -\tilde{\lambda}_i$.

These are the same symmetries possessed by the ABJM theory with $n = 1$ [7].

The approximate equation of motion (3.10) then becomes

$$\begin{aligned} -\frac{\partial F}{\partial \lambda_i} &\simeq \left(1 - \frac{2}{n+1} - \frac{n-1}{n+1}\right) \sum_{j=1}^{N} \operatorname{sgn}(\xi_i - \xi_j) \ , \\ &= 0 \ , \end{aligned} \tag{4.11}$$

where we have assumed that $\Re\lambda_i = \Re\tilde{\lambda}_i = \xi_i$, as in section 3.2. In the case at hand, this is also related to a *symmetry*, namely $\mathbb{Z}_2^{\text{flip}}$. Since the action is invariant under this symmetry, it is reasonable to expect that the same is true of the saddle point solution. From the comments 1 and 2 above, this implies that $\tilde{\lambda}_i = \bar{\lambda}_i$, which in particular implies that $\Re\lambda_i = \Re\tilde{\lambda}_i$. The vanishing of the coefficient in (4.11) is of course precisely the single non-trivial NSVZ beta function relation, as expected from (3.26).

4.4 Evaluating the free energy

In this section we would like to evaluate the free energy in the M-theory large N limit. In order to do so, it is convenient to use the $\mathbb{Z}_2^{\text{flip}}$ symmetry of the theory and action. From the comments made at the end of the previous subsection, this implies that

$$\lambda_i = N^{\alpha} x_i + \mathrm{i} y_i \ , \qquad \tilde{\lambda}_i = N^{\alpha} x_i - \mathrm{i} y_i \ . \tag{4.12}$$

In the continuum limit, this becomes $y(x) \equiv y^1(x) = -y^2(x)$. We may then use the general results in (3.35), (3.37), (3.38) to write the total

$$\begin{aligned} F_2 &= \frac{4N^{2-\alpha}}{\pi} \int_{x_1}^{x_2} (\rho(x))^2 \mathrm{d}x \sum_{m=1}^{\infty} \left[\frac{2}{m^3} \sin\frac{2\pi m}{n+1} \cos 2my + \frac{1}{m^3} \sin\frac{2\pi m(n-1)}{n+1} \right. \\ &\quad + \frac{\pi}{m^2} + \frac{2y}{m^2} \sin\frac{2\pi m}{n+1} \sin 2my - \frac{\pi(n-1)}{m^2(n+1)} \cos\frac{2\pi m(n-1)}{n+1} \\ &\quad \left. - \frac{2\pi}{m^2(n+1)} \cos\frac{2\pi m}{n+1} \cos 2my \right] \ . \end{aligned} \tag{4.13}$$

The sums in this expression are simple Fourier expansions, and it is elementary to sum them explicitly. For example, using

$$\frac{1}{3}w^3 \;=\; 4\sum_{m=1}^{\infty}\frac{(-1)^m}{m^3}\sin mw + \frac{\pi^2}{3}w\ , \qquad -\pi < w < \pi\ , \tag{4.14}$$

one easily shows that the first term in (3.35) is

$$\begin{aligned}\sum_{m=1}^{\infty}\frac{2}{m^3}\sin\frac{2\pi m}{n+1}\cos 2my \;&=\; -\frac{1}{12}\left(\tfrac{\pi(n-1)}{n+1}+2y-2\pi\epsilon\right)\left[\left(\tfrac{(n-1)\pi}{n+1}+2y-2\pi\epsilon\right)^2-\pi^2\right]\\ &\quad -\frac{1}{12}\left(\tfrac{\pi(n-1)}{n+1}-2y\right)\left[\left(\tfrac{\pi(n-1)}{n+1}-2y\right)^2-\pi^2\right]\ .\end{aligned} \tag{4.15}$$

The range on y for the first term in (4.15) is

$$\begin{aligned} -\frac{\pi n}{n+1} <\ y\ &< \frac{\pi}{n+1}\ , \qquad \epsilon = 0\ ,\\ \frac{\pi}{n+1} <\ y\ &< \frac{\pi(n+2)}{n+1}\ , \qquad \epsilon = 1\ ,\end{aligned} \tag{4.16}$$

while for the second term it is

$$-\frac{\pi}{n+1} < y < \frac{\pi n}{n+1}\ . \tag{4.17}$$

Notice that the range (4.17) has non-empty overlap with *both* choices of range in (4.16).

Using similar arguments it is straightforward to derive

$$\begin{aligned} F_2 \;&=\; \frac{4N^{2-\alpha}}{\pi}\int_{x_1}^{x_2}(\rho(x))^2\mathrm{d}x\left\{\frac{2\pi}{(n+1)^3}\left[n^2\pi^2-(n+1)^2y^2\right]\right.\\ &\quad\left.+\frac{\pi^2\epsilon(\epsilon-1)}{3}\left[\frac{\pi}{n+1}\left[2\epsilon(n+1)+2-n\right]-3y\right]\right\}\ .\end{aligned} \tag{4.18}$$

Remarkably, for $\epsilon=0$ *or* $\epsilon=1$ this dramatically simplifies to the same expression

$$F_2 \;=\; \frac{4N^{2-\alpha}}{\pi}\int_{x_1}^{x_2}(\rho(x))^2\mathrm{d}x\frac{2\pi}{(n+1)^3}\left[n^2\pi^2-(n+1)^2y^2\right]\ . \tag{4.19}$$

We see that the classical term (3.32) scales as $N^{1+\alpha}$ to leading order, while the one-loop term F_2 in (4.19) scales as $N^{2-\alpha}$. In order to obtain non-trivial critical points in the large N limit we thus need $1+\alpha = 2-\alpha$, or $\alpha=\frac{1}{2}$. Altogether, the free energy is then, to leading order in the M-theory large N limit, given by

$$F \;=\; N^{3/2}\left[\frac{k}{\pi}\int \mathrm{d}x x\rho(x)y(x) + \int\mathrm{d}x(\rho(x))^2h[y(x)] - \frac{\mu}{2\pi}\left(\int\mathrm{d}x\rho(x)-1\right)\right]\ . \tag{4.20}$$

Here we have introduced a Lagrange multiplier μ for the density, and defined

$$h[y] \equiv \frac{8}{(n+1)^3}\left[n^2\pi^2 - (n+1)^2 y^2\right] . \tag{4.21}$$

The Euler-Lagrange equations for a critical point are simply

$$4\pi\rho(x)h[y(x)] = \mu - 2kxy(x) , \tag{4.22}$$
$$\pi\rho(x)h'[y(x)] = -kx . \tag{4.23}$$

Computing $h' = -16y/(n+1)$ one easily obtains the solution

$$\rho(x) = \frac{(n+1)^3\mu}{32n^2\pi^3} , \qquad y(x) = \frac{2kn^2\pi^2 x}{(n+1)^2\mu} . \tag{4.24}$$

Noting that the action is invariant under $x \leftrightarrow -x$, it follows that $x_1 = -x_2$ and the constraint equation is

$$\int_{-x_*}^{x_*} \rho(x)\mathrm{d}x = 1 \quad \Rightarrow \quad \mu = \frac{16n^2\pi^3}{(n+1)^3 x_*} . \tag{4.25}$$

In turn this gives

$$y = \frac{k(n+1)x_*}{8\pi} x . \tag{4.26}$$

A sketch of these functions is shown in Figure 2.

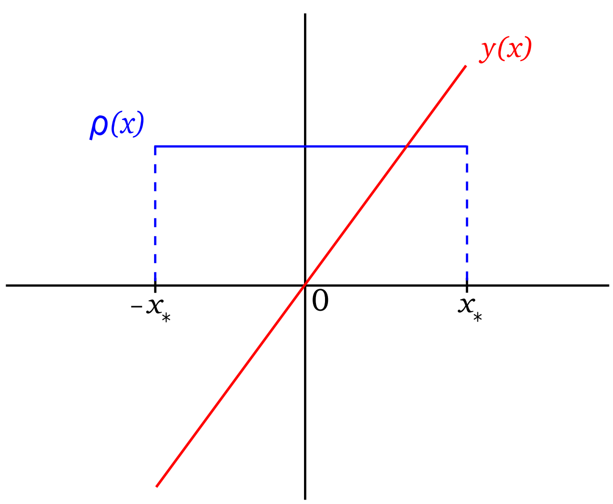

Figure 2: Sketch of $\rho(x)$, which is zero for $|x| > x_*$ and takes the constant value $\rho = \frac{1}{2x_*}$ for $|x| < x_*$. The function $y(x)$, shown in red, is linear in this latter region.

It is now a simple matter to substitute this back into the action and integrate, to obtain

$$F = N^{3/2}\left[\frac{4n^2\pi^2}{(n+1)^3 x_*} + \frac{k^2(n+1)}{48\pi^2}x_*^3\right] . \tag{4.27}$$

Finally, we must extremize over the endpoints of the eigenvalue distribution, so that $\mathrm{d}F/\mathrm{d}x_* = 0$ implies

$$x_* = \frac{2\pi}{n+1}\sqrt{\frac{2n}{k}} , \qquad y(x_*) = \frac{n\pi}{n+1} . \tag{4.28}$$

Looking back to (4.16), (4.17), we see that the full solution for $y(x)$ lies in the range of validity of the Fourier expansions we have made, provided we take $\epsilon = 1$ for the $y > \pi/(n+1)$ region in Figure 2, and $\epsilon = 0$ for the $y < \pi/(n+1)$ region. The leading saddle point free energy is hence

$$F = N^{3/2}k^{1/2}\frac{8\pi n^{3/2}}{3\sqrt{2}(n+1)^2} . \tag{4.29}$$

For $n = 1$ this is the ABJM result of [34, 5, 7]. We then see that

$$\frac{F(n=1)}{F(n)} = \sqrt{\frac{(n+1)^4}{16n^3}} , \tag{4.30}$$

which is exactly the expected square root of the volume given by (2.15), (4.5).

4.5 The superconformal theory for $n > 2$

As discussed in section 4.1, for $n > 2$ in fact the superpotential (4.1) cannot be marginal with $s \neq 0$, due to the Lichnerowicz/unitarity bound. In [24] we therefore conjectured that for $n > 2$ the coupling $s\Psi_I^{n+1}$ is *irrelevant* in the IR, and thus one should set $s = 0$ at the conformal fixed point. This then alters the above computation. The constraint $\Delta[W] = 2$ imposes

$$\Delta_{\mathrm{Ad}} = 2(1-\Delta) , \tag{4.31}$$

where we have set $\Delta \equiv \Delta_{\mathrm{Bi}}$. Equation (4.31) is also equivalent to (3.26), which is the leading order saddle point equation. It is then straightforward to redo the computation of the previous section, with the *weaker* condition (4.31). One finds the free energy is still given by (4.20), but where the function $h[y]$ in (4.21) is

$$h[y] = 8(1-\Delta)(\pi^2\Delta^2 - y^2) . \tag{4.32}$$

Of course, setting $\Delta = \frac{n}{n+1}$ reproduces the function (4.21). The rest of the computation proceeds in much the same way, and one finds the free energy

$$F(\Delta) \;=\; N^{3/2}k^{1/2}\frac{4\pi\sqrt{2\Delta^3(1-\Delta)}}{3}\;. \tag{4.33}$$

As in [20] this is a function of Δ, which one should regard as a trial R-charge. Following the latter reference, extremizing F with respect to Δ gives for the conformal field theory

$$\Delta \;=\; \Delta_{\mathrm{Bi}} \;=\; \frac{3}{4}\;, \qquad \Delta_{\mathrm{Ad}} \;=\; \frac{1}{2}\;. \tag{4.34}$$

Notice that this is formally equal to the previous result for $n = 3$.

We may now compare with the dual gravity analysis. As shown in [24], the vacuum moduli space of the theory with $s = 0$ is $\mathbb{C} \times \mathrm{Con}/\mathbb{Z}_k$. This certainly admits a Ricci-flat Kähler cone metric, namely the product of the flat metric on $\mathbb{C}$ times the conifold metric. More generally, one can compute the volume of a (singular) Sasakian metric on the link Y using the results of [17]. Realizing the conifold as the quadric hypersurface $\mathrm{Con} = \{z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\} \subset \mathbb{C}^4$, we may compute the character of the action of $(\mathbb{C}^*)^2$ on $\mathbb{C}\times\mathrm{Con}$, where the first copy of $\mathbb{C}^*$ acts with weight one on $\mathbb{C}$ and the second $\mathbb{C}^*$ acts with weight one on each z_i, $i = 1, \ldots, 4$. In the notation of [17], this gives

$$C(q_0, q, \mathbb{C}\times\mathrm{Con}) \;=\; \frac{1-q^2}{(1-q_0)(1-q)^4}\;, \tag{4.35}$$

where (q_0, q) are coordinates on $(\mathbb{C}^*)^2$. The volume, as a function of the Reeb vector field, is then given by setting $q_0 = \mathrm{e}^{-t\xi_0}$, $q = \mathrm{e}^{-t\xi}$ and taking the limit of t^{-4} times (4.35) as $t \to 0$. This gives

$$\mathrm{Vol}(\xi_0, \xi) \;=\; \frac{2}{\xi_0\xi^3}\mathrm{Vol}(S^7)\;, \tag{4.36}$$

where $\mathrm{Vol}(S^7)$ denotes the volume of the round metric on S^7. One can easily check that the holomorphic $(4,0)$-form on $\mathbb{C}\times\mathrm{Con}$ has charge 4 if and only if

$$\xi_0 \;=\; 4 - 2\xi\;, \tag{4.37}$$

which is precisely the geometric analogue of (4.31). Indeed, in the field theory $\Delta_{\mathrm{Ad}} = \xi_0/2$, $\Delta = \xi/2$, which follows straightforwardly from the field theory description of the vacuum moduli space. We thus obtain the geometric formula

$$\mathrm{Vol}(\Delta) \;=\; \frac{1}{16\Delta^3(1-\Delta)}\mathrm{Vol}(S^7)\;. \tag{4.38}$$

Remarkably, this precisely agrees with (4.33) and our conjecture (1.1); that is, the free energy is related to the Sasakian volume of Y, even before extremizing with respect to the trial R-charges.

This result adds further support to the conjecture made in [24] for the $n > 2$ theories. Notice that at this conformal fixed point $\Delta_{\rm Ad} = \frac{1}{2}$. Naively this contradicts the Lichnerowicz obstruction of [17], since it is stated there that $\Delta_{\rm Ad} = \frac{1}{2}$ can hold only for the round metric on S^7. However, the Lichnerowicz theorem of [17] is here circumvented precisely because the link Y is *singular*. In fact certainly there is a Sasaki-Einstein metric on Y, which is known explicitly; but it has an S^1 locus of conical singularities.

5 The $U(N)^3$ SPP theory

In this section we study a different non-chiral Chern-Simons-matter theory, this time with $U(N)^3$ gauge group. We compute the partition function, as a function of the trial R-charges [20], explicitly in the large N limit, and verify that it satisfies our general Sasakian volume conjecture (1.1). Note that although the Sasaki-Einstein metric is not known in explicit form in this case, its existence is guaranteed by the results of [38].

5.1 The quiver theory

The theory of interest is a $\mathcal{G} = U(N)_{2k} \times U(N)_{-k} \times U(N)_{-k}$ Chern-Simons-quiver theory, where the vector of Chern-Simons levels is $(2k, -k, -k)$. Thus $G = 3$. The matter content consists of the following bifundamental fields:

$$\begin{aligned} A_1 &: \mathbf{N} \otimes \overline{\mathbf{N}} \otimes 1 \, , \qquad & A_2 &: \overline{\mathbf{N}} \otimes \mathbf{N} \otimes 1 \, , \\ B_1 &: 1 \otimes \mathbf{N} \otimes \overline{\mathbf{N}} \, , \qquad & B_2 &: 1 \otimes \overline{\mathbf{N}} \otimes \mathbf{N} \, , \\ C_1 &: \mathbf{N} \otimes 1 \otimes \overline{\mathbf{N}} \, , \qquad & C_2 &: \overline{\mathbf{N}} \otimes 1 \otimes \mathbf{N} \, . \end{aligned} \tag{5.1}$$

We also include an adjoint scalar field Ψ for the first gauge group factor. The superpotential is

$$W \;=\; \mathrm{Tr}\left[\Psi\left(A_1A_2 - C_1C_2\right) - A_2A_1B_1B_2 + C_2C_1B_2B_1\right] \, . \tag{5.2}$$

The quiver diagram is shown in Figure 3.

As a four-dimensional $\mathcal{N} = 1$ quiver gauge theory, this describes the low energy dynamics of N D3-branes at the suspended pinch point (SPP) singularity [39]. The latter is the (non-isolated) three-fold hypersurface singularity given by $\{x^2y = uv\} \subset$

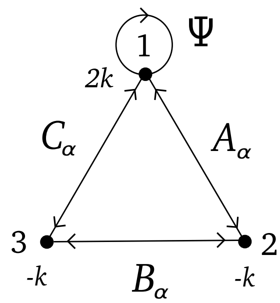

Figure 3: The SPP Chern-Simons quiver.

$\mathbb{C}^4$. From the general results of [22], [30], the corresponding three-dimensional Chern-Simons-quiver theory is expected to describe N M2 branes probing a related toric Calabi-Yau four-fold singularity. This Abelian moduli space was studied in both [30] and [25].

Assuming the theory has a superconformal fixed point at which (5.2) is marginal, it follows from $\Delta[W] = 2$ that

$$2 \;=\; 2\Delta[A] + 2\Delta[B]\;, \qquad 2 \;=\; 2\Delta[C] + 2\Delta[B]\;, \qquad 2 \;=\; 2\Delta[A] + \Delta[\Psi]\;. \tag{5.3}$$

Here we have taken, by symmetry, $\Delta[A_1] = \Delta[A_2] = \Delta[A]$, *etc.* This leads to

$$\Delta[B] \;=\; \Delta\;, \qquad \Delta[A] \;=\; \Delta[C] \;=\; (1-\Delta)\;, \qquad \Delta[\Psi] \;=\; 2\Delta\;. \tag{5.4}$$

In fact in this example the superpotential constraints (5.3) are precisely equivalent to the vanishing of the four-dimensional NSVZ beta functions (3.26). Recall that the latter are equivalent to the leading order saddle point equations for the partition function.

5.2 The Sasakian volume function

Until now it has not been possible to determine Δ at the superconformal fixed point using a purely field theory computation. However, one can determine Δ using the AdS/CFT correspondence together with the volume minimization of [15]. In the latter reference it is shown how to uniquely determine the volume of a Sasaki-Einstein metric on the link of a toric Calabi-Yau singularity by minimizing a certain rational function. The latter is the volume of a general Sasakian metric as a function of the Reeb vector field ξ, which is holographically dual to the R-symmetry. This volume function is easily computed using the toric data of the Calabi-Yau singularity, and referring[6] to equation

[6] In [25] the authors have normalized the Reeb vector field so that the holomorphic $(4,0)$-form on the cone over Y has charge 2, so that the superpotential then also has charge 2 (see the discussion

(5.39) of [25] we see that in the present example (with $k=1$)

$$\mathrm{Vol}(Y)[\Delta] \;=\; \frac{4-3\Delta}{32\Delta(1-\Delta)^2(2-\Delta)^2}\mathrm{Vol}(S^7)\ , \tag{5.5}$$

where we have simplified somewhat the expression given in [25]. Here Δ is geometrically parametrizing the choice of Reeb vector field in the Sasakian metric on Y. However, this may be related to the field theoretic Δ using the correspondence between bifundamental fields in the Chern-Simons-quiver theory and M5 branes wrapped on (links of) certain toric divisors. More precisely, the AdS/CFT correspondence gives

$$\Delta[\Phi] \;=\; \frac{N\pi\mathrm{Vol}(\Sigma_\Phi)}{6\mathrm{Vol}(Y)}\ , \tag{5.6}$$

where geometrically a bifundamental field Φ defines a line bundle over the moduli space, which is equivalent to a toric divisor $C(\Sigma_\Phi)$ with $\Sigma_\Phi \subset Y$ a codimension two subspace of Y. Again, the volume of Σ_Φ may be computed using [15], and this identifies Δ in (5.5) with the dimension $\Delta = \Delta[B]$.

We note that in this case minimizing (5.5) gives [25] the conformal dimension[7]

$$\Delta \;=\; \frac{1}{18}\left[19 - \frac{37}{\left(431-18\sqrt{417}\right)^{1/3}} - \left(431-18\sqrt{417}\right)^{1/3}\right] \simeq 0.319\ . \tag{5.7}$$

5.3 Evaluating the free energy

Our goal in this section is to reproduce the geometric formula (5.5), and hence following [20] also (5.7), from a purely field theoretic computation. We simply apply the general results described in section 3.

Recall that with $G=3$ gauge group factors we will have three sets of eigenvalues λ_i^1, λ_i^2, λ_i^3, where $i=1,\ldots,N$. Our choice of Chern-Simons levels in this example was determined by requiring the $\mathbb{Z}_2$ symmetry of the matter content and superpotential to extend to the whole Chern-Simons-matter theory. This $\mathbb{Z}_2$ acts by exchanging A and C fields and correspondingly the second and third $U(N)$ factors. Assuming that the saddle point solution is invariant under this $\mathbb{Z}_2$ symmetry of the theory, we may thus

after their equation (4.3)). In fact in order to obtain the correctly normalized volume this should be charge 4 [15], [16]. Since the Sasakian volume function in this dimension is homogeneous degree -4, one should thus rescale all volumes in [25] by $2^{-4}=\frac{1}{16}$ in order to obtain the correct normalization. This factor will be crucial later when we compare to the large N partition function for this model.

[7] Note that although $\Delta < 1/2$, this does not violate the unitarity bound because there is no gauge invariant operator with this dimension.

take

$$y_i^1 \equiv y_i \ , \qquad y_i^2 = y_i^3 \equiv w_i \ , \tag{5.8}$$

where recall that the eigenvalues are $\lambda_i^I = N^\alpha x_i + \mathrm{i} y_i^I$. It it then straightforward to apply the general results (3.32), (3.35), (3.37), (3.38) to obtain the leading order partition function at large N. In particular, the one-loop contribution is

$$\begin{aligned} F_2 \ = \ & \frac{2N^{2-\alpha}}{\pi}\int_{x_1}^{x_2}(\rho(x))^2 \mathrm{d}x \sum_{m=1}^{\infty}\left[\frac{2}{m^3}\sin 2\pi m(1-\Delta) - \frac{2\pi}{m^2}(1-\Delta)\cos 2\pi m(1-\Delta)\right. \\ & + \frac{1}{m^3}\sin 2\pi m(1-2\Delta) - \frac{\pi}{m^2}(1-2\Delta)\cos 2\pi m(1-2\Delta) + \frac{3\pi}{m^2} \\ & \left. + \frac{4}{m^3}\sin 2\pi m\Delta\cos mu - \frac{4\pi}{m^2}\Delta\cos 2\pi m\Delta\cos mu + \frac{2}{m^2}u\sin 2\pi m\Delta\sin mu\right] \ . \end{aligned} \tag{5.9}$$

Here the first line comes from the B fields, the second line from the adjoint Ψ and the gauge sector (the last term), while the last line comes from the A and C fields. We have also introduced the quantity

$$u \ \equiv \ y - w \ . \tag{5.10}$$

One can sum the Fourier series as before. Again, the classical and one-loop contributions are at the same order, thus leading to non-trivial solutions, only for $\alpha = \frac{1}{2}$. This leads to

$$F \ = \ N^{3/2}\left[\frac{k}{\pi}\int \mathrm{d}x\, x\rho(x)u(x) + \int \mathrm{d}x\,(\rho(x))^2 h[u(x)] - \frac{\mu}{2\pi}\left(\int \mathrm{d}x\rho(x) - 1\right)\right] , \tag{5.11}$$

where

$$h[u] \ = \ 2\Delta\left[2\pi^2(\Delta-1)(\Delta-2) - u^2\right] \ . \tag{5.12}$$

Importantly, as we discuss in Appendix B, this is valid only for u in the range

$$-2\pi(1-\Delta) \ < \ u \ < \ 2\pi(1-\Delta) \ . \tag{5.13}$$

Solving the Euler-Lagrange equations following from (5.11) we find the solution

$$\rho \ = \ \frac{\mu}{16\pi^3\Delta(1-\Delta)(2-\Delta)} \ , \qquad u(x) \ = \ \frac{4\pi^2 k(1-\Delta)(2-\Delta)x}{\mu} \ , \tag{5.14}$$

where without loss of generality let us take $k > 0$. Notice that $\rho > 0$ implies $\mu > 0$. Now, if we assume that this solution is valid in an interval $[-x_*, x_*] \subset [-x_\Delta, x_\Delta]$, where

$$u(x_\Delta) \equiv u_\Delta = 2\pi(1-\Delta) , \tag{5.15}$$

we get a contradiction, since it then turns out that $u(x_*) > u_\Delta$. Hence this solution can be valid only in the interval $[-x_\Delta, x_\Delta]$, while in $[-x_*, -x_\Delta] \cup [x_\Delta, x_*]$ we necessarily have a different solution. Following [7], in fact $u(x)$ is frozen to the constant boundary value $u = u_\Delta$ in the interval $[x_\Delta, x_*]$, and correspondingly frozen to the other boundary value $u = -u_\Delta$ in the interval $[-x_*, -x_\Delta]$. The resulting continuous, piecewise-linear function is shown in red in Figure 4. Thus F is not extremized with respect to $u(x)$ in the range $[-x_*, -x_\Delta] \cup [x_\Delta, x_*]$, but only with respect to $\rho(x)$ at fixed $u = \pm u_\Delta$. One finds the Euler-Lagrange equation solution

$$\rho(x) = \frac{\mu - 4\pi k(1-\Delta)|x|}{16\pi^3\Delta^2(1-\Delta)} \qquad \text{for} \qquad x \in [-x_*, -x_\Delta] \cup [x_\Delta, x_*] . \tag{5.16}$$

The value of x_Δ is easily determined from (5.15), while x_* can be fixed by demanding that $\rho(x_*) = 0$, giving

$$x_\Delta = \frac{\mu}{2\pi k(2-\Delta)} > 0 , \qquad x_* = \frac{\mu}{4\pi k(1-\Delta)} > 0 , \tag{5.17}$$

respectively.

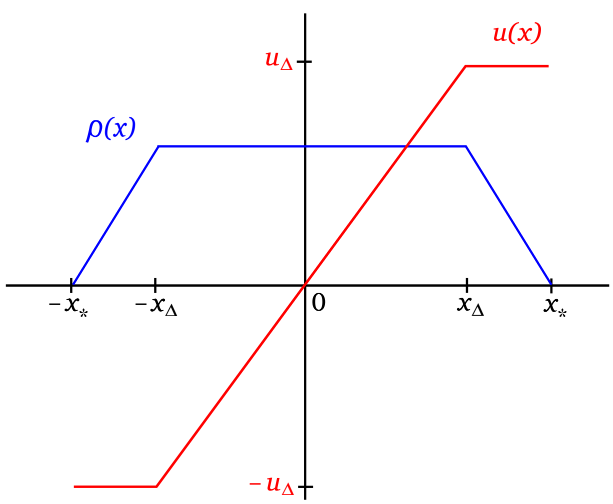

Figure 4: Sketch of the piecewise-linear functions $u(x)$, $\rho(x)$.

We can then compute the on-shell F as a function of μ:

$$F = \frac{N^{3/2}\mu^3(4-3\Delta)}{192k\pi^5(2-\Delta)^2(1-\Delta)^2\Delta} . \tag{5.18}$$

Next, μ may be determined by imposing the normalization of $\rho(x)$, giving

$$\mu^2 \;=\; \frac{64k\pi^4\Delta(1-\Delta)^2(2-\Delta)^2}{(4-3\Delta)} \;. \tag{5.19}$$

Finally, substituting this back into F we obtain

$$F \;=\; N^{3/2}k^{1/2}\frac{8\pi}{3}(2-\Delta)(1-\Delta)\sqrt{\frac{\Delta}{4-3\Delta}}\;, \tag{5.20}$$

which, remarkably, agrees with the conjectured relation (1.1) and the Sasakian volume (5.5).

6 Discussion

In this paper we have reproduced for the first time the expected $N^{3/2}$ scaling of the number of degrees of freedom in $\mathcal{N}=2$ superconformal field theories arising on a large number N of M2 branes, where the conformal dimensions of matter fields are *different* from the canonical value $\Delta=\frac{1}{2}$. We have also reproduced the volumes of certain Sasaki-Einstein seven-manifolds, thereby providing non-trivial tests of some conjectured $\mathrm{AdS}_4/\mathrm{CFT}_3$ dualities [25, 24]. More generally, we conjectured a relation between the large N limit of the partition function, interpreted as a function of trial R-charges, and the volumes of Sasakian metrics on links of Calabi-Yau four-fold singularities.

The method [7] that we have developed here applies to the general class of non-chiral quiver theories. These share some essential properties with $\mathcal{N}=3$ theories, albeit with superpotentials that are more arbitrary and not restricted to quartic interactions. Consequently, the scaling dimensions of the matter fields at the superconformal point are not restricted to be $\frac{1}{2}$. However, we believe that this is merely a technical difficulty and that an extension of the methods of this paper will allow one to tackle general $\mathcal{N}=2$ Chern-Simons-matter quivers. In particular, it will be important to justify more carefully some of the assumptions we and [7] have made about the scaling of eigenvalues in the large N limit. Although this ansatz reproduces the correct gravity results in the cases studied so far, and is supported by numerical results [7], clearly a more detailed understanding is desirable. This ansatz might need modification in more general classes of Chern-Simons-matter theories.

It would be interesting to distill a simple general procedure for determining the R-charges of an arbitrary $\mathcal{N}=2$ Chern-Simons-quiver theory. This will presumably involve extracting an expression for the large N free energy, as a function of the trial

R-charges. Recall that in a-maximization for four-dimensional theories one is given a simple function (a cubic polynomial) of the trial R-charges to begin with. Again, the results of [15, 16] imply, via the AdS/CFT correspondence, that the square of the large N free energy must be a polynomial function of the trial R-charges, at least in models with candidate gravity duals. It would be very interesting to prove our conjecture, at least in sub-classes such as toric theories, using strategies analogous to those used in [18, 19]. Finally, it would be interesting to apply the matrix model techniques of [5, 8] to the theories studied in this paper.

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A Expansions

Recall the definition

$$\ell(z) = -z \log\left(1 - \mathrm{e}^{2\pi \mathrm{i} z}\right) + \frac{\mathrm{i}}{2}\left[\pi z^2 + \frac{1}{\pi}\mathrm{Li}_2\left(\mathrm{e}^{2\pi i z}\right)\right] - \frac{\mathrm{i}\pi}{12} \, . \tag{A.1}$$

It is convenient to introduce the variable

$$\zeta = \mathrm{e}^{2\pi \mathrm{i} z} = \mathrm{e}^{-2\pi \Im z}\left(\cos 2\pi \Re z + \mathrm{i} \sin 2\pi \Re z\right) \, . \tag{A.2}$$

Then for $\Im z > 0$ we have the expansions

$$\begin{aligned} \log(1-\zeta) &= -\sum_{m=1}^{\infty} \frac{\zeta^m}{m} \, , \\ \mathrm{Li}_2(\zeta) &= \sum_{m=1}^{\infty} \frac{\zeta^m}{m^2} \, , \end{aligned} \tag{A.3}$$

so that

$$\ell(z) = \frac{\mathrm{i}\pi}{2}\left(z^2 - \tfrac{1}{6}\right) + \sum_{m=1}^{\infty}\left(\frac{z}{m} + \frac{\mathrm{i}}{2\pi m^2}\right)\mathrm{e}^{2\pi \mathrm{i} m z} \, . \tag{A.4}$$

On the other hand, for $\Im z < 0$ we have the expansions

$$\begin{aligned} \log(1-\zeta) &= \mathrm{i}\pi + \log\zeta - \sum_{m=1}^{\infty} \frac{1}{m\zeta^m} , \\ \mathrm{Li}_2(\zeta) &= \frac{\pi^2}{3} - \mathrm{i}\pi\log\zeta - \frac{1}{2}(\log\zeta)^2 - \sum_{m=1}^{\infty} \frac{1}{m^2\zeta^m} , \end{aligned} \tag{A.5}$$

so that

$$\ell(z) = -\frac{\mathrm{i}\pi}{2}\left(z^2 - \tfrac{1}{6}\right) + \sum_{m=1}^{\infty}\left(\frac{z}{m} - \frac{\mathrm{i}}{2\pi m^2}\right)\mathrm{e}^{-2\pi\mathrm{i}mz} . \tag{A.6}$$

In summary, we have the following two series expansions for the function $\ell(z)$:

$$\begin{aligned} \ell_+(z) &= \frac{\mathrm{i}\pi}{2}\left(z^2 - \tfrac{1}{6}\right) + \sum_{m=1}^{\infty}\left(\frac{z}{m} + \frac{\mathrm{i}}{2\pi m^2}\right)\mathrm{e}^{2\pi\mathrm{i}mz} , \qquad \text{for } \Im z > 0 , \\ \ell_-(z) &= -\frac{\mathrm{i}\pi}{2}\left(z^2 - \tfrac{1}{6}\right) + \sum_{m=1}^{\infty}\left(\frac{z}{m} - \frac{\mathrm{i}}{2\pi m^2}\right)\mathrm{e}^{-2\pi\mathrm{i}mz} , \quad \text{for } \Im z < 0 . \end{aligned} \tag{A.7}$$

We also note the following expansion. Writing $2\sinh\frac{w}{2} = \mathrm{e}^{w/2}(1-\mathrm{e}^{-w})$, we have

$$\log\left(2\sinh\tfrac{w}{2}\right) = \frac{w}{2} - \sum_{m=1}^{\infty}\frac{1}{m}\mathrm{e}^{-mw} , \qquad \text{for } \Re w > 0 . \tag{A.8}$$

B More on the range of u in section 5.3

By a reasoning similar to that in section 4.4, we find that the Fourier series in (5.10) may be resummed to the expression

$$\begin{aligned} h[u] = \frac{2}{3}\Big[&2\pi^2\Big(3\Delta^3 - 9\Delta^2 + 3\Delta(2 + p(1+p) + q(1+q)) - \\ &(1+p+q)(p+q+2p^2+2q^2-2pq)\Big) + 3\pi u(q(1+q) - p(1+p)) - 3\Delta u^2\Big] , \end{aligned} \tag{B.1}$$

where *a priori* p and q are arbitrary integers. This is valid only if both conditions

$$\begin{aligned} -\pi < \pi(1-2\Delta) + u + 2\pi p < \pi , \\ -\pi < \pi(1-2\Delta) - u + 2\pi q < \pi , \end{aligned} \tag{B.2}$$

hold simultaneously. Notice that when (p,q) take the values $(0,0)$, $(-1,0)$ and $(0,-1)$ the expression (B.1) simplifies to

$$h[u] = 2\Delta\left[2\pi^2(\Delta-1)(\Delta-2) - u^2\right] . \tag{B.3}$$

Rewriting (B.2) as

$$
\begin{aligned}
-2\pi(1-\Delta+p) &< u < 2\pi(\Delta-p) \ , \\
-2\pi(\Delta-q) &< u < 2\pi(1-\Delta+q) \ ,
\end{aligned}
\tag{B.4}
$$

the analysis can be split into two cases, and we have

$$
-2\pi(\Delta-q) < u < 2\pi(\Delta-p) \qquad \text{for} \ \ p+q+1>2\Delta \ , \tag{B.5}
$$

$$
-2\pi(1-\Delta+p) < u < 2\pi(1-\Delta+q) \qquad \text{for} \ \ p+q+1<2\Delta \ , \tag{B.6}
$$

where the two cases coincide when $2\Delta = p+q+1$. In order for the interval in (B.5) to be as large as possible, the integers p and q should be "as negative as possible". However, if for example $p=0, q=-1$ we have $\Delta<0$, hence they cannot be negative. The largest interval is thus obtained for $p=q=0$ and we have

$$
-2\pi\Delta < u < 2\pi\Delta \qquad \text{for} \ \ p=0,\ q=0 \ . \tag{B.7}
$$

Similarly, in order for the interval in (B.6) to be as large as possible, the integers p and q should be as large (and positive) as possible. At this point, let us take a short-cut and use the information that $\Delta \sim 0.32 < 1/2$ for the superconformal fixed point. Then notice that if $q=p=0$ we would have $\Delta>1/2$, so at least one of p, q must be negative and we have the two solutions

$$
2\pi\Delta < u < 2\pi(1-\Delta) \qquad \text{for} \ \ p=-1,\ q=0 \ , \tag{B.8}
$$

$$
-2\pi(1-\Delta) < u < -2\pi\Delta \qquad \text{for} \ \ p=0,\ q=-1 \ . \tag{B.9}
$$

Remarkably, in all cases the function $h[u]$ simplifies to the expression (B.3). Therefore, putting everything together, we conclude that the range of validity (at least when $\Delta<1/2$) of (B.3) is

$$
-2\pi(1-\Delta) < u < 2\pi(1-\Delta) \ , \tag{B.10}
$$

as claimed in section 5.3.

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