

The character of the supersymmetric Casimir energy

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ABSTRACT: We study the supersymmetric Casimir energy E_{susy} of $\mathcal{N} = 1$ field theories with an R-symmetry, defined on rigid supersymmetric backgrounds $S^1 \times M_3$, using a Hamiltonian formalism. These backgrounds admit an ambi-Hermitian geometry, and we show that the net contributions to E_{susy} arise from certain twisted holomorphic modes on $\mathbb{R} \times M_3$, with respect to both complex structures. The supersymmetric Casimir energy may then be identified as a limit of an index-character that counts these modes. In particular this explains a recent observation relating E_{susy} on $S^1 \times S^3$ to the anomaly polynomial. As further applications we compute E_{susy} for certain secondary Hopf surfaces, and discuss how the index-character may also be used to compute generalized supersymmetric indices.

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Contents

1	Introduction	2
2	Supersymmetric backgrounds	4
2.1	Background geometry	4
2.2	Hopf surfaces	7
2.2.1	Primary Hopf surfaces	7
2.2.2	Secondary Hopf surfaces	9
2.3	Flat connections	10
3	Supersymmetric Casimir energy	12
3.1	Path integral formulation	12
3.2	Hamiltonian formulation	13
3.3	Twisted variables	14
3.4	Unpaired modes on $\mathbb{R} \times M_3$	15
4	Primary Hopf surfaces	17
4.1	Solving for the unpaired modes	17
4.2	The character	19
4.3	Zeta function versus heat kernel regularization	20
4.4	Rewriting as a Dirac character	22
5	Secondary Hopf surfaces and generalizations	24
5.1	Lens spaces	24
5.2	Fixed point formula	25
5.3	More general M_3	26
5.3.1	Poincaré Hopf surface	27
5.3.2	Homogeneous hypersurface singularities	27
5.4	Full supersymmetric Casimir energy	29
6	Discussion	31
A	Supersymmetric index from the character	32
A.1	Primary Hopf surfaces	32
A.2	Secondary Hopf surfaces	34

1 Introduction

In recent years the technique of localization [1] has provided access to a host of exact results in supersymmetric field theories defined on certain curved backgrounds. This method can be used to compute a number of observables in strongly coupled field theories. These in general depend on the background geometry, leading to a richer structure than in flat space. In this paper we will consider the supersymmetric Casimir energy, introduced in [2] and further studied in [3–5]. We will focus on four-dimensional $\mathcal{N} = 1$ theories with an R-symmetry, defined on manifolds $S^1 \times M_3$, with M_3 a compact three-manifold. These arise as rigid supersymmetric backgrounds admitting two supercharges of opposite R-charge, which are ambi-Hermitian, with integrable complex structures I_{\pm} [6, 7]. Moreover, the backgrounds are equipped with a complex Killing vector field K of Hodge type $(0, 1)$ for both complex structures. Denoting this as $K = \frac{1}{2}(\xi - i\partial_{\tau})$, where $\tau \in [0, \beta)$ parametrizes $S^1 = S^1_{\beta}$, ξ is a nowhere zero vector on M_3 (the *Reeb* vector field), generating a transversely holomorphic foliation. When all orbits of ξ close, this means that M_3 is a Seifert fibred three-manifold, with ξ generating the fibration.

On such a background, one can consider the partition function of an $\mathcal{N} = 1$ theory with supersymmetric boundary conditions for the fermions. As is familiar from finite temperature field theory, this computes

$$Z_{S^1_{\beta} \times M_3} = \text{Tr} e^{-\beta H_{\text{susy}}}, \tag{1.1}$$

where the Hamiltonian H_{susy} generates time-translations along ∂_{τ} . Supposing this has a spectrum of energies $\{E_i\}_{i \in I}$, with $H_{\text{susy}} |i\rangle = E_i |i\rangle$, then the minimum energy is $E_0 \equiv E_{\text{susy}}$ where evidently

$$E_{\text{susy}} = - \lim_{\beta \rightarrow \infty} \frac{d}{d\beta} Z_{S^1_{\beta} \times M_3}. \tag{1.2}$$

Thus the supersymmetric Casimir energy is given by $E_{\text{susy}} = \langle 0 | H_{\text{susy}} | 0 \rangle$, where $|0\rangle$ is the vacuum state. Unlike the usual Casimir energy on $S^1 \times M_3$ (proportional to the integral of the energy-momentum tensor $T_{\tau\tau}$ over M_3), this has been argued to be a well-defined observable of the theory, i.e. it is scheme-independent, in any supersymmetric regularization [4].

We will be interested in computing $\langle H_{\text{susy}} \rangle = E_{\text{susy}}$ via canonical quantization. This approach was initiated in [3] for the conformally flat $S^1 \times S^3$ background, and further elaborated on in [4]. One can dimensionally reduce the one-loop operators on M_3 to obtain a supersymmetric quantum mechanics on \mathbb{R}_{τ} , where the $\beta \rightarrow \infty$ limit effectively decompactifies the circle S^1_{β} . Most of the modes of the one-loop operators are paired by supersymmetry, and these combine into long multiplets that do not contribute to $\langle H_{\text{susy}} \rangle$ in the supersymmetric quantum mechanics [4]. In this paper we will show that the unpaired modes are certain (twisted) holomorphic functions on $\mathbb{R} \times M_3$, where there is one set of modes for each of the two complex structures I_{\pm} . More precisely, here we will restrict attention to the contribution of the chiral multiplet. We expect that the vector multiplet contributions will

also arrange into short multiplets, and will similarly be related to (twisted) holomorphic functions. However, we will not perform this analysis in this paper.

When $\mathbb{R} \times M_3 \cong X \setminus \{o\}$ is the complement of an isolated singularity o in a Gorenstein canonical singularity X , one can elegantly solve for these unpaired modes that contribute to the supersymmetric Casimir energy. These include of course $M_3 = S^3$, as well as $M_3 = L(p, 1) = S^3/\mathbb{Z}_p$ (i.e. a Lens space), for which $X = \mathbb{C}^2$ and $X = \mathbb{C}^2/\mathbb{Z}_p$ is an A_{p-1} singularity, previously studied in the literature; but this construction also includes many other interesting three-manifolds. A large class may be constructed from homogeneous hypersurface singularities. Here X comes equipped with a \mathbb{C}^* action, which is generated by the complex vector field K , and $X \setminus \{o\}$ fibres over a compact orbifold Riemann surface Σ_2 . Then $X_+ \cong X_- \cong X$ are isomorphic as complex varieties, but the relative sign of the complex structures on fibre and base are opposite in the two complex structures I_\pm . We will show that the modes that contribute to the supersymmetric Casimir energy in a chiral matter multiplet take the form

$$\Phi_\pm = \left| \frac{\mathcal{P}_\pm}{\Omega_\pm} \right|^{\pm k_\pm/2} \mathcal{F}_\pm, \tag{1.3}$$

where \mathcal{P}_\pm are the globally defined nowhere zero $(2,0)$ -forms defined by the Hermitian structures for I_\pm , while Ω_\pm are the globally defined nowhere zero holomorphic $(2,0)$ -forms of definite Reeb weight under ξ , that exist because $X_+ \cong X_-$ is Gorenstein. Furthermore, k_\pm denote the R-charges of the relevant fields; in particular, $k_+ = r - 2$, $k_- = r$, where $r \in \mathbb{R}$ is the R-charge of the top component of a chiral multiplet. These correspond to fermionic (Φ_+) and bosonic (Φ_-) modes, respectively. The essential point in (1.3) is that \mathcal{F}_\pm are simply *holomorphic functions* on X_\pm . More precisely, in general the path integral (1.1) splits into different topological sectors, labelled by flat gauge connections, and for the trivial flat connection \mathcal{F}_\pm are holomorphic functions; more generally they are holomorphic sections of the associated flat holomorphic bundles. For example, for quotients of $M_3 \cong S^3$, such as the Lens spaces $L(p, 1) = S^3/\mathbb{Z}_p$, the relevant holomorphic modes may be obtained as a projection of the holomorphic functions on the covering space.

The supersymmetric Casimir energy is computed by “counting” these holomorphic functions according to their charge under the Reeb vector ξ . As such, E_{susy} is closely related to the index-character of [8]. In this reference, it was shown that the volume of a Sasakian manifold Y can be obtained from a certain limit of the equivariant index of the $\bar{\partial}$ operator on the associated Kähler cone singularity $X = C(Y)$. In a similar vein, here we will show that the supersymmetric Casimir energy is obtained from a limit of an index-character counting holomorphic functions on $\mathbb{R} \times M_3$. In the case of $M_3 \cong S^3$, this explains a conjecture/observation made in [5], where it was proposed that E_{susy} may be computed using the equivariant anomaly polynomial.

The rest of the paper is organized as follows. In section 2 we review and expand on the relevant background geometry, emphasizing the role of the ambi-Hermitian structure. In section 3, after recalling how the supersymmetric Casimir energy arises, we formulate the conditions for (un-)pairing of modes on $\mathbb{R} \times M_3$. In section 4 we discuss the index-character counting holomorphic functions, and make the connection with [5] in the case

of primary Hopf surfaces. Extensions to secondary Hopf surfaces, and more general M_3 realized as links of homogeneous hypersurface singularities, are discussed in section 5. We conclude in section 6. We have included an appendix A, where we discuss the relation of the index-character to the supersymmetric index [9] and its generalizations.

2 Supersymmetric backgrounds

2.1 Background geometry

We are interested in studying four-dimensional $\mathcal{N} = 1$ theories with an R-symmetry on $M_4 = S^1 \times M_3$, where M_3 is a compact three-manifold. In Euclidean signature, the relevant supersymmetry conditions are the two independent first-order differential equations

$$(\nabla_\mu \mp iA_\mu)\zeta_\pm + iV_\mu\zeta_\pm + iV^\nu(\sigma_\pm)_{\mu\nu}\zeta_\pm = 0, \tag{2.1}$$

where ζ_\pm are spinors of opposite chirality. Here we use the spinor conventions¹ of [7], in which ζ_\pm are two-component spinors with corresponding Clifford algebra generated by $(\sigma_\pm)^a = (\pm\vec{\sigma}, -i\mathbb{1}_2)$, where $a = 1, \dots, 4$ is an orthonormal frame index and $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ are the Pauli matrices. In particular the generators of $SU(2)_\pm \subset Spin(4) = SU(2)_+ \times SU(2)_-$ are

$$(\sigma_\pm)^{ab} = \frac{1}{4} \left(\sigma_\pm^a \sigma_\mp^b - \sigma_\pm^b \sigma_\mp^a \right). \tag{2.2}$$

The field V_μ is assumed to be a globally defined one-form obeying $\nabla^\mu V_\mu = 0$, and will not play a role in this paper. The field A_μ is associated to local R-symmetry transformations, with all matter fields being charged under this via appropriate covariant derivatives.

The Killing spinors ζ_\pm equip M_4 with two commuting integrable complex structures²

$$(I_\pm)^\mu{}_\nu \equiv -\frac{2i}{|\zeta_\pm|^2} \zeta_\pm^\dagger (\sigma_\pm)^\mu{}_\nu \zeta_\pm. \tag{2.3}$$

The metric g_{M_4} is Hermitian with respect to both I_\pm , but where the induced orientations are opposite, which means the geometry is by definition *ambi-Hermitian*. This structure also equips M_4 with a complex Killing vector field

$$K^\mu \equiv \zeta_+ \sigma_+^\mu \zeta_- . \tag{2.4}$$

This has Hodge type $(0, 1)$ for both complex structures, and satisfies $K^\mu K_\mu = 0$. We assume that K commutes with its complex conjugate K^* , $[K, K^*] = 0$.³ It then follows

¹Differently from previous literature, we denote the Killing spinors and associated complex structures with \pm subscripts. This emphasizes the fact that the two spinors and complex structures are on an equal footing.

²We adopt the same sign conventions as [10, 11] for the complex structures. Our main motivation for this choice of convention is that the modes that contribute to the supersymmetric Casimir energy will turn out to be (twisted) *holomorphic*, whereas if we reversed the signs of the complex structures they would be (twisted) *anti-holomorphic*.

³If $[K, K^*] \neq 0$ the metric is locally isometric to $\mathbb{R} \times S^3$ with the standard round metric on S^3 [7].

that we may write $K = \frac{1}{2}(\xi - i\partial_\tau)$, where $\tau \in [0, \beta)$ parametrizes $S^1 = S^1_\beta$ and ξ is a nowhere zero vector field on M_3 .

Following [2], we assume the metric on $M_4 = S^1 \times M_3$ to take the form

$$g_{M_4} = \Omega^2 (d\tau^2 + g_{M_3}) , \tag{2.5}$$

where the local form of the metric on M_3 may be written as

$$g_{M_3} = (d\psi + a)^2 + c^2 dzd\bar{z} . \tag{2.6}$$

Here $\xi = \partial_\psi$ generates a transversely holomorphic foliation of M_3 , with z a local transverse complex coordinate. Since ∂_τ and ∂_ψ are both Killing vectors the positive conformal factor is $\Omega = \Omega(z, \bar{z})$, while $c = c(z, \bar{z})$ is a locally defined non-negative function and $a = a_z(z, \bar{z})dz + \bar{a}_{\bar{z}}(z, \bar{z})d\bar{z}$ is a local real one-form. Notice that any Riemannian three-manifold admitting a unit length Killing vector $\xi = \partial_\psi$ may be put into the local form (2.6). Notice also that this geometry is precisely the rigid three-dimensional supersymmetric geometry of [12, 13], for which there are two three-dimensional supercharges of opposite R-charge.

We shall refer to $\xi = \partial_\psi$ as the *Reeb* vector field. Globally the foliation of M_3 that it induces splits into three types: regular, quasi-regular and irregular. In the first two cases all the leaves are closed, and hence ξ generates a U(1) isometry of M_3 . If this U(1) action is free, the foliation is said to be regular. In this case M_3 is the total space of a circle bundle over a compact Riemann surface Σ_2 , which can have arbitrary genus $g \geq 0$. The local metric $c^2 dzd\bar{z}$ then pushes down to a (arbitrary) Riemannian metric on Σ_2 , while the one-form a is a connection for the circle bundle over Σ_2 . More generally, in the quasi-regular case since ξ is nowhere zero the U(1) action on M_3 is necessarily locally free, and the base $\Sigma_2 \equiv M_3/U(1)$ is an orbifold Riemann surface. Topologically this is a Riemann surface of genus g , with some number \mathbb{M} of orbifold points which are locally modelled on $\mathbb{C}/\mathbb{Z}_{k_i}$, $k_i \in \mathbb{N}$, $i = 1, \dots, \mathbb{M}$. The induced metric on Σ_2 then has a conical deficit around each orbifold point, with total angle $2\pi/k_i$. The three-manifold M_3 is the total space of a circle orbibundle over Σ_2 . Such three-manifolds are called *Seifert fibred three-manifolds*, and they are classified.

In the irregular case ξ has at least one open orbit. Since the isometry group of a compact manifold is compact, this means that M_3 must have at least U(1) \times U(1) isometry, with ξ being an irrational linear combination of the two generating vector fields. Notice that M_3 is still a Seifert manifold, by taking a rational linear combination, and that the corresponding base Σ_2 inherits a U(1) isometry. There are then two cases: either this U(1) action is Hamiltonian, meaning there is an associated moment map, or else $\pi_1(\Sigma_2)$ is non-trivial. In the first case $\Sigma_2 \cong \mathbb{WCP}^2_{[p,q]}$ is necessarily a weighted projective space [14], while in the second case instead $\Sigma_2 \cong T^2$. In particular in the first case M_3 is either $S^1 \times S^2$, or it has finite fundamental group with simply-connected covering space S^3 .

In addition to the local complex coordinate z , we may also introduce

$$w \equiv \psi - i\tau + P(z, \bar{z}) , \tag{2.7}$$

where $P(z, \bar{z})$ is a local complex function. Taking this to solve

$$\partial_z \bar{P} = a_z, \tag{2.8}$$

where recall that a is the local one-form appearing in the metric (2.6), and defining

$$h = h(z, \bar{z}) \equiv -2i \partial_z \text{Im} P, \tag{2.9}$$

the metric (2.5) may be rewritten as

$$g_{M_4} = \Omega^2 [(dw + h dz)(d\bar{w} + \bar{h} d\bar{z}) + c^2 dz d\bar{z}]. \tag{2.10}$$

In these complex coordinates we have the complex vector fields

$$K = \partial_{\bar{w}}, \quad Y = \frac{s}{\Omega^2 c} (\partial_{\bar{z}} - \bar{h} \partial_{\bar{w}}). \tag{2.11}$$

Here s is a complex-valued function which appears in the Killing spinors ζ_{\pm} , where the vector Y , like K in (2.4), is defined as a spinor bilinear via

$$Y^\mu \equiv \frac{1}{2|\zeta_-|^2} \zeta_-^\dagger \sigma_-^\mu \zeta_+. \tag{2.12}$$

Following [10], we also define

$$\bar{K} \equiv \frac{1}{\Omega^2} \partial_w = \frac{1}{\Omega^2} K^*, \quad \bar{Y} \equiv \frac{1}{sc} (\partial_z - h \partial_w), \tag{2.13}$$

which again have natural expressions as bilinears. The dual one-forms to K and Y are

$$K^b = \Omega^2 (dw + h dz), \quad Y^b = sc dz. \tag{2.14}$$

These both have Hodge type $(1, 0)$ with respect to I_+ , showing that z and w are local holomorphic coordinates for this complex structure. In fact K^b, Y^b form a basis for $\Lambda_+^{1,0}$. On the other hand $K^b, (Y^b)^*$ form a basis for $\Lambda_-^{1,0}$. It follows that K generates a complex transversely holomorphic foliation of M_4 , where the transverse complex structure has opposite sign for I_{\pm} , while the complex structure of the leaves is the same for both I_{\pm} . In other words, z is a transverse holomorphic coordinate for I_+ , but it is \bar{z} that is a transverse holomorphic coordinate for I_- . In the quasi-regular and regular cases, this means that the induced complex structure on the (orbifold) Riemann surface $\Sigma_2 = M_3/U(1)$ has the opposite sign for I_{\pm} .

Finally, let us introduce the complex two-form bilinears

$$\mathcal{P}_{\pm} \equiv \frac{1}{2} \zeta_{\pm} (\sigma_{\pm})_{\mu\nu} \zeta_{\pm} dx^\mu \wedge dx^\nu. \tag{2.15}$$

These are nowhere zero sections of $\Lambda_{\pm}^{2,0} \otimes L_{\pm}^2$, where $L_{\pm} \cong (\Lambda_{\pm}^{2,0})^{-1/2}$ are spin^c line bundles for the Killing spinors ζ_{\pm} . We shall consider a class of geometries in which the background Abelian gauge field A_μ that couples to the R-symmetry is real. In this case we may write

$$\begin{aligned} \mathcal{P}_+ &= (\det g_{M_4})^{1/4} s (dw + h dz) \wedge dz = \Omega^3 c e^{-i\omega} dw \wedge dz, \\ \mathcal{P}_- &= (\det g_{M_4})^{1/4} \frac{\Omega^2}{s} (dw + h dz) \wedge d\bar{z} = \Omega^3 c e^{i\omega} (dw + h dz) \wedge d\bar{z}, \end{aligned} \tag{2.16}$$

where $(\det g_{M_4})^{1/4} = \Omega^2 c$ and $s = \Omega e^{-i\omega}$, with ω real [2]. Notice that the latter implies $\bar{Y} = Y^*$ in (2.13), where the star denotes complex conjugation. By definition

$$d\mathcal{P}_\pm = -i\mathcal{Q}_\pm \wedge \mathcal{P}_\pm, \tag{2.17}$$

where \mathcal{Q}_\pm are the associated Chern connections. We calculate

$$\mathcal{Q}_\pm = d_\pm^c \log(\Omega^3 c) \pm d\omega, \tag{2.18}$$

where $d_\pm^c \equiv i(\bar{\partial}^\pm - \partial^\pm)$. The background Abelian gauge field A_μ is then

$$A = -\frac{1}{2}\mathcal{Q}_+ = \frac{1}{2}\mathcal{Q}_-. \tag{2.19}$$

It follows that $\pm A$ is a connection on $\mathcal{K}_\pm^{-1/2}$, where $\mathcal{K}_\pm \equiv \Lambda_\pm^{2,0}$ is the canonical bundle for the I_\pm complex structure. Notice that dA in fact has Hodge type $(1, 1)$ for both I_\pm , and thus \mathcal{K}_\pm are both holomorphic line bundles (with respect to their relevant complex structures).

2.2 Hopf surfaces

In most of the paper we will focus on backgrounds $M_4 = S^1 \times M_3$, where the three-manifold M_3 has finite fundamental group. This means that the universal covering space of M_3 is a three-sphere S^3 , and moreover $M_3 \cong S^3/\Gamma$, where $\Gamma \subset \text{SO}(4)$.⁴ These so-called *spherical three-manifolds* are classified: Γ is either cyclic, or is a central extension of a dihedral, tetrahedral, octahedral, or icosahedral group. The cyclic case corresponds to Lens spaces $L(p, q)$, with fundamental group $\Gamma \cong \mathbb{Z}_p$. Another particularly interesting case is when Γ is the binary icosahedral group: here M_3 is the famous Poincaré homology sphere. Being a homology sphere means that Γ is a perfect group (equal to its commutator subgroup), and hence has trivial Abelianization. In fact $\pi_1(M_3) \cong \Gamma$ has order 120, while $H_1(M_3, \mathbb{Z})$ is trivial. Of course our three-manifold M_3 also comes equipped with extra structure, and $M_4 = S^1 \times M_3$ must be ambi-Hermitian with respect to I_\pm . As we shall see, one can realise such supersymmetric $S^1 \times M_3$ backgrounds as *Hopf surfaces*, at least for $\Gamma \subset \text{SU}(2) \subset \text{U}(2) \subset \text{SO}(4)$.

2.2.1 Primary Hopf surfaces

Let us first describe this structure in the case when $M_3 \cong S^3$. Here M_4 is by definition a *primary Hopf surface* — a compact complex surface obtained as a quotient of $\mathbb{C}^2 \setminus \{0\}$ by a free \mathbb{Z} action. These were studied in detail in [2], and in what follows we shall review and extend the analysis in this reference.

In the I_+ complex structure global complex coordinates (z_1^+, z_2^+) on the covering space $\mathbb{C}^2 \setminus \{0\}$ are expressed in terms of the local complex coordinates z, w defined in the previous subsection via

$$\begin{aligned} z_1^+ &= e^{|b_1|(iw-z)}, \\ z_2^+ &= e^{|b_2|(iw+z)}. \end{aligned} \tag{2.20}$$

⁴This is Thurston’s elliptization conjecture, now a theorem.

The Hopf surface $M_4 = S^1 \times S^3$ is the quotient of $\mathbb{C}^2 \setminus \{0\}$ by the \mathbb{Z} action generated by

$$(z_1^+, z_2^+) \rightarrow (\mathbf{p}_+ z_1^+, \mathbf{q}_+ z_2^+), \quad (2.21)$$

where the complex structure parameters are $\mathbf{p}_+ \equiv e^{\beta|b_1|}$, $\mathbf{q}_+ \equiv e^{\beta|b_2|}$.⁵ Notice that we may equivalently reverse the sign of the generator in (2.21), with $(z_1^+, z_2^+) \rightarrow (\mathbf{p}_- z_1^+, \mathbf{q}_- z_2^+)$ and $\mathbf{p}_- \equiv \mathbf{p}_+^{-1}$, $\mathbf{q}_- \equiv \mathbf{q}_+^{-1}$.

We may further express these complex coordinates in terms of four real coordinates $\varrho, \tau, \psi_1, \psi_2$ via

$$w = \frac{1}{2|b_1|}\psi_1 + \frac{1}{2|b_2|}\psi_2 - i\tau - iQ(\varrho), \quad z = u(\varrho) - i\left(\frac{1}{2|b_1|}\psi_1 - \frac{1}{2|b_2|}\psi_2\right), \quad (2.22)$$

where in the notation of section 2.1 we have that $Q = iP$ is real. We have introduced a polar coordinate $\varrho \in [0, 1]$ on S^3 , so that the real functions $u = u(\varrho)$, $Q = Q(\varrho)$; these obey equations that may be found in [2], although their precise form won't be relevant in what follows.⁶ We then have

$$\begin{aligned} z_1^+ &= e^{|b_1|\tau} e^{|b_1|(Q-u)} e^{i\psi_1}, \\ z_2^+ &= e^{|b_2|\tau} e^{|b_2|(Q+u)} e^{i\psi_2}, \end{aligned} \quad (2.23)$$

and the quotient by (2.21) simply sets $\tau \sim \tau + \beta$, with τ a coordinate on $S^1 = S^1_\beta$. In [2] a general class of metrics on $M_3 \cong S^3$ was studied, with $U(1) \times U(1)$ isometry. The latter has standard generators $\partial_{\psi_1}, \partial_{\psi_2}$, and the Reeb vector field is

$$\xi = \partial_\psi = |b_1|\partial_{\psi_1} + |b_2|\partial_{\psi_2}. \quad (2.24)$$

The complex structure I_- also equips $M_4 = S^1 \times S^3$ with the structure of a Hopf surface. Global complex coordinates on the covering space $\mathbb{C}^2 \setminus \{0\}$ are now

$$\begin{aligned} z_1^- &= e^{-|b_1|[i(w+2iQ)+\bar{z}]} = e^{-|b_1|\tau} e^{|b_1|(Q-u)} e^{-i\psi_1}, \\ z_2^- &= e^{-|b_2|[i(w+2iQ)-\bar{z}]} = e^{-|b_2|\tau} e^{|b_2|(Q+u)} e^{-i\psi_2}. \end{aligned} \quad (2.25)$$

In particular notice in these coordinates the complex structure parameters are $\mathbf{p}_- \equiv e^{-\beta|b_1|} = \mathbf{p}_+^{-1}$, $\mathbf{q}_- \equiv e^{-\beta|b_2|} = \mathbf{q}_+^{-1}$. Notice also that $w + 2iQ$ and \bar{z} are local complex coordinates for I_- , the former following from $dw + 2idQ = (dw + hdz) + 2i\partial_{\bar{z}}Qd\bar{z}$, both of which have Hodge type $(1, 0)$ with respect to I_- . The fact that (z_1^-, z_2^-) cover $\mathbb{C}^2 \setminus \{0\}$ follows from an analysis similar to that in [2] for the I_+ complex structure.

Another fact that we need from [2], that will be particularly important when we come to solve globally for the modes in section 4, is that

$$\omega = -\psi_1 - \psi_2. \quad (2.26)$$

⁵For a general primary Hopf surface these parameters may be complex.

⁶Compared to reference [2] we have defined $\psi_i = \text{sgn}(b_i)\varphi_i$, $i = 1, 2$, and recall from footnote 2 that we have also reversed the overall sign of the two complex structures I_\pm compared with that reference, meaning that $z_i^\pm|_{\text{here}} = z_i^\mp|_{\text{there}}$.

Recall here that $s = \Omega e^{-i\omega}$, which for example enters the Chern connections (2.18), and hence the background R-symmetry gauge field (2.19). This choice of phase in s is fixed uniquely by requiring that A is a *global one-form* on $M_3 \cong S^3$. The Killing spinors ζ_{\pm} are then globally defined as sections of trivial rank 2 bundles over $M_4 = S^1 \times S^3$. Gauge transformations $A \rightarrow A + d\lambda$ of course shift $\omega \rightarrow \omega - 2\lambda$.

2.2.2 Secondary Hopf surfaces

More generally, if M_3 has finite fundamental group Γ then $M_4 = S^1 \times M_3$ is a quotient of a primary Hopf surface by Γ . These are examples of *secondary Hopf surfaces*.

Let us first look at cyclic $\Gamma \cong \mathbb{Z}_p$. In order that the quotient by Γ preserves supersymmetry, in particular s must be invariant. In terms of either I_{\pm} complex structures, this means that $\Gamma \cong \mathbb{Z}_p \subset U(1) \subset SU(2)$, with $SU(2)$ acting on \mathbb{C}^2 in the standard two-dimensional representation **2**. The generator of this $U(1)$ subgroup of the isometry group $U(1) \times U(1)$ is the Killing vector $\chi = \partial_{\psi_1} - \partial_{\psi_2}$, and from (2.26) we see that $\mathcal{L}_{\chi}s = 0$. It follows that $M_4 = S^1 \times M_3$ is isomorphic to the secondary Hopf surface $(\mathbb{C}^2 \setminus \{0\})/(\mathbb{Z} \times \mathbb{Z}_p)$, in both complex structures. The three-manifold M_3 is the Lens space $L(p, 1)$ in this case. Notice that χ commutes with the Reeb vector field ξ , and hence $|b_1|, |b_2|$ (which determine the complex structure parameters $\mathbf{p}_{\pm}, \mathbf{q}_{\pm}$) can be arbitrary.

We may also realise supersymmetric backgrounds with non-Abelian fundamental groups. Here we may take $\Gamma \subset SU(2)$ to act on \mathbb{C}^2 in the representation **2**. In order for Γ to act isometrically we assume the isometry group to be enlarged to $U(2) \cong U(1) \times_{\mathbb{Z}_2} SU(2)$, with the Reeb vector embedded along $U(1)$. This means $|b_1| = |b_2|$. The metric on $M_3 \cong S^3$ is then that of a Berger sphere

$$ds_{M_3}^2 = d\theta^2 + \sin^2\theta d\varphi^2 + v^2(d\varsigma + \cos\theta d\varphi)^2, \tag{2.27}$$

where $v > 0$ is a squashing parameter, and $\varsigma = \psi_1 + \psi_2$, $\varphi = \psi_1 - \psi_2$. This special case of a Hopf surface background was studied in appendix C of [2], and has $b_1 = -b_2 = 1/2v$, and I_+ complex coordinates

$$z_1^+ = \sqrt{2} e^{\frac{\tau}{2v}} \cos \frac{\theta}{2} e^{i\psi_1}, \quad z_2^+ = \sqrt{2} e^{\frac{\tau}{2v}} \sin \frac{\theta}{2} e^{i\psi_2}. \tag{2.28}$$

In particular $|z_1^+|^2 + |z_2^+|^2 = 2e^{\tau/v}$ is invariant under $SU(2)$. The I_- complex coordinates are

$$(z_1^-, z_2^-) = e^{-\frac{\tau}{v}} (z_1^+, z_2^+)^*, \tag{2.29}$$

meaning that the $SU(2)$ group acts in the complex conjugate representation $\bar{\mathbf{2}}$ in the I_- complex structure. As is well known, $\mathbf{2} \cong \bar{\mathbf{2}}$, and thus again $M_4 = S^1 \times M_3$ is isomorphic to the secondary Hopf surface $(\mathbb{C}^2 \setminus \{0\})/(\mathbb{Z} \times \Gamma)$ in both complex structures. Of course finite subgroups $\Gamma \subset SU(2)$ have an ADE classification, where the A series are precisely the Abelian $\Gamma \cong \mathbb{Z}_p$ quotients of primary Hopf surfaces described at the beginning of this subsection, while the D and E groups are the dihedral series and tetrahedral E_6 , octahedral E_7 and icosahedral E_8 groups, respectively.

We may also describe the complex geometry of the associated Hopf surfaces algebraically. Consider the polynomials

$$\begin{aligned} f_{A_{p-1}} &= Z_1^p + Z_2^2 + Z_3^2, & f_{D_{p+1}} &= Z_1^p + Z_1 Z_2^2 + Z_3^2, \\ f_{E_6} &= Z_1^3 + Z_2^4 + Z_3^2, & f_{E_7} &= Z_1^3 + Z_1 Z_2^3 + Z_3^2, & f_{E_8} &= Z_1^3 + Z_2^5 + Z_3^2, \end{aligned} \tag{2.30}$$

on \mathbb{C}^3 with coordinates (Z_1, Z_2, Z_3) . The zero sets

$$X \equiv \{f(Z_1, Z_2, Z_3) = 0\} \subset \mathbb{C}^3, \tag{2.31}$$

have an isolated singularity at the origin o of \mathbb{C}^3 . These are all weighted homogeneous hypersurface singularities, meaning they inherit a \mathbb{C}^* action from the \mathbb{C}^* action $(Z_1, Z_2, Z_3) \rightarrow (q^{w_1} Z_1, q^{w_2} Z_2, q^{w_3} Z_3)$ on \mathbb{C}^3 , where $w_i \in \mathbb{N}$ are the weights, $i = 1, 2, 3$, and $q \in \mathbb{C}^*$. For example, $f_{A_{p-1}}$ has degree $d = 2p$ under the weights $(w_1, w_2, w_3) = (2, p, p)$, while f_{E_8} has degree $d = 30$ under the weights $(w_1, w_2, w_3) = (10, 6, 15)$. The smooth locus $X \setminus \{o\} \cong \mathbb{R} \times M_3$, where $M_3 = S^3/\Gamma_{ADE}$, while the quotients $(X \setminus \{o\})/\mathbb{Z}$ are precisely the ADE secondary Hopf surfaces described above. Here $\mathbb{Z} \subset \mathbb{C}^*$ is embedded as $n \rightarrow q^n$ for some fixed $q > 1$. The Reeb vector field action is quasi-regular, generated by $q \in U(1) \subset \mathbb{C}^*$. The quotient $\Sigma_2 = M_3/U(1)$ is in general an orbifold Riemann surface of genus $g = 0$.

2.3 Flat connections

The path integral of any four-dimensional $\mathcal{N} = 1$ theory with an R-symmetry on one of the supersymmetric backgrounds $S^1 \times M_3$ of section 2.1 localizes. In particular, the supercharges generated by ζ_{\pm} localize the vector multiplet onto instantons and anti-instantons, respectively [2], which intersect on the *flat connections*. In the Hamiltonian formalism for computing the supersymmetric Casimir energy, we will then need to study flat connections on the covering space $\mathbb{R} \times M_3$. The two spaces $S^1 \times M_3$ and $\mathbb{R} \times M_3$ have respectively τ periodic with period β , and $\tau \in \mathbb{R}$.

Recall that flat connections on M_4 with gauge group G are in one-to-one correspondence with

$$\text{Hom}(\pi_1(M_4) \rightarrow G)/\text{conjugation} . \tag{2.32}$$

In particular a flat G -connection is determined by its holonomies, which define a homomorphism $\varrho : \pi_1(M_4) \rightarrow G$, while gauge transformations act by conjugation. In the path integral on $M_4 = S^1 \times M_3$ we have $\pi_1(M_4) \cong \mathbb{Z} \times \pi_1(M_3)$, with $\pi_1(S^1) \cong \mathbb{Z}$. A flat connection is then the sum of pull-backs of flat connections on S^1 and M_3 , and we denote the former by \mathcal{A}_0 . On the other hand in the Hamiltonian formalism instead $M_4 = \mathbb{R} \times M_3$, so that $\pi_1(M_4) \cong \pi_1(M_3)$, and a flat connection on M_4 is simply the pull-back of a flat connection on M_3 .

When $\pi_1(M_3)$ is finite, which is the case for the primary and secondary Hopf surfaces in section 2.2, the number of inequivalent flat connections on M_3 is also finite. The path integral on $S^1 \times M_3$ correspondingly splits into a finite sum over these topological sectors,

together with a matrix integral over the holonomy of \mathcal{A}_0 . In the Hamiltonian formalism on $\mathbb{R} \times M_3$, instead for each flat connection on M_3 we will obtain a different supersymmetric quantum mechanics on \mathbb{R} .

A matter multiplet will be in some representation \mathcal{R} of the gauge group G . In the presence of a non-trivial flat connection on $M_4 = \mathbb{R} \times M_3$, this matter multiplet will be a section of the associated flat vector bundle, tensored with $\mathcal{K}_+^{-k/2}$ if the matter field has R-charge k . The latter follows since recall that the background R-symmetry gauge field A is a connection on $\mathcal{K}_+^{-1/2} \cong \mathcal{K}_+^{+1/2}$. For the Hopf surface cases of interest this will always be a trivial bundle, albeit with a generically non-flat connection, and we hence suppress this in the following discussion. Concretely then, composing $\varrho : \pi_1(M_4) \rightarrow G$ with the representation \mathcal{R} of G determines a corresponding flat connection in the representation \mathcal{R} , and the scalar field in the matter multiplet is a section of the vector bundle

$$\mathcal{V}_{\text{matter}} = (\widetilde{M}_4 \times V) / \pi_1 . \tag{2.33}$$

Here \widetilde{M}_4 is the universal cover of M_4 (which is $\mathbb{R} \times S^3$ for Hopf surfaces), $\pi_1 = \pi_1(M_4)$ is the fundamental group, $V \cong \mathbb{C}^M$ is the vector space associated to \mathcal{R} , and the action of π_1 on V is determined by the flat \mathcal{R} -connection described above. The scalar field in the matter multiplet is then a section of the bundle (2.33), which is a \mathbb{C}^M vector bundle over M_4 .

To illustrate, let us focus on the simplest non-trivial example, namely the Lens space $M_3 = S^3/\mathbb{Z}_p = L(p, 1)$. For a $G = \text{U}(1)$ gauge theory the flat connections on $\mathbb{R} \times M_3$ may be labelled by an integer $0 \leq m < p$, which determines the holonomy

$$\exp \left(i \int_{\gamma} \mathcal{A} \right) = e^{2\pi i m / p} . \tag{2.34}$$

Here \mathcal{A} is the dynamical $\text{U}(1)$ gauge field, while the circle γ generates $\pi_1(\mathbb{R} \times M_3) \cong \mathbb{Z}_p$. The associated homomorphism $\varrho : \mathbb{Z}_p \rightarrow \text{U}(1)$ is generated by $\varrho(\omega_p) = \omega_p^m$, where $\omega_p \equiv e^{2\pi i / p}$ is a primitive p th root of unity. For a $\text{U}(N)$ gauge theory the flat connections are similarly labelled by $0 \leq m_i < p$, where $i = 1, \dots, N$ runs over the generators of the Cartan $\text{U}(1)^N$ subgroup of $\text{U}(N)$. These are permuted by the Weyl group, so without loss of generality one may choose to order $m_1 \leq m_2 \leq \dots \leq m_N$, and label the flat $\text{U}(N)$ connection by a vector $\mathbf{m} = (m_1, \dots, m_N)$. Now $\varrho : \mathbb{Z}_p \rightarrow \text{U}(N)$ is generated by $\varrho(\omega_p) = \text{diag}(\omega_p^{m_1}, \dots, \omega_p^{m_N}) \in \text{U}(N)$.

An irreducible representation of $\text{U}(1)$ is labelled by the charge $\sigma \in \mathbb{Z}$, so $\mathcal{R} = \mathcal{R}_\sigma$. In the presence of the flat connection (2.34), a matter field in this representation becomes a section of the line bundle L over $\mathbb{R} \times L(p, 1)$ with first Chern class $c_1(L) \in H^2(\mathbb{R} \times L(p, 1), \mathbb{Z}) \cong \mathbb{Z}_p$ given by $c_1(L) \equiv \sigma m \pmod{p}$. Equivalently, on the universal covering space $\widetilde{M}_4 \cong \mathbb{R} \times S^3 \cong \mathbb{C}^2 \setminus \{0\}$, the relevant sections of $\mathcal{V}_{\text{matter}}$ may be identified with functions on \widetilde{M}_3 which pick up a phase $e^{2\pi i c_1(L)/p}$ under the generator of the \mathbb{Z}_p action. More generally, for a $\text{U}(N)$ gauge group we may decompose the representation $\mathcal{R} = \oplus_{\rho} V_{\rho}$ into weight spaces, with weights ρ . This then essentially reduces to the line bundle case above, with the part of the matter field in V_{ρ} now being a section of L with $c_1(L) \equiv \rho(\mathbf{m}) \pmod{p}$. For example, the fundamental representation of $\text{U}(N)$ has weights $\rho_i(\mathbf{m}) = m_i$, $i = 1, \dots, N$, the adjoint representation has weights $\rho_{ij} = m_i - m_j$, etc.

3 Supersymmetric Casimir energy

In this section we review the two approaches to define the supersymmetric Casimir energy E_{susy} , involving the path integral formulation on a compact manifold $S^1 \times M_3$, and the Hamiltonian formalism on its covering space $\mathbb{R} \times M_3$, respectively. We also present a geometric interpretation of the shortening conditions previously discussed in [4, 10].

3.1 Path integral formulation

On general grounds [11], the localized path integral of a four-dimensional $\mathcal{N} = 1$ theory with an R-symmetry on $M_4 = S^1 \times M_3$ is expected to depend on the background geometry only via the complex structure(s) of M_4 . For example, for the primary Hopf surfaces described in section 2.2.1 the complex structure parameters are $\mathbf{p}_\pm = e^{\pm\beta|b_1|}$, $\mathbf{q}_\pm = e^{\pm\beta|b_2|}$, which may equivalently be thought of as specified by the choice of Reeb vector field ξ in (2.24) (together with β). For a secondary Hopf surface $S^1 \times M_3$, the localized partition function also carries information about the finite fundamental group $\Gamma = \pi_1(M_3)$. Of course the partition function will also depend on the choice of $\mathcal{N} = 1$ theory, through the choice of gauge group, matter representation, and in particular on the R-charges of the matter fields.

In analogy with the usual zero point energy of a field theory, the supersymmetric Casimir energy was defined in [2] as a limit of the supersymmetric partition function $Z_{S^1_\beta \times M_3}^{\text{susy}}$, namely the path integral with periodic boundary conditions for the fermions along S^1_β . More precisely,

$$E_{\text{susy}} \equiv - \lim_{\beta \rightarrow \infty} \frac{d}{d\beta} \log Z_{S^1_\beta \times M_3}^{\text{susy}} . \tag{3.1}$$

This may be computed using localization. As already mentioned in section 2.3, the vector multiplet localizes onto flat connections for the gauge group G , while at least for primary Hopf surfaces the matter multiplet localizes to zero. The localized partition function comprises the contributions of one-loop determinants for the vector and chiral multiplets of the theory, evaluated around each such BPS locus, and one then integrates/sums over the space of flat connections. For primary Hopf surfaces ($M_3 \cong S^3$), the only non-trivial gauge field holonomy is for the flat connection \mathcal{A}_0 along S^1_β [2]. On the other hand, if $\pi_1(M_3)$ is non-trivial one should also sum or integrate over flat connections on M_3 , in the cases that $\pi_1(M_3)$ is finite, or infinite, respectively [15].

For primary Hopf surfaces the partition function factorises $Z_{S^1_\beta \times S^3}^{\text{susy}} = e^{-\beta E_{\text{susy}}(|b_1|, |b_2|)} \mathcal{I}$, where \mathcal{I} is a matrix integral over the gauge field holonomies on S^1_β , known as the supersymmetric index [9]. The latter does not contribute to the limit (3.1), and thus in order to compute E_{susy} one can effectively set the gauge field $\mathcal{A}_0 = 0$ in the one-loop determinants. The regularization of these determinants is rather delicate and it was proved in [4] that regularizations respecting supersymmetry give rise to a partition function with large and small β limits consistent with general principles [16]. See appendix C of [4].

For secondary Hopf surfaces the partition function is a sum of contributions over sectors with a fixed flat connection on M_3 . Let us label these sectors as $\alpha \in \mathcal{M}_{\text{flat}}$. Recall that in the special case that $M_3 = L(p, 1) = S^3/\mathbb{Z}_p$ is a Lens space and $G = \text{U}(N)$ we

may identify $\mathcal{M}_{\text{flat}}$ with the space of vectors $\mathbf{m} = (m_1, \dots, m_N)$, where $0 \leq m_i < p$ and $m_1 \leq m_2 \leq \dots \leq m_N$. Then from the definition (3.1) it is clear that the supersymmetric Casimir energy is given by

$$E_{\text{susy}} = \min_{\alpha \in \mathcal{M}_{\text{flat}}} \{E_{\text{susy}, \alpha}\}, \tag{3.2}$$

where for each α we have defined a ‘‘supersymmetric Casimir energy in the sector α ’’ as

$$E_{\text{susy}, \alpha} = - \lim_{\beta \rightarrow \infty} \frac{d}{d\beta} \log Z_{\alpha}. \tag{3.3}$$

In the Lens space case $M_3 = L(p, 1) = S^3/\mathbb{Z}_p$ the partition functions Z_{α} , which include the Casimir contributions $E_{\text{susy}, \alpha}$, have been computed in [15].

3.2 Hamiltonian formulation

Because the geometries of interest are of the form $M_4 = S^1 \times M_3$, with ∂_{τ} the Killing vector generating translations on S^1 , we can consider the theories on the covering space $M_4 = \mathbb{R} \times M_3$, employing the Hamiltonian formalism.⁷ These two approaches have been shown to yield equivalent results for both the supersymmetric Casimir energy, as well as the index \mathcal{I} , for primary Hopf surfaces, $M_3 \cong S^3$. It was argued in [9] that the supersymmetric index cannot depend on continuous couplings of the theory or the RG scale, and therefore may be computed in the free limit (assuming this exists). We return to discussing the supersymmetric index in appendix A. The supersymmetric Casimir energy can also be obtained as the vacuum expectation value of the supersymmetric (Weyl ordered) Hamiltonian H_{susy} , and again it can be reliably computed in a free theory [4]. This can be further Kaluza-Klein reduced on M_3 to give a supersymmetric quantum mechanics on \mathbb{R} , with an infinite number of fields, organised into multiplets of one-dimensional supersymmetry. Then $E_{\text{susy}} = \langle H_{\text{susy}} \rangle$, where H_{susy} is the total Hamiltonian for this supersymmetric quantum mechanics. If supersymmetric regularizations are employed, then this definition has been shown to agree with (3.1) in the primary Hopf surface case $M_3 \cong S^3$ [4].

This formalism can also be utilised when $\pi_1(M_3)$ is non-trivial (and finite), as we will see in more detail later in the paper. In this case there is a supersymmetric quantum mechanics for *each* flat connection on M_3 . This leads to a definition of ‘‘supersymmetric Casimir energy in the sector α ’’ that will depend on the flat connection $\alpha \in \mathcal{M}_{\text{flat}}$, thus

$$E_{\text{susy}, \alpha} = \langle H_{\text{susy}, \alpha} \rangle. \tag{3.4}$$

We will see that this quantum-mechanical definition of $E_{\text{susy}, \alpha}$ coincides with the path integral definition given previously, in any sector $\alpha = \mathbf{m}$, for Lens space secondary Hopf surfaces with $M_3 = L(p, 1) = S^3/\mathbb{Z}_p$. Of course, the actual supersymmetric Casimir energy of the theory will be given by the minimum $E_{\text{susy}, \alpha}$ among all flat connections.

⁷On $M_4 = \mathbb{R} \times M_3$ one usually works in Lorentzian signature. In this paper, however, we will always remain in Euclidean signature. One can then take the point of view that the Wick rotation ($t = i\tau$) to pass from Euclidean to Lorentzian signature can be done after the reduction to one dimension. In practice, we will never need to perform this last step.

In the simplest case, where $M_3 = S_{\text{round}}^3$, the Hamiltonian formalism can be used to obtain explicitly all of the modes and their eigenvalues [3, 4, 17]. Only a subset of *unpaired* modes contribute to E_{susy} [3]. These modes were shown in [4] to correspond to *short* 1d supersymmetry multiplets (chiral and Fermi multiplets). This feature extends to more general geometries, where the unpaired modes obey shortening conditions taking the form of linear first order differential equations [10].

3.3 Twisted variables

In what follows we will focus attention on a chiral multiplet. Using a set of “twisted variables” [10], the fermion of a chiral multiplet can be replaced by a pair of anticommuting complex scalar fields B and C . Thus such a multiplet comprises the four scalar fields (ϕ, B, C, F) , with R-charges $(r, r - 2, r, r - 2)$, respectively. There is also a set of tilded fields $(\tilde{\phi}, \tilde{B}, \tilde{C}, \tilde{F})$ with opposite sign R-charges, that are eventually simply related to the untilded fields by complex conjugation. The localizing deformation⁸ in these variables takes the simple form

$$\mathcal{L}_{\text{loc}} = 4\tilde{\phi}\Delta_{\text{bos}}\phi + 2\tilde{\Psi}\Delta_{\text{fer}}\Psi - \tilde{F}F, \tag{3.5}$$

where $\tilde{\Psi} = (\tilde{B}, \tilde{C})$, $\Psi = (B, C)^T$, and we have defined the operators

$$\Delta_{\text{bos}} \equiv -(\hat{\mathcal{L}}_{\bar{K}}\hat{\mathcal{L}}_K + \hat{\mathcal{L}}_Y\hat{\mathcal{L}}_{\bar{Y}}), \quad \Delta_{\text{fer}} \equiv i \begin{pmatrix} \hat{\mathcal{L}}_K & \hat{\mathcal{L}}_{\bar{Y}} \\ -\hat{\mathcal{L}}_Y & \hat{\mathcal{L}}_{\bar{K}} \end{pmatrix}, \tag{3.6}$$

with the first order operators

$$\hat{\mathcal{L}}_U = U^\mu(\partial_\mu - ikA_\mu - i\mathcal{A}_\mu). \tag{3.7}$$

Here U is one of the four complex vector fields K, \bar{K}, Y, \bar{Y} , defined in section 2.1, k is the R-charge of the field on which the operator is acting, and \mathcal{A}_μ denotes the localized flat gauge connection, acting on the field in the appropriate representation \mathcal{R} . As discussed in section 2.3, such matter fields may equivalently be identified with functions on the covering space that transform appropriately under the action of $\pi_1 = \pi_1(M_3)$ determined by the flat connection \mathcal{A}_μ . This action commutes with K and \bar{K} , as was necessary to preserve supersymmetry. We note the following relations

$$[\hat{\mathcal{L}}_K, \hat{\mathcal{L}}_{\bar{K}}] = 0, \quad [\hat{\mathcal{L}}_K, \hat{\mathcal{L}}_Y] = 0, \quad [\hat{\mathcal{L}}_K, \hat{\mathcal{L}}_{\bar{Y}}] = 0. \tag{3.8}$$

These were proven in [10] in a fixed (local) R-symmetry gauge where $s = s(z, \bar{z})$ (and without the flat connection), although it is obvious that they are valid in any gauge. In particular they are valid in the unique global non-singular gauge (2.26), relevant for Hopf surfaces.

The unpaired modes were shown in [10] to satisfy the shortening conditions

$$\begin{aligned} \hat{\mathcal{L}}_Y\check{B} &= 0, & i\hat{\mathcal{L}}_K\check{B} &= \lambda^B\check{B}, \\ \hat{\mathcal{L}}_{\bar{Y}}\check{\phi} &= 0, & i\hat{\mathcal{L}}_K\check{\phi} &= \lambda^\phi\check{\phi}, \end{aligned} \tag{3.9}$$

⁸This coincides with the standard chiral multiplet Lagrangian for a particular choice of the parameter κ .

where we have denoted the modes \check{B} , $\check{\phi}$, to distinguish them from the closely related modes to be introduced momentarily. It is worth emphasizing that these equations are valid both on $S^1 \times M_3$ and on $\mathbb{R} \times M_3$; however, the eigenvalues λ^B , λ^ϕ are different in the two cases. In particular, on $S^1 \times M_3$ one expands all fields in Kaluza-Klein modes over S^1 [2], thus $\check{\Phi}(\tau, x_i) = \check{\Phi}(x_i) e^{-in\tau}$, where $n \in \mathbb{Z}$ and x_i , $i = 1, 2, 3$ are coordinates on M_3 . Correspondingly we have $\lambda^\Phi = -\frac{i}{2}n + \lambda_\xi^\Phi$, where $\lambda_\xi^\Phi \in \mathbb{R}$ is the Reeb charge of the modes on M_3 :

$$i\hat{\mathcal{L}}_{\frac{1}{2}\xi}\check{\Phi}(x_i) = \lambda_\xi^\Phi\check{\Phi}(x_i). \tag{3.10}$$

On the other hand, using the equations (3.9) in the context of the Hamiltonian formalism on $\mathbb{R} \times M_3$ [4], one has effectively to set $n = 0$, and therefore in this case $\lambda^\Phi = \lambda_\xi^\Phi$.

In order to compute E_{susy} in principle one should consider the Hamiltonian canonically conjugate to (3.5), insert all modes obeying their (free) equations of motion, and then reduce the problem to one dimension [3]. Alternatively, one can focus on the unpaired modes, giving rise to short 1d multiplets, and determine their Σ -charge, for example by analysing the reduced supersymmetry transformations [4]. Here Σ is the Hermitian operator appearing in the one-dimensional supersymmetry algebra

$$\begin{aligned} \{Q, Q^\dagger\} &= 2(H_{\text{susy}} - \Sigma), & Q^2 &= 0, \\ [H_{\text{susy}}, Q] &= [\Sigma, Q] = 0. \end{aligned} \tag{3.11}$$

Then E_{susy} is determined using the fact that for every multiplet $\langle H_{\text{susy}} \rangle = \langle \Sigma \rangle$ [4].

3.4 Unpaired modes on $\mathbb{R} \times M_3$

In the path integral formalism, localization reduces the problem to computing the one-loop determinant associated to (3.5). Correspondingly, in the Hamiltonian formulation, we consider modes obeying the equations of motion following from (3.5), namely

$$\Delta_{\text{bos}}\phi = 0, \quad \Delta_{\text{fer}}\Psi = 0. \tag{3.12}$$

It is simple to show that modes satisfying the equations in (3.12) are paired by supersymmetry. Indeed, if ϕ is a bosonic zero mode, $\Delta_{\text{bos}}\phi = 0$, one can check using (3.8) that $\Psi = (\hat{\mathcal{L}}_{\bar{Y}}\phi, -\hat{\mathcal{L}}_K\phi)^T$ is a fermionic zero mode, so $\Delta_{\text{fer}}\Psi = 0$. Conversely, if $(B, C)^T$ is a fermionic zero mode, $\Delta_{\text{fer}}(B, C)^T = 0$, one can check that $\phi \equiv C$ is a bosonic zero mode, so $\Delta_{\text{bos}}\phi = 0$. Modes that are paired this way form long multiplets that do not contribute to the supersymmetric Casimir energy. Notice that a fermionic zero mode satisfies

$$\hat{\mathcal{L}}_{\bar{Y}}C = -\hat{\mathcal{L}}_KB, \quad \hat{\mathcal{L}}_{\bar{K}}C = \hat{\mathcal{L}}_YB. \tag{3.13}$$

The net contribution to E_{susy} comes from unpaired modes. These are bosonic/fermionic zero modes for which the putative fermionic/bosonic partner is identically zero. Thus these are fermionic $(B, 0)$ modes satisfying (using (3.13))

$$\hat{\mathcal{L}}_YB = 0 = \hat{\mathcal{L}}_KB, \tag{3.14}$$

and bosonic ϕ modes satisfying

$$\hat{\mathcal{L}}_{\bar{Y}}\phi = 0 = \hat{\mathcal{L}}_K\phi. \tag{3.15}$$

Recalling the definition (3.7) and using the preliminaries in section 2.1, one recognises the two operators $\hat{\mathcal{L}}_Y$ and $\hat{\mathcal{L}}_K$ as the components of the twisted $\bar{\partial}_{A,\mathcal{A}}^+$ differential. This denotes the $(0,1)_+$ part of $d - ikA - i\mathcal{A}$, where the twisting is determined by the R-symmetry connection A in (2.19) and flat connection \mathcal{A} . In particular, the unpaired B modes in (3.14) obey

$$\bar{\partial}_{A,\mathcal{A}}^+ B = 0, \tag{3.16}$$

and are therefore (twisted) holomorphic in the I_+ complex structure. Similarly, one can show the unpaired ϕ modes in (3.15) satisfy

$$\bar{\partial}_{A,\mathcal{A}}^- \phi = 0, \tag{3.17}$$

where $\bar{\partial}_{A,\mathcal{A}}^-$ denotes the $(0,1)_-$ part of $d - ikA - i\mathcal{A}$, and are therefore (twisted) holomorphic in the I_- complex structure. Notice that more precisely the unpaired B and ϕ modes are sections of $\mathcal{V}_{\text{matter}} \otimes \mathcal{K}_+^{-(r-2)/2}$ and $\mathcal{V}_{\text{matter}} \otimes \mathcal{K}_-^{r/2}$, respectively, where $\mathcal{V}_{\text{matter}}$ is the flat matter vector bundle (2.33).

It is simple to see that the above holomorphic modes may be decomposed into modes on M_3 which have definite charge under the (twisted) Reeb vector field $i\hat{\mathcal{L}}_{\frac{1}{2}\xi}$. In particular, writing a mode as

$$\Phi(\tau, x_i) = e^{-2\lambda_\xi^\Phi \tau} \check{\Phi}(x_i), \tag{3.18}$$

and using $i\hat{\mathcal{L}}_K = \mathcal{L}_{\frac{1}{2}\partial_\tau} + i\hat{\mathcal{L}}_{\frac{1}{2}\xi}$, one sees that

$$\hat{\mathcal{L}}_K \Phi(\tau, x_i) = 0 \iff i\hat{\mathcal{L}}_{\frac{1}{2}\xi} \check{\Phi}(x_i) = \lambda_\xi^\Phi \check{\Phi}(x_i). \tag{3.19}$$

This shows that the modes on $\mathbb{R} \times M_3$ defined by (3.9) were indeed independent of τ , and therefore defined on M_3 , as already remarked below equation (3.9). Thus we can think of the modes (3.18) as the “lifting to the cone” of the modes in the previous section. In fact setting $r = e^{-\tau}$ one sees that the metric on $\mathbb{R} \times M_3$ is conformally related to the metric on the cone $C(M_3)$: $g_{C(M_3)} = dr^2 + r^2 g_{M_3}$. Notice also that upon the Wick rotation $t = i\tau$, these become $\Phi(t, x_i) = e^{2i\lambda_\xi^\Phi t} \check{\Phi}(x_i)$, as expected for modes solving the free equations of motion on $\mathbb{R} \times M_3$ in Lorentzian signature [3]. These have to be contrasted with the modes on $S^1 \times M_3$ discussed earlier, namely $\check{\Phi}(\tau, x_i) = e^{-in\tau} \check{\Phi}(x_i)$.

Recall that the supersymmetry algebra acting on fields contains the anti commutation relation [7]

$$\{\delta_{\zeta_+}, \delta_{\zeta_-}\} = 2i\hat{\mathcal{L}}_K = 2\left(\mathcal{L}_{\frac{1}{2}\partial_\tau} + i\hat{\mathcal{L}}_{\frac{1}{2}\xi}\right), \tag{3.20}$$

where δ_{ζ_\pm} denote supersymmetry variations with respect to the ζ_\pm Killing spinors, respectively. Comparing this with the anti-commutator in (3.11), one can identify the eigenvalues of the quantum mechanical operators H_{susy} and Σ with those of the operators⁹

⁹After performing the Wick rotation $t = i\tau$ to go to Lorentzian signature.

$\mathcal{L}_{\frac{1}{2}\partial_\tau}$ and $-i\hat{\mathcal{L}}_{\frac{1}{2}\xi}$, acting on the classical modes, respectively. Therefore, the condition $\hat{\mathcal{L}}_K\Phi = 0$ obeyed by the holomorphic modes (on the cone) may be interpreted as showing that the Hamiltonian eigenvalues are equal to their Reeb charge, and is the counterpart of $\langle H_{\text{susy}} \rangle = \langle \Sigma \rangle$ in the supersymmetric quantum mechanics.

To summarise, the supersymmetric Casimir energy is computed by summing the Reeb charges of (twisted) holomorphic modes on $\mathbb{R} \times M_3$, with fermionic and bosonic modes corresponding to each complex structure I_\pm , respectively.

4 Primary Hopf surfaces

In this section we re-examine the supersymmetric Casimir energy for the primary Hopf surfaces $S^1 \times S^3$ in the above formalism. This was first defined and computed in the path integral approach in [2]. Since $M_3 \cong S^3$ there are no flat connections on M_3 .

4.1 Solving for the unpaired modes

Recall that the unpaired B and ϕ modes, that contribute to the supersymmetric Casimir energy, are zero modes on $\mathbb{C}^2 \setminus \{0\}$ of the twisted holomorphic differentials $\bar{\partial}_A^+$, $\bar{\partial}_A^-$, respectively, where the background R-symmetry gauge field A is given by (2.19) and the operators are understood to act on fields of R-charge k . The curvature of A has Hodge type $(1, 1)$ with respect to both I_\pm , and thus both differentials are nilpotent.

Using the global complex coordinates defined in section 2.2.1, it is straightforward to solve explicitly for these zero modes. In what follows we assume that we are working in a weight space decomposition of the matter representation \mathcal{R} , so that for a fixed weight ρ we have $B = B_\rho$ is a single scalar field. For the unpaired B modes we first note from (2.19) that the $(0, 1)_+$ part of A is

$$A_{(0,1)_+} = -\frac{i}{2}\bar{\partial}^+ \log(\Omega^3 c) - \frac{1}{2}\bar{\partial}^+ \omega . \tag{4.1}$$

The equation $\bar{\partial}_A^+ B = 0$ may thus be rewritten as

$$\bar{\partial}^+ \left[(\Omega^3 c)^{-k/2} |z_1^+ z_2^+|^{k/2} B \right] = 0 . \tag{4.2}$$

In particular notice that we have used

$$(z_1^+ z_2^+)^{k/2} = |z_1^+ z_2^+|^{k/2} e^{-i(k/2)\omega} , \tag{4.3}$$

where $\omega = -\psi_1 - \psi_2$. Recall that Ω is globally a nowhere zero function, while near the complex axes (i.e. $z_1^+ = 0$ and $z_2^+ = 0$) the real function c behaves to leading order as $|c| \sim |z_1^+|$, $|c| \sim |z_2^+|$, respectively. This is required for regularity of the metric [2]. It follows that the factor in front of B inside the square bracket in (4.2) is a real nowhere zero function on $\mathbb{C}^2 \setminus \{0\}$. A basis of regular solutions is hence

$$B = B_{n_1, n_2} \equiv \left(\frac{\Omega^3 c}{|z_1^+ z_2^+|} \right)^{k/2} (z_1^+)^{n_1} (z_2^+)^{n_2} , \tag{4.4}$$

where $n_1, n_2 \in \mathbb{Z}_{\geq 0}$.

One may similarly solve for the unpaired ϕ zero modes. Since from (2.19) we now have

$$A_{(0,1)-} = \frac{i}{2} \bar{\partial}^- \log(\Omega^3 c) - \frac{1}{2} \bar{\partial}^- \omega, \tag{4.5}$$

and one obtains a basis of regular solutions given by

$$\phi = \phi_{n_1, n_2} \equiv \left(\frac{\Omega^3 c}{|z_1^- z_2^-|} \right)^{-k/2} (z_1^-)^{n_1} (z_2^-)^{n_2}. \tag{4.6}$$

The prefactors in front of the holomorphic monomials in the modes (4.4), (4.6) also have a simple geometric interpretation. Recall that the Hermitian structure (g_{M_4}, I_+) equips $\mathbb{C}^2 \setminus \{0\}$ with the $(2, 0)_+$ -form

$$\mathcal{P}_+ \equiv \frac{1}{2} \zeta_+ (\sigma_+)_{\mu\nu} \zeta_+ dx^\mu \wedge dx^\nu = \Omega^3 c e^{-i\omega} dw \wedge dz. \tag{4.7}$$

On the other hand, \mathbb{C}^2 has the global holomorphic $(2, 0)_+$ -form¹⁰

$$\Omega_+ \equiv \frac{1}{2|b_1||b_2|} dz_1^+ \wedge dz_2^+ = iz_1^+ z_2^+ dw \wedge dz, \tag{4.8}$$

where we have used (2.20). Then

$$\frac{\Omega^3 c}{|z_1^+ z_2^+|} = \left| \frac{\mathcal{P}_+}{\Omega_+} \right|, \tag{4.9}$$

is simply the modulus of the ratio of these two canonically defined $(2, 0)_+$ -forms. A similar computation shows that

$$\frac{\Omega^3 c}{|z_1^- z_2^-|} = \left| \frac{\mathcal{P}_-}{\Omega_-} \right|, \tag{4.10}$$

where we define

$$\Omega_- \equiv \frac{1}{2|b_1||b_2|} dz_1^- \wedge dz_2^- = iz_1^- z_2^- (dw + hdz) \wedge d\bar{z}. \tag{4.11}$$

To summarize: the unpaired B modes are $|\mathcal{P}_+/\Omega_+|^{k/2}$ times a holomorphic function on \mathbb{C}^2 with respect to the I_+ complex structure, while the unpaired ϕ modes are $|\mathcal{P}_-/\Omega_-|^{-k/2}$ times a holomorphic function on \mathbb{C}^2 with respect to the I_- complex structure. Here $k = r - 2$ for B , while $k = r$ for ϕ , where r is the R-charge of the matter multiplet.

As discussed in section 3.4, the contributions of these modes to the supersymmetric Casimir energy is determined by their eigenvalues under $i\hat{\mathcal{L}}_{\frac{1}{2}\xi}$, where recall that acting on scalars

$$i\hat{\mathcal{L}}_{\frac{1}{2}\xi} = \frac{i}{2} \mathcal{L}_\xi + \frac{k}{2} \gamma, \tag{4.12}$$

where

$$\gamma \equiv -\mathcal{L}_{\frac{1}{2}\xi} \omega = \frac{1}{2} (|b_1| + |b_2|). \tag{4.13}$$

¹⁰This is not to be confused with the conformal factor Ω , especially in the following formulae.

The eigenvalues are then easily computed:

$$\begin{aligned} i\hat{\mathcal{L}}_{\frac{1}{2}\xi} B_{n_1, n_2} &= \frac{1}{2} [-n_1|b_1| - n_2|b_2| + k\gamma] B_{n_1, n_2} = \lambda_{n_1, n_2}^B B_{n_1, n_2}, \\ i\hat{\mathcal{L}}_{\frac{1}{2}\xi} \phi_{n_1, n_2} &= \frac{1}{2} [n_1|b_1| + n_2|b_2| + k\gamma] \phi_{n_1, n_2} = \lambda_{n_1, n_2}^\phi \phi_{n_1, n_2}, \end{aligned} \quad (4.14)$$

where we have used that the Reeb vector is given by (2.24), and hence

$$\mathcal{L}_\xi z_i^\pm = \pm i|b_i|z_i^\pm, \quad i = 1, 2. \quad (4.15)$$

We may now further reinterpret the eigenvalues λ^B, λ^ϕ , using our earlier description of the holomorphic volume forms Ω_\pm . Recall that A is a connection on $\mathcal{K}_+^{-1/2}$, so that $\bar{\partial}_A^+$ acts on sections of $\mathcal{K}_+^{-k/2}$. In the case at hand $\mathcal{K}_+ \cong \Lambda_+^{2,0}$ is a trivial bundle over $\mathbb{C}^2 \setminus \{0\}$, but the holomorphic section Ω_+ of \mathcal{K}_+ leads to a canonical lifting of the $U(1) \times U(1)$ action, with generators $\partial_{\psi_1}, \partial_{\psi_2}$, to the fibre. Specifically, since Ω_+ satisfies $\mathcal{L}_{\partial_{\psi_i}} \Omega_+ = i\Omega_+, i = 1, 2$, the generators $(q_1, q_2) \in U(1) \times U(1)$ acting on $\mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}$ as $(z_1^+, z_2^+) \rightarrow (q_1 z_1^+, q_2 z_2^+)$ act as multiplication by $q_1 q_2$ on Ω_+ . With this understanding, the eigenvalue λ^B is the eigenvalue of the *ordinary* Lie derivative $i\mathcal{L}_{\frac{1}{2}\xi}$ acting on holomorphic sections of $\mathcal{K}_+^{-k/2}$. Here the action on the fibre contributes precisely $-\frac{k}{2}(-\gamma) = \frac{k}{2}\gamma$ to $i\mathcal{L}_{\frac{1}{2}\xi}$, since $i\mathcal{L}_{\frac{1}{2}\xi} \Omega_+ = -\gamma \Omega_+$.

A similar reasoning applies to the ϕ modes. Here $-A$ is a connection on $\mathcal{K}_-^{-1/2}$, so that $\bar{\partial}_A^-$ acts on sections of $\mathcal{K}_-^{k/2}$. Again the canonical bundle is trivial, but the action of $(q_1, q_2) \in U(1) \times U(1)$ above on the fibre is now $(q_1 q_2)^{-1}$. This follows from the relative minus signs in the phases in (2.23), (2.25). The action on the fibre then again contributes precisely $\frac{k}{2}\gamma$ to $i\mathcal{L}_{\frac{1}{2}\xi}$, since now $i\mathcal{L}_{\frac{1}{2}\xi} \Omega_- = \gamma \Omega_-$.

Notice that with these definitions $\mathcal{K}_+ \cong (\mathcal{K}_-)^{-1}$ as equivariant holomorphic line bundles under $U(1) \times U(1)$.

4.2 The character

The supersymmetric Casimir energy is (formally, before regularization)

$$E_{\text{susy}}^{\text{matter}} = \sum_{n_1, n_2 \in \mathbb{Z}_{\geq 0}} \lambda_{n_1, n_2}^B + \sum_{n_1, n_2 \in \mathbb{Z}_{\geq 0}} \lambda_{n_1, n_2}^\phi, \quad (4.16)$$

where the eigenvalues are those on the right hand side of (4.14). Here we have introduced the superscript ‘‘matter’’ to emphasize that in what follows we focus on the contribution of a single weight ρ in a weight space decomposition of the chiral matter representation \mathcal{R} . We have seen that the eigenvalues in (4.16) are precisely Reeb charges, under $i\mathcal{L}_{\frac{1}{2}\xi}$, of holomorphic sections of $\mathcal{K}_+^{-k/2}$ and $\mathcal{K}_-^{k/2}$, respectively, where $k = r - 2$ for the B modes and $k = r$ for the ϕ modes. Thus it is natural at this point to introduce the index-character of [8] that counts such holomorphic sections according to their $U(1) \times U(1)$ charges. We take the $U(1) \times U(1)$ generators to be (q_1, q_2) , which act as

$$(z_1^\pm, z_2^\pm) \rightarrow (q_1^{\pm 1} z_1^\pm, q_2^{\pm 1} z_2^\pm). \quad (4.17)$$

For the B modes we have the associated index-character

$$C(\bar{\partial}_{\mathcal{K}_+^{-k/2}}, (q_1, q_2)) = \sum_{n_1, n_2 \geq 0} (q_1 q_2)^{-k/2} \cdot q_1^{n_1} q_2^{n_2}. \quad (4.18)$$

The left hand side is defined as the trace of the action of (q_1, q_2) on the zero modes of the operator $\bar{\partial}_{\mathcal{K}_+^{-k/2}}$. The right hand side of (4.18) is a divergent series for $|q_1| = |q_2| = 1$, but by analytically continuing to $|q_1|, |q_2| < 1$ the series converges to give

$$C(\bar{\partial}_{\mathcal{K}_+^{-k/2}}, (q_1, q_2)) = \frac{(q_1 q_2)^{-k/2}}{(1 - q_1)(1 - q_2)}. \quad (4.19)$$

This then effectively regularizes the eigenvalue sum. Indeed, setting $q_1 = e^{t|b_1|}$, $q_2 = e^{t|b_2|}$ and formally expanding (4.18) in a Taylor series around $t = 0$, the coefficient of $-t$ is precisely $2\lambda_{n_1, n_2}^B = -n_1|b_1| - n_2|b_2| + k\gamma$. Recalling that the B modes have $k = r - 2$, we hence see that according to this ‘‘character regularization’’ their contribution to the supersymmetric Casimir energy is

$$\begin{aligned} E_{\text{susy}}^B &= \frac{1}{2} \left[\frac{(q_1 q_2)^{-(r-2)/2}}{(1 - q_1)(1 - q_2)} \Big|_{q_1 = e^{t|b_1|}, q_2 = e^{t|b_2|}} \right]_{\text{coefficient of } -t} \\ &= \frac{1}{96|b_1||b_2|} (|b_1| + |b_2|)(r - 1) [(|b_1| + |b_2|)^2 (r - 1)^2 - (b_1^2 + b_2^2)] \\ &= \frac{1}{2} \cdot \frac{4u^3 - (b_1^2 + b_2^2)u}{24|b_1||b_2|} \Big|_{u = (r-1)\gamma}, \end{aligned} \quad (4.20)$$

where the second equality is by a simple direct computation. This is indeed the correct contribution of the unpaired B modes to the supersymmetric Casimir energy!

The ϕ modes work similarly. The relevant character is now

$$C(\bar{\partial}_{\mathcal{K}_-^{k/2}}, (q_1, q_2)) = \sum_{n_1, n_2 \geq 0} (q_1 q_2)^{-k/2} \cdot q_1^{-n_1} q_2^{-n_2}. \quad (4.21)$$

Summing the series for $|q_1|, |q_2| > 1$ we obtain

$$C(\bar{\partial}_{\mathcal{K}_-^{k/2}}, (q_1, q_2)) = \frac{(q_1 q_2)^{-k/2}}{(1 - q_1^{-1})(1 - q_2^{-1})} = \frac{(q_1 q_2)^{-(k-2)/2}}{(1 - q_1)(1 - q_2)}. \quad (4.22)$$

Recalling that ϕ has R-charge r , we see that their contribution is also precisely the right hand side of the first line of (4.20). Thus they contribute equally to the supersymmetric Casimir energy, as expected, $E_{\text{susy}}^\phi = E_{\text{susy}}^B$.

4.3 Zeta function versus heat kernel regularization

At first sight the result just obtained is somewhat remarkable, because we regularized the eigenvalue sum (4.16) using the index-character (via analytic continuation to a simple geometric series), while in previous work the sum in (4.16) is regularized using the Barnes double zeta function. The two regularization schemes lead to the same result.

This may be explained as follows. In order to regularize each sum in (4.16) in a supersymmetric fashion one should replace¹¹

$$\sum_n \lambda_n \rightarrow \sum_n \lambda_n f(\lambda_n, t), \tag{4.23}$$

with $f(x, t)$ a function chosen so that the sum converges. Requiring that $f(x, 0) = 1$, the value of the regularized sum is given by the *finite part* in the limit that the parameter $t \rightarrow 0$. Indeed, supersymmetric counterterms exist that may be added to remove divergences appearing as poles in t^{-2} and t^{-1} . However, the fact that finite supersymmetric counterterms do not exist [18] implies that the finite part is unambiguous, and therefore independent of the details of the regularization. There are two natural choices. Picking $f(\lambda_n, t) = \lambda_n^{-t}$ leads to the spectral zeta function regularization, while the choice $f(\lambda_n, t) = e^{-t\lambda_n}$ leads to the heat kernel regularization, which as we shall see is the “character regularization” we have used above. It is well known that these two are related to each other via the Mellin transform.

In the case of interest the sums in (4.16) were regularized in [4] using the Barnes double zeta function, defined as

$$\zeta_2(t; |b_1|, |b_2|, x) \equiv \sum_{n_1, n_2 \in \mathbb{Z}_{\geq 0}} (|b_1|n_1 + |b_2|n_2 + x)^{-t}, \tag{4.24}$$

where $x = r\gamma$ for the physical case of interest. Here we have focused on the ϕ modes. The sum in (4.24) converges for $\text{Re } t > 1$ and one analytically continues to $t = -1$ obtaining [19]

$$E_{\text{susy}}^{\text{matter}} = \frac{u^3}{6|b_1||b_2|} - \frac{(b_1^2 + b_2^2)u}{24|b_1||b_2|}, \tag{4.25}$$

where we have defined $u = (r - 1)\gamma = x - \gamma$. Note that

$$\frac{1}{2}\zeta_2(-1; |b_1|, |b_2|, u + \gamma) = -\frac{1}{2}\zeta_2(-1; |b_1|, |b_2|, -u + \gamma), \tag{4.26}$$

so that the contributions to $E_{\text{susy}}^{\text{matter}}$ of the modes ϕ and B are indeed identical.

Alternatively, in the heat kernel regularization we are led to consider

$$S(t; |b_1|, |b_2|, x) \equiv \sum_{n_1, n_2 \in \mathbb{Z}_{\geq 0}} e^{-t(|b_1|n_1 + |b_2|n_2 + x)}, \tag{4.27}$$

and we extract $E_{\text{susy}}^{\text{matter}}$ from the coefficient of $-t$ in a series around $t = 0$. This is precisely the character regularization we introduced above. Concretely,

$$\begin{aligned} S(t; |b_1|, |b_2|, x) &= \frac{e^{-tx}}{(1 - e^{-t|b_1|})(1 - e^{-t|b_2|})} \\ &= \frac{(q_1 q_2)^{-r/2}}{(1 - q_1^{-1})(1 - q_2^{-1})} \Bigg|_{q_1 = e^{t|b_1|}, q_2 = e^{t|b_2|}}, \end{aligned} \tag{4.28}$$

where recall that $x = r\gamma$, and in the second line we have precisely the character (4.22) for the ϕ modes.

¹¹Below n denotes a multi-index.

4.4 Rewriting as a Dirac character

In the above discussion we saw that both the B and ϕ unpaired modes lead to the same contribution to the supersymmetric Casimir energy. However, the discussion is not quite symmetric because B has R-charge $k = r - 2$, while ϕ has R-charge $k = r$. One can put these on the same footing, with overall R-charge $r - 1$, by effectively further twisting the $\bar{\partial}$ operators, thus viewing them as (part of) a Dirac operator.

Let us begin with the ϕ zero modes. The relevant operator is

$$\bar{\partial}_{\mathcal{K}_-^{r/2}} \cong \bar{\partial}_{\mathcal{K}_-^{1/2} \otimes \mathcal{L}_-}, \quad (4.29)$$

where we have defined

$$\mathcal{L}_- \equiv \mathcal{K}_-^{(r-1)/2}. \quad (4.30)$$

Let us denote the weight on \mathcal{L}_- as $\lambda = (q_1 q_2)^{-(r-1)/2}$. Then the relevant character is

$$C(\bar{\partial}_{\mathcal{K}_-^{r/2}}, (q_1, q_2)) = \frac{(q_1 q_2)^{-1/2}}{(1 - q_1^{-1})(1 - q_2^{-1})} \lambda = \frac{(q_1 q_2)^{1/2}}{(1 - q_1)(1 - q_2)} \lambda, \quad (4.31)$$

where the $(q_1 q_2)^{-1/2}$ in the numerator comes from the twisting by $\mathcal{K}_-^{1/2}$.

Similarly for the B zero modes the operator is

$$\bar{\partial}_{\mathcal{K}_+^{-(r-2)/2}} \cong \bar{\partial}_{\mathcal{K}_+^{1/2} \otimes \mathcal{L}_+}, \quad (4.32)$$

where

$$\mathcal{L}_+ \equiv \mathcal{K}_+^{-(r-1)/2}. \quad (4.33)$$

Notice that the weight on \mathcal{L}_+ is also $\lambda = (q_1 q_2)^{-(r-1)/2}$, and indeed $\mathcal{L}_+ \cong \mathcal{L}_-$. Thus the relevant character is

$$C(\bar{\partial}_{\mathcal{K}_+^{-(r-2)/2}}, (q_1, q_2)) = \frac{(q_1 q_2)^{1/2}}{(1 - q_1)(1 - q_2)} \lambda. \quad (4.34)$$

This makes manifest that the two modes have the *same* character. In both cases the operator is $\bar{\partial}^\pm$ twisted by $\mathcal{K}_\pm^{1/2} \otimes \mathcal{L}_\pm$, which may be viewed as part of a Dirac-type operator twisted by \mathcal{L}_\pm . From this point of view, the explicit $(q_1 q_2)^{1/2}$ factors come from the fact that the modes transform as spinors under the $U(1) \times U(1)$ action.

We may thus define

$$C(\text{Dirac}, (q_1, q_2, \lambda)) \equiv \frac{(q_1 q_2)^{1/2}}{(1 - q_1)(1 - q_2)} \lambda. \quad (4.35)$$

Setting $q_1 = e^{t|b_1|}$, $q_2 = e^{t|b_2|}$, $\lambda = e^{-tu}$, we may expand in a Laurent series around $t = 0$ as above:

$$\begin{aligned} C(\text{Dirac}, (q_1, q_2, \lambda)) &= \frac{1}{|b_1||b_2|t^2} - \frac{u}{|b_1||b_2|t} + \left(\frac{u^2}{2!|b_1||b_2|} - \frac{b_1^2 + b_2^2}{24|b_1||b_2|} \right) \\ &\quad - \left(\frac{u^3}{3!|b_1||b_2|} - \frac{(b_1^2 + b_2^2)u}{24|b_1||b_2|} \right) t \\ &\quad + \left(\frac{u^4}{4!|b_1||b_2|} - \frac{(b_1^2 + b_2^2)u^2}{24 \cdot 2!|b_1||b_2|} + \frac{7(b_1^2 + b_2^2) - 4b_1^2 b_2^2}{5760|b_1||b_2|} \right) t^2 + O(t^3). \end{aligned} \quad (4.36)$$

We immediately see that the divergent “index”, which is given by setting $t = 0$, arises as a second order pole, while the coefficient of the linear term in $-t$ precisely reproduces the regularized supersymmetric Casimir energy (setting u to its physical value of $u = (r - 1)\gamma$). Of course this is simply equivalent to the computation in (4.20), although now the equal contribution of the B and ϕ modes is manifest.

The appearance of the A -roof class in the expansion (4.36) is explained by the following identity:

$$\frac{\hat{A}(i\theta_1, i\theta_2)}{\chi(i\theta_1, i\theta_2)} = \frac{1}{(q_1^{1/2} - q_1^{-1/2})(q_2^{1/2} - q_2^{-1/2})} = \frac{(q_1 q_2)^{1/2}}{(1 - q_1)(1 - q_2)}, \quad (4.37)$$

where $q_1 = e^{i\theta_1}$, $q_2 = e^{i\theta_2}$. Here the numerator on the left hand side is the A -roof class, which in general is defined as

$$\hat{A}(x_1, \dots, x_n) = \prod_{i=1}^n \frac{x_i}{e^{x_i/2} - e^{-x_i/2}} = \prod_{i=1}^n \frac{x_i}{2 \sinh x_i/2}, \quad (4.38)$$

while the denominator is the Euler class

$$\chi(x_1, \dots, x_n) = \prod_{i=1}^n x_i. \quad (4.39)$$

In the usual index theorem the x_i would be the first Chern classes of the line bundles that arise on application of the splitting principle. In the equivariant setting these are replaced by $x_i + i\theta_i$, where the group action on the complex line fibre is multiplication by $e^{i\theta_i}$. The Euler class cancels against the numerator of (4.38), which leads to the first equality in (4.37). The A -roof class may be expanded as

$$\hat{A} = 1 - \frac{1}{24}p_1 + \frac{1}{5760}(7p_1^2 - 4p_2) + \dots, \quad (4.40)$$

where the Pontryagin classes p_I are the I th elementary symmetric functions in the x_i^2 . Thus in particular for complex dimension $n = 2$ we have $p_1 = x_1^2 + x_2^2$, $p_2 = x_1^2 x_2^2$. These comments of course explain the structure of the right hand side of (4.36). Analytically continuing $q_1 = e^{t|b_1|}$, $q_2 = e^{t|b_2|}$ amounts to sending $i\theta_i \rightarrow t|b_i|$ above. Then (4.36) may be rewritten as

$$\begin{aligned} C(\text{Dirac}, (q_1, q_2, \lambda)) &= \frac{e^{-tu}}{4 \sinh(t|b_1|/2) \sinh(t|b_2|/2)} \\ &= \frac{1}{|b_1||b_2|t^2} \left(1 - \frac{b_1^2 + b_2^2}{24}t^2 + \frac{7(b_1^2 + b_2^2)^2 - 4b_1^2 b_2^2}{5760}t^4 + \dots \right) e^{-tu}. \end{aligned} \quad (4.41)$$

The middle term in brackets is the contribution from the A -roof class. This of course explains the observation in [5] that the supersymmetric Casimir energy on the primary Hopf surface is obtained (formally) by an equivariant integral on \mathbb{R}^4 associated to the Dirac operator. This arises naturally in the way we have formulated the problem. Here the supersymmetric Casimir energy is the coefficient of $-t$ in an expansion of the index-character of the Dirac operator, where the latter is regularized by analytically continuing a divergent geometric series into its domain of convergence. Mathematically, this arises as a heat kernel regularization, as opposed to a (Barnes) zeta function regularization.

5 Secondary Hopf surfaces and generalizations

5.1 Lens spaces

The simplest way to generalize the primary Hopf surfaces studied in the previous section is to take a $\Gamma \cong \mathbb{Z}_p$ quotient. These secondary Hopf surfaces were described at the beginning of section 2.2.2. With respect to either complex structure I_{\pm} the \mathbb{Z}_p action is generated by $(z_1, z_2) \rightarrow (e^{2\pi i/p} z_1, e^{-2\pi i/p} z_2)$, where $z_i = z_i^{\pm}$, $i = 1, 2$. This action preserves the Killing spinors ζ_{\pm} , and hence in particular the function s and holomorphic volume forms on \mathbb{C}^2 . The quotient $M_3 = S^3/\mathbb{Z}_p = L(p, 1)$ is then a Lens space.

Since $\pi_1(M_3) \cong \mathbb{Z}_p$, the space $\mathbb{R} \times M_3$ now supports non-trivial flat connections. As discussed in section 3, the localized partition function on $S^1 \times M_3$ splits into associated topological sectors, which are summed over. In the Hamiltonian approach, each such sector leads to a distinct supersymmetric quantum mechanics on \mathbb{R} . Following the end of section 2.3, here we consider a $U(N)$ gauge theory with matter in a representation \mathcal{R} in a weight space decomposition. The modes $B = B_{\rho}$, $\phi = \phi_{\rho}$ then become sections of $\mathcal{K}_+^{-k/2} \otimes L$ and $\mathcal{K}_-^{k/2} \otimes L$, respectively, where the line bundle L over $\mathbb{R} \times L(p, 1)$ has first Chern class $c_1(L) \equiv \rho(\mathbf{m}) \bmod p$. In the Hamiltonian approach, and for fixed topological sector \mathbf{m} , we thus want to compute a *twisted character*, which counts holomorphic sections of $\mathcal{K}_+^{-k/2} \otimes L$ and $\mathcal{K}_-^{k/2} \otimes L$ according to their $U(1) \times U(1)$ charges (where as usual $k = r - 2$ for B and $k = r$ for ϕ).

Recall that holomorphic functions on \mathbb{C}^2 are counted by

$$C(\bar{\partial}, (q_1, q_2), \mathbb{C}^2) = \frac{1}{(1 - q_1)(1 - q_2)}. \quad (5.1)$$

The Dirac index-character (4.35) is constructed from this by multiplying by $(q_1 q_2)^{1/2} \lambda$, which takes account of the lifting of the $U(1) \times U(1)$ action to $\mathcal{K}_{\pm}^{\mp k/2}$. More generally, holomorphic sections of L over $\mathbb{C}^2/\mathbb{Z}_p$, where $c_1(L) \equiv \nu \bmod p$, are counted by the twisted character

$$C(\bar{\partial}_L, (q_1, q_2), \mathbb{C}^2/\mathbb{Z}_p) = \frac{q_1^{\nu}(1 - (q_1 q_2)^{p-\nu}) + q_2^{p-\nu}(1 - (q_1 q_2)^{\nu})}{(1 - (q_1 q_2))(1 - q_1^p)(1 - q_2^p)}. \quad (5.2)$$

Here ν is understood to lie in the range $0 \leq \nu < p$, and as usual one expands the denominator in a geometric series, for $|q_1|, |q_2| < 1$. Perhaps the simplest way to derive (5.2) is via an appropriate projection of (5.1). Recall that the \mathbb{Z}_p action on \mathbb{C}^2 is generated by $(z_1, z_2) \rightarrow (\omega_p z_1, \omega_p^{-1} z_2)$, where $\omega_p \equiv e^{2\pi i/p}$. The twisted character is then

$$C(\bar{\partial}_L, (q_1, q_2), \mathbb{C}^2/\mathbb{Z}_p) = \frac{1}{p} \sum_{j=0}^{p-1} \frac{\omega_p^{-j\nu}}{(1 - \omega_p^j q_1)(1 - \omega_p^{-j} q_2)}. \quad (5.3)$$

One easily verifies that this may be simplified to give (5.2). For zero twist, meaning $\nu = 0$, we are simply counting holomorphic functions on $\mathbb{C}^2/\mathbb{Z}_p$, and (5.2) reads

$$\begin{aligned} C(\bar{\partial}, (q_1, q_2), \mathbb{C}^2/\mathbb{Z}_p) &= \frac{(1 - (q_1 q_2)^p)}{(1 - (q_1 q_2))(1 - q_1^p)(1 - q_2^p)} \\ &= \frac{1 + q_1 q_2 + (q_1 q_2)^2 + \dots + (q_1 q_2)^{p-1}}{(1 - q_1^p)(1 - q_2^p)}. \end{aligned} \quad (5.4)$$

This is the index-character of an $A_{p-1} = \mathbb{C}^2/\mathbb{Z}_p$ singularity.

Thus the contribution of a matter field, for weight ρ and fixed flat connection \mathbf{m} , leads to a supersymmetric Casimir energy (in the sector $\alpha = \mathbf{m} \in \mathcal{M}_{\text{flat}}$) given by the character

$$\frac{(q_1 q_2)^{1/2} \left[q_1^\nu (1 - (q_1 q_2)^{p-\nu}) + q_2^{p-\nu} (1 - (q_1 q_2)^\nu) \right]}{(1 - (q_1 q_2))(1 - q_1^p)(1 - q_2^p)} \lambda. \tag{5.5}$$

As in section 4, the Casimir energy is obtained by substituting $q_1 = e^{t|b_1|}$, $q_2 = e^{t|b_2|}$, $\lambda = e^{-tu}$, and extracting the coefficient of $-t$ in a Laurent series around $t = 0$. This is easily done, and we find

$$E_{\text{susy}, \mathbf{m}}^{\text{matter}} = \frac{1}{24|b_1||b_2|p} \left[4u^3 - (b_1^2 + b_2^2 - 2|b_1||b_2|(p^2 - 6\nu p + 6\nu^2 - 1))u + 2|b_1||b_2|(|b_1| - |b_2|)\nu(\nu - p)(2\nu - p) \right], \tag{5.6}$$

where $\nu = \widehat{\rho(\mathbf{m})}$. Here the hat indicates that ν is understood to lie in the range $0 \leq \nu < p$, and thus $\rho(\mathbf{m}) \in \mathbb{Z}$ should be reduced mod p to also lie in this range. Recall that we fixed the convention that $0 \leq m_i < p$, and ordered $m_1 \leq \dots \leq m_N$. As usual we should also put $u = (r - 1)\gamma$, where $\gamma = (|b_1| + |b_2|)/2$. This is the contribution from the weight ρ ; one should of course then sum over weights to get the total contribution of the matter field, in the sector \mathbf{m} .

The partition function on $S^1 \times L(p, 1)$ has been computed in [15], and fixing the sector \mathbf{m} one can check that indeed

$$E_{\text{susy}, \mathbf{m}}^{\text{matter}} = - \lim_{\beta \rightarrow \infty} \frac{d}{d\beta} \log Z_{\mathbf{m}}^{\text{matter}}. \tag{5.7}$$

See equations (5.32)–(5.34) of [15]. Thus the Hamiltonian approach does indeed correctly reproduce the supersymmetric Casimir energy, defined in terms of the partition function, for each topological sector.

5.2 Fixed point formula

In [8] it was explained that the index-character may be computed for a general isolated singularity by first resolving the singularity, and using a fixed point formula. In the case at hand $\mathbb{C}^2/\mathbb{Z}_p = A_{p-1}$ is well-known to admit a crepant resolution, meaning that the holomorphic $(2, 0)$ -form extends smoothly to the resolved space, by blowing up $p - 1$ two-spheres. The action of $U(1) \times U(1)$ on $\mathbb{C}^2/\mathbb{Z}_p$ extends to the resolution, which is hence *toric*, with p isolated fixed points. Each such fixed point is of course locally modelled by \mathbb{C}^2 , and the general formula in [8] expresses the index-character of $\mathbb{C}^2/\mathbb{Z}_p = A_{p-1}$ in terms of a sum of the index-characters for \mathbb{C}^2 , for each fixed point. Labelling the fixed points by $j = 0, \dots, p - 1$, explicitly we have

$$C(\bar{\partial}, (q_1, q_2), \mathbb{C}^2/\mathbb{Z}_p) = \sum_{j=0}^{p-1} \frac{1}{(1 - q_1^{u_1^{(j)}} q_2^{u_2^{(j)}})(1 - q_1^{v_1^{(j)}} q_2^{v_2^{(j)}})}. \tag{5.8}$$

Here the action of $U(1) \times U(1)$ on each fixed origin of \mathbb{C}^2 is specified by the two vectors $\mathbf{u}^{(j)} = (u_1^{(j)}, u_2^{(j)})$, $\mathbf{v}^{(j)} = (v_1^{(j)}, v_2^{(j)}) \in \mathbb{Z}^2$ as

$$(z_1, z_2) \rightarrow (q_1^{u_1^{(j)}} q_2^{u_2^{(j)}} z_1, q_1^{v_1^{(j)}} q_2^{v_2^{(j)}} z_2). \tag{5.9}$$

One finds (for example using toric geometry methods) that

$$\mathbf{u}^{(j)} = (p - j, -j), \quad \mathbf{v}^{(j)} = (-p + j + 1, j + 1), \quad (5.10)$$

and (5.8) reads

$$C(\bar{\partial}, (q_1, q_2), \mathbb{C}^2/\mathbb{Z}_p) = \sum_{j=0}^{p-1} \frac{1}{(1 - q_1^{p-j} q_2^{-j})(1 - q_1^{-p+j+1} q_2^{j+1})}, \quad (5.11)$$

which one can verify agrees with (5.4).

Let us define the matter contribution to the supersymmetric Casimir energy for S^3 as

$$E_{\text{susy}}^{\text{matter}}[S^3; b_1, b_2] = \frac{4u^3 - (b_1^2 + b_2^2)u}{24b_1b_2}. \quad (5.12)$$

Then (5.8) leads to the following fixed point formula for the Casimir for $S^1 \times L(p, 1)$ (with trivial flat connection):

$$\begin{aligned} E_{\text{susy}}^{\text{matter}}[L(p, 1); b_1, b_2] &= \sum_{j=0}^{p-1} E_{\text{susy}}^{\text{matter}}[S^3; b_1^{(j)}, b_2^{(j)}] \\ &= \frac{4u^3 - [(|b_1| + |b_2|)^2 - 2|b_1||b_2|p^2]u}{24|b_1||b_2|p}. \end{aligned} \quad (5.13)$$

Here we have defined

$$b_1^{(j)} \equiv p|b_1| - j(|b_1| + |b_2|), \quad b_2^{(j)} \equiv -p|b_1| + (j + 1)(|b_1| + |b_2|). \quad (5.14)$$

In fact $(b_1^{(j)}, b_2^{(j)})$, $j = 0, \dots, p - 1$, are precisely the Reeb weights at the p fixed points. In this precise sense, we may write the supersymmetric Casimir energy for the secondary Hopf surface $(S^1 \times S^3)/\mathbb{Z}_p$ as the sum of p Casimir energies for primary Hopf surfaces $S^1 \times S^3$, where each fixed point contribution has a different complex structure, determined by (5.14). This data is in turn determined by the equivariant geometry of the resolved space.

5.3 More general M_3

In section 2.2.2 we discussed more general classes of secondary Hopf surfaces, realised as $\Gamma = \Gamma_{ADE} \subset \text{SU}(2)$ quotients of primary Hopf surfaces. The A series is precisely the Lens space case discussed in the previous subsection, while the D and E series result in non-Abelian fundamental groups. The formalism we have described gives a prescription for computing the supersymmetric Casimir energy E_{susy} (or at least the matter contribution $E_{\text{susy}}^{\text{matter}}$) for such backgrounds. One first needs to classify the inequivalent flat G -connections on $M_3 = S^3/\Gamma$, via their corresponding homomorphisms $\varrho : \Gamma \rightarrow G$. A given matter representation \mathcal{R} of G then gives a corresponding flat \mathcal{R} -connection, from which one constructs the matter bundle (2.33). For each such flat connection one then needs to compute the index-character of this bundle, namely one counts holomorphic sections via their Reeb charges. The supersymmetric Casimir energy, in this topological sector, is then obtained as a limit of this index-character.

5.3.1 Poincaré Hopf surface

In practice, one thus first needs to understand the representation theory of the relevant non-Abelian groups, before one can compute the associated index-characters. An interesting but simple example is provided by the exceptional group $\Gamma = \Gamma_{E_8}$: this is the binary icosahedral group, which has order 120. The quotient $M_3 = S^3/\Gamma$ is the famous Poincaré sphere, which has the homology groups of S^3 , despite the very large fundamental group. This follows since Γ_{E_8} is equal to its commutator subgroup, and hence its Abelianization (which equals $H_1(M_3, \mathbb{Z})$) is trivial. Related to this fact is that consequently any homomorphism into an Abelian group is necessarily trivial. This is easy to see: since any group element $g \in \Gamma$ may be written as $g = hvh^{-1}v^{-1}$, then for any homomorphism $\varrho : \Gamma \rightarrow G$ we have $\varrho(g) = \varrho(h)\varrho(v)\varrho(h)^{-1}\varrho(v)^{-1} = \text{identity}$, where in the last step we used that G is Abelian. This shows that, for example, any flat U(1) connection over the Poincaré sphere is necessarily trivial. Because of this, to compute the supersymmetric Casimir energy we need only the index-character of \mathbb{C}^2/Γ . But this is easily computed by realizing the latter as a homogeneous hypersurface singularity

$$\mathbb{C}^2/\Gamma_{E_8} \cong \{f_{E_8} \equiv Z_1^3 + Z_2^5 + Z_3^2 = 0\} \subset \mathbb{C}^3. \quad (5.15)$$

Here the polynomial f_{E_8} has degree $d = 30$ under the weighted \mathbb{C}^* action on \mathbb{C}^3 with weights $(w_1, w_2, w_3) = (10, 6, 15)$. From the general formula in [20] we thus compute the index-character

$$C(\bar{\partial}, q, \mathbb{C}^2/\Gamma_{E_8}) = \frac{1 - q^{30}}{(1 - q^6)(1 - q^{10})(1 - q^{15})} = 1 + q^6 + q^{10} + q^{12} + q^{15} + \dots \quad (5.16)$$

Here $q \in \mathbb{C}^*$ acts diagonally on $\mathbb{C}^2/\Gamma_{E_8}$ as $(z_1, z_2) \rightarrow (q^{1/2}z_1, q^{1/2}z_2)$. Notice that the centre of Γ_{E_8} is \mathbb{Z}_2 , which acts as multiplication on (z_1, z_2) by -1 . The holomorphic $(2, 0)$ -form thus has weight q under the \mathbb{C}^* action, and the supersymmetric Casimir energy for an Abelian gauge theory on the ‘‘Poincaré Hopf surface’’ is

$$\begin{aligned} E_{\text{susy}}^{\text{matter}} &= \left[q^{1/2} \lambda \cdot \frac{1 - q^{30}}{(1 - q^{10})(1 - q^6)(1 - q^{15})} \Big|_{q=e^{t|b|}, \lambda=e^{-tu}} \right]_{\text{coefficient of } -t} \\ &= \frac{4u^3 + 539b^2u}{720b^2}. \end{aligned} \quad (5.17)$$

The Reeb vector field acting on $(z_1 = |z_1|e^{i\psi_1}, z_2 = |z_2|e^{i\psi_2})$ is

$$\xi = \partial_\psi = \frac{|b|}{2}(\partial_{\psi_1} + \partial_{\psi_2}), \quad (5.18)$$

while for a matter multiplet of R-charge r we have $u = (r - 1)|b|/2$.

5.3.2 Homogeneous hypersurface singularities

For an Abelian gauge theory on the Poincaré Hopf surface just discussed, any flat connection over S^3/Γ_{E_8} is trivial, and thus the index-character that counts holomorphic functions on $\mathbb{C}^2/\Gamma_{E_8}$ is sufficient to compute the supersymmetric Casimir energy. However, more

generally we may easily extend the above discussion to compute E_{susy} for \mathbb{Z} quotients of homogeneous hypersurface singularities in the sector with trivial flat connection. These are compact complex surfaces of the form $M_4 = S^1 \times M_3$, where M_3 is the link of the singularity.

Consider a general weighted homogeneous hypersurface singularity in \mathbb{C}^3 . Here the weighted \mathbb{C}^* action on \mathbb{C}^3 is $(Z_1, Z_2, Z_3) \rightarrow (q^{w_1} Z_1, q^{w_2} Z_2, q^{w_3} Z_3)$, where $w_i \in \mathbb{N}$ are the weights, $i = 1, 2, 3$, and $q \in \mathbb{C}^*$. The hypersurface is the zero set $X \equiv \{f = 0\} \subset \mathbb{C}^3$ of a weighted homogeneous polynomial $f = f(Z_1, Z_2, Z_3)$, where

$$f(qZ_1, qZ_2, qZ_3) = q^d f(Z_1, Z_2, Z_3), \tag{5.19}$$

which defines the degree $d \in \mathbb{N}$. We assume that f is such that $X \setminus \{o\} \cong \mathbb{R} \times M_3$ is smooth, where o is the origin $Z_1 = Z_2 = Z_3 = 0$. The associated compact complex surface is obtained as a free \mathbb{Z} quotient of $X \setminus \{o\}$, where $\mathbb{Z} \subset \mathbb{C}^*$ is embedded as $n \rightarrow q^n$ for some fixed $q > 1$. The Reeb vector field action is quasi-regular, generated by $q \in U(1) \subset \mathbb{C}^*$, and the quotient $\Sigma_2 = M_3/U(1)$ is in general an orbifold Riemann surface. This construction of course includes all the spherical three-manifolds in section 2.2.2, for which $M_3 \cong S^3/\Gamma_{ADE}$ and Σ_2 has genus $g = 0$, but it also includes many other Seifert three-manifolds. For example, taking weights $(w_1, w_2, w_3) = (1, 1, 1)$ and f to have degree d , then M_3 is the total space of a circle bundle over a Riemann surface Σ_2 of genus $g = (d - 1)(d - 2)/2$.

Such homogeneous hypersurface singularities are Gorenstein canonical singularities, meaning they admit a global holomorphic $(2, 0)$ -form Ω_0 , defined on the complement of the isolated singularity at $Z_1 = Z_2 = Z_3 = 0$. With respect to the I_+ complex structure, so that we identify $\Omega_0 = \Omega_+$, we may then write

$$\Omega_0 = \kappa dz \wedge dw, \tag{5.20}$$

where z and w are the local coordinates defined by supersymmetry on $\mathbb{R} \times M_3$, defined in section 2.1, and $\kappa = \kappa(z, w)$ is a local holomorphic function. The argument in section 4.1 then generalizes to give that the unpaired B modes that contribute to the supersymmetric Casimir energy are

$$B = \left| \frac{\mathcal{P}_+}{\Omega_+} \right|^{k/2} \mathcal{F}, \tag{5.21}$$

where $|\mathcal{P}_+/\Omega_+| = \Omega^3 c/|\kappa|$ is a real, globally defined, nowhere zero function on $X \setminus \{o\}$, and \mathcal{F} is a holomorphic function on X . This follows since \mathcal{P}_+ and Ω_+ are both globally defined, and being both $(2, 0)$ -forms are necessarily proportional. The holomorphic functions \mathcal{F} on X are spanned by monomials $Z_1^{n_1} Z_2^{n_2} Z_3^{n_3}$, where $n_i \in \mathbb{Z}_{\geq 0}$, modulo the ideal generated by the defining polynomial f . The index-character that counts such holomorphic functions according to their weights under $q \in \mathbb{C}^*$ is

$$C(\bar{\partial}, q, X) = \frac{1 - q^d}{(1 - q^{w_1})(1 - q^{w_2})(1 - q^{w_3})}. \tag{5.22}$$

The ϕ modes work similarly, with respect to the second complex structure I_- . This may be defined globally in this setting as follows. The singularity X may be viewed as a complex

cone over the orbifold Riemann surface $\Sigma_2 = M_3/U(1)$. Here $\mathbb{R} \times M_3$ may be identified with a (orbifold) \mathbb{C}^* fibration over Σ_2 , with the isolated singularity arising by contracting the whole space to a point. In terms of the coordinates defined by supersymmetry, the \mathbb{C}^* action is generated by the complex vector field K . The I_- complex structure is then obtained by reversing the sign of the complex structure on the base Σ_2 , while keeping that of the \mathbb{C}^* fibre. This leads to the same complex manifold, although of course the map between the two copies is not holomorphic. As for the primary Hopf surfaces in section 4, the unpaired ϕ modes then give an identical contribution to the B modes above.

It follows that the relevant character is

$$C(q, \lambda, X) \equiv q^{(-d + \sum_{i=1}^3 w_i)/2} \lambda \cdot C(\bar{\partial}, q, X), \quad (5.23)$$

where $C(\bar{\partial}, q, X)$ is the index-character (5.22). Here the power of q is precisely $\frac{1}{2}$ the charge of the holomorphic $(2, 0)$ -form (arising as usual since A is a connection on $\mathcal{K}_+^{-1/2}$), and $q \in \mathbb{C}^*$ is the generator of the \mathbb{C}^* action. The supersymmetric Casimir energy in this case is obtained as usual by setting $q = e^{t|b|}$, $\lambda = e^{-tu}$, and extracting the coefficient of $-t$ in a Laurent series about $t = 0$. A simple calculation shows that this leads to the supersymmetric Casimir energy

$$E_{\text{susy}}^{\text{matter}} = \frac{4du^3 - (w_1^2 + w_2^2 + w_3^2 - d^2)db^2u}{24b^2w_1w_2w_3}. \quad (5.24)$$

Here $u = (r - 1)\gamma$ for a matter multiplet of R-charge r , where now $1/2$ the Reeb charge of the holomorphic $(2, 0)$ form is $\gamma = (-d + \sum_{i=1}^3 w_i)|b|/2$. For example, the Lens space case $L(p, 1)$ in sections 5.1, 5.2 is $w_1 = 2$, $w_2 = w_3 = p$, $d = 2p$ (with $|b_1| = |b_2| = |b|$), while the Poincaré Hopf surface in section 5.3.1 is $w_1 = 10$, $w_2 = 6$, $w_3 = 15$, $d = 30$. We stress again that (5.24) gives the matter contribution to the supersymmetric Casimir energy in the topological sector with trivial flat gauge connection. For non-trivial flat connections one would instead need to compute the index-character of the relevant (flat) matter bundle.

5.4 Full supersymmetric Casimir energy

As in much of the previous literature, in this paper we have focused attention on the contribution of a matter multiplet to the supersymmetric Casimir energy. However, we expect that the vector multiplet contribution will also arrange into short multiplets, and will similarly be related to (twisted) holomorphic functions. At least for primary Hopf surfaces, and secondary Hopf surfaces with $M_3 = L(p, 1)$, previous results in the literature imply that the contribution of a vector multiplet to the supersymmetric Casimir energy is (formally) obtained from the contribution of a matter multiplet by (i) setting the R-charge $r = 0$ (since the dynamical gauge field has zero R-charge), (ii) replacing weights ρ by roots α of the gauge group G , and finally (iii) reversing the overall sign. In this subsection we will simply *conjecture* this is true more generally, at least in the sector with trivial flat connection on which we focus.

Given this conjecture, it is straightforward to combine the matter multiplet result (5.24) for a general homogeneous hypersurface singularity with the vector multiplet

result, and sum over relevant weights/roots. Remarkably, we find the following simple formula for the total supersymmetric Casimir energy

$$E_{\text{susy}} = \frac{2|b|}{27} \frac{dc_1^3}{w_1 w_2 w_3} (3c - 2a) + \frac{|b|}{3} \frac{dc_1}{w_1 w_2 w_3} (c_1^2 - c_2)(a - c). \quad (5.25)$$

Here we have defined

$$c_1 \equiv -d + \sum_{i=1}^3 w_i, \quad c_2 \equiv -d^2 + \sum_{i=1}^3 w_i^2, \quad (5.26)$$

which depend on the weights (w_1, w_2, w_3) and degree d of the hypersurface singularity, while a and c denote the usual trace anomaly coefficients,

$$a = \frac{3}{32} (3\text{Tr } R^3 - \text{Tr } R) = \frac{3}{32} \left[2|G| + \sum_{\rho} (3(r_{\rho} - 1)^3 - (r_{\rho} - 1)) |\mathcal{R}_{\rho}| \right], \quad (5.27)$$

$$c = \frac{1}{32} (9\text{Tr } R^3 - 5\text{Tr } R) = \frac{1}{32} \left[4|G| + \sum_{\rho} (9(r_{\rho} - 1)^3 - 5(r_{\rho} - 1)) |\mathcal{R}_{\rho}| \right],$$

with R being the R-symmetry charge, and the trace running over all fermions.

By setting $(w_1, w_2, w_3) = (2, p, p)$, $d = 2p$, which correspond to A_{p-1} singularities with corresponding secondary Hopf surfaces $S^1 \times L(p, 1)$, one sees that (5.25) reduces to

$$E_{\text{susy}} = \frac{16|b|}{27p} (3c - 2a) + \frac{4|b|p}{3} (a - c). \quad (5.28)$$

This agrees with the $\beta \rightarrow \infty$ limit of the partition function in [15], and reproduces the original primary Hopf surface result of [2] when $p = 1$.

One can make a number of interesting observations about the general formula (5.25). Firstly, it depends on the choice of supersymmetric gauge theory only via a and c . Secondly, the coefficient of the term $(3a - 2c)$ is related to the *Sasakian volume* of M_3 via

$$\text{vol}(M_3) = \frac{d}{w_1 w_2 w_3} \cdot \frac{1}{|b|^2} \cdot \text{vol}(S^3). \quad (5.29)$$

Here $\text{vol}(S^3) = 2\pi^2$ is the volume of the standard round metric on the unit sphere, and the Reeb vector is normalized as $\xi = |b|\zeta$, where ζ generates the canonical $U(1) \subset \mathbb{C}^*$ action on the hypersurface singularity. M_3 is the link of this singularity, and any compatible Sasakian metric on M_3 has volume given by (5.29), as follows from the general formula in [20]. The metric on M_3 is not in general Sasakian, but the point is that M_3 is equipped in general with an (almost) contact one-form $\eta = d\psi + a$. The corresponding contact volume $\frac{1}{2} \int_{M_3} \eta \wedge d\eta$ depends only on the Reeb vector, and thus agrees with the Sasakian volume. We shall briefly comment further on this in the discussion section. We also note that in (5.25) $c_1 = -d + \sum_{i=1}^3 w_i$ is the first Chern class (number) of the (orbifold) anti-canonical bundle of the orbifold Riemann surface $\Sigma_2 = M_3/U(1)$ (more precisely, global sections of $K_{\Sigma_{\text{orb}}}^{-1}$ are given by weighted homogeneous polynomials of degree c_1). Thirdly, we have suggestively denoted $c_2 = -d^2 + \sum_{i=1}^3 w_i^2$. Of course this is not supposed to suggest the second Chern class/number of a line bundle, which is zero, but rather is a quadratic invariant of the singularity that takes a similar form to c_1 . It would be interesting to understand the geometric interpretation of the second term, proportional to $(a - c)$, in (5.25).

6 Discussion

In this paper we have shown that the supersymmetric Casimir energy E_{susy} of four-dimensional $\mathcal{N} = 1$ field theories defined on $S^1 \times M_3$ is computed by a limit of the index-character counting holomorphic functions on (or more generally holomorphic sections over) the space $\mathbb{R} \times M_3$. In particular, the latter is equipped with an ambi-Hermitian structure, and the short multiplets contributing to the supersymmetric Casimir energy are in one-to-one correspondence with (twisted) holomorphic functions, with respect to either complex structure. As examples of Seifert three-manifolds M_3 we considered S^3 , as well as the links S^3/Γ_{ADE} of ADE hypersurface singularities in \mathbb{C}^3 . For $M_3 \cong S^3$ our analysis explains the relation of the supersymmetric Casimir energy to the anomaly polynomial, pointed out in [5]. In the case of $M_3 \cong L(p, 1)$ we obtained formulas that may independently be derived using the path integral results of [15]; while, to our knowledge, the formulas for the D and E singularities have not appeared before. We have also presented a formula (5.25) for the supersymmetric Casimir energy when M_3 is the link of a general homogeneous hypersurface singularity, in the trivial flat connection sector, and assuming a conjecture for the vector multiplet contribution.

Our analysis can be extended in various directions. The localization results of [2, 15] strongly suggest that in the supersymmetric quantum mechanics the contributions of the vector multiplet will also arrange into short multiplets. One should show explicitly that these are indeed related to (twisted) holomorphic functions, and therefore ultimately to the index-character we have studied (and in particular hence prove (5.25)). In this paper we have explained how to incorporate the contributions of discrete flat connections on M_3 , considering $M_3 \cong L(p, 1)$ as concrete example. It may be interesting to work out more examples. Moreover, here we have not addressed the role of continuous flat connections arising when $\pi_1(M_3)$ is infinite. Ultimately, the complete supersymmetric Casimir energy of a theory should be obtained by appropriately minimizing over the set of all flat connections, and it would be nice to see whether this quantity may be used as a new test of dualities between different field theories and/or geometries.

Using the formulas presented in appendix A one can also easily obtain new supersymmetric indices for theories defined on $S^1 \times M_3$, where M_3 is the Seifert link of the D and E type hypersurface singularities. It would be interesting to explore their properties, as they involve a generalization of the elliptic gamma function appearing for $M_3 \cong S^3$ [2] and $M_3 \cong L(p, 1)$ [15, 23, 24].

We close our discussion by recalling that it is not clear how to reproduce the supersymmetric Casimir energy with a holographic computation in a supergravity solution, even for $M_3 = S^3_{\text{round}}$. See for example [28, 29] for some attempts and further discussion. Let us point out that the formula (5.25) shows that in the large N limit the supersymmetric Casimir energy (in the trivial flat connection sector) is proportional to $N^2 \cdot \text{vol}(M_3)$. We expect that it should be possible to reproduce this result from a dual holographic computation, and indeed we will report on this in [30].

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A Supersymmetric index from the character

In this appendix we return to the supersymmetric index \mathcal{I} [9], clarifying its relation to the index-character, that is the main subject of this paper.

A.1 Primary Hopf surfaces

We begin with the case $M_3 \cong S^3$ and consider the modifications needed for the extension to more general M_3 in the next subsection. Following [9], we can work on $M_4 = \mathbb{R} \times S_{\text{round}}^3$, with the complex structure parameters of the Hopf surfaces emerging as fugacities associated to two commuting global symmetries [2, 11]. The supersymmetric index may be defined quite generally for any theory that admits the superalgebra (3.11), in terms of a trace over states in the Hilbert space, as

$$\mathcal{I}(x) = \text{Tr}(-1)^F x^\Sigma, \quad (\text{A.1})$$

where F is the fermion number. A standard argument then shows that the net contribution to the trace arises from states obeying $\Xi \equiv H_{\text{susy}} - \Sigma = 0$. As this quantity does not depend on continuous parameters it can be computed in the free theory, where it takes the form of a plethystic exponential

$$\mathcal{I}(x) = \text{Pexp}(f(x)) \equiv \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} f(x^k)\right). \quad (\text{A.2})$$

Physically, this is the grand-canonical partition function written in terms of the *single particle* partition function $f(x)$, counting single particle states (annihilated by Ξ) of the free theory. In practice, the operator Σ appearing in the superalgebra is given by $\Sigma = -(2J_3^L + R)$, where R is the R-symmetry and J_3^L is the angular momentum associated to rotations in $U(1) \subset SU(2)_L \subset SU(2)_L \times SU(2)_R$. One can introduce a second fugacity y conjugated to the angular momentum J_3^R associated to rotations in $U(1) \subset SU(2)_R \subset SU(2)_L \times SU(2)_R$. After changing variables,¹² setting $p_1 = xy$ and $p_2 = x/y$, the single particle index for a chiral multiplet is given by [21]

$$f^{\text{matter}}(p_1, p_2) = \frac{(p_1 p_2)^{\frac{r}{2}} - (p_1 p_2)^{\frac{2-r}{2}}}{(1-p_1)(1-p_2)}, \quad (\text{A.3})$$

and the contribution of a chiral multiplet to the supersymmetric index then reads

$$\mathcal{I}^{\text{matter}}(p_1, p_2) = \prod_{n_1, n_2 \geq 0}^{\infty} \frac{1 - (p_1 p_2)^{-r/2} p_1^{n_1+1} p_2^{n_2+1}}{1 - (p_1 p_2)^{r/2} p_1^{n_1} p_2^{n_2}} = \Gamma((p_1 p_2)^{r/2}; p_1, p_2), \quad (\text{A.4})$$

where $\Gamma(z; p_1, p_2)$ is the elliptic gamma function.

¹²In this section we will denote p_1, p_2 the variables in which the index is written naturally in terms of elliptic gamma functions. We will later make contact with the variables q_1, q_2 used in the previous sections.

It was noticed in [2, 22] that the supersymmetric Casimir energy can be extracted from the single particle index by setting $p_1 = e^{t|b_1|}$, $p_2 = e^{t|b_2|}$, and taking the finite part of the limit

$$E_{\text{susy}}(|b_1|, |b_2|) = \frac{1}{2} \lim_{t \rightarrow 0} \frac{d}{dt} f(p_1, p_2) . \quad (\text{A.5})$$

Below we will clarify the reason why this limit reproduces the supersymmetric Casimir energy by relating $f^{\text{matter}}(p_1, p_2)$ to the index-character counting holomorphic functions.

For the computation of $f^{\text{matter}}(p_1, p_2)$ we can use the ingredients worked out in [3, 17]. In particular, the expressions for the operators $H_{\text{susy}}, R, J_3^L, J_3^R$ can be found in these references,¹³ written in terms of bosonic and fermionic oscillators. For example, writing $\Xi = \Xi_{\text{bos}} + \Xi_{\text{fer}}$, we have

$$\Xi_{\text{bos}} = \frac{1}{2} \sum_{\ell=0}^{\infty} \sum_{m, n = -\frac{\ell}{2}}^{\frac{\ell}{2}} \Xi_{\ell m}^a \left(a_{\ell m n} a_{\ell m n}^\dagger + a_{\ell m n}^\dagger a_{\ell m n} \right) + \Xi_{\ell m}^b \left(b_{\ell m n} b_{\ell m n}^\dagger + b_{\ell m n}^\dagger b_{\ell m n} \right) , \quad (\text{A.6})$$

and

$$\begin{aligned} \Xi_{\text{fer}} = & \frac{1}{2} \sum_{\ell=0}^{\infty} \sum_{n = -\frac{\ell}{2}}^{\frac{\ell}{2}} \sum_{m = -\frac{\ell}{2} - 1}^{\frac{\ell}{2}} \Xi_{\ell m}^c \left(c_{\ell m n} c_{\ell m n}^\dagger - c_{\ell m n}^\dagger c_{\ell m n} \right) \\ & - \frac{1}{2} \sum_{\ell=1}^{\infty} \sum_{n = -\frac{\ell}{2}}^{\frac{\ell}{2}} \sum_{m = -\frac{\ell}{2}}^{\frac{\ell}{2} - 1} \Xi_{\ell m}^d \left(d_{\ell m n} d_{\ell m n}^\dagger - d_{\ell m n}^\dagger d_{\ell m n} \right) , \end{aligned} \quad (\text{A.7})$$

with

$$\begin{aligned} \Xi_{\ell m}^a &= \ell + 2 + 2m , & \Xi_{\ell m}^b &= \ell + 2m , \\ \Xi_{\ell m}^c &= -(\ell + 2 + 2m) , & \Xi_{\ell m}^d &= -\ell + 2m , \end{aligned}$$

and similar expressions for the other operators. There are four types of single particle states in the Fock space, namely $|a_{\ell, m, n}\rangle = a_{\ell m n}^\dagger |0\rangle$, $|b_{\ell, m, n}\rangle = b_{\ell m n}^\dagger |0\rangle$, $|c_{\ell, m, n}\rangle = c_{\ell m n}^\dagger |0\rangle$, and $|d_{\ell, m, n}\rangle = d_{\ell m n}^\dagger |0\rangle$. However, the only zero-modes of Ξ are

$$|b_{\ell, -\frac{\ell}{2}, n}\rangle , \quad |c_{\ell, -\frac{\ell}{2} - 1, n}\rangle , \quad (\text{A.8})$$

while there are no zero-modes of the a -type and d -type states. These have $m = -\frac{\ell}{2}$ and $m = -\frac{\ell}{2} - 1$, respectively, which are precisely the shortening conditions obeyed by the ϕ and B modes, in the special case of the round three-sphere [4]. These two sets of modes are contributing non-trivially to (A.3). Let us now show this explicitly. From the definition

$$f^{\text{matter}}(x, y) = \text{tr}(-1)^F x^\Sigma y^{2J_3^R} = f_{\text{bos}}(x, y) - f_{\text{fer}}(x, y) , \quad (\text{A.9})$$

¹³We use the notation of [3]. For simplicity, and to make contact with [21], we are setting the parameters κ, ϵ in [3] to $\kappa = -1, \epsilon = 1$.

where here the trace is over the single particle states in (A.8), and we have

$$\begin{aligned}
 f_{\text{bos}}(x, y) &= \sum_{\ell=0}^{\infty} x^{r+\ell} \sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}} y^{2n} = \frac{x^r}{(1-xy)(1-\frac{x}{y})}, \\
 f_{\text{fer}}(x, y) &= \sum_{\ell=0}^{\infty} x^{\ell-r+2} \sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}} y^{2n} = \frac{x^{2-r}}{(1-xy)(1-\frac{x}{y})}.
 \end{aligned}
 \tag{A.10}$$

To derive these we used¹⁴

$$\Sigma |b_{\ell, -\frac{\ell}{2}, n}\rangle = (r + \ell) |b_{\ell, -\frac{\ell}{2}, n}\rangle, \quad \Sigma |c_{\ell, -\frac{\ell}{2}-1, n}\rangle = -(r - 2 - \ell) |c_{\ell, -\frac{\ell}{2}-1, n}\rangle,
 \tag{A.11}$$

and

$$J_3^R |b_{\ell, -\frac{\ell}{2}, n}\rangle = n |b_{\ell, -\frac{\ell}{2}, n}\rangle, \quad J_3^R |c_{\ell, -\frac{\ell}{2}-1, n}\rangle = n |c_{\ell, -\frac{\ell}{2}-1, n}\rangle.
 \tag{A.12}$$

Notice that the R-charge of the bosonic modes $|b_{\ell, -\frac{\ell}{2}, n}\rangle$ is r , while that of the fermionic modes $|c_{\ell, -\frac{\ell}{2}-1, n}\rangle$ is $-(r - 2)$. Thus $f^{\text{matter}}(x, y)$ is counting the bosonic particles minus the fermionic anti-particles [21].

In order to make contact with the main part of the paper, one can see that upon making the identifications¹⁵ $p_1 = q_1^{-1}$, $p_2 = q_2^{-1}$, the first term in (A.3) is precisely the character $C(\bar{\partial}_{\mathcal{K}_-^{r/2}}, (q_1, q_2))$ in (4.22), counting ϕ modes. On the other hand, the second term is equal to the character $C(\partial_{\mathcal{K}_+^{-(r-2)/2}}, (q_1, q_2))$, namely it can be identified with the character counting \tilde{B} modes. Notice that

$$C(\partial_{\mathcal{K}_+^{-(r-2)/2}}, (q_1, q_2)) = C(\bar{\partial}_{\mathcal{K}_-^{r/2}}, (q_1^{-1}, q_2^{-1})).
 \tag{A.13}$$

On taking the limit (A.5), the opposite signs in front of the fermionic part and in its exponent cancel each other, effectively giving the *same* result as the limit of the character, or Dirac character, that we considered before.

A.2 Secondary Hopf surfaces

Let us now discuss secondary Hopf surfaces $M_4 = S^1 \times M_3$, starting with the case that the fundamental group of M_3 is $\Gamma \cong \mathbb{Z}_p$. Thus $M_3 = L(p, 1)$ is a Lens space. The supersymmetric index in this case was studied in [15, 23, 24]. We can work on the space with a round metric on $S^1 \times S^3/\mathbb{Z}_p$ and obtain the modes by *projecting* from those on the covering space $S^1 \times S^3$. In the absence of a flat connection the modes on $L(p, 1)$ are precisely the \mathbb{Z}_p -invariant modes on S^3 . For example, for the scalar field ϕ , these are given by the S^3 hyperspherical harmonics Y_ℓ^{mn} satisfying $2n \equiv 0 \pmod p$. More generally, in the

¹⁴Here the operators are *normal ordered* [21].

¹⁵The need for this change of variables originates from our definition of the complex structures. See footnote 2. This is of course just a convention.

presence of a flat connection with first Chern class $c_1(L)$, the modes that descend to the Lens space from S^3 obey the condition [23, 25]

$$2n \equiv c_1(L) \pmod{p} . \quad (\text{A.14})$$

Since the flat connection can be removed locally by a gauge transformation, the eigenvalues of the operators $H_{\text{susy}}, R, J_3^L, J_3^R$ are unchanged. One can then compute the generating function by restricting the sums in (A.9) to the single particle states annihilated by Ξ of the previous subsection, and further obeying the projection (A.14), with $c_1(L) = \rho(\mathbf{m}) = \nu$. Accordingly, the bosonic part is then given by

$$f_{\text{bos}}(x, y) = x^r \sum_{\ell=0}^{\infty} x^\ell \sum_{n \in P} y^{2n} , \quad (\text{A.15})$$

where $P = \{n \in \{-\frac{\ell}{2}, \dots, \frac{\ell}{2}\} : 2n \equiv \nu \pmod{p}\}$. The sums are then computed exactly as in section 5.1, and we have

$$f_{\text{bos}}(x, y) = x^r \frac{(xy)^\nu (1 - x^{2(p-\nu)}) + (\frac{x}{y})^{p-\nu} (1 - x^{2\nu})}{(1 - x^2)(1 - (xy)^p)(1 - (\frac{x}{y})^p)} . \quad (\text{A.16})$$

Expressing this in terms of the variables $p_1 = xy$ and $p_2 = x/y$, we obtain

$$f_{\text{bos}}^{p,\nu}(p_1, p_2) = (p_1 p_2)^{\frac{r}{2}} C(\bar{\partial}_L, (p_1, p_2), \mathbb{C}^2/\mathbb{Z}_p) . \quad (\text{A.17})$$

For the fermions in the complex conjugate multiplet, the projection condition has to be modified as [24]

$$2n \equiv -\nu \pmod{p} . \quad (\text{A.18})$$

This effectively swaps n_1 and n_2 , or equivalently, p_1 and p_2 . Therefore, the index counting antifermions is given by

$$f_{\text{fer}}^{p,\nu}(p_1, p_2) = (p_1 p_2)^{\frac{2-r}{2}} C(\bar{\partial}_L, (p_2, p_1), \mathbb{C}^2/\mathbb{Z}_p) , \quad (\text{A.19})$$

Again, it can be checked explicitly that $f_{\text{fer}}^{p,\nu}(p_1, p_2) = f_{\text{bos}}^{p,\nu}(p_1^{-1}, p_2^{-1})$, showing the character contributing to the fermions is counting *anti*-holomorphic sections, as opposed to the bosonic character, which counts holomorphic sections. Of course, the result of the limit (A.5) reproduces precisely the supersymmetric Casimir energy in (5.6).

In order to compute the supersymmetric index using the plethystic exponential, it is convenient to write the twisted Lens space character as

$$C(\bar{\partial}_L, (p_1, p_2), \mathbb{C}^2/\mathbb{Z}_p) = \frac{p_1^\nu}{(1 - p_1 p_2)(1 - p_1^p)} + \frac{p_2^{p-\nu}}{(1 - p_1 p_2)(1 - p_2^p)} . \quad (\text{A.20})$$

Using this, it is immediate to obtain the index in the factorised form [24], namely

$$\mathcal{I}_{p,\nu}^{\text{matter}}(p_1, p_2) = \Gamma((p_1 p_2)^{\frac{r}{2}} p_2^{p-\nu}; p_2^p, p_1 p_2) \Gamma((p_1 p_2)^{\frac{r}{2}} p_1^\nu; p_1^p, p_1 p_2) , \quad (\text{A.21})$$

where notice that this does not contain any Casimir energy contribution.

The reasoning that led to the expression of the single particle index above should be valid more generally for a theory defined on $M_4 = \mathbb{R} \times M_3$ (where $\pi_1(M_3)$ is finite), with a fixed flat connection in a sector $\alpha \in \mathcal{M}_{\text{flat}}$. In particular, we expect that this is always given by

$$f^{\text{matter}}(p_1, p_2) = (p_1 p_2)^{\frac{r}{2}} C(\bar{\partial}_\alpha, (p_1, p_2), M_4) - (p_1 p_2)^{-\frac{r}{2}} C(\bar{\partial}_\alpha, (p_1^{-1}, p_2^{-1}), M_4). \quad (\text{A.22})$$

However, we will not pursue this direction further here. To illustrate our prescription, below we will derive expressions for the (chiral multiplet contribution to the) supersymmetric index in the class of homogeneous hypersurface singularities, in the sector without flat connection.

As before, to evaluate the bosonic single letter partition function, we can start from the theory on $\mathbb{R} \times S^3$, and evaluate the sums as in (A.15) by projecting out the modes not invariant under $\Gamma \subset \text{SU}(2)$. This is equivalent to counting holomorphic functions on \mathbb{C}^2 that are invariant under Γ . For $\Gamma = \mathbb{Z}_p$ this is of course the case of the Lens space, yielding (A.20). Let us then discuss the remaining D and E singularities. Implementing the projection on the modes, we find

$$f_{\text{bos}}^{DE}(x) = x^r \frac{1 - x^{2d}}{(1 - x^{2w_1})(1 - x^{2w_2})(1 - x^{2w_3})}, \quad (\text{A.23})$$

where the weights and degrees of the singularities can be read off from the defining equations given in (2.30). For example, for the E_8 singularity, corresponding to the Poincaré Hopf surface, the (minimal) set of weights is $(w_1, w_2, w_3) = (10, 6, 15)$, with degree $d = 30$. For the D_{p+1} series the weights are $(w_1, w_2, w_3) = (2, p - 1, p)$ and the degree is $d = 2p$. Notice that in all cases the series expansion of (A.23) does not contain odd powers of x . This is because for $\Gamma = \Gamma_D$ and $\Gamma = \Gamma_E$, $\Gamma \supset \mathbb{Z}_2$, where this acts as $\mathbb{Z}_2 : (z_1, z_2) \rightarrow -(z_1, z_2)$.

Changing variable setting $x = q^{1/2}$, we indeed find that

$$f_{\text{bos}}^{DE}(q) = q^{r/2} \frac{1 - q^d}{(1 - q^{w_1})(1 - q^{w_2})(1 - q^{w_3})} = q^{r/2} C(\bar{\partial}, q, \mathbb{C}^2/\Gamma). \quad (\text{A.24})$$

Moreover, using that

$$w_1 + w_2 + w_3 - d = 1, \quad (\text{A.25})$$

we compute

$$f_{\text{fer}}^{DE}(q) = f_{\text{bos}}^{DE}(q^{-1}) = q^{(2-r)/2} C(\bar{\partial}, q, \mathbb{C}^2/\Gamma). \quad (\text{A.26})$$

Thus the single particle index for the chiral multiplet reads

$$f_{DE}^{\text{matter}}(q) = \frac{(q^{r/2} - q^{(2-r)/2})(1 - q^d)}{(1 - q^{w_1})(1 - q^{w_2})(1 - q^{w_3})}, \quad (\text{A.27})$$

and taking the plethystic exponential it results in the following *triple* infinite products

$$\mathcal{I}_{DE}^{\text{matter}}(q) = \frac{\prod_{n_1, n_2, n_3 \geq 0}^\infty (1 - q^{1-r/2} q^{n_1 w_1 + n_2 w_2 + n_3 w_3}) (1 - q^{r/2+d} q^{n_1 w_1 + n_2 w_2 + n_3 w_3})}{\prod_{n_1, n_2, n_3 \geq 0}^\infty (1 - q^{r/2} q^{n_1 w_1 + n_2 w_2 + n_3 w_3}) (1 - q^{1-r/2+d} q^{n_1 w_1 + n_2 w_2 + n_3 w_3})}. \quad (\text{A.28})$$

Notice that this cannot be expressed in term of the ordinary elliptic gamma functions. However, interestingly, using the condition (A.25), valid for the D and E singularities, we find that this can be written as

$$\mathcal{I}_{DE}^{\text{matter}}(q) = \frac{\Gamma(q^{r/2+d}; q^{w_1}, q^{w_2}, q^{w_3})}{\Gamma(q^{r/2}; q^{w_1}, q^{w_2}, q^{w_3})}, \quad (\text{A.29})$$

where

$$\Gamma(z; q_1, q_2, q_3) = \prod_{n_1, n_2, n_3 \geq 0}^{\infty} (1 - z^{-1} q_1^{n_1+1} q_2^{n_2+1} q_3^{n_3+1}) (1 - z q_1^{n_1} q_2^{n_2} q_3^{n_3}) \quad (\text{A.30})$$

is a generalization of the elliptic gamma function [26, 27].

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