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Notes on toric Sasaki-Einstein seven-manifolds and AdS_4/CFT_3

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Notes on toric Sasaki-Einstein seven-manifolds and AdS_4/CFT_3

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Abstract: We study the geometry and topology of two infinite families $Y^{p,k}$ of Sasaki-Einstein seven-manifolds, that are expected to be $\mathrm{AdS}_4/\mathrm{CFT}_3$ dual to families of $\mathcal{N}=2$ superconformal field theories in three dimensions. These manifolds, labelled by two positive integers p and k, are Lens space bundles $S^3/\mathbb{Z}_p$ over $\mathbb{C}P^2$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$, respectively. The corresponding Calabi-Yau cones are toric. We present their toric diagrams and gauged linear sigma model charges in terms of p and k, and find that the $Y^{p,k}$ manifolds interpolate between certain orbifolds of the homogeneous spaces $S^7, M^{3,2}$ and $Q^{1,1,1}$.

Keywords: AdS-CFT Correspondence, Superstring Vacua, Flux compactifications, Differential and Algebraic Geometry.

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1. Introduction

There has recently been renewed interest in supersymmetric three-dimensional conformal field theories in the context of the AdS/CFT correspondence [1]. The reasons for this interest are diverse. One motivation is that three-dimensional CFTs describe the low-energy world-volume theory of coincident M2-branes. Until recently, the understanding of these theories had been rather rudimentary. However, a breakthrough was made in the work of [2], where an $\mathcal{N} = 8$ supersymmetric Chern-Simons theory was constructed. The authors of [2] proposed that this theory is related to the theory on coincident M2-branes. A careful study of the vacuum moduli space [3] subsequently led to a more precise interpretation of this theory as describing two M2-branes at an $\mathbb{R}^8/\mathbb{Z}_2$ orbifold singularity. A number of papers have further studied the proposal of [2], culminating in the results of [4] (ABJM). In the latter reference, the theory of [2] was recast in terms of an $\mathrm{SU}(2) \times \mathrm{SU}(2)$ Chern-Simons quiver gauge theory, which allowed for a generalisation[1] of the construction to an arbitrary number N of M2-branes, with Chern-Simons level k. The authors of [4] also discussed the gravity duals of these theories, showing that they are $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$

[1] More precisely, in this generalisation the gauge group is taken to be $\mathrm{U}(N) \times \mathrm{U}(N)$.

backgrounds of M-theory, with N units of four-form flux. These works open the way for a systematic study of AdS_4 M-theory backgrounds in terms of three-dimensional conformal field theories, using the AdS/CFT correspondence.

The analogous problems in the context of type IIB string theory are understood rather well. In this case the gauge theories arise as the low-energy limit of D3-brane world-volume theories. The maximally supersymmetric case is the $\mathcal{N}=4$ SYM theory. One can obtain $\mathcal{N}=1$ SCFTs by placing the D3-branes at a Calabi-Yau singularity. In the case of orbifolds or toric singularities the technology to construct these gauge theories is now standard. The gravity duals of these theories are type IIB $\mathrm{AdS}_5\times Y_5$ backgrounds, where Y_5 is a Sasaki-Einstein five-manifold [5−8] (or orbifold) with N units of five-form flux. For some time an obstacle in the study of $\mathrm{AdS}_5/\mathrm{CFT}_4$ duals was the lack of examples — specifically, there existed only two (non-orbifold) examples where the metric was known explicitly, namely S^5 and $T^{1,1}$. The discovery of the $Y^{p,q}$ Sasaki-Einstein 5-manifolds in [9, 10] radically improved this situation.

In dimension seven, the classification of manifolds Y_7, not locally isometric to S^7, admitting Killing spinors falls into 3 types: weak G_2 manifolds, Sasaki-Einstein manifolds, and tri-Sasakian manifolds. These admit 1, 2 and 3 Killing spinors, respectively. The Killing spinor equation immediately implies that the metric is Einstein with positive Ricci curvature, and that $\mathrm{AdS}_4\times Y_7$, with N units of four-form flux, is a supersymmetric solution to eleven-dimensional supergravity. The AdS/CFT dual theories then have $\mathcal{N}=1,2$ and 3 supersymmetry, respectively. The metric cones $\mathrm{d}r^2+r^2\mathrm{d}s_7^2$ are Ricci-flat and are correspondingly Spin(7), Calabi-Yau and hyper-Kähler cones, respectively. We note that toric tri-Sasakian manifolds are extremely well-studied — see, for example, [11]. In particular, the Einstein metric on a toric tri-Sasakian manifold is the induced metric one obtains from a hyper-Kähler quotient construction.

In this paper we focus on Sasaki-Einstein manifolds that are not tri-Sasakian. In dimension seven, the list of known explicit Sasaki-Einstein manifolds in the literature, before [12], consisted of the following: $M^{3,2}, Q^{1,1,1}$ and $V_{5,2}$. For a review of these manifolds see, for example, [13]. The manifolds $M^{3,2}$ and $Q^{1,1,1}$ are natural generalisations of $T^{1,1}$ in dimension five. In particular, the corresponding Calabi-Yau cones are toric. Proposals for the AdS/CFT duals of these homogeneous Sasaki-Einstein seven-manifolds were given in [14−16]. In [12] the construction of [9, 10] was generalised to arbitrary dimension, thus providing infinite families of Sasaki-Einstein manifolds in all odd dimensions. The main result of [12] shows that for any positive curvature Kähler-Einstein manifold B_{2n} there is a countably infinite class of associated Sasaki-Einstein manifolds $Y_{2n+3}(B_{2n})$.[2] Here we will analyse two families in seven dimensions, where B_4 is either $\mathbb{C}P^2$ or $\mathbb{C}P^1\times\mathbb{C}P^1$. The seven-dimensional Sasaki-Einstein manifolds will be denoted $Y^{p,k}(\mathbb{C}P^2)$ and $Y^{p,k}(\mathbb{C}P^1\times\mathbb{C}P^1)$, respectively. Following [18], we will give a presentation of the Calabi-Yau cones in terms of a Kähler quotient, also known as a gauged linear sigma model description [19]. In fact, note that the results of this analysis were anticipated in [18] (see the introduction of the latter reference).

[2]This construction has been subsequently generalised in [17] to the case where B_{2n} is a product of Kähler-Einstein manifolds.

The description of the toric Calabi-Yau cones associated to the $Y^{p,q}$ metrics, presented in [18], gave important clues that aided the identification of the dual gauge theories. In particular, the Calabi-Yau singularity is (part of) the moduli space of supersymmetric vacua of these gauge theories. It was also observed that the family of $Y^{p,q}$ singularities interpolates between two limiting cases: $\mathbb{C}^3/\mathbb{Z}_{2p}$ and a $\mathbb{Z}_p$ orbifold of the conifold. For these the gauge theories are simple orbifolds of the $\mathcal{N}=4$ SYM and Klebanov-Witten theories, respectively. The geometric information in [18] was then used in [20] (see also [21]) to identify the general family of quiver gauge theories. Moduli spaces of orbifolds of the ABJM theory are currently under investigation [22 − 25]. Thus, the results presented here should be useful for identifying the $\mathcal{N}=2$ conformal field theory duals to the families of $\mathrm{AdS}_4 \times Y_7^{p,k}$ backgrounds [12].

In the regime of parameters where $p^5 \gg N \gg p$, the backgrounds that we discuss are better described as type IIA solutions of the form $\mathrm{AdS}_4 \times M_6$, with non-trivial dilaton, F_4 and F_2 RR fluxes. The non-trivial dilaton comes from the norm of the Killing vector along which we reduce, and is naturally dictated by the construction of the metrics in [12]. The reduction preserves the $\mathcal{N}=2$ supersymmetries. In particular, this is a different reduction to that considered in [4].

The rest of the paper is organised as follows. In section 2 we review the Sasaki-Einstein metrics presented in [12], and for the cases of interest determine explicitly their dependence on the two integers p and k. In section 3 we compute the toric data of the $Y^{p,k}(\mathbb{C}P^2)$ and $Y^{p,k}(\mathbb{C}P^1\times\mathbb{C}P^1)$ Calabi-Yau four-folds. We present the toric diagrams and GLSM charges. In section 4 we compute the homology of the manifolds and discuss supersymmetric five-submanifolds. In section 5 we discuss the $\mathrm{AdS}_4 \times Y_7^{p,k}$ M-theory backgrounds and their reduction to type IIA supergravity. The limiting case $\mathrm{AdS}_4 \times Y^{p,3p}(\mathbb{C}P^2)$ is described in some detail. We conclude in section 6.

2. Metrics and volumes

2.1 Review of the metrics of [12]

In this section we briefly recall the construction of the metrics of [12], and compute the volumes of the corresponding manifolds. We initially keep the Kähler-Einstein manifold $(B_{2n}, \tilde{g})$ general, specialising to the two examples of interest, $B_4 = \mathbb{C}P^2$ and $B_4 = \mathbb{C}P^1 \times \mathbb{C}P^1$, only when it is necessary.

Take any complete $2n$-dimensional positive curvature Kähler-Einstein manifold B_{2n}, with line element $\mathrm{d}\tilde{s}^2$ and Kähler form[3] $\tilde{J} = \mathrm{d}A/2$. The metric is normalised so that $\widetilde{\mathrm{Ric}} = \lambda\tilde{g}$. Given any such B_{2n}, there is a countably infinite family of associated Sasaki-Einstein metrics on the total space of certain Lens space bundles $S^3/\mathbb{Z}_p$ over B_{2n}. The local metrics were presented in [12] in the following form

$$\mathrm{d}s^2 = \rho^2\mathrm{d}\tilde{s}^2 + U(\rho)^{-1}\mathrm{d}\rho^2 + q(\rho)(\mathrm{d}\psi + A)^2 + w(\rho)\left[\mathrm{d}\alpha + f(\rho)(\mathrm{d}\psi + A)\right]^2 \qquad (2.1)$$

[3] Note that the one-form A is only defined locally. In fact A is a connection on the anti-canonical line bundle of B_{2n}.

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where the function $U(\rho)$ is conveniently written as

$$U(\rho) = \frac{\lambda}{2(n+1)(n+2)} \frac{1}{x^{n+1}} P(x;\kappa)\,, \qquad x = \frac{\Lambda}{\lambda}\rho^2 \tag{2.2}$$

and

$$P(x;\kappa) = -(n+1)x^{n+2} + (n+2)x^{n+1} + \kappa\,. \tag{2.3}$$

The remaining metric functions are then

$$\begin{aligned} w(\rho) &= \rho^2 U(\rho) + (\rho^2 - \lambda/\Lambda)^2 \\ q(\rho) &= \frac{\lambda^2}{\Lambda^2}\frac{\rho^2 U(\rho)}{w(\rho)} \\ f(\rho) &= \frac{\rho^2(U(\rho) + \rho^2 - \lambda/\Lambda)}{w(\rho)}\,. \end{aligned} \tag{2.4}$$

In [12] (see also [26]) it was shown that for

$$-1 < \kappa < 0 \tag{2.5}$$

one can take the ranges of the coordinates $0 \le \psi \le 4\pi/\lambda$ and $\rho_1 \le \rho \le \rho_2$ so that the "base" M_{2n+2} (excluding the α direction in (2.1)) is the total space of an S^2 bundle over B_{2n}. In particular, ρ_i are the two positive roots of the equation $U(\rho) = 0$ and satisfy the inequalities

$$0 < \rho_1 < \sqrt{\frac{\lambda}{\Lambda}} < \rho_2 < \sqrt{\frac{\lambda(n+2)}{\Lambda(n+1)}}\,. \tag{2.6}$$

The S^2 fibre is then coordinatised by the polar coordinate ρ and the axial coordinate ψ. Without loss of generality we now set $\lambda = 2$ and $\Lambda = 2(n+2)$, so that the Sasaki-Einstein metric has Ricci curvature $2n+2$ times the metric.

For appropriate values of κ in the above range, one can periodically identify the α coordinate so as to obtain a principal $\mathrm{U}(1)$ bundle over the space M_{2n+2}. Recall that the group $H_2(M_{2n+2};\mathbb{Z})$ of two-cycles on M_{2n+2} is naturally $\mathbb{Z} \oplus H_2(B_{2n};\mathbb{Z})$ where the first factor is generated by a copy Σ of the fibre S^2, and the generators Σ_i of $H_2(B_{2n};\mathbb{Z})$ are pushed forward into M_{2n+2} by the map $\sigma^N : B_{2n} \to M_{2n+2}$, which denotes the section of $\pi : M_{2n+2} \to B_{2n}$ corresponding to the "north pole" $\rho = \rho_2$ of the S^2 fibres. One can then periodically identify α to obtain a principal $\mathrm{U}(1)$ bundle over M_{2n+2} provided $B \equiv f(\rho)(\mathrm{d}\psi + A)$ is proportional to a connection one-form. This is true if and only if the periods of $\frac{1}{2\pi}\mathrm{d}B$ over the representative basis $\{\Sigma, \sigma^N\Sigma_i\}$ are rationally related. Equivalently, one ensures that the periods of $\frac{\ell^{-1}}{2\pi}\mathrm{d}B$ are all integers, for some positive constant $\ell \in \mathbb{R}$.

The periods are easily computed[4] to be

$$f(\rho_2) - f(\rho_1) = \int_\Sigma \frac{\mathrm{d}B}{2\pi} \equiv \ell p \tag{2.7}$$

$$f(\rho_2)c_{(i)} = \int_{\sigma^N\Sigma_i} \frac{\mathrm{d}B}{2\pi} \equiv \ell\frac{k}{h}c_{(i)}\,, \tag{2.8}$$

[4] The definitions here are slightly different to those in [12].

where

$$c_{(i)} \;=\; \int_{\Sigma_i} \frac{\mathrm{d}A}{2\pi} \;=\; \langle c_1(\mathcal{L}), [\Sigma_i]\rangle \in \mathbb{Z} \tag{2.9}$$

are Chern numbers of the anti-canonical bundle[5] $\mathcal{L}$ over B_{2n} and we have defined $h = \mathrm{hcf}\{c_{(i)}\}$. Thus we see that, if $f(\rho_1)/f(\rho_2)$ is rational and hence $p, k \in \mathbb{Z}$, α can be periodically identified with period $2\pi\ell$. The $\mathrm{U}(1)$ principal bundle, with coordinate $\gamma\ell \equiv \alpha$, then has Chern numbers $\{p, kc_{(i)}/h\}$ with respect to the basis $\{\Sigma, \sigma^N\Sigma_i\}$. The range of k is fixed so that

$$\frac{hp}{2} \;<\; k \;<\; hp\,, \tag{2.10}$$

as follows from the bound (2.5) on κ.

Note that the resulting Sasaki-Einstein manifold is indeed a Lens space bundle $S^3/\mathbb{Z}_p$ over B_{2n} — this follows since the Chern number of the $\mathrm{U}(1)$ principal bundle over the fibre S^2 is p, which thus forms a Lens space fibre. We will see later that these Chern numbers may be re-interpreted as units of RR fluxes in a dual type IIA picture. For further details on the construction of [12], see also [26].

2.2 Volumes

We now turn to an analysis of the volumes of these manifolds. First, it will be useful to note the following formulae

$$f(\rho_i) \;=\; \frac{\rho_i^2}{\rho_i^2 - \frac{1}{n+2}}\,, \tag{2.11}$$

where ρ_1, ρ_2 are roots of the $(n+2)$-order polynomial. One also easily derives the following relations

$$h\rho_1^2 \;=\; (k-hp)\ell\left(\rho_1^2 - \frac{1}{n+2}\right)\,, \qquad\qquad h\rho_2^2 \;=\; k\ell\left(\rho_2^2 - \frac{1}{n+2}\right)\,. \tag{2.12}$$

Defining $x_i = (n+2)\rho_i^2$, the integrated volume may be written as

$$\mathrm{vol}(Y^{p,k}_{2n+3}(B_{2n})) \;=\; \mathrm{vol}(B_{2n})\frac{2\pi^2}{(n+1)(n+2)^{n+2}}\ell(x_2^{n+1} - x_1^{n+1}) \tag{2.13}$$

where

$$\ell \;=\; \frac{x_2 - x_1}{p(x_2-1)(1-x_1)}\,. \tag{2.14}$$

It is interesting to compute the formal limiting values of the volume formula (2.13) in the limit that k approaches the endpoints of the interval (2.10). The case $k \to hp$ corresponds to $\kappa \to 0$. It follows that $\ell \to \frac{n+2}{p}$, and the volume approaches

$$\mathrm{vol}(Y^{p,k}_{2n+3}) \;\stackrel{k\to hp}{\longrightarrow}\; \mathrm{vol}(B_{2n})\frac{2\pi^2}{p\,(n+1)^{n+2}}\,. \tag{2.15}$$

[5] Note, in particular, that for $B_4 = \mathbb{C}P^2$, this is $\mathcal{L} = \mathcal{O}(3)_{\mathbb{C}P^2}$.

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The case $k \to hp/2$ corresponds to $\kappa \to -1$. The limiting value of the volume is easily computed to be

$$\mathrm{vol}(Y^{p,k}_{2n+3}) \stackrel{k\to hp/2}{\longrightarrow} \mathrm{vol}(B_{2n})\frac{8\pi^2}{p\,(n+2)^{n+2}} \ . \tag{2.16}$$

Notice that in both cases the volumes are rational multiples of the volume of the round sphere S^{2n+3}. For $n=1$, from (2.15) and (2.16) we correctly obtain[6] the values $\frac{\pi^3}{2p}$ and $\frac{16\pi^3}{27p}$, respectively [18]. To be more explicit one should determine the roots x_i in terms of the integer parameters p,k. To this end, let us define the polynomial

$$Z(x_1,x_2) \ = \ \sum_{i=0}^{n} x_1^i x_2^{n-i} \ . \tag{2.17}$$

The defining equation of the roots x_i is then generally

$$(n+1)Z_{n+1}(x_1,x_2) \ = \ (n+2)Z_n(x_1,x_2) \ . \tag{2.18}$$

This is an $(n+1)$-th order equation in the two variables x_1,x_2. To determine the roots we combine (2.18) with another relation that may be obtained by eliminating ℓ from the equations (2.7), (2.8) defining the periods. This yields

$$\frac{x_1(x_2-1)}{x_2(x_1-1)} \ = \ 1-\frac{hp}{k} \ . \tag{2.19}$$

After solving for one of the roots and substituting back into (2.18), one obtains the final equation from which the roots may be extracted. In the case $n=1$ one can check that the quadratic equation in [10] is reproduced. For our purposes, it suffices to analyse the case of $n=2$. We obtain cubic equations defining the two roots:

$$\begin{aligned} 3p^3\,x_1^3+2p^2(6b-5p)\,x_1^2+p(18b^2-28pb+11p^2)\,x_1+4(3b^3+4p^2b-6pb^2-p^3) \ &= \ 0 \\ 3p^3\,x_2^3+2p^2(p-6b)\,x_2^2+p(18b^2-8pb+p^2)\,x_2+4b(3pb-3b^2-p^2) \ &= \ 0\,, \end{aligned} \tag{2.20}$$

where we have defined $b=k/h$. These may be solved analytically, although the resulting expressions are lengthy. However, it is interesting to note that the volumes are written in terms of cubic irrational numbers.

Note that for $B_4=\mathbb{C}P^2$, the first non-trivial example has $p=1$, $k=2$. In this case one easily computes

$$x_1 \ = \ \frac{1}{9}\left[2+\left(53+6\sqrt{78}\right)^{1/3}+\frac{1}{\left(53+6\sqrt{78}\right)^{1/3}}\right] \ \approx \ 0.77 \tag{2.21}$$

$$x_2 \ = \ \frac{1}{9}\left[6+\left(27+3\sqrt{78}\right)^{1/3}+\frac{3}{\left(27+3\sqrt{78}\right)^{1/3}}\right] \ \approx \ 1.17\,, \tag{2.22}$$

[6]Note that the volume of the Kähler-Einstein base B_{2n} is normalised so that $\widetilde{\mathrm{Ric}}=2\tilde{g}$.

giving the volume formula

$$\mathrm{vol}\left(Y_7^{1,2}(\mathbb{C}P^2)\right) = \frac{3\pi^4}{64}\left[\frac{107}{27}+\left(\frac{521}{54}-\sqrt{78}\right)\left(53+6\sqrt{78}\right)^{1/3}\right.$$
$$\left.+\left(\frac{2341}{54}-\frac{44}{9}\sqrt{78}\right)\left(53+6\sqrt{78}\right)^{2/3}\right]. \tag{2.23}$$

It would be nice to reproduce these numbers from a field theory calculation.

3. Toric description

Provided the base Kähler-Einstein manifold $(B_{2n}, \tilde{g})$ is toric, the Calabi-Yau cones in two complex dimensions higher are also toric. One can analyse these explicitly following the techniques described in [18] for the case of $n = 1$. The general idea is simple. The Calabi-Yau cones

$$\mathrm{d}s^2(\mathrm{CY}_{2n+4}) = \mathrm{d}r^2 + r^2\mathrm{d}s^2(Y_{2n+3}) \tag{3.1}$$

have a Hamiltonian torus action by $\mathbb{T}^{n+2}$, and so by definition are toric. Here the Kähler form ω of the Calabi-Yau may be regarded as a symplectic form, and one can then introduce a moment map $\mu : C(Y_{2n+3}) \to \mathbb{R}^{n+2}$. The image is always a convex rational polyhedral cone, of a special type, and the moment map exhibits the Calabi-Yau as a $\mathbb{T}^{n+2}$ fibration over this polyhedral cone. Writing the symplectic form of B_{2n} as

$$\tilde{J} = \mathrm{d}\phi_i \wedge \mathrm{d}\mu^i_{B_{2n}}, \tag{3.2}$$

the symplectic form of the Calabi-Yau cones may be written as

$$\omega = \mathrm{d}\phi_i \wedge \mathrm{d}\Big[r^2\rho^2\mu^i_{B_{2n}}\Big] + \mathrm{d}\psi \wedge \mathrm{d}\Big[-\frac{1}{2}r^2\rho^2\Big] + \mathrm{d}\gamma \wedge \Big[\frac{\ell}{2}r^2\Big(\frac{1}{n+2}-\rho^2\Big)\Big]. \tag{3.3}$$

From this it is fairly immediate to read off the moment map. However, a remaining problem is to determine a choice of angular coordinates, and correspondingly the choice of moment map coordinates, such that the associated vector fields generate an effectively acting $\mathbb{T}^{n+2}$. This coordinate basis will be unique up to $\mathrm{SL}(4;\mathbb{Z})$. In the remainder of this section we compute the toric and linear sigma model descriptions of $Y^{p,k}(\mathbb{C}P^2)$ and $Y^{p,k}(\mathbb{C}P^1 \times \mathbb{C}P^1)$ using the techniques described in [18], to which we refer for further details. For the time being we assume $\mathrm{hcf}(p, k) = 1$.

3.1 $Y^{p,k}(\mathbb{C}P^2)$ family

Recall that $\mathbb{C}P^2$ equipped with its Fubini-Study metric is a toric Kähler-Einstein manifold. In terms of homogeneous coordinates the torus action is

$$[z_0, z_1, z_2] \to [z_0, \exp(i\phi_1)z_1, \exp(i\phi_2)z_2] \tag{3.4}$$

which has moment map $\mu_{\mathrm{FS}} : \mathbb{C}P^2 \to \mathbb{R}^2$ given by

$$\mu_{\mathrm{FS}} = -\frac{3}{2}\left(\frac{|z_1|^2}{|z_0|^2+|z_1|^2+|z_2|^2}, \frac{|z_2|^2}{|z_0|^2+|z_1|^2+|z_2|^2}\right). \tag{3.5}$$

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Here we have normalised the metric so that $\mathrm{Ric} = 2g_{\mathrm{FS}}$. As is well-known, the image in $\mathbb{R}^2$ is a triangle with vertices $(0,0)$, $(-3/2,0)$, $(0,-3/2)$. The canonical bundle over $\mathbb{C}P^2$ has Chern class -3, and hence $h = 3$. Note we may take

$$A = -2\mu^i_{\mathrm{FS}} \mathrm{d}\phi_i \ . \tag{3.6}$$

The following is a basis for an effectively acting $\mathbb{T}^4$ on $Y^{p,k}(\mathbb{C}P^2)$:

$$e_1 = \frac{\partial}{\partial\phi_1} - \frac{\partial}{\partial\psi} + \frac{k}{3}\frac{\partial}{\partial\gamma}, \quad e_2 = \frac{\partial}{\partial\phi_2} - \frac{\partial}{\partial\psi} + \frac{k}{3}\frac{\partial}{\partial\gamma}, \quad e_3 = \frac{\partial}{\partial\psi} - \frac{k}{3}\frac{\partial}{\partial\gamma}, \quad e_4 = \frac{\partial}{\partial\gamma}. \tag{3.7}$$

The appearance of the fractional terms $k/3$ is crucial in order that the orbits of the group action close, giving an effective action of the torus on the Calabi-Yau cone. This issue was discussed in [18], and there is a straightforward way to fix a good basis of angular coordinates. Consider, for example, the fixed complex ray $\mathbb{C}^*$ given by $\{z_1 = z_2 = 0, \rho = \rho_2\}$. The induced metric is

$$\mathrm{d}r^2 + r^2\ell^2 w(\rho_2)\left(\mathrm{d}\gamma + \frac{k}{3}\mathrm{d}\psi\right)^2 \ . \tag{3.8}$$

Thus we define the new coordinates

$$\phi_3 = \psi, \qquad \phi_4 = \gamma + \frac{k}{3}\psi \,, \tag{3.9}$$

and note that ϕ_4 is a periodic coordinate on the $\mathbb{C}^*$, and that ϕ_1, ϕ_2 and ϕ_3 coordinatise the $\mathbb{T}^3$ that fixes this line. Thus ϕ_a, $a = 1, \ldots, 4$, may be taken to be standard coordinates on $\mathbb{T}^4$. Note that

$$\frac{\partial}{\partial\phi_3} = \frac{\partial}{\partial\psi} - \frac{k}{3}\frac{\partial}{\partial\gamma}, \qquad \frac{\partial}{\partial\phi_4} = \frac{\partial}{\partial\gamma} \ . \tag{3.10}$$

This basis is unique only up to $\mathrm{SL}(4;\mathbb{Z})$ transformations. For example, one easily checks that the natural induced bases at the other rays are equivalent to the one above. In (3.7) we have chosen a slightly different, but particularly convenient, basis. The moment map in this basis is then

$$\begin{aligned}\mu = r^2\Bigg[&\rho^2\mu^1_{\mathrm{FS}} + \frac{1}{2}\rho^2 - \frac{1}{6}k\ell\left(\rho^2 - \frac{1}{4}\right), \rho^2\mu^2_{\mathrm{FS}} + \frac{1}{2}\rho^2 - \frac{1}{6}k\ell\left(\rho^2 - \frac{1}{4}\right),\\ &-\frac{1}{2}\rho^2 + \frac{1}{6}k\ell\left(\rho^2 - \frac{1}{4}\right), -\frac{1}{2}\ell\left(\rho^2 - \frac{1}{4}\right)\Bigg] \ .\end{aligned} \tag{3.11}$$

This is easily computed using the Kähler form (3.3) of the Calabi-Yau cone.

We now identify the half-lines which form the polyhedral cone. These are submanifolds of Y_7 over which a $\mathbb{T}^3$ collapses. They are precisely the collection of 6 circles given by the vanishing of all but one of the 3 homogeneous coordinates on $\mathbb{C}P^2$, together with $\rho = \rho_1, \rho_2$. Noting that $\rho_1^2 - \frac{1}{n+2} < 0$ and $\rho_2^2 - \frac{1}{n+2} > 0$, these half-lines are spanned by the vectors in $\mathbb{R}^4$:

$$\begin{aligned} &u_1 = [p, p, -p, 1]\,, \quad u_2 = [-2p + k, p, -p, 1]\,, \quad u_3 = [p, -2p + k, -p, 1]\,,\\ &u_4 = [0, 0, 0, -1]\,, \quad u_5 = [-k, 0, 0, -1]\,, \qquad\quad u_6 = [0, -k, 0, -1]\,, \end{aligned} \tag{3.12}$$

where the first 3 vectors correspond to $\rho = \rho_1$ and the remaining 3 correspond to $\rho = \rho_2$. These vectors form a convex rational polyhedral cone, and it is simple to compute the outward pointing primitive normal vectors to the facets of this cone. There are 5 facets with normal vectors

$$v_1 = [0,0,1,0], \; v_2 = [0,0,1,p], \; v_3 = [1,0,1,0], \; v_4 = [0,1,1,0], \; v_5 = [-1,-1,1,k] \; . \quad (3.13)$$

As in [18], it will be useful to obtain a gauged linear sigma model description of the geometry. Here, in order to keep the paper relatively self-contained, we give a lightning review of gauged linear sigma models and Delzant's theorem [27], referring to [18] for further details. Let $z_1, \ldots, z_d$ denote complex coordinates on $\mathbb{C}^d$. In physics terms, these will be the lowest components of chiral superfields Φ_i, $i = 1, \ldots, d$. We may specify an action of the group $\mathbb{T}^r \cong \mathrm{U}(1)^r$ on $\mathbb{C}^d$ by giving the integral charge matrix $Q = \{Q_a^i \mid i = 1, \ldots, d; \; a = 1, \ldots, r\}$; here the ath copy of $\mathrm{U}(1)$ acts on $\mathbb{C}^d$ as

$$(z_1, \ldots, z_d) \;\rightarrow\; (\lambda^{Q_a^1} z_1, \ldots, \lambda^{Q_a^d} z_d) \quad (3.14)$$

where $\lambda \in \mathrm{U}(1)$. We may then perform the so-called Kähler quotient $X = \mathbb{C}^d // \mathrm{U}(1)^r$ by imposing the r constraints

$$\sum_{i=1}^{d} Q_a^i |z_i|^2 \;=\; t_a \qquad\quad a \;=\; 1, \ldots, r \, , \quad (3.15)$$

where t_a are constants, and then quotienting by $\mathrm{U}(1)^r$. The resulting space X has complex dimension $m = d - r$ and inherits a Kähler, and hence also symplectic, structure from that of $\mathbb{C}^d$. In physics terms, the constraints (3.15) correspond to setting the D-terms of the gauged linear sigma model to zero to give the vacuum, where t_a are FI parameters. The quotient by $\mathbb{T}^r$ then removes the gauge degrees of freedom. Thus the Kähler quotient of the gauged linear sigma model precisely describes the classical vacuum of the theory. For the cases of interest in this paper, we set $t_a = 0$ so that the resulting quotient space is a cone. It is also an important fact that $c_1(X) = 0$ is equivalent to the statement that the sum of the $\mathrm{U}(1)$ charges is zero for each $\mathrm{U}(1)$ factor. Thus

$$\sum_{i=1}^{d} Q_a^i \;=\; 0 \qquad a = 1, \ldots, r \; . \quad (3.16)$$

The sigma model is then Calabi-Yau, although note that the metric induced by the Kähler quotient is *not* in general Ricci-flat.

In order to go from the moment map description to the gauged linear sigma model description above, one can apply the Delzant theorem of [27]. We begin by considering the linear map $\pi : \mathbb{R}^d \rightarrow \mathbb{R}^m$ which maps the standard basis vectors E_i of $\mathbb{R}^d$ to the outward normal vectors v_i of the moment polytope. Thus $\pi(E_i) = v_i$ for each $i = 1, \ldots, d$. Moreover, since the map maps lattice vectors to lattice vectors, one also obtains an induced map of tori

$$\tilde{\pi} : \mathbb{T}^d \rightarrow \mathbb{T}^m \; . \quad (3.17)$$

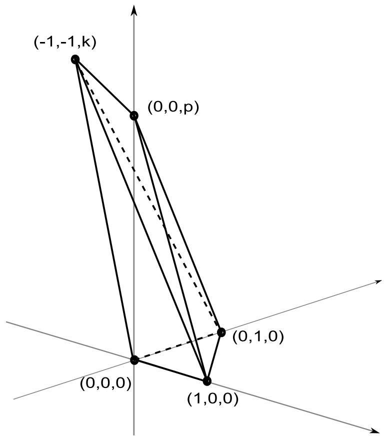

Figure 1: Toric diagram for $Y^{p,k}(\mathbb{C}P^2)$. The polytope is bounded by 6 triangular faces.

The Delzant theorem is that the gauged linear sigma model gauge group is the kernel of the map $\tilde{\pi}$. Note this may contain discrete factors, so that the kernel is not connected. This will occur, for example, for the orbifold $\mathbb{C}^4/\mathbb{Z}_{3p}$ discussed below.

In the case at hand, we must compute the kernel of the map

$$\mathbb{R}^5 \to \mathbb{R}^4 : \quad E_a \mapsto v_a \ . \tag{3.18}$$

Thus $d = 5$, $m = 4$, in the above notation. The kernel is generated by the primitive vector in the integral lattice $\mathbb{Z}^5$ given by

$$(-3p + k, -k, p, p, p) \ . \tag{3.19}$$

These are thus the charges of the gauged linear sigma model.

Note that the vectors (3.13) are coplanar, all lying on the plane $\{E_3 = 1\}$. This is a result of the Calabi-Yau condition. We may hence represent the toric data as a set of vectors in $\mathbb{Z}^3$:

$$w_1 = [0,0,0], \quad w_2 = [0,0,p], \quad w_3 = [1,0,0], \quad w_4 = [0,1,0], \quad w_5 = [-1,-1,k] \ . \tag{3.20}$$

It is a general result that these vectors form the vertices of a compact convex lattice polytope in $\mathbb{Z}^3 \subset \mathbb{R}^3$. The corresponding diagram is usually called the toric diagram in the physics literature. The polytope for $Y^{p,k}(\mathbb{C}P^2)$ is shown in figure 1.

This information allows for a simple identification of the limits $k = 3p$ and $2k = 3p$. In the former case one can show that the vector w_2 lies on the plane defined by w_3, w_4, w_5. We may hence discard this vector to give a minimal presentation of the singularity. Thus this limit is necessarily an orbifold of $\mathbb{C}^4$. Using the Delzant theorem one easily finds the orbifold action is generated by

$$(\omega_{3p},\ \omega_{3p},\ \omega_{3p},\ \omega_{3p}^{-3}) \in \mathrm{SU}(4) \tag{3.21}$$

where ω_{3p} is a $3p$-th root of unity. We thus obtain the orbifold $\mathbb{C}^4/\mathbb{Z}_{3p} = (\mathbb{C}^4/\mathbb{Z}_3)/\mathbb{Z}_p$. The toric diagram is shown in figure 2. We shall return to consider this orbifold in more detail later.

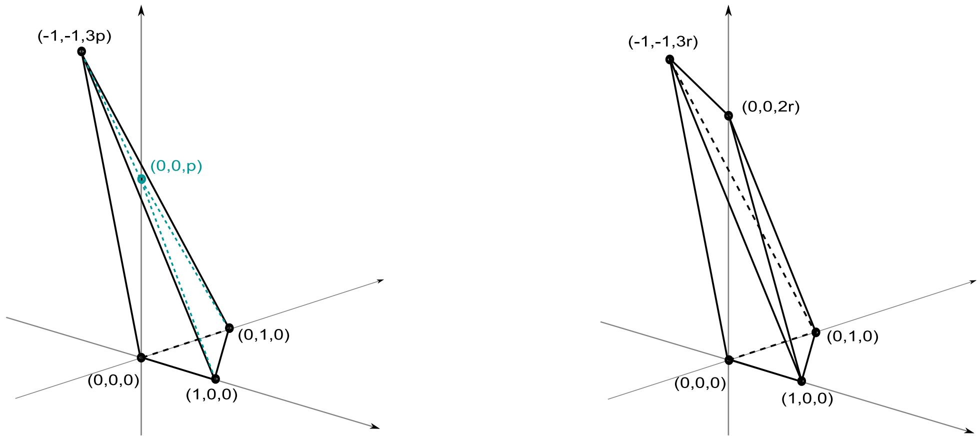

Figure 2: On the left hand side: Toric diagram for the orbifold $Y^{p,3p}(\mathbb{C}P^2) = \mathbb{C}^4/\mathbb{Z}_{3p}$. The polytope is bounded by 4 triangular faces. On the right hand side: Toric diagram for $Y^{2r,3r}(\mathbb{C}P^2) = M^{3,2}/\mathbb{Z}_r$.

The limit $2k = 3p$ clearly requires $p = 2r$ even. One then has a $\mathbb{Z}_r$ orbifold of the gauged linear sigma model with charges

$$(-3, -3, 2, 2, 2) \ . \tag{3.22}$$

In fact this is just the complex cone over $\mathbb{C}P^2 \times \mathbb{C}P^1$. The Sasaki-Einstein metric is the homogeneous metric known as $M^{3,2}$. The finite quotient is given by $\mathbb{Z}_r \subset \mathrm{SU}(2)$ which acts on the $\mathbb{C}P^1$, thus breaking the isometry group to $\mathrm{SU}(3) \times \mathrm{U}(1)^2$ which is the isometry group of the general $Y^{p,k}(\mathbb{C}P^2)$ manifold. The toric diagram is shown in figure 2. Equations[7] (2.15) and (2.16) show that the volume of a generic $Y^{p,k}(\mathbb{C}P^2)$ lies within the following range

$$\frac{9\pi^4}{128r} = \mathrm{vol}(M^{3,2}/\mathbb{Z}_r) > \mathrm{vol}(Y^{p,k}(\mathbb{C}P^2)) > \mathrm{vol}(S^7/\mathbb{Z}_{3p}) = \frac{\pi^4}{9p} \tag{3.23}$$

where the volume[8] of $M^{3,2}$ is easily computed using the topological formula

$$\mathrm{vol}(M^{3,2}) = \frac{\pi^4}{768} \int_{\mathbb{C}P^2 \times \mathbb{C}P^1} c_1^3 \ . \tag{3.24}$$

It is interesting to notice that, at fixed p, the volume is a monotonically decreasing function of k in the range (3.23).

3.2 $Y^{p,k}(\mathbb{C}P^1 \times \mathbb{C}P^1)$ family

Since the canonical bundle over $\mathbb{C}P^1 \times \mathbb{C}P^1$ has both Chern numbers equal to -2, we have $h = 2$. The following is a basis for an effectively acting $\mathbb{T}^4$ on $Y^{p,k}(\mathbb{C}P^1 \times \mathbb{C}P^2)$:

$$e_1 = \frac{\partial}{\partial \phi_1} - \frac{k}{2}\frac{\partial}{\partial \gamma}, \quad e_2 = \frac{\partial}{\partial \phi_2} - \frac{k}{2}\frac{\partial}{\partial \gamma}, \quad e_3 = \frac{\partial}{\partial \psi} - \frac{k}{2}\frac{\partial}{\partial \gamma}, \quad e_4 = \frac{\partial}{\partial \gamma} \ . \tag{3.25}$$

[7] Recall that, in our normalisation for the Kähler-Einstein base, $\mathrm{vol}(\mathbb{C}P^2) = 9\pi^2/2$.

[8] This volume was also computed in [15].

Here ϕ_j, $j = 1, 2$, are azimuthal coordinates on the two copies of $\mathbb{C}P^1$, respectively. The argument that leads to the basis (3.25) is similar to that in the previous subsection. The moment map in this basis is

$$\mu = r^2\left[\frac{1}{2}\rho^2\cos\theta_1 + \frac{1}{4}k\ell\left(\rho^2 - \frac{1}{4}\right), \frac{1}{2}\rho^2\cos\theta_2 + \frac{1}{4}k\ell\left(\rho^2 - \frac{1}{4}\right),\right.$$
$$\left. -\frac{1}{2}\rho^2 + \frac{1}{4}k\ell\left(\rho^2 - \frac{1}{4}\right), -\frac{1}{2}\ell\left(\rho^2 - \frac{1}{4}\right)\right] . \tag{3.26}$$

Here θ_1, θ_2 are usual polar coordinates on the two two-spheres.

We now identify the half-lines which form the polyhedral cone. These are precisely the collection of 8 circles given by all 2^3 combinations of $\theta_1 = 0, \pi$; $\theta_2 = 0, \pi$; $\rho = \rho_1, \rho_2$. These half-lines are spanned by the following vectors in $\mathbb{R}^4$:

$$\begin{aligned}
&u_1 = [-k+p, -k+p, -p, 1], \quad u_2 = [-k+p, -p, -p, 1], \\
&u_3 = [-p, -k+p, -p, 1], \qquad u_4 = [-p, -p, -p, 1], \qquad u_5 = [k, k, 0, -1], \\
&u_6 = [k, 0, 0, -1], \qquad\qquad u_7 = [0, k, 0, -1], \qquad\quad u_8 = [0, 0, 0, -1] ,
\end{aligned}$$

where the first 4 vectors correspond to $\rho = \rho_1$ and the remaining 4 correspond to $\rho = \rho_2$. There are 6 facets for this polyhedral cone with normal vectors

$$\begin{aligned}
&v_1 = [0, 0, 1, 0], \quad v_2 = [0, 0, 1, p], \quad v_3 = [-1, 0, 1, 0], \\
&v_4 = [1, 0, 1, k], \quad v_5 = [0, -1, 1, 0], \quad v_6 = [0, 1, 1, k] .
\end{aligned} \tag{3.27}$$

Each of these vectors has zero dot products with precisely four of the u_i and has negative dot products with the remaining four.

We may now apply the Delzant theorem of [27]. Thus we compute the kernel of the map

$$\mathbb{R}^6 \to \mathbb{R}^4 : \quad E_a \mapsto v_a \tag{3.28}$$

where E_a, $a = 1, \ldots, 6$ is the standard orthonormal basis for $\mathbb{R}^6$. This kernel is generated by the primitive vectors in the lattice $\mathbb{Z}^6$

$$\begin{aligned}
&(-2p+k, -k, p, p, 0, 0) \\
&(-2p+k, -k, 0, 0, p, p)
\end{aligned} \tag{3.29}$$

which give the charges of the gauged linear sigma model. Again note that the vectors (3.27) lie on the plane $\{E_3 = 1\}$ and thus we may project onto this plane to obtain

$$\begin{aligned}
&w_1 = [0, 0, 0], \quad w_2 = [0, 0, p], \quad w_3 = [-1, 0, 0], \\
&w_4 = [1, 0, k], \quad w_5 = [0, -1, 0], \quad w_6 = [0, 1, k] .
\end{aligned} \tag{3.30}$$

The corresponding toric diagram for $Y^{p,k}(\mathbb{C}P^1 \times \mathbb{C}P^1)$ is shown in figure 3.

We now identify the limits $k = 2p$, $k = p$. The former is a $\mathbb{Z}_p$ quotient of the gauged linear sigma model with charges

$$\begin{aligned}
&(0, -2, 1, 1, 0, 0) \\
&(0, -2, 0, 0, 1, 1) .
\end{aligned} \tag{3.31}$$

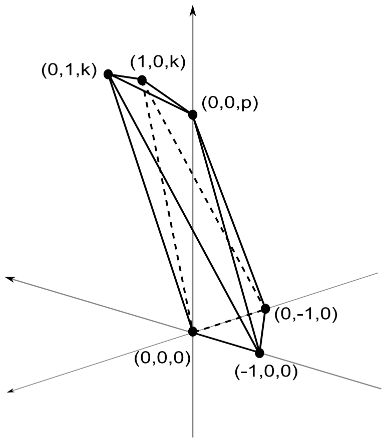

Figure 3: Toric diagram for $Y^{p,k}(\mathbb{C}P^1 \times \mathbb{C}P^1)$. The polytope is bounded by 8 triangular faces.

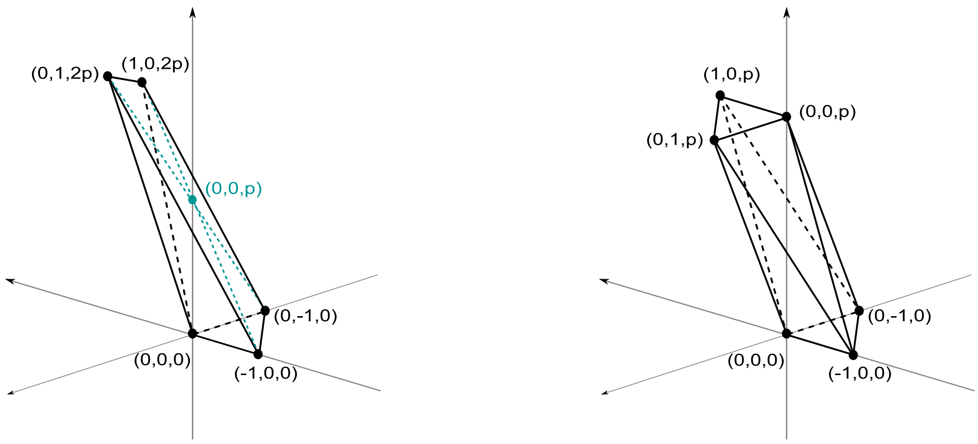

Figure 4: On the left hand side: Toric diagram for $Y^{p,2p}(\mathbb{C}P^1 \times \mathbb{C}P^1)$. This is bounded by 4 triangles and a parallelogram, implying that the link of the singularity has worse-than-orbifold singularities. On the right hand side: Toric diagram for $Y^{p,p}(\mathbb{C}P^1 \times \mathbb{C}P^1) = Q^{1,1,1}/\mathbb{Z}_p$.

In fact this describes $\mathbb{C} \times \mathcal{C}_{\mathbb{C}}(\mathbb{C}P^1 \times \mathbb{C}P^1)$, where $\mathcal{C}_{\mathbb{C}}(\mathbb{C}P^1 \times \mathbb{C}P^1)$ denotes the complex cone over $\mathbb{C}P^1 \times \mathbb{C}P^1$. Thus the boundary of this space has worse-than-orbifold singularities.[9] One can also see this from the toric diagram, shown in figure 4. For general p and k, the vertices w_3, w_5, w_4, w_6 form a parallelogram with edge vectors $(1,1,k)$ and $(1,-1,0)$. When $k = 2p$, the vertex $w_2 = (0,0,p)$ lies in this parallelogram. Thus the parallelogram itself becomes a bounding face of the polytope; the fact that this is not a triangle implies that one has worse-than-orbifold singularities on the link of the singularity at the apex of the cone.

The limit $k = p$ is instead a $\mathbb{Z}_p$ quotient of the gauged linear sigma model with charges

$$\begin{aligned} &(-1,-1,1,1,0,0) \\ &(-1,-1,0,0,1,1)\ . \end{aligned} \tag{3.32}$$

This space is the circle bundle over $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ with Chern numbers 1 over each

[9]The complex cone over $\mathbb{C}P^2$ is an orbifold, which is why projective spaces are exceptional in this limit.

$\mathbb{C}P^1$, and the corresponding homogeneous Sasaki-Einstein manifold is known as $Q^{1,1,1}$. The finite quotient is given by $\mathbb{Z}_p \subset \mathrm{SU}(2)$ in the first $\mathbb{C}P^1$ which thus breaks the isometry group to $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)^2$. This is the isometry group of the general $Y^{p,k}(\mathbb{C}P^1\times\mathbb{C}P^1)$ manifold. The toric diagram is shown in figure 4. Using equations (2.15) and (2.16), we find that the volume of $Y^{p,k}(\mathbb{C}P^1\times\mathbb{C}P^1)$ lies within the following range

$$\frac{\pi^4}{8p} = \mathrm{vol}(Q^{1,1,1}/\mathbb{Z}_p) > \mathrm{vol}(Y^{p,k}(\mathbb{C}P^1\times\mathbb{C}P^1)) > \mathrm{vol}(\partial L/\mathbb{Z}_p) = \frac{8\pi^4}{81p}\,, \tag{3.33}$$

where the volume of $Q^{1,1,1}$ is easily computed using a topological formula similar to (3.24), and is also given for instance in [15]. Again, at fixed p, the volume is a monotonically decreasing function of k in the range (3.33).

4. Homology and supersymmetric submanifolds

4.1 Homology

In this subsection we make some comments on the homology of $Y = Y^{p,k}(B_{2n})$. We begin by keeping B_{2n} general, specialising to the two cases of interest only when it is necessary.

Since for $(p,k)\equiv \mathrm{hcf}(p,k)=1$ the Chern numbers of the α circle bundle are relatively prime, it follows that Y is simply-connected. More generally we have $\pi_1(Y^{p,k})\cong\mathbb{Z}_{(p,k)}$. Using the Gysin sequence for the circle fibration, it is also easy to see that $H_2(Y)\cong\mathbb{Z}^{b_2(B_{2n})}$ where $b_2(B_{2n})$ is the second Betti number of B_{2n}. In fact these topological invariants are also easily deduced from the toric data [28]. Specifically,

$$\pi_1(Y)\;\cong\;\mathbb{Z}^{n+2}/\langle v_a\rangle\,, \qquad\qquad \pi_2(Y)\;\cong\;\mathbb{Z}^{d-(n+2)}\,, \tag{4.1}$$

where d is the number of normals $\{v_a\}$. Thus, in particular, the number of gauge groups in the gauged linear sigma model is always given by $b_2(B_{2n})$.

The remaining homology groups are easily computed using the Gysin sequence of the α circle bundle. The result for the two cases studied in this paper is

$$\begin{array}{llll} H_0\;\cong\;\mathbb{Z}, & H_1\;\cong\;\mathbb{Z}_{(p,k)}, & H_2\cong\mathbb{Z}^{b_2(B_{2n})}, & H_3\;\cong\;\Gamma, \\ H_4\;\cong\;0, & H_5\;\cong\;\mathbb{Z}^{b_2(B_{2n})}\oplus\mathbb{Z}_{(p,k)}, & H_6\;\cong\;0, & H_7\;\cong\;\mathbb{Z}\,. \end{array} \tag{4.2}$$

Here the finite group Γ is

$$\Gamma\;\cong\;\begin{cases}\mathbb{Z}^2/\langle(0,-3p+k),(k,p)\rangle \\ \mathbb{Z}^3/\langle(0,-2p+k,-2p+k),(k,p,0),(k,0,p)\rangle\end{cases} \tag{4.3}$$

in the case $B_4=\mathbb{C}P^2$ and $B_4=\mathbb{C}P^1\times\mathbb{C}P^1$, respectively. To derive these last results it is useful to note that the cohomology ring of M_6 is given by the polynomial ring

$$H^*(M_6)\;\cong\;H^*(B_4)[z]/(z^2-c_1(\mathcal{L})z) \tag{4.4}$$

where z generates the cohomology of the fibre S^2. This follows since, topologically, M_6 is the projectivisation of the bundle $\mathcal{O}\oplus\mathcal{L}\to B_{2n}$. The cohomology ring of M_6 is then standard — see [29]. Then the Gysin sequence gives that

$$H^4(Y_7)\;\cong\;H^4(M_6)/[c_1\cup H^2(M_6)] \tag{4.5}$$

where $c_1 = pz + (k/h)\pi^* c_1(\mathcal{L})$ is the first Chern class of the α circle bundle.

We conclude by summarising the non-zero Betti numbers for the two cases of interest:

$$\begin{aligned} Y^{p,k}(\mathbb{C}P^2): \quad & b_0 = b_7 = 1\,, \quad b_2 = b_5 = 1\ . \\ Y^{p,k}(\mathbb{C}P^1 \times \mathbb{C}P^1): \quad & b_0 = b_7 = 1\,, \quad b_2 = b_5 = 2\ . \end{aligned} \tag{4.6}$$

This implies that in the dual gauge theories one expects to find one or two global "baryonic" U(1) symmetries, respectively.[10] These are associated to massless gauge fields in AdS_4, coming from Kaluza-Klein reduction of the M-theory six-form (dual to the three-form) on the internal five-cycles. In fact, the values above are also valid for the limiting cases $Y^{2,3}(\mathbb{C}P^2) = M^{3,2}$ and $Y^{1,1}(\mathbb{C}P^1 \times \mathbb{C}P^1) = Q^{1,1,1}$ [15].

4.2 Supersymmetric submanifolds

We now discuss supersymmetric 5-submanifolds. By definition, one may wrap M5-branes on these submanifolds and preserve supersymmetry. These should correspond to BPS baryon-like operators in the dual SCFT_3. In particular, the conformal dimensions (and R-charges) of these operators are proportional to the corresponding volumes of the submanifolds, and provide important checks on the conjectured dual field theories. Specifically, the conformal dimension of such operators is given by [30]

$$\Delta = \frac{\pi N}{6} \frac{\mathrm{vol}(\Sigma_5)}{\mathrm{vol}(Y_7)}\,, \tag{4.7}$$

where N denotes the number of M2 branes, and should also be related to the rank of the gauge group in the dual CFT_3.

These submanifolds are the bases of six-dimensional cones which are divisors in the Calabi-Yau. The toric divisors are the inverse images under the moment map of the facets of the polyhedral cone. However, here we will characterise the submaniolds using specific features of the construction of [12]. As we reviewed in section 2, all Sasaki-Einstein manifolds constructed in [12] arise as principal $\mathrm{U}(1)_\alpha$ bundles over certain manifolds M_{2n+2}, which are themselves S^2 bundles over Kähler-Einstein manifolds B_{2n}. It is easy to show (see [18]) that taking a section $\{\rho = \rho_i\}$ of the S^2 fibre and fibering with $\mathrm{U}(1)_\alpha$ gives rise to two supersymmetric $(2n+1)$-submanifolds Ξ_1, Ξ_2. The volumes of these are given by

$$\mathrm{vol}(\Xi_i) = \mathrm{vol}(B_{2n}) 2\pi\ell \frac{x_i^n}{(n+2)^{n+1}} |x_i - 1| \qquad i = 1,2\ . \tag{4.8}$$

For the $n = 2$ cases discussed in this paper it is also easy to determine their topology:

$$Y^{p,k}(\mathbb{C}P^2): \quad \begin{cases} \Xi_1 \cong S^5/\mathbb{Z}_{3p-k} \\ \Xi_2 \cong S^5/\mathbb{Z}_k \end{cases} \tag{4.9}$$

$$Y^{p,k}(\mathbb{C}P^1 \times \mathbb{C}P^1): \quad \begin{cases} \Xi_1 \cong (S^2 \times S^3)/\mathbb{Z}_{2p-k} \\ \Xi_2 \cong (S^2 \times S^3)/\mathbb{Z}_k \end{cases} \tag{4.10}$$

[10] Notice that, although S^7 has no five-cycles, the ABJM quiver theory has a global "baryonic" symmetry.

The finite quotients are along the fibres of the principal circle bundles $S^1 \hookrightarrow S^5 \to \mathbb{C}P^2$, $S^1 \hookrightarrow T^{1,1} \to \mathbb{C}P^1 \times \mathbb{C}P^1$, respectively.

Now, if B_{2n} is toric it will admit a number of $(2n-2)$-dimensional toric divisors $\{\sigma_i, i = 1, \dots g\}$. These lift to non-compact toric divisors on the Calabi-Yau $(n+2)$-fold whose boundaries are g additional supersymmetric $(2n+1)$-submanifolds Θ_i of Y_{2n+3}. Their volumes are given by

$$\mathrm{vol}(\Theta_i) \;=\; \mathrm{vol}(\sigma_i)\frac{2\pi^2}{n(n+2)^{n+1}}\ell(x_2^n - x_1^n)\,, \qquad i = 1, \dots, g\ . \tag{4.11}$$

For $B_4 = \mathbb{C}P^1 \times \mathbb{C}P^1$, notice that these are in fact topologically four copies of $Y_5^{p,q}$, where $k = p + q$. For $B_4 = \mathbb{C}P^2$, the projection to M_6 gives topologically three copies of the third Hirzebruch surface $\mathbb{F}_3$ — that is, a $\mathbb{C}P^1$ bundle over $\mathbb{C}P^1$ with twist 3. In fact this space is diffeomorphic to $\mathbb{F}_1$. $\mathbb{F}_3$ is not a spin manifold and, in fact, depending on p and k, neither is the total space of the α circle bundle over this. Thus, in these cases, these supersymmetric submanifolds are not spin. However, note that both D-branes and M5-branes may still be wrapped supersymmetrically on non-spin manifolds.[11] For reference, we write down the volumes

$$\begin{aligned} Y^{p,k}(\mathbb{C}P^2) \ &: \ \ \mathrm{vol}(\Theta_i) = \frac{3\pi^3\ell}{4^3}(x_2^2 - x_1^2) \quad i = 1, 2, 3 \\ Y^{p,k}(\mathbb{C}P^1 \times \mathbb{C}P^1) \ &: \ \ \mathrm{vol}(\Theta_i) = \frac{2\pi^3\ell}{4^3}(x_2^2 - x_1^2) \quad i = 1, \dots, 4\ . \end{aligned} \tag{4.12}$$

In general, the x_i are cubic roots and the expressions for these volumes are rather lengthy. However, it may be useful to record the values in the orbifold limits. We do this for the case of $Y^{p,k}(\mathbb{C}P^2)$. We have

$$\begin{aligned} Y^{p,3p}(\mathbb{C}P^2) \ &: \ \ \mathrm{vol}(\Xi_2) \;=\; \mathrm{vol}(\Theta_i) \;=\; \frac{\pi^3}{3p} \\ Y^{2r,3r}(\mathbb{C}P^2) \ &: \ \ \mathrm{vol}(\Xi_1) \;=\; \mathrm{vol}(\Xi_2) \;=\; \frac{9\pi^3}{64r}\,, \quad \mathrm{vol}(\Theta_i) \;=\; \frac{3\pi^3}{16r}\ . \end{aligned}$$

Notice that in the case $Y^{p,3p}(\mathbb{C}P^2) = S^7/\mathbb{Z}_{3p}$ the volume of Ξ_1 is formally zero. The fact that one submanifold disappears in this limit may be also understood from the fact that the number of external points in the toric diagram jumps from five to four, as discussed around equation (3.20).

Notice that the volumes given above satisfy the relation

$$\sum_{i=1}^{2}\mathrm{vol}(\Xi_i) + \sum_{i=1}^{g}\mathrm{vol}(\Theta_i) \;=\; \frac{12}{\pi}\,\mathrm{vol}(Y_7)\ . \tag{4.13}$$

In fact, this follows from specialising a general formula (cf. equation (2.88)) given in the first reference in [36]. Using (4.7), with $N = 1$, this may be rewritten as

$$\sum_{a=1}^{d}\Delta_a \;=\; 2\ . \tag{4.14}$$

[11] Although this may introduce additional subtleties. For example, the Freed-Witten anomaly shifts the periods of the world-volume gauge field to half-integer values on a non-spin manifold.

In the context of $\mathrm{AdS}_5/\mathrm{CFT}_4$ this formula is interpreted as the constraint that the R-charges of the fields entering in a superpotential term sum to two [31], and it is natural to give the same interpretation in the context of $\mathrm{AdS}_4/\mathrm{CFT}_3$.

5. Supergravity solutions

We now turn to a discussion of the $\mathrm{AdS}_4 \times Y_7$ M-theory backgrounds and their reduction to type IIA string theory. We then describe in more detail the orbifold $S^7/\mathbb{Z}_{3p}$ and its cone $\mathbb{C}^4/\mathbb{Z}_{3p}$.

5.1 M-theory and type IIA backgrounds

We use the notation of [32, 4]. The M-theory backgrounds of interest take the form

$$\begin{aligned} \mathrm{d}s^2 &= R^2\left(\frac{1}{4}\mathrm{d}s^2(\mathrm{AdS}_4) + \mathrm{d}s^2(Y_7)\right) , \\ G_4 &= \frac{3}{8}R^3 \mathrm{dvol}(\mathrm{AdS}_4) , \end{aligned} \tag{5.1}$$

where the Einstein metrics on AdS_4 and Y_7 obey

$$\mathrm{Ric}_{\mathrm{AdS}_4} = 3\, g_{\mathrm{AdS}_4} \qquad \mathrm{Ric}_{Y_7} = 6\, g_{Y_7} , \tag{5.2}$$

respectively. The radius R is determined by the quantisation of the G_4 flux

$$N = \frac{1}{(2\pi l_p)^6}\int_{Y_7} *G_4 , \tag{5.3}$$

where l_p is the eleven-dimensional Planck length, given by

$$R^6 = \frac{(2\pi l_p)^6 N}{6\mathrm{vol}(Y_7)} . \tag{5.4}$$

Recall that Sasaki-Einstein metrics may be canonically written as

$$\mathrm{d}s^2(Y_7) = \mathrm{d}s^2(B_6) + (\mathrm{d}\varphi + \sigma)^2 , \tag{5.5}$$

where $\mathrm{d}s^2(B_6)$ is in general only a local Kähler-Einstein metric (with $\mathrm{Ric}_{B_6} = 8\, g_{B_6}$) and $\mathrm{d}\sigma/2 = \omega_{B_6}$ is the corresponding Kähler two-form. When the Sasaki-Einstein manifold Y_7 is of (quasi-) regular type, meaning that B_6 is a manifold (orbifold), one may then quotient by the $\mathrm{U}(1)$ action generated by the Reeb vector field ∂_φ. Thus, in these cases one can reduce to type IIA supergravity along this particular direction. This is the reduction discussed in [4] for the case of $Y_7 = S^7$, or more generally $Y_7 = S^7/\mathbb{Z}_k$. The $\mathbb{Z}_k$ action discussed by ABJM divides by a factor of k the periodicity of φ. The radius of the M-theory circle is in this case $R_\varphi = R/k \sim (N/k^5)^{1/6}$, and thus the M-theory description is valid for $N \gg k^5$ [4]. On the other hand, when $N \ll k^5$ the circle becomes small and

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one should pass to a type IIA description. The resulting type IIA supergravity solution preserves $\mathcal{N}=6$ supersymmetry at the supergravity level [33], and gives the background

$$\mathrm{d}s^2_{\mathrm{st}} \;=\; \frac{R^3}{k}\left(\frac{1}{4}\mathrm{d}s^2(\mathrm{AdS}_4)+\mathrm{d}s^2(\mathbb{C}P^3)\right)\,, \tag{5.6}$$

$$\mathrm{e}^{2\Phi} \;=\; \frac{R^3}{k^3}\,, \qquad F_4 \;=\; \frac{3}{8}R^3\mathrm{dvol}(\mathrm{AdS}_4)\,, \qquad F_2 \;=\; 2k\,\omega_{\mathbb{C}P^3}\,, \tag{5.7}$$

where the metric is in the string frame. There are then N units of F_4 flux through AdS_4, and k units of F_2 flux through the linearly embedded $\mathbb{C}P^1\subset\mathbb{C}P^3$. Note here that, due to the normalisation of the Kähler-Einstein metric on $\mathbb{C}P^3$, the Ricci form of the latter is given by $\rho=8\omega_{\mathbb{C}P^3}$, and thus

$$\int_{\mathbb{C}P^1}\frac{\omega_{\mathbb{C}P^3}}{2\pi} \;=\; \frac{1}{8}\int_{\mathbb{C}P^1}c_1(\mathbb{C}P^3) \;=\; \frac{1}{2}\,. \tag{5.8}$$

Here we have used the fact that the first Chern class of the tangent bundle of $\mathbb{C}P^3$ is equal to 4 times the hyperplane class. The radius of curvature of this background is $R^2_{\mathrm{st}}=R^3/k\sim(N/k)^{1/2}$, and thus the type IIA supergravity approximation is valid for $N\gg k$ [4].

One might consider performing a similar reduction of one of the homogeneous Sasaki-Einstein manifolds to a solution of type IIA supergravity. However, because one starts with $\mathcal{N}=2$ supersymmetry only, now all supersymmetries are broken in the reduction [33]. Moreover, for generic $Y_7^{p,k}$ manifolds, there is no way to make sense of the quotient space, even locally, as a manifold.

However, from the construction of [12] reviewed in section 2.1, we see that one may consider a different reduction along the α-circle, obtaining perfectly smooth $\mathcal{N}=2$ supersymmetric[12] type IIA backgrounds. These are *warped* products $\mathrm{AdS}_4\times M_6$, with RR fields and a non-trivial dilaton. The manifolds M_6 are S^2 bundles over the Kähler-Einstein manifold B_4. In fact this bundle is obtained from the canonical bundle[13] $\mathcal{L}$ over B_4 by replacing the $\mathbb{C}$ fibre by $\mathbb{C}P^1$. Note that in [12] it was shown that M_6 are always spin manifolds. The topology of M_6 was discussed earlier. To perform the reduction we write $\mathrm{d}s^2(Y_7)=\mathrm{d}s^2(M_6)+w(\rho)\ell^2(\mathrm{d}\gamma+\ell^{-1}B)^2$, obtaining

$$\mathrm{d}s^2_{\mathrm{st}} \;=\; \sqrt{w(\rho)}\,\ell R^3\left(\frac{1}{4}\mathrm{d}s^2(\mathrm{AdS}_4)+\mathrm{d}s^2(M_6)\right) \tag{5.9}$$

$$\mathrm{e}^{2\Phi} \;=\; \ell^3R^3(w(\rho))^{3/2} \qquad F_4 \;=\; \frac{3}{8}R^3\mathrm{dvol}(\mathrm{AdS}_4) \qquad F_2 \;=\; \ell^{-1}\mathrm{d}B\,, \tag{5.10}$$

where $w(\rho)=(1-8\rho^2/3+\kappa/(48\rho^4))/16$ is a bounded function on M_6. From section (2.1) we find that the RR two-form flux has quantised periods, namely

$$\int_\Sigma\frac{F_2}{2\pi} \;=\; p\,, \qquad\qquad \int_{\sigma^N\Sigma_i}\frac{F_2}{2\pi} \;=\; k\,. \tag{5.11}$$

[12]This follows since both Killing spinors of the Sasaki-Einstein seven-manifolds are invariant under this $\mathrm{U}(1)_\alpha$ action. See e.g. [32] for an explicit calculation.

[13]However, one should note that the natural complex structure here is *different* from the one associated to the Calabi-Yau cone.

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Here $\Sigma \cong S^2$, and $\sigma^N \Sigma_i$ is either a copy of $\mathbb{C}P^1 \subset \mathbb{C}P^2$, or one of the two copies of $\mathbb{C}P^1 \subset \mathbb{C}P^1 \times \mathbb{C}P^1$, in the two examples, respectively. Notice that $\kappa \sim 1$, so $w(\rho)$ is of order 1 in p and k. The radius of the M-theory circle is

$$R_\gamma = \ell R \sim \frac{\ell N^{1/6}}{\mathrm{vol}(Y_7)^{1/6}} , \tag{5.12}$$

and we should pass to a type IIA description when this is small. The radius of curvature in the type IIA solution is

$$R^2_{\mathrm{st}} = \ell R^3 \sim \frac{\ell N^{1/2}}{\mathrm{vol}(Y_7)^{1/2}} . \tag{5.13}$$

Recall that $\mathrm{vol}(Y_7)$ and ℓ are determined in terms of p and k through (2.13), (2.14), and the range of k is constrained by the value of p. Thus, we can consider the limit $p \gg 1$, $k \gg 1$, at fixed p/k. Since $x_i \sim 1$, we have both $\ell \sim 1/p$, $\mathrm{vol}(Y_7) \sim 1/p$, thus we obtain a behaviour qualitatively similar to the orbifold case, reviewed above. In particular, the M-theory description is valid when $N \gg p^5$, while type IIA supergravity is a good approximation in the regime $p^5 \gg N \gg p$.

5.2 The orbifolds $S^7/\mathbb{Z}_{3p}$ and $\mathbb{C}^4/\mathbb{Z}_{3p}$

For the five-dimensional $Y^{p,q}$ manifolds, understanding the limiting case $Y^{p,p} = S^5/\mathbb{Z}_{2p}$ was a key step for constructing the complete family of quiver gauge theories [20]. We hence now discuss in more detail the analogous case of the $Y^{p,3p} = \mathbb{C}^4/\mathbb{Z}_{3p}$ orbifold.

In terms of standard complex coordinates on $\mathbb{C}^4$, the orbifold action (3.21) is

$$(z_1, z_2, z_3, z_4) \quad \rightarrow \quad (\mathrm{e}^{\frac{2\pi i}{3p}} z_1, \mathrm{e}^{\frac{2\pi i}{3p}} z_2, \mathrm{e}^{\frac{2\pi i}{3p}} z_3, \mathrm{e}^{-\frac{2\pi i}{p}} z_4) . \tag{5.14}$$

The orbifold therefore preserves $\mathcal{N} = 2$ supersymmetry [22, 24]. We begin by noting that after the following non-holomorphic change of coordinates

$$w_1 = z_1, \quad w_2 = z_2, \quad w_3 = z_3, \quad w_4 = \bar{z}_4 , \tag{5.15}$$

the above orbifold acts as the $3p$-th roots of unity $\mathbb{Z}_{3p} \subset \mathrm{U}(1)$ acting on $\widetilde{\mathbb{C}^4}$ with weights $(1, 1, 1, 3)$. The quotient by the latter realises S^7 as a $\mathrm{U}(1)$ orbi-bundle over the weighted projective space $W\mathbb{C}P^3_{[1,1,1,3]}$. For $p = 1$, one divides by $\mathbb{Z}_3$ along the fibre, resulting in the solution $\mathrm{AdS}_4 \times W\mathbb{C}P^3_{[1,1,1,3]}$, with three units of of RR F_2 flux through the $\mathbb{C}P^1$ and one unit of flux at a $\mathbb{Z}_3$ orbifold singularity — see (5.11). For general p these are replaced by $3p$ units and p units, respectively.

We may also understand this orbifold via the canonical Hopf fibration (5.5) of S^7 over $\mathbb{C}P^3$. Note that the orbifold acts as a subgroup of $\mathrm{SU}(4)$ acting on $\mathbb{C}^4$, which descends to an action on $\mathbb{C}P^3$ itself. For simplicity, we discuss the case $p = 1$ — the general $p > 1$ case is a further $\mathbb{Z}_p$ quotient of this geometry. The action is

$$(\omega_3, \omega_3, \omega_3, 1) . \tag{5.16}$$

On $\mathbb{C}^4$ this fixes the complex line $(0,0,0,z_4)$. The action on the copy of $\mathbb{C}^3$ given by $(z_1,z_2,z_3,0)$ is the usual diagonal Lens space action, with the $\mathbb{Z}_3 \subset \mathrm{U}(1)$ acting along the Hopf fibre of $S^5 \to \mathbb{C}P^2$. The action is thus free away from the origin. We now descend to $\mathbb{C}P^3$. We obtain in this way a $\mathrm{U}(1)$ bundle over $\mathbb{C}P^3/\mathbb{Z}_3$. The orbifold action has fixed points at a point and the linearly embedded $\mathbb{C}P^2$. Indeed, where $z_4 \neq 0$ we may introduce homogeneous coordinates

$$x_1 = \frac{z_1}{z_4}, \quad x_2 = \frac{z_2}{z_4}, \quad x_3 = \frac{z_3}{z_4} \ . \tag{5.17}$$

The $\mathbb{Z}_3$ action is simply the diagonal action, which thus has an isolated $\mathbb{Z}_3$ fixed point $\{x_1 = x_2 = x_3 = 0\}$. Similarly, the $\mathbb{C}P^2$ at $z_4 = 0$ is also fixed by the orbifold action. In fact the orbifold action acts on the Hopf fibre over this $\mathbb{C}P^2$, as mentioned above. Thus the $\mathrm{U}(1)$ bundle restricted to $\mathbb{C}P^2$ is $O(-3)$.

The resulting orbifold of $\mathbb{C}P^3$ may be viewed as follows. We begin by viewing $\mathbb{C}P^3$ as $O(1)_{\mathbb{C}P^2}$ glued to an open ball in $\mathbb{C}^3$ — both have boundary S^5. We may also think of this as collapsing the boundary of $O(1)_{\mathbb{C}P^2}$ to a point p_∞. This is the point $\{x_1 = x_2 = x_3 = 0\}$ above. The $\mathbb{Z}_3$ action is along the fibre of $O(1)_{\mathbb{C}P^2}$, which is also the Hopf fibre of the S^5. Thus we see explicitly that the $\mathbb{C}P^2$ zero section and the point p_∞ are fixed. We may construct the same space by instead starting with $O(3)_{\mathbb{C}P^2}$. The boundary is $S^5/\mathbb{Z}_3$, which collapsing to a point in the same way means that p_∞ is now an isolated $\mathbb{Z}_3$ singularity. Note that originally the $\mathbb{C}P^2$ zero section was a fixed locus of the $\mathbb{Z}_3$ action. However, $\mathbb{C}/\mathbb{Z}_3 \cong \mathbb{C}$, and thus the two spaces we have described are diffeomorphic, although not equivalent as orbifolds.

The discussion in the above paragraph is precisely analogous to the discussion of the orbifold $S^5/\mathbb{Z}_2$ in [6]. Following the latter reference, we may thus ask what happens when we blow up the isolated $\mathbb{Z}_3$ singularity at the point p_∞. This results in the space

$$\mathbb{C}P^1 \times_{\mathrm{U}(1)} O(3)_{\mathbb{C}P^2} \ . \tag{5.18}$$

This is a $\mathbb{C}P^1$ bundle over $\mathbb{C}P^2$, and in fact is precisely the base space M_6 in the construction of section 2. However, unlike [6], we cannot interpret this as the base of the homogeneous space $M^{3,2}$, since (5.18) is not diffeomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^2$. This suggests that we cannot view the $M^{3,2}$ theory as the IR fixed point of a deformation of the orbifold $S^7/\mathbb{Z}_3$, in the same way that $T^{1,1}$ arises as a relevant deformation of $S^5/\mathbb{Z}_2$ [6].

It is also clear in this description that the four supersymmetric 5-submanifolds are in this case copies of $S^5/\mathbb{Z}_{3p}$. Note that one of these is a smooth Lens space, with action generated by $(\omega_{3p}, \omega_{3p}, \omega_{3p})$, whereas the other three are isomorphic to each other, being *singular* quotients $(\omega_{3p}, \omega_{3p}, \omega_{3p}^{-3})$. In fact these latter quotients are similar to the $S^5/\mathbb{Z}_2$ quotient, mentioned above.

Note that when p is even the orbifold action contains elements that act diagonally along the Hopf $\mathrm{U}(1)$. To see this, note that the condition for an element to act along the Hopf diagonal is

$$\frac{l}{3p} \cong -\frac{l}{p} \mod 1 \tag{5.19}$$

which implies

$$4l = 3pn \tag{5.20}$$

where, without loss of generality, we take $0 < l < 3p$ so that $n \in \{1, 2, 3\}$. Clearly for p odd this has no solution. However, for $p = 2r$ even we may in general take $l = 3r$, $n = 2$, which leads to the diagonal $\mathbb{Z}_2$ action on $\mathbb{C}^4$

$$(z_1, z_2, z_3, z_4) \rightarrow -(z_1, z_2, z_3, z_4) . \tag{5.21}$$

This is precisely the $k = 2$ orbifold action considered by ABJM [4]. On the other hand, if p is divisible by 4, so $p = 4m$, we may take $l = 3m$, $n = 1$, leading to the diagonal $\mathbb{Z}_4$ action generated by

$$(z_1, z_2, z_3, z_4) \rightarrow \omega_4 \cdot (z_1, z_2, z_3, z_4) . \tag{5.22}$$

This is the $k = 4$ orbifold action considered by ABJM. In these latter two cases we may view the orbifold instead as $(\mathbb{C}^4/\mathbb{Z}_2)/\mathbb{Z}_{3r}$ and $(\mathbb{C}^4/\mathbb{Z}_4)/\mathbb{Z}_{3m}$, respectively, where the first quotient is the ABJM quotient.

Notice that in the discussion above one has to be careful about which complex structure one is using on $\mathbb{C}^4$. Recall that the $\mathbb{Z}_k$ action considered by ABJM is actually a discrete subgroup of the baryonic $\mathrm{U}(1)_B$, acting as follows on the bifundamental fields

$$A_i \rightarrow \mathrm{e}^{i\alpha} A_i , \qquad B_i \rightarrow \mathrm{e}^{-i\alpha} B_i . \tag{5.23}$$

Setting $\alpha = 2\pi/k$, we see that $\mathbb{Z}_k \subset \mathrm{U}(1)_B$. Thus, on the natural GLSM coordinates[14] z_i on $\mathbb{C}^4$, the ABJM $\mathbb{Z}_k$ quotient acts as

$$(z_1, z_2, z_3, z_4) \rightarrow (\mathrm{e}^{i2\pi/k} z_1, \mathrm{e}^{i2\pi/k} z_2, \mathrm{e}^{-i2\pi/k} z_3, \mathrm{e}^{-i2\pi/k} z_4) . \tag{5.24}$$

The coordinates on $\mathbb{C}^4$ used in [4] are related to the above coordinates by a non-holomorphic change of variable: $z'_1 = z_1$, $z'_2 = z_2$, $z'_3 = \bar{z}_3$, $z'_4 = \bar{z}_4$. Notice that for $k = 2$ (and only for this value) the action on z_i and z'_i is obviously the same. To construct $\mathcal{N} = 2$ orbifold quivers of the ABJM theory, it seems more appropriate to use the orbifold action on the z_i coordinates above. However, it is not clear that the standard rules ([34]) for constructing four-dimensional orbifold quivers will apply.

6. Discussion

In this paper we have studied in detail two of the families of Sasaki-Einstein seven-manifolds constructed in [12]. These are the simplest examples, with the largest isometry groups. In particular, we have given gauged linear sigma model descriptions of these manifolds, discussed their topology, and also described relevant supersymmetric submanifolds and their volumes. As is the case for the five-dimensional $Y^{p,q}$ manifolds [18], we have shown

[14]The GLSM description gives the conifold as a $\mathbb{C}^4 // \mathrm{U}(1)_B$ quotient.

that these families interpolate between certain orbifolds of homogeneous Sasaki-Einstein manifolds. In particular, the family $Y^{p,k}(\mathbb{C}P^2)$ has a limit $Y^{p,3p}(\mathbb{C}P^2) = \mathbb{C}^4/\mathbb{Z}_{3p}$, and we discussed this orbifold in some detail. The geometric results of this paper should be a useful first step in constructing candidate $\mathrm{AdS}_4/\mathrm{CFT}_3$ dual superconformal field theories. We conclude by discussing some of the issues involved in pursuing this programme.

As a general comment, note that a key ingredient in $\mathrm{AdS}_5/\mathrm{CFT}_4$ duality involving Sasaki-Einstein five-manifolds is a-maximisation [35]. Among the consequences of a-maximisation is the fact that the central charges, as well as the R-charges of a given SCFT, are necessarily algebraic numbers, i.e. roots of polynomials with integer coefficients. It was proven in [36] that the volumes, and volumes of supersymmetric submanifolds, of Sasaki-Einstein manifolds are always algebraic numbers, in *any* dimension. For the examples discussed in this paper we obtain cubic irrational numbers. This strongly suggests that there should be some type of analogue of a-maximisation for three-dimensional conformal field theories with $\mathcal{N} = 2$ supersymmetry. Note that the field theoretic τ-minimisation of [37] applies to such theories, although it is currently not known how to use this to obtain exact field theory results.

The Calabi-Yau cones $C(Y_7^{p,k})$ we have discussed admit explicit Calabi-Yau resolutions, or partial resolutions where there are residual orbifold singularities [26]. This fact might be useful for obtaining further insight into these theories [38, 39]. Note that such resolutions would also allow the BPS "mesonic" spectrum to be read off [40] from the index-character of [36]. Indeed, such generating functions have already been computed for the handful of currently-known orbifold duals in [41].

Since the geometries are toric, there will also be a dual brane web description. In this case the Calabi-Yau cones may be described as Special Lagrangian $\mathbb{T}^3 \times \mathbb{R}$ fibrations over $\mathbb{R}^4$, with certain types of degeneration of the fibres encoded combinatorially in terms of toric data. Reduction and two T-dualities leads to a dual description in terms of prq-4-branes in type IIA [42]. The configuration of these 4-branes may be read off from the toric data we presented earlier. This leads to a three-dimensional "web diagram", describing the locus of the prq-4-branes. The problem of finding the dual gauge theory then becomes translated into a problem of understanding the effective theory of such webs of 4-branes. Again, the toric nature of these manifolds also implies that one can write down associated M-theory crystals [43]. These are analogues of dimer configurations, although it is not clear to us how these are related to the recent Chern-Simons gauge theory construction of [4], and various follow-up papers.

A possible avenue of research is to try to construct a Chern-Simons-matter theory that is dual to the orbifold $\mathbb{C}^4/\mathbb{Z}_{3p}$. Similar orbifold theories have recently been constructed and discussed in [22 − 25]. This should be, in some sense, a limiting theory of the theories dual to $Y^{p,k}(\mathbb{C}P^2)$. The aforementioned orbifold constructions simply apply the standard methods to construct the orbifold theories. However, the reasoning for this is currently obscure. In particular, the ABJM orbifold $S^7/\mathbb{Z}_k$ is not simply a standard orbifold projection of the theory for $k = 1$ — instead one changes the Chern-Simons level from $k = 1$ to k. A systematic understanding of how to construct orbifold theories is currently lacking. However, note that a necessary condition for a candidate theory to be dual to a particular

$AdS_4 \times Y_7$ background is that its vacuum moduli space contains the Nth symmetric product of $C(Y_7)$ as a subvariety. This is because the latter is the moduli space of N M2-branes that are transverse to the Calabi-Yau singularity $C(Y_7)$. This problem, for general classes of $d = 3$, $\mathcal{N} = 2$ Chern-Simons quiver gauge theories, will be addressed in [44].

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