Resumming instantons in $\mathcal{N} = 2^\star$ theories with arbitrary gauge groups

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We discuss the modular anomaly equation satisfied by the the prepotential of 4-dimensional $\mathcal{N} = 2^\star$ theories and show that its validity is related to S-duality. The recursion relations that follow from the modular anomaly equation allow one to write the prepotential in terms of (quasi)-modular forms, thus resumming the instanton contributions. These results can be checked against the microscopic multi-instanton calculus in the case of classical algebras, but are valid also for the exceptional $E_{6,7,8}$, F_4 and G_2 algebras, where direct computations are not available.

Keywords: $\mathcal{N} = 2$ SYM theories; recursion relations; instantons.

1. Introduction

These proceedings are based on the papers[1] where we studied $\mathcal{N} = 2^\star$ SYM theories with a gauge algebra $\mathfrak{g} \in \{\tilde{A}_r, B_r, C_r, D_r, E_{6,7,8}, F_4, G_2\}$, extending previous results obtained in[2] for the unitary groups.[a] Our motivation is to shed light on the general structure of $\mathcal{N} = 2^\star$ SYM theories at low energy and show that the constraints imposed by S-duality take the form of a recursion relation which allows one to determine the prepotential at a non-perturbative level and resum all instanton contributions.

The $\mathcal{N} = 2^\star$ theories arise as deformations of the $\mathcal{N} = 4$ theories when the adjoint hypermultiplet acquires a mass m. Their low-energy effective dynamics is entirely encoded in the prepotential, which we denote as $F^{\mathfrak{g}}$ and which is a holomorphic function of the coupling constant

$$\tau = \frac{\theta}{2\pi} + \mathrm{i}\frac{4\pi}{g^2} , \tag{1}$$

and of the vacuum expectation value a of the scalar field in the adjoint vector

[a] Here and in the following we denote by $\tilde{A}_r$ the algebra of the unitary group $\mathrm{U}(r+1)$.

multiplet. For definiteness, we take a along the Cartan directions of $\mathfrak{g}$, namely

$$a = \mathrm{diag}\big(a_1, a_2, \cdots, a_{\mathfrak{r}}\big) \tag{2}$$

where $\mathfrak{r} = \mathrm{rank}(\mathfrak{g})$.[b] To treat all algebras simultaneously it is convenient to introduce the parameter

$$n_{\mathfrak{g}} = \frac{\alpha_{\mathrm{L}} \cdot \alpha_{\mathrm{L}}}{\alpha_{\mathrm{S}} \cdot \alpha_{\mathrm{S}}} \tag{3}$$

where α_{L} and α_{S} are, respectively, the long and the short roots of $\mathfrak{g}$. For the root system $\Psi_{\mathfrak{g}}$, we follow the standard conventions[1] (see also the Appendix), so that

$$\begin{aligned} n_{\mathfrak{g}} &= 1 \quad \text{for } \mathfrak{g} = \tilde{A}_r, D_r, E_{6,7,8}\ , \\ n_{\mathfrak{g}} &= 2 \quad \text{for } \mathfrak{g} = B_r, C_r, F_4\ , \\ n_{\mathfrak{g}} &= 3 \quad \text{for } \mathfrak{g} = G_2\ . \end{aligned} \tag{4}$$

Using this, one finds that

$$F^{\mathfrak{g}}(\tau, a) = n_{\mathfrak{g}} \mathrm{i}\pi\tau a^2 + f^{\mathfrak{g}}(\tau, a) \tag{5}$$

where the first term is the classical contribution while $f^{\mathfrak{g}}$ is the quantum part. The latter has a τ-independent one-loop term

$$f^{\mathfrak{g}}_{1-\mathrm{loop}} = \frac{1}{4} \sum_{\alpha \in \Psi_{\mathfrak{g}}} \Big[-(\alpha \cdot a)^2 \log\Big(\frac{\alpha \cdot a}{\Lambda}\Big)^2 + (\alpha \cdot a + m)^2 \log\Big(\frac{\alpha \cdot a + m}{\Lambda}\Big)^2 \Big] \tag{6}$$

where Λ is an arbitrary scale, and a series of non-perturbative corrections at instanton number k proportional to q^k, where $q = \exp(2\pi\mathrm{i}\tau)$.

The quantum prepotential can be expanded in even powers of m as

$$f^{\mathfrak{g}}(\tau, a) = \sum_{n \geq 1} f^{\mathfrak{g}}_n(\tau, a) \tag{7}$$

with $f^{\mathfrak{g}}_n$ proportional to m^{2n}. The first coefficient $f^{\mathfrak{g}}_1$ receives only a contribution at one-loop and, thus, is independent of τ. For $n > 1$, instead, the coefficients $f^{\mathfrak{g}}_n$ receive contributions also from the instanton sectors. When $\mathfrak{g} \in \{\tilde{A}_r, B_r, C_r, D_r\}$, these non-perturbative terms can be computed using localization techniques[3–6] as we will show in Section 4, but for the exceptional algebras they have to be derived with other methods. As a by-product, our analysis provides also an explicit derivation of all instanton contributions to the prepotential for the exceptional algebras $E_{6,7,8}$, F_4 and G_2, at least for the first few values of n. The key ingredient for this is S-duality.

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[b]The special unitary case, corresponding to the algebra A_r is recovered by simply imposing the tracelessness condition on a.

2. S-duality

In $\mathcal{N}=4$ SYM theories with gauge algebra $\mathfrak{g}$, the duality group is generated by

$$S=\begin{pmatrix}0 & -1/\sqrt{n_{\mathfrak{g}}}\\ \sqrt{n_{\mathfrak{g}}} & 0\end{pmatrix}\quad\text{and}\quad T=\begin{pmatrix}1&1\\0&1\end{pmatrix}\,, \tag{8}$$

which, on the coupling constant τ, act projectively as follows

$$S(\tau)=-\frac{1}{n_{\mathfrak{g}}\tau}\quad\text{and}\quad T(\tau)=\tau+1\,. \tag{9}$$

The matrices (8) satisfy the constraints

$$S^2=-1\quad\text{and}\quad (ST)^{p_{\mathfrak{g}}}=-1\quad\text{with}\ \ n_{\mathfrak{g}}=4\cos^2\left(\tfrac{\pi}{p_{\mathfrak{g}}}\right)\,, \tag{10}$$

and generate a subgroup of SL(2,$\mathbb{R}$) which is known as the Hecke group H($p_{\mathfrak{g}}$). For the simply laced algebras, *i.e.* $n_{\mathfrak{g}}=1$, we have $p_{\mathfrak{g}}=3$, and the duality group H(3) is just the modular group $\Gamma=\mathrm{SL}(2,\mathbb{Z})$. For the non-simply laced algebras, the duality groups H(4) and H(6), corresponding respectively to $n_{\mathfrak{g}}=2$ and $n_{\mathfrak{g}}=3$, are clearly different from the modular group but contain subgroups which are also congruence subgroups of Γ. Indeed, one can show that the following H($p_{\mathfrak{g}}$) elements

$$V=STS=\begin{pmatrix}-1&0\\ n_{\mathfrak{g}}&-1\end{pmatrix}\quad\text{and}\quad T=\begin{pmatrix}1&1\\0&1\end{pmatrix} \tag{11}$$

generate

$$\Gamma_0(n_{\mathfrak{g}})=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma\ :\ c=0\mod n_{\mathfrak{g}}\right\}\subset\Gamma\,. \tag{12}$$

As we will see, the modular forms of $\Gamma_0(n_{\mathfrak{g}})$, which are known and classified, and have a simple behavior also under S-duality, play an important role for the $\mathcal{N}=2^\star$ SYM theories.[c]

Another important feature is that the duality transformations exchange electric states of the theory with gauge algebra $\mathfrak{g}$ with magnetic states of the theory with the GNO dual algebra $\mathfrak{g}^\vee$, which is obtained from $\mathfrak{g}$ by exchanging (and suitably rescaling) the long and the short roots[8]. The correspondence between $\mathfrak{g}$ and $\mathfrak{g}^\vee$ is given in the following table

$\mathfrak{g}$	$\tilde{A}_r$	B_r	C_r	D_r	$E_{6,7,8}$	F_4	G_2
$\mathfrak{g}^\vee$	$\tilde{A}_r$	C_r	B_r	D_r	$E_{6,7,8}$	F_4'	G_2'

where for F_4 and G_2, the $'$ in the last two columuns means that the dual root systems are equivalent to the original ones up to a rotation.

[c]It is interesting to observe that for $n_{\mathfrak{g}}=3$, the matrices T and V^2 generate the subgroup $\Gamma_1(3)$, whose modular forms play a role in the $\mathcal{N}=2$ SYM theory with gauge group SU(3) and six fundamental hypermultiplets[7].

This duality structure remains and gets actually enriched when the $\mathcal{N} = 4$ SYM theories are deformed into the corresponding $\mathcal{N} = 2^\star$ ones. Here the S transformation (8) relates the electric variable a of the $\mathfrak{g}$ theory with the magnetic variable $a_{\rm D}$ of the dual $\mathfrak{g}^\vee$ theory

$$a_{\rm D} \equiv \frac{1}{2\pi i n_{\mathfrak{g}}}\frac{\partial F^{\mathfrak{g}^\vee}}{\partial a} = \tau\Big(a + \frac{1}{2\pi i n_{\mathfrak{g}}\tau}\frac{\partial f^{\mathfrak{g}^\vee}}{\partial a}\Big)\ , \tag{13}$$

according to

$$S\begin{pmatrix} a_{\rm D} \\ a \end{pmatrix} = \begin{pmatrix} 0 & -1/\sqrt{n_{\mathfrak{g}}} \\ \sqrt{n_{\mathfrak{g}}} & 0 \end{pmatrix}\begin{pmatrix} a_{\rm D} \\ a \end{pmatrix} = \begin{pmatrix} -a/\sqrt{n_{\mathfrak{g}}} \\ \sqrt{n_{\mathfrak{g}}}\, a_{\rm D} \end{pmatrix}\ . \tag{14}$$

In other words, the S transformation exchanges the description based on a with its Legendre-transformed one, based on $a_{\rm D}$:

$$S[F^{\mathfrak{g}}] = \mathcal{L}[F^{\mathfrak{g}^\vee}]\ , \tag{15}$$

where the Legendre transform is defined as

$$\mathcal{L}\big[F^{\mathfrak{g}^\vee}\big] \equiv F^{\mathfrak{g}^\vee} - a\cdot\frac{\partial F^{\mathfrak{g}^\vee}}{\partial a} = -n_{\mathfrak{g}}\pi i\tau a^2 - a\cdot\frac{\partial f^{\mathfrak{g}^\vee}}{\partial a} + f^{\mathfrak{g}^\vee}\ . \tag{16}$$

Thus, as is clear from (15), S-duality is not a symmetry of the effective theory since it changes the gauge algebra; nevertheless, as we shall see, it is powerful enough to constrain the form of the prepotential at the non-perturbative level.

3. The modular anomaly equation

If one uses eqs. (9), (13) and (14) to evaluate $S[F^{\mathfrak{g}}]$, the requirement (15) can be recast in the following form:

$$f^{\mathfrak{g}}\Big(-\tfrac{1}{n_{\mathfrak{g}}\tau}, \sqrt{n_{\mathfrak{g}}}a_D\Big) = \frac{1}{4\pi i n_{\mathfrak{g}}\tau}\Big(\frac{\partial f^{\mathfrak{g}^\vee}}{\partial a}\Big)^2 + f^{\mathfrak{g}^\vee} \tag{17}$$

where the r.h.s. is evaluated in τ and a.

Eq. (17) can be solved assuming that the coefficients f_n in the mass expansion (7) of the quantum prepotential depend on τ only through quasi-modular forms of $\Gamma_0(n_{\mathfrak{g}})$. The ring of these quasi-modular forms is generated by

$$\begin{aligned} &\{E_2, E_4, E_6\} && \text{for } n_{\mathfrak{g}} = 1\ ,\\ &\{E_2, H_2, E_4, E_6\} && \text{for } n_{\mathfrak{g}} = 2, 3\ , \end{aligned} \tag{18}$$

where $E_n(\tau)$ are the Eisenstein series while

$$H_2(\tau) = \left[\Big(\frac{\eta^{n_{\mathfrak{g}}}(\tau)}{\eta(n_{\mathfrak{g}}\tau)}\Big)^{\lambda_{\mathfrak{g}}} + \lambda_{\mathfrak{g}}^{n_{\mathfrak{g}}}\Big(\frac{\eta^{n_{\mathfrak{g}}}(n_{\mathfrak{g}}\tau)}{\eta(\tau)}\Big)^{\lambda_{\mathfrak{g}}}\right]^{1-\frac{1}{n_{\mathfrak{g}}}} \tag{19}$$

where η is the Dedekind η-function and $\lambda_{\mathfrak{g}} = n_{\mathfrak{g}}^{\overline{\frac{6}{n_{\mathfrak{g}}(n_{\mathfrak{g}}-1)}}}$. Thus, $\lambda_{\mathfrak{g}} = 8, 3$ for $n_{\mathfrak{g}} = 2, 3$ respectively. All these forms admit a Fourier expansion in terms of the instanton weight q, which starts as $1+O(q)$. This means that their perturbative part is just 1. Being able to express the prepotential in terms of quasi-modular forms entails resumming its istanton expansion.

The modular forms (18) transform in a simple way also under S; in fact

$$H_2\big(-\tfrac{1}{n_{\mathfrak{g}}\tau}\big) = -\big(\sqrt{n_{\mathfrak{g}}}\,\tau\big)^2 H_2\ , \tag{20a}$$

$$E_2\big(-\tfrac{1}{n_{\mathfrak{g}}\tau}\big) = \big(\sqrt{n_{\mathfrak{g}}}\,\tau\big)^2 \Big[E_2 + (n_{\mathfrak{g}}-1)H_2 + \delta\Big]\ , \tag{20b}$$

$$E_4\big(-\tfrac{1}{n_{\mathfrak{g}}\tau}\big) = \big(\sqrt{n_{\mathfrak{g}}}\,\tau\big)^4 \Big[E_4 + 5(n_{\mathfrak{g}}-1)H_2^2 + (n_{\mathfrak{g}}-1)(n_{\mathfrak{g}}-4)E_4\Big]\ , \tag{20c}$$

$$\begin{aligned} E_6\big(-\tfrac{1}{n_{\mathfrak{g}}\tau}\big) = \big(\sqrt{n_{\mathfrak{g}}}\,\tau\big)^6 \Big[&E_6 + \tfrac{7}{2}(n_{\mathfrak{g}}-1)(3n_{\mathfrak{g}}-4)H_2^3 \\ &- \tfrac{1}{2}(n_{\mathfrak{g}}-1)(n_{\mathfrak{g}}-2)(7E_4\,H_2 + 2E_6)\Big]\ , \end{aligned} \tag{20d}$$

where $\delta = \frac{6}{\pi \mathrm{i}\tau}$. Thus a quasi-modular form of $\Gamma_0(n_{\mathfrak{g}})$ with weight w is mapped under S to a form of the same weight with a prefactor $(\sqrt{n_{\mathfrak{g}}}\tau)^w$, up to the δ-shift introduced by E_2.

Suppose moreover that the coefficients $f_n^{\mathfrak{g}}$ enjoy the following property:

$$f_n^{\mathfrak{g}}\Big(-\tfrac{1}{n_{\mathfrak{g}}\tau}, a\Big) = \big(\sqrt{n_{\mathfrak{g}}}\,\tau\big)^{2n-2}\ f_n^{\mathfrak{g}^\vee}(\tau,a)\Big|_{E_2\to E_2+\delta}\ . \tag{21}$$

If we use this relation in the l.h.s. of eq. (17) and take into account eq. (13), upon formally expanding in δ we obtain

$$\frac{\partial f^{\mathfrak{g}^\vee}}{\partial E_2} + \frac{1}{24\,n_{\mathfrak{g}}}\frac{\partial f^{\mathfrak{g}^\vee}}{\partial a}\cdot\frac{\partial f^{\mathfrak{g}^\vee}}{\partial a} = 0\ ; \tag{22}$$

of course, since we considered a generic case, we could have equivalently written it in terms of $f^{\mathfrak{g}}$. This equation governs the appearance in the quantum prepotential of terms containing the second Eisenstein series E_2, which is the only source of a *quasi*-modular behaviour. Using the mass expansion (7), this "modular anomaly" equation becomes a recursion relation

$$\frac{\partial f_n^{\mathfrak{g}}}{\partial E_2} = -\frac{1}{24\,n_{\mathfrak{g}}}\sum_{\ell=1}^{n-1}\frac{\partial f_\ell^{\mathfrak{g}}}{\partial a}\cdot\frac{\partial f_{n-\ell}^{\mathfrak{g}}}{\partial a}\ . \tag{23}$$

3.1. *Exploiting the modular anomaly*

Starting from $f_1^{\mathfrak{g}}$, we can use the relation (23) to determine the parts of the $f_n^{\mathfrak{g}}$'s which explicitly contain E_2. The remaining terms of $f_n^{\mathfrak{g}}$ are strictly modular; we fix them by comparison with the result of the explicit computation of $f_n^{\mathfrak{g}}$ via localization techniques, when available, up to instanton order $(d_{2n-2}-1)$ where d_{2n-2} is the number of independent modular forms of weight $(2n-2)$. Once this is done, the resulting expression is valid at *all* istanton orders. We stress that the modular anomaly implements a symmetry requirement and does not eliminate the need of a dynamical input; yet it is extremely powerful as it greatly reduces it.

The mass expansion of the one-loop prepotential (6) reads

$$f^{\mathfrak{g}}_{1-\text{loop}} = \frac{m^2}{4} \sum_{\alpha\in\Psi_{\mathfrak{g}}} \log\left(\frac{\alpha\cdot a}{\Lambda}\right)^2 - \sum_{n=2}^{\infty} \frac{m^{2n}}{4n(n-1)(2n-1)} \left(L^{\mathfrak{g}}_{2n-2} + S^{\mathfrak{g}}_{2n-2}\right) \tag{24}$$

where we introduced the sums

$$\begin{aligned} L^{\mathfrak{g}}_{n;\, m_1\cdots m_\ell} &= \sum_{\alpha\in\Psi^{\mathrm{L}}_{\mathfrak{g}}} \sum_{\beta_1\neq\cdots\beta_\ell\in\Psi_{\mathfrak{g}}(\alpha)} \frac{1}{(\alpha\cdot a)^n(\beta_1\cdot a)^{m_1}\cdots(\beta_\ell\cdot a)^{m_\ell}} \ , \\ S^{\mathfrak{g}}_{n;\, m_1\cdots m_\ell} &= \sum_{\alpha\in\Psi^{\mathrm{S}}_{\mathfrak{g}}} \sum_{\beta_1\neq\cdots\beta_\ell\in\Psi^{\vee}_{\mathfrak{g}}(\alpha)} \frac{1}{(\alpha\cdot a)^n(\beta^{\vee}_1\cdot a)^{m_1}\cdots(\beta^{\vee}_\ell\cdot a)^{m_\ell}} \ , \end{aligned} \tag{25}$$

which are crucial in expressing the results of the recursion procedure. Here $\Psi^{\mathrm{L}}_{\mathfrak{g}}$ and $\Psi^{\mathrm{S}}_{\mathfrak{g}}$ denote, respectively, the sets of long and short roots of $\mathfrak{g}$, and for any root α we have defined

$$\begin{aligned} \Psi_{\mathfrak{g}}(\alpha) &= \{\beta\in\Psi_{\mathfrak{g}} \,:\, \alpha^{\vee}\cdot\beta = 1\} \ , \\ \Psi^{\vee}_{\mathfrak{g}}(\alpha) &= \{\beta\in\Psi_{\mathfrak{g}} \,:\, \alpha\cdot\beta^{\vee} = 1\} \end{aligned} \tag{26}$$

with $\alpha^{\vee}$ being the coroot of α. For the ADE algebras ($n_{\mathfrak{g}} = 1$) all roots are long and only the sums of type $L^{\mathfrak{g}}_{n;\, m_1\cdots m_\ell}$ exist. Thus, in all subsequent formulae the sums $S^{\mathfrak{g}}_{n;\, m_1\cdots m_\ell}$ are to be set to zero in these cases.

The initial condition for the recursion relation (23) is $f^{\mathfrak{g}}_1$. Since this receives contribution only at one-loop, it can be read from the term of order m^2 in eq. (24). Then, the first step of the recursion reads

$$\frac{\partial f^{\mathfrak{g}}_2}{\partial E_2} = -\frac{1}{24 n_{\mathfrak{g}}} \frac{\partial f^{\mathfrak{g}}_1}{\partial a}\cdot\frac{\partial f^{\mathfrak{g}}_1}{\partial a} = -\frac{m^4}{96 n_{\mathfrak{g}}} \sum_{\alpha,\beta\in\Psi_{\mathfrak{g}}} \frac{\alpha\cdot\beta}{(\alpha\cdot a)(\beta\cdot a)} = -\frac{m^4}{24}\left(L^{\mathfrak{g}}_2 + \frac{1}{n_{\mathfrak{g}}} S^{\mathfrak{g}}_2\right) \tag{27}$$

where the last equality follows from the properties of the root system $\Psi_{\mathfrak{g}}$.

For $n_{\mathfrak{g}} = 1$ there are no forms of weight 2 other than E_2 (see (18)), and thus $f^{\mathfrak{g}}_2$ only depends on E_2. For $n_{\mathfrak{g}} = 2, 3$, instead, $f^{\mathfrak{g}}_2$ may contain also the other modular form of degree 2 that exists in these cases, namely H_2 . The coefficient of H_2 in $f^{\mathfrak{g}}_2$ is fixed by matching the perturbative term with the m^4 term in eq. (24), namely $-\frac{m^4}{24}(L^{\mathfrak{g}}_2 + S^{\mathfrak{g}}_2)$. In this way we completely determine the expression of $f^{\mathfrak{g}}_2$. The process can be continued straightforwardly to higher orders in the mass expansion, though of course the structure gets rapidly more involved. In[1] we gave the results up to order m^{10} for the simply-laced algebras, and up to m^8 for the non simply-laced ones. Here, for the sake of brevity we only report the results up to order m^6, namely $f^{\mathfrak{g}}_2$ and $f^{\mathfrak{g}}_3$:

$$f^{\mathfrak{g}}_2 = -\frac{m^4}{24} E_2\, L^{\mathfrak{g}}_2 - \frac{m^4}{24 n_{\mathfrak{g}}}\Big[E_2 + (n_{\mathfrak{g}} - 1) H_2\Big]\, S^{\mathfrak{g}}_2 \ , \tag{28}$$

$$
\begin{aligned}
f_3^{\mathfrak{g}} = &-\frac{m^6}{720}\Big[5E_2^2+E_4\Big]\,L_4^{\mathfrak{g}}-\frac{m^4}{576}\Big[E_2^2-E_4\Big]\,L_{2;11}^{\mathfrak{g}}\\
&-\frac{m^6}{720n_{\mathfrak{g}}^2}\Big[5E_2^2+E_4+10(n_{\mathfrak{g}}-1)E_2H_2\\
&\qquad+5n_{\mathfrak{g}}(n_{\mathfrak{g}}-1)H_2^2+(n_{\mathfrak{g}}-1)(n_{\mathfrak{g}}-4)E_4\Big]\,S_4^{\mathfrak{g}}\\
&-\frac{m^6}{576n_{\mathfrak{g}}^2}\Big[E_2^2-E_4+2(n_{\mathfrak{g}}-1)E_2H_2\\
&\qquad+(n_{\mathfrak{g}}-1)(n_{\mathfrak{g}}-6)H_2^2-(n_{\mathfrak{g}}-1)(n_{\mathfrak{g}}-4)E_4\Big]\,S_{2;11}^{\mathfrak{g}}\ .
\end{aligned}
\tag{29}
$$

Consistency requires that the $f_n^{\mathfrak{g}}$'s obtained from the recursion procedure satisfy eq. (21). For the ADE algebras ($n_{\mathfrak{g}}=1$), using the modular properties of the Eisenstein series, it is not difficult to show that they do. On the other hand, for the non-simply laced algebras ($n_{\mathfrak{g}}=2,3$), using the properties of the root systems, one can prove that

$$
\begin{aligned}
L^{\mathfrak{g}}_{n;m_1\cdots m_\ell} &= \Big(\frac{1}{\sqrt{n_{\mathfrak{g}}}}\Big)^{n+m_1+\cdots+m_\ell}\,S^{\mathfrak{g}^\vee}_{n;m_1\cdots m_\ell}\ ,\\
S^{\mathfrak{g}}_{n;m_1\cdots m_\ell} &= \big(\sqrt{n_{\mathfrak{g}}}\big)^{n+m_1+\cdots+m_\ell}\,L^{\mathfrak{g}^\vee}_{n;m_1\cdots m_\ell}\ .
\end{aligned}
\tag{30}
$$

These duality relations, together with the modular transformations (20), ensure that the expressions in eq.s (28) and (29), as well as those arising at higher mass orders, indeed obey eq. (21).

3.2. *One-instanton contributions*

By considering the instanton expansion of the modular forms appearing in the expression of the $f_n^{\mathfrak{g}}$'s, one can see that at the one-instanton order, *i.e.* at order q, the only remaining terms involve the sums of type $L^{\mathfrak{g}}_{2;1\cdots1}$. In fact it can be argued from the recursion relation that this is the case at any order in the mass expansion. Thus, the one-instanton prepotential reads

$$
\begin{aligned}
F^{\mathfrak{g}}_{k=1} &= m^4\sum_{\ell\geq0}\frac{m^{2\ell}}{\ell!}\,L^{\mathfrak{g}}_{2;\underbrace{1\ldots1}_{\ell}}\\
&=\sum_{\alpha\in\Psi^{\mathrm{L}}_{\mathfrak{g}}}\frac{m^4}{(\alpha\cdot a)^2}\sum_{\ell\geq0}\frac{m^{2\ell}}{\ell!}\sum_{\beta_1\neq\cdots\neq\beta_\ell\in\Psi_{\mathfrak{g}}(\alpha)}\frac{1}{(\beta_1\cdot a)\cdots(\beta_\ell\cdot a)}\\
&=\sum_{\alpha\in\Psi^{\mathrm{L}}_{\mathfrak{g}}}\frac{m^4}{(\alpha\cdot a)^2}\prod_{\beta\in\Psi_{\mathfrak{g}}(\alpha)}\Big(1+\frac{m}{\beta\cdot a}\Big)
\end{aligned}
\tag{31}
$$

where the intermediate step follows from the definition (25) of the sums $L^{\mathfrak{g}}_{2;1\cdots1}$. The number of factors in the product above is given by the order of $\Psi_{\mathfrak{g}}(\alpha)$. When α is a long root, this is $(2h^\vee_{\mathfrak{g}}-4)$ where $h^\vee_{\mathfrak{g}}$ is the dual Coxeter number of $\mathfrak{g}$ (see the

Appendix). Thus, in (31) the highest power of the mass is $m^{2h^\vee_{\mathfrak{g}}}$. This is precisely the only term which survives in the decoupling limit

$$q \to 0 \quad \text{and} \quad m \to \infty \quad \text{with} \quad q\, m^{2h^\vee_{\mathfrak{g}}} \equiv \widehat{\Lambda}^{2h^\vee_{\mathfrak{g}}} \quad \text{fixed} \ , \tag{32}$$

in which the $\mathcal{N} = 2^\star$ theory reduces to the pure $\mathcal{N} = 2$ SYM theory. Indeed, $2h^\vee_{\mathfrak{g}}$ is the one-loop β-function coefficient for the latter. In this case the one-instanton prepotential is

$$q\, F_{k=1}\Big|_{\mathcal{N}=2} = \widehat{\Lambda}^{2h^\vee_{\mathfrak{g}}} \sum_{\alpha \in \Psi^{\mathrm{L}}_{\mathfrak{g}}} \frac{1}{(\alpha \cdot a)^2} \prod_{\beta \in \Psi_{\mathfrak{g}}(\alpha)} \frac{1}{\beta \cdot a} \ . \tag{33}$$

This expression perfectly coincides with the known results present in the literature (see for example[9] and in particular[10]), while (31) represents the generalization thereof to the $\mathcal{N} = 2^\star$ theories with any gauge algebra $\mathfrak{g}$.

4. Multi-instanton results from localization

For a classical algebra $\mathfrak{g} \in \{\tilde{A}_r, B_r, C_r, D_r\}$ one can efficiently apply the equivariant localization methods[3–6] to compute the instanton prepotential, order by order in the instanton number k. Even if straightforward in principle, these methods become computationally quite involved as k increases, and thus they are practical only for the first few values of k. Nonetheless the information obtained in this way is extremely useful since it provides a benchmark against which one can test the results predicted using the recursion relation and S-duality.

The essential ingredient is the instanton partition function

$$Z^{\mathfrak{g}}_k = \oint \prod_{i=1}^{K_{\mathfrak{g}}} \frac{d\chi_i}{2\pi \mathrm{i}}\ z^{\text{gauge}}_k\ z^{\text{matter}}_k \tag{34}$$

where $K_{\mathfrak{g}}$ is the number of integration variables given by

$$K_{\mathfrak{g}} = \begin{cases} k & \text{for } \mathfrak{g} = \tilde{A}_r, B_r, D_r \ , \\ \left[\frac{k}{2}\right] & \text{for } \mathfrak{g} = C_r \ , \end{cases} \tag{35}$$

while z^{gauge}_k and z^{matter}_k are, respectively, the contributions of the gauge vector multiplet and the matter hypermultiplet in the adjoint representation of $\mathfrak{g}$. These factors, which are different for the different algebras, depend on the vacuum expectation value a and on the deformation parameters $\epsilon_1, \cdots, \epsilon_4$, and are typically meromorphic functions of the integration variables χ_i. The integrals in (34) are computed by closing the contours in the upper-half complex χ_i-planes after giving the ϵ-parameters an imaginary part with the following prescription

$$\mathrm{Im}(\epsilon_4) \gg \mathrm{Im}(\epsilon_3) \gg \mathrm{Im}(\epsilon_2) \gg \mathrm{Im}(\epsilon_1) > 0 \ . \tag{36}$$

In this way all ambiguities are removed and we obtain the instanton partition function

$$Z^{\mathfrak{g}}_{\text{inst}} = 1 + \sum_{k \geq 1} q^k Z^{\mathfrak{g}}_k \ . \tag{37}$$

At the end of the calculations we have to set

$$\epsilon_3 = m - \frac{\epsilon_1+\epsilon_2}{2}\ , \quad \epsilon_4 = -m - \frac{\epsilon_1+\epsilon_2}{2} \tag{38}$$

in order to express the result in terms of the hypermultiplet mass m in the normalization of the previous sections. Finally, the non-perturbative prepotential of the $\mathcal{N}=2^\star$ SYM theory is given by

$$F^{\mathfrak{g}}_{\text{inst}} = \lim_{\epsilon_1,\epsilon_2\to 0}\Big(-\epsilon_1\epsilon_2\,\log Z^{\mathfrak{g}}_{\text{inst}}\Big) = \sum_{k\geq 1} q^k F^{\mathfrak{g}}_k\ . \tag{39}$$

We now provide the explicit expressions of z_k^{gauge} and z_k^{matter} for all classical algebras. The details on the derivation of these expressions can be found in[1,2] (see also, for example,[9] and[5]).

- **The unitary algebras $\tilde{A}_r$.** In this case the localization techniques yield

$$z_k^{\text{gauge}} = \frac{(-1)^k}{k!}\,\frac{(\epsilon_1+\epsilon_2)^k}{(\epsilon_1\epsilon_2)^k}\,\frac{\Delta(0)\,\Delta(\epsilon_1+\epsilon_2)}{\Delta(\epsilon_1)\,\Delta(\epsilon_2)}\prod_{i=1}^k \frac{1}{P\big(\chi_i+\frac{\epsilon_1+\epsilon_2}{2}\big)P\big(\chi_i-\frac{\epsilon_1+\epsilon_2}{2}\big)}\ , \tag{40a}$$

$$z_k^{\text{matter}} = \frac{(\epsilon_1+\epsilon_3)^k(\epsilon_1+\epsilon_4)^k}{(\epsilon_3\epsilon_4)^k}\,\frac{\Delta(\epsilon_1+\epsilon_3)\,\Delta(\epsilon_1+\epsilon_4)}{\Delta(\epsilon_3)\,\Delta(\epsilon_4)}\prod_{i=1}^k P\big(\chi_i+\tfrac{\epsilon_3-\epsilon_4}{2}\big)P\big(\chi_i-\tfrac{\epsilon_3-\epsilon_4}{2}\big) \tag{40b}$$

where

$$P(x) = \prod_{u=1}^{r+1}\big(x-a_u\big)\ , \qquad \Delta(x) = \prod_{i<j}^k\big(x^2-(\chi_i-\chi_j)^2\big)\ . \tag{41}$$

- **The orthogonal algebras B_r and D_r.** In these cases we find

$$z_k^{\text{gauge}} = \frac{(-1)^k}{2^k\,k!}\,\frac{(\epsilon_1+\epsilon_2)^k}{(\epsilon_1\epsilon_2)^k}\,\frac{\Delta(0)\,\Delta(\epsilon_1+\epsilon_2)}{\Delta(\epsilon_1)\,\Delta(\epsilon_2)}\prod_{i=1}^k \frac{4\chi_i^2\big(4\chi_i^2-(\epsilon_1+\epsilon_2)^2\big)}{P\big(\chi_i+\frac{\epsilon_1+\epsilon_2}{2}\big)\big(\chi_i-\frac{\epsilon_1+\epsilon_2}{2}\big)}\ , \tag{42a}$$

$$z_k^{\text{matter}} = \frac{(\epsilon_1+\epsilon_3)^k(\epsilon_1+\epsilon_4)^k}{(\epsilon_3\epsilon_4)^k}\,\frac{\Delta\big(\epsilon_1+\epsilon_3\big)\Delta\big(\epsilon_1+\epsilon_4\big)}{\Delta\big(\epsilon_3\big)\Delta\big(\epsilon_4\big)} \times \prod_{i=1}^k \frac{P\big(\chi_i+\frac{\epsilon_3-\epsilon_4}{2}\big)P\big(\chi_i-\frac{\epsilon_3-\epsilon_4}{2}\big)}{\big(4\chi_i^2-\epsilon_3^2\big)\big(4\chi_i^2-\epsilon_4^2\big)}\ , \tag{42b}$$

where

$$\begin{aligned} \Delta(x) &= \prod_{i<j}^k\big(x^2-(\chi_i-\chi_j)^2)\big)\big(x^2-(\chi_i+\chi_j)^2\big)\ , \\ P(x) &= x\prod_{u=1}^r\big(x^2-2a_u^2\big)\ \text{for}\ B_r\ , \quad P(x) = \prod_{u=1}^r\big(x^2-a_u^2\big)\ \text{for}\ D_r\ . \end{aligned} \tag{43}$$

- **The symplectic algebras C_r.** Finally, for the symplectic algebras we have

$$z_k^{\text{gauge}} = \frac{(-1)^k}{2^{k+\nu}\,k!}\,\frac{(\epsilon_1+\epsilon_2)^k}{(\epsilon_1\epsilon_2)^{k+\nu}}\,\frac{\Delta(0)\,\Delta(\epsilon_1+\epsilon_2)}{\Delta(\epsilon_1)\,\Delta(\epsilon_2)}\,\frac{1}{P\left(\frac{\epsilon_1+\epsilon_2}{2}\right)^{\nu}} \times \prod_{i=1}^{[\frac{k}{2}]}\frac{1}{P\big(\chi_i+\frac{\epsilon_1+\epsilon_2}{2}\big)P\big(\chi_i-\frac{\epsilon_1+\epsilon_2}{2}\big)(4\chi_i^2-\epsilon_1^2)\big(4\chi_i^2-\epsilon_2^2\big)}\,, \tag{44a}$$

$$z_k^{\text{matter}} = \frac{(\epsilon_1+\epsilon_3)^{k+\nu}(\epsilon_1+\epsilon_4)^{k+\nu}}{(\epsilon_3\epsilon_4)^k}\,\frac{\Delta\big(\epsilon_1+\epsilon_3\big)\Delta\big(\epsilon_1+\epsilon_4\big)}{\Delta\big(\epsilon_3\big)\Delta\big(\epsilon_4\big)}\,P\big(\tfrac{\epsilon_3-\epsilon_4}{2}\big)^{\nu} \times \prod_{i=1}^{[\frac{k}{2}]}P\big(\chi_i+\tfrac{\epsilon_3-\epsilon_4}{2}\big)P\big(\chi_i-\tfrac{\epsilon_3-\epsilon_4}{2}\big)\big(4\chi_i^2-(\epsilon_1+\epsilon_3)^2\big)\big(4\chi_i^2-(\epsilon_1+\epsilon_4)^2\big)\,, \tag{44b}$$

where $\nu = k - 2\big[\frac{k}{2}\big]$ and

$$\begin{aligned} P(x) &= \prod_{u=1}^{r}\big(x^2-a_u^2\big)\,, \\ \Delta(x) &= \prod_{i<j}^{[\frac{k}{2}]}\big(x^2-(\chi_i-\chi_j)^2)\big)\big(x^2-(\chi_i+\chi_j)^2\big)\prod_{i=1}^{[\frac{k}{2}]}\big(x^2-\chi_i^2\big)^{\nu}. \end{aligned} \tag{45}$$

Using these expressions we have computed the non-perturbative prepotential of the $\mathcal{N}=2^\star$ theories up to $k=5$ for the unitary and simplectic algebras, and up to $k=2$ for the orthogonal algebras. These explicit results, once rewritten in terms of the root lattice sums (25), are in perfect agreement with those obtained using the recursion relation presented in the previous section. This agreement provides a highly non-trivial consistency check on the entire construction.

5. Conclusions

We have shown that the S-duality of $\mathcal{N}=2^\star$ theories allows the recursive determination of the terms in the mass expansion of the prepotential in terms of (quasi-)modular forms of a suitable subgroup of the S-duality group; this yields expressions valid at all instanton numbers with very little input from microscopic computations. Our results agree with those obtained from localization techniques when $\mathfrak{g}$ is a classical algebra but, beeing based only on the formal properties of the root systems, they represent a solid prediction for the gauge theories based on exceptional groups, where no ADHM costruction of instantons and no localization methods are avaliable. The original papers[1] also discuss the recursion procedure in an Ω-background with generic ϵ parameters.

Appendix

Here we give our conventions for the root system of all algebras $\mathfrak{g}$ in terms of an orthonormal basis $\{\mathbf{e}_i\,;\,1\leq i\leq \mathfrak{r}\}$ in $\mathbb{R}^{\mathfrak{r}}$ where $\mathfrak{r}=\text{rank}(\mathfrak{g})$.

- $\tilde{A}_r$ The roots of $\tilde{A}_r$ are:

$$\left\{ \pm (\mathbf{e}_i - \mathbf{e}_j) \,;\, 1 \leq i < j \leq r+1 \right\} . \tag{46}$$

- B_r The long and short roots of B_r are, respectively:

$$\left\{\pm\sqrt{2}\,\mathbf{e}_i \pm \sqrt{2}\,\mathbf{e}_j \,;\, 1 \leq i < j \leq r\right\} \quad \text{and} \quad \left\{\pm\sqrt{2}\,\mathbf{e}_i \,;\, 1 \leq i \leq r\right\} . \tag{47}$$

- C_r The long and short roots of C_r are, respectively:

$$\{\pm 2\,\mathbf{e}_i \,;\, 1 \leq i \leq r\} \quad \text{and} \quad \{\pm\mathbf{e}_i \pm \mathbf{e}_j \,;\, 1 \leq i < j \leq r\} . \tag{48}$$

- D_r The roots of D_r are:

$$\left\{ \pm \mathbf{e}_i \pm \mathbf{e}_j \,;\, 1 \leq i < j \leq r \right\} , \tag{49}$$

- E_6 The roots of E_6 are:

$$\left\{ \pm \mathbf{e}_i \pm \mathbf{e}_j \,;\, 1 \leq i < j \leq 5 \right\} \cup \left\{ \pm \tfrac{1}{2}\,\mathbf{e}_1 \cdots \pm \tfrac{1}{2}\,\mathbf{e}_5 \pm \tfrac{\sqrt{3}}{2}\,\mathbf{e}_6 \right\} , \tag{50}$$

where the elements of the second set must have an even number of minus signs.

- E_7 The roots of E_7 are:

$$\left\{ \pm \mathbf{e}_i \pm \mathbf{e}_j \,;\, 1 \leq i < j \leq 6 \right\} \cup \left\{ \pm \sqrt{2}\,\mathbf{e}_7 \right\} \cup \left\{ \pm \tfrac{1}{2}\,\mathbf{e}_1 \cdots \pm \tfrac{1}{2}\,\mathbf{e}_6 \pm \tfrac{1}{\sqrt{2}}\,\mathbf{e}_7 \right\} , \tag{51}$$

where the elements of the third set must have an odd (even) number of minus signs in the $(\mathbf{e}_1, \cdots, \mathbf{e}_6)$ components if the $\mathbf{e}_7$ is positive (negative).

- E_8 The roots of E_8 are:

$$\left\{ \pm \mathbf{e}_i \pm \mathbf{e}_j \,;\, 1 \leq i < j \leq 8 \right\} \cup \left\{ \pm \tfrac{1}{2}\,\mathbf{e}_1 \cdots \pm \tfrac{1}{2}\,\mathbf{e}_8 \right\} , \tag{52}$$

where the element of the second set must have an even number of minus signs.

- F_4 The long roots of F_4 are:

$$\left\{\pm\sqrt{2}\,\mathbf{e}_i \pm \sqrt{2}\,\mathbf{e}_j \,;\, 1 \leq i < j \leq 4\right\} , \tag{53}$$

while the short roots are:

$$\left\{\pm\sqrt{2}\,\mathbf{e}_1, \pm\sqrt{2}\,\mathbf{e}_2, \pm\sqrt{2}\,\mathbf{e}_3, \pm\sqrt{2}\,\mathbf{e}_4,\ \pm\tfrac{1}{\sqrt{2}}\,\mathbf{e}_1 \pm \tfrac{1}{\sqrt{2}}\,\mathbf{e}_2 \pm \tfrac{1}{\sqrt{2}}\,\mathbf{e}_3 \pm \tfrac{1}{\sqrt{2}}\,\mathbf{e}_4\right\} . \tag{54}$$

- G_2 The long and short roots of G_2 are, respectively:

$$\left\{\pm\tfrac{3}{\sqrt{2}}\,\mathbf{e}_1 \pm \sqrt{\tfrac{3}{2}}\,\mathbf{e}_2 \,,\, \pm\sqrt{6}\,\mathbf{e}_2\right\} \quad \text{and} \quad \left\{\pm\sqrt{2}\,\mathbf{e}_1 \,,\, \pm\tfrac{1}{\sqrt{2}}\,\mathbf{e}_1 \pm \sqrt{\tfrac{3}{2}}\,\mathbf{e}_2\right\} . \tag{55}$$

Finally, in the following table we collect the main properties for the various algebras that are useful for the calculations presented in the main text:

$\mathfrak{g}$	dim	rank	$h^\vee$	$\mathrm{ord}(\Psi_{\mathfrak{g}}^{\mathrm{L}})$	$\mathrm{ord}(\Psi_{\mathfrak{g}}^{\mathrm{S}})$	$\mathrm{ord}(\Psi_{\mathfrak{g}}(\alpha_{\mathrm{L}}))$	$\mathrm{ord}(\Psi_{\mathfrak{g}}^{\vee}(\alpha_{\mathrm{S}}))$
A_r	$(r+1)^2$	$r+1$	$r+1$	$r(r+1)$	–	$2r-2$	–
B_r	$r(2r+1)$	r	$2r-1$	$2r(r-1)$	$2r$	$4r-6$	$2r-2$
C_r	$r(2r+1)$	r	$r+1$	$2r$	$2r(r-1)$	$2r-2$	$4r-6$
D_r	$r(2r-1)$	r	$2r-2$	$2r(r-1)$	–	$4r-8$	–
E_6	78	6	12	72	–	20	–
E_7	133	7	18	126	–	32	–
E_8	248	8	30	240	–	56	–
F_4	52	4	9	24	24	14	14
G_2	14	2	4	6	6	4	4

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