

Resolutions of non-regular Ricci-flat Kähler cones

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ABSTRACT

We present explicit constructions of complete Ricci-flat Kähler metrics that are asymptotic to cones over non-regular Sasaki–Einstein manifolds. The metrics are constructed from a complete Kähler–Einstein manifold (V, g_V) of positive Ricci curvature and admit a Hamiltonian two-form of order two. We obtain Ricci-flat Kähler metrics on the total spaces of (i) holomorphic $\mathbb{C}^2/\mathbb{Z}_p$ orbifold fibrations over V , (ii) holomorphic orbifold fibrations over weighted projective spaces \mathbb{WCP}^1 , with generic fibres being the canonical complex cone over V , and (iii) the canonical orbifold line bundle over a family of Fano orbifolds. As special cases, we also obtain smooth complete Ricci-flat Kähler metrics on the total spaces of (a) rank two holomorphic vector bundles over V , and (b) the canonical line bundle over a family of geometrically ruled Fano manifolds with base V . When $V = \mathbb{CP}^1$ our results give Ricci-flat Kähler orbifold metrics on various toric partial resolutions of the cone over the Sasaki–Einstein manifolds $Y^{p,q}$.

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1. Introduction and summary

1.1. Introduction

A Sasaki–Einstein manifold (L, g_L) is a complete Riemannian manifold whose metric cone

$$C(L) = \mathbb{R}_+ \times L, \quad g_{C(L)} = dr^2 + r^2 g_L \quad (1.1)$$

is Ricci-flat Kähler. The metric in (1.1) is singular at $r = 0$, unless (L, g_L) is the round sphere, and it is natural to ask whether there exists a resolution i.e. a complete Ricci-flat Kähler metric on a non-compact manifold X which is asymptotic to the cone (1.1). More generally, one can consider *partial* resolutions in which X also has singularities. There are particularly strong physical motivations for studying such partial resolutions; for example, certain types of orbifold singularity are well-studied in String Theory, and may give rise to interesting phenomena, such as non-abelian gauge symmetry.

These geometrical structures are of particular interest in the AdS/CFT correspondence [1]. In complex dimension three or four, a Ricci-flat Kähler cone $C(L)$ is AdS/CFT dual to a supersymmetric conformal field theory in dimension four or three, respectively. Resolutions of such conical singularities are then of interest for a number of different physical applications. For example, in AdS/CFT such resolutions correspond to certain deformations of the conformal field theory.

On a Kähler cone $(C(L), g_{C(L)})$ there is a canonically defined vector field, the Reeb vector field:

$$\xi = J \left(r \frac{\partial}{\partial r} \right) \quad (1.2)$$

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where J denotes the complex structure tensor on the cone. ξ is a holomorphic Killing vector field, and has unit norm on the link $L = \{r = 1\}$ of the singularity at $r = 0$. If the orbits of ξ all close then ξ generates a $U(1)$ isometry of (L, g_L) , which necessarily acts locally freely since ξ is nowhere zero, and the Sasakian structure is said to be either regular or quasi-regular if this action is free or not, respectively. The orbit space is in general a Kähler–Einstein orbifold (M, g_M) of positive Ricci curvature, which is a smooth manifold in the regular case. More generally, the orbits of ξ need not all close, in which case the Sasakian structure is said to be irregular.

Suppose that (L, g_L) is a regular Sasaki–Einstein manifold. In this case L is a $U(1)$ fibration over a Kähler–Einstein manifold (M, g_M) , which we assume² is simply-connected. Let K_M denote the canonical line bundle of M , and let I denote the Fano index of M . The latter is the largest positive integer such that

$$K_M^{-1/I} \in \text{Pic}(M) = H^2(M; \mathbb{Z}) \cap H^{1,1}(M; \mathbb{C}). \quad (1.3)$$

It is then well-known that the simply-connected cover of L is diffeomorphic to the unit circle bundle in the holomorphic line bundle $K_M^{1/I}$. Taking the quotient of L by $\mathbb{Z}_m \subset U(1)$ gives instead a smooth Sasaki–Einstein manifold diffeomorphic to the unit circle bundle in $K_M^{m/I}$. For example, suppose (M, g_M) is \mathbb{CP}^2 equipped with its Fubini–Study metric. Then the Fano index is $I = 3$, and the canonical line bundle is $K_{\mathbb{CP}^2} = \mathcal{O}(-3)$. The total space of the associated circle bundle is thus S^5/\mathbb{Z}_3 , whereas the simply-connected cover of (L, g_L) is S^5 equipped with its round metric.

When $m = I$, there is a canonical way of resolving the above Ricci-flat Kähler cone: there exists a smooth complete Ricci-flat Kähler metric on the total space of the canonical line bundle K_M over M . The metric is in fact explicit, up to the Kähler–Einstein metric g_M on M , and is constructed using the Calabi ansatz [3]. These metrics were constructed in the mathematics literature in [4], and in the physics literature in [5]. More generally, there may exist other resolutions. The simplest example is perhaps given by $M = \mathbb{CP}^1 \times \mathbb{CP}^1$ (also known as zeroth Hirzebruch surface, denoted \mathbb{F}_0) with its standard Kähler–Einstein metric. Here $I = 2$, and for $m = 2$ the construction of [4,5] produces a complete metric on the total space of $K_{\mathbb{F}_0}$, which is asymptotic to a cone over the homogeneous Sasaki–Einstein manifold $T^{1,1}/\mathbb{Z}_2$. On the other hand, the cone over the Sasaki–Einstein manifold with $m = 1$ instead has a small resolution: there is a smooth complete Ricci-flat Kähler metric on the total space of the rank two holomorphic vector bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over \mathbb{CP}^1 , which is asymptotic to a cone over $T^{1,1}$. This is known in the physics literature as the resolved conifold metric [6].

More generally, there are the existence results of Tian and Yau [7,8]. In the latter reference it is proven that, under certain mild assumptions, $X = \bar{X} \setminus D$ admits a complete Ricci-flat Kähler metric that is asymptotic to a cone, provided that the divisor $D \subset \bar{X}$ in the compact Kähler manifold (or orbifold) \bar{X} admits a Kähler–Einstein metric of positive Ricci curvature. These metrics are therefore also asymptotic to cones over regular, or quasi-regular, Sasaki–Einstein manifolds. However, the metrics that we shall present in this paper lie outside this class, and their existence was not guaranteed by any theorem.

In [9–11] infinite families of explicit Sasaki–Einstein manifolds were constructed, in all odd dimensions, in both the quasi-regular and irregular classes. In particular, these were the first examples of irregular Sasaki–Einstein manifolds. The construction produces, for each complete Kähler–Einstein manifold (V, g_V) of positive Ricci curvature, an infinite family $Y^{p,k}(V)$ of associated complete Sasaki–Einstein manifolds. Here p and k are positive integers satisfying $pl/2 < k < pl$, where l is the Fano index of V . Given the above results, it is natural to investigate whether or not there exist resolutions of the corresponding Ricci-flat Kähler cones. In fact examples of such resolutions have recently been constructed in [12,13]. In this paper we significantly generalise these results; the results of [12,13] are recovered by substituting $(V, g_V) = \mathbb{CP}^1$ (in particular, $l = 2, m = 1$) with its standard metric, or $(V, g_V) = \text{product of complex projective spaces}$, into Corollary 1.5, respectively.

Quite recently, Futaki [14] has used the Calabi ansatz to construct complete Ricci-flat Kähler metrics on the canonical line bundles (i.e. $m = I$, in the above notation) over toric Fano manifolds. A key point in the construction is the general existence result of [15] for toric Sasaki–Einstein metrics on links of isolated toric Gorenstein singularities.

1.2. Summary

Our constructions are based on a class of explicit local Kähler metrics that have appeared recently in the mathematics literature [16–18] and have been independently discovered in the physics literature in [19]. The metrics we study all admit a *Hamiltonian two-form*, in the sense of [16], of order two. As noted in [16], the Calabi ansatz is a special case of a local Kähler metric admitting a Hamiltonian two-form of order one. More generally, a Kähler metric admitting a Hamiltonian two-form of order one locally fibres over a product of Kähler manifolds: this ansatz was in fact used in the paper [20] to construct complete Ricci-flat Kähler metrics on various holomorphic vector bundles over products of Kähler–Einstein manifolds; the asymptotic cones are again all regular, however. For simplicity, we study here only a single Kähler–Einstein manifold (V, g_V) , rather than a product of Kähler–Einstein manifolds³. The local metrics depend on two real parameters. In Sections 2.3 and 3 we establish that it is possible to choose one metric parameter ν in such a way that the metric asymptotes to a cone over one of the non-regular Sasaki–Einstein metrics constructed in [11]; there are a countably infinite number of

² $b_1(M) = 0$ necessarily [2].

³ The product case was in fact discussed briefly in [11], with some global analysis of the corresponding Sasaki–Einstein metrics appearing in [21].

such choices for ν . The remainder of the paper is then devoted to analysing in detail the various choices for the second metric parameter μ . We obtain Ricci-flat Kähler metrics on *partial resolutions*, with singularities that we carefully describe, as well as various *smooth complete Ricci-flat Kähler* metrics, that provide distinct resolutions of the conical singularities. When the Fano manifold (V, g_V) is toric, the resolutions we construct are all toric resolutions. In particular, when $V = \mathbb{CP}^1$ equipped with its standard round metric, our results may be described in terms of the toric geometry of the family $\mathcal{C}(Y^{p,q})$ of isolated toric Gorenstein singularities [22]. Such a description, together with the AdS/CFT interpretation of the metrics constructed here, will appear elsewhere [23].

In Section 4 we investigate two classes of (partial) resolution that we shall refer to as *small resolutions*. This nomenclature is motivated by the fact that these metrics may be thought of as two different generalisations of the resolved conifold metric on $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{CP}^1$. First, we obtain complete Ricci-flat Kähler orbifold metrics on the total spaces of certain holomorphic $\mathbb{C}^2/\mathbb{Z}_p$ fibrations over V . When $p = 1$ this leads to smooth Ricci-flat Kähler metrics on the total spaces of certain rank two holomorphic vector bundles over the Fano V , as summarised in Corollary 1.2 below. For instance, taking $V = \mathbb{CP}^2$ with its standard Kähler–Einstein metric, we obtain a smooth complete metric on the total space of the rank two holomorphic vector bundle $\mathcal{O}(-2) \oplus \mathcal{O}(-1) \rightarrow \mathbb{CP}^2$. On the other hand, these results also produce an infinite family of *partial* small resolutions of the cones over the non-regular Sasaki–Einstein manifolds $Y^{p,q}$ [10]. The resolution is in general only partial since the normal fibre to the blown-up \mathbb{CP}^1 is $\mathbb{C}^2/\mathbb{Z}_p$. The fibres are non-trivially twisted over \mathbb{CP}^1 , with the form of the twisting depending on the integer q . When $p = 1$ we recover precisely the resolved conifold metric. In Section 4.3 we will describe a second type of partial resolution, where one instead blows up a weighted projective space. The partial resolution is a fibration over this weighted projective space with generic fibres being the singular canonical complex cone over the Fano V (cf. Theorem 1.3 below). In particular, when $V = \mathbb{CP}^1$, the latter fibres are simply copies of $\mathbb{C}^2/\mathbb{Z}_2$, which is the blow-down of $\mathcal{O}(-2) \rightarrow \mathbb{CP}^1$. More precisely, in this case we obtain a $\mathbb{C}^2/\mathbb{Z}_2$ orbifold fibration over $\mathbb{WCP}_{[d,p-d]}^1$, where $d = k/2$ implies that $k = p + q$ must be even.

In Section 5 we investigate a class of complete Ricci-flat Kähler orbifold metrics on the total space of the canonical line bundle over a family of Fano orbifolds. These are a direct generalisation of the work of [4,5], which was based on the Calabi ansatz. Our Fano orbifolds are $\mathbb{WCP}_{[r,p-r]}^1$ fibrations over V , where $0 < r < k/l$. The induced orbifold metric on M , which is the zero-section of the canonical line bundle, is Kähler, but (M, g_M) is not Kähler–Einstein. M is smooth if and only if $p = 2$, $r = 1$ and in this case M is a \mathbb{CP}^1 fibration over the Fano V , of the form $M = \mathbb{P}_V(\mathcal{O} \oplus K_V^{m/l})$ where $0 < m < l$. For instance, when $V = \mathbb{CP}^1$, M is the first del Pezzo surface, which is well known to have non-vanishing Futaki invariant, the latter being an obstruction to the existence of a Kähler–Einstein metric. More generally we obtain smooth complete metrics on the total space of the canonical line bundle K_M over M , generalising [4,5] to the case of non-regular Sasaki–Einstein boundaries.

We summarise our results more formally by the following Theorems. Note that for general p and k the Sasaki–Einstein manifolds $Y^{p,k}(V)$ [11] are irregular:

Theorem 1.1. *Let (V, g_V) be a complete Kähler–Einstein manifold of positive Ricci curvature with canonical line bundle K_V and Fano index l . Then for every $p, k \in \mathbb{N}$ positive integers with $pl/2 < k < pl$ there is an explicit complete Ricci-flat Kähler orbifold metric on the total space of a $\mathbb{C}^2/\mathbb{Z}_p$ bundle over V . Here $\mathbb{Z}_p \subset U(1) \subset SU(2)$ acts on \mathbb{C}^2 in the standard way, and the bundle is given by*

$$\left[K_V \oplus K_V^{k/l} \right] \times_{\lambda} \mathbb{C}^2/\mathbb{Z}_p \quad (1.4)$$

where

$$\begin{aligned} \lambda : S^1 \times S^1 \times \mathbb{C}^2/\mathbb{Z}_p &\rightarrow \mathbb{C}^2/\mathbb{Z}_p \\ (\theta_1, \theta_2; z_1, z_2) &\mapsto (\exp(i\theta_1 - i\theta_2/p)z_1, \exp(i\theta_2/p)z_2) \end{aligned} \quad (1.5)$$

and z_1, z_2 are standard complex coordinates on \mathbb{C}^2 . The metric asymptotes to a cone over the Sasaki–Einstein manifold $Y^{p,k}(V)$.

When $p = 1$ we obtain a finite number of completely smooth resolutions, for each (V, g_V) . These may be regarded as higher-dimensional versions of the small resolution of the conifold, which are asymptotic to non-regular Ricci-flat Kähler cones. Setting $p = 1$, $m = l - k$ in Theorem 1.1 gives

Corollary 1.2. *Let (V, g_V) be a complete Kähler–Einstein manifold of positive Ricci curvature with canonical line bundle K_V and Fano index l . Then for every $m \in \mathbb{N}$ with $0 < m < l/2$ there is an explicit smooth complete asymptotically conical Ricci-flat Kähler metric on the total space of the rank two holomorphic vector bundle $K_V^{m/l} \oplus K_V^{(l-m)/l}$ over V . The metric asymptotes to a cone over the Sasaki–Einstein manifold $Y^{1,l-m}(V)$.*

We also obtain

Theorem 1.3. *Let (V, g_V) be a complete Kähler–Einstein manifold of positive Ricci curvature with canonical line bundle K_V and Fano index l . Then for each $p, d \in \mathbb{N}$ with $p/2 < d < p$ there is an explicit complete Ricci-flat Kähler orbifold metric on the total*

space of the canonical complex cone \mathbb{C}_V over V , fibred over the weighted projective space $\mathbb{WCP}_{[d,p-d]}^1$. The fibration structure is given by the orbifold fibration

$$K_{\mathbb{WCP}_{[d,p-d]}^1} \times_{U(1)} \mathbb{C}_V. \quad (1.6)$$

Here the $U(1) \subset \mathbb{C}^*$ action is the standard one on the complex cone \mathbb{C}_V . The metric is completely smooth away from the tip of the complex cone fibres, and asymptotes to a cone over the Sasaki–Einstein manifold $Y^{p,k}(V)$.

In Section 5 we prove

Theorem 1.4. *Let (V, g_V) be a complete Kähler–Einstein manifold of positive Ricci curvature with canonical line bundle K_V and Fano index I . Then for every $p, k, r \in \mathbb{N}$ positive integers with $p/2 < k/I < p$, $0 < r < k/I$, there is an explicit smooth complete Ricci-flat Kähler orbifold metric on the total space of the canonical line bundle K_M over the Fano orbifold*

$$M = K_V^{m/I} \times_{U(1)} \mathbb{WCP}_{[r,p-r]}^1, \quad (1.7)$$

where $m = k - rI$. Here we use the standard effective action of $U(1)$ on the weighted projective space $\mathbb{WCP}_{[r,p-r]}^1$, with orientation fixed so that the section with normal fibre $\mathbb{C}/\mathbb{Z}_{p-r}$ has normal bundle $K_V^{m/I}$. The metric asymptotes, for every r , to a cone over the Sasaki–Einstein manifold $Y^{p,k}(V)$.

Setting $p = 2$, $r = 1$ in Theorem 1.4 effectively blows up the zero section of the orbifold metric in Theorem 1.1 to again obtain a finite number of completely smooth resolutions, for each (V, g_V) :

Corollary 1.5. *Let (V, g_V) be a complete Kähler–Einstein manifold of positive Ricci curvature with canonical line bundle K_V and Fano index I . Then for each $m \in \mathbb{N}$ with $0 < m < I$ there is an explicit smooth complete Ricci-flat Kähler metric on the total space of the canonical line bundle K_M over the geometrically ruled Fano manifold $M = \mathbb{P}_V(\mathcal{O} \oplus K_V^{m/I})$. The metric asymptotes to a cone over the Sasaki–Einstein manifold $Y^{2,m+I}(V)$.*

We note that $I \leq n + 1$ with equality if and only if $V = \mathbb{CP}^n$. In fact also $I = n$ if and only if $V = \mathbb{Q}^n$ is the quadric in \mathbb{CP}^{n+1} —see, for example, [24]. Both of these examples admit homogeneous Kähler–Einstein metrics.

2. Local metrics

In this section we introduce the class of explicit local Kähler metrics that we wish to study. These metrics all admit a Hamiltonian two-form [16]. In Section 2.1 we give a brief review of local Kähler metrics admitting Hamiltonian two-forms, focusing on the relevant cases of order one and order two, and present the local form of the metrics used throughout the remainder of the paper. In Section 2.2 we introduce local complex coordinates. Finally, Section 2.3 demonstrates that, in a certain limit, the local metrics are asymptotic to a cone over the local class of Sasaki–Einstein metrics studied in [11].

2.1. Kähler metrics with Hamiltonian two-forms

If (X, g, J, ω) is a Kähler structure, then a Hamiltonian two-form ϕ is a real $(1, 1)$ -form that solves non-trivially the equation [16]

$$\nabla_Y \phi = \frac{1}{2} (d \operatorname{tr}_\omega \phi \wedge JY^\flat - J d \operatorname{tr}_\omega \phi \wedge Y^\flat). \quad (2.1)$$

Here Y is any vector field, ∇ denotes the Levi-Civita connection, and $Y^\flat = g(Y, \cdot)$ is the one-form dual to Y .

The key result for our purposes is that the existence of ϕ leads to an ansatz for the Kähler metric g such that Ricci-flatness is equivalent to solving a simple set of decoupled ordinary differential equations. We therefore merely sketch the basic ideas that lead to this result; for a full exposition on Hamiltonian two-forms, the reader is referred to [16]. We note that many of these ansätze had been arrived at prior to the work of [16], both in the mathematics literature (as pointed out in [16]), and also in the physics literature. The theory of Hamiltonian two-forms unifies these various approaches.

One first notes that if ϕ is a Hamiltonian two-form, then so is $\phi_t = \phi - t\omega$ for any $t \in \mathbb{R}$. One then defines the momentum polynomial of ϕ to be

$$p(t) = \frac{(-1)^N}{N!} * \phi_t^N. \quad (2.2)$$

Here N is the complex dimension of the Kähler manifold and $*$ is the Hodge operator with respect to the metric g . It is then straightforward to show that $\{p(t)\}$ are a set of Poisson-commuting Hamiltonian functions for the one-parameter family of Killing vector fields $K(t) = J \operatorname{grad}_g p(t)$. For a fixed point in the Kähler manifold, these Killing vectors will span a vector subspace of the tangent space of the point; the maximum dimension of this subspace, taken over all points, is called the order s of ϕ . This leads to a Hamiltonian \mathbb{T}^s action, at least locally, on the Kähler manifold, and one may then take a (local)

Kähler quotient by this torus action. The reduced Kähler metric depends on the moment map level at which one reduces, but only very weakly: the reduced Kähler metric is a direct product of S Kähler manifolds $(V_i, c_i(\mu)g_{V_i})$, $i = 1, \dots, S$, where $c_i(\mu)$ are functions of the moment map coordinates μ . The $2s$ -dimensional fibres turn out to be *orthotoric*, which is a rather special type of toric Kähler structure. For further details, we refer the reader to reference [16].

The simplest non-trivial case is a Hamiltonian two-form of order one, with $S = 1$. This turns out to be precisely the *Calabi ansatz* [3]. The local metric and Kähler form may be written in the form

$$\begin{aligned} g &= (\beta - y)g_V + \frac{dy^2}{4Y(y)} + Y(y)(d\psi + A)^2, \\ \omega &= (\beta - y)\omega_V - \frac{1}{2}dy \wedge (d\psi + A). \end{aligned} \quad (2.3)$$

Here A is a local one-form on the Kähler manifold (V, g_V, ω_V) satisfying $dA = 2\omega_V$. The Killing vector field $\partial/\partial\psi$ generates the Hamiltonian action, with Hamiltonian function y . The momentum polynomial is given by $p(t) = (t - y)(t - \beta)^{N-1}$ where $\beta \in \mathbb{R}$ is a constant. Calabi used this ansatz to produce an explicit family of so-called extremal Kähler metrics on the blow-up of \mathbb{CP}^2 at a point. One of these metrics is conformal to Page's Einstein metric [25], which is perhaps more well-known to physicists. The same ansatz was used in [4,5] to produce explicit constructions of complete non-compact Kähler metrics; indeed, this leads to the construction of complete Ricci-flat Kähler metrics on K_M , where (M, g_M) is a complete Kähler–Einstein manifold of positive Ricci curvature. The general form of a Kähler metric with a Hamiltonian two-form of order one allows one to replace (V, g_V) by a direct product of $S > 1$ Kähler manifolds, as mentioned above. In fact precisely this ansatz was used in section 5 of [20], before the work of [16], to produce a number of examples of complete non-compact Ricci-flat Kähler manifolds. The same general form was thoroughly investigated in [18], and used to give explicit constructions of compact extremal Kähler manifolds.

In this paper we study the case of a Hamiltonian two-form of order two, with $S = 1$. A Kähler structure (X, g, ω) admitting such a two-form may be written in the form

$$\begin{aligned} g &= \frac{(\beta - x)(\beta - y)}{\beta}g_V + \frac{y - x}{4X(x)}dx^2 + \frac{y - x}{4Y(y)}dy^2 \\ &\quad + \frac{X(x)}{y - x} \left[d\tau + \frac{\beta - y}{\beta}(d\psi + A) \right]^2 + \frac{Y(y)}{y - x} \left[d\tau + \frac{\beta - x}{\beta}(d\psi + A) \right]^2, \end{aligned} \quad (2.4)$$

$$\omega = \frac{(\beta - x)(\beta - y)}{\beta}\omega_V - \frac{1}{2}dx \wedge \left[d\tau + \frac{\beta - y}{\beta}(d\psi + A) \right] - \frac{1}{2}dy \wedge \left[d\tau + \frac{\beta - x}{\beta}(d\psi + A) \right]. \quad (2.5)$$

Here (V, g_V, ω_V) is again a Kähler manifold with, locally, $dA = 2\omega_V$. The momentum polynomial is now given by $p(t) = (t - x)(t - y)(t - \beta)^n$, where we denote $n = \dim_{\mathbb{C}} V = N - 2$. The Hamiltonian action is generated by the Killing vector fields $\partial/\partial\tau, \partial/\partial\psi$.

A computation shows that the metric (2.4) is Ricci-flat if (V, g_V) is a Kähler–Einstein manifold of positive⁴ Ricci curvature and the metric functions are given by

$$\begin{aligned} X(x) &= \beta(x - \beta) + \frac{n+1}{n+2}c(x - \beta)^2 + \frac{2\mu}{(x - \beta)^n} \\ Y(y) &= \beta(\beta - y) - \frac{n+1}{n+2}c(\beta - y)^2 - \frac{2\nu}{(\beta - y)^n}. \end{aligned} \quad (2.6)$$

Here β, c, μ and ν are real constants and, without loss of generality, we have normalised the metric g_V so that $\text{Ric}_V = 2(n+1)g_V$.

Note that, provided $\beta \neq 0$, one may define $x = \beta\hat{x}$, $y = \beta\hat{y}$, multiply g by $1/\beta$, and then relabel $\hat{x} = x$, $\hat{y} = y$ to obtain (2.4) with $\beta = 1$. Similarly, provided $c \neq 0$, one may define $x' = 1 + c(x - 1)$, $y' = 1 + c(y - 1)$, $\tau' = c\tau$, multiply g by c^2 , and then relabel $x' = x$, $y' = y$, $\tau' = \tau$ to obtain (2.4) with $c = 1$. The cases $c = 0$ and $\beta = 0$ (accompanied by a suitable scaling of the coordinates) are treated in the Appendix, where the parameter β is also further discussed. Henceforth we set $\beta = c = 1$.

2.2. Complex structure

In this section we introduce a set of local complex coordinates on the (local) Kähler manifold (X, g, ω) . We first define the complex one-forms

$$\begin{aligned} \eta_1 &= \frac{dx}{2X(x)} - \frac{dy}{2Y(y)} - id\psi \\ \eta_2 &= \frac{1-x}{2X(x)}dx - \frac{1-y}{2Y(y)}dy + id\tau. \end{aligned} \quad (2.7)$$

⁴ More generally one might also consider zero or negative Ricci curvature.

The following is then a closed $(n + 2, 0)$ -form:

$$\Omega = \kappa \sqrt{X(x)Y(y)} [(1-x)(1-y)]^{n/2} (\eta_1 - iA) \wedge \eta_2 \wedge \Omega_V \quad (2.8)$$

where

$$\kappa = \exp[i(n+1)(\tau + \psi)] \quad (2.9)$$

and Ω_V is the n -form on V satisfying

$$d\Omega_V = i(n+1)A \wedge \Omega_V. \quad (2.10)$$

More precisely, we may introduce local complex coordinates z_1, \dots, z_n on V and locally write

$$\Omega_V = f_V dz_1 \wedge \dots \wedge dz_n. \quad (2.11)$$

Globally, f_V is a holomorphic section of the anti-canonical line bundle of V ; on the overlaps of local complex coordinate patches this transforms oppositely to $dz_1 \wedge \dots \wedge dz_n$, giving a globally defined n -form Ω_V on V . So $f_V \in H^0(V, K_V^{-1})$. The holomorphicity of f_V may be seen by comparing with (2.10), which implies

$$[d \log f_V - i(n+1)A] \wedge \Omega_V = 0. \quad (2.12)$$

The local one-form $(n+1)A$ is a connection on the holomorphic line bundle K_V^{-1} , since $(n+1)dA = \rho_V$ is the Ricci form of V . Eq. (2.12) then says that f_V is a holomorphic section. Note that Ω_V has constant norm, even though f_V necessarily has zeroes on V .

We may then introduce the local complex coordinates

$$\begin{aligned} Z_1 &= \exp \left[-i\psi + \int \frac{dx}{2X(x)} - \frac{dy}{2Y(y)} \right] f_V^{-1/(n+1)} \\ Z_2 &= \exp \left[i\tau + \int \frac{(1-x)dx}{2X(x)} - \frac{(1-y)dy}{2Y(y)} \right] \end{aligned} \quad (2.13)$$

satisfying

$$d \log Z_1 = \eta_1 - \frac{1}{(n+1)} d \log f_V, \quad d \log Z_2 = \eta_2. \quad (2.14)$$

2.3. Asymptotic structure

The metric (2.4) is symmetric in x and y . We shall later break this symmetry by choosing one coordinate to be a radial coordinate and the other to be a polar coordinate. Without loss of generality, we may take x to be the radial coordinate. We analyse the metric in the limit $x \rightarrow \pm\infty$. Setting

$$x = \pm \frac{n+1}{n+2} r^2 \quad (2.15)$$

we obtain $\mp g \rightarrow dr^2 + r^2 g_L$ where g_L is the Sasaki–Einstein metric

$$g_L = \left(\frac{n+1}{n+2} d\tau + \sigma \right)^2 + g_T. \quad (2.16)$$

Note in particular that for $x \rightarrow \infty$ it is $-g$ that is positive definite. In (2.16) we have defined

$$\sigma = \frac{n+1}{n+2} (1-y)(d\psi + A) \quad (2.17)$$

with $d\sigma = 2\omega_T$, and g_T is a local Kähler–Einstein metric given by

$$g_T = \frac{n+1}{n+2} \left[(1-y)g_V + \frac{dy^2}{4Y(y)} + Y(y)(d\psi + A)^2 \right]. \quad (2.18)$$

Note this is of the Calabi form (2.3). The vector field

$$\frac{n+2}{n+1} \frac{\partial}{\partial \tau} \quad (2.19)$$

is thus asymptotically the Reeb vector field: locally the metric (2.4) asymptotes, for large $\pm x$, to a cone over the local Sasaki–Einstein metric of [11].

3. Global analysis: $\pm x > \pm x_{\pm}$

We begin by making the following change of angular coordinates

$$\tau = -\alpha, \quad \psi = \alpha + \frac{\gamma}{n+1}. \quad (3.1)$$

The metric (2.4) becomes

$$g = (1-x)(1-y)g_V + \frac{y-x}{4X(x)}dx^2 + \frac{y-x}{4Y(y)}dy^2 + \frac{v(x,y)}{(n+1)^2} [d\gamma + (n+1)A]^2 \\ + w(x,y) \left[d\alpha + \frac{f(x,y)}{n+1} [d\gamma + (n+1)A] \right]^2 \quad (3.2)$$

where we have defined

$$w(x,y) = \frac{1}{y-x} [y^2X(x) + x^2Y(y)] \quad (3.3)$$

$$f(x,y) = 1 - \frac{[yX(x) + xY(y)]}{y^2X(x) + x^2Y(y)} \quad (3.4)$$

$$v(x,y) = \frac{X(x)Y(y)}{w(x,y)}. \quad (3.5)$$

The strategy for extending the local metric (2.4) to a complete metric on a non-compact manifold is as follows. We shall take

$$y_1 \leq y \leq y_2 \quad (3.6)$$

where y_1, y_2 are two appropriate adjacent zeroes of $Y(y)$ satisfying

$$y_1 < y_2 < 1. \quad (3.7)$$

On the other hand, we take x to be a non-compact coordinate, with

$$-\infty < x \leq x_- \leq y_1, \quad \text{or} \quad 1 \leq x_+ \leq x < +\infty. \quad (3.8)$$

Here x_- is the smallest zero of $X(x)$ and x_+ is the largest zero; thus $X(x) > 0$ for all $x < x_-$ or $x > x_+$. First, we examine regularity of the metric (3.2) for $\pm x > \pm x_{\pm}$. Following a strategy similar to [10,11], we show that the induced metric at any constant x , that is not a zero of $X(x)$, may be extended to a complete metric on the total space of a $U(1)$ principal bundle (with local fibre coordinate $\tilde{\alpha}$) over a smooth compact base space $Z(V)$. In particular, this will fix the metric parameter v . The analysis essentially carries over from that presented in [11]. Note, however, that the results of Section 3.3 complete the discussion in reference [11]. The remaining sections of the paper will deal with regularity of the metric at $x = x_{\pm}$.

3.1. Zeroes of the metric functions

Recall that

$$Y(y) = \frac{p(y)}{(1-y)^n} \quad (3.9)$$

where

$$p(y) = (1-y)^{n+1} - \frac{(n+1)}{(n+2)}(1-y)^{n+2} - 2v. \quad (3.10)$$

One easily verifies that $p'(y) = 0$ if and only if $y = 0$ or $y = 1$. The former is a local maximum of $p(y)$, whereas the latter is a local minimum or a point of inflection depending on whether n is odd or even, respectively. Defining

$$v_{\max} = \frac{1}{2(n+2)} \quad (3.11)$$

we also see that $p(0) \leq 0$ for $v \geq v_{\max}$ and $p(1) \geq 0$ for $v \leq 0$. Since for regularity we require two adjacent real zeroes y_1, y_2 of $Y(y)$, with $1 \notin (y_1, y_2)$, it follows that we must take

$$0 \leq v \leq v_{\max}. \quad (3.12)$$

Since $p'(0) = 0$, the roots then satisfy $y_1 \leq 0, y_2 \geq 0$, and we have $Y(y) > 0$ for $y \in (y_1, y_2)$. We also note that for any zero y_i of $Y(y)$ we have

$$Y'(y_i) = -(n+1)y_i. \quad (3.13)$$

The metric with $\nu = 0$ is the local Ricci-flat Kähler metric one obtains by using the Calabi ansatz with local Kähler–Einstein metric g_T in (2.18). Thus this metric admits a Hamiltonian two-form of order one, with $S = 1$. The local metric g_T extends to a smooth Kähler–Einstein metric on a complete manifold only when $V = \mathbb{CP}^n$, in which case g_T is the Kähler–Einstein metric on \mathbb{CP}^{n+1} . More generally, the quasi-regular Sasaki–Einstein metrics constructed in [11] lead to smooth complete orbifold metrics. For $\nu = \nu_{\max}$ one finds that $y_1 = y_2 = 0$. One can verify that the metric also reduces to the Calabi ansatz in this limit, with the product Kähler–Einstein metric on $\mathbb{CP}^1 \times V$ as base. Hence one obtains a complete Ricci-flat Kähler metric on the canonical line bundle over $\mathbb{CP}^1 \times V$. Henceforth we take $\nu \in (0, \nu_{\max})$.

We now turn to an analysis of the zeroes of $X(x)$. Recall that

$$X(x) = \frac{q(x)}{(x-1)^n} \quad (3.14)$$

where

$$q(x) = (x-1)^{n+1} + \frac{(n+1)}{(n+2)}(x-1)^{n+2} + 2\mu. \quad (3.15)$$

Any root $x_0(\mu)$ of $q(x)$ therefore satisfies

$$\frac{dx_0}{d\mu} = -\frac{2}{(n+1)x_0(x_0-1)^n}. \quad (3.16)$$

Note that $x_0 = 0$ is a root of $q(x)$ when

$$\mu = \bar{\mu} = \frac{(-1)^n}{2(n+2)} \quad (3.17)$$

and that $x_0 = 1$ is a root of $q(x)$ when $\mu = 0$. In our later analysis we shall require there to exist either a smallest zero x_- of $X(x)$, with $x_- \leq y_1 < 0$, or a largest zero x_+ , with $1 \leq x_+$. It is easy to see that $q'(x) = 0$ if and only if $x = 0$ or $x = 1$. The former is local maximum for n odd and a local minimum for n even, while the latter is local minimum for n odd and a point of inflection for n even. The behaviour of x_- is summarised by the following

Lemma 3.1. *For each $x_- \in (-\infty, 0]$ there exists a unique μ such that x_- is the smallest zero of $X(x)$. Moreover, $x_-(\mu)$ is monotonic.*

Proof. The proof depends on the parity of n . For n odd, $x_- \rightarrow -\infty$ as $\mu \rightarrow \infty$. Since $x_-(\bar{\mu}) = 0$, (3.16) shows that $x_-(\mu)$ is monotonic decreasing in $[\bar{\mu}, \infty)$. For n even, instead $x_- \rightarrow -\infty$ as $\mu \rightarrow -\infty$. Eq. (3.16) now shows that $x_-(\mu)$ is monotonic increasing in $(-\infty, \bar{\mu}]$. \square

For x_+ we similarly have

Lemma 3.2. *For each $x_+ \in [1, \infty)$ there exists a unique $\mu \leq 0$ such that x_+ is the largest zero of $X(x)$. Moreover, $x_+(\mu)$ is monotonic decreasing.*

Proof. Noting that $q(1) = 2\mu$, and $q'(x) > 0$ for $x > 1$, we see that for a zero $x_+ \geq 1$ of $X(x)$ to exist, we must require $\mu \leq 0$ (independently of the parity of n). Moreover, (3.16) immediately implies that $x_+(\mu)$ is monotonic decreasing in μ . \square

3.2. Regularity for $\pm x > \pm x_{\pm}$

Let us fix x with $\pm x > \pm x_{\pm}$ and consider the positive definite⁵ metric h_x given by

$$\mp h_x = (1-x)(1-y)g_V + \frac{y-x}{4Y(y)}dy^2 + \frac{v(x,y)}{(n+1)^2} [d\gamma + (n+1)A]^2. \quad (3.18)$$

Near to a root y_i of $Y(y)$ we have

$$Y(y) = Y'(y_i)(y-y_i) + \mathcal{O}((y-y_i)^2). \quad (3.19)$$

Defining

$$\mp R_i^2 = \frac{(y_i-x)(y-y_i)}{Y'(y_i)} \quad (3.20)$$

for each $i = 1, 2$, one easily obtains, near to $R_i = 0$,

$$h_x = \pm(x-1)(1-y_i + \mathcal{O}(R_i^2))g_V + dR_i^2 + R_i^2 [d\gamma + (n+1)A]^2 + \mathcal{O}(R_i^4). \quad (3.21)$$

⁵ For $x > x_+$ the metric (3.2) is negative definite. Henceforth all metrics we write will be positive definite.

Fixing a point on V , we thus see that the metric is regular near to either zero provided that we take the period of γ to be 2π . This ensures that $y = y_i$ is merely a coordinate singularity, resembling the origin of \mathbb{R}^2 in polar coordinates (R_i, γ) . The one-form

$$d\gamma + (n+1)A \quad (3.22)$$

is precisely the global angular form on the unit circle bundle in the canonical line bundle K_V over V . Indeed, recall that

$$(n+1)dA = 2(n+1)\omega_V = \rho_V \quad (3.23)$$

is the curvature two-form of the anti-canonical line bundle K_V^{-1} . It follows that h_x extends to a smooth metric on the manifold

$$Z(V) = K_V \times_{U(1)} S^2 \quad (3.24)$$

for all v with $0 < v < v_{\max}$. That is, $Z(V)$ is the total space of the S^2 bundle over V obtained using the $U(1)$ transition functions of K_V , with the natural action of $U(1) \subset SO(3)$ on the S^2 fibres.

The induced metric on $\pm x > \pm x_{\pm}$ is in fact

$$g_x = h_x \mp w(x, y) \left[d\alpha + \frac{f(x, y)}{n+1} [d\gamma + (n+1)A] \right]^2. \quad (3.25)$$

One first notes that $w(x, y) < 0$ for all $x > x_+$, and $w(x, y) > 0$ for all $x < x_-$, $y \in [y_1, y_2]$, so that g_x has positive definite signature. We then consider the one-form $d\alpha + B$ where

$$B = \frac{f(x, y)}{n+1} [d\gamma + (n+1)A]. \quad (3.26)$$

As in [11], the strategy now is to show that, for appropriate v , there exists $\ell \in \mathbb{R}_+$ such that $\ell^{-1}B$ is locally a connection one-form on a $U(1)$ principal bundle over $Z(V)$. By periodically identifying α with period $2\pi\ell$, we thereby obtain a complete metric on the total space of this $U(1)$ principal bundle.

Assuming that V is simply-connected⁶, one can show that $Z(V)$ has no torsion in $H^2(Z(V); \mathbb{Z})$. The isomorphism class of a complex line bundle over $Z(V)$ is then determined completely by the integral of a curvature two-form over a basis of two-cycles. Such a basis is provided by $\{\Sigma, \sigma_*(\Sigma_i)\}$. Here Σ is represented by a copy of the fibre S^2 at any fixed point on V ; $\sigma: V \rightarrow Z(V)$ is the section $y = y_1$, and $\{\Sigma_i\}$ are a basis of two-cycles for $H^2(V; \mathbb{Z})$, which similarly is torsion-free.

One may then compute the periods

$$\begin{aligned} \int_{\Sigma} \frac{dB}{2\pi} &= \frac{1}{n+1} (f(x, y_1) - f(x, y_2)) \\ \int_{\sigma_*(\Sigma_i)} \frac{dB}{2\pi} &= \frac{f(x, y_1)}{n+1} \langle c_1(K_V^{-1}), \Sigma_i \rangle. \end{aligned} \quad (3.27)$$

We now note that

$$f(x, y_i) = \frac{y_i - 1}{y_i} \equiv f(y_i) \quad (3.28)$$

is independent of the choice of x . Notice that $f(y_1) > 0$ since $y_1 < 0$. Defining

$$\ell = \frac{If(y_1)}{k(n+1)} \quad (3.29)$$

it follows that the periods of $\ell^{-1}dB/2\pi$ over the two-cycles $\{\Sigma, \sigma_*(\Sigma_i)\}$ are

$$\left\{ \frac{k(f(y_1) - f(y_2))}{If(y_1)}, \frac{k}{I} \langle c_1(K_V^{-1}), \Sigma_i \rangle \right\}. \quad (3.30)$$

We now choose v so that

$$\frac{f(y_1) - f(y_2)}{f(y_1)} = \frac{pl}{k} \in \mathbb{Q} \quad (3.31)$$

is rational. In particular $p, k \in \mathbb{N}$ are positive integers; no coprime condition is assumed, so the rational number (3.31) is not assumed to be expressed in lowest terms. We note the following useful identities

$$\begin{aligned} \frac{y_2(1 - y_1)}{y_2 - y_1} &= \frac{k}{pl} \\ \frac{y_1(1 - y_2)}{y_2 - y_1} &= \frac{k}{pl} - 1, \end{aligned} \quad (3.32)$$

⁶ Again, $b_1(V) = 0$ necessarily.

which we will use repeatedly in the remainder of the paper. We shall return to which values of p and k are allowed momentarily. The periods (3.30) are then

$$\left\{ p, k \left(c_1(K_V^{-1/I}), \Sigma_i \right) \right\}. \quad (3.33)$$

Notice that, by definition of the Fano index I , $c_1(K_V^{-1/I}) \in H^2(V; \mathbb{Z})$ is primitive. Defining the new angular variable

$$\tilde{\alpha} = \ell^{-1} \alpha, \quad (3.34)$$

it follows that if we periodically identify $\tilde{\alpha}$ with period 2π , then the metric g_x at fixed x is a smooth complete metric on a $U(1)$ principal bundle over $Z(V)$, where the Chern numbers of the circle fibration over the two-cycles $\{\Sigma, \sigma_*(\Sigma_i)\}$ are given by (3.33). We denote the total space by $L^{p,k}(V)$. Note that $L^{p,k}(V)$ is simply a \mathbb{Z}_h quotient of $L^{p,k}(V)$, where $\mathbb{Z}_h \subset U(1)$ acts on the fibres of the $U(1)$ principal bundle $L^{p,k}(V) \rightarrow Z(V)$. We may also think of this manifold as a Lens space $L(1, p) = S^3/\mathbb{Z}_p$ fibration over V . Since the regularity analysis was essentially independent of x , we see that the Ricci-flat Kähler metric g extends to a smooth asymptotically conical metric on $\mathbb{R}_+ \times L^{p,k}(V)$, where $x - x_+ > 0$ or $x_- - x > 0$ is a coordinate on \mathbb{R}_+ . The asymptotic cone has Sasaki–Einstein base $Y^{p,k}(V)$, constructed originally in [11].

3.3. Allowed values of p and k

We will now determine the allowed values of p and k in (3.31). We begin by defining

$$Q(v) \equiv \frac{f(y_1) - f(y_2)}{f(y_1)} = \frac{y_2 - y_1}{y_2(1 - y_1)}, \quad (3.35)$$

regarding the roots y_i of $p(y)$ as functions of the metric parameter v . In the remainder of this section we shall prove

Proposition 3.3. *The function $Q : [0, v_{\max}] \rightarrow \mathbb{R}$ is a continuous monotonic increasing function with $Q(0) = 1$, $Q(v_{\max}) = 2$.*

Given (3.31), this implies that

$$\frac{pI}{2} < k < pI \quad (3.36)$$

and that, for each p and k , there is a corresponding *unique* metric⁷. From the defining equations of the roots we have

$$2v = (1 - y_1)^{n+1} - \frac{n+1}{n+2}(1 - y_1)^{n+2} = (1 - y_2)^{n+1} - \frac{n+1}{n+2}(1 - y_2)^{n+2}, \quad (3.37)$$

and from (3.37) one easily obtains the following useful identity

$$\frac{(1 - y_2)^{n+1}}{(1 - y_1)^{n+1}} = \frac{1 + (n+1)y_1}{1 + (n+1)y_2}. \quad (3.38)$$

From (3.37) one also computes

$$\frac{dy_i}{dv} = -\frac{2}{(n+1)y_i(1 - y_i)^n}. \quad (3.39)$$

One may then use this formula to prove the following

Lemma 3.4. *$y_1(v)$ (respectively $y_2(v)$) is monotonically increasing (respectively decreasing) in the interval $[0, v_{\max}]$. In particular, in the open interval $(0, v_{\max})$ the following bounds hold:*

$$-\frac{1}{n+1} < y_1 < 0 \quad (3.40)$$

$$0 < y_2 < 1. \quad (3.41)$$

Proof. We have $y_1(0) = -1/(n+1)$ and $y_1(v) < 0$ for all $v \in (0, v_{\max})$ since $p'(y=0) = 0$ for all v . Thus from (3.39) y_1 is monotonic increasing in this range. A similar argument applies for y_2 on noting that $y_2 > 0$ and $y_2(0) = 1$. \square

We now define

$$R(v) = \frac{y_1(1 - y_2)}{y_2(1 - y_1)}, \quad (3.42)$$

⁷ Note this analysis completes the argument presented in [11].

so that $Q = 1 - R$. Using (3.38) and (3.39) one easily obtains

$$\frac{dR}{dv} = \frac{-2(y_2 - y_1)}{(n+1)y_1y_2^3(1-y_1)(1-y_2)^n(1+(n+1)y_2)} D \quad (3.43)$$

where we have defined

$$D \equiv y_1 + y_2 + (n+1)y_1y_2. \quad (3.44)$$

Making use of the above identities, we also compute

$$\frac{dD}{dv} = -\frac{2(1+(n+1)y_2)}{(n+1)y_1(1-y_1)^n} (1+R). \quad (3.45)$$

The above computations, and Lemma 3.4, result in

Lemma 3.5. *In the open interval $(0, v_{\max})$ the sign of D' is correlated with that of $1+R$, and the sign of R' is correlated with that of D . In particular, we have*

$$R = -1 \quad \text{iff} \quad \frac{dD}{dv} = 0 \quad (3.46)$$

$$D = 0 \quad \text{iff} \quad \frac{dR}{dv} = 0. \quad (3.47)$$

Next, we turn to analysing the behaviour of R, D and their derivatives at the endpoints of the interval. It is easily checked that $R(0) = 0, D(0) = -1/(n+1)$ and $D(v_{\max}) = 0$. In order to compute $R(v_{\max})$ we write $v = v_{\max} - (n+1)\delta^2$, where the factor of $(n+1)$ is inserted for later convenience. We may solve for y in a power series in δ ; the first two terms suffice for our purposes:

$$y_1 = -2\delta + \frac{4}{3}n\delta^2 + \mathcal{O}(\delta^3), \quad y_2 = 2\delta + \frac{4}{3}n\delta^2 + \mathcal{O}(\delta^3). \quad (3.48)$$

With these one then computes

$$R = -1 + \left(4 + \frac{4}{3}n\right)\delta + \mathcal{O}(\delta^2), \quad (3.49)$$

which proves that $R \rightarrow -1$ as $v \rightarrow v_{\max}$. Before turning to the proof of Proposition 3.3 we shall need another

Lemma 3.6. $R'(v) \rightarrow -\infty$ for $v \rightarrow 0^+$ and $v \rightarrow v_{\max}^-$.

Proof. Near $v = 0$ this is easily checked; near $v = v_{\max}$ the result follows from (3.49). \square

Proof of Proposition 3.3. It is enough to show that $R(v)$ is a monotonically decreasing function in the interval $(0, v_{\max})$. Suppose this is not so. Then $R'(v_1) = 0$ for some least $v_1 \in (0, v_{\max})$. By Lemma 3.5, this is also the first time that D crosses zero, $D(v_1) = 0$. There are then three cases. We make repeated use of Lemma 3.5:

- Suppose $R(v_1) > -1$. Then $D'(v_1) > 0$, and D is positive in the range $(v_1, v_1 + \epsilon)$ for some $\epsilon > 0$. Since $D(v_{\max}) = 0$, there must be a turning point $D'(v_2) = 0$ for some smallest $v_2 \in (v_1, v_{\max})$. We then have $R(v_2) = -1$. But $R'(v) > 0$ for all $v \in (v_1, v_2)$ since $D > 0$ in this interval, which implies that R is strictly monotonic increasing in this range. This is a contradiction since we assumed $R(v_1) > -1$.
- Suppose $R(v_1) < -1$. Then $D'(v_1) < 0$. This is an immediate contradiction, since $D(0) = -1/(n+1)$ is negative and v_1 is the first zero of D ; hence $D'(v_1)$ must be non-negative.
- Suppose $R(v_1) = -1$. Either $R(v_1) = -1$ is a local minimum or a point of inflection:
 - Suppose $R(v_1) = -1$ is a local minimum of R . Since $R(v_{\max}) = -1$ also, there must be a turning point $R'(v_3) = 0$ for some least $v_3 \in (v_1, v_{\max})$. Then $D(v_1) = D(v_3) = 0$. Since $R(v) > -1$ for $v \in (v_1, v_3)$, it follows that $D' > 0$ in the same range, a contradiction.
 - Suppose $R(v_1) = -1$ is a point of inflection of R . Then $D(v_1) = 0$ is a local maximum of D . But since $D(v_{\max}) = 0$ also, D must have a turning point $D'(v_5) = 0$ for some least $v_5 \in (v_1, v_{\max})$. Thus $R(v_5) = -1$. But $D(v) < 0$ for all $v \in (v_1, v_5)$ implies that $R' < 0$ in the same range, a contradiction.

This proves that there is no $v_1 \in (0, v_{\max})$ where $R'(v_1) = 0$, and hence R is strictly monotonic decreasing in this range. \square

3.4. Summary

We end the section by summarising what we have proven so far:

Proposition 3.7. *Let (V, g_V) be a complete Kähler–Einstein manifold of positive Ricci curvature with Fano index l . Then for every $p, k \in \mathbb{N}$ positive integers with $pl/2 < k < pl$ there is an asymptotically conical Ricci-flat Kähler metric on $\mathbb{R}_+ \times L^{p,k}(V)$, with local form (2.4)–(2.6) and \mathbb{R}_+ coordinate either $x - x_+ > 0$ or $x_- - x > 0$. All metric parameters and ranges of coordinates are fixed uniquely for a given p and k , except for the constant μ . The metric is asymptotically a cone over the Sasaki–Einstein manifold $Y^{p,k}(V)$.*

In the remainder of the paper we examine regularity of the above metric at $x = x_{\pm}$ for the cases $x_- = y_1, x_+ = 1$ and $x_- < y_1, x_+ > 1$.

4. Small resolutions

In this section we consider the cases $x_- = y_1$ and $x_+ = 1$. Equivalently, these are the special cases $\mu = \pm\nu$ and $\mu = 0$. These will give rise to *partial* small resolutions, where one blows up the Fano V or a weighted projective space \mathbb{WCP}^1 , respectively. In particular, we prove Theorems 1.1 and 1.3. As a simple consequence of Theorem 1.1, we shall also obtain *smooth* small resolutions, as summarised in Corollary 1.2. The remaining cases where $x_- < y_1$ or $x_+ > 1$ will be the subject of Section 5.

4.1. Partial resolutions I: $x_- = y_1$

In this section we analyse regularity in the case that $x_- = y_1$. This special case arises since the function $y - x$ appears in the metric (2.4); when $x_- < y_1$ this function is strictly everywhere positive, whereas when $x_- = y_1$ the function has a vanishing locus. From Section 3.1 one easily deduces that $x_- = y_1$ corresponds to $\mu = -\nu$ when n is odd, and $\mu = \nu$ when n is even.

The first remarkable point to note is that, due to the symmetry in x and y , the analysis of the collapse at $x = x_-$, for $y_1 < y < y_2$, is identical to that of the collapse at $y = y_i$ for $x < x_-$. Thus for fixed $y \in (y_1, y_2)$ we deduce immediately that the metric collapses smoothly at $x = x_-$ for all μ . It thus remains to check the behaviour of the metric at $\{x = x_-, y = y_1\}$ and $\{x = x_-, y = y_2\}$.

The induced metric on $x = y_1$ is

$$g_{\{x=y_1\}} = (1 - y_1)(1 - y)g_V + \frac{dy^2}{4W(y)} + y_1^2 W(y) \left[d\alpha + \frac{y_1 - 1}{(n + 1)y_1} [d\gamma + (n + 1)A] \right]^2 \quad (4.1)$$

where we have defined

$$W(y) = \frac{Y(y)}{y - y_1}. \quad (4.2)$$

Since $y = y_1$ is a simple zero of $Y(y)$, for $y_1 \neq 0$, we have $W(y_1) \neq 0$. Thus $y = y_2$ is the only zero of $W(y)$ for $y_1 \leq y \leq y_2$, with $W(y) > 0$ for $y_1 \leq y < y_2$. We thus see that the above metric is regular at $y = y_1$.

In general it will turn out that $\{x = y_1, y = y_2\}$ is a locus of orbifold singularities. Although one can analyse the behaviour of the metric here using (4.1), it will turn out that $x = y_1$ is a rather unusual type of coordinate singularity; this might have been anticipated from the above factorisation of $Y(y)$ into $W(y)$. We will therefore introduce a new set of coordinates, resembling polar coordinates⁸ on $\mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R}^2$, in which this coordinate singularity is more easily understood.

We begin by defining

$$\begin{aligned} R_1^2 &= a_1(x - y_2)(y - y_2) \\ R_2^2 &= a_2(x - y_1)(y - y_1) \end{aligned} \quad (4.3)$$

where a_1, a_2 are constants that will be fixed later. The induced change in the $x - y$ part of the metric is then

$$\begin{aligned} \frac{y - x}{4X(x)} dx^2 + \frac{y - x}{4Y(y)} dy^2 &= \frac{1}{(y - x)(y_1 - y_2)^2} \left\{ \left[\frac{(x - y_1)^2}{X(x)} + \frac{(y - y_1)^2}{Y(y)} \right] \frac{R_1^2}{a_1^2} dR_1^2 + \left[\frac{(x - y_2)^2}{X(x)} + \frac{(y - y_2)^2}{Y(y)} \right] \frac{R_2^2}{a_2^2} dR_2^2 \right. \\ &\quad \left. - \frac{2R_1 R_2}{a_1 a_2} \left[\frac{(x - y_1)(x - y_2)}{X(x)} + \frac{(y - y_1)(y - y_2)}{Y(y)} \right] dR_1 dR_2 \right\}. \end{aligned} \quad (4.4)$$

⁸ As we shall see later, the global structure is in fact $\mathbb{C}^2/\mathbb{Z}_p$.

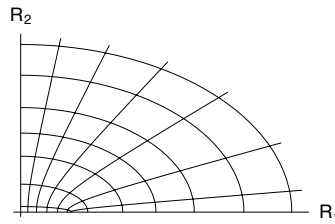


Fig. 1. An illustration of the change of coordinates (4.8). Curves of constant x and constant y are depicted.

Let us expand this near to $x = y_1$, $y = y_2$, which is $R_2 = 0$, $R_1 = 0$. Using

$$\begin{aligned} y_2 - y &= \frac{R_1^2}{a_1(y_2 - y_1)} + \mathcal{O}(R_1^2, R_2^2) \\ y_1 - x &= -\frac{R_2^2}{a_2(y_2 - y_1)} + \mathcal{O}(R_1^2, R_2^2) \end{aligned} \quad (4.5)$$

one finds

$$\frac{y-x}{4X(x)}dx^2 + \frac{y-x}{4Y(y)}dy^2 = -\frac{1}{a_1Y'(y_2)}[1 + \mathcal{O}(R^2)]dR_1^2 + \frac{1}{a_2X'(y_1)}[1 + \mathcal{O}(R^2)]dR_2^2 + \mathcal{O}(R_1, R_2)dR_1dR_2, \quad (4.6)$$

where $\mathcal{O}(R^2)$ denotes terms of order $\mathcal{O}(R_1^2)$ or order $\mathcal{O}(R_2^2)$. We thus set

$$a_1 = -\frac{1}{Y'(y_2)} = \frac{1}{(n+1)y_2} \quad a_2 = \frac{1}{X'(y_1)} = \frac{1}{(n+1)y_1}. \quad (4.7)$$

The change of coordinates (4.3) now becomes

$$\begin{aligned} R_1^2 &= \frac{1}{(n+1)y_2}(y_2 - x)(y_2 - y) \\ R_2^2 &= -\frac{1}{(n+1)y_1}(y_1 - x)(y - y_1) \end{aligned} \quad (4.8)$$

and these relations imply

$$\begin{aligned} \frac{y_2}{y_2 - x}R_1^2 + \frac{y_1}{x - y_1}R_2^2 &= \frac{y_2 - y_1}{n+1} \\ \frac{y_2}{y_2 - y}R_1^2 + \frac{y_1}{y - y_1}R_2^2 &= \frac{y_2 - y_1}{n+1}. \end{aligned} \quad (4.9)$$

Despite the symmetry in x and y , the curves of constant x and constant y are different due to the difference in ranges of the variables. Recall that $y_1 \leq y \leq y_2$ and $x \leq x_- = y_1$. The constant x curves are ellipses, while the constant y curves are hyperbolae. This behaviour is depicted in Fig. 1. Notice that both sets of curves degenerate on the R_1 -axis. Indeed, note that $R_1 = 0$ if and only if $y = y_2$; but $R_2 = 0$ if $x = y_1$ or $y = y_1$. In particular, when $x = y_1$ we have $R_1^2 = a_1(y_2 - y)(y_2 - y_1)$ and thus this branch of the R_1 -axis is coordinatised by y . On the other hand, when $y = y_1$ we have $R_1^2 = a_1(y_2 - x)(y_2 - y_1)$, and thus this branch of the R_1 -axis is coordinatised by x .

Fixing a point on V , to leading order the induced metric near $R_1 = R_2 = 0$ is

$$g_{\text{fibre}} = dR_1^2 + \frac{(n+1)^2}{(y_2 - y_1)^2}y_2^2R_1^2 \left[y_1 d\alpha + \frac{(y_1 - 1)}{(n+1)} d\gamma \right]^2 + dR_2^2 + \frac{(n+1)^2}{(y_2 - y_1)^2}y_1^2R_2^2 \left[y_2 d\alpha + \frac{(y_2 - 1)}{(n+1)} d\gamma \right]^2. \quad (4.10)$$

We then define

$$\begin{aligned} \phi_1 &= -\frac{(n+1)}{(y_2 - y_1)}y_2 \left[y_1 \alpha + \frac{(y_1 - 1)}{(n+1)} \gamma \right] \\ \phi_2 &= -\frac{(n+1)}{(y_2 - y_1)}y_1 \left[y_2 \alpha + \frac{(y_2 - 1)}{(n+1)} \gamma \right] \end{aligned} \quad (4.11)$$

so that (4.10) becomes

$$g_{\text{fibre}} = dR_1^2 + R_1^2 d\phi_1^2 + dR_2^2 + R_2^2 d\phi_1^2. \quad (4.12)$$

In terms of the variable $\tilde{\alpha} = \ell^{-1}\alpha$, the change of coordinates (4.11) becomes

$$\begin{aligned}\phi_1 &= \frac{1}{p}\tilde{\alpha} + \frac{k}{pl}\gamma \\ \phi_2 &= \frac{1}{p}\tilde{\alpha} + \left(\frac{k}{pl} - 1\right)\gamma\end{aligned}\quad (4.13)$$

on using the identities (3.32). Notice that the Jacobian of the transformation (4.13) is $1/p$. Recall also from Section 3.2 that $\tilde{\alpha}$ and γ are periodically identified with period 2π . It follows from (4.12) that a neighbourhood of $R_1 = R_2 = 0$, at a fixed point on V , is diffeomorphic to $\mathbb{R}^4/\mathbb{Z}_p$. Indeed, as mentioned in Section 3, the surfaces of constant $x < x_-$ are Lens space fibrations $L(1, p) = S^3/\mathbb{Z}_p$ over V . These are then constant radius surfaces in the $\mathbb{R}^4/\mathbb{Z}_p$ fibration over V . The set of points $\{x = x_-, y = y_2\}$ are the zero-section, which is a copy of V and locus of orbifold singularities. In fact, the possible existence of such metrics was raised at the end of reference [10]. The fibres must of course be complex submanifolds, and one easily checks that the complex structure is such that each fibre is $\mathbb{C}^2/\mathbb{Z}_p$.

One may work out the precise fibration structure as follows. Setting $y = y_1$ and $y = y_2$ gives two different \mathbb{C}/\mathbb{Z}_p fibrations over V . It is enough to determine the fibration structure of these bundles. Of course, the unit circle bundle in each is a $U(1)$ principal bundle over V . These $U(1)$ bundles are determined from the analysis in Section 3. The $U(1)$ bundle at fixed $x < x_-$ and $y = y_1$ has first Chern class $kc_1(K_V)/I$. The associated complex line bundle is thus $K_V^{k/I}$. The $U(1)$ bundle at fixed $x < x_-$ and $y = y_2$ is determined from the periods of $-\ell^{-1}dB/2\pi$ over the image of cycles Σ in V at $y = y_2$. We denote these as $\tau_*(\Sigma_i)$. The periods are given by

$$\begin{aligned}-\int_{\tau_*(\Sigma_i)} \ell^{-1} \frac{dB}{2\pi} &= -\ell^{-1} \frac{f(y_2)}{n+1} \langle c_1(K_V^{-1}), \Sigma_i \rangle \\ &= \left(p - \frac{k}{I}\right) \langle c_1(K_V^{-1}), \Sigma_i \rangle.\end{aligned}\quad (4.14)$$

This implies that the $U(1)$ bundle has first Chern class $(pl - k)c_1(K_V)/I$, and thus the associated line bundle is $K_V^{(pl-k)/I}$.

It is now a simple matter to determine the twisting of the $\mathbb{C}^2/\mathbb{Z}_p$ fibres themselves. The $\mathbb{Z}_p \subset U(1) \subset SU(2)$ acts via the standard action of $SU(2)$ on \mathbb{C}^2 . Define $\mathcal{L}_1 = K_V$ and $\mathcal{L}_2 = K_V^{k/I}$. Let $\{U_\alpha\}$ be a trivialising open cover of V , and let $g_{\alpha\beta}^i$, $i = 1, 2$, denote the transition functions of the above bundles. Thus $g_{\alpha\beta}^1 : U_\alpha \cap U_\beta \rightarrow S^1$. Let (z_1, z_2) denote standard complex coordinates on \mathbb{C}^2 . These are identified via the action of $\mathbb{Z}_p \subset U(1) \subset SU(2)$. We must specify precisely how $U_\alpha \times \mathbb{C}^2/\mathbb{Z}_p$ is glued to $U_\beta \times \mathbb{C}^2/\mathbb{Z}_p$ over the overlap $U_{\alpha\beta} = U_\alpha \cap U_\beta$. To do this, we define the following action of $\mathbb{T}^2 = S^1 \times S^1$ on $\mathbb{C}^2/\mathbb{Z}_p$:

$$\begin{aligned}\lambda : S^1 \times S^1 \times \mathbb{C}^2/\mathbb{Z}_p &\rightarrow \mathbb{C}^2/\mathbb{Z}_p \\ \lambda(\theta_1, \theta_2; z_1, z_2) &= (\exp(i\theta_1 - i\theta_2/p)z_1, \exp(i\theta_2/p)z_2).\end{aligned}\quad (4.15)$$

Note that this indeed defines an action of $S^1 \times S^1$ on $\mathbb{C}^2/\mathbb{Z}_p$. Note also that the standard action of $U(1) \subset SU(2)$ on \mathbb{C}^2 descends to a non-effective action of $U(1)$ on the quotient $\mathbb{C}^2/\mathbb{Z}_p$ —this factors p times through the effective $U(1)$ action in (4.15). The $\mathbb{C}^2/\mathbb{Z}_p$ bundle is then constructed using the gluing functions

$$\begin{aligned}F_{\alpha\beta} : U_{\alpha\beta} \times \mathbb{C}^2/\mathbb{Z}_p &\rightarrow U_{\alpha\beta} \times \mathbb{C}^2/\mathbb{Z}_p \\ F_{\alpha\beta}[u; z_1, z_2] &= [u; \lambda(g_{\alpha\beta}^1(u), g_{\alpha\beta}^2(u); z_1, z_2)].\end{aligned}\quad (4.16)$$

To check this is correct, we simply set $z_1 = 0$ and $z_2 = 0$ separately. This should be equivalent to setting $y = y_1$ and $y = y_2$, respectively, to give \mathbb{C}/\mathbb{Z}_p fibrations over V . From (4.15) we see that $z_1 = 0$ has $U(1)$ principal bundle given by $K_V^{k/I}$. On the other hand, setting $z_2 = 0$, the corresponding $U(1)$ principal bundle is given by $K_V^p \otimes K_V^{-k/I}$. These are precisely the same \mathbb{C}/\mathbb{Z}_p fibrations determined above using the metric. This completes the proof of Theorem 1.1.

4.2. Smooth resolutions: $p = 1$

Setting $p = 1$ in the last subsection gives a family of smooth complete Ricci-flat Kähler metrics for each choice of (V, g_V) . These are all holomorphic \mathbb{C}^2 fibrations over V . From (4.15) this is easily seen to be a direct sum of two complex line bundles over V , namely $[K_V \otimes K_V^{-k/I}] \oplus K_V^{k/I}$. Setting $m = I - k$ this is $K_V^{m/I} \oplus K_V^{(I-m)/I}$, as stated in Corollary 1.2. The range of k is given by (3.36) with $p = 1$, which implies that $0 < m < I/2$.

For example, we may take $V = \mathbb{CP}^n$ with its standard Kähler–Einstein metric. In this case $I = n+1$ and $K_V = \mathcal{O}(-(n+1))$, so that $K_V^{-1/I} = \mathcal{O}(1)$. We have $0 < m < (n+1)/2$ and the metrics are defined on the total space of the rank two holomorphic vector bundle $\mathcal{O}(-m) \oplus \mathcal{O}(-(n-m+1))$ over \mathbb{CP}^n . Note that $m = n = 1$ is the small resolution of the conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{CP}^1$, which is understood as a limiting case.

As another simple example, one might take V to be a product of complex projective spaces, equipped with the natural product Kähler–Einstein metric:

$$V = \prod_{a=1}^M \mathbb{CP}^{d_a-1} \quad (4.17)$$

where

$$\sum_{a=1}^M d_a = n + M. \quad (4.18)$$

In this case $I = \text{hcf}\{d_a\}$ and the rank two holomorphic vector bundle is given by

$$\mathcal{O}(-md_1/I, \dots, -md_M/I) \oplus \mathcal{O}(-(I-m)d_1/I, \dots, -(I-m)d_M/I). \quad (4.19)$$

4.3. Partial resolutions II: $x_+ = 1$

In this subsection we consider a different small partial resolution, where the Fano V shrinks to zero size, while a weighted projective space \mathbb{WCP}^1 is blown up. The analysis of this subsection is summarised by [Theorem 1.3](#). Notice that when $V = \mathbb{CP}^1$, the process of going from the round \mathbb{CP}^1 to the weighted \mathbb{WCP}^1 may be understood as a form of generalised flop transition (called a *flip*) in the Kähler moduli space of the family of toric Gorenstein singularities $C(Y^{p,q})$ [22]. This will be discussed elsewhere [23].

From the general form of the metric (2.4) it is simple to see that in order for V to collapse one requires $x_+ = 1$, and this implies that $\mu = 0$. We then have

$$X(x) = \frac{x-1}{n+2} (1 + (n+1)x). \quad (4.20)$$

We choose the $x \geq 1$ branch of x . To analyse the metric near $x = 1$ it is useful to change coordinates, defining

$$x = 1 + r^2. \quad (4.21)$$

Expanding the metric near $r = 0$, and keeping terms up to order r^2 , we find

$$g = (1-y) \left\{ dr^2 + r^2 \left[g_V + \frac{1}{(n+1)^2} [d\gamma + (n+1)(A + F(y)d\alpha)]^2 + \frac{dy^2}{4Y(y)(1-y)} + G(y)d\alpha^2 \right] \right. \\ \left. + \frac{1-y}{4Y(y)} dy^2 + \frac{Y(y)}{1-y} d\alpha^2 + \mathcal{O}(r^4) \right\}, \quad (4.22)$$

where we have defined

$$F(y) = \frac{Y(y)}{(1-y)^2} - \frac{y}{1-y} \quad (4.23)$$

$$G(y) = \frac{1}{(1-y)^4} [Y(y)^2 + (1-y) - 2Y(y)(1-y)]. \quad (4.24)$$

We first analyse the induced metric on $r = 0$, which is given by

$$g_W = \frac{1-y}{4Y(y)} dy^2 + \frac{Y(y)}{1-y} d\alpha^2. \quad (4.25)$$

As usual, near each root y_i we introduce the coordinates

$$R_i^2 = \frac{(y_i - 1)}{(n+1)y_i} (y - y_i) \quad (4.26)$$

from which we see that

$$g_W = dR_i^2 + \left[\frac{I y_i (1 - y_i) R_i}{k y_i (1 - y_i)} \right]^2 d\tilde{\alpha}^2 + \mathcal{O}(R_i^4). \quad (4.27)$$

Thus

$$g_W = \begin{cases} dR_1^2 + \frac{I^2}{k^2} R_1^2 d\tilde{\alpha}^2 + \mathcal{O}(R_1^4) & \text{near } y = y_1 \\ dR_2^2 + \frac{I^2}{(pI - k)^2} R_2^2 d\tilde{\alpha}^2 + \mathcal{O}(R_2^4) & \text{near } y = y_2. \end{cases}$$

Recall now that $\tilde{\alpha}$ has period 2π . In order to obtain an orbifold singularity near to $y = y_1$, it is therefore necessary that the Fano index I divides k . Thus we assume this, and define $k = Id$ with d a positive integer. It follows that g_W smoothly

approaches, in an orbifold sense, the flat metric on \mathbb{C}/\mathbb{Z}_d , where $y = y_1$ is the origin. Similarly, at $y = y_2$ the metric smoothly approaches the flat metric on $\mathbb{C}/\mathbb{Z}_{p-d}$. It follows that, provided $k = Id$, the induced metric on $r = 0$ is a smooth Kähler orbifold metric on the weighted projective space $W = \mathbb{WCP}^1_{[d,p-d]}$.

Now fix any smooth point (y, α) on W , so $y_1 < y < y_2$. Setting $R^2 = (1 - y)r^2$, the induced metric near to $R = 0$ is

$$g = dR^2 + R^2 \left[g_V + \frac{1}{(n+1)^2} [d\gamma + (n+1)A]^2 \right] + \mathcal{O}(R^4). \quad (4.28)$$

From Section 3, γ has period 2π , and the induced metric (4.28) is simply the canonical complex cone \mathbb{C}_V over V . Equivalently, fixed R gives the associated circle bundle in the canonical line bundle over (V, g_V) , and near to $R = 0$ the whole metric is a real cone over this regular Sasaki–Einstein manifold. Thus near to $R = 0$ the fibre metric over a smooth point on W itself approaches a Ricci-flat Kähler cone.

One needs to consider what happens over the roots $y = y_i$ separately. These are the singular points of the weighted projective space $W = \mathbb{WCP}^1_{[d,p-d]}$. To determine the period of γ in (4.28) over these points one may simply compute the volume of $\{y = y_i, x = 1 + r^2\}$ with r^2 small and fixed, and compare with Section 3. Each space is a $U(1)$ principal bundle over V , namely that associated to the complex line bundles $K_V^{k/I}, K_V^{(pl-k)/I}$, respectively. From Section 3 we have the induced metric

$$g = (1 - y_i)r^2 g_V + \frac{y_i^2}{1 - y_i} r^2 d\alpha^2 + \mathcal{O}(r^4) \quad (4.29)$$

where α has period $2\pi\ell$. Comparing with (4.28), we see that γ must be identified with period $\Delta\gamma_i$ over each pole, where

$$\Delta\gamma_i = \frac{(n+1)y_i\ell}{1 - y_i} 2\pi = \begin{cases} -\frac{2\pi I}{k} & i = 1 \\ \frac{2\pi I}{pl - k} & i = 2. \end{cases} \quad (4.30)$$

Thus γ has period $2\pi/d$ over $y = y_1$ and period $2\pi/(p-d)$ over $y = y_2$. This implies that the fibres over the singular points of W are complex cones over V associated to K_V^d and K_V^{p-d} , respectively; the generic fibre is the complex cone \mathbb{C}_V associated to K_V . This gives fibres $\mathbb{C}_V/\mathbb{Z}_d$ and $\mathbb{C}_V/\mathbb{Z}_{p-d}$ over the singular points of W , respectively.

In fact this latter behaviour of the fibres could have been deduced differently, by considering the fibration structure. Recall we have now checked that the metric is smooth away from $W = \mathbb{WCP}^1_{[d,p-d]}$, that W is itself a smooth orbifold, and that each fibre over W is a complex cone over V . We now compute the twisting of this fibration. The twisting is determined via the one-form $d\gamma + (n+1)(A + F(y)d\alpha)$ in the metric (4.22). The integral of the corresponding curvature two-form is

$$\frac{n+1}{2\pi} \int_W dF(y)d\alpha = \frac{p}{d(p-d)}. \quad (4.31)$$

Fixing a point on V , and fixing $r > 0$, we obtain a circle orbibundle over W . The right hand side of (4.31) is minus the Chern number⁹ of this orbibundle, and corresponds to the canonical line orbibundle over $W = \mathbb{WCP}^1_{[d,p-d]}$. This is given by $K_W = \mathcal{O}(-p)$. One way to see this is via the Kähler quotient description of the weighted projective space together with its canonical line bundle over it. This is $\mathbb{C}^3//U(1)$ where the $U(1)$ action has weights $(d, p-d, -p)$. The weighted projective space itself is $z_3 = 0$, in standard complex coordinates on \mathbb{C}^3 .

The above fibration structure immediately implies the earlier statements about the period of γ over the singular points of W . In order to see this one needs to know some facts about orbibundles and orbifold fibrations. Suppose W is an orbifold, with local orbifold charts $\{U_\alpha = \tilde{U}_\alpha/\Gamma_\alpha\}$, where \tilde{U}_α is an open set in \mathbb{R}^N and Γ_α is a finite subgroup of $GL(N, \mathbb{R})$. The data that defines an orbibundle over W with structure group G includes elements $h_\alpha \in \text{Hom}(\Gamma_\alpha, G)$ for each α , subject to certain gluing conditions. In particular, if F denotes a fibre over a smooth point of W , on which G acts, then the fibre over a singular point with orbifold structure group Γ_α is $F/h_\alpha(\Gamma_\alpha)$. Thus an orbibundle is generally not a fibration in the usual sense, since not all fibres are isomorphic.

In the present situation it is particularly simple to work out the representations h_α , since the orbibundle we require is the canonical line bundle K_W over W . Since W is a complex orbifold of dimension one, this is the holomorphic cotangent orbibundle. The orbifold structure groups are of the form $\mathbb{Z}_d, \mathbb{Z}_{p-d} \subset U(1)$, and then the maps $h_1 : \mathbb{Z}_d \rightarrow U(1)$, $h_2 : \mathbb{Z}_{p-d} \rightarrow U(1)$ are just the standard embeddings into $U(1)$. This implies that the metrics above are defined on

$$K_{\mathbb{WCP}^1_{[d,p-d]}} \times_{U(1)} \mathbb{C}_V. \quad (4.32)$$

Here the $U(1) \subset \mathbb{C}^*$ action is the standard one on the canonical complex cone \mathbb{C}_V . The fibres over the poles of $W = \mathbb{WCP}^1_{[d,p-d]}$ are then $\mathbb{C}_V/\mathbb{Z}_d$ and $\mathbb{C}_V/\mathbb{Z}_{p-d}$, where the cyclic groups are embedded in $U(1)$ in the standard way. Here we have used the above maps $h_\alpha, \alpha = 1, 2$. This completes the proof of Theorem 1.3.

⁹ For an explanation, see Section 5.1.

Note that, in contrast to the previous section, we only obtain metrics for which $k = l$ is divisible by l . Note also that the weighted projective space is a smooth \mathbb{CP}^1 if and only if $p = 2$, $k = l$, which is a limiting case of the solutions considered here.

5. Canonical resolutions

In this section we turn our attention to complete Ricci-flat Kähler orbifold metrics, where the conical singularity gets replaced by a divisor M with at worst orbifold singularities. In Section 3.2 we addressed regularity of the metrics for $\pm x > \pm x_{\pm}$, and this fixed uniquely the value of the parameter ν in terms of the pair of integers p and k , in the range (3.36). The strategy here will be to show that one can choose appropriate values for the parameter μ so that the metrics collapse smoothly, in an orbifold sense, to a divisor M at $x = x_+$ or $x = x_-$, provided $x_+ > 1$ and $x_- < y_1$. In fact for each p and k we shall find a family of values of μ , indexed by an integer r with $0 < r < k/l$. M is then a Fano orbifold of complex dimension $n + 1$ which is a $\mathbb{WCP}^1_{[r, p-r]}$ fibration over V . The Ricci-flat Kähler metric is defined on the total space of the canonical line bundle over M . The induced metric on M is Kähler, though in general the Kähler class is irrational. In order that M be smooth ones requires $p = 2$, $r = 1$, and this leads to Corollary 1.5.

5.1. Partial resolutions III

Again, due to the symmetry in x and y , the analysis of the collapse at $x = x_{\pm}$, for $y_1 < y < y_2$, is identical to that of the collapse at $y = y_i$ for $\pm x > \pm x_{\pm}$. Thus for fixed $y \in (y_1, y_2)$ we deduce that the metric collapses smoothly at $x = x_{\pm}$ for all μ . It thus remains to check the behaviour of the metric at $\{x = x_{\pm}, y = y_1\}$ and $\{x = x_{\pm}, y = y_2\}$.

We begin by writing the induced metric on $x = x_{\pm}$

$$\mp g_M = (1 - x_{\pm})(1 - y)g_V + \frac{y - x_{\pm}}{4Y(y)}dy^2 + \frac{x_{\pm}^2 Y(y)}{y - x_{\pm}} \left[d\alpha + \frac{x_{\pm} - 1}{(n + 1)x_{\pm}} [d\gamma + (n + 1)A] \right]^2. \quad (5.1)$$

Near to a root $y = y_i$ we define

$$\mp R_i^2 = \frac{(y_i - x_{\pm})(y - y_i)}{Y'(y_i)}, \quad (5.2)$$

so that near each root we have the positive definite metric

$$g_M = \pm(x_{\pm} - 1)(1 - y_i + \mathcal{O}(R_i^2))g_V + dR_i^2 + \left[\frac{(n + 1)x_{\pm}y_i R_i}{y_i - x_{\pm}} \right]^2 \left[d\alpha + \frac{x_{\pm} - 1}{(n + 1)x_{\pm}} [d\gamma + (n + 1)A] \right]^2 + \mathcal{O}(R_i^4). \quad (5.3)$$

Let us define

$$\varphi_i = \frac{(n + 1)x_{\pm}y_i}{y_i - x_{\pm}} \left[\alpha + \frac{x_{\pm} - 1}{(n + 1)x_{\pm}} \gamma \right]. \quad (5.4)$$

In order to allow for orbifold singularities, we impose the periodicities

$$\Delta\varphi_1 = \frac{2\pi}{r} \quad \Delta\varphi_2 = \frac{2\pi}{s} \quad (5.5)$$

for r, s positive integers. This implies the necessary condition

$$-\frac{ry_1}{y_1 - x_{\pm}} = \frac{sy_2}{y_2 - x_{\pm}} \quad (5.6)$$

where the minus sign ensures that both sides of the equation have the same sign. This then gives

$$x_{\pm} = \frac{(r + s)y_1y_2}{ry_1 + sy_2}. \quad (5.7)$$

We shall return to this formula in a moment. A calculation using (5.7) and (3.32) shows that

$$\begin{aligned} \varphi_1 &= \left(1 + \frac{s}{r}\right) \frac{1}{p} \tilde{\alpha} + \frac{k(1 + \frac{s}{r}) - pl}{pl} \gamma \\ -\varphi_2 &= \left(1 + \frac{r}{s}\right) \frac{1}{p} \tilde{\alpha} + \frac{k(1 + \frac{r}{s}) - pl}{pl} \gamma. \end{aligned} \quad (5.8)$$

Recall that $\tilde{\alpha}$ and γ have period 2π . In order to satisfy (5.5) we must then require that $p = s + r$, which gives

$$\begin{aligned}\varphi_1 &= \frac{1}{r}\tilde{\alpha} + \left(\frac{k}{rl} - 1\right)\gamma \\ -\varphi_2 &= \frac{1}{s}\tilde{\alpha} + \left(\frac{k}{sl} - 1\right)\gamma.\end{aligned}\quad (5.9)$$

Let us now examine (5.7). Since the numerator is negative definite, for $\nu \in (0, \nu_{\max})$, this implies that

$$\begin{aligned}x_- &= \frac{(r+s)y_1y_2}{ry_1 + sy_2}, \quad \text{for } ry_1 + sy_2 > 0 \\ x_+ &= \frac{(r+s)y_1y_2}{ry_1 + sy_2}, \quad \text{for } ry_1 + sy_2 < 0.\end{aligned}\quad (5.10)$$

Note that $ry_1 + sy_2 = 0$ implies from (5.6) that $y_1 = y_2$, which is impossible for $\nu \in (0, \nu_{\max})$. In particular we have

$$x_- - y_1 = \frac{ry_1(y_2 - y_1)}{ry_1 + sy_2} < 0 \quad (5.11)$$

since the numerator is negative. By Lemma 3.1 there is therefore a unique μ such that $X(x)$ has x_- as its smallest zero. On the other hand, using (3.32) it is easy to compute

$$\frac{y_1(x_+ - 1)}{y_1 - x_+} = \frac{k - pl}{pl} + \frac{s}{r} \frac{k}{pl}. \quad (5.12)$$

Since $y_1/(y_1 - x_+)$ is certainly positive, this implies that $x_+ > 1$ if and only if the right hand side of (5.12) is positive. Using $r + s = p$, this easily becomes

$$x_+ > 1 \quad \text{iff} \quad k - rl > 0. \quad (5.13)$$

Thus when $ry_1 + sy_2 < 0$ and $k - rl > 0$, by Lemma 3.2 there is a unique $\mu < 0$ such that $x_+ > 1$ is the largest zero of $X(x)$. We now define

$$m = k - rl \quad (5.14)$$

and compute

$$ry_1 + sy_2 = \frac{1}{l} [(ky_1) + (pl - k)y_2 + m(y_2 - y_1)]. \quad (5.15)$$

Here we have substituted $s = p - r$. Using

$$\begin{aligned}y_1 &= \left(\frac{k}{pl} - 1\right)(y_2 - y_1) + y_1y_2 \\ y_2 &= \frac{k}{pl}(y_2 - y_1) + y_1y_2,\end{aligned}\quad (5.16)$$

which is a rewriting of (3.32), we thus have

$$ry_1 + sy_2 = py_1y_2 + \frac{m}{l}(y_2 - y_1). \quad (5.17)$$

Suppose that $m > 0$. Then either $ry_1 + sy_2 > 0$ and we are on the x_- branch, with $x_- < y_1$; or else $ry_1 + sy_2 < 0$ and by (5.13) we are on the x_+ branch, with $x_+ > 1$. If $m < 0$ then from (5.17) $ry_1 + sy_2 < 0$ and hence we are on the x_+ branch; but by (5.13) $x_+ < 1$ and hence the metric cannot be regular. When $m = 0$ we formally obtain $x_+ = 1$, which was the special case considered in the previous section. We thus conclude that we obtain regular orbifold metrics if and only if $m > 0$.

We have now shown that the metric g_M extends to a smooth orbifold metric, for all p, k, r positive integers with

$$\frac{p}{2} < \frac{k}{l} < p, \quad 0 < r < \frac{k}{l}. \quad (5.18)$$

It remains simply to check the fibration structure of M and thus describe its topology. Defining

$$\varphi = \frac{(n+1)ky_1}{l(y_1-1)} \left[\alpha + \frac{x_{\pm}-1}{(n+1)x_{\pm}} \gamma \right] = \tilde{\alpha} + \frac{k-rl}{l} \gamma \quad (5.19)$$

we see that the one-form in the second line of the metric (5.1) is proportional to

$$d\varphi + \frac{k-rl}{l}(n+1)A. \quad (5.20)$$

Since φ has canonical period 2π , this is a global angular form on the associated circle bundle to $K_V^{m/l}$, where recall $m = k - rl$. Thus M may be described as follows. One takes the weighted projective space $\mathbb{WCP}_{[r,p-r]}^1$ and fibres it over V . The transition functions are precisely those for $K_V^{m/l}$, using the standard effective $U(1)$ action on $\mathbb{WCP}_{[r,p-r]}^1$. Thus M may be written

$$M = K_V^{m/l} \times_{U(1)} \mathbb{WCP}_{[r,p-r]}^1. \quad (5.21)$$

The Ricci-flat Kähler metric is defined on the total space of an orbifold line bundle over M , which is necessarily the canonical line orbibundle.

Notice that M is singular precisely along the two divisors D_1, D_2 , located at $y = y_1, y = y_2$, respectively. D_1 has normal fibre \mathbb{C}/\mathbb{Z}_r , and D_2 has normal fibre $\mathbb{C}/\mathbb{Z}_{p-r}$. The normal bundles are $K_V^{-m/l}, K_V^{m/l}$, respectively. Due to the fact that the only orbifold singularities are in complex codimension one, M is in fact completely smooth as a manifold, and as an algebraic variety.¹⁰ In either case, M is a \mathbb{CP}^1 fibration over V . One must then be extremely careful when making statements such as “ M is Fano”: the anti-canonical line bundle and anti-canonical orbifold line bundle are different objects.

Let $\pi : M \rightarrow V$ denote the projection. Then the canonical line bundle is

$$K_M = \pi^* K_V - 2D_1 - m\pi^*(K_V/I). \quad (5.22)$$

Recall here that the divisor D_1 at $y = y_1$ has normal bundle $K_V^{-m/l}$. Note in (5.22) we have switched to an additive notation, rather than the multiplicative notation we have been using so far throughout the paper; this is simply so that the equations are easier to read. On the other hand, the orbifold canonical line bundle is

$$K_M^{\text{orb}} = K_M + \left(1 - \frac{1}{r}\right) D_1 + \left(1 - \frac{1}{p-r}\right) D_2. \quad (5.23)$$

This may be argued simply by the following computation, taken largely from [26]. Let U be an open set in M containing some part of a divisor D with normal fibre \mathbb{C}/\mathbb{Z}_r . We suppose that $\tilde{U} \subset \mathbb{C}^n$ is the local covering chart, and that the preimage of the divisor D is given locally in \tilde{U} by $x_1 = 0$. This is called the ramification divisor; we denote this divisor in \tilde{U} by R . We also complete x_1 to a set of local complex coordinates on \tilde{U} , (x_1, \dots, x_n) . The orbifold structure group is $\Gamma = \mathbb{Z}_r$, and the map $\phi : \tilde{U} \rightarrow U$ near D looks like

$$\phi : (x_1, x_2, \dots, x_n) \rightarrow (z_1 = x_1^r, z_2 = x_2, \dots, z_n = x_n) \quad (5.24)$$

where (z_1, \dots, z_n) are complex coordinates on U , which is also biholomorphic to an open set in \mathbb{C}^n . In particular, we may compute

$$\phi^*(dz_1 \wedge \dots \wedge dz_n) = rx_1^{r-1} dx_1 \wedge \dots \wedge dx_n. \quad (5.25)$$

Now, the orbifold line bundle K_M^{orb} is defined as the canonical line bundle of \tilde{U} in each covering chart \tilde{U} , i.e. as the top exterior power of the holomorphic cotangent bundle. These naturally glue together on the orbifold M to give an orbifold line bundle over M . However, we see from (5.25) that

$$K_{\tilde{U}} = \pi^* K_U \otimes [(r-1)R]. \quad (5.26)$$

Since $\pi^* D = rR$, this gives the general formula

$$K_M^{\text{orb}} = K_M + \sum \left(1 - \frac{1}{r_i}\right) D_i \quad (5.27)$$

where D_i is a so-called branched divisor, with multiplicity r_i . This rather formal expression may be understood more concretely as follows. M is a complex manifold with divisors K_M and D_i defining complex line bundles over M . For each line bundle, by picking a connection we obtain a curvature two-form whose cohomology class lies in the image of $H^2(M; \mathbb{Z})$ in $H^2(M; \mathbb{R})$. The corresponding cohomology class of the right hand side of (5.27) is thus in $H^2(M; \mathbb{Q})$. This in fact represents the cohomology class of the curvature of a connection on the orbifold line bundle K_M^{orb} .

Returning to (5.23), we obtain

$$\begin{aligned} K_M^{\text{orb}} &= -\frac{p}{r(p-r)} D_1 + \left(I - \frac{m}{p-r}\right) \pi^*(K_V/I) \\ &= -\frac{p}{r(p-r)} D_2 + \left(I + \frac{m}{r}\right) \pi^*(K_V/I) \end{aligned} \quad (5.28)$$

where note that

$$D_1 - D_2 = -m\pi^*(K_V/I). \quad (5.29)$$

¹⁰ Note that V is a smooth Fano manifold, and hence is projective.

Note in (5.28) the first Chern class of the weighted projective space $\mathbb{WCP}_{[r,p-r]}^1$ appearing. Indeed, by the above comments the integral of this orbifold first Chern class is

$$\int_{\mathbb{WCP}_{[r,p-r]}^1} c_1^{\text{orb}}(\mathbb{WCP}_{[r,p-r]}^1) = 2 - \left(1 - \frac{1}{r}\right) - \left(1 - \frac{1}{p-r}\right) = \frac{p}{r(p-r)}, \quad (5.30)$$

a formula we encountered earlier in Eq. (4.31).

Let $\Sigma \subset V$ be a holomorphic curve in V . We may map Σ into M via the sections $s_i : V \rightarrow M$ at $y = y_i$. Using (5.28) we then compute

$$\begin{aligned} \langle c_1(K_M^{\text{orb}}), s_1(\Sigma) \rangle &= \left[\frac{p}{r(p-r)}m + \left(1 - \frac{m}{p-r}\right) \right] \langle c_1(K_V^{1/l}), \Sigma \rangle \\ &= \frac{k}{r} \langle c_1(K_V^{1/l}), \Sigma \rangle, \end{aligned} \quad (5.31)$$

$$\begin{aligned} \langle c_1(K_M^{\text{orb}}), s_2(\Sigma) \rangle &= \left[-\frac{p}{r(p-r)}m + \left(1 + \frac{m}{r}\right) \right] \langle c_1(K_V^{1/l}), \Sigma \rangle \\ &= \frac{pl - k}{p-r} \langle c_1(K_V^{1/l}), \Sigma \rangle. \end{aligned} \quad (5.32)$$

Here we have used that e.g. $\langle D_1, s_1(\Sigma) \rangle = -m \langle c_1(K_V^{1/l}), \Sigma \rangle$, since $y = y_1$ has normal bundle $K_V^{-m/l}$. Since V is Fano, $\langle c_1(K_V^{1/l}), \Sigma \rangle < 0$, and hence M Fano implies that

$$k - pl < 0. \quad (5.33)$$

Of course, this condition is indeed satisfied by the explicit metrics we have constructed. This completes the proof of Theorem 1.4.

5.2. Smooth resolutions: $p = 2$

Setting $p = 2, r = 1$ in the last subsection gives a family of smooth complete Ricci-flat Kähler metrics for each choice of (V, g_V) , leading to Corollary 1.5. These are all defined on the canonical line bundle over M , where $M = \mathbb{P}_V(\mathcal{O} \oplus K^{m/l})$, $m = k - l$, and $0 < m < l$.

For example, we may take $V = \mathbb{CP}^n$ with its standard Kähler–Einstein metric. In this case $0 < m < n + 1$ and the metrics are defined on the total space of the canonical line bundle over $\mathbb{P}_{\mathbb{CP}^n}(\mathcal{O}(0) \oplus \mathcal{O}(-m))$. Note that $n = m = 1$ is precisely the metric found in reference [12].

On the other hand, taking V to be a product of complex projective spaces, as in (4.17), reproduces the metrics discussed in reference [13]. In the latter reference the authors considered each product separately; this was necessary, given the method they use to analyse regularity of the metric. The number of smooth metrics found is $l - 1 = \text{hcf}\{d_a\} - 1$; one can easily verify that this number agrees with the number of smooth resolutions found in the various cases considered in [13].

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Appendix. Limits

In this section we briefly analyse various special limits of the metrics (2.4). Recall that these depend on four real parameters: c, β, ν, μ . First, note that setting $c = 0$ in the functions (2.6) implies that the $g_{\tau\tau}$ component of the metric asymptotes to a constant ($\mp\beta$) as $x \rightarrow \pm\infty$. Therefore the metric is not asymptotically conical. When $c \neq 0$ we may then set $c = 1$ by a diffeomorphism and rescaling of the metric, as we have assumed throughout the paper.

If β is different from zero, it may also be scaled out as an overall coefficient of the metric (2.4), where $\beta = 1$. However, we may also consider asymmetric scalings of the variables x and y , before letting $\beta \rightarrow 0$, with the result depending on which variable goes to zero faster. There are then two cases to consider. Thus, let us first make the substitution $y \rightarrow \beta y$. The resulting metric reads

$$\begin{aligned} g &= \frac{\beta y - x}{4X_\beta(x)} dx^2 + \frac{\beta y - x}{4Y(y)} dy^2 + \frac{X_\beta(x)}{\beta y - x} [d\tau + (1 - y)(d\psi + A)]^2 \\ &\quad + \frac{Y(y)}{\beta y - x} [\beta d\tau + (\beta - x)(d\psi + A)]^2 + (\beta - x)(1 - y)g_V, \end{aligned} \quad (A.1)$$

where the parameter ν in $Y(y)$ has been redefined, and we have introduced the notation

$$X_\beta(x) = \beta(x - \beta) + \frac{n+1}{n+2}(x - \beta)^2 + \frac{2\mu}{(x - \beta)^n} \quad (\text{A.2})$$

to emphasize that $X_\beta(x)$ depends β , as opposed to $Y(y)$. Setting $\beta = 0$ in (A.1) and introducing the change of variable

$$x = \pm \frac{n+1}{n+2} r^2 \quad (\text{A.3})$$

as in Section 2.3, we obtain the positive definite metric

$$g = \frac{1}{H(r)} dr^2 + H(r) r^2 \left(\frac{n+1}{n+2} d\tau + \sigma \right)^2 + r^2 g_T, \quad (\text{A.4})$$

where

$$H(r) = 1 + 2\mu \left(\frac{n+1}{n+2} \right)^{n+3} \frac{(-1)^n}{r^{2n+4}}. \quad (\text{A.5})$$

This is precisely the Calabi ansatz of [4,5]. Note from the metric (A.1) that β plays the role of a resolution parameter in all cases considered in the paper. The parameter ν is fixed by regularity at $x > x_+$ or $x < x_-$, as discussed in Section 3.2, and one is left with two parameters μ and β . This is analogous to the two-parameter family of Ricci-flat metrics on the canonical line bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1$ found in [27], which has regular asymptotic boundary metric $T^{1,1}/\mathbb{Z}_2$. However, in the present case, regularity (in the orbifold sense) of the metrics at $x = x_\pm$ imposes a relation. This leaves β as the only free parameter, measuring the size of the blown-up cycles. It would be interesting to investigate generalisations of the ansatz (2.4), allowing for more than one resolution parameter.

Finally, it is straightforward to repeat the previous analysis setting instead $x \rightarrow \beta x$, by exchanging the roles of x and y . However, in this case if we set $\beta = 0$, the metric becomes

$$g = \frac{y}{4X(x)} dx^2 + \frac{y}{4Y_0(y)} dy^2 + X(x)Y(y)(d\psi + A)^2 + \frac{Y_0(y)}{y} [d\tau + (1-x)(d\psi + A)]^2 - y(1-x)g_V, \quad (\text{A.6})$$

where $Y_0(y)$ denotes the function $Y(y)$ in (2.6), evaluated at $\beta = 0$. We see that the metric is again not asymptotically conical.

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