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ON SOME PROPERTIES OF RANK 2 REFLEXIVE SHEAVES ON A SMOOTH THREEFOLD

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ABSTRACT. We show that some properties of rank 2 reflexive sheaves true on $\mathbb{P}^3$ can be extended to a wide class of smooth projective threefolds. In particular, we establish some cohomological conditions in order that a rank 2 reflexive sheaf is locally free or a split bundle, or, equivalently, that an equidimensional, locally Cohen-Macaulay and generically local complete intersection curve lying on the threefold is subcanonical or a complete intersection.

1. Introduction

It is well known that there exists a correspondence, also called "Hartshorne-Serre" correspondence, between rank 2 reflexive sheaves on the projective space $\mathbb{P}^3$ on the one hand and equidimensional, locally Cohen-Macaulay and generically local complete intersection curves embedded in $\mathbb{P}^3$ on the other hand. In fact we have a short exact sequence of type

$$0 \to \mathscr{O}_{\mathbb{P}^3} \to \mathscr{F} \to \mathscr{I}_C(c_1) \to 0$$

linking a rank 2 reflexive sheaf $\mathscr{F}$ on $\mathbb{P}^3$ and a curve C zero locus of a global section of $\mathscr{F}$, where c_1 is the first Chern class of $\mathscr{F}$ considered as an integer. This result, due to Hartshorne (1980), is the generalization of the so called Serre correspondence between rank 2 vector bundles, *i.e.*, locally free sheaves, on $\mathbb{P}^3$ and subcanonical curves embedded in $\mathbb{P}^3$, where a curve is called subcanonical if its dualizing sheaf is isomorphic to a suitable twist of its structure sheaf. Thanks to the above exact sequence we obtain that cohomological properties of a reflexive sheaf and of a corresponding curve are in very close connection. This interaction between reflexive sheaves, or vector bundles, and curves in $\mathbb{P}^3$ allows us to work on the ones or on the others to get as a consequence results on both. It is also known that the Hartshorne-Serre correspondence can be extended, with little changes, from the 3–dimensional projective space $\mathbb{P}^3$ to a smooth algebraic threefold X (see Valenzano 2004)).

In the present paper we study rank 2 reflexive sheaves on a smooth algebraic projective polarized threefold $(X, \mathscr{O}_X(1))$ verifying some technical conditions, where $\mathscr{O}_X(1)$ is a fixed very ample invertible sheaf on X, with the aim of extending some results true for rank 2 reflexive sheaves on $\mathbb{P}^3$. We assume that the threefold X satisfies two conditions: the Picard group of X is isomorphic to $\mathbb{Z}$ and the 1–cohomology, and so also the 2–cohomology, of $\mathscr{O}_X(n) = \mathscr{O}_X(1)^{\otimes n}$ vanishes for every $n \in \mathbb{Z}$.

Under these technical conditions we extend some properties of rank 2 reflexive sheaves known on $\mathbb{P}^3$ [see, for example, Hartshorne (1978, 1980), Okonek *et al.* (1980), Roggero (1988), Geramita *et al.* (1989), and Roggero and Valabrega (1994), and also the survey paper by Roggero *et al.* (2001) for a partial overview on the matter]. In fact we establish cohomological conditions in order that a reflexive sheaf is locally free or, equivalently, that an equidimensional, locally Cohen-Macaulay and generically local complete intersection curve in X is subcanonical, where subcanonical means that the dualizing sheaf of the curve is isomorphic to the restriction to the curve of some invertible sheaf on the threefold. The cohomological condition is the vanishing of one 2–cohomology group of $\mathscr{F}$ for a suitable shift in a given range, which is the best one. Then we use a result by Madonna (1998), which extends the Chiantini-Valabrega splitting criterion for rank 2 vector bundles on $\mathbb{P}^3$ (see Chiantini and Valabrega 1984), to get some cohomological conditions in order that a reflexive sheaf is a split bundle or that a curve in X is a complete intersection. These cohomological conditions in the case $X = \mathbb{P}^3$ were studied by Roggero (1988). For all general facts not explicitly mentioned we refer to Hartshorne (1977).

2. Preliminaries

We consider a smooth polarized threefold $(X, \mathscr{O}_X(1))$ defined over an algebraically closed field k of characteristic zero. This means that X is a nonsingular irreducible projective algebraic variety of dimension three and $\mathscr{O}_X(1)$ is a fixed very ample invertible sheaf on X.

Given a coherent sheaf $\mathscr{F}$ on X we use the following notations: $\mathscr{O}_X(n) = \mathscr{O}_X(1)^{\otimes n}$ for each $n \in \mathbb{Z}$, $H^i(\mathscr{F}) = H^i(X, \mathscr{F})$ and $h^i(\mathscr{F}) = \dim_k H^i(X, \mathscr{F})$, also $\mathscr{F}^\vee$ denote the dual of $\mathscr{F}$ as a sheaf, while V^* is the dual of V as a k-vector space. We call vector bundle a locally free sheaf of finite rank. We assume that the threefold $(X, \mathscr{O}_X(1))$ verifies the following technical conditions:

- $\mathrm{Pic}(X) \cong \mathbb{Z}$ (generated by $\mathscr{O}_X(1)$),
- $H^1_* \mathscr{O}_X = \bigoplus_{n \in \mathbb{Z}} H^1(\mathscr{O}_X(n)) = 0.$

By the first assumption every invertible sheaf on X is, up to isomorphism, of type $\mathscr{O}_X(a)$ with $a \in \mathbb{Z}$, so we set $\omega_X = \mathscr{O}_X(\varepsilon)$. Furthermore by Serre duality the second assumption implies that the 2-cohomology $H^2(\mathscr{O}_X(n))$ vanishes for all $n \in \mathbb{Z}$.

Some smooth threefolds which verify the above conditions are:

- the projective space $\mathbb{P}^3$;
- the smooth hypersurfaces in $\mathbb{P}^4$;
- the smooth complete intersections of dimension 3;
- some Fano threefolds, like the intersection of the Grassmannian of lines of $\mathbb{P}^5$, in its Plücker embedding, with five general hyperplanes of $\mathbb{P}^{14}$, or the intersection of the Grassmannian of lines of $\mathbb{P}^4$, in its Plücker embedding, with two general hyperplanes and a general quadric of $\mathbb{P}^9$.

Let $A(X) = \oplus_{i=0}^3 A^i(X)$ be the Chow ring of the threefold X, with $A^1(X) = \mathrm{Pic}(X)$ and $A^0(X) \cong \mathbb{Z}$. For every rank r coherent sheaf $\mathscr{F}$ on X it is defined the i-th Chern class $c_i(\mathscr{F}) \in A^i(X)$ for $i = 0, 1, \dots, r$, and the Chern polynomial of $\mathscr{F}$ is $c_t(\mathscr{F}) = c_0(\mathscr{F}) + c_1(\mathscr{F})t + \cdots + c_r(\mathscr{F})t^r$. We denote with $h = c_1(\mathscr{O}_X(1))$ the class of the "hyperplane" divisor in $A^1(X)$. Given a cycle Z on X of codimension i, that is $Z \in A^i(X)$, we define the degree of Z with respect to $\mathscr{O}_X(1)$ as

$$\deg(Z; \mathscr{O}_X(1)) = Z \cdot h^{3-i}$$

having identified codimension 3 cycles with integers through the degree map. So we denote with $\delta = h^3$ the degree of X with respect to $\mathscr{O}_X(1)$, and with $\bar{c}_i(\mathscr{F}) = \deg(c_i(\mathscr{F}); \mathscr{O}_X(1))$ the degree of the i-th Chern class of a coherent sheaf $\mathscr{F}$ on X.

We are interested in rank 2 reflexive sheaves on the smooth threefold $(X, \mathscr{O}_X(1))$. The main references for reflexive sheaves are the papers by Hartshorne (1980) and Okonek *et al.* (1980). Let $\mathscr{F}$ be a rank 2 reflexive sheaf on X. The *singular locus* of $\mathscr{F}$ is the closed set

$$\mathrm{S}(\mathscr{F}) = \{x \in X \mid \mathscr{F}_x \text{ is not a free } \mathscr{O}_{X,x}\text{-module}\}$$

which has codimension ≥ 3 (see Okonek *et al.* 1980, Lemma 1.1.10), thus $\mathrm{S}(\mathscr{F})$ is a finite number of points or is empty, and in the latter case the sheaf $\mathscr{F}$ is actually locally free. Since $\mathscr{F}$ is reflexive we have

$$\mathrm{S}(\mathscr{F}) = \mathrm{Supp}\,(\mathscr{E}xt^1_X(\mathscr{F}, \mathscr{L})) \qquad \text{with} \quad \mathscr{L} \in \mathrm{Pic}(X).$$

We denote with $\lambda(\mathscr{F})$ the length of the zero-dimensional scheme $\mathrm{S}(\mathscr{F})$, therefore we have

$$\lambda(\mathscr{F}) = h^0(\mathscr{E}xt^1_X(\mathscr{F}, \mathscr{L}))$$

and it holds trivially the following

Proposition 2.1. *Let $\mathscr{F}$ be a rank 2 reflexive sheaf on X. Then $\mathscr{F}$ is locally free if and only if $\lambda(\mathscr{F}) = 0$.*

Proposition 2.2. *Let $\mathscr{F}$ be a rank 2 reflexive sheaf on X. Then $\mathscr{F}^\vee \simeq \mathscr{F} \otimes (\det \mathscr{F})^{-1}$.*

Proof. See Hartshorne (1980, Proposition 1.10). □

For the following results see Valenzano (2004).

Proposition 2.3. *Let $\mathscr{F}$ be a rank 2 reflexive sheaf on X and $l \in \mathbb{Z}$. Then the Chern classes of $\mathscr{F}(l)$ are*

$$\begin{aligned} c_1(\mathscr{F}(l)) &= c_1(\mathscr{F}) + 2l \in \mathbb{Z} \\ c_2(\mathscr{F}(l)) &= c_2(\mathscr{F}) + c_1(\mathscr{F})lh^2 + l^2h^2 \in A^2(X) \\ c_3(\mathscr{F}(l)) &= c_3(\mathscr{F}) \end{aligned}$$

and the degree of the second Chern class of $\mathscr{F}(l)$ is

$$\bar{c}_2(\mathscr{F}(l)) = \bar{c}_2(\mathscr{F}) + c_1(\mathscr{F})\delta l + \delta l^2 \in \mathbb{Z}.$$

Proposition 2.4. *Let $\mathscr{F}$ be a rank 2 reflexive sheaf on X with first Chern class c_1. Then $\mathscr{F}^\vee \simeq \mathscr{F}(-c_1)$.*

Proposition 2.5 (Serre duality)**.** *Let $\mathscr{F}$ be a reflexive sheaf on X with first Chern class c_1. Then for every $l \in \mathbb{Z}$ there are isomorphisms*

$$H^0(\mathscr{F}(m)) \cong H^3(\mathscr{F}(l))^*$$
$$H^3(\mathscr{F}(m)) \cong H^0(\mathscr{F}(l))^*$$

and an exact sequence

$$0 \to H^1(\mathscr{F}(m)) \to H^2(\mathscr{F}(l))^* \to H^0(\mathscr{E}xt^1_X(\mathscr{F}(l), \omega_X)) \to H^2(\mathscr{F}(m)) \to H^1(\mathscr{F}(l))^* \to 0$$

where $m = -l - c_1 + \varepsilon$.

Remark **2.6**. By Serre's vanishing theorem (see Hartshorne 1977, III Theorem 5.2), it holds that $H^i(\mathscr{F}(l)) = 0$ for $i > 0$ and $l \gg 0$. If $\mathscr{F}$ is locally free on X this implies that $H^i(\mathscr{F}(l)) = 0$ for $i < 3$ and $l \ll 0$. If $\mathscr{F}$ is reflexive, the above version of Serre duality shows that $H^i(\mathscr{F}(l)) = 0$ for $i = 0, 1$ and $l \ll 0$, and that $H^2(\mathscr{F}(l))$ is of constant dimension $\lambda(\mathscr{F}) = h^0(\mathscr{E}xt^1_X(\mathscr{F}, \omega_X))$ for $l \ll 0$.

Proposition 2.7 (Riemann-Roch)**.** *Let $\mathscr{F}$ be a rank 2 reflexive sheaf on X with Chern classes c_1, c_2 and c_3. Then the Euler-Poincaré characteristic of $\mathscr{F}$ is*

$$\chi(\mathscr{F}) = \frac{1}{6}\left(c_1^3\delta - 3c_1\bar{c}_2 + 3c_3\right) + \frac{1}{4}\left(2\bar{c}_2 - c_1^2\delta\right)\varepsilon + \frac{1}{12}c_1\left(\varepsilon^2\delta + \tau\right) - \frac{1}{12}\varepsilon\tau$$

where $\tau = \bar{c}_2(X)$.

Corollary 2.8. *Let $\mathscr{F}$ be a rank 2 reflexive sheaf on X and $l \in \mathbb{Z}$. Then the Euler-Poincaré characteristic of $\mathscr{F}(l)$ is*

$$\begin{aligned}\chi(\mathscr{F}(l)) =& \frac{1}{3}\delta l^3 + \frac{1}{2}(c_1 - \varepsilon)\delta l^2 + \left(\frac{1}{2}c_1^2\delta - \bar{c}_2 - \frac{1}{2}c_1\varepsilon\delta + \frac{1}{6}\varepsilon^2\delta + \frac{1}{6}\tau\right) l + \\ &+ \frac{1}{6}\left(c_1^3\delta - 3c_1\bar{c}_2 + 3c_3\right) + \frac{1}{4}\left(2\bar{c}_2 - c_1^2\delta\right)\varepsilon + \\ &+ \frac{1}{12}c_1\left(\varepsilon^2\delta + \tau\right) - \frac{1}{12}\varepsilon\tau\end{aligned}$$

Theorem 2.9 (Hartshorne-Serre correspondence)**.** *Fix an integer c_1. Then there is a bijective correspondence between*

(i) *pairs $(\mathscr{F}, s)$, where $\mathscr{F}$ is a rank 2 reflexive sheaf on X with $c_1(\mathscr{F}) = c_1$ and $s \in H^0(\mathscr{F})$ is a global section whose zero locus has codimension 2, and*

(ii) *pairs (Y, ξ), where Y is a closed subscheme of X of pure dimension 1, locally Cohen-Macaulay and generically local complete intersection and $\xi \in H^0(Y, \omega_Y(-\varepsilon - c_1))$ is a global section which generates the sheaf $\omega_Y(-\varepsilon - c_1)$ except at finitely many points.*
If the pairs $(\mathscr{F}, s)$ and (Y, ξ) are in correspondence, then there is the exact sequence

$$0 \to \mathscr{O}_X(-c_1) \to \mathscr{F}^\vee \to \mathscr{I}_Y \to 0$$

or

$$0 \to \mathscr{O}_X \to \mathscr{F} \to \mathscr{I}_Y(c_1) \to 0.$$

Remark **2.10**. If the sheaf $\mathscr{F}$ corresponds to the curve Y we have the above exact sequence which gives a cohomological connection between the reflexive sheaf $\mathscr{F}$ and the curve Y zero locus of a section of $\mathscr{F}$. In fact we get in cohomology the exact sequence

$$0 \to H^0(\mathscr{O}_X(l)) \to H^0(\mathscr{F}(l)) \to H^0(\mathscr{I}_Y(l+c_1)) \to 0$$

and the isomorphism

$$H^1(\mathscr{F}(l)) \cong H^1(\mathscr{I}_Y(l+c_1))$$

for all integer l.

Theorem 2.11 (Serre correspondence)**.** *Fix an integer c_1. Then there is a bijective correspondence between*

(i) *pairs $(\mathscr{E}, s)$, where $\mathscr{E}$ is a rank 2 locally free sheaf on X with $c_1(\mathscr{E}) = c_1$ and $s \in H^0(\mathscr{E})$ is a global section whose zero locus has codimension 2, and*

(ii) *pairs (Y, ξ), where Y is a closed subscheme of X of pure dimension 1, locally complete intersection with $\omega_Y \simeq \mathscr{O}_Y(\varepsilon + c_1)$ and $\xi \in H^0(Y, \omega_Y(-\varepsilon - c_1))$ is a global section which generates the sheaf $\omega_Y(-\varepsilon - c_1)$ everywhere.*

Corollary 2.12. *Let $\mathscr{E}$ be a locally free sheaf on X corresponding to the curve Y. Then Y is a complete intersection if and only if $\mathscr{E}$ splits.*

Proposition 2.13. *Let $\mathscr{F}$ be a rank 2 reflexive sheaf on X and Y a curve in X of degree d and arithmetic genus p_a which corresponds to $\mathscr{F}$. Then it holds*

$$\bar{c}_2(\mathscr{F}) = d$$
$$\lambda(\mathscr{F}) = 2p_a - 2 - d(\varepsilon + c_1).$$

3. Cohomological conditions

In the present section we look for cohomological conditions which characterize a rank 2 reflexive sheaf on X to be locally free or to split. Through the Hartshorne-Serre correspondence this is equivalent to find conditions such that a curve in X is either subcanonical or a complete intersection. This is an extension of some results by Roggero (1988) on rank 2 reflexive sheaves on the projective space $\mathbb{P}^3$.

Definition 3.1. Given a rank 2 reflexive sheaf $\mathscr{F}$ on X we define

$$G(n) = h^2(\mathscr{F}(n)) - h^1(\mathscr{F}(-n-c_1+\varepsilon)) \qquad \forall n \in \mathbb{Z},$$

where $\mathscr{F}(-n-c_1+\varepsilon) \simeq \mathscr{F}(n)^\vee \otimes \omega_X$.

Lemma 3.2. *It holds:*

1. *$G(n) \geq 0$ for all $n \in \mathbb{Z}$;*
2. *$G(n) + G(-n-c_1+\varepsilon) = \lambda$ for all $n \in \mathbb{Z}$, where $\lambda = \lambda(\mathscr{F})$;*
3. *if $h^2(\mathscr{F}(n)) = 0$, then $G(n) = 0$ and $h^1(\mathscr{F}(-n-c_1+\varepsilon)) = 0$.*

Proof. Given $n \in \mathbb{Z}$ we set $m = -n - c_1 + \varepsilon$. Then it holds $\mathscr{F}(n)^\vee \otimes \omega_X \simeq \mathscr{F}(m)$, so by Serre duality (see Proposition **2.5**) we have the exact sequence

$$0 \to H^1(\mathscr{F}(m)) \to H^2(\mathscr{F}(n))^* \to H^0(\mathscr{E}xt^1_X(\mathscr{F}(n), \omega_X)) \to H^2(\mathscr{F}(m)) \to H^1(\mathscr{F}(n))^* \to 0$$

from which we get

$$G(n) = h^2(\mathscr{F}(n)) - h^1(\mathscr{F}(m)) \geq 0$$

and also

$$\begin{aligned} G(n) + G(m) &= h^2(\mathscr{F}(n)) - h^1(\mathscr{F}(m)) + h^2(\mathscr{F}(m)) - h^1(\mathscr{F}(n)) \\ &= h^0(\mathscr{E}xt^1_X(\mathscr{F}(n), \omega_X)) = \lambda. \end{aligned}$$

If $h^2(\mathscr{F}(n)) = 0$, then $G(n) = -h^1(\mathscr{F}(-n - c_1 + \varepsilon)) \leq 0$, but, by point 1, $G(n) \geq 0$, so $G(n) = 0$ and $h^1(\mathscr{F}(-n - c_1 + \varepsilon)) = 0$. □

Proposition 3.3. *A rank 2 reflexive sheaf $\mathscr{F}$ on X is locally free if and only if $G(n) = 0$ for all $n \in \mathbb{Z}$.*

Proof. If $\mathscr{F}$ is locally free, then by Serre duality for vector bundles we have

$$H^i(\mathscr{F}) \cong H^{3-i}(\mathscr{F}^\vee \otimes \omega_X)^* \qquad i = 0, 1, 2, 3$$

and hence $h^2(\mathscr{F}(n)) = h^1(\mathscr{F}(-n - c_1 + \varepsilon))$ for all $n \in \mathbb{Z}$, that is $G(n) = 0$ for all $n \in \mathbb{Z}$. Conversely, if $G(n) = 0$ for all $n \in \mathbb{Z}$, then, by the above Lemma, $\lambda = 0$ and so $\mathscr{F}$ is locally free (see Proposition **2.1**). □

Proposition 3.4. *A rank 2 reflexive sheaf $\mathscr{F}$ on X is locally free if and only if $G(n) = G(m) = 0$ for a pair of integers n and m such that $n + m = \varepsilon - c_1$.*

Proof. If $\mathscr{F}$ is locally free, then $G(n) = 0$ for every $n \in \mathbb{Z}$ (see Proposition **3.3**).
For the converse, assume that there is two integers n and m, with $n + m = \varepsilon - c_1$, such that $G(n) = G(m) = 0$. Then by Lemma **3.2**(2) $\lambda = G(n) + G(m) = 0$ and so $\mathscr{F}$ is locally free (see Proposition **2.1**). □

Proposition 3.5. *Let C be a curve on X zero locus of a section of $\mathscr{F}(t)$ and let n and m be integers such that $n + m = \varepsilon - c_1$. Then:*

1. $h^1(\mathscr{O}_C(n + t + c_1)) - h^0(\mathscr{O}_C(m + t + c_1)) = G(n)$;
2. $(n - m)d = 2h^0(\mathscr{O}_C(n + t + c_1)) - 2h^0(\mathscr{O}_C(m + t + c_1)) - G(n) + G(m)$;
3. *if $h^2(\mathscr{F}(m)) = 0$, then $h^1(\mathscr{I}_C(n + t + c_1)) = 0$ and* $(n - m)d = 2h^0(\mathscr{O}_C(n + t + c_1)) - 2h^0(\mathscr{O}_C(m + t + c_1)) - \lambda$;

where d is the degree of C.

Proof. 1. Using the exact sequences

$$0 \to \mathscr{I}_C \to \mathscr{O}_X \to \mathscr{O}_C \to 0$$

$$0 \to \mathscr{O}_X(-2t - c_1) \to \mathscr{F}(-t - c_1) \to \mathscr{I}_C \to 0$$

and Serre duality we obtain in cohomology the equalities

$$\begin{aligned}
h^1(\mathscr{O}_C(n+t+c_1)) &= h^2(\mathscr{I}_C(n+t+c_1))\\
&= h^2(\mathscr{F}(n)) + h^3(\mathscr{O}_X(n-t)) - h^3(\mathscr{F}(n)) + h^3(\mathscr{I}_C(n+t+c_1))\\
&= h^2(\mathscr{F}(n)) + h^0(\mathscr{O}_X(m+t+c_1)) - h^0(\mathscr{F}(m)) + h^0(\mathscr{O}_X(m-t))\\
&= h^2(\mathscr{F}(n)) + h^0(\mathscr{O}_X(m+t+c_1)) - h^0(\mathscr{I}_C(m+t+c_1))\\
&= h^2(\mathscr{F}(n)) + h^0(\mathscr{O}_C(m+t+c_1)) - h^1(\mathscr{I}_C(m+t+c_1))\\
&= h^2(\mathscr{F}(n)) + h^0(\mathscr{O}_C(m+t+c_1)) - h^1(\mathscr{F}(m))
\end{aligned}$$

from which

$$h^1(\mathscr{O}_C(n+t+c_1)) - h^0(\mathscr{O}_C(m+t+c_1)) = h^2(\mathscr{F}(n)) - h^1(\mathscr{F}(m)) = G(n).$$

2. Computing the Hilbert polynomial of C for $n+t+c_1$ and $m+t+c_1$, we find the expressions

$$\begin{aligned}
d(n+t+c_1)+1-p_a &= h^0(\mathscr{O}_C(n+t+c_1)) - h^1(\mathscr{O}_C(n+t+c_1))\\
d(m+t+c_1)+1-p_a &= h^0(\mathscr{O}_C(m+t+c_1)) - h^1(\mathscr{O}_C(m+t+c_1))
\end{aligned}$$

where d is the degree and p_a the arithmetic genus of C. Then subtracting the second one from the first one and using the above formula we get the thesis.
3. If $h^2(\mathscr{F}(m)) = 0$, then, by Lemma **3.2**, $G(m) = 0$ and $h^1(\mathscr{I}_C(n+t+c_1)) = h^1(\mathscr{F}(n)) = 0$, so $G(n) = \lambda$, always by Lemma **3.2**. Substituting these values in the above formula we get the thesis. □

Proposition 3.6. *Let C be an integral curve contained in X. Then for all integer n it holds*

$$h^0(\omega_C(n)) - h^0(\mathscr{O}_C(e+n)) \leq h^0(\omega_C(n+1)) - h^0(\mathscr{O}_C(e+n+1))$$

where $e = \max\{t \in \mathbb{Z} \mid h^1(\mathscr{O}_C(t)) = h^0(\omega_C(-t)) \neq 0\}$ is the index of speciality of C.

Proof. The same proof by Roggero (1988, Proposition 2.2) works in this case too. □

Lemma 3.7. *Let $\mathscr{F}$ be a rank 2 reflexive sheaf on X with first Chern class c_1. Then for all $t \gg 0$ the sheaf $\mathscr{F}(t)$ has a section whose zero locus is an integral curve C with index of speciality $e = 2t + c_1 + \varepsilon$.*

Proof. For $t \gg 0$ the general section of $\mathscr{F}(t)$ gives rise to an integral curve C (see Roggero 1985, Teorema 3), hence $e \geq 2t + c_1 + \varepsilon$ since $\omega_C(-2t - c_1 - \varepsilon)$ has a non zero global section which corresponds to the reflexive sheaf $\mathscr{F}(t)$ (see Theorem **2.9**). Moreover we have

$$\begin{aligned}
h^0(\omega_C(-2t-c_1-\varepsilon-1)) &= h^1(\mathscr{O}_C(2t+c_1+\varepsilon+1))\\
&= h^2(\mathscr{I}_C(2t+c_1+\varepsilon+1))\\
&= h^2(\mathscr{F}(t+\varepsilon+1))
\end{aligned}$$

where the last equality follows from the exact sequence

$$0 \to \mathscr{O}_X \to \mathscr{F}(t) \to \mathscr{I}_C(2t+c_1) \to 0$$

because $h^3(\mathscr{O}_X(\varepsilon+1)) = h^0(\mathscr{O}_X(-1)) = 0$, so for $t \gg 0$ we get $h^0(\omega_C(-2t-c_1-\varepsilon-1)) = 0$, that is $e = 2t + c_1 + \varepsilon$. □

Proposition 3.8. *Let $\mathscr{F}$ be a rank 2 reflexive sheaf on X, then $G(n) \geq G(n+1)$ for all $n \in \mathbb{Z}$.*

Proof. By the above Lemma there is an integer t such that $\mathscr{F}(t)$ has a section whose zero locus is a reduced and irreducible curve C with index of speciality $e = 2t + c_1 + \varepsilon$. Setting $r + 1 = -n - t - c_1$, then

$$e + r + 1 = 2t + c_1 + \varepsilon - n - t - c_1 = -n + \varepsilon + t = m + t + c_1$$

where $m = -n - c_1 + \varepsilon$, so by Proposition **3.5**(1) we get

$$\begin{aligned} G(n) &= h^1(\mathscr{O}_C(n+t+c_1)) - h^0(\mathscr{O}_C(m+t+c_1)) \\ &= h^1(\mathscr{O}_C(-r-1)) - h^0(\mathscr{O}_C(e+r+1)) \\ &= h^0(\omega_C(r+1)) - h^0(\mathscr{O}_C(e+r+1)) \\ G(n+1) &= h^1(\mathscr{O}_C(n+1+t+c_1)) - h^0(\mathscr{O}_C(m-1+t+c_1)) \\ &= h^1(\mathscr{O}_C(-r)) - h^0(\mathscr{O}_C(e+r)) \\ &= h^0(\omega_C(r)) - h^0(\mathscr{O}_C(e+r)) \end{aligned}$$

and then by Proposition **3.6**

$$G(n+1) = h^0(\omega_C(r)) - h^0(\mathscr{O}_C(e+r)) \leq h^0(\omega_C(r+1)) - h^0(\mathscr{O}_C(e+r+1)) = G(n).$$

□

Remark **3.9**. Given a reflexive sheaf $\mathscr{F}$ on X, the function $G\colon \mathbb{Z} \to \mathbb{Z}$ of Definition **3.1** is, by Proposition **3.8**, non increasing, and also $G(n) = 0$ for $n \gg 0$ and $G(n) = \lambda$ for $n \ll 0$ (see Remark **2.6**). Moreover the graph of G, obtained by connecting with segments the points with integer coordinates $(n, G(n))$, is simmetric with respect to the point $P = ((\varepsilon - c_1)/2, \lambda/2)$ (which not necessarily has integer coordinates).
Note also that if $c_1 + \varepsilon$ is even, then also $\varepsilon - c_1$ is even, so $(\varepsilon - c_1)/2$ is a whole number. Setting $n = (\varepsilon - c_1)/2$ and $m = -n - c_1 + \varepsilon$, we get that $m = n$ and therefore, by Lemma **3.2**(2) it results $G(n) = \lambda/2 \in \mathbb{Z}$, *i.e.*, λ is even. In othe words: $c_1 + \varepsilon$ even implies λ even, and also c_3 even (being $c_3 = \lambda\delta$ by Valenzano (2004, Proposition 10). In the particular case $X = \mathbb{P}^3$ we have $\varepsilon = -4$, so $c_1 + \varepsilon$ is even if and only if c_1 is even, moreover $c_3 = \lambda$ because $\delta = 1$, therefore it holds: c_1 even implies c_3 even.

Theorem 3.10. *Let $\mathscr{F}$ be a rank 2 reflexive sheaf on X with first Chern class c_1. Then the following facts are equivalent:*

1. *$\mathscr{F}$ is locally free;*
2. *$h^2(\mathscr{F}(n)) = 0$ for all $n \ll 0$;*
3. *$h^2(\mathscr{F}(n)) = 0$ for some $n \leq (\varepsilon - c_1)/2$;*
4. *$h^2(\mathscr{F}(n)) = h^1(\mathscr{F}(-n - c_1 + \varepsilon))$ for some $n \leq (\varepsilon - c_1)/2$;*
5. *$G(n) = 0$ for some $n \leq (\varepsilon - c_1)/2$;*
6. *$G(m) = \lambda$ for some $m \geq (\varepsilon - c_1)/2$.*

Proof. 1. $\Rightarrow$ 2. See Hartshorne (1977, Chapter III, Theorem 7.6).
2. $\Rightarrow$ 3. Obvious.
3. $\Rightarrow$ 4. If there is an integer $n \leq (\varepsilon - c_1)/2$ such that $h^2(\mathscr{F}(n)) = 0$, then by Lemma **3.2**(3) $h^1(\mathscr{F}(-n-c_1+\varepsilon)) = 0$, so we have $h^2(\mathscr{F}(n)) = h^1(\mathscr{F}(-n-c_1+\varepsilon))$ for some $n \leq (\varepsilon - c_1)/2$.
4. $\Rightarrow$ 5. By the hypothesis and the definition of the function G we have $G(n) = 0$ for some $n \leq (\varepsilon - c_1)/2$.
5. $\Rightarrow$ 6. If $G(n) = 0$ for some $n \leq (\varepsilon - c_1)/2$, then by Lemma **3.2**(2) $G(m) = \lambda$ for some $m \geq (\varepsilon - c_1)/2$, where $m = -n - c_1 + \varepsilon$.
6. $\Rightarrow$ 1. If $G(m) = \lambda$ for some $m \geq (\varepsilon - c_1)/2$, then by Lemma **3.2**(2) $G(n) = 0$ for some $n \leq (\varepsilon - c_1)/2$, where $n = -m - c_1 + \varepsilon$. As it holds $m \geq n$, by Proposition **3.8** we get $0 = G(n) \geq G(m) = \lambda$, therefore $\lambda = 0$, so $\mathscr{F}$ is locally free. □

Translating the above result in the language of curves, we obtain the following

Theorem 3.11. *Let C be a curve in X, equidimensional, locally Cohen-Macaulay and generically local complete intersection. Then C is a-subcanonical, i.e., $\omega_C \simeq \mathscr{O}_C(a)$, if and only if $\omega_C(-a)$ has a global section which generates it almost everywhere and $h^1(\mathscr{O}_C(n)) = h^0(\mathscr{O}_C(a-n))$ for some $n \leq a/2$. If, moreover, C is reduced and irreducible, then C is e-subcanonical, with e the index of speciality of C, if and only if $h^1(\mathscr{O}_C(n)) = h^0(\mathscr{O}_C(e-n))$ for some $n \leq e/2$.*

Proof. If C is a-subcanonical, then $\omega_C \simeq \mathscr{O}_C(a)$, hence the sheaf $\omega_C(-a)$ is generated by a global section and by duality we have $h^1(\mathscr{O}_C(n)) = h^0(\omega_C(-n)) = h^0(\mathscr{O}_C(a-n))$ for all integer n.
Conversely, let $\mathscr{F}$ be the reflexive sheaf on X corresponding to C, that exists by the hypothesis on $\omega_C(-a)$, where $a = c_1 + \varepsilon$. Let n such that $h^1(\mathscr{O}_C(n)) = h^0(\mathscr{O}_C(a-n))$, set $p = n - c_1$ and $q = -p - c_1 + \varepsilon = \varepsilon - n$, then by Proposition **3.5**(1)

$$\begin{aligned} G(p) &= h^1(\mathscr{O}_C(p+c_1)) - h^0(\mathscr{O}_C(q+c_1)) \\ &= h^1(\mathscr{O}_C(n)) - h^0(\mathscr{O}_C(a-n)) \end{aligned}$$

and so $G(p) = 0$ for some

$$p = n - c_1 \leq \frac{a}{2} - c_1 = \frac{\varepsilon - c_1}{2}.$$

By Theorem **3.10** $\mathscr{F}$ is locally free and therefore C is a-subcanonical. □

Example 3.12. Let X be a smooth quadric threefold in $\mathbb{P}^4$. Then $\omega_X \simeq \mathscr{O}_X(-3)$ so $\varepsilon = -3$. Let C be either a line or a conic lying on X, then C is a nonsingular irreducible curve of degree $d = 1$ or 2 which is a-subcanonical with $a = \varepsilon + d$. Let $\mathscr{F}$ be the reflexive sheaf corresponding to C through a non zero section of $\omega_C(1-\varepsilon-d) = \omega_C(1-a) \simeq \mathscr{O}_C(1)$, then $c_1 = d - 1$. Note that the sheaf we are considering is not the one "canonically" associated to the curve C, that is the reflexive sheaf, or more precisely the vector bundle, corresponding to C through a non zero section of $\omega_C(-a)$. From the exact sequence $0 \to \mathscr{O}_X \to \mathscr{F} \to \mathscr{I}_C(c_1) \to 0$ we get

$$h^2(\mathscr{F}(n)) \leq h^2(\mathscr{I}_C(n+c_1)) = h^1(\mathscr{O}_C(n+c_1)) = h^0(\omega_C(-n-c_1)) = h^0(\mathscr{O}_C(\varepsilon+1-n))$$

and so
$$h^2(\mathscr{F}(n)) = 0 \quad \text{for every } n > \varepsilon + 1 = -2$$
that is
$$h^2(\mathscr{F}(n)) = 0 \quad \text{for every } n > \frac{\varepsilon - c_1}{2}.$$
Note that in both cases the sheaf $\mathscr{F}$ is not locally free, in fact it has $\lambda = 1$ in the case of the line and $\lambda = 2$ in the case of the conic. The above examples show that the integers $n \leq (\varepsilon - c_1)/2$ are the only ones for which it holds the equivalence between 1. and 3. in Theorem **3.10**. For other examples in the case $X = \mathbb{P}^3$ see Roggero (1988, Example 2.4).

Now we want to determine a cohomological condition which assures that a rank 2 reflexive sheaf is a split bundle. First we recall the definition of the first relevant level α of a reflexive sheaf and the splitting criterion given by Madonna (1998).

Definition 3.13. Let $\mathscr{F}$ be a rank 2 reflexive sheaf on X. We define the *first relevant level* of $\mathscr{F}$ as the integer
$$\alpha = \min\{l \in \mathbb{Z} \mid h^0(\mathscr{F}(l)) \neq 0\},$$
i.e., α is the minimum shift for which $\mathscr{F}$ has a non zero global section.

Theorem 3.14. *Let $\mathscr{F}$ be a rank 2 locally free sheaf on X with first Chern class c_1 and first relevant level α. Then $\mathscr{F}$ splits if and only if*
$$\begin{cases} H^1\left(\mathscr{F}\left(\frac{\varepsilon - c_1 + 2}{2}\right)\right) = 0 \textit{ if } c_1 + \varepsilon \textit{ is even},\\ H^1\left(\mathscr{F}\left(\frac{\varepsilon - c_1 + 1}{2}\right)\right) = 0 \textit{ if } c_1 + \varepsilon \textit{ is odd}, \end{cases}$$
unless $-\varepsilon - 3 < c_1 + 2\alpha < \varepsilon + 5$.

Proof. See Madonna (1998, Theorem 7). □

Now we are able to state and prove a splitting criterion for a rank 2 reflexive sheaf on a threefold X.

Theorem 3.15. *Let $\mathscr{F}$ be a rank 2 reflexive sheaf on X with first Chern class c_1 and first relevant level α. Assume that does not hold $-\varepsilon - 3 < c_1 + 2\alpha < \varepsilon + 5$.*
If $c_1 + \varepsilon$ is even we set
$$n = \frac{\varepsilon - c_1 - 2}{2} \quad \textit{and} \quad m = \frac{\varepsilon - c_1 + 2}{2};$$
if $c_1 + \varepsilon$ is odd we set
$$n = \frac{\varepsilon - c_1 - 1}{2} \quad \textit{and} \quad m = \frac{\varepsilon - c_1 + 1}{2}.$$
Then the following facts are equivalent:

1. *$\mathscr{F}$ is a split bundle,*
2. *$h^2(\mathscr{F}(n)) = 0$,*
3. *$h^2(\mathscr{F}(l)) = h^1(\mathscr{F}(m)) = 0$ for some $l \leq (\varepsilon - c_1)/2$,*
4. *$G(l) = h^1(\mathscr{F}(m)) = 0$ for some $l \leq (\varepsilon - c_1)/2$.*

Proof. 1. $\Rightarrow$ 2. Obvious.
2. $\Rightarrow$ 3. In both cases, $c_1+\varepsilon$ even or odd, it holds $m=-n-c_1+\varepsilon$, hence by the hypothesis it follows by Lemma **3.2**(3) that $h^1(\mathscr{F}(m))=0$. So we have $h^2(\mathscr{F}(l))=h^1(\mathscr{F}(m))=0$ with $l=n\le(\varepsilon-c_1)/2$.
3. $\Rightarrow$ 4. If $h^2(\mathscr{F}(l))=0$, then by Lemma **3.2**(3) we have $G(l)=0$.
4. $\Rightarrow$ 1. The hypothesis $G(l)=0$ for some $l\le(\varepsilon-c_1)/2$ implies by Theorem **3.10** that $\mathscr{F}$ is locally free. Then the other hypothesis $h^1(\mathscr{F}(m))=0$ implies by Theorem **3.14** that $\mathscr{F}$ splits. □

Remark **3.16**. Note that in Theorem **3.15** we have $m=n+2$ if $c_1+\varepsilon$ is even, and $m=n+1$ if $c_1+\varepsilon$ is odd.

Theorem 3.17. *Let $\mathscr{F}$ be a rank 2 reflexive sheaf on X with first Chern class c_1 and first relevant level α. Assume that does not hold $-\varepsilon-3<c_1+2\alpha<\varepsilon+5$. Then $\mathscr{F}$ splits if and only if $h^2(\mathscr{F}(l))=0$ for some l with $\alpha+\varepsilon\le l\le n$, where either $n=(\varepsilon-c_1-2)/2$ if $c_1+\varepsilon$ is even or $n=(\varepsilon-c_1-1)/2$ if $c_1+\varepsilon$ is odd.*

Proof. If $\mathscr{F}$ is a split bundle, then $h^2(\mathscr{F}(l))=0$ for all integer l.
For the converse, assume that $h^2(\mathscr{F}(l))=0$ for some l such that $\alpha+\varepsilon\le l\le n$, then $n\le(\varepsilon-c_1)/2$ so by Theorem **3.10** $\mathscr{F}$ is locally free, *i.e.*, $\lambda=0$ (see Proposition **2.1**). If $l=n$, then $\mathscr{F}$ splits by Theorem **3.15**. Now let $\alpha+\varepsilon\le l\le n-1$. Suppose $\mathscr{F}$ does not split and let C be a curve corresponding to a global section of $\mathscr{F}(\alpha)$ (existing by Valenzano 2004, Proposition 14). We set $p=-l-c_1+\varepsilon$, so by Proposition **3.5**(3) we have $h^1(\mathscr{I}_C(p+\alpha+c_1))=0$ and $(p-l)d=2h^0(\mathscr{O}_C(p+\alpha+c_1))-2h^0(\mathscr{O}_C(l+\alpha+c_1))$, where d is the degree of C. From the exact sequence $0\to\mathscr{I}_C\to\mathscr{O}_X\to\mathscr{O}_C\to 0$ we get in cohomology

$$0\to H^0(\mathscr{I}_C(p+\alpha+c_1))\to H^0(\mathscr{O}_X(p+\alpha+c_1))\to H^0(\mathscr{O}_C(p+\alpha+c_1))\to 0$$

from which

$$(p-l)d\le 2h^0(\mathscr{O}_X(p+\alpha+c_1))$$

where $p-l=-c_1+\varepsilon-2l\ge 3$, because $l\le n-1\le(\varepsilon-c_1-3)/2$, and

$$p+\alpha+c_1=\alpha+\varepsilon-l\le 0.$$

Then $3d\le 2$, absurd. □

Remark **3.18**. Obviously the above Theorem makes sense if $\alpha\le n-\varepsilon$, that is if $c_1+2\alpha\le -\varepsilon-2$ in the case $c_1+\varepsilon$ even or if $c_1+2\alpha\le-\varepsilon-1$ in the case $c_1+\varepsilon$ odd. So if $c_1+\varepsilon$ is even it must exclude the case $c_1+2\alpha=-\varepsilon-2$, while if $c_1+\varepsilon$ is odd it must avoid the case $c_1+2\alpha=-\varepsilon-1$ (if $c_1+\varepsilon$ is odd it cannot hold $c_1+2\alpha=-\varepsilon-2$).

Corollary 3.19. *Let $\mathscr{F}$ be a non split rank 2 reflexive sheaf on X. Set $n=(\varepsilon-c_1-2)/2$ if $c_1+\varepsilon$ is even and $n=(\varepsilon-c_1-1)/2$ if $c_1+\varepsilon$ is odd. Then $h^2(\mathscr{F}(l))\ne 0$ for $\alpha+\varepsilon+1\le l\le n$, moreover $h^2(\mathscr{F}(\alpha+\varepsilon))\ne 0$ unless $-\varepsilon-3<c_1+2\alpha<\varepsilon+5$.*

Proof. If $\alpha+\varepsilon+1\le n$, then it does not hold $-\varepsilon-3<c_1+2\alpha<\varepsilon+5$, so by Theorem **3.17** we have $h^2(\mathscr{F}(l))\ne 0$ for $\alpha+\varepsilon+1\le l\le n$. Moreover we have also $h^2(\mathscr{F}(\alpha+\varepsilon))\ne 0$ if $c_1+2\alpha<-\varepsilon-2$ (see Remark **3.18**). □

If we interpret the result of Theorem **3.15** in the language of curves we obtain the following

Proposition 3.20. *Let C be an equidimensional, locally Cohen-Macaulay and generically local complete intersection curve in X. Then C is a complete intersection curve if and only if there exists an integer a, with* $a+2\alpha \le -3$ *or* $a+2\alpha \ge 2\varepsilon+5$ *where* α *is the first relevant level of the sheaf corresponding to C, which satisfies the following conditions:*

i) $\omega_C(-a)$ *has a global section which generates it almost everywhere;*

ii) $h^1(\mathscr{O}_C(n)) = h^0(\mathscr{O}_C(a-n))$ *for some* $n \le a/2$*;*

iii) $h^1(\mathscr{I}_C(\frac{a+r}{2})) = 0$ *where* $r=2$ *if a is even and* $r=1$ *if a is odd.*

Proof. If C is a complete intersection, then C is a-subcanonical with $a = r+s+\varepsilon$, where r and s are the degrees of the surfaces that cut C, then by Theorem **3.11** conditions i) and ii) are satisfied, moreover C is arithmetically Cohen-Macaulay, that is $h^1(\mathscr{I}_C(t)) = 0$ for all t, so also condition iii) is verified.
Conversely, assume there exists an integer a which satisfies the hypothesis of the statement. Conditions i) and ii) imply by Theorem **3.11** that C is a-subcanonical with $a = c_1+\varepsilon$, where c_1 is the first Chern class of the reflexive sheaf $\mathscr{F}$ which corresponds to C. Set $p = n-c_1$ and $q = -p-c_1+\varepsilon$, where n is the integer of condition ii), then by Proposition **3.5**(1)

$$\begin{aligned} G(p) &= h^1(\mathscr{O}_C(p+c_1)) - h^0(\mathscr{O}_C(q+c_1)) \\ &= h^1(\mathscr{O}_C(n)) - h^0(\mathscr{O}_C(a-n)) = 0 \end{aligned}$$

with

$$p = n - c_1 \le \frac{a}{2} - c_1 = \frac{\varepsilon - c_1}{2}.$$

On the other hand from the exact sequence $0 \to \mathscr{O}_X \to \mathscr{F} \to \mathscr{I}_C(c_1) \to 0$ it follows, by condition iii),

$$h^1\left(\mathscr{F}\left(\frac{a+r}{2} - c_1\right)\right) = h^1\left(\mathscr{I}_C\left(\frac{a+r}{2}\right)\right) = 0$$

where

$$\frac{a+r}{2} - c_1 = \frac{\varepsilon - c_1 + r}{2}$$

with $r=2$ if $c_1+\varepsilon$ is even and $r=1$ if $c_1+\varepsilon$ is odd. Moreover

$$a+2\alpha \le -3 \quad \Rightarrow \quad c_1+2\alpha \le -\varepsilon-3$$

and

$$a+2\alpha \ge 2\varepsilon+5 \quad \Rightarrow \quad c_1+2\alpha \ge \varepsilon+5.$$

Therefore, by Theorem **3.15**, $\mathscr{F}$ splits and so C is a complete intersection. □

References

Chiantini, L. and Valabrega, P. (1984). "Subcanonical curves and complete intersections in projective 3-space". *Annali di Matematica Pura ed Applicata* **138**(1), 309–330. DOI: 10.1007/BF01762550.

Geramita, A., Roggero, M., and Valabrega, P. (1989). "Subcanonical curves with the same postulation as Q skew complete intersections in projective 3-space". *Istituto Lombardo. Accademia di Scienze e Lettere. Rendiconti. Scienze Matematiche e Applicazioni. A* **123**, 111–121.

Hartshorne, R. (1977). *Algebraic Geometry*. Vol. 52. Graduate Texts in Mathematics. Springer Verlag. URL: https://www.springer.com/us/book/9780387902449.

Hartshorne, R. (1978). "Stable vector bundles of rank 2 on $\mathbb{P}^3$". *Mathematische Annalen* **238**(3), 229–280. DOI: 10.1007/BF01420250.

Hartshorne, R. (1980). "Stable reflexive sheaves". *Mathematische Annalen* **254**(2), 121–176. DOI: 10.1007/BF01467074.

Madonna, C. (1998). "A splitting criterion for rank 2 vector bundles on hypersurfaces in $\mathbb{P}^4$". *Rendiconti del Seminario Matematico di Università e Politecnico di Torino* **56**(2), 43–54. URL: http://www.seminariomatematico.unito.it/rendiconti/cartaceo/56-2.html.

Okonek, C., Schneider, M., and Spindler, H. (1980). *Vector Bundles on Complex Projective Spaces*. Vol. 3. Progress in Mathematics. Springer US. URL: https://www.springer.com/la/book/9781475714623.

Roggero, M. (1985). "On sections of a rank-2 reflexive sheaf defining integral subvarieties". *Bollettino dell'Unione Matematica Italiana D Serie VI Algebra e Geometria* **4**(1), 57–61.

Roggero, M. (1988). "On the rank 2 reflexive sheaves and the subcanonical curves in $\mathbb{P}^3$". *Communications in Algebra* **16**(9), 1779–1790. DOI: 10.1080/00927878808823661.

Roggero, M. and Valabrega, P. (1994). "Some vanishing properties of the intermediate cohomology of a reflexive sheaf on $\mathbb{P}^n$". *Journal of Algebra* **170**(1), 307–321. DOI: 10.1006/jabr.1994.1340.

Roggero, M., Valabrega, P., and Valenzano, M. (2001). "Rank Two Bundles and Reflexive Sheaves on $\mathbb{P}^3$ and Corresponding Curves: An Overview". In: *Geometric and Combinatorial Aspects of Commutative Algebra*. Ed. by J. Herzog and G. Restuccia. Vol. 217. Lecture Notes in Pure and Applied Mathematics. Marcel Dekker, pp. 327–343. URL: https://www.crcpress.com/Geometric-And-Combinatorial-Aspects-Of-Commutative-Algebra/Herzog-Restuccia/p/book/9780824705671.

Valenzano, M. (2004). "Rank 2 reflexive sheaves on a smooth threefold". *Rendiconti del Seminario Matematico di Università e Politecnico di Torino* **62**(3), 235–254. URL: http://www.seminariomatematico.unito.it/rendiconti/62-3.html.

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