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ON ALMOST REVLEX IDEALS WITH HILBERT FUNCTION OF COMPLETE INTERSECTIONS

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Abstract. In this paper, we investigate the behavior of almost reverse lexicographic ideals with the Hilbert function of a complete intersection. More precisely, over a field K, we give a new constructive proof of the existence of the almost revlex ideal $J \subset K[x_1, \dots, x_n]$, with the same Hilbert function as a complete intersection defined by n forms of degrees $d_1 \leq \dots \leq d_n$. Properties of the reduction numbers for an almost revlex ideal have an important role in our inductive and constructive proof, which is different from the more general construction given by Pardue in 2010. We also detect several cases in which an almost revlex ideal having the same Hilbert function as a complete intersection corresponds to a singular point in a Hilbert scheme. This second result is the outcome of a more general study of lower bounds for the dimension of the tangent space to a Hilbert scheme at stable ideals, in terms of the number of minimal generators.

Introduction

In this paper, we investigate the behavior of almost reverse lexicographic ideals with the Hilbert function of a complete intersection. It is already known that if an almost revlex ideal with a prescribed Hilbert function exists, then it is unique [18, Remark 11].

Referring to [21], recall that a proper ideal I in a Noetherian ring is called a *complete intersection* if the length of the shortest system of minimal generators of I is equal to the height of I. A proper ideal I that is generated by a regular sequence in a Noetherian ring is a complete intersection and the converse holds if the ring is Cohen-Macaulay, like a polynomial ring over a field. Moreover every ideal in a Noetherian ring has a system of generators containing a complete intersection with the same dimension.

The existence of the almost reverse lexicographic ideal with a given Hilbert function is interesting in the study of general schemes with a given Hilbert function in Algebraic Geometry. We recall to the reader two famous conjectures involving almost revlex ideals.

Moreno-Socías' conjecture states that the generic initial ideal (with respect to degrevlex term order) of a polynomial ideal $I \subset R := K[x_1, \dots, x_n]$ that is generated by r generic forms over an infinite field K is the almost reverse lexicographic ideal J such that the Hilbert function of R/J is the same as that of R/I (see [25, Conjecture 4.1] and also [1, 25, 17, 8, 7, 9] for some partial solutions to this conjecture).

It is noteworthy that Moreno-Socías' conjecture implies Fröberg's conjecture, which states that if $d_1, \dots, d_r$ are the degrees of the r generic forms generating the above ideal $I \subseteq R$, then the Hilbert series of R/I is

$$h_{R/I} = \left| \frac{\prod_{i=1}^{r}(1-t^{d_i})}{(1-t)^n} \right|,$$

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where $|\sum_{\ell=0}^{\infty} a_\ell t^\ell|$ is the series whose ℓ-th coefficient is a_ℓ, if $a_i > 0$ for every $0 \leq i \leq \ell$, and 0 otherwise (see [16, 28, 8] and the references therein, and [26, 33] for some very recent contributions on the latter conjecture).

Artinian almost reverse lexicographic ideals in n variables also have a significant role in the study of the Lefschetz properties, because they have the maximal possible Betti numbers among all Artinian polynomial ideals that satisfy these properties with order n [11, 18].

Our investigation focuses on almost reverse lexicographic ideals (Definition 1.5) with the Hilbert function of a complete intersection and starts from the following more general observation. Let J be an Artinian strongly stable ideal in the polynomial ring R. It is straightforward that there exists an integer ℓ such that, for every $t \geq \ell$, every term of degree t outside J is divisible by the smallest variable. As a consequence, the Hilbert function of R/J is decreasing from ℓ on. The minimal integer ℓ with this property is strictly connected with the first reduction number of an Artinian K-algebra.

In case we deal with the Hilbert function H of a complete intersection defined by n forms of degrees $d_1 \leq \cdots \leq d_n$ in R, we describe an explicit construction of the almost reverse lexicographic ideal $J \subseteq R$ such that H is the Hilbert function of R/J (Theorem 4.1). The minimal integer ℓ with the above property has an important role in the proof of this result. More precisely, properties of reduction numbers of almost reverse lexicographic ideals are crucial in the inductive and constructive proof we provide, together with the combinatorial properties of the first expansion of the sous-escalier of a stable ideal (see [24]) and the particular structure of the Hilbert function of a complete intersection (see [31, 28]). Partial results of our construction have been presented in [4].

In [28, Theorems 4 and 5, Corollary 6], K. Pardue gave a complete characterization of the Hilbert functions that admit almost reverse lexicographic ideals, and among them there are the Hilbert functions of complete intersections. Our proof follows a different path from that used by Pardue thus still providing a new insight into the case of complete intersections.

From our study of the reduction numbers for an almost revlex ideal J, a closed formula for the number of minimal generators of J arises (see Theorem 3.10). Using this formula, and a lower bound on the dimension of the Zariski tangent space to a Hilbert scheme at a point corresponding to a stable ideal (Corollary 5.4), we exhibit several cases in which the point corresponding to an Artinian almost reverse lexicographic ideal with the Hilbert function of a complete intersection is singular in the Hilbert scheme (see Section 6). The main tools for this result are taken from the more general study of marked schemes over quasi-stable ideals [5], similarly to [10, Section 6] for reverse lexicographic ideals with the Hilbert function of general points.

1. Background

Let $R := K[x_1, \dots, x_n]$ be the polynomial ring over a field K in n variables. For every term $\tau := x_1^{\alpha_1} \dots x_n^{\alpha_n} \neq 1$ we let $\deg(\tau) := \sum_{i=1}^{n} \alpha_i$ be its degree.

We denote by $\succ$ the *degree reverse lexicographic* (degrevlex, for short) term order, with $x_1 \succ \cdots \succ x_n$. Recall that, for every couple of terms $\tau = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\sigma = x_1^{\beta_1} \cdots x_n^{\beta_n}$, τ is greater than σ ($\tau \succ \sigma$) with respect to degrevlex if either $\deg(\tau) > \deg(\sigma)$, or $\deg(\tau) = \deg(\sigma)$ and there is an integer $j \in \{1, \dots, n\}$ such that $\alpha_i = \beta_i$ for every $j < i \leq n$ and $\alpha_j < \beta_j$.

In this setting, we define $\min(\tau) := \min\{x_i \mid \alpha_i \neq 0\}$, which is the minimal variable that appears in τ with a non-null exponent. If $\min(\tau) = x_i$, we say that "τ has minimal variable x_i". Similarly, we define $\max(\tau) := \max\{x_i \mid \alpha_i \neq 0\}$. Let $\mathbb{T}$ be the set of the terms of R.

For every integer t, we denote by R_t the K-vector space of homogeneous polynomials of R of degree t and, for every subset $\Gamma \subset R$, we let $\Gamma_t := \Gamma \cap R_t$.

If $I \subset R$ is a homogeneous ideal of R, we denote by $I_{\leq t}$ the ideal generated in R by the homogeneous polynomials of I of degree $\leq t$. Moreover, we denote by $\mathrm{reg}(I)$ the *regularity of* I, that is the minimal integer m such that the h-th syzygy module of I is generated in degrees $\leq m + h$, for every $h \geq 0$.

We refer to [32, 34] for definitions and results about Hilbert functions of standard graded K-algebras. When the K-algebra is R/I for a homogeneous ideal $I \subseteq R$, we denote by $H_{R/I}$ its Hilbert function. We define $\Delta^0 H_{R/I}(t) := H_{R/I}(t)$ and, for every $1 \leq i \leq n$, $\Delta^i H_{R/I}(0) := 1$ and $\Delta^i H_{R/I}(t) := \Delta^{i-1} H_{R/I}(t) - \Delta^{i-1} H_{R/I}(t-1)$ for $t > 0$. We call $\Delta^i H(t)$ the *i-th derivative of* H.

Given a monomial ideal $J \subset R$, we denote by B_J the minimal monomial basis of J and by $\mathcal{N}(J)$ the *sous-escalier* of J, that is the set of terms of R outside J. Recall that, for every integer t, the cardinality of $\mathcal{N}(J)_t$ coincides with the value of the Hilbert function of R/J at the degree t.

Definition 1.1. A monomial ideal $J \subseteq R$ is

- *quasi-stable* if for every term $\tau \in J$ and for every variable $x_j \succ \min(\tau)$, there is some integer $s > 0$ such that the term $x_j^s \tau / \min(\tau)$ belongs to J;
- *stable* if for every term $\tau \in J$ and variable $x_j \succ \min(\tau)$, the term $x_j \tau / \min(\tau)$ belongs to J;
- *strongly stable* if, for every term $\tau \in J$, variable x_i by which τ is divisible and variable $x_j \succ x_i$, the term $x_j \tau / x_i$ belongs to J.

In the hypothesis that the ideal J is stable, we have the following result, that will be used in Section 5.

Lemma 1.2. *Let $J \subset R$ be a stable ideal. Then*

$$|\mathcal{N}(J) \cap (J : x_n)| = |\{\tau \in B_J : \tau / x_n \in \mathbb{T}\}|.$$

Proof. It is enough to observe that $x^\beta \in \mathcal{N}(J) \cap (J : x_n)$ if and only if $x_n x^\beta \in B_J$ (for example, see [3, Lemma 3(ii)]). □

Let $J \subset R$ be any monomial ideal and t an integer. Then, the *first expansion* of $\mathcal{N}(J)_t$ is $\mathcal{E}(\mathcal{N}(J)_t) := \mathbb{T}_{t+1} \setminus (\{x_1, \ldots, x_n\} \cdot J_t) = \mathcal{N}(J_{\leq t})_{t+1}$ (see for instance [24]).

If J is a stable ideal, for every integer t the first expansion of $\mathcal{N}(J)_t$ can be directly computed without repetitions and in increasing order with respect to the reverse lexicographic order as follows, where in square brackets we denote a list of terms of $\mathbb{T}_t$ that is increasingly ordered with respect to $\succ$:

$$\mathcal{E}(\mathcal{N}(J)_t) = \bigsqcup_{i=0}^{n-1} x_{n-i} \cdot [\tau \in \mathcal{N}(J)_t : \min(\tau) \succeq x_{n-i}]. \tag{1.1}$$

Thus, if ℓ is an integer such that $\mathcal{N}(J)_\ell \cap K[x_1, \ldots, x_{n-1}] = \emptyset$ and H is the Hilbert function of R/J, then for every $t \geq \ell$ we immediately obtain:

(i) $\mathcal{N}(J)_t \cap K[x_1,\dots,x_{n-1}] = \emptyset$,
(ii) $H(t) \geq H(t+1)$.

Definition 1.3. A subset $L \subset \mathbb{T}_t$ is a *reverse lexicographic segment* (*revlex segment*, for short) if, for every $\tau \in L$ and $\tau' \in \mathbb{T}_t$, $\tau' \succ \tau$ implies that τ' belongs to L.
A monomial ideal $J \subset R$ is a *reverse lexicographic ideal* (*revlex ideal, for short*) if $J_t \cap \mathbb{T}$ is a revlex segment, for every degree t.

Remark 1.4. There are several types of so-called *segments* (see for instance [10]). Among them, a reverse lexicographic segment has a special place. For example, the generic initial ideal of general points with respect to the degree reverse lexicographic term order is a revlex ideal [23].

Definition 1.5. [13] A monomial ideal $J \subset R$ is an *almost reverse lexicographic ideal* (or *weakly reverse lexicographic ideal* or *almost revlex ideal*, for short) if, for every minimal generator $\tau \in B_J$ of J and $\tau' \in \mathbb{T}_{\deg(\tau)}$, $\tau' \succ \tau$ implies that τ' belongs to J.

Example 1.6. A revlex ideal is almost revlex, but an almost revlex ideal is not in general revlex. The ideal $J = (x_2^3, x_2^2x_1, x_2x_1^2, x_1^3, x_3^2x_1^2) \subset K[x_1,\dots,x_4]$ is not a revlex ideal because $J_4 \cap \mathbb{T}$ is not a revlex segment.

Remark 1.7. [18, Definitions 8 and 10, and Remark 11]
(i) An almost revlex ideal is strongly stable.
(ii) If J is an almost revlex ideal in the polynomial ring $K[x_1,\dots,x_{n-1}] \subset R$, then the ideal JR is an almost revlex ideal too.

We say that a Hilbert function H *admits an almost revlex ideal* if there exists the almost revlex ideal J such that H is the Hilbert function of R/J. Indeed, if $J, J' \subset R$ are almost revlex ideals with the same Hilbert function, in the sense that $H_{R/J}(t) = H_{R/J'}(t)$ for all t, then $J = J'$ [18, Remark 11].

2. Preliminaries on Hilbert functions of complete intersections

The Hilbert function of a complete intersection is well known, as well as the minimal free resolution, which is a Koszul complex. In this section, we collect some known properties of the Hilbert function of a complete intersection that have important consequences for our aims.

We start with a very classical result that connects the Hilbert function of a K-algebra with the Hilbert function of a general hypersurface section.

Lemma 2.1. *If $I \subset R$ is a homogeneous ideal and $F \in R$ is a form of degree d that is not a zero-divisor in R/I, then $H_{R/I}(t) - H_{R/I}(t-d) = H_{R/(I,F)}(t)$, for every t.*

Proof. It is enough to apply the additive property of a Hilbert function on the short exact sequence $0 \longrightarrow (R/I)_{t-d} \xrightarrow{\cdot F} (R/I)_t \longrightarrow (R/(I+(F)))_t \longrightarrow 0$. □

Let $d_1 \leq \cdots \leq d_{n-1} \leq d_n$ be n positive integers. We assume $d_1 \geq 2$ because, if $d_1 = 1$, we can rephrase the framework we are interested in using one less variable and forms of degrees $d_2 \leq \cdots \leq d_n$.

For every $1 \leq i \leq n$, let $H^{[i]}$ be the Hilbert function of a complete intersection generated by a regular sequence of i forms of degrees $d_1 \leq \cdots \leq d_i$ in $K[x_1,\dots,x_i]$. Moreover, for every $1 \leq i \leq n$, we let $m_i := (\sum_{j=1}^{i} d_j) - i$. Note that $m_i + 1$ is the regularity of the ideal generated by a regular sequence of polynomials of degrees $d_1 \leq \cdots \leq d_i$ in $K[x_1,\dots,x_i]$.

Theorem 2.2. [12, Theorem (Hilbert Functions under Liaison)] *The Hilbert function* $H^{[i]}$ *is symmetric and* $\max\{t \mid H^{[i]}(t) \neq 0\} = m_i$.

Proposition 2.3. *For every* $2 \leq i \leq n$*, we have*

$$H^{[i]}(t) = \sum_{j=0}^{t} H^{[i-1]}(j) - \sum_{j=0}^{t-d_i} H^{[i-1]}(j).$$

In particular, $\Delta H^{[i]}(t) = H^{[i-1]}(t) - H^{[i-1]}(t-d_i)$.

Proof. It is enough to apply Lemma 2.1 to regular sequences. □

The following result is a weaker version of [31, Theorem 1 and Corollary 2], where the behavior of the Hilbert function of a complete intersection is precisely described. Here, we only recall the properties we need. We define $\bar{u}_1 := 0$ and, for every $i > 1$, $\bar{u}_i := \min\left\{\lfloor \frac{m_i}{2} \rfloor, m_{i-1}\right\}$.

Theorem 2.4. [31, Theorem 1 and Corollary 2] *The Hilbert function* $H^{[i]}$ *is strictly increasing in the range* $[0, \bar{u}_i]$ *and is decreasing in the range* $[\bar{u}_i, m_i]$.

For a Hilbert function H, let δ be the Krull dimension of a graded K-algebra having Hilbert function H. Then, for every $s \geq \delta$, we let

$$c_s(H) := \max\{c \mid \Delta^s H(j) > 0,\ \forall\ 0 \leq j \leq c\} \text{ (see [28, Theorem 5]).}$$

We write c_s if it is clear from the context which Hilbert function is involved. If $H = H^{[i]}$, we let $c_s^{[i]} := c_s(H^{[i]})$.

In Theorem 2.4 a decreasing behavior of the Hilbert function of a complete intersection is described. The following relevant result, which is due to Pardue, highlights that also the derivatives of such a function have a decreasing behavior.

Theorem 2.5. [28, Theorem 5] *If* $0 \leq t \leq c_s^{[n]}$ *and* $\Delta^{s+1} H^{[n]}(t) \leq 0$*, then we also have* $\Delta^{s+1} H^{[n]}(t+1) \leq 0$.

Example 2.6. Let $n = 4$. For $d_1 = 4$, $d_2 = 5$, $d_3 = 7$, $d_4 = 8$, we obtain $m_1 = 3$, $m_2 = 7$, $m_3 = 13$, $m_4 = 20$, $\bar{u}_2 = 3$, $\bar{u}_3 = 6$, $\bar{u}_4 = 10$ and

t	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$H^{[4]}(t)$	1	4	10	20	34	51	70	89	105	116	120	116	105	89	70	...
$H^{[3]}(t)$	1	3	6	10	14	17	19	19	17	14	10	6	3	1	0	...
$\Sigma_{j=0}^{t} H^{[3]}(j)$	1	4	10	20	34	51	70	89	106	120	130	136	139	140	140	...
$\Delta H^{[4]}(t)$	1	3	6	10	14	17	19	19	16	11	4	-4	-9	-16	-11	...

Observe that $H^{[4]}(t) = \sum_{j=0}^{t} H^{[3]}(t)$ for every $t < d_4$. Furthermore in this case $c_0^{[4]} = m_4 = 20$, $c_1^{[4]} = 10$, $c_2^{[4]} = 6$.

3. Almost revlex ideals: reduction numbers and minimal generators

Using [20, Corollary 1.4], we can consider the following definition for reduction numbers for strongly stable ideals (about the more general definition and properties of reduction numbers see [20] and the references therein).

Definition 3.1. Let $J \subset R = K[x_1, \dots, x_n]$ be a strongly stable ideal and δ the Krull dimension of R/J. For any $s \geq \delta$, we denote by $r_s(R/J)$ the *s-reduction number* of R/J, that is $\min\{t \mid x_{n-s}^{t+1} \in J\}$. If from the context it is clear what strongly stable ideal is involved, we write r_s only.

Remark 3.2. Let $J \subset R$ be a strongly stable ideal and δ the Krull dimension of R/J.

(i) $r_s \leq r_{s-1}$, for every $s > \delta$.
(ii) If J is also almost revlex, for every $n-\delta+2 \leq j \leq n$ the variable x_j is not a zero-divisor on R/J.

In this section, we highlight some results about reduction numbers for an almost revlex ideal J that can be deduced from the combinatorial structure of J. Moreover, we deduce a closed formula for the number of minimal generators.

Lemma 3.3. *Let $J \subset R$ be an almost revlex ideal, δ the Krull dimension and H the Hilbert function of R/J. For every $s > \delta$, the Hilbert function of $R/(J+(x_{n-s+1},\dots,x_n))$ coincides with $\Delta^s H(t)$ at every $t \leq r_s$.*

Proof. We first consider the Artinian case $\delta = 0$. Recall that $r_0 \geq r_1 \geq \dots \geq r_{s-1} \geq r_s$. We argue by induction on s.

Let $s = 1$ and consider $A = R/J$. Suppose that, for some integer $t \leq r_1$, the term $x_n\tau$ belongs to J_t, while x_n and τ do not belong to J. Since J is strongly stable, this means that $x_n\tau$ is a minimal monomial generator of J. Thanks to Definition 1.5, this would imply that $x_{n-1}^t \in J$, because $x_{n-1}^t \succ x_n\tau$ for every $\tau \in \mathbb{T}_{t-1}$, in contradiction with the definition of r_1. This means that the variable x_n is not a zero-divisor on A_t, for every $t \leq r_1$. Hence, we can conclude by Lemma 2.1 because the short sequence $0 \longrightarrow A_{t-1} \xrightarrow{\cdot x_n} A_t \longrightarrow (A/(x_n))_t \longrightarrow 0$ is exact for every $t \leq r_1$.

For every $s > 1$, we apply the same argument to $A = R/(J+(x_{n-s+2},\dots,x_n))$ observing that the variable x_{n-s+1} is not a zero-divisor on A_j, for every $j \leq r_s \leq r_{s-1}$.

Let us now pass to the case $\delta > 0$. If $\delta = 1$ then consider $A = R/J$, if $\delta > 1$ then consider $A = R/(J+(x_{n-\delta+2},\dots,x_n))$. In both cases A has Hilbert function $\Delta^{\delta-1}H$ thanks to Remark 3.2(ii) and Lemma 2.1. Thus, we can proceed like in the Artinian case. □

Lemma 3.4. *Let $J \subset R$ be an almost revlex ideal, δ the Krull dimension and H the Hilbert function of R/J. If $r_s < r_{s-1}$ for some $s > \delta$, then $\Delta^s H(t) \leq 0$ for every $r_s < t \leq r_{s-1}$.*

Proof. If $\delta = 0$ and $s = 1$, consider $A = K[x_1,\dots,x_n]/J$ and observe that, for every $t > r_1$, $(K[x_1,\dots,x_n]/(J+(x_n)))_t$ vanishes. So, we have the short exact sequence

$$0 \longrightarrow (K[x_1,\dots,x_n]/J:(x_n))_{t-1} \longrightarrow A_{t-1} \xrightarrow{\cdot x_n} A_t \longrightarrow 0$$

and obtain the thesis for every $r_1 < t \leq r_0$. If $\delta = 0$ and $s > 1$, we can apply the same argument to $A = K[x_1,\dots,x_n]/(J+(x_{n-s+2},\dots,x_n))$, which has Hilbert function $\Delta^{s-1}H(t)$ for every $t \leq r_{s-1}$ by Lemma 3.3, because for every $t > r_s$ the quotient $(K[x_1,\dots,x_n]/(J+(x_{n-s+1},\dots,x_n)))_t$ vanishes. If $\delta > 0$ we argue in the same way considering: $A = R/J$, if $\delta = 1$ and $s = 2$; $A = K[x_1,\dots,x_n]/(J+x_{n-\delta+2},\dots,x_n)$, which has Hilbert function $\Delta^{\delta-1}H$ thanks to Remark 3.2(ii) and Lemma 2.1, if $\delta > 1$ and $s = \delta+1$; $A = K[x_1,\dots,x_n]/(J+(x_{n-s+2},\dots,x_n))$, otherwise. □

Remark 3.5. At a first glance, the result of Lemma 3.4 could seem similar to that of Theorem 2.5, but the context and the aim are different. Statement of Theorem 2.5 (and its proof in [28]) considers the Hilbert function of a complete intersection and describes the decreasing behavior of its derivatives, without assuming that this function admits an almost revlex ideal. Statement of Lemma 3.4 considers an almost revlex ideal and relates the reduction numbers of the almost revlex ideals to the integers $c_s(H)$.

Proposition 3.6. *Let $J \subset R$ be an almost revlex ideal, δ the Krull dimension and H the Hilbert function of R/J. For every $s \geq \delta$, we have $r_s = c_s$.*

Proof. First, we observe that:

(a) the result of Lemma 3.3 implies $r_s \leq c_s$ for every $s > \delta$;
(b) item(a) and the result of Lemma 3.4 imply $r_s = c_s$ if $r_s < r_{s-1}$ for some $s > \delta$.

We proceed by induction on s. For the base of induction we distinguish several cases.

If $s = \delta = 0$ then $r_0 = \mathrm{reg}(J) - 1 = c_0$.

If $s = \delta = 1$ then we consider $A = R/J$ and $\Delta H(r_1 + 1) = H(r_1 + 1) - H(r_1)$. Recall that $H(t)$ counts the number of terms in $\mathcal{N}(J)_t$. The $H(r_1 + 1)$ terms in $\mathcal{N}(J)_{r_1+1}$ are all divisible by x_n because $x_{n-1}^{r_1+1}$ belongs to J. Moreover, from (1.1) we have that all terms in $\mathcal{N}(J)_{r_1+1}$ are obtained multiplying terms in $\mathcal{N}(J)_{r_1}$ by x_n. Thus, $H(r_1 + 1) = |\mathcal{N}(J)_{r_1+1}| \leq |\mathcal{N}(J)_{r_1}| = H(r_1)$. Hence, $c_1 < r_1 + 1$ and by item (a) we have $r_1 = c_1$.

If $s = \delta > 1$ then we consider $A = R/(J + (x_{n-\delta+2}, \ldots, x_n))$ which has Hilbert function $\Delta^{\delta-1}H$ thanks to Remark 3.2(ii) and Lemma 2.1, like in the proof of Lemma 3.3. Hence, we can proceed as in the case $s = \delta = 1$ replacing the variable x_n by $x_{n-\delta+1}$.

Assume now $s > \delta$. If $r_s < r_{s-1}$, then we can apply item (b). If $r_s = r_{s-1}$, we have $r_s = r_{s-1} = c_{s-1}$ by the inductive hypothesis and, hence, $\Delta^{s-1}H(r_s) > 0$ and $\Delta^{s-1}H(r_s + 1) \leq 0$, so that $\Delta^s H(r_s + 1) = \Delta^{s-1}H(r_s + 1) - \Delta^{s-1}H(r_s) < 0$. We can now conclude that $r_s = c_s$. □

Remark 3.7. For every strongly stable ideal J with the Hilbert function of an Artinian complete intersection, we immediately obtain $c_1 = \bar{u}_n$ from Theorem 2.4. If J is also almost revlex ideal, then $r_1 = c_1 = \bar{u}_n$ from Proposition 3.6.

Example 3.8. Let I be the defining ideal of the general space rational curve of degree 5. In $R = K[x_1, x_2, x_3]$, for the strongly stable ideals $J = (x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, x_1^2x_3^2)$ and $J' = (x_1^3, x_1^2x_2, x_1x_2^2, x_1^2x_3, x_3x_2^3, x_2^4)$ the K-algebras R/J and R/J' have the same Hilbert function H as $K[x_1, x_2, x_3, x_4]/(I, x_4)$:

t	0	1	2	3	4	5	6
$H(t)$	1	3	6	6	5	5	...

We have $c_1(H) = c_2(H) = 2$. For the ideal J, which is almost revlex (see Example 1.6), we have $r_1 = r_2 = 2 = c_1(H) = c_2(H)$ and for the ideal J' we have $r'_1 = 3$ and $r'_2 = 2$.

Remark 3.9. Let $J \subset R$ be an almost revlex ideal. For every integer $s > \delta$, consider the ideal $\bar{J} = (J + (x_{n-s+1}, \ldots, x_n))/(x_{n-s+1}, \ldots, x_n)$ in $K[x_1, \ldots, x_{n-s}]$. From Proposition 3.6, the Hilbert function of $K[x_1, \ldots, x_{n-s}]/\bar{J}$ is the function $|\Delta^s H(t)|$ that is defined in [28, Section 3] in the following way: $|\Delta^s H(t)| = \Delta^s H(t)$, if $\Delta^s H(j) > 0$ for $0 \leq j \leq t$, and $|\Delta^s H(t)| = 0$ otherwise.

From the above considerations, an exact closed formula for the number of the minimal generators of an almost revlex ideal follows in terms of the Hilbert function only.

Theorem 3.10. *Let $J \subset R$ be an almost revlex ideal and B_J its minimal monomial basis, δ the Krull dimension and H the Hilbert function of R/J. Then,*

$$|B_J| = \begin{cases} \sum_{s=0}^{n-1} \Delta^s H(c_{s+1}), & \text{if } \delta = 0 \\ \sum_{s=\delta}^{n-1} \Delta^s H(c_{s+1}) + \Delta^{\delta-1} H(c_\delta) - \Delta^{\delta-1} H(\varrho), & \text{if } \delta > 0 \end{cases} \tag{3.1}$$

where $\varrho = \min\{t : \Delta^{\delta-1}H(j) = \Delta^{\delta-1}H(j+1), \forall j \geq t\}$.

Proof. We start detecting the minimal generators with minimal variables $x_1, \dots, x_{n-\delta}$, respectively. For every $\delta \leq s \leq n-1$, the minimal generators with minimal variable x_{n-s} have degree between $r_{s+1}+1$ and $r_s + 1$.

Consider first the case $s = \delta$. If $\delta = 0$ then let $A := R/J$, and if $\delta > 0$ then let $A := R/(J + (x_{n-\delta+1}, \dots, x_n))$ which has Hilbert function $\Delta^\delta H(t)$, for every $t \leq r_\delta$, because the variables $x_{n-\delta+1}, \dots, x_n$ form a regular sequence for $A_{\leq r_\delta}$. For every $t \geq r_{\delta+1}+1$, we have the short exact sequence

$$0 \to (A/(0 :_A (x_{n-\delta})))_{t-1} \to A_{t-1} \xrightarrow{\cdot x_{n-\delta}} A_t \to 0$$

because $(A/(x_{n-\delta}))_t = 0$, for every $t \geq r_{\delta+1}+1$. Then, we find that the minimal generators of degree t with minimal variable $x_{n-\delta}$ are $\Delta^\delta H(t-1) - \Delta^\delta H(t) = -\Delta^{\delta+1}H(t)$ for every $r_{\delta+1}+1 \leq t \leq r_\delta$ (also see Lemma 3.4) and $\Delta^\delta H(r_\delta)$ at degree $r_\delta + 1$, because the short exact sequence becomes

$$0 \to (A/(0 :_A (x_{n-\delta})))_{r_\delta} \to A_{r_\delta} \to 0.$$

So, the number of minimal generators of J with minimal variable $x_{n-\delta}$ is $\Delta^\delta H(r_{\delta+1})$ because

$$\begin{aligned} &-\Delta^{\delta+1}H(r_{\delta+1}+1) - \dots - \Delta^{\delta+1}H(r_\delta) + \Delta^\delta H(r_\delta) = \\ &\quad = \Delta^\delta H(r_{\delta+1}) - \Delta^\delta H(r_{\delta+1}+1) + \dots + \Delta^\delta H(r_\delta - 1) - \Delta^\delta H(r_\delta) + \Delta^\delta H(r_\delta) = \\ &\qquad = \Delta^\delta H(r_{\delta+1}). \end{aligned}$$

We can repeat the above argument for every $\delta < s \leq n-1$, applying Lemma 3.3, so that the minimal generators with minimal variable x_{n-s} are $\Delta^s H(r_{s+1})$. We can then conclude thanks to Proposition 3.6. For the case $\delta = 0$, this gives the statement on $|B_J|$.

For what concerns the case $\delta > 0$, we have to carefully consider the fact that B_J in general contains terms with minimal variable $x_{n-\delta+1}$. If $\delta > 1$, by definition of almost revlex ideal the sequence of variables $x_{n-\delta+2}, \dots, x_n$ is a regular sequence for R/J. Then, we let $A := R/(J + (x_{n-\delta+2}, \dots, x_n))$, which has Hilbert function $\Delta^{\delta-1}H$, thanks to Lemma 2.1. If $\delta = 1$, we let $A := R/J$. The $(\delta-1)$-th derivative $\Delta^{\delta-1}H$ has constant Hilbert polynomial $p(z) = d$. Thus, if ϱ is the minimal value such that $\Delta^{\delta-1}H(t) = d$, for every $t \geq \varrho$, then $\Delta^\delta H(t) = 0$ for every $t \geq \varrho + 1$.

In this case, the ideal J can have minimal generators with minimal variable $x_{n-\delta+1}$ in the degrees $t \geq r_\delta + 1 = c_\delta + 1$. Moreover, for every $t \geq r_\delta + 1 = c_\delta + 1$, we have $(A/(x_{n-\delta+1}))_t = 0$, because $x_{n-\delta}^{r_\delta+1} \in J$. Hence, like in the proof of Lemma 3.4, for every $t \geq r_\delta + 1$ we can consider the short exact sequence

$$0 \longrightarrow (A/(0 :_A (x_{n-\delta+1})))_{t-1} \longrightarrow A_{t-1} \xrightarrow{\cdot x_{n-\delta+1}} A_t \longrightarrow 0.$$

Thus, for every $t \geq r_\delta + 1 = c_\delta + 1$, the possible minimal generators of degree t with minimal variable $x_{n-\delta+1}$ are $\Delta^{\delta-1}H(t-1) - \Delta^{\delta-1}H(t) = -\Delta^\delta H(t)$. In conclusion, the number of possible minimal generators of J with minimal variable $x_{n-\delta+1}$ is 0 if $r_\delta+1=c_\delta+1>\varrho$, and is $\sum_{j=c_\delta+1}^{\varrho} -\Delta^\delta H(j) = \Delta^{\delta-1}H(c_\delta)-\Delta^{\delta-1}H(\varrho)$ if $r_\delta+1=\varrho$. □

Remark 3.11. We highlight that the case $\delta=0$ of formula (3.1), as well as the summation from δ to $n-1$ of the other case, can be also deduced from the construction of almost revlex ideals that was given by K. Pardue in [28, Theorem 4]. For the Artinian case in characteristic 0, see also [18]. As a consequence of Theorem 3.10, formulas for the Betti numbers of J can be also obtained, because J is strongly stable (for example, see [18, Section 4]).

Example 3.12. Going back to Example 3.8, we now apply Theorem 3.10 to the ideal $J=(x_1^3,x_1^2x_2,x_1x_2^2,x_2^3,x_1^2x_3^2)\subseteq K[x_1,x_2,x_3]$. In this case we have $\delta=1$, $\varrho=4$, $c_1=c_2=2$ and $c_3=0$. Thus, $|B_J|=\Delta H(2)+\Delta^2H(0)+H(2)-H(4)=3+1+6-5$.

4. Construction of the almost revlex ideal for $H^{[n]}$

Let $2\leq d_1\leq\cdots\leq d_n$ be integers. In this section, we describe our construction of the almost revlex ideal that is admitted by the Hilbert function $H^{[n]}$. Then, we make a comparison with the construction given in [28, Theorem 4].

Theorem 4.1. *For every n, the Hilbert function $H^{[n]}$ admits an almost revlex ideal.*

Proof. We proceed by induction on n.

For $n=1$, it is sufficient to consider the ideal $(x_1^{d_1})\subseteq K[x_1]$.

For every $n>1$, by inductive hypothesis the Hilbert function $H^{[n-1]}$ admits an almost revlex ideal, that we denote by $J^{[n-1]}\subseteq K[x_1,\dots,x_{n-1}]$. Consider the almost revlex ideal

$$J' := \left(J^{[n-1]}\right)_{\leq d_n}\cdot K[x_1,\dots,x_n].$$

The Hilbert function of $K[x_1,\dots,x_n]/J'$ is $H'(t)=\sum_{j=0}^{t}H^{[n-1]}(j)$, for every $t\leq d_n$, moreover $H'(t)=H^{[n]}(t)$ for every $t\leq d_n-1$ and $H'(d_n)=H^{[n]}(d_n)+1$. We set $c_i'=c_i(H')$ and $r_i'=r_i(R/J')$.

Let τ be the highest term of degree d_n, with respect to the degree reverse lexicographic term order, in $\mathcal{N}(J')_{d_n}$. Replace the ideal J' by $J'+(\tau)$, so that J' now is an almost revlex ideal with Hilbert function H' such that $H'(j)=H^{[n]}(j)$ for every $j\leq d_n$.

For $t>d_n$, assume there is an ideal $J'=J'_{\leq t-1}$ which is an almost revlex ideal with Hilbert function H' such that $H'(j)=H^{[n]}(j)$ for every $j\leq t-1$. Let s be the maximum integer such that there exists a term in $\mathcal{N}(J')_{t-1}$ that is divisible by the variable x_{n-s}. Then, $c_{s+1}^{[n]}=c_{s+1}'=r_{s+1}'<t-1\leq r_s'=c_s'\leq c_s^{[n]}$, because J' is almost revlex, hence $\Delta^sH^{[n]}(t-1)=\Delta^sH'(t-1)>0$ and $\Delta^{s+1}H^{[n]}(t-1)=\Delta^{s+1}H'(t-1)\leq 0$.

If $s>0$, from Lemma 3.3 and (1.1) the first expansion of $\mathcal{N}(J')_{t-1}$ consists of

$H^{[n]}(t-1)$ terms with minimal variable x_n
$\Delta H^{[n]}(t-1)$ terms with minimal variable x_{n-1}
⋮
$\Delta^sH^{[n]}(t-1)$ terms with minimal variable x_{n-s},

that are $\sum_{i=0}^{s}\Delta^iH^{[n]}(t-1)$ terms. Observing that

$$H^{[n]}(t)=\Delta^sH^{[n]}(t)+\sum_{j=0}^{s-1}\Delta^jH^{[n]}(t-1),$$

we obtain

$$|\mathcal{E}(\mathcal{N}(J')_{t-1})| = H^{[n]}(t) - \Delta^{s+1}H^{[n]}(t).$$

Since $\Delta^{s+1}H^{[n]}(t-1) \leq 0$, thanks to Theorem 2.5 we have $\Delta^{s+1}H^{[n]}(t) \leq 0$, from which it follows that the cardinality of the first expansion of $\mathcal{N}(J')_{t-1}$ is higher than or equal to $H^{[n]}(t)$. In this case, let $\tau_1, \dots, \tau_h$ be the highest $h = \sum_{i=0}^{s} \Delta^i H^{[n]}(t-1) - H^{[n]}(t) = -\Delta^{s+1}H^{[n]}(t) \geq 0$ terms of degree t, with respect to the degree reverse lexicographic term order, in the first expansion of $\mathcal{N}(J')_{t-1}$. Replace the ideal J' by $J' + (\tau_1, \dots, \tau_h)$, so that J' becomes an almost revlex ideal with Hilbert function H' such that $H'(j) = H^{[n]}(j)$ for every $j \leq t$.

We can repeat this construction until we find that the sous-escalier of J' at degree $t-1$ only consists of $H^{[n]}(t-1)$ terms with minimal variable x_n, that is $s = 0$. In this case, the first expansion of $\mathcal{N}(J')_{t-1}$ also consists of $H^{[n]}(t-1)$ terms of degree t with minimal variable x_n. So, we have $c_1^{[n]} = c_1' = r_1' < t-1 \leq r_0' = c_0' \leq c_0^{[n]}$. By Theorem 2.4, we know that $c_1^{[n]} = \bar{u}_n$, because $H^{[n]}$ is strictly decreasing from $\bar{u}_n$ on, and we can continue this construction up to degree $t = d_1 + \cdots + d_{n-1} + d_n - n$, obtaining an almost revlex ideal J' having Hilbert function $H^{[n]}$. □

From now, for every $1 \leq i \leq n$, we denote by $J^{[i]}$ the almost revlex such that the Hilbert function of $K[x_1, \dots, x_i]/J^{[i]}$ is $H^{[i]}$. Moreover, we let $r_s^{[i]} := r_s(K[x_1, \dots, x_i]/J^{[i]})$.

The explicit construction of $J^{[n]}$ in the proof of Theorem 4.1 can be algorithmically improved observing that, for every $i \in \{2, \dots, n-1\}$, it is sufficient to compute the generators of $J^{[i]}$ up to degree d_{i+1}.

Summing up, we obtain Algorithm 1, for which we assume that the following procedures are available:

- HFunction($[d_1, \dots, d_j], h$) takes in input an increasingly ordered list of j positive integers $[d_1, \dots, d_j]$ and an integer $h > 0$, and returns the list of values of the Hilbert function $H^{[j]}(t)$, for every $t \leq h$. Precisely, for $j = 1$, the output is the function that assumes value 1 for every $t < d_1$ and 0 otherwise, for $j > 1$ the output can be computed for instance by Proposition 2.3.
- Greatest($\mathcal{N}, h$) takes in input a set of terms $\mathcal{N}$ of degree t and returns the greatest h terms w.r.t. degrevlex.

Algorithm 1 ALMOSTREVLEX($n, [d_1, \dots, d_n]$)

Input: A positive integer n
Input: an increasingly ordered list of n positive integers $[d_1, \dots, d_n]$, with $d_1 \geq 2$.
Output: The ideal $J^{[n]}$.

1: $J' \leftarrow (x_1^{d_1})$;
2: **if** $n > 1$ **then**
3: $\quad J' \leftarrow (J')_{\leq d_2}$;
4: $\quad d_{n+1} \leftarrow (\sum_i d_i) - n + 1$;
5: $\quad$ **for** $i = 2, \dots, n$ **do**
6: $\quad\quad H^{[i]} \leftarrow$ HFUNCTION$([d_1, \dots, d_i], d_{i+1})$;
7: $\quad\quad J' \leftarrow J' \cdot K[x_1, \dots, x_i]$;
8: $\quad\quad \tau \leftarrow \max_{degrevlex} \mathcal{N}(J')_{d_i}$;
9: $\quad\quad J' \leftarrow J' + (\tau)$;
10: $\quad\quad$ **for** $t = d_i, \dots, d_{i+1}$ **do**
11: $\quad\quad\quad x_{n-s} \leftarrow \min_{degrevlex}\{\min(\tau) | \tau \in \mathcal{N}(J')_{t-1}\}$;
12: $\quad\quad\quad h = -\Delta^{s+1} H^{[i]}(t)$;
13: $\quad\quad\quad [\tau_1, \dots, \tau_h] \leftarrow$ GREATEST$(\mathcal{N}(J')_t, h)$;
14: $\quad\quad\quad J' \leftarrow J' + (\tau_1, \dots, \tau_h)$;
15: $\quad\quad$ **end for**
16: $\quad$ **end for**
17: **end if**
18: **return** J'

Remark 4.2. In general, the ideal $J' = (J^{[i]})_{\leq d_i+1}$ that we obtain at line 14 of Algorithm 1 is not Artinian in $K[x_1, \dots, x_i]$, hence the Hilbert function of $K[x_1, \dots, x_i]/J'$ is neither $H^{[i]}$ nor $\Delta^i H^{[n]}$.

Example 4.3. The Hilbert function $H^{[3]}$ of a complete intersection generated by 3 forms of degrees $d_1 = 3$, $d_2 = d_3 = 4$ admits the following almost revlex ideal $J^{[3]} \subset K[x_1, x_2, x_3]$:

$$J^{[3]} = (x_1^3, x_1^2x_2^2, x_1x_2^3, x_2^5, x_2^4x_3, x_1^2x_2x_3^3, x_1x_2^2x_3^3, x_2^3x_3^3, x_1^2x_3^5, x_1x_2x_3^5, x_2^2x_3^5, x_1x_3^7, x_2x_3^7, x_3^9)$$

which is constructed by Algorithm 1 in the following way.

n=1: $J^{[1]} = (x_1^3)$;

n=2: In order to compute $J^{[2]}$ up to degree d_3, we add to $J' = (J^{[1]})_{\leq 4} K[x_1, x_2]$ only the term $x_1^2x_2^2$, and we do not need to explicitly compute the other generators of $J^{[2]}$. Observe indeed that $K[x_1, x_2]/J'$, with $J' = (x_1^3, x_1^2x_2^2)$, does not have Hilbert function $H^{[2]}$, because $H_{R/J'}(t) = H^{[2]}(t)$ only up to $t = d_3$.

n=3: Let now $J' := (x_1^3, x_1^2x_2^2) \cdot K[x_1, x_2, x_3]$. In order to compute $J^{[3]}$ up to degree $d_1 + d_2 + d_3 - 3 + 1 = 9$, we add to J' the term $x_1x_2^3$. We consider the highest term $\tau = x_1x_2^3$ of $\mathcal{N}(J')_4$ with respect to degrevlex and update J' to the ideal $(J')_{\leq 4} + (x_1x_2^3) \subset K[x_1, x_2, x_3]$. In this way, $H'(4) = H^{[3]}(4)$, while $H'(5) - H^{[3]}(5) = 11 - 9 = 2$. The first expansion $\mathcal{E}(\mathcal{N}(J')_4)$ contains $H^{[3]}(4) = 9$ terms with minimal variable x_3 and $\Delta H^{[3]}(4) = 1$ term with minimal variable x_2:

$$x_2^5, x_2^4x_3, x_1^2x_2x_3^2, x_1x_2^2x_3^2, x_2^3x_3^2, x_1^2x_3^3, x_1x_2x_3^3, x_2^2x_3^3, x_2x_3^4, x_1x_3^4, x_3^5.$$

Then we update J' to $J'+(x_2^5, x_2^4x_3)$, so that $H'(5)=H^{[3]}(5)=9$, while $H'(6)-H^{[3]}(6)=9-6=3$. We then consider $\mathcal{E}(\mathcal{N}(J')_5)$, whose terms are all divisible by x_3, because $\Delta H^{[3]}(6)<0$. The greatest 3 terms of $\mathcal{E}(\mathcal{N}(J')_5)$ are $x_1^2x_2x_3^3, x_1x_2^2x_3^3, x_2^3x_3^3$. Again, we update J' to $J'+(x_1^2x_2x_3^3, x_1x_2^2x_3^3, x_2^3x_3^3)$, so that $H'(6)=H^{[3]}(6)=6$, while $H'(7)-H^{[3]}(7)=3$. Iterating this process, we add to J' the terms $x_3^5x_2^2, x_3^5x_2x_1, x_3^5x_1^2$, so that $H'(7)=H^{[3]}(7)$, and the terms $x_3^7x_2, x_3^7x_1$, so that $H'(8)=H^{[3]}(8)$. Finally, the first expansion $\mathcal{E}(\mathcal{N}(J)_8)$ contains only the term x_3^9, which we add to J', obtaining $J'=J^{[3]}$.

In [28, Theorem 4], K. Pardue characterizes all the Hilbert functions that admit an almost revlex ideal. The symmetry of a Hilbert function is not a sufficient condition for admitting an almost revlex ideal, indeed. For example, the symmetric sequence $\mathbf{h}=(1,13,12,13,1)$ is the h-vector of Gorenstein ideals (see [32, 2]), but this Hilbert function does not admit an almost revlex ideal. We can explain this fact either observing that $\mathbf{h}$ does not satisfy the conditions of [28, Theorem 4] or looking at the structure of the expansion of a stable ideal (see formula (1.1)).

In [28, Theorem 4], K. Pardue considers a Hilbert function H, the behavior of which has been recalled in Theorem 2.5, and constructs an almost revlex ideal J such that H is the Hilbert function of R/J. He proceeds by induction performing a hyperplane section by x_n, so that the new generators to be added have minimal variable x_n. Then, he uses the formula of Eliahou and Kervaire [14] for the Hilbert series of a stable ideal, which is based on the shape of the minimal generators of the ideal, in order to guarantee the correctness of his construction.

The construction given by K. Pardue in [28, proof of Theorem 4, (3) $\to$ (1)] is more general than ours, since it concerns not only the Hilbert functions of complete intersections. Nevertheless, the Hilbert functions and almost revlex ideals that one has to consider following Pardue's induction are different from those involved in the proof of Theorem 4.1. Indeed, in order to construct an almost revlex ideal in $K[x_1,\dots,x_n]$ having Hilbert function H, Pardue considers the almost revlex ideal in $K[x_1,\dots,x_{n-1}]$ having Hilbert function $|\Delta^1 H(t)|$ (see Remark 3.9). Our construction starts from the ideal $J^{[n-1]}$ which has Hilbert function $H^{[n-1]}$. In general, if $H=H^{[n]}$ is the Hilbert function of the complete intersection with integers $d_1\le\dots\le d_n$, then $H^{[n-s]}(t)\neq|\Delta^s H^{[n]}(t)|$, hence the corresponding almost revlex ideals are different. Furthermore, following Pardue's construction for $H^{[n]}$, the new generators to be added with minimal variable x_n have in general a degree that is higher than d_n.

Example 4.4. The Hilbert function $H^{[3]}$ of a complete intersection generated by 3 forms of degrees $d_1=3$, $d_2=d_3=4$ admits the almost revlex ideal $J^{[3]}\subset K[x_1,x_2,x_3]$, whose construction is given in Example 4.3 according to Algorithm 1. Take into account that $J^{[2]}=(x_1^3,x_2^2x_1^2,x_1x_2^4,x_2^6)\subset K[x_1,x_2]$. We denote by H' the Hilbert function of $K[x_1,x_2,x_3]/J'$ when $J'=(J^{[2]})_{\le d_3}\cdot K[x_1,x_2,x_3]$ and by H'' that of $K[x_1,x_2]/(J^{[2]})_{\le d_3}$. Observe that $H'(t)=\sum_{j=0}^t H^{[2]}(j)$ for every $t\le d_3$, and $H''(t)=H^{[2]}(t)$ for every $t\le d_3$.

t	0	1	2	3	4	5	6	7	8	9	10
$H^{[3]}(t)$	1	3	6	9	10	9	6	3	1	0	...
$H'(t)$	1	3	6	9	11	13	15	17	19	21	...
$H^{[2]}(t)$	1	2	3	3	2	1	0	0	0	0	...
$H''(t)$	1	2	3	3	2	2	2	2	2	2	...

We highlight that the path one follows by Pardue's construction is different from the path of our construction: using Theorem 4.1 for every $i \le n-1$ one considers the Hilbert function $H^{[i]}(t)$ and constructs $(J^{[i]})_{\le d_{i+1}}$, while using [28, proof of Theorem 4, (3) $\to$ (1)] one considers the Hilbert function $|\Delta^i H^{[n]}(t)|$. In particular, following [28, proof of Theorem 4, (3) $\to$ (1)] in order to construct the almost revlex ideal $J^{[3]} \subset K[x_1, x_2, x_3]$, we consider the almost revlex ideal $I = (x_1^3, x_1^2x_2^2, x_1x_2^3, x_2^5)$ in $K[x_1, x_2]$ having Hilbert function $|\Delta H^{[3]}(t)|$:

t	0	1	2	3	4	5	6
$\lvert\Delta H^{[3]}(t)\rvert$	1	2	3	3	1	0	$\cdots$

Observe that $|\Delta H^{[3]}(t)|$ is not the Hilbert function of a complete intersection because it is not symmetric.

Remark 4.5. The proof of Theorem 4.1 gives rise to the following formula for the number of minimal generators of $J^{[n]}$, which is of course equivalent to that of Theorem 3.10:

$$|B_{J^{[n]}}| = \sum_{s=0}^{n-1} \sum_{j=c_{s+1}+1}^{c_s} -\Delta^{s+1} H^{[n]}(j+1). \tag{4.1}$$

We obtain (4.1) in the following way. If $n = 1$ then $c_0 = d_1$ and $c_1 = 0$. Further, $\Delta H^{[1]}(t) = 0$ for every $1 \le t \le c_0$ and $\Delta H^{[1]}(t) = -1$. Hence, (4.1) holds for $n = 1$. If $n > 1$ and $J' := (J^{[n-1]})_{\le d_n} K[x_1, \dots, x_n]$ like in the proof of Theorem 4.1, then for every $t < d_n$, we have $\Delta^{s+1} H^{[n]}(t) = \Delta^s H^{[n-1]}(t)$ and $\Delta^{s+1} H^{[n]}(d_n) = \Delta^s H^{[n-1]}(d_n) - 1$ thanks to Proposition 2.3; further, for every $s \ge 1$, if $c_s^{[n]} < d_n$, then $c_s^{[n]} = c_{s-1}^{[n-1]}$. Moreover, up to degree d_n the ideal J' has the same number of minimal generators of $J^{[n-1]}$, that is

$$\sum_{s=0}^{n-2} \sum_{j=\min\{c_{s+1}^{[n-1]}+1, d_n-1\}}^{\min\{c_s^{[n-1]}, d_n-1\}} -\Delta^{s+1} H^{[n-1]}(j+1) = -1 + \sum_{s=0}^{n-2} \sum_{j=\min\{c_{s+2}^{[n]}+1, d_n-1\}}^{\min\{c_{s+1}^{[n]}, d_n-1\}} -\Delta^{s+2} H^{[n]}(j+1) =$$

$$= -1 + \sum_{s=1}^{n-1} \sum_{j=\min\{c_{s+1}^{[n]}+1, d_n-1\}}^{\min\{c_s^{[n]}, d_n-1\}} -\Delta^{s+1} H^{[n]}(j+1).$$

At degree d_n the ideal $J^{[n]}$ has one more generator with respect to J'. Then, we conclude adding the minimal generators of degrees $j \ge d_n$ as in the proof of Theorem 4.1.

5. A lower bound for the dimension of the tangent space to a punctual Hilbert scheme at stable ideals

In the present section, we start considering Artinian monomial ideals J in R. It is quite straightforward that every Artinian monomial ideal is *quasi-stable*.

Every quasi-stable ideal J has a special set $\mathcal{P}(J)$ of monomial generators, in general non-minimal, that is called the *Pommaret basis* of J and allows a unique decomposition of every term $\tau \in J$ in the following sense:

($\star$) for every $\tau \in J$, there is a unique $x^\alpha \in \mathcal{P}(J)$ so that $\tau = x^\alpha x^\delta$ with $\max(x^\delta) \preceq \min(x^\alpha)$.

Stable ideals are quasi-stable, and strongly stable ideals are stable. If J is stable, then $\mathcal{P}(J) = B_J$.

Let H be the Hilbert function of R/J and $S := R[x_{n+1}]$. The Hilbert polynomial $p(z)$ of $S/(JS)$ is the constant $D := \sum_{j\geq 0} H(j)$, because J is Artinian. We can then identify J with the point $\mathrm{Proj}(S/(JS))$ of the Hilbert scheme Hilb^n_D, which parameterizes flat families of closed subschemes in $\mathbb{P}^n_K$ with Hilbert polynomial D. Hence, we will say that J is (or corresponds to) a point of Hilb^n_D.

Our aim is to give a lower bound for the dimension of the Zariski tangent space $\mathcal{T}_J$ to Hilb^n_D at the point J, using techniques and results that have been developed in [5, 6]. A similar investigation has been given in [10, Lemma 6.1 and Theorem 6.2] under the more restrictive hypotheses that JS is a *hilb-segment ideal* with respect to a suitable term order and the field K has characteristic zero. Although we will use the Jacobian criterion in order to compute $\mathcal{T}_J$, for the moment we can consider any field K because J is a K-valued point (K-point, for short) of the Hilbert scheme, i.e. a closed point with residue field K.

Referring to [5, 6, 22], first we briefly recall how one can obtain a set of equations defining the Zariski tangent space to Hilb^n_D at J, but more generally at any point of a suitable open subset of Hilb^n_D. Also recall that J is a monomial Artinian (hence, quasi-stable) ideal and $\mathcal{P}(J)$ denotes its Pommaret basis.

For every $x^\gamma \in \mathcal{P}(J)$, we define the *marked polynomial*

$$f_\gamma = x^\gamma + \sum_{x^\beta \in \mathcal{N}(J)} C_{\gamma\beta} x^\beta \in K[C][x_1, \dots, x_n],$$

where x^γ is called the *head term* of f_γ and $C = \{C_{\alpha\beta} | x^\alpha \in \mathcal{P}(J), x^\beta \in \mathcal{N}(J)\}$ is the set of all parameters appearing in the marked polynomials over $\mathcal{P}(J)$. In this context, the coefficient of a term in the variables $x_1, \dots, x_n$ will be called a *x-coefficient*.

For every $x^\gamma \in \mathcal{P}(J)$ and $x_j \succ \min(x^\gamma)$, we consider the polynomial $x_j f_\gamma$ and, by a suitable Noetherian confluent reduction process that is based on property ($\star$) (see [5, Definition 4.2]), compute the polynomials that are involved in the following equality

$$x_j f_\gamma = \sum p_{\alpha'\delta'} x^{\delta'} f_{\alpha'} + h_{j\gamma}, \tag{5.1}$$

where $x^{\alpha'}$ belongs to $\mathcal{P}(J)$, $\max(x^{\delta'}) \preceq \min(x^{\alpha'})$, $h_{j\gamma}$ is supported on $\mathcal{N}(J)$, and $p_{\alpha'\delta'}$ belongs to $K[C]$. All polynomials and terms in (5.1) are uniquely determined (see [5, Proposition 4.3]).

Let $\mathcal{U} \subset K[C]$ be the set consisting of all the x-coefficients appearing in the polynomials $h_{j\alpha}$. The ideal generated by $\mathcal{U}$ in $K[C]$ defines an open affine subscheme of Hilb^n_D, that is called *J-marked scheme* and usually denoted by $\mathrm{Mf}(J)$ [5, Propositions 5.6 and 6.13(ii)].

Being $\mathrm{Mf}(J) \subset \mathbb{A}^{|C|}$ an open subscheme of Hilb^n_D, we can explicitely compute the Zariski tangent space to Hilb^n_D at any point belonging to $\mathrm{Mf}(J)$ using the polynomials in the set $\mathcal{U}$, as explained in [6, Corollary 1.9 and Remark 1.10]. For what concerns the quasi-stable ideal J, we can simply take the linear part of the polynomials in $\mathcal{U}$.

Remark 5.1. For an efficient way to only compute the linear part of the polynomials in $\mathcal{U}$, one can use [22, Algorithm 2], which also applies to marked schemes. At http://wpage.unina.it/cioffifr/MaterialeAlmostRevLex an implementation of this algorithm for marked schemes on an Artinian quasi-stable ideal is available.

In order to finally obtain a lower bound for $\dim \mathcal{T}_J$, we consider now an Artinian *stable* ideal J and focus on the parameters $C_{\alpha\beta}$ that never appear in the linear part of the polynomials in $\mathcal{U}$. It is clear that, if $\bar{C}$ is a set of such parameters, then $|C| \geq \dim_K \mathcal{T}_J \geq |\bar{C}|$ indeed.

Theorem 5.2. *If $J \subset R$ is an Artinian stable ideal, then for every $x^\beta \in \mathcal{N}(J) \cap (J : x_n)$ and for every $x^\alpha \in B_J$, the parameter $C_{\alpha\beta}$ does not appear in the linear part of the polynomials in $\mathcal{U}$.*

Proof. We first observe that $x_n x^\beta \in J$ implies $x_j x^\beta \in J$ for every j, because J is a stable ideal. Let x^α belong to B_J. Our aim is to prove that for every $x^\gamma \in B_J$, for every $x_j \succ \min(x^\gamma)$, in the writing (5.1) for $x_j f_\gamma$, the coefficient $C_{\alpha\beta}$ does not appear linearly in the x-coefficients of $h_{j\gamma}$.

We first consider the case $\gamma = \alpha$. Recall that $x_j x^\beta$ belongs to J. Hence, following the procedure in [5, Section 4] in order to compute the writing (5.1) for $x_j f_\alpha$ with $x_j \succ \min(x^\alpha)$, we find some $x^{\delta'} f_{\alpha'}$ ($\alpha' \neq \alpha$) such that $x_j x^\beta = x^{\alpha'} x^{\delta'}$. Then, $C_{\alpha\beta} x_j x^\beta$ can be rewritten by $C_{\alpha\beta} x^{\delta'} f_{\alpha'}$ and $C_{\alpha\beta}$ appears in $p_{\alpha'\delta'} \in K[C]$ in the right-hand side of (5.1), but not in the x-coefficients of the polynomials $h_{j\alpha}$, in this case.

We now assume that $\gamma \neq \alpha$ and f_α is used in the writing (5.1) for $x_j f_\gamma$, with $x_j \succ \min(x^\gamma)$. We first observe that $x_j x^\gamma$ does not belong to B_J, otherwise the term x^γ of B_J would divide another minimal monomial generator of B_J. Hence, according to [5, Section 4], if f_α is used for rewriting $x_j x^\gamma$, that is $x_j x^\gamma = x^\delta x^\alpha$ with $x^\delta \neq 1$, we conclude as in the previous case that $C_{\alpha\beta}$ does not appear linearly in the x-coefficients of $h_{j\gamma}$. If f_α is used for rewriting an other term τ, then τ must to have a non-constant coefficient in $K[C]$ and $C_{\alpha\beta}$ appears in the *non-linear* part either of some $p_{\alpha\delta'} \in K[C]$ in the right-hand side of (5.1) or of the x-coefficients of $h_{j\gamma}$. □

Remark 5.3. Observe that if $x^\beta \in \mathcal{N}(J)$ and $x_\ell x^\beta \in J$ for some $\ell < n$, we cannot identify any $x^\alpha \in B_J$ such that $C_{\alpha\beta}$ does not appear in the linear part of the x-coefficients of the polynomials $h_{j\gamma}$. Indeed, if $x^\delta f_\alpha$ appears in the right-hand side of (5.1) for some $x^\gamma \in B_J$, then $x^\delta <_{lex} x_j$, hence there is no way to guarantee that $C_{\alpha\beta}$ does not appear linearly in the x-coefficients of $h_{j\gamma}$.

Corollary 5.4. *If $J \subset R$ is an Artinian stable ideal, then*

$$|B_J| \cdot |\mathcal{N}(J)| \geq \dim \mathcal{T}_J \geq |B_J| \cdot |\{\tau \in B_J : \tau/x_n \in \mathbb{T}\}|.$$

Proof. The first inequality is a consequence of the construction of $\mathcal{T}_J$ by means of marked schemes. Indeed, the marked scheme is embedded in an affine space of dimension $|C| = |B_J||\mathcal{N}(J)|$. The other inequality is a consequence of Theorem 5.2 and of Lemma 1.2. □

Thanks to Corollary 5.4 we now obtain a sufficient condition for J being a singular point in Hilb^n_D when J is stable and also Borel-fixed over an infinite field K. From now, we assume that the field K is infinite, because we will use notions and results that need this hypothesis.

Recall that Borel-fixed ideals are fixed points of an algebraic group action on the Hilbert scheme. More precisely, an ideal $J \subset R$ is Borel-fixed (Borel, for short) if $g(I) = I$ for every element g of the Borel subgroup consisting of the upper triangular matrices on K of order n. A Borel ideal is always a monomial quasi-stable ideal (not stable in general), but the property to be Borel depends on the characteristic of the field. In characteristic 0, Borel ideals and strongly stable ideals coincide. In general, strongly stable ideals are Borel regardless of the characteristic of the field K, however there are quasi-stable (resp. stable) ideals that are not Borel, for every characteristic of the field. In the study of Hilbert schemes, Borel ideals have a very important role (see for instance [19]).

Theorem 5.5. *Let K be an infinite field. With the above notation, if $J \subset R$ is an Artinian stable Borel-fixed ideal, then*

$$|B_J| \cdot |\{\tau \in B_J : \tau/x_n \in \mathbb{T}\}| > n \cdot D \;\Rightarrow\; J \textit{ corresponds to a singular point in } \mathrm{Hilb}^n_D.$$

Proof. From the proof of [27, Corollary 19] it follows that for every Artinian Borel ideal J there is a component of Hilb^n_D that contains JS and the lex-segment ideal. This component must be the same for all these Borel ideals (see also [30] in characteristic zero), because there exists a unique component containing the lex-segment ideal. Indeed, recall that the lex segment ideal is a smooth point in the Hilbert scheme, hence it lies on a unique component of the Hilbert scheme, which has dimension $n \cdot D$ (see [29]).

Hence if $|B_J| \cdot |\{\tau \in B_J : \tau/x_n \in \mathbb{T}\}| > n \cdot D$, then by Corollary 5.4 the dimension of the Zariski tangent space to Hilb^n_d at J is strictly bigger than the dimension of the lex component. So, J is a singular point of this component and of Hilb^n_D. □

Remark 5.6. Observe that the condition that is given in Theorem 5.5 is sufficient only. For instance, the ideal $J^{[3]}$ of Example 4.4 does not satisfy the numerical condition of Theorem 5.5, being $|B_{J^{[3]}}| = 14$ and $|\{\tau \in B_{J^{[3]}} : \tau/x_3 \in \mathbb{T}\}| = 10$. Nevertheless, $J^{[3]}$ corresponds to a singular point of Hilb^3_{48} because a direct computation gives $\dim \mathcal{T}_{J^{[3]}} = 286 > 144$, where $144 = 48 \cdot 3$ is the dimension of the lex component of Hilb^3_{48}. We will find some analogous situations in Examples 6.8 and 6.9.

6. Some almost revlex singular points in a Hilbert scheme

In this section, over an infinite field K we specialize the results of Section 5 to Artinian almost revlex ideals and find several classes of almost revlex ideals with the Hilbert function of a complete intersection that are singular points in a Hilbert scheme. We assume $n \geq 3$, because Hilb^2_D is irreducible and smooth [15, Theorem 2.4].

An almost revlex ideal is strongly stable, hence it is stable and Borel-fixed in every characteristic. Using Theorems 5.5 and 3.10, we obtain a sufficient condition for an Artinian almost revlex ideal J to be a singular point in the Hilbert scheme Hilb^n_D in terms of the Hilbert function H of R/J only, where recall that $D = \sum_{j\geq 0} H(j)$. We can re-state Corollary 5.4 and Theorem 5.5 in the following way, indeed.

Lemma 6.1. *If H is a Hilbert function admitting an Artinian almost revlex ideal $J \subset R$, then $|\{\tau \in B_J : \tau/x_n \in \mathbb{T}\}| = H(c_1)$.*

Proof. The statement follows from the arguments of the proof of Theorem 3.10. □

Theorem 6.2. *Let H be a Hilbert function admitting an Artinian almost revlex ideal $J \subset R$ and $D = \sum_{j\geq 0} H(j)$.*

(i) $\dim \mathcal{T}_J \geq \left(\sum_{s=0}^{n-1} \Delta^s H(c_{s+1})\right) \cdot H(c_1) > H(c_1)^2$.
(ii) if either $\left(\sum_{s=0}^{n-1} \Delta^s H(c_{s+1})\right) \cdot H(c_1) > n \cdot D$ *or* $H(c_1)^2 \geq n \cdot D$*, then J is a singular point in* Hilb^n_D.

Proof. We can use Theorem 3.10 to write $|B_J|$ in terms of H and its derivatives, because H admits the almost revlex ideal J. So, the first inequality of item (i) follows from Corollary 5.4 and Lemma 6.1. For the second inequality it is enough to observe that $|B_J| \geq H(c_1) + n - 1$, where $n - 1$ counts the minimal generators of J that are powers of the variable $x_1, \ldots, x_{n-1}$. For item (ii), thanks to Theorem 5.5 and item (i) we have the thesis. □

As in the previous sections, we denote by $H^{[n]}$ the Hilbert function of the Artinian complete intersection generated by polynomials of degrees $2 \leq d_1 \leq \cdots \leq d_n$ and by $J^{[n]}$ the almost revlex ideal in R such that the Hilbert function of $R/J^{[n]}$ is $H^{[n]}$. In this case, we have $D = \sum_{j\geq 0} H^{[n]}(j) = d_1 \cdots d_n$.

Theorem 6.2 gives a sufficient condition for $J^{[n]}$ to be a singular point of Hilb^n_D, that only involves $H^{[n]}$. We now collect some technical results in order to reach our aim.

Lemma 6.3. *Let $H^{[n]}$ be the Hilbert function of the Artinian complete intersection defined by the positive integers $d_1 \leq \cdots \leq d_n$. Then*

$$H^{[n]}\left(c_1^{[n]}\right) > \frac{d_1 \cdots d_n}{\sum_{i=1}^n d_i}.$$

Proof. From Theorem 2.4, for every $d_1 \leq \cdots \leq d_n$, the value $H^{[n]}\left(c_1^{[n]}\right)$ is the maximum that is assumed by the Hilbert function $H^{[n]}$. Remembering that $(\sum_{i=1}^n d_i) - n$ is the maximum integer at which $H^{[n]}$ assumes a non-null value and $\sum_{j\geq 0} H^{[n]}(j) = d_1 \cdots d_n$, we obtain

$$H^{[n]}\left(c_1^{[n]}\right) \cdot \left(\sum_{i=1}^n d_i - n + 1\right) \geq d_1 \cdots d_n.$$

Using the fact $\sum_{i=1}^n d_i - n + 1 < \sum_{i=1}^n d_i$, we conclude. □

Remark 6.4. We can refine the statement of Lemma 6.3 in the following way. Observe that $H^{[n]}(j) = \binom{n-1+j}{j}$, for every $0 \leq j < d_1$. So, thanks to the symmetry of $H^{[n]}$ we have:

$$H^{[n]}\left(c_1^{[n]}\right) \geq \frac{d_1 \cdots d_n - 2\binom{n+d_1-1}{d_1-1}}{\sum_{i=1}^n d_i - n + 1 - 2d_1}. \tag{6.1}$$

Proposition 6.5. *Let $d_1 \leq \cdots \leq d_n$ be positive integers and $D = d_1 \cdots d_n$. The almost revlex ideal $J^{[n]}$ is a singular point of the Hilbert scheme Hilb^n_D, if either of the following numerical conditions hold*

(i) $d_1 \cdots d_n > n\left(\sum_{i=1}^n d_i\right)^2$;
(ii) $d_1 \cdots d_{n-1} > n^3 d_n$;
(iii) $d_{n-1} = d_n$ *and* $d_1 \cdots d_{n-2} \geq n^3$.

Proof. By Theorem 6.2 it is sufficient to prove that $\left(H^{[n]}\left(c_1^{[n]}\right)\right)^2 > n \cdot D$ if either of (i), (ii) or (iii) holds. If (i) holds, by Lemma 6.3 we immediately have the thesis. If (ii) holds, observing that $(\sum_{i=1}^n d_i)^2 \leq n^2 d_n^2$ and using Lemma 6.3 we have

$$\left(H^{[n]}\left(c_1^{[n]}\right)\right)^2 > \frac{(d_1 \ldots d_{n-1} d_n)^2}{n^2 d_n^2} \geq \frac{d_1 \cdots d_n n^3 d_n^2}{n^2 d_n^2} = n \cdot D.$$

Finally, if (iii) holds, we obtain the thesis by the same arguments of the previous case. □

Corollary 6.6. *For all integers $d_1 \leq \cdots \leq d_{n-1} \leq d_n$, if $d_{n-1} = d_n$ then the almost revlex ideal $J^{[n]}$ corresponds to a singular point in the Hilbert scheme in the following cases*

(i) $d_1 \geq 2$ *and* $n \geq 14$;
(ii) $d_1 \geq 3$ *and* $n \geq 8$;
(iii) $d_1 \geq 4$ *and* $n \geq 6$;
(iv) $d_1 \geq 5$ *and* $n \geq 5$;

(v) $d_1 \geq 8$ *and* $n \geq 4$*;*
(vi) $d_1 \geq 27$ *and* $n \geq 3$.

Proof. By induction on n, we obtain $d_1 \cdots d_{n-2} \geq d_1^{n-2} \geq n^3$ in all the cases that are listed in the statement. Then, it is enough to apply Proposition 6.5(iii). □

Corollary 6.7. *For every* $n \geq 3$ *and* $2 \leq d = d_1 = d_n$*,* $J^{[n]}$ *corresponds to a singular point in the Hilbert scheme* $\mathrm{Hilb}^n_{d^n}$*.*

Proof. For $d = 2, 3, 4$ the statement holds thanks to items (i), (ii), (iii) of Corollary 6.6 and by direct computations that are collected in next Examples 6.8, 6.9, 6.10. For $d \geq 5$ and $n \geq 5$ the statement holds due to item (iv) of Corollary 6.6. So, the case $d \geq 5$ with $n = 3, 4$ remains open.

More precisely, thanks to items (v) and (vi) of Corollary 6.6 we have to focus on $d = 5, 6, 7$ with $n = 4$ and $5 \leq d \leq 26$ with $n = 3$. Except for $d = 5$ with $n = 3$, in each of these cases we obtain the thesis using formula (6.1) and Theorem 6.2. For $d = 5$ with $n = 3$, we compute $|B_{J^{[3]}}| \cdot H^{[3]}(c_1^{[3]}) = 25 \cdot 19 = 475 > 375 = 3 \cdot 5^3$ and conclude by Theorem 6.2 □

Example 6.8. Let $2 = d_1 = d_n$. We show that $J^{[n]}$ is a singular point for every $3 \leq n \leq 13$.

If n is even, then $c_1 = \frac{n}{2}$ and $H^{[n]}(c_1) = \binom{n}{\frac{n}{2}}$ because

t	0	1	2	…	$\frac{n}{2}$	$\frac{n}{2}+1$	…	$n-1$	n
$H^{[n]}(t)$	1	$\binom{n}{1}$	$\binom{n}{2}$	…	$\binom{n}{\frac{n}{2}}$	$\binom{n}{\frac{n}{2}-1}$	…	$\binom{n}{1}$	1

If n is odd, then $c_1 = \frac{n-1}{2}$ and $H^{[n]}(c_1) = \binom{n}{\frac{n-1}{2}}$ because

t	0	1	2	…	$\frac{n-1}{2}$	$\frac{n+1}{2}$	…	$n-1$	n
$H^{[n]}(t)$	1	$\binom{n}{1}$	$\binom{n}{2}$	…	$\binom{n}{\frac{n-1}{2}}$	$\binom{n}{\frac{n-1}{2}}$	…	$\binom{n}{1}$	1

If $6 \leq n \leq 13$, we obtain $H^{[n]}(c_1^{[n]})^2 > n \cdot 2^n$ by an explicit computation.
If $n = 5$, we find $|B_{J^{[5]}}| \cdot H^{[5]}(c_1^{[5]}) = 18 \cdot 10 = 180 > 160 = 5 \cdot 2^5$.
If $n = 4$, we find $|B_{J^{[4]}}| \cdot H^{[4]}(c_1^{[4]}) = 12 \cdot 6 = 72 > 64 = 4 \cdot 2^4$.
If $n = 3$, we have $|B_{J^{[3]}}| \cdot H^{[3]}(c_1^{[3]}) = 6 \cdot 3 = 18 < 3 \cdot 2^3$. Hence, in order to prove that $J^{[3]}$ corresponds to a singular point in the Hilbert scheme $\mathrm{Hilb}^3_{2^3}$, we need to make a direct computation which gives $\dim \mathcal{T}_{J^{[3]}} = 36 > 24 = 3 \cdot 2^3$.

Example 6.9. Let $3 = d_1 = d_n$. For $3 \leq n \leq 7$ we have the following Hilbert functions, respectively:

t	0	1	2	3	4	5	6	7	8	
$H^{[3]}(t)$	1	3	6	7	6	3	1			
$H^{[4]}(t)$	1	4	10	16	19	16	…			
$H^{[5]}(t)$	1	5	15	30	45	51	45	…		
$H^{[6]}(t)$	1	6	21	50	90	126	141	126	…	
$H^{[7]}(t)$	1	7	28	77	161	266	357	393	357	…

If $n = 3$, then $H^{[3]}(c_1) = 7$, $\Delta H^{[3]}(c_2) = 3$, $\Delta^2 H^{[3]}(c_3) = 1$ and $(7 + 3 + 1) \cdot 7 = 77 < 3 \cdot 3^3$, so that in this case we cannot apply Theorem 6.2. Nevertheless, we conclude that $J^{[3]}$ corresponds to a singular point of the Hilbert scheme Hilb^3_{27} because by a direct computation we obtain $\dim \mathcal{T}_J = 147 > 3 \cdot 27$.
If $n = 4$, then $H^{[4]}(c_1)^2 = 19^2 > 324 = 4 \cdot 3^4$.
If $n = 5$, then $H^{[5]}(c_1)^2 = 51^2 > 1215 = 5 \cdot 3^5$.

If $n = 6$, then $H^{[6]}(c_1)^2 = 141^2 > 4374 = 6 \cdot 3^6$.
If $n = 7$, then $H^{[7]}(c_1)^2 = 393^2 > 7 \cdot 3^7$.
In conclusion, if $3 = d_1 = d_n$ and $3 \leq n \leq 7$, then $J^{[n]}$ is a singular point thanks to either Theorem 6.2 or a direct computation of the dimension of the Zariski tangent space.

Example 6.10. Let $4 = d_1 = d_n$. For $n = 3, 4, 5$ we have the following Hilbert functions, respectively:

t	0	1	2	3	4	5	6	7	8	9	...
$H^{[3]}(t)$	1	3	6	10	12	12	10	...			
$H^{[4]}(t)$	1	4	10	20	31	40	44	40	...		
$H^{[5]}(t)$	1	5	15	35	65	101	135	155	155	135	...

If $n = 3$, then $H^{[3]}(c_1) = 12$, $\Delta H^{[3]}(c_2) = 4$, $\Delta^2 H^{[3]}(c_3) = 1$ and $(12+4+1)12 > 192 = 3 \cdot 4^3$.
If $n = 4$, then $H^{[4]}(c_1)^2 = 44^2 > 1024 = 4 \cdot 4^4$.
If $n = 5$, then $H^{[5]}(c_1)^2 = 155^2 > 5 \cdot 4^5$.
In conclusion, if $4 = d_1 = d_n$ and $n = 3, 4, 5$, $J^{[n]}$ is a singular point due to Theorem 6.2.

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