

UNIVERSITÀ
DEGLI STUDI
DI TORINO

ALMA UNIVERSITAS
TAURINENSIS



Founded in 1404

DEPARTMENT OF
ECONOMICS AND STATISTICS
WORKING PAPER SERIES

Quaderni del Dipartimento di Scienze
Economico-Sociali e Matematico-Statistiche

ISSN 2279-7114

STATIC USE OF OPTIONS IN DYNAMIC PORTFOLIO OPTIMIZATION UNDER TRANSACTION COSTS AND SOLVENCY CONSTRAINTS

A grayscale photograph of a multi-story building facade with arched windows and classical architectural details, serving as a background for the author's name.

STEFANO BACCARIN

Working paper No. 63 - May 2019

Static use of options in dynamic portfolio optimization under transaction costs and solvency constraints

Stefano Baccarin

Università degli Studi di Torino, Dipartimento di Scienze Economico-Sociali e Matematico-Statistiche, Corso Unione Sovietica 218/bis, 10134 Torino, Italy.

Abstract

We study a dynamic portfolio optimization problem where it is possible to invest in a risk-free bond, in a risky stock modeled by a lognormal diffusion and in call options written on the stock. The use of the options is limited to static strategies at the beginning of the investment period. The investor faces transaction costs with a fixed component and solvency constraints and the objective is to maximize the expected utility of the final wealth. We characterize the value function as a constrained viscosity solution of the associated quasi-variational inequality and we prove the local uniform convergence of a Markov chain approximation scheme to compute numerically the optimal solution. Because of transaction costs and solvency constraints the options cannot be perfectly replicated and despite the restriction to static policies our numerical results show that in most cases the investor will keep a significant part of his portfolio invested in options.

1 Introduction

The original formulation of option pricing theory by Black and Scholes [4] has a paradoxical consequence: options, and more general contingent claims, are redundant assets because they can be perfectly replicated. If markets are arbitrage-free, options prices are determined by the cost of the replicating portfolios and apparently options markets have no reason to exist because investors can trade the hedging portfolios instead of the options. However perfect hedging is impossible in real markets and options are not redundant assets. This paradoxical consequence depends on the completeness property of the Black-Scholes market model, that is any contingent claim admits a self-financing replicating strategy. It is sufficient to introduce the jump processes to describe the evolution of the stock prices that the completeness of the market model is lost, not all risk can be hedged away and the notion of pricing by replication is no longer valid (see [8], part III). The most basic fact which does not allow perfect

hedging is that the continuous trading necessary to replicate an option is impossible in the presence of market frictions, in particular of transaction costs. The literature on option pricing and hedging under transaction costs is vast. To avoid prohibitively expensive trading costs many articles formulate the hedging problem in discrete time, allowing for portfolio rebalancing only at fixed time intervals (see [18], [5], [3], [13]). However as the revision interval becomes shorter, decreasing the hedging error, the price of a replicating, or super-replicating portfolio, increases steadily, revealing an unavoidable trade-off between the hedging error and the transaction costs incurred to reduce this error. This trade-off is particularly evident if we consider a fixed component in the trading costs. But even with only proportional transaction costs it has been shown in [24] that in the continuous time limit the cheapest way to super-replicate a European call option is the trivial strategy of buying and holding one share of the underlying, giving the largest possible upper bound for the option price. To balance the risk against the costs of replication an utility indifference approach has been proposed. The indifference, or reservation, buying price is defined as the price to pay which makes the investor indifferent to buy or not the option when he solves a portfolio optimization problem (see for this approach [15], [12], [6], [9], [21], [25]). There is a symmetrical definition for the indifference, or reservation, selling price and the two unit indifference prices turn out to be different. We want to stress here that reservation prices are subjective and not market prices for options. The reservation buying or selling price depends on the choice of the utility function and within a class of utility functions varies greatly by changing the risk-aversion parameter (see, for instance, [9], [21], [25]). Moreover indifference pricing leads to subjective nonlinear pricing rules where the unit buying or selling price depends on how many options the investor decides to buy or sell. More importantly indifference prices (also the marginal ones) depend on the initial position of the investor. It has been argued that the buying price is always below the writing price (see on this point [12], [9]) but this result is valid only for a given utility function, risk aversion and initial endowment. Even if we use the exponential utility (when the reservation prices do not depend on the initial cash) it is not difficult to build an example where the writing price of one investor is less than the buying price of another investor, because of different initial endowments in stocks. This fact has a nice implication: there certainly are situations where a buyer and a writer with different holdings can agree on a common price to trade an option.

In this paper we follow the utility based approach but we look at the absence of market completeness due to transaction costs from a different perspective. Since perfect replication of options is not possible, and super-replication too expensive, we intend to investigate if it is profitable to use options as a specific asset class in a portfolio optimization problem. As a first step in this direction we consider a simplified setting. Our agent invests only in three financial assets, a risky security, which we also call the stock, a risk-free asset, which we refer to as the bond and a call option with underlying asset the risky security. We model the price evolution of the risky security by a lognormal diffusion while the risk free asset grows at a fixed continuously compounded rate. The objec-

tive is to maximize the expected utility from the portfolio liquidation at a given final horizon. This simplified setting can describe a passive investment strategy where an agent invests in an index fund, or ETF, tracking a broad market index, in a money instrument, such as Treasury Bills, and in call options written on the market index. We are considering a small investor whose trading has no influence on market prices and who pays significant transaction costs whenever he trades. Moreover his overall financial position is subject to solvency constraints such as the margin requirements required by brokers and exchanges to shortsell securities, buy on margin and write options. The transaction costs are assumed of a fixed plus proportional type and we formulate the portfolio model as an impulse control problem. Most of the literature considers only proportional transaction costs because this assumption simplifies the analysis (see, for instance, [11], [23], [19]). However considering only proportional costs leads to singular control problems where the trading policies have a rather unrealistic nature. The optimal strategy is to make the minimal effort, in terms of transactions, to maintain the portfolio inside a no-trading region. When the portfolio reaches the boundary of this region the investor trades continuously making an infinite number of infinitesimally small transactions to prevent the portfolio to cross the borders. This kind of policy, which is a local time process, is not a feasible one in the real world. Fixed transaction costs and impulse control lead to more realistic policies where the investor intervenes a finite number of times in any time interval, trading a finite amount of the assets.

We will limit the use of options to static policies: the investor buys or writes the call options, with maturity the final horizon, at the beginning of his investment period and hold them until expiration. This limitation has the advantage that it is not necessary to specify the dynamics of the option price to solve our problem, we need only to know its initial price. Despite this restriction the main result of this paper is to show that a static use of options is profitable in portfolio optimization even if we do not have a theory of the market prices of options. In fact, we will show numerically that the only presence of transaction costs induces in most cases our agent to keep a significant part of his portfolio invested in a long, or in a short, position in these securities. The size and the sign of the investment will depend on the investor's initial holdings, his risk aversion and the initial price of the option. It is convenient to use options also if the initial price is equal to the Black and Scholes price. The reason lies in the form of the optimal strategy without transaction costs: the so-called Merton's problem. In this model the agent trades continuously in order to keep constant the fraction of wealth invested in the different assets. The presence of even small transaction costs has a deep impact in this kind of policy reducing strongly the frequency of trading and transforming it to an almost "buy and hold" trading strategy (see [19], [1]). Options can mitigate this effect. If the agent starts with a large fraction of wealth invested in cash it is better to buy options rather than stocks, and viceversa it is more profitable to sell options rather than stocks if the initial wealth is mainly in stocks. These derivatives can be profitably used as substitutes of the underlying stock reducing the transaction costs in a leveraged way and improving the efficiency of the optimal policy. Of course if the initial

price is not aligned with the Black and Scholes price the investor will also use options to exploit the difference in prices. However the unit reservation buying (selling) price will decrease (increase) with the number of traded options. It will not be possible to build a deterministic arbitrage because the transaction costs introduce an unavoidable element of risk in the final position making perfect hedging impossible.

The paper is organized as follows. In section 2 we give a precise formulation of our portfolio model as a parabolic impulse control problem with state constraints. We propose a solution method using the dynamic programming approach by considering the auxiliary problem where the initial number of options is different from zero but it is not possible to trade the options. The rest of the paper focuses on the value function V of this auxiliary problem because the optimal solution of our model is derived by solving a static optimization problem on the values assumed by V at the initial time. In section 3 we characterize V as the unique constrained viscosity solution of the associated Hamilton-Jacobi-Bellman quasi-variational inequality verifying certain boundary conditions. Moreover we describe in a formal way the optimal policy which has the typical iterative structure of impulse control problems. In section 4 we propose a Markov chain approximation method to solve numerically the Hamilton-Jacobi-Bellman quasi-variational inequality. We present a discrete time dynamic programming scheme for the controlled Markov chain approximating the diffusion, which can be solved backwards in time from the final condition. The main result of this section is to show the local uniform convergence of the solution of this approximation scheme to the value function of the continuous problem. Section 5 presents some numerical results of our numerical procedure for investors whose preferences are modeled by a power utility function which has a constant relative risk aversion index. We show the form of the optimal control when a long or short position in options is present in the portfolio. Moreover we focus on how the investment in options depends on the investor initial holdings in bonds and stocks and how it varies with the strikes prices of the options for a given initial stock price. A case where the call price is not aligned with the Black and Scholes price is also considered. Finally section 6 concludes the paper with some final remarks and suggestions for future research.

2 The model

We describe the dynamics of the price P of the risky security by the lognormal diffusion

$$dP(t) = \mu P(t)dt + \sigma P(t)dW(t) \quad (1)$$

where W_t is a Wiener process on the filtered probability space $(\Omega, F, P, \mathbb{F}_t)$ and \mathbb{F}_t is the P -augmentation to the natural filtration generated by W_t . Let α and β denote respectively the number of stocks and the number of call options and B the amount of money invested in the risk-free asset. Without control B grows

at a fixed instantaneous rate r

$$dB(t) = rB(t)dt.$$

The initial portfolio at $t = 0^-$, that is before any intervention, is denoted by $\pi_0 = (B_0, \alpha_0, \beta_0)$. At any time the investor can buy ($\xi > 0$) or sell ($\xi < 0$) the number $\xi \in \mathbb{R}$ of stocks, reducing (or increasing) correspondingly the investment in the risk-free asset. Moreover, but only at $t = 0$, the investor can decide to buy ($\rho > 0$) or sell ($\rho < 0$) the number $\rho \in \mathbb{R}$ of call options, with underlying asset the risky security and strike price P_{str} . Let $P^c < P_0$ be the given market price of a call option at $t = 0$. To buy or sell the risky securities (stocks or options) it is necessary to pay transaction costs which are drawn immediately from the risk-free asset. We assume these costs of a fixed plus proportional type

$$C(x, P) = K + c|x|P \quad K > 0, 0 \leq c < 1$$

where x is the number of securities bought ($x > 0$) or sold ($x < 0$) and P is their unit price. A portfolio control policy $p = \{(\rho, \xi_0), (\tau_i, \xi_i)\}, i = 1, 2, \dots$, is made of a couple (ρ, ξ_0) , representing the first transactions in options and stocks at $t = 0$, and a sequence of stopping times τ_i and corresponding random variables ξ_i , which represent the subsequent number of stocks bought (or sold) at the stopping times τ_i . We denote by $\bar{B}_0, \bar{\alpha}_0, \bar{\beta}_0$ the values of B, α, β after the initial transactions (ρ, ξ_0)

$$\begin{cases} \bar{B}_0 = B_0 - \xi_0 P_0 - C(\xi_0, P_0) - \rho P^c - C(\rho, P^c) \\ \bar{\alpha}_0 = \alpha_0 + \xi_0 \\ \bar{\beta}_0 = \beta_0 + \rho \end{cases} \quad (2)$$

A policy p is said to be feasible if it verifies the following conditions ($i \geq 1$):

$$\begin{cases} \tau_i \text{ is a } \mathbb{F}_t \text{ stopping time} \\ 0 \leq \tau_i \leq \tau_{i+1} \quad \forall i \\ \lim_{i \rightarrow +\infty} \tau_i = +\infty \text{ almost surely} \\ \xi_i \text{ is } F_{\tau_i} \text{ measurable.} \end{cases}$$

The dynamics of the portfolio $\pi^p(t) = (B^p(t), \alpha^p(t), \beta^p(t))$, controlled by policy p , is given by the constant number of options $\beta^p = \beta_0 + \rho$, and by the following set of stochastic differential equations:

$$\begin{cases} dB^0(t) = rB^0(t)dt, \quad B^0(0) = \bar{B}_0 \\ d\alpha^0(t) = 0, \quad \alpha^0(0) = \bar{\alpha}_0 \end{cases} \quad \text{for } t \in [0, \tau_1]$$

and $\forall i \geq 1$

$$\begin{cases} dB^i(t) = rB^i(t)dt \\ B^i(\tau_i) = B^{i-1}(\tau_i) - \xi_i P(\tau_i) - C(\xi_i, P(\tau_i)) \\ d\alpha^i(t) = 0 \\ \alpha^i(\tau_i) = \alpha^{i-1}(\tau_i) + \xi_i \end{cases} \quad \text{for } t \in [\tau_i, \tau_{i+1}]$$

When $\tau_i < \tau_{i+1}$ we set $(B^p(t), \alpha^p(t)) = (B^i(t), \alpha^i(t))$ for $t \in [\tau_i, \tau_{i+1})$ and if, for instance, $\tau_{i-1} < \tau_i = \tau_{i+1} = \dots = \tau_{i+n} < \tau_{i+n+1}$, then we set

$$\begin{cases} (B^p(\tau_{i+n}^-), \alpha^p(\tau_{i+n}^-)) = (B^{i-1}(\tau_i), \alpha^{i-1}(\tau_i)) \\ (B^p(\tau_{i+n}), \alpha^p(\tau_{i+n})) = (B^{i+n}(\tau_{i+n}), \alpha^{i+n}(\tau_{i+n})) \end{cases}$$

where $(B^p(\tau_{i+n}^-), \alpha^p(\tau_{i+n}^-))$ are the left limits at $\tau_i = \dots = \tau_{i+n}$. The resulting process $\pi^p(t)$ is cadlag and adapted to the filtration \mathbb{F}_t . Besides the transaction costs our investor must face solvency constraints on his overall financial position that correspond to the margin requirements required by brokers to buy on margin, shortsell securities and trade options. We denote by

$$L(\alpha, P) = \begin{cases} \max\{\alpha P - C(\alpha, P), 0\} & \text{if } \alpha \geq 0 \\ \alpha P - C(\alpha, P) & \text{if } \alpha < 0 \end{cases}$$

the liquidation value of α stocks, which is set to zero if selling a long position in stocks is not sufficient to cover the fixed cost K . The solvency level of the portfolio (B, α, β) , when the stock price is P , is defined by

$$Sol(B, \alpha, \beta, P) = B + L(\alpha - (\frac{|\beta| + \frac{K}{P}}{1-c})1_{\beta < 0}, P)$$

where 1 is the indicator function. We will require that $Sol(B, \alpha, \beta, P) \geq 0$.

Note that the options influence the portfolio solvency level only if the agent writes them. As shown in [24] in the presence of transaction costs the cheapest super replicating strategy of a long position in one option is to buy one unit of the underlying. When $\beta < 0$ the condition $Sol(B, \alpha, \beta, P) \geq 0$ means that at any time the portfolio's liquidation value must stay positive after allowing the investor with a short position to perform the cheapest super replicating strategy. Since we assume cash settlement of the options, taking account of the transaction costs it is necessary to buy $\frac{|\beta| + \frac{K}{P}}{1-c}$ units of stocks because $L(\frac{|\beta| + \frac{K}{P}}{1-c}, P) = |\beta|P$. On the contrary the cheapest way to super replicate a short position of one option is simply to invest nothing and thus a long position in the options has no impact on the portfolio's solvency level (see on this point [9]). Therefore to satisfy these solvency constraints the agent's portfolio $\pi = (B, \alpha, \beta)$ must verify $(\pi, P) \in \overline{SOL}$ where $\overline{SOL} \subset \mathbb{R}^4$ is the closed solvency region

$$\overline{SOL} \equiv \{(B, \alpha, \beta, P) \in \mathbb{R}^3 \times \mathbb{R}_+ : Sol(B, \alpha, \beta, P) \geq 0\}.$$

The initial portfolio π_0 is supposed to satisfy this condition. We call a feasible policy p admissible if $(B^p(t), \alpha^p(t), \beta^p, P(t)) \in \overline{SOL}$, $\forall t \in [0, T]$ and we denote by $A_0(\pi_0, P_0, P^c)$ the set of admissible policies given the initial condition (π_0, P_0, P^c) . The agent's preferences are described by an utility function $U : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $U(0) = 0$. We assume that U is continuous, increasing and that it satisfies for some $C > 0$ and $0 < \gamma < 1$ the sublinear growth

$$U(L) \leq CL^\gamma.$$

The investor's objective is to maximize the expected utility of the liquidation value of his portfolio at the fixed terminal horizon T .

Assuming cash settlement of the options, without additional costs, the liquidation value L_T of a final portfolio (B, α, β) is given by

$$L_T(B, \alpha, \beta, P) = B + L(\alpha, P) + \beta \times (P - P_{str})^+.$$

Note that we certainly have $L_T(B^p(T), \alpha^p(T), \beta^p, P(T)) \geq 0$ if p is admissible. The payoff functional J^p associated to policy p is therefore

$$J^p = E_{\pi_0, P_0, P^c} [U(L_T(B^p(T), \alpha^p(T), \beta^p, P(T)))] .$$

Our problem is to find M and, if it exists p^* , such that

$$M = \sup_{p \in A_0(\pi_0, P_0, P^c)} J^p = J^{p^*}$$

We will solve this problem by using the dynamic programming methodology. Let $F_0(\pi_0, P_0, P^c) \subset \mathbb{R}^2$ be the set of the first two interventions at $t = 0$ that are admissible

$$F_0(B_0, \alpha_0, \beta_0, P_0, P^c) = \{(\xi_0, \rho) \in \mathbb{R}^2 : (\bar{B}_0, \bar{\alpha}_0, \bar{\beta}_0, P_0) \in \overline{SOL}\}$$

where $\bar{B}_0, \bar{\alpha}_0, \bar{\beta}_0$ are defined in (2). Since, for given values $B_0, \alpha_0, \beta_0, P_0, P^c$, the variables $\bar{B}_0, \bar{\alpha}_0, \bar{\beta}_0, L(\bar{\alpha}_0 - (\frac{|\bar{\beta}_0| + \frac{\kappa}{P}}{1-c})1_{\bar{\beta}_0 < 0}, P_0)$ and $Sol(\bar{B}_0, \bar{\alpha}_0, \bar{\beta}_0, P_0)$ are upper semicontinuous functions of (ρ, ξ_0) it follows that $F_0(\pi_0, P_0, P^c)$ is a closed subset of \mathbb{R}^2 . Moreover it is also bounded, because of the variable part of the transaction costs, and consequently compact. Now, we consider the auxiliary problem where the process starts in $t \in [0, T]$, with the initial number of options β_t in the portfolio but it is not possible to trade the options, that is $\beta_s = \beta_t, \forall s \in [t, T]$. For this problem a policy \tilde{p} is made only of a sequence $(\tau_i, \xi_i), i = 1, 2, \dots$, representing the number of stocks bought (or sold) at the stopping times $\tau_i \geq t$. One such policy is admissible if $(B_s, \alpha_s, \beta_t, P_s) \in \overline{SOL}, \forall s \in [t, T]$. We set $\bar{Q} \equiv [0, T] \times \overline{SOL}$ and we denote by $A(t, B, \alpha, \beta, P)$ the set of these admissible policies when the system starts in $(t, B, \alpha, \beta, P) \in \bar{Q}$. We define the value function $V : \bar{Q} \rightarrow \mathbb{R}$ by

$$V(t, B, \alpha, \beta, P) = \sup_{\tilde{p} \in A(t, B, \alpha, \beta, P)} J^{\tilde{p}}(t, B, \alpha, \beta, P)$$

where

$$J^{\tilde{p}}(t, B, \alpha, \beta, P) = E_{t, B, \alpha, \beta, P} [U(L_T(B^{\tilde{p}}(T), \alpha^{\tilde{p}}(T), \beta, P(T)))] .$$

Considering the first interventions (ρ, ξ_0) at $t = 0$ we have clearly

$$M = \sup_{p \in A_0(\pi_0, P_0, P^c)} J^p = \sup_{(\rho, \xi_0) \in F_0(\pi_0, P_0, P^c)} V(0, \bar{B}_0, \bar{\alpha}_0, \bar{\beta}_0, P_0) . \quad (3)$$

Moreover if V is upper semicontinuous in \overline{SOL} , then an optimal (ρ^*, ξ_0^*) certainly exists because $F_0(\pi_0, P_0, P^c)$ is compact and then it follows

$$M = V(0, \bar{B}_0^*, \bar{\alpha}_0^*, \bar{\beta}_0^*, P_0) .$$

Therefore in the sequel we will focus our attention on the value function V as M can be derived by V by considering a static optimization problem. We assume that V verifies the following dynamic programming principle (see [24] chapter five, and [20]), which holds for any $(t, B, \alpha, \beta, P) \in \bar{Q}$ and $\{\mathbb{F}_s\}$ – stopping time $\theta \in [t, T]$:

$$V(t, B, \alpha, \beta, P) = \sup_{\bar{p} \in A(t, B, \alpha, \beta, P)} E_{t, B, \alpha, \beta, P} [V(\theta, B(\theta), \alpha(\theta), \beta, P(\theta))].$$

We denote by $F(B, \alpha, \beta, P)$ the set of admissible purchases or sales of stocks when the agent's position is $(B, \alpha, \beta, P) \in \overline{SOL}$

$$F(B, \alpha, \beta, P) \equiv \{\xi \in \mathbb{R} : (B - \xi P - C(\xi, P), \alpha + \xi, \beta, P) \in \overline{SOL}\}$$

and by F the subset of \overline{SOL} where $F(B, \alpha, \beta, P) \neq \emptyset$.

Now, we consider the non local operator \mathcal{M} that applied to V will give us the value function after the best possible intervention. For any function $Z : \bar{Q} \rightarrow \mathbb{R}$ we define $\mathcal{M}Z : \bar{Q} \rightarrow \mathbb{R}$ by

$$\mathcal{M}Z(t, \pi, P) \equiv \begin{cases} \sup_{\xi \in F(\pi, P)} Z(t, B + \xi P - C(\xi, P), \alpha + \xi, \beta, P) & \text{if } (\pi, P) \in F \\ -1 & \text{if } (\pi, P) \notin F . \end{cases}$$

When Z is upper-semicontinuous it can be shown (see, for instance, [22]) that there exists a Borel measurable function $\xi_Z^* : F \rightarrow \mathbb{R}$ such that for any $(\pi, P) \in F$ we have

$$\mathcal{M}Z(t, \pi, P) \equiv Z(t, B + \xi_Z^*(\pi, P) \times P - C(\xi_Z^*(\pi, P), P), \alpha + \xi_Z^*(\pi, P), \beta, P) . \quad (4)$$

If at time t it is optimal to transact it holds $V(t, \pi(t), P(t)) = \mathcal{M}V(t, \pi(t), P(t))$ but in general we have $V \geq \mathcal{M}V$ because at any time t it is also possible to let the system evolve freely. In this second case it is easy to show formally that the application of the dynamic programming principle for an infinitesimal interval leads to the following condition

$$-\frac{\partial V}{\partial t} - \mathcal{L}V \geq 0$$

where the second order differential operator

$$\mathcal{L}V(t, B, \alpha, \beta, P) = rB \frac{\partial V}{\partial B} + \mu P \frac{\partial V}{\partial P} + \frac{1}{2} \sigma^2 P^2 \frac{\partial^2 V}{\partial P^2}$$

corresponds to the infinitesimal generator of the uncontrolled process. Since only one of the two possible decisions, to trade or not to trade, must be taken

optimally at any time t , we can argue that V is a solution of the following Hamilton-Jacobi-Bellman quasi-variational inequality (HJBQVI) in \bar{Q}

$$\min \left\{ -\frac{\partial V}{\partial t} - \mathcal{L}V, V - \mathcal{M}V \right\} = 0. \quad (5)$$

This is indeed the case but, as it is usual in impulse control problems, V is not regular enough to be a classical solution of (5). It is not even continuous in some points of \bar{Q} , such as, for instance, the points along the line $\alpha = 0$ in $t = T$. In the next section we characterize V as the unique constrained viscosity solution of (5) verifying certain boundary conditions and we describe the form of the optimal strategy.

3 Viscosity characterization and optimal trading strategy

Let SOL be the interior of \overline{SOL} . We denote by Q the set $Q \equiv [0, T) \times SOL$ and by ∂^*Q the boundary

$$\partial^*Q \equiv ([0, T) \times \partial SOL) \cup (T \times \overline{SOL}).$$

When it is convenient we will denote $(B, \alpha, \beta, P) \in \overline{SOL}$ by $x = (B, \alpha, \beta, P)$. To characterize V as a solution of (5) we are interested in the values assumed by V on ∂^*Q and to determine some bounds for V in $\bar{Q} \equiv [0, T] \times \overline{SOL}$. At the final date T we have, $\forall (B, \alpha, \beta, P) \in \overline{SOL}$

$$\begin{aligned} V(T, B, \alpha, \beta, P) &= U(L_T(B, \alpha, \beta, P)) = \\ &= U(B + L(\alpha, P) + \beta \times (P - P_{str})^+). \end{aligned} \quad (6)$$

For $t < T$ the behaviour of V at the boundary ∂^*Q depends on the sign of β , that is if the options are written or bought.

If $\beta \geq 0$ and $\min(B, \alpha) < 0$ it is necessary to clear the position in stocks and bonds and keep only the options, otherwise the process could leave \overline{SOL} with a positive probability. In this case we have

$$\begin{aligned} V(t, B, \alpha, \beta, P) &= \mathcal{M}V(t, B, \alpha, \beta, P) = V(t, 0, 0, \beta, P) = \\ &= E_{t,P}[U(\beta \times (P_T - P_{str})^+)] \end{aligned}$$

that is V is the expected value of the β options in T . If $\beta \geq 0$, $x \in \partial SOL$ but $\min(B, \alpha) \geq 0$ the agent stays solvent and it is not possible to intervene clearing the position in stocks because of the fixed cost K . Except for the case $B = \alpha = 0$ the value of V is not known a priori but it is greater than $E_{t,P}[U(\beta \times (P_T - P_{str})^+)]$.

If $\beta < 0$ and $Sol(B, \alpha, \beta, P) = 0$ it is necessary to perform the cheapest super

replication strategy of β options by buying $-\beta$ stocks and clearing out the position in bonds. We have ($\beta < 0$)

$$\begin{aligned} V(t, B, \alpha, \beta, P) &= V(t, 0, -\beta, \beta, P) = \\ &= E_{t,P}[U(-\beta \times P_T + \beta \times (P_T - P_{str})^+)]. \end{aligned}$$

with $V(t, B, \alpha, \beta, P) = MV(t, B, \alpha, \beta, P)$ if $\alpha \neq -\beta$.

It is not difficult to see that V is locally bounded in \bar{Q} and it verifies a sublinear growth condition.

Proposition 1 *We have, $\forall (t, B, \alpha, \beta, P) \in \bar{Q}$,*

$$0 \leq V(t, B, \alpha, \beta, P) \leq Ce^{\delta(T-t)}(B + \alpha P + \beta P_{BS})^\gamma \quad (8)$$

where P_{BS} is the Black-Scholes price (given $P, P_{str}, r, \sigma, t, T$) of one call option and

$$\delta = \gamma \left(r + \frac{(\mu - r)^2}{2\sigma^2(1 - \gamma)} \right).$$

Proof. Since $J^{\bar{p}}(t, B, \alpha, \beta, P) \geq 0, \forall \bar{p} \in A(t, B, \alpha, \beta, P)$ it follows immediately $V(t, B, \alpha, \beta, P) \geq 0$. Moreover $V(t, B, \alpha, \beta, P)$ is less than the value function of the same problem without transaction costs and solvency constraints. In this case the optimal policy is to perfectly replicate a long (or short) position in options and to invest the remaining value of the portfolio according to the optimal policy of a Merton problem, see [16]. Given an initial portfolio (B, α) the optimal expected final value of a Merton problem (without consumption, finite horizon and power utility Cx^γ) is equal to $Ce^{\delta(T-t)}(B + \alpha P)^\gamma$. Since the replicating portfolio to build in t has a price of $\beta \times P_{BS}$ the upper bound in (8) follows. ■

From (8) it follows in particular that V is continuous in $(t, 0, 0, 0, P)$ verifying

$$\lim_{\substack{(t', x') \in \bar{Q} \\ (t', x') \rightarrow (t, 0, 0, 0, P)}} V(t', x') = V(t, 0, 0, 0, P) = 0 \quad \forall (t, 0, 0, 0, P) \in \bar{Q}. \quad (9)$$

However V is not continuous in many other points of \bar{Q} and it is necessary to consider the notion of discontinuous viscosity solutions of the HJBQVI. We recall now the definitions of discontinuous viscosity subsolutions and supersolutions and of constrained viscosity solutions of (5) in $Q = [0, T] \times SOL$. Let $USC(\bar{Q})$ and $LSC(\bar{Q})$ be the sets of upper-semicontinuous (usc) and lower-semicontinuous (lsc) functions defined on \bar{Q} . Given a locally bounded function $u : \bar{Q} \rightarrow \mathbb{R}$ let u^* and u_* be respectively the upper-semicontinuous and lower-semicontinuous envelope of u

$$\left| \begin{array}{ll} u^*(t, x) = \limsup_{\substack{(t', x') \in \bar{Q} \\ (t', x') \rightarrow (t, x)}} u(t', x') & \forall (t, x) \in \bar{Q} \\ u_*(t, x) = \liminf_{\substack{(t', x') \in \bar{Q} \\ (t', x') \rightarrow (t, x)}} u(t', x') & \forall (t, x) \in \bar{Q}. \end{array} \right.$$

Definition 2 Given $\mathcal{O} \subseteq \overline{SOL}$, a locally bounded function $u : \overline{Q} \rightarrow \mathbb{R}$ is called a viscosity subsolution (resp. supersolution) of (5) in $[0, T) \times \mathcal{O}$ if for all $(\bar{t}, \bar{x}) \in [0, T) \times \mathcal{O}$ and $\varphi(t, x) \in C^{1,2}(\overline{Q})$ such that $(u^* - \varphi)(\bar{t}, \bar{x}) = 0$ (resp. $(u_* - \varphi)(\bar{t}, \bar{x}) = 0$) and (\bar{t}, \bar{x}) is a maximum of $u^* - \varphi$ (resp. a minimum of $u_* - \varphi$) on \overline{Q} , we have

$$\min \left\{ -\frac{\partial \varphi}{\partial t}(\bar{t}, \bar{x}) - \mathcal{L}\varphi(\bar{t}, \bar{x}), u^*(\bar{t}, \bar{x}) - \mathcal{M}u^*(\bar{t}, \bar{x}) \right\} \leq 0$$

(resp. u_* and ≥ 0).

Definition 3 We say that a locally bounded function $u : \overline{Q} \rightarrow \mathbb{R}$ is a constrained viscosity solution of (5) in Q if it is a viscosity supersolution in $Q = [0, T) \times SOL$ and a viscosity subsolution in $[0, T) \times \{SOL \cup \partial SOL\}$.

We state now the viscosity property of the value function.

Theorem 4 The value function $V(t, B, \alpha, \beta, P)$ is a constrained viscosity solution of (5) in Q .

The proof of the theorem follows along the lines of theorem 5.3 in [20]. The state constraint $(B_s, \alpha_s, \beta_s, P_s) \in \overline{SOL}$, $\forall s \in [t, T]$, makes it possible to extend the subsolution property to the lateral boundary $[0, T) \times \partial SOL$. See also the proof of theorem 3.7 in [22]. However there can be many constrained viscosity solutions of (5) and in order to characterize the value function we need to show that the boundary conditions satisfied by V are sufficient to determine a unique constrained viscosity solution of (5). An essential step to prove uniqueness is to show a comparison principle between sub and supersolutions and to look at V^* as a subsolution and at V_* as a supersolution. However, because V is discontinuous at some points on the lateral boundary, we state only a weak comparison principle which does not necessarily hold on $[0, T) \times \partial SOL$. Moreover V can be discontinuous whenever $\alpha = 0$ and we distinguish the cases $\alpha > 0$ and $\alpha < 0$. Let $SOL^+, Q^+, SOL^-, Q^-, D$ be the sets

$$\begin{cases} SOL^+ \equiv \{(B, \alpha, \beta, P) \in SOL : \alpha > 0\}, & Q^+ \equiv [0, T) \times SOL^+ \\ SOL^- \equiv \{(B, \alpha, \beta, P) \in SOL : \alpha < 0\}, & Q^- \equiv [0, T) \times SOL^- \\ D \equiv \{(t, B, \alpha, \beta, P) \in \overline{Q} : \alpha = 0\} \end{cases}$$

and $\overline{SOL}^+, \overline{SOL}^-$ the closures of SOL^+, SOL^- .

Theorem 5 (Weak comparison principle) Let $u \in USC(\overline{Q})$ be a viscosity subsolution of (5) in $[0, T) \times \{SOL \cup \partial SOL\}$ and $v \in LSC(\overline{Q})$ be a viscosity

supersolution of (5) in Q . Assume that

$$\left| \begin{array}{l} \limsup_{\substack{(t',x') \in Q^+ \\ (t',x') \rightarrow (T,x)}} u(t',x') \leq \liminf_{\substack{(t',x') \in Q^+ \\ (t',x') \rightarrow (T,x)}} v(t',x') \quad \forall (T,x) \in (T \times \overline{SOL}^+) \\ \limsup_{\substack{(t',x') \in Q^- \\ (t',x') \rightarrow (T,x)}} u(t',x') \leq \liminf_{\substack{(t',x') \in Q^- \\ (t',x') \rightarrow (T,x)}} v(t',x') \quad \forall (T,x) \in (T \times \overline{SOL}^-) \\ \limsup_{\substack{(t',x') \in Q \\ (t',x') \rightarrow (t,0,0,P)}} u(t',x') \leq \liminf_{\substack{(t',x') \in Q \\ (t',x') \rightarrow (t,0,0,P)}} v(t',x') \quad \forall t \in [0, T] \end{array} \right. \quad (10)$$

and that for some constant $0 < C_1 < \infty$

$$\left| \begin{array}{l} u(t, B, \alpha, \beta, P) \geq -C_1(B + \alpha P + \beta P_{BS})^\gamma \quad \forall (t, B, \alpha, \beta, P) \in \bar{Q} \\ v(t, B, \alpha, \beta, P) \leq C_1(B + \alpha P + \beta P_{BS})^\gamma \quad \forall (t, B, \alpha, \beta, P) \in \bar{Q}. \end{array} \right. \quad (11)$$

Then $u \leq v$ on $Q \setminus D$.

The proof of this theorem follows along the lines of Theorem 3.2 in [1]. See also theorem 3.8 in [22] and theorem 5.6 in [20]. As SOL is unbounded the comparison is given in the set of functions satisfying the sublinear growth (11). In order to use this comparison principle we need to know the behavior of V approaching the final boundary $T \times \overline{SOL}$. It is possible to show (as in Lemma 3.2 in [1]) that V verifies the following limit conditions near the boundary $T \times \overline{SOL}$

$$\left| \begin{array}{l} \lim_{\substack{(t',x') \in Q \\ (t',x') \rightarrow (T,x)}} V(t',x') = U(L_T(x)) \quad \forall (T,x) \in (T \times \overline{SOL}) \setminus D \\ \lim_{\substack{(t',x') \in Q^+ \\ (t',x') \rightarrow (T,B,0,\beta,P)}} V(t',x') = U(B + \beta(P - P_{str})^+) \quad \forall (T,B,0,\beta,P) \in D \\ \lim_{\substack{(t',x') \in Q^- \\ (t',x') \rightarrow (T,B,0,\beta,P)}} V(t',x') = U(B - K + \beta(P - P_{str})^+) \quad \forall (T,B,0,\beta,P) \in D. \end{array} \right. \quad (12)$$

The next theorem gives a complete viscosity characterization of the value function using the continuity in $(t, 0, 0, P)$, the final boundary condition and the sublinear growth of V , also taking account of the discontinuity in D .

Theorem 6 *The value function V is continuous in $Q \setminus D$ and it is the unique, in $Q \setminus D$, constrained viscosity solution of (5) in Q which verifies the boundary conditions (9), (12) and the sublinear growth (8).*

Proof. We apply Theorem 5 to the viscosity subsolution V^* and to the viscosity supersolution V_* . The conditions (10) and (11) are verified because of (9), (12)

and (8). Therefore $V^* \leq V_*$ on $Q \setminus D$ and since by definition $V^* \geq V_*$ it follows that $V^* = V_*$ and V is continuous in $Q \setminus D$. Moreover suppose \bar{V} is another constrained viscosity solution of (5) in Q which verifies the boundary conditions (9), (12) and (8). By the comparison principle we have $\bar{V}^* \leq V_* = V^* \leq \bar{V}_*$ and consequently $V = \bar{V}$ in $Q \setminus D$. ■

Now we describe, in a formal way, the optimal strategy. Let $A \subset \bar{Q}$ and $B \equiv \bar{Q} \setminus A$ be the regions defined by

$$\left\{ \begin{array}{l} A \equiv \{(t, \pi, P) \in \bar{Q} : V(t, \pi, P) = \mathcal{M}V(t, \pi, P)\} \\ B \equiv \{(t, \pi, P) \in \bar{Q} : V(t, \pi, P) > \mathcal{M}V(t, \pi, P)\} . \end{array} \right.$$

If $(t, \pi(t), P(t)) \in A$ then it is optimal to intervene. A is called the intervention region and the best action is given by $\xi_V^*(\pi(t), P(t))$, where ξ_V^* is the function defined in (4). When $(t, \pi(t), P(t)) \in B$, the continuation region, it is not optimal to intervene and the system evolves freely. Therefore it is natural to guess that, for the initial condition $(t, B, \alpha, \beta, P) \in \bar{Q}$, the optimal policy $\tilde{p}^* = \{(\tau_i^*, \xi_i^*)\} = \tilde{p}^*(t, B, \alpha, \beta, P)$ will be defined recursively by $(\tau_0^* \equiv t, i \in N)$

$$\left\{ \begin{array}{l} \tau_{i+1}^* = \inf \{s \geq \tau_i^* \mid (s, \pi^{\tilde{p}^*}(s), P(s)) \in A\} \\ \xi_{i+1}^* = \begin{cases} \xi_V^*(\pi^{\tilde{p}^*}(\tau_{i+1}^*), P(\tau_{i+1}^*)) & \text{if } \tau_{i+1}^* < \infty \\ \text{arbitrary} & \text{if } \tau_{i+1}^* = +\infty . \end{cases} \end{array} \right.$$

4 Markov chain numerical approximation

As usual in stochastic control problems the Hamilton-Jacobi-Bellman quasi-variational inequality (5) can only be solved by a numerical procedure. We will use the Markov chain approximation method due to Kushner (see [17] and for an application to options reservation prices [12], [10], [25]). The basic idea is to approximate the controlled Markov diffusion by a controlled Markov chain and to apply the discrete time dynamic programming principle. We propose a discrete dynamic programming scheme for this controlled Markov chain, which corresponds to the HJBQVI and can be solved backward in time from the final condition (6). To show that the solution V^h of this discrete inequality converges to the value function V of the continuous problem we will use the Barles-Souganidis viscosity solution method (see [2]). We will show that the usc envelope of V^h converges, as the step size h goes to zero, to a viscosity subsolution (supersolution) of (5). Then by applying the comparison principle it follows that V^h converges locally uniformly to V .

Consider the partition $0 = t_0 < t_1^h < \dots < t_{n-1}^h < t_n = T$ of the time interval $[0, T]$ where $t_i^h = ih$ for $i = 0, \dots, n$ and $h = \frac{T}{n}$ is the time step. The Markov chain describing the discrete stock price process $P^h(t_i)$ is modeled by the difference

equation

$$P^h(t_{i+i}) = \begin{cases} P^h(t_i) \times u & \text{with probability } p_u \\ P^h(t_i) \times d & \text{with probability } p_d = 1 - p_u \end{cases} \quad (13)$$

where the up and down movements and the corresponding probabilities p_u, p_d are obtained by equating the first and second moments of the chain with those of the continuous process. There are several ways u, d and the corresponding probabilities p_u, p_d , can be chosen in such a way that the discrete process $P^h(t_i)$ converges in distribution to $P(t)$ as $h \rightarrow 0$ (see, for a proof of convergence [7]). We used $p_u = p_d = \frac{1}{2}$ and

$$\begin{cases} u = e^{(\mu - \frac{1}{2}\sigma^2)h + \sigma\sqrt{h}} \\ d = e^{(\mu - \frac{1}{2}\sigma^2)h - \sigma\sqrt{h}} \end{cases}$$

which are an approximation of order $\mathcal{O}(\sqrt{h^3})$ of the exact values which are obtained by matching the first and second moments of $P^h(t_i)$ and $P(t)$ when $p_u = p_d = \frac{1}{2}$. We denote by PT^h the price binomial tree generated by (13) in the interval $[0, T]$ and by $PT_{t_i}^h$ the possible values in t_i . By analogy with the price process we discretize the bond space B by using the deterministic difference equation

$$B^h(t_{i+i}) = B^h(t_i)e^{rh} \quad (14)$$

starting from a set of initial conditions

$$BG_0^h = \{l \times \lambda_B h : l = 0, \pm 1, \pm 2, \dots\} \quad (15)$$

BG^h will denote the bond grid generated by (14), (15) and $BG_{t_i}^h$ the possible values in t_i . We discretize the α and β spaces by the one-dimensional lattices

$$\begin{aligned} \Sigma_\alpha^h &= \{\alpha = j \times \lambda_\alpha h : j = 0, \pm 1, \pm 2, \dots\} \\ \Sigma_\beta^h &= \{\beta = j \times \lambda_\beta h : j = 0, \pm 1, \pm 2, \dots\} \end{aligned}$$

where $\lambda_\alpha h > 0$ and $\lambda_\beta h > 0$ are respectively the spatial steps of stocks and options. The approximated solvency region is defined by $(i = 0, \dots, n)$

$$SOL_{t_i}^h = \{(B, \alpha, \beta, P) \in BG_{t_i}^h \times \Sigma_\alpha^h \times \Sigma_\beta^h \times PT_{t_i}^h : Sol(B, \alpha, \beta, P) \geq 0\}$$

and the computational domain Q^h is

$$Q^h = \{(t, B, \alpha, \beta, P) : t = ih, i = 0, 1, \dots, n; (B, \alpha, \beta, P) \in SOL_t^h\}.$$

It is immediate to see that

$$\lim_{h \downarrow 0} dist \{(t, x), Q^h\} = 0 \text{ for all } (t, x) \in \bar{Q}.$$

Given $B_{t_i} \in BG_{t_i}^h$, $P_{t_i} \in PT_{t_i}^h$, $\xi \in \Sigma_\alpha^h$ we denote by $B^\xi(B_{t_i}, P_{t_i}, \xi)$ the new value of B in t_i after a transaction of ξ stocks

$$B^\xi(B_{t_i}, P_{t_i}, \xi) = \arg \min_{\hat{B} \in BG_{t_i}^h} \text{dist}(\hat{B}, B_{t_i} - \xi P_{t_i} - C(\xi, P_{t_i}))$$

The set $F^h(B_{t_i}, \alpha, \beta, P_{t_i})$ of admissible purchases or sales of stocks when the agent's position is $(B_{t_i}, \alpha, \beta, P_{t_i}) \in SOL_{t_i}^h$ is defined by

$$F^h(B_{t_i}, \alpha, \beta, P_{t_i}) = \{ \xi \in \Sigma_\alpha^h : (B^\xi(B_{t_i}, P_{t_i}, \xi), \alpha + \xi, \beta, P_{t_i}) \in SOL_{t_i}^h \}.$$

Let us define the non local operator $\mathcal{M}^h V^h(t_i, x_{t_i})$ (here $x_{t_i} = (B_{t_i}, \alpha, \beta, P_{t_i}) \in SOL_{t_i}^h$, $i = 0, 1, \dots, n$)

$$\mathcal{M}^h V^h(t_i, x_{t_i}) \equiv \begin{cases} \sup_{\xi \in F^h(x_{t_i})} E_{t_i, x_{t_i}} [V^h(t_{i+1}, B^\xi(B_{t_i}, P_{t_i}, \xi) e^{r_h}, \alpha + \xi, \beta, P_{t_{i+1}})] \\ -1 & \text{if } F^h(x_{t_i}) = \emptyset. \end{cases}$$

We consider the following discrete dynamic programming scheme to approximate V (here $x_{t_i} = (B_{t_i}, \alpha, \beta, P_{t_i}) \in SOL_{t_i}^h$, $i = 0, 1, \dots, n$)

$$V^h(t_i, x_{t_i}) = \text{Max} \begin{cases} E_{t_i, x_{t_i}} [V^h(t_{i+1}, x_{t_{i+1}})] \\ \mathcal{M}^h V^h(t_i, x_{t_i}) \end{cases} \quad (16)$$

which has to be solved backwards in time from the final condition in $t_n = T$

$$V^h(t_n, B_{t_n}, \alpha, \beta, P_{t_n}) = U(L_T(B_{t_n}, \alpha, \beta, P_{t_n})) \quad (17)$$

which holds for every $(B_{t_n}, \alpha, \beta, P_{t_n}) \in SOL_{t_n}^h$. We show now that V^h converges locally uniformly to the value function V .

Theorem 7 *Let $V^h(t_i, x_{t_i})$ be the solution to (16)-(17). Then*

$$\lim_{\substack{(t_i, x_{t_i}) \rightarrow (t, x) \\ h \downarrow 0}} V^h(t_i, x_{t_i}) = V(t, x) \quad \forall (t, x) \in Q \setminus D \cup (T \times \overline{SOL}) \quad (18)$$

and the convergence is uniform on any compact subset of $Q \setminus D \cup (T \times \overline{SOL})$.

Proof (Sketch). Let, $\forall (t, x) \in Q$,

$$\left| \begin{aligned} \bar{V}(t, x) &= \limsup_{\substack{(t_i, x_{t_i}) \rightarrow (t, x) \\ h \downarrow 0}} V^h(t_i, x_{t_i}) \\ \underline{V}(t, x) &= \liminf_{\substack{(t_i, x_{t_i}) \rightarrow (t, x) \\ h \downarrow 0}} V^h(t_i, x_{t_i}). \end{aligned} \right.$$

We need to show that \bar{V} is a viscosity subsolution in $Q \cup [0, T] \times \partial SOL$ and \underline{V} is a viscosity supersolution in Q . Then by using (17) and the weak comparison

principle we obtain $\bar{V} \leq V \leq \underline{V}$ in $Q \setminus D$. By the definition of \bar{V} and \underline{V} it follows that (18) holds true and the convergence is uniform on any compact subset of $Q \setminus D \cup (T \times \overline{SOL})$. We only show that \underline{V} is a supersolution as the proof for \bar{V} uses the same arguments. Consider $(\bar{t}, \bar{x}) \in Q$ and $\varphi(t, x) \in C^{1,2}(\bar{Q})$ such that $(\underline{V} - \varphi)(\bar{t}, \bar{x}) = 0$ and (\bar{t}, \bar{x}) is a minimum of $\underline{V} - \varphi$ on \bar{Q} . We have to show that

$$\min \left\{ -\frac{\partial \varphi}{\partial t}(\bar{t}, \bar{x}) - \mathcal{L}\varphi(\bar{t}, \bar{x}), \underline{V}(\bar{t}, \bar{x}) - \mathcal{M}\underline{V}(\bar{t}, \bar{x}) \right\} \geq 0 .$$

Without loss of generality we can assume that the minimum is strict. Then there is a sequence h_n , converging to zero, such that $V^{h_n} - \varphi$ has a minimum in Q^{h_n} at (s_n, y_n) and $(s_n, y_n) \rightarrow (\bar{t}, \bar{x})$ and $V^{h_n}(s_n, y_n) \rightarrow \underline{V}(\bar{t}, \bar{x})$ as $h_n \downarrow 0$. Considering (16) at the point (s_n, y_n) we obtain

$$V^{h_n}(s_n, y_n) \geq \mathcal{M}^{h_n} V^{h_n}(s_n, y_n).$$

As $n \rightarrow \infty$ and $h_n \rightarrow 0$ we have

$$\begin{cases} \lim_{n \rightarrow \infty} V^{h_n}(s_n, y_n) = \underline{V}(\bar{t}, \bar{x}) \\ \lim_{n \rightarrow \infty} \mathcal{M}^{h_n} V^{h_n}(s_n, y_n) = \mathcal{M}\underline{V}(\bar{t}, \bar{x}) \end{cases}$$

and therefore $\underline{V}(\bar{t}, \bar{x}) \geq \mathcal{M}\underline{V}(\bar{t}, \bar{x})$. To complete the proof we need to show now that

$$\frac{\partial \varphi}{\partial t}(\bar{t}, \bar{x}) + \mathcal{L}\varphi(\bar{t}, \bar{x}) \leq 0 .$$

Let $y_n = (B_n, \alpha_n, \beta_n, P_n)$ and $\bar{x} = (\bar{B}, \bar{\alpha}, \bar{\beta}, \bar{P})$. As (s_n, y_n) is a minimum of $V^{h_n} - \varphi$ in Q^{h_n} we have

$$\begin{aligned} & E_{s_n, y_n} [V^{h_n}(s_n + h_n, B_n e^{r h_n}, \alpha_n, \beta_n, P_n e^{(\mu - \frac{1}{2}\sigma^2)h_n + \theta\sigma\sqrt{h_n}})] \\ & - E_{s_n, y_n} [\varphi(s_n + h_n, B_n e^{r h_n}, \alpha_n, \beta_n, P_n e^{(\mu - \frac{1}{2}\sigma^2)h_n + \theta\sigma\sqrt{h_n}})] \\ & \geq V^{h_n}(s_n, B_n, \alpha_n, \beta_n, P_n) - \varphi(s_n, B_n, \alpha_n, \beta_n, P_n) \end{aligned} \quad (19)$$

where θ is a random variable taking values ± 1 with equal probability $p = \frac{1}{2}$. From (16) considered at (s_n, y_n) and (19) it follows

$$\begin{aligned} 0 & \geq E_{s_n, y_n} [\varphi(s_n + h_n, B_n e^{r h_n}, \alpha_n, \beta_n, P_n e^{(\mu - \frac{1}{2}\sigma^2)h_n + \theta\sigma\sqrt{h_n}})] \\ & - \varphi(s_n, B_n, \alpha_n, \beta_n, P_n) \end{aligned}$$

and since P^{h_n} converges in distribution to P , as $h_n \rightarrow 0$ we obtain (here Z is the standard normal distribution)

$$\begin{aligned} 0 & \geq \liminf_{n \rightarrow \infty} \frac{E_{s_n, y_n} [\varphi(s_n + h_n, B_n e^{r h_n}, \alpha_n, \beta_n, P_n e^{(\mu - \frac{1}{2}\sigma^2)h_n + \theta\sigma\sqrt{h_n}})] - \varphi(s_n, y_n)}{h_n} \\ & \geq \liminf_{h \rightarrow 0} \frac{E_{\bar{t}, \bar{x}} [\varphi(\bar{t} + h, \bar{B} e^{r h}, \bar{\alpha}, \bar{\beta}, \bar{P} e^{(\mu - \frac{1}{2}\sigma^2)h + Z\sigma\sqrt{h}})] - \varphi(\bar{t}, \bar{x})}{h} \\ & = \frac{\partial \varphi}{\partial t}(\bar{t}, \bar{x}) + \mathcal{L}\varphi(\bar{t}, \bar{x}) . \end{aligned}$$

■

5 Numerical results

In this section we present some numerical results of our scheme for an investor whose preferences are modeled by the constant relative risk aversion utility function

$$U(L) = \frac{1}{\gamma} L^\gamma$$

with $0 < \gamma < 1$. Most of the papers on options indifference prices assume an exponential utility function which exhibits a constant absolute risk aversion (see [15], [12], [6], [21], [25]). The choice of the exponential utility is basically due to the fact that in this case the reservation prices do not depend on the initial cash holdings. This decrease in dimensionality simplifies the analysis and reduces the computational load of the numerical solution. However the exponential utility has some serious drawbacks to describing the investor's behaviour. Without transaction costs the optimal strategy of a portfolio problem with exponential utility is to maintain constant the discounted value invested in the risky asset, independently of the level of wealth. Changing only the amount of money in the bank account, this strategy leads to extremely leveraged position for low levels of wealth. Investors behave in a more risky way when they are poor than when they are rich and this appears unrealistic. In our numerical experiments we have used the following values of the model parameters: $\mu = 0.06$, $\sigma = 0.4$, $r = 0.02$, $K = 0.1$, $c = 0.005$, $\gamma = 0.3$, $T = 1$. The finite-difference discretization was implemented with a rectangular mesh in the B, α variables of approximately 12,000 points and a time grid of 40 steps for the discrete stock price process. The boundary conditions for the value function were set according to the analysis of section 3. We fixed the initial price of the risky asset at $P_0 = 100$ and considered the three possible values 90, 100, 110 for the call option strike price P_{str} . As for the option initial price P^c the most natural choice has been to set $P^c = P_{BS}$, the Black-Scholes price, in order to exclude arbitrage opportunities and to focus on the use of options because of transaction costs and solvency constraints. Our first numerical experiments were intended to show the form of the optimal policy when an open long or short position in options is present in the portfolio. As shown in ([11], [23], [22], [1]), without options the no trade region of the optimal policy in the presence of transaction costs and solvency constraints is described by a cone with its vertex in the origin. In Fig. 1 it is depicted the optimal policy of our model without options. The two continuous lines inside the cone represent the recalibrated portfolios. The upper (lower) line is the set of target portfolios where it is optimal to move when the investor's position is in the upper (lower) part of the trading area, that is above (below) the cone. Some optimal transactions have been depicted by the straight lines connecting the threshold portfolios in the transaction region to the corresponding target portfolios inside the no trade area. The dotted line inside the cone represents the Merton line of optimal portfolios for a problem with power utility and without

transaction costs.

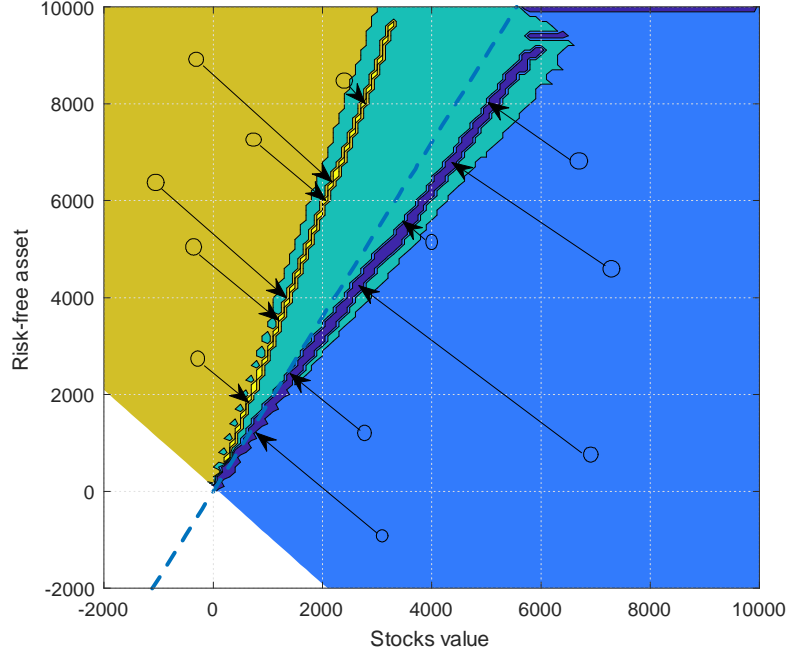


Figure 1. Transaction regions in the plane $(\alpha P, B)$ at $t = 0$ without options.

When there are β options in the portfolio, by using the results in [16], the optimal policy without market frictions is to perfectly replicate the long or short position in options and to invest the remaining value according to the optimal policy of a Merton problem. That is the optimal portfolio must be continuously recalibrated to verify

$$(B - B_{BS}^\beta) = \frac{1 - M}{M}(\alpha - \alpha_{BS}^\beta)P \quad (20)$$

where $M = \frac{\mu - r}{\sigma^2(1 - \gamma)}$ is the Merton proportion and $(B_{BS}^\beta, \alpha_{BS}^\beta)$ is the portfolio perfectly replicating β options. Considering transaction costs and solvency constraints the no trade region of a problem with options seems to be still a cone with two continuous lines of target portfolios inside the cone, but now its vertex

is located along the straight line (20).

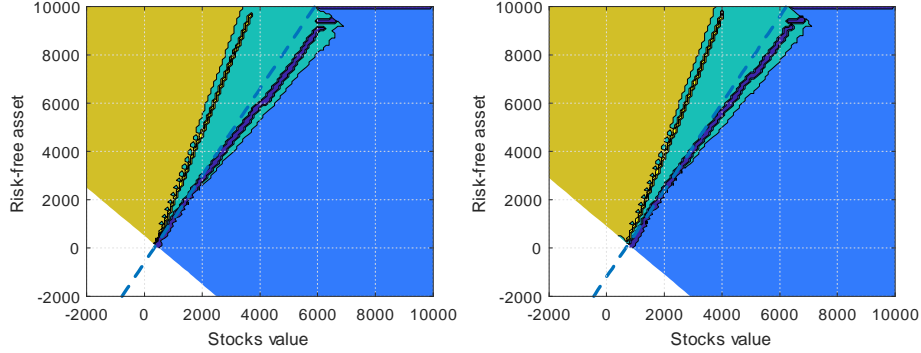


Figure 2. Transaction regions for a writer of 2 (left) and 8 (right) call options.
 $P_{str} = 100, P^C = P_{BS}$.

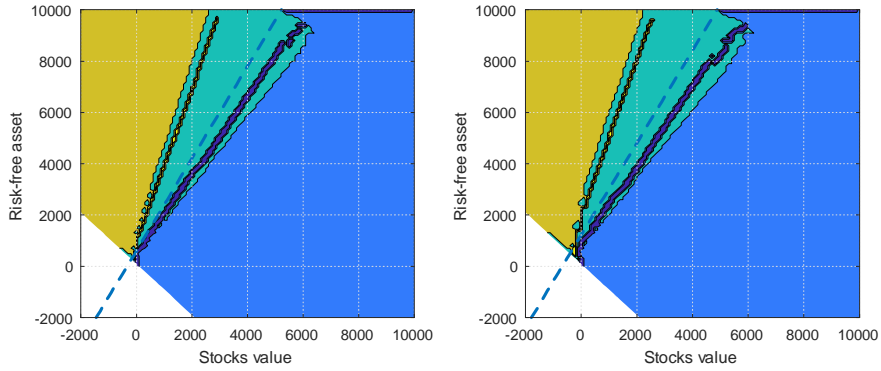


Figure 3. Transaction regions for a buyer of 2 (left) and 8 (right) call options.
 $P_{str} = 100, P^C = P_{BS}$.

In Fig. 2 and Fig. 3 the optimal control regions and the recalibrated portfolios at $t = 0$ are shown for, respectively, a writer and a buyer of 2 and 8 at the money call options with initial price equal to the Black and Scholes price (the other model parameters remaining the same). The dotted lines inside the no trade regions represent equation (20) in the four cases, that is the optimal portfolios without frictions. Some numerical experiments have been tried varying the model parameters concerning the transaction costs, the risk aversion, and the final horizon. The effects on the optimal regions have been in line with what expected and similar to to the case without options. In particular increasing the transaction costs coefficients or decreasing the time to expiration widens the no trade area, reducing the convenience and therefore the frequency of trading. The

main purpose of our numerical investigation was to show that in the presence of market frictions a significant use of options becomes profitable in our model. We have considered the following set of possible values for the number of options held in the portfolio:

$$\beta = 0, \pm 1, \pm 11, \pm 21, \pm 31, \pm 41, \pm 51.$$

The most important results of our computations are illustrated in Figs. 4-12 below. We have focused our attention on how the trading in the calls depends on the initial portfolio of the investor and how it depends on the strike prices and market prices of the options. Fig. 4 shows how many at the money call options ($P_{str} = P_0$) an investor, having no derivative at $t = 0$, will buy (positive values) or sell (negative values) at the beginning of the investment period. The sign and size of the trade are shown as a function of the investor's initial holdings in stocks and bonds. We see that if the initial wealth is mainly in stocks the agent will sell options instead of stocks and vice versa he will buy options when a large fraction of wealth is invested in bonds. The number of traded options increases with the level of wealth but the selling region of at the money call options is much smaller than the buying region. Fig. 5 and Fig. 6 have the same meaning of Fig. 4 but with strike prices respectively of 90 and 110. We can see by these figures that increasing the strike price (for fixed $P_0 = 100$) the option buying region decreases and the selling region increases. However the buying region is fairly stable while the selling region is very sensitive to a large increase in P_{str} . This different sensitivity comes from the different amount of risk taken by the writer compared to the buyer of a call option. Fig. 6 shows an extensive use at $t = 0$ of deep out of the money call options quoted at their Black and Scholes price. In Fig. 7 and Fig. 8 the investor is allowed to choose which one of the three call options to buy or sell at $t = 0$, at its specific Black and Scholes price. We see in Fig. 7 that he will choose the out of the money option ($P_{str} = 110$) when he is a writer and the option in the money ($P_{str} = 90$), when he is a buyer. In this figure the Merton's dotted line of optimal portfolios without market frictions is also depicted. Fig. 8 shows the percentage of the value of the initial portfolio which is invested in these two options at $t = 0$ for the different initial holdings in stocks and bonds. Under our assumptions on the model parameters the long position in options amounts up to 10% of the value for some initial portfolios while the short position is always below 5% of the initial wealth. If the call price is not equal to the Black and Scholes price then the investor also exploits the difference in prices. In Fig. 9 and Fig. 10 we assume that the price of the call at the money is 2% greater than the Black and Scholes price. We see that the short positions are now made of this option, they are bigger and the selling region is more extended. The use of options is very sensitive to call prices which are lower than the Black and Scholes prices because of the smaller risk taken when the calls are bought. In Fig. 11 and Fig. 12 the price of the call at the money is 2% smaller than Black and Scholes price. The long positions are made of this option, the buying region extends to the right of the Merton line and it is much larger than the selling region. In both cases the investment in options

amounts up to 15% of the portfolio value for some initial conditions and we see that the investor uses options for almost all initial portfolios.

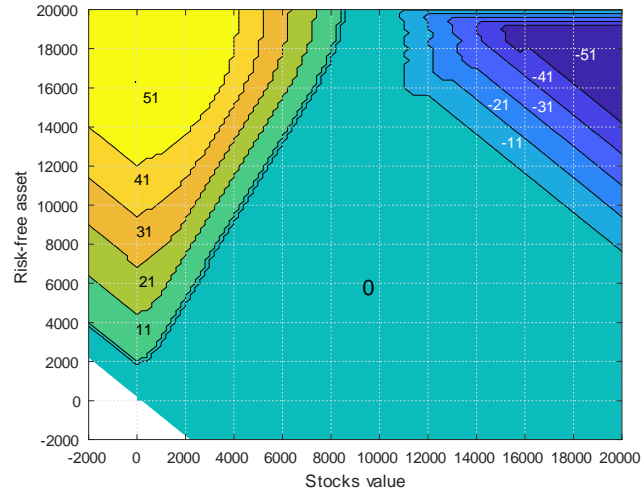


Fig 4. Number of call options bought (+) or sold (-) at $t = 0$ for different initial holdings $(\alpha_0 P_0, B_0)$.
 $P_{str} = P_0 = 100, P^C = P_{BS}$.

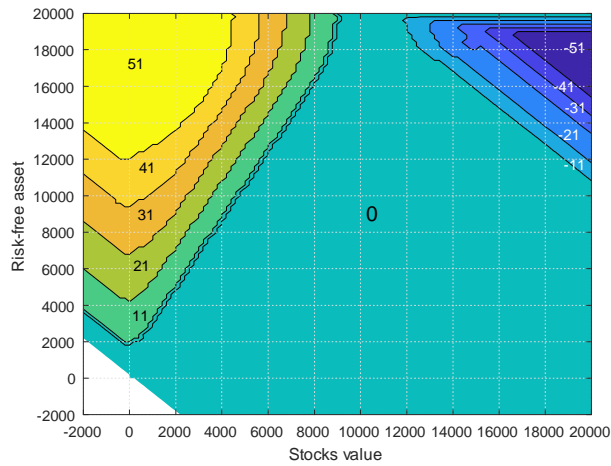


Fig 5. Number of call options bought (+) or sold (-) at $t = 0$ for different initial holdings $(\alpha_0 P_0, B_0)$.
 $P_{str} = 90, P^C = P_{BS}$

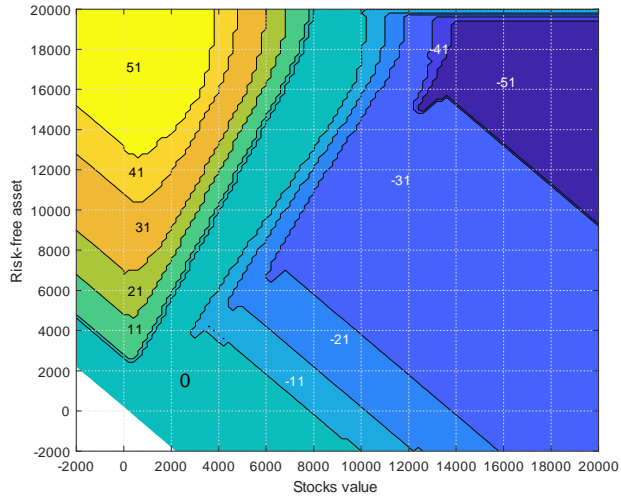


Fig 6. Number of call options bought (+) or sold (-) at $t = 0$ for different initial holdings $(\alpha_0 P_0, B_0)$.
 $P_{str} = 110, P^C = P_{BS}$.

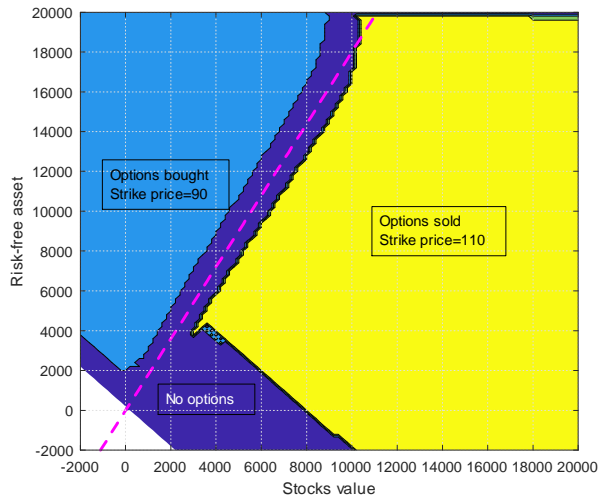


Fig. 7. Type of options bought or sold at $t = 0$ for different initial conditions.
 $P^C = P_{BS}$.

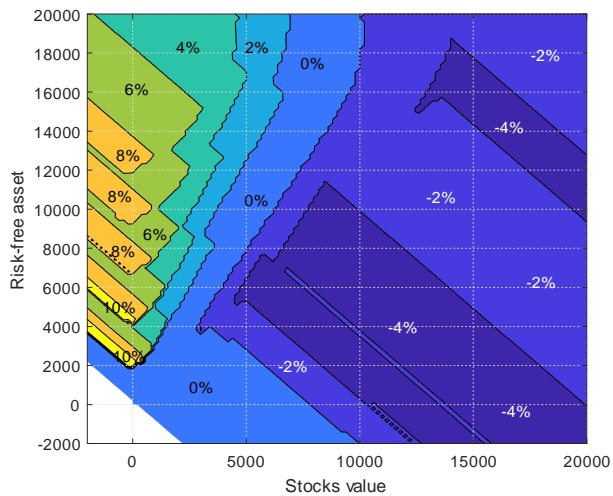


Fig. 8. Percentage of initial wealth invested in options at $t = 0$.

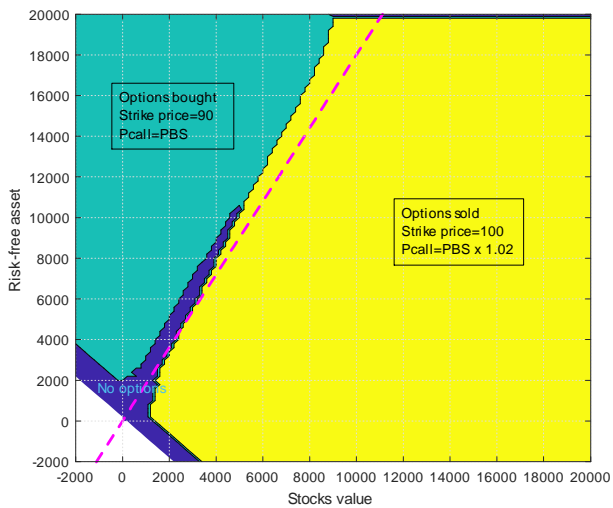


Fig. 9. Type of options bought or sold at $t = 0$ for different initial conditions.

$$P^C = P_{BS} \times 1.02 \text{ if } P_{str} = 100.$$

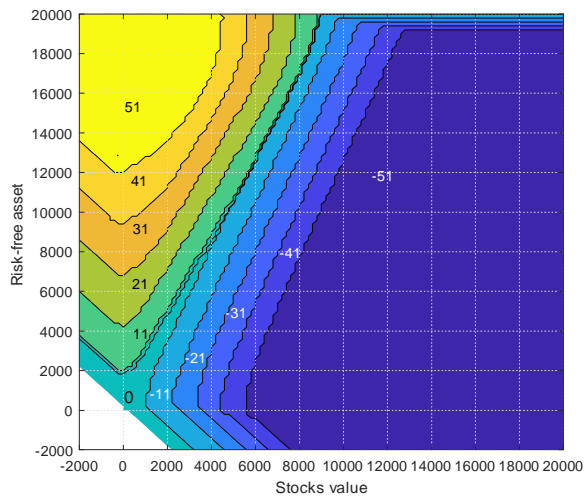


Fig. 10. Number of options bought or sold at $t = 0$ for different initial conditions.
 $P^C = P_{BS} \times 1.02$ if $P_{str} = 100$.

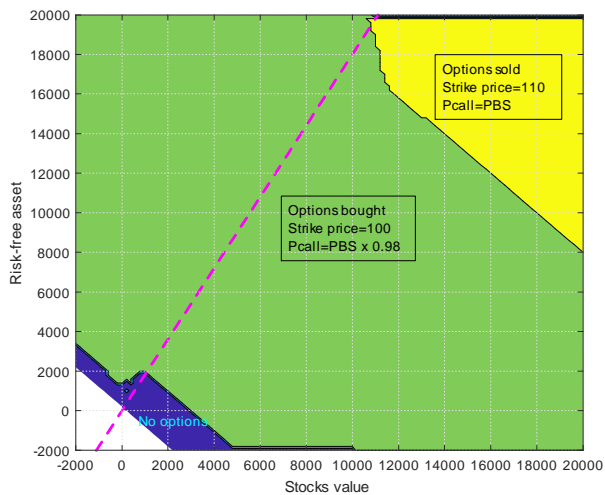


Fig. 11. Type of options bought or sold at $t = 0$ for different initial conditions.
 $P^C = P_{BS} \times 0.98$ if $P_{str} = 100$.

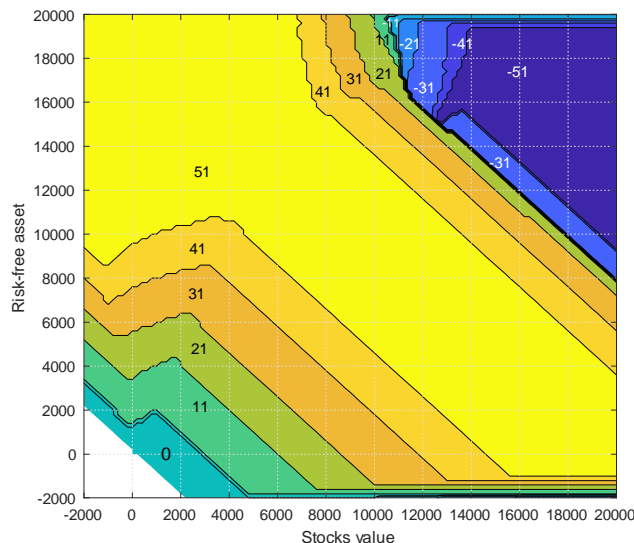


Fig. 12. Number of options bought or sold at $t = 0$ for different initial conditions.
 $P^C = P_{BS} \times 0.98$ if $P_{str} = 100$.

6 Concluding remarks

Most of the articles on options in the mathematical finance literature consider the problem of pricing and hedging these derivatives. In this paper we have formulated a model where options can play an active role in portfolio management because real financial markets are not complete and these derivatives cannot be perfectly replicated. Our numerical experiments have shown that if we take account of transaction costs and solvency constraints it is convenient to add static policies in options to the set of the trading strategies. When we assume the Black and Scholes price the agent will take a long (short) position in call options if his initial portfolio is sufficiently on the left (right) of his Merton line. He buys in the money and sells out of the money calls. The number of call options sold tends always to be smaller because of the greater amount of risk present in a short position. It is sufficient a small difference between the market and the Black and Scholes price to increase greatly the use of these derivatives, especially when the call is underpriced. The investor exploits the difference in prices but the investment in options remains bounded because a perfect arbitrage is not possible. There are several directions in which our approach can be further investigated. It is possible to add other derivatives, such as the put

options, to the set of available securities and to allow to invest in more than one derivative at the same time. However since the optimal policy essentially hedges the derivatives we don't believe that this extension will increase significantly the optimal solution. One can think of enlarging the class of admissible policies considering dynamic strategies which involve options. In this case it is necessary to specify the process of the option market price and the most natural choice could be still to use the Black and Scholes price process. Furthermore we can consider a more general stochastic process for the stock price dynamics, such as a jump-diffusion process. Considering jump processes the market is incomplete simply because not all of the stock price risk can be hedged away. In this case it is interesting to investigate if introducing options can improve the solution of our utility maximization problem by reducing the unhedgeable risk present in the portfolio.

References

- [1] S. Baccharin and D. Marazzina, Passive portfolio management over a finite horizon with a target liquidation value under transaction costs and solvency constraints, *Ima Journal of Management Mathematics* **27**, (2016) 471-504.
- [2] G. Barles and P.E. Souganidis, Convergence of approximation schemes for fully nonlinear second order equations, *Journal of Asymptotic Analysis* **4**, (1991) 271-283.
- [3] B. Bensaid, J.P. Lesne, H. Pagès and H. Scheinkman, Derivative asset pricing with transaction costs, *Mathematical Finance* **2**, (1992) 63-86.
- [4] F. Black and M. Scholes, The pricing of options and corporate liabilities, *Journal of Political Economy* **81**, (1973) 637-654.
- [5] P.P. Boyle and T. Vorst, Option replication in discrete time with transaction costs, *Journal of Finance* **47**, (1992) 271-293.
- [6] L. Clewlow and S. Hodges, Optimal delta-hedging under transactions costs, *Journal of Economic Dynamics & Control* **21**, (1997) 1353-1376.
- [7] J.C. Cox and M. Rubinstein, *Options Markets*, Prentice Hall, Englewood Cliffs, NJ, 1985
- [8] R. Cont and P. Tankov, *Financial Modelling with jump processes*. Chapman & Hall/CRC 2004.
- [9] A. Damgaard, Utility based option evaluation with proportional transaction costs, *Journal of Economic Dynamics & Control* **27**, (2003) 667-700.
- [10] A. Damgaard, Computation of reservation prices of options with proportional transaction costs, *Journal of Economic Dynamics & Control* **30**, (2006) 415-444.

- [11] M. Davis and A. Norman, Portfolio selection with transaction costs, *Mathematics of Operations Research* **15** (1990) 676-713
- [12] M. Davis, V. Panas and T. Zariphopoulou, European option pricing with transaction costs, *SIAM Journal on Control and Optimization* **31**, (1993) 470-493
- [13] C. Ederisinghe, V. Naik and R. Uppal, Optimal replication of options with transactions costs and trading restrictions, *Journal of Finance and Quantitative Analysis* **28**, (1993) 117-138
- [14] W.H. Fleming and M. Soner, Controlled Markov processes and viscosity solutions, Springer, 1993.
- [15] S.D. Hodges and A. Neuberger, Optimal replication of contingent claims under transaction costs, *Review of Futures Markets* **8**, (1989) 222-239
- [16] I. Karatzas, Optimization problems in the theory of continuous trading, *SIAM Journal on Control and Optimization* **27**, (1989) 1221-1259.
- [17] H.J. Kushner and P.G. Dupuis, Numerical methods for Stochastic Control Problems in Continuous Time, Springer New York, 1992
- [18] H.E. Leland, Option pricing and replication with transaction costs, *Journal of Finance* **40**, (1985), 1283-1301.
- [19] H. Liu and M. Loewenstein, Optimal portfolio selection with transaction costs and finite horizon, *The Review of Financial Studies* **15**, (2002) 805-835
- [20] V. Ly Vath, M. Mnif and H. Pham, A model of portfolio selection under liquidity risk and price impact, *Finance and Stochastics* **11**, (2007) 51-90.
- [21] M. Monoyios, Option pricing with transaction costs using a Markov chain approximation, *Journal of Economic Dynamics & Control* **28**, (2004) 889-913
- [22] B. Øksendal and A. Sulem, Optimal consumption and portfolio with both fixed and proportional transaction costs, *SIAM Journal on Control and Optimization* **40**, (2002) 1765-1790.
- [23] S.E. Shreve and H.M. Soner, Optimal investment and consumption with transaction costs, *The Annals of Applied Probability* **4**, (1994) 609-692.
- [24] H.M. Soner, S.E. Shreve and J. Cvitanic, There is no nontrivial hedging portfolio for option pricing with transaction costs, *The Annals of Applied Probability* **5**, (1995) 327-355.
- [25] V.I. Zakamouline, European option pricing and hedging with both fixed and proportional transaction costs, *Journal of Economic Dynamics & Control* **30**, (2006) 1-25

DEPARTMENT OF ECONOMICS AND STATISTICS
UNIVERSITY OF TORINO
Corso Unione Sovietica 218 bis - 10134 Torino (ITALY)
Web page: <http://esomas.econ.unito.it/>
