Modeling learning and teaching interaction by a map with vanishing denominators: Fixed points stability and bifurcations

Published version:

DOI:10.1016/j.chaos.2019.06.008

Manuscript Number: CHAOS-D-19-00023R2

Title: Modeling learning and teaching interaction by a map with vanishing denominators: Fixed points stability and bifurcations

Article Type: VSI: DyMoEcSoSc (MDEF18)

Keywords: maps with vanishing denominator
focal points
developmental psychology
learning and teaching coupling

Corresponding Author: Professor Ugo Merlone, PhD

Corresponding Author's Institution: University of Torino

First Author: Ugo Merlone, PhD

Order of Authors: Ugo Merlone, PhD; Anastasiia Panchuk, Ph.D; Paul van Geer, Ph.D

Manuscript Region of Origin: ITALY

Abstract: Among the different dynamical systems which have been considered in psychology, those modeling the dynamics of learning and teaching interaction are particularly important. In this paper we consider a well known model of proximal development and analyze some of its mathematical properties. The dynamical system we study belongs to a class of 2D noninvertible piecewise smooth maps characterized by vanishing denominators in both components. We determine focal points, among which the origin is particular since its prefocal set contains this point itself.
We also find fixed points of the map and investigate their stability properties. Finally, we consider map dynamics for two sample parameter sets, providing plots of basins of attraction for coexisting attractors in the phase plane. We emphasize that in the first example there exists a set of initial conditions of non-zero measure, whose orbits asymptotically approach the focal point at the origin.

Dear Editors,

Please find the final version of the paper with all the figures.

Kind Regards

Ugo Merlone, Anastasiia Panchuk and Paul van Geert

***Response to Reviewers**

Dear Reviewers,

Please find the final version of the paper with all the figures.

Kind Regards

Ugo Merlone, Anastasiia Panchuk and Paul van Geert

Modeling learning and teaching interaction by a map with vanishing denominators: Fixed points stability and bifurcations

Ugo Merlone[a,*], Anastasiia Panchuk[b], Paul van Geert[c]

[a] *Department of Psychology, Center for Cognitive Science, via Verdi 10,*
University of Torino, Torino, Italy, I 10124
[b] *Institute of Mathematics, NAS of Ukraine, 3 Tereshchenkivska str., Kiev, 01601, Ukraine*
[c] *University of Groningen Heymans Institute for Psychological Research, Grote Kruisstraat*
2/1, 9712 TS Groningen, Netherlands.

Abstract

Among the different dynamical systems which have been considered in psychology, those modeling the dynamics of learning and teaching interaction are particularly important. In this paper we consider a well known model of proximal development and analyze some of its mathematical properties. The dynamical system we study belongs to a class of 2D noninvertible piecewise smooth maps characterized by vanishing denominators in both components. We determine focal points, among which the origin is particular since its prefocal set contains this point itself. We also find fixed points of the map and investigate their stability properties. Finally, we consider map dynamics for two sample parameter sets, providing plots of basins of attraction for coexisting attractors in the phase plane. We emphasize that in the first example there exists a set of initial conditions of non-zero measure, whose orbits asymptotically approach the focal point at the origin.

Keywords: maps with vanishing denominator, focal points, developmental psychology, learning and teaching coupling

---

[☆] Fully documented templates are available in the elsarticle package on CTAN.

[*] Corresponding author

Email addresses: `ugo.merlone@unito.it` (Ugo Merlone), `anastasiia.panchuk@gmail.com` (Anastasiia Panchuk), `p.l.c.van.geert@ppsw.rug.nl` (Paul van Geert)

2010 MSC: 37G10, 37N99, 91E99

---

1. Introduction

The application of dynamical systems in the social and behavioral sciences [1], developmental psychology [2] although being a relatively new approach, has provided interesting contributions. In particular, a promising line of research has examined changing interaction between the learner (the child or student) and the helper (the teacher or tutor). In fact, provided that an adult or more competent peer has given a particular form of help, guidance or collaboration, and that a certain amount of change has occurred in the learner's actual level (e. g., it has moved a little bit towards the objective or goal level in terms of his independent performance), the next help, guidance or assistance given must reckon with this change, because they must be adaptive to the changed actual level of the learner. Hence, the level of help that lead to optimal change in the learner, must be a different one than the preceding level of help. And this means that change not only occurs in the learner, but that it also occurs in the helper, and that the helper must be capable of adequately adapting the level of help according to the actual level of learner. That is to say, there is not only developmental or learning change in the person receiving help, but there is also change in the level of the help given. Help that exceeds the current capabilities of learning and understanding of the learner or that remains too close to the learner's current level of independent performance, will greatly hamper learning or development. The helper must therefore find the level of help, relative to the learner's current level of independent performance, that results in maximal learning given the learner's possibilities. In this sense, the process of socially mediated learning is a process of *co-adaptation* [3].

It is quite natural to formalize developmental processes as dynamical systems [4, 5] given the importance of time in any psychological process. As a matter of fact, important pioneers in Mathematical Psychology claimed that "[t]he observation that psychological processes occur in time is trite" in [6, p.231].

In this paper we consider a version of the model of proximal development presented in [4, 7, 3, 8]. This model is inspired by ideas and principles of L. S. Vygotsky [9], in particular, his well-known *zone of proximal development.* By definition this zone represents the range between a learner's performance on his/her *actual developmental level* (where the learner can do without dedicated help) and the level of the learner's performance under conditions of adequate help from the teacher, referred to as *potential developmental level.* Another keynote concept forming the basis of the model studied below is *the principle of scaffolding.* The widespread use of this term started with an article [10], in which the authors presented a model of effective helping that was consistent with the Vygotskyan approach, although the article makes no mention of Vygotsky's work. The main idea of the scaffolding principle is as follows: only those forms of help or assistance that the learner can *understand as being functional* are actually effective in causing learning to occur (see [7] for details and a dynamic model). Both mentioned approaches suggest that the crucial dynamic aspect of the learning process is existence of optimal distance between the learner's actual developmental level and the level of performance with help and assistance (potential developmental level). And this optimal distance results in an optimal learning effect under the current help and assistance given.

The resulting model is represented by a 2D noninvertible piecewise smooth map, both components of which have the form of a rational function. This implies that the map is not defined in the whole space possessing the *set of nondefinition* being the locus of points in which at least one denominator vanishes. Maps of such kind are called *maps with vanishing denominator* and have been extensively investigated by many researchers. See, for instance, the trilogy [11, 12, 13] and references therein, for a detailed description of peculiar properties of such maps, related to particular bifurcations and changes in structure of the phase space. One may also refer to [14, 15], where the authors survey several models coming from economics, biology and ecology defined by maps with vanishing denominator and investigate the global properties of their dynamics.

Two distinguishing concepts related to maps with vanishing denominator are notions of a *prefocal set* and a *focal point*. Roughly speaking a prefocal set is a locus of points that is mapped (or often said “is focalized”) into a single point (focal point) by one of the map inverses. In a certain sense, the focal point can be considered as the preimage of the prefocal set with using a particular inverse of the map. At the focal point at least one component of the map takes the form of uncertainty $0/0$, and hence, the focal point can be derived as a root of a 2D system of algebraic equations. If it is a simple root, the focal point is called *simple*.

Presence of focal points and prefocal curves has an important influence on the global dynamics of the map. There may occur certain global bifurcations related to contacts of prefocal sets with invariant sets (such as basin boundaries) or critical curves. Such bifurcations usually lead to qualitative changes in structure of attracting sets or basins of attraction. In particular, one may observe creation of basin structures specific to maps with denominator, called lobes and crescents, sometimes resembling feather fans centered at focal points.

For the map investigated in the current paper we determine focal points and their prefocal sets. We show that among three focal points only one is simple. Moreover, a focal point at the origin denoted SP_0 is rather particular, since its prefocal set coincides with the set of nondefinition including the point SP_0 itself. In a certain sense the focal point SP_0 plays a role similar to that of a fixed point of the map. After analyzing focal points, we examine fixed points of the map and derive analytic expressions for their computation. Some fixed points can be obtained in explicit form, while the others are identified by finding the roots of certain cubic equations. We also investigate stability properties of the fixed points and for some of them derive conditions for their stability in form of analytic expressions. Finally, we consider map dynamics for two sample parameter sets, providing plots of basins of attraction for coexisting attractors in the phase plane. Noteworthy, in one of the examples there exists a set of initial conditions of non-zero measure, whose orbits asymptotically approach the focal point at the origin.

The paper is organized as follows. Section 2 presents a brief description of the main concepts, the terms and the model. Section 3 concerns determining focal points and the associated prefocal sets. In Section 4 we discuss some preliminary analytical results concerning the map and find all possible fixed points. In Section 5 we study their stability and in Section 6 two numerical examples of map dynamics are provided. Section 7 concludes.

2. A Model of Learning and Teaching Coupling

When modeling an educational process one usually distinguishes three main objects involved: a person to be educated (a student), a person who imparts specific knowledge or skills (a teacher or tutor), and the final educational goal. Formally speaking, the educational goal can be considered as a stock of information and skills K, which is a real positive parameter (as shown below, it is not restrictive to fix $K = 1$). Moreover, the information can be ordered according to its level of intricacy and has to be expounded complying with this order. For instance, it is useless to explain methods for solving a system of linear equations to a person (e. g., a child) who does not have any idea about neither numbers nor arithmetic operations. The latter concepts have to be learned before mastering more complex things.

Formally speaking, a student can be also represented by a certain amount of knowledge (information and skills) A that he has already picked up, and that can be expressed in perspective to the educational goal to be attained, specified by the level K. Now, the process of learning can be considered as a flow from the goal stock, K, to the individual stock, A, that is, can be modeled by a dynamic equation over the variable A. The speed of knowledge assimilation or skill learning depends on a variety of different factors, for instance how much effort the student makes to learn, as well as on his individual flairs and abilities. However, it suffices to specify this speed or rate by a single parameter, without reference to the host of factors that form its psychological basis. Note that this is only a formal representation of the process, which is not intended to serve as some sort of picture of the psychological processes that take place. Depending on personal capabilities and actual developmental or learning level, A, the teacher must foresee what new information or which new performance the student can comprehend, that is, the teacher must foresee what the nature of the appropriate help will be, at any moment in the teaching-learning process. That is to say, the teacher continuously estimates the student's potential level of development, P. As the student is learning, i. e. is progressing towards the educational goal level represented by K, the teacher must adapt the complexity of the help and assistance given, which in practice means that the level of help and assistance is progressively coming closer to K. The rate with which the teacher adapts this level of help and assistance given, contingent upon any progress in the student's learning, i. e. contingent upon any change in the level A, is a teacher-specific parameter.

According to [16] the dynamical system approach has emerged as one of the most prevalent and dominant in developmental psychology both in terms of the number of proponents and volume of direct empirical tests. In particular, the interaction between the actual and potential developmental levels has been modeled in [17, 4] as a two-dimensional map and has inspired several other contributions: for example, see [18] for a dynamical system studying second language acquisition and [19, 20] for empirical analyses of mediated learning experience and the role of zone of proximal development in terms of peer interaction, respectively. However, despite its importance, an analysis of the mathematical properties of the model proposed in [4] is missing. Below we perform first steps towards understanding dynamics of the aforementioned model from theoretical viewpoint. For this we consider the two-dimensional map $\Phi : (A, P) \in \mathbb{R}^2 \to (A', P') \in \mathbb{R}^2$ defined by

$$\begin{cases} A' = A\left[1 + R_a(A,P)\left(1 - \dfrac{A}{P}\right)\right] \overset{\text{def}}{=} \Phi_1(A,P)\ , \\ P' = P\left[1 + R_p(A,P)\left(1 - \dfrac{P}{K}\right)\right] \overset{\text{def}}{=} \Phi_2(A,P)\ , \end{cases} \tag{1}$$

where functions $R_a(A, P)$ and $R_p(A, P)$ (change rates of the actual and the potential developmental levels, respectively) are given by

$$R_a(A,P) \overset{\text{def}}{=} R_a = r_a - \left|\frac{P}{A} - O_a\right| b_a \left(1 - \frac{A}{K}\right) , \tag{2a}$$

$$R_p(A,P) \overset{\text{def}}{=} R_p = r_p - \left(\frac{P}{A} - O_p\right) b_p \left(1 - \frac{P}{K}\right) . \tag{2b}$$

Here parameter $r_a > 0$ denotes the so-called *maximum individual rate of learning* that differs among students. The parameter $O_a > 0$ reflects the *optimal distance* (also individual for a student) between the actual, A, and the potential, P, developmental levels. If there is $P/A = O_a$, then the growth rate R_a attains its maximum (r_a) and the learning proceeds the fastest. The value b_a is a student-dependent damping/moderating parameter. For instance, with $b_a \ll 1$, even if the current ratio P/A differs essentially from the optimal distance O_a, it does not influence much the learning rate. On the contrary, for values $b_a > 1$ the student's degree of comprehension is rather sensitive to the deviation of P/A from its optimal.

As for the change rate R_p, the argument is different. The constant growth factor $r_p > 0$ corresponds to what one may call a 'default property' of a teacher (like teaching manner, training methods, character traits, etc.). The actual rate of change of P can be greater or smaller than this default (or habitual) value r_p. Indeed, optimum of R_p cannot be considered as only a teacher-specific property but is also influenced by the learner. Namely, the rate of change R_p is optimal if it guarantees that P/A equals O_a. Clearly, such an optimum cannot be defined uniquely and usually changes with changing A and/or P. The meaning of the remaining two parameters is as follows. The parameter $O_p > 0$ represents the teacher's *estimation* for the optimal value of the ratio P/A, and hence, also depends on him/her. In general, the value O_p may differ from O_a, but the closer they are, the more efficient the educational process is. And $b_p > 0$ is the damping/moderating parameter, whose influence is similar to that of b_a.

We remark that due to modulus function in the expression for R_a the map (1) is piecewise smooth.[1] Hence, the phase space is divided into two regions;

---

[1] For the detailed overview of piecewise smooth maps occurring in different applications

namely, D_+ for that $P/A > O_a$ and D_- for that $P/A < O_a$ (see Figs. 1). The lines $P = O_a A$ and $A = 0$ constitute the *switching set*. Recall that the *switching set* is a locus of points where the map changes its definition, that is, on either side of the switching set the map is defined by different functions.

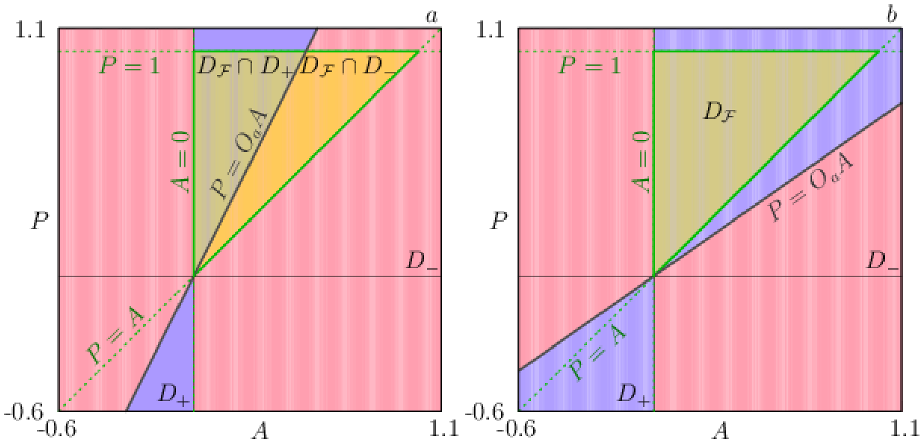

Figure 1: Schematic representation of the phase space (A, P). Switching set given by $P = O_a A$ and $A = 0$ separates the phase space into regions D_- (pink) and D_+ (blue). Green line marks the boundaries of the feasible domain $D_{\mathcal{F}}$. For $O_a > 1$ as in (a) the feasible domain $D_{\mathcal{F}}$ has intersections with both D_- and D_+. For $O_a \leq 1$ as in (b) $D_{\mathcal{F}} \subset D_+$.

Let us consider for sake of shortness the set of all parameters as a point in a 7-dimensional space

$$\mu = (r_a, r_p, b_a, b_p, O_a, O_p, K) \in \mathbb{R}^7_+ \tag{3}$$

with $\mathbb{R}_+$ denoting the positive semi-axis of real numbers. For a certain representative of the map family (1) we then use the notation Φ_μ.

Recall that from the application viewpoint, A is the actual developmental level of the student, P is the potential developmental level, and K is the final educational goal. It follows that the inequalities

$$A \leq K, \quad P \leq K, \quad A \leq P \tag{4}$$

---

and associated dynamical peculiarities see, for instance, [21, 22] and references therein.

confine the feasible domain $D_{\mathcal{F}}$ (outlined green in Figs. 1) for the states of the system (1). The boundary of $D_{\mathcal{F}}$ is denoted $\partial D_{\mathcal{F}}$. Notice that if $O_a > 1$, then the feasible domain $D_{\mathcal{F}}$ is divided into two parts, that is, $D_{\mathcal{F}} = (D_{\mathcal{F}} \cap D_-) \cup (D_{\mathcal{F}} \cap D_+)$ (see Fig. 1a). Otherwise, it is completely contained inside D_+ (see Fig. 1b).

The domain $D_{\mathcal{F}}$ constitutes quite a limited area in the $\mathbb{R}^2$ space, and moreover, $D_{\mathcal{F}}$ is not invariant under Φ_μ. It is important then to distinguish between feasible orbits, which completely belong to $D_{\mathcal{F}}$, and nonfeasible ones, which eventually leave the feasible domain. Although from applied context we have to restrict our studies to the orbits located completely inside $D_{\mathcal{F}}$, we consider larger part of the phase space. The main reason is that, in general, dynamic phenomena occurring outside $D_{\mathcal{F}}$ may influence also the feasible part of the phase space. For example, suppose that some homoclinic bifurcation occurs outside $D_{\mathcal{F}}$ and this changes the complete structure of basins, including those related to attractors belonging to $D_{\mathcal{F}}$. In other words, considering orbits that are located outside $D_{\mathcal{F}}$ may shed light on the feasible dynamics of map (1). And this way we also obtain a better understanding of the map dynamics in cases in which some of the conditions in (4) are relaxed.

It seems that conditions (4) must always hold in reality, though in some cases their violation can be explained in applied context. Let us suppose, for instance, that $A > P$. It means that the actual student's developmental level is greater than the potential developmental level estimated by the teacher, that is, the student already knows what he is expected to learn. Generally speaking, in the real learning process this may happen. For instance, if the current level of the student's knowledge is evaluated incorrectly. As a matter of fact, this may happen when evaluating gifted-children, as the definition of giftedness has a multifaced-nature [23] and identification process is not immediate [24] and often poses some problems [25]. In such cases the potential level P has to be updated accordingly (so that $P > A$ is restored) before the student gets bored by the training. With respect to dynamics of (1), it means that transient states are allowed to fall below the line $P = A$, but eventually an orbit must come back in the interior of $D_{\mathcal{F}}$ and stay there forever. Similarly, violation of other inequalities in (4) may be the result of incorrect decisions made by the teacher. One may certainly argue that a qualified and experienced teacher will never put the estimated level P greater than the final educational goal K. However, reality suggests that not all teachers are qualified or experienced enough, and hence, it may happen that $P > K$. As for $A > K$, it may mean that the student is rather smart. Theoretically, in such cases the learning process has to be stopped, since the final goal has been achieved. Though in reality it might not happen, as it is well known that evaluating and measuring the potential of a student, as well as his/her actual mastering level, is a complex task that involves using several assessment tools [26, 27].

An alternative interpretation of the inequalities inverse to (4), namely, $A > P$, $P > K$, $A > K$, is that they represent a case where a person has to unlearn something, for instance, a bad or unhealthy or unwanted habit. This is a sort of situation we find as typical clinical settings, or clinical-educational settings, such as children who are overly aggressive, where the goal is to reduce the level of aggressiveness to normal proportions.

In the following, as parameter K denotes the final educational goal represented by the stock of information and skills, it is not restrictive to normalize K to unity (or assume any other positive value). Mathematically it can be achieved by showing topological conjugacy between any two maps from the family (1), Φ_{μ_1} and Φ_{μ_2}, with two different values K_1 and K_2, respectively, and the other parameters being identical. The related homeomorphism is given by

$$h(A, P) = \left(\frac{K_1}{K_2} A, \frac{K_1}{K_2} P \right),$$

so that

$$\Phi_{\mu_1} \circ h = h \circ \Phi_{\mu_2}.$$

Without loss of generality we can assume that the set of parameters belongs to the six-dimensional hyperplane $\mu \in \mathbb{R}^6_+ \times \{K = 1\}$.

3. Focal Points

As has been already mentioned in the Introduction, one of the particular characteristics of the map Φ_μ is that both its components assume the form of a rational function. Indeed, (1) can be rewritten in the following form:

$$A' = \frac{N_1(A,P)}{D_1(A,P)} = \frac{A(|A|P + (r_a|A| - |O_aA - P|b_a(1-A))(P-A))}{|A|P}\ , \tag{5a}$$

$$P' = \frac{N_2(A,P)}{D_2(A,P)} = \frac{P(A + (r_pA - (P - O_pA)b_p(1-P))(1-P))}{A}\ . \tag{5b}$$

Clearly, at points belonging to the set $\delta_s \overset{\text{def}}{=} \{(A,P) : A = 0\} \cup \{(A,P) : P = 0\}$, at least one of the denominators $D_1(A,P)$ or $D_2(A,P)$ vanishes. Hence, the set δ_s represents the *set of nondefinition* of Φ_μ. Maps of similar kind are called *maps with vanishing denominator* and have been studied by many researchers (see, e. g., [11, 12, 13, 14, 15] to cite a few). Particular feature of such maps is possibility of having *focal points* and associated *prefocal sets/curves.* Due to contact between phase curves and these prefocal sets or a set of nondefinition, certain bifurcations can occur, which are peculiar for maps with denominator.

Recall that a point $Q(A_0, P_0)$ is called a *focal point* if

(i) at least one component of Φ_μ takes the form of uncertainty zero over zero at Q, that is, $N_i(A_0,P_0) = D_i(A_0,P_0) = 0$ for $i = 1$ or $i = 2$;

(ii) there exist smooth simple arcs $\gamma(\tau)$ with $\gamma(0) = Q$ such that $\lim_{\tau\to 0} \Phi_\mu(\gamma(\tau))$ is finite.

The set of all such finite values, obtained by taking different arcs $\gamma(\tau)$ through Q, is called the *prefocal set* δ_Q. Note that not every point at which Φ_μ takes the form $0/0$ is a focal point.

Suppose that $\Phi_i(A,P)$, $i = 1,2$, takes the form $0/0$ at the focal point Q. The point Q is called *simple* if $N_{iA}D_{iP} - N_{iP}D_{iA} \neq 0$, where N_{iA}, N_{iP}, D_{iA} and D_{iP} are the respective partial derivatives over A and P. Otherwise, Q is called *nonsimple.*

For any smooth simple arc $\gamma(\tau) = (\gamma_1(\tau), \gamma_2(\tau))$ its both components can be represented as Taylor series:

$$\gamma_1(\tau) = \xi_0 + \xi_1\tau + \xi_2\tau^2 + \dots, \tag{6a}$$

$$\gamma_2(\tau) = \eta_0 + \eta_1\tau + \eta_2\tau^2 + \dots \tag{6b}$$

If a focal point is simple, then there exists a one-to-one correspondence between the slope $m = \eta_1/\xi_1$ of a curve $\gamma(\tau)$ at this focal point and the limit point $\lim_{\tau\to 0} \Phi_\mu(\gamma(\tau))$. In case of a nonsimple focal point this generically does not hold.

At first, we consider the points with $A = 0$ and arbitrary P and consider arcs $\gamma(\tau)$ through this point implying $\xi_0 = 0$, $\eta_0 = P$. The function $\Phi_1(0, P)$ assumes uncertainty $0/0$, while $\Phi_2(0, P) = -P^2 b_p(1 - P)^2/0$. If $P \neq 0, 1$, the limit of $\Phi_\mu(\gamma(\tau))$ with $\tau \to 0$ is $(-b_a P \mathrm{sgn}(P), \infty)$, where ∞ means either $+\infty$ or $-\infty$ depending on whether limit is taken from the left or from the right, respectively. Hence, the point $(0, P)$, $P \neq 0, 1$, is not a focal point.

Let us check whether $SP_0 = SP_0(0, 0)$ and $SP_1 = SP_1(0, 1)$ are the focal points. Note that now also the function $\Phi_2(0, P)$ assumes uncertainty $0/0$. For SP_0, clearly, $\xi_0 = \eta_0 = 0$. First, we suppose that $\xi_1 \neq 0$ and $\eta_1 \neq 0$. The limit is then $\lim_{\tau\to 0} \Phi_\mu(\gamma(\tau)) = (0, 0)$ regardless of the arc $\gamma(\tau)$. It means that the focal point SP_0 belongs to its prefocal set δ_{SP_0}. It also implies that whatever is the slope $m = \eta_1/\xi_1$ of $\gamma(\tau)$ at SP_0, the image $\Phi_\mu(\gamma(\tau))$ always intersects δ_{SP_0} at the same point, namely, SP_0 itself. In a certain sense the focal point SP_0 plays a role similar to that of a fixed point of Φ_μ. However, the set δ_{SP_0} contains also other points. Indeed, if we put $\xi_1 = 0$, $\eta_1 \neq 0$ then

$$\lim_{\tau\to 0} \Phi_\mu(\gamma(\tau)) = \left(0, -\frac{\eta_1^2 b_p}{\xi_2}\right),$$

while if $\eta_1 = 0$, $\xi_1 \neq 0$ then

$$\lim_{\tau\to 0} \Phi_\mu(\gamma(\tau)) = \left(\frac{\pm\xi_1^2(r_a \pm O_a b_a)}{\eta_2}, 0\right),$$

where '$+$' and '$-$' are chosen depending on the signs of A and $(P - O_a A)$. Hence, prefocal set

$$\delta_{SP_0} = \{(A, P)\ :\ A = 0\} \cup \{(A, P)\ :\ P = 0\},$$

which coincides with the set of nondefinition δ_s. Note that, $N_{iA} = N_{iP} = D_{iP} = D_{1A} = 0$, $i = 1, 2$, $D_{2A} = 1$, and therefore, the focal point SP_0 is nonsimple.

Similarly, we get that the prefocal set of SP_1 is

$$\delta_{SP_1} = \{(A, P) \,:\, A = -b_a\}.$$

For SP_1 there holds $N_{iP} = D_{iP} = 0$, $i = 1, 2$, and this focal point is nonsimple as well.

Finally, $\Phi_1(A, P)$ also assumes uncertainty $0/0$, if $A = 1 - r_a/(O_a b_a)$ and $P = 0$, while $\Phi_2(A, P)$ is finite. The prefocal set of the focal point $SP_a = SP_a(1 - r_a/(O_a b_a), 0)$ is the line

$$\delta_{SP_a} = \{(A, P) \,:\, P = 0\} \subset \delta_s.$$

The point SP_a is simple provided that $r_a \neq O_a b_a$. If $r_a = O_a b_a$ then $SP_a \equiv SP_0$. The point SP_a belongs to its prefocal set δ_{SP_a}, similarly to SP_0. However, there exists only one slope $m = \eta_1/\xi_1$ for which the image $\Phi_\mu(\gamma(\tau))$ intersects δ_{SP_a} at SP_a, since SP_a is simple.

4. Fixed Points

The system equations are not only polynomials of the variables A and P but the latter also appear in denominators, therefore evaluating fixed points seems not so trivial at the first sight. Fixed points can be defined by solving the following equations:

$$\begin{cases} A = A\left(1 + R_a \cdot \left(1 - \dfrac{A}{P}\right)\right), \\ P = P\left(1 + R_p \cdot (1 - P)\right). \end{cases} \tag{7}$$

This is equivalent to

$$\begin{cases} f_1(A, P) \overset{\text{def}}{=} AR_a \cdot \left(1 - \dfrac{A}{P}\right) = 0\,, & \text{(8a)} \\ f_2(A, P) \overset{\text{def}}{=} PR_p \cdot (1 - P) = 0\,. & \text{(8b)} \end{cases}$$

Each of the equations (8) defines a geometrical locus of points in the (A, P)-plane. Every intersection of the two loci of points is a (potential) fixed point of (1). We use the word 'potential' here because some of intersections may correspond to focal points, as for instance, the point $SP_0(0, 0)$.

4.1. Locus of points $f_1(A, P) = 0$

From (8a) the function f_1 of the two variables A and P equals zero when one of the following holds:

$$P = A, \quad AR_a(A, P) = 0. \tag{9}$$

The values $A = 0$ are omitted since they correspond to the set of nondefinition δ_s as seen above. Let us solve the remaining equation $R_a(A, P) = 0$. Expanding the modulus we get two different equations:

$$\frac{P}{A} - O_a = \frac{r_a}{b_a(1-A)} \quad \text{and} \quad \frac{P}{A} - O_a = -\frac{r_a}{b_a(1-A)},$$

where one has to require $A < 1$. This implies the following two functions

$$P = \frac{-r_a - O_a b_a + b_a O_a A}{b_a(1-A)} A = \frac{A^2 - B_{\mathrm{I}} A}{A-1} O_a \overset{def}{=} P_{\mathrm{I}}(A), \tag{10a}$$

$$P = \frac{-r_a + O_a b_a - b_a O_a A}{b_a(1-A)} A = \frac{A^2 - B_{\mathrm{II}} A}{A-1} O_a \overset{def}{=} P_{\mathrm{II}}(A) \tag{10b}$$

with

$$B_{\mathrm{I}} = 1 + \frac{r_a}{O_a b_a}, \quad B_{\mathrm{II}} = 1 - \frac{r_a}{O_a b_a}. \tag{11}$$

In general, both equations (10a) and (10b) define curves in the (A, P)-plane consisting of two branches each (one for $A < 1$ and the other for $A > 1$): $P_{\mathrm{I}}^{\mathrm{L}}, P_{\mathrm{I}}^{\mathrm{R}}$ and $P_{\mathrm{II}}^{\mathrm{L}}, P_{\mathrm{II}}^{\mathrm{R}}$ (see Fig. 2). However, only branches $P_{\mathrm{I}}^{\mathrm{L}}$ and $P_{\mathrm{II}}^{\mathrm{L}}$ reduce $R_a(A, P)$ to zero.

Note that the curve $P = P_{\mathrm{I}}^{\mathrm{L}}(A)$ is strictly increasing and have two asymptotes: $A = 1$ and $P = O_a A - r_a/b_a$. As for $P = P_{\mathrm{II}}^{\mathrm{L}}(A)$, it has a local maximum at

$$A = 1 - \sqrt{\frac{r_a}{b_a O_a}} \overset{\mathrm{def}}{=} A_{\mathrm{II}}^{\max}, \quad P_{\mathrm{II}}(A_{\mathrm{II}}^{\max}) = O_a \cdot (A_{\mathrm{II}}^{\max})^2. \tag{12}$$

Obviously, $A_{\mathrm{II}}^{\max} < 1$ for any parameter values. Additionally, if $r_a < b_a O_a$ then $A_{\mathrm{II}}^{\max} > 0$, otherwise $A_{\mathrm{II}}^{\max} < 0$. The function $P = P_{\mathrm{II}}^{\mathrm{L}}(A)$ also has two asymptotes: $A = 1$ and $P = O_a A + r_a/b_a$.

For sake of shortness, we omit the upper indices $^{\mathrm{L}}$ writing simply $P_{\mathrm{I}}(A)$ and $P_{\mathrm{II}}(A)$, except for the cases where it is necessary to distinguish between the two different branches.

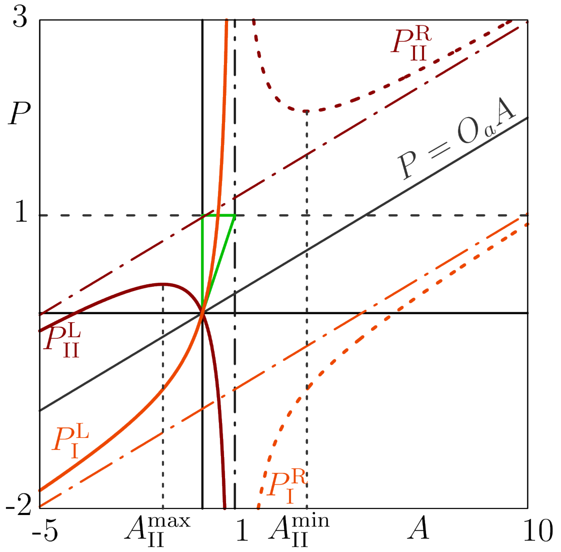

Figure 2: The functions $P = P_{\mathrm{I}}(A)$ and $P = P_{\mathrm{II}}(A)$. The branches $P_{\mathrm{I}}^{\mathrm{L}}$ (solid orange curve) and $P_{\mathrm{II}}^{\mathrm{L}}$ (solid red curve) reduce $R_a(A, P)$ to zero, while the branches $P_{\mathrm{I}}^{\mathrm{R}}$ and $P_{\mathrm{II}}^{\mathrm{R}}$ (dashed curves of respective colors) do not. Green line marks the feasible domain $D_{\mathcal{F}}$. The parameters are $r_a = 0.098$, $r_p = 0.09$, $b_a = b_p = 0.1$, $O_a = 0.2$, $O_p = 0.11$.

4.2. Locus of points $f_2(A, P) = 0$

From (8b) the function f_2 equals zero when one of the following holds:

$$P = 0, \quad P = 1, \quad R_p(A, P) = 0, \tag{13}$$

where the first line $P = 0$ belongs to the set of nondefinition δ_s as discussed above. The last equation of (13) is equivalent to

$$P = \frac{1 + O_p A \pm \sqrt{(1 - O_p A)^2 - 4A\frac{r_p}{b_p}}}{2} \overset{\text{def}}{=} P_{\pm}(A), \quad A \neq 0. \tag{14}$$

Notice that the curves $P_{\pm}(A)$ are defined only for those values of A which guarantee positive discriminant

$$(1 - O_p A)^2 - 4A\frac{r_p}{b_p} \geq 0.$$

Solving this inequality gives

$$A < A_{\text{lim}}^{\text{L}} \quad \text{or} \quad A > A_{\text{lim}}^{\text{R}}$$

with

$$A_{\text{lim}}^{\text{L}} = \frac{b_p O_p + 2r_p - 2\sqrt{b_p O_p r_p + r_p^2}}{b_p O_p^2}, \tag{15a}$$

$$A_{\text{lim}}^{\text{R}} = \frac{b_p O_p + 2r_p + 2\sqrt{b_p O_p r_p + r_p^2}}{b_p O_p^2}. \tag{15b}$$

Both $A_{\text{lim}}^{\text{L}}$, $A_{\text{lim}}^{\text{R}}$ are always positive and may be less or greater than one.

Further, each curve $P_-(A)$ and $P_+(A)$ consists of two branches, one defined for $A \leq A_{\text{lim}}^{\text{L}}$ (denoted $P_-^{\text{L}}(A)$ and $P_+^{\text{L}}(A)$, resp.) and the other for $A \geq A_{\text{lim}}^{\text{R}}$ ($P_-^{\text{R}}(A)$ and $P_+^{\text{R}}(A)$, resp.). Both curves have also two asymptotes (see Fig. 3):

$$\mathcal{L}_1 = \left\{(A, P) \ : \ P = 1 + \frac{r_p}{b_p O_p}\right\}, \tag{16}$$

$$\mathcal{L}_2 = \left\{(A, P) \ : \ P = O_p A - \frac{r_p}{b_p O_p}\right\}. \tag{17}$$

4.3. Intersection of the two loci

Finally we find the fixed points of the map Φ_μ as intersections of $f_1(A, P) = 0$ (8a) and $f_2(A, P) = 0$ (8b). Figures 4 show the (A, P)-plane with the two corresponding geometrical loci of points. The curves along each of that f_1 becomes zero are plotted dark-red, while the branches reducing f_2 to zero are plotted blue. Left and right panels show different parameter sets.

As one can deduce from the figure, the branches of $f_1(A, P) = 0$ and $f_2(A, P) = 0$ cross at several points, whose number may change depending on the parameter values. And they always intersect at the point $SP_0(0, 0)$, which is a focal point.

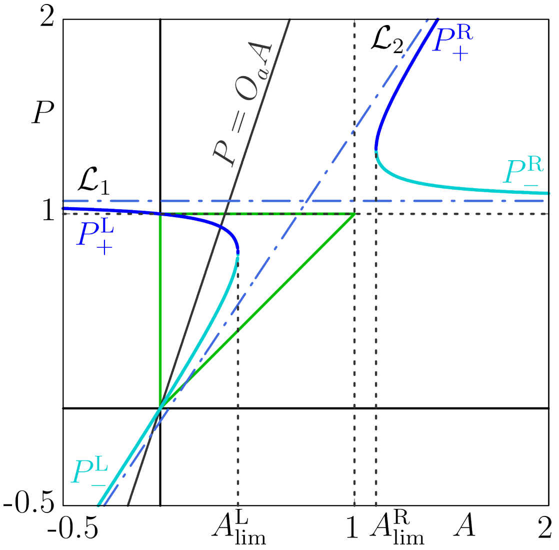

Figure 3: The functions $P = P_-(A)$ (light-blue curve) and $P = P_+(A)$ (dark-blue curve) and their asymptotes $\mathcal{L}_1$ and $\mathcal{L}_2$ (dash-dot lines). Green line marks the feasible domain $D_{\mathcal{F}}$. The parameters are $r_a = 0.03$, $r_p = 0.01$, $b_a = b_p = 0.1$, $O_a = 1.5$, $O_p = 3$.

The detailed analysis of different intersections is reported in Appendix A.

We can see that, the map Φ_μ can have from 2 to 11 coexisting fixed points. Namely, the two points $FP_1(1,1) = \{P = A\} \cap \{P \equiv 1\}$ (the application target fixed point) and $FP_2(A_{\mathrm{I},1}^-, 1) = \{P = P_{\mathrm{I}}(A)\} \cap \{P \equiv 1\}$ (with $A_{\mathrm{I},1}^-$ given in (A.2)) always exist, the point $FP_5(A_{\mathrm{d}}, A_{\mathrm{d}}) = \{P = A\} \cap \{P = P_\pm(A)\}$ (with A_{d} defined in (A.11)) exists for almost any parameter values except for the set of measure zero given in (A.24). The pair $FP_3(A_{\mathrm{II},1}^-, 1)$ and $FP_4(A_{\mathrm{II},1}^+, 1)$ (with $A_{\mathrm{II},1}^-$, $A_{\mathrm{II},1}^+$ given in (A.5)), being the intersection of $P = P_{\mathrm{II}}(A)$ and $P \equiv 1$, appears due to the fold bifurcation at $r_a = b_a \left(1 - \sqrt{O_a}\right)^2$ if $O_a > 1$ and exists for $r_a < b_a \left(1 - \sqrt{O_a}\right)^2$. Finally, existence of the triples FP_6, FP_7, FP_8 (intersections of $P = P_{\mathrm{I}}(A)$ with $P = P_\pm(A)$) and FP_9, FP_{10}, FP_{11} (intersections of $P = P_{\mathrm{II}}(A)$ with $P = P_\pm(A)$) depends on the sign of discriminant of the related cubic equation (see Appendix A, Eqs. (A.16) and (A.17), (A.20)) and whether the roots of this equation are less or greater than one.

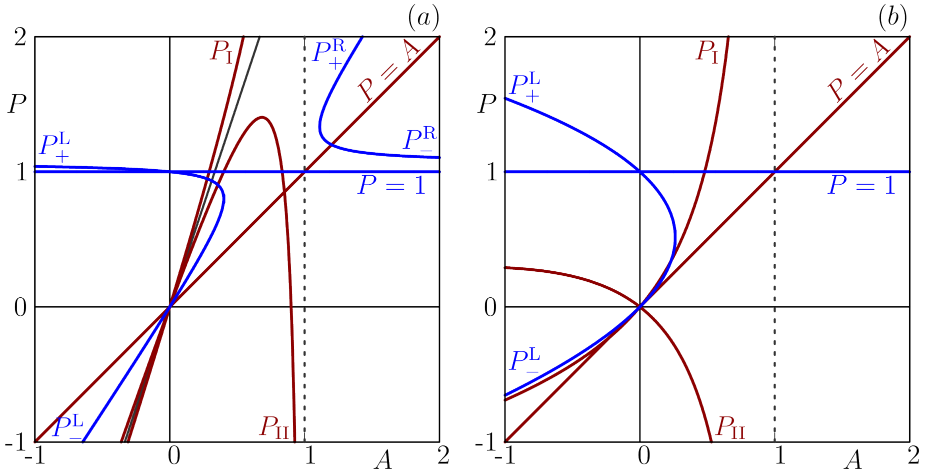

Figure 4: Loci of points reducing $f_1(A, P)$ (dark-red) and $f_2(A, P)$ (blue) to zero. The parameters are (a) $r_a = 0.03$, $r_p = 0.01$, $b_a = b_p = 0.1$, $O_a = 3$, $O_p = 1.5$; (b) $r_a = 0.098$, $r_p = 0.09$, $b_a = b_p = 0.1$, $O_a = 0.2$, $O_p = 0.11$.

5. Fixed Points Stability

Since the map Φ_μ is piecewise smooth, the Jacobian matrix for an arbitrary point (A, P) is defined differently depending on whether $(A, P) \in D_-$ $(P/A < O_a)$ or $(A, P) \in D_+$ $(P/A > O_a)$. However, in particular cases these two matrices coincide. 300

5.1. FP_1

The Jacobian matrix for the fixed point FP_1 is defined as

$$J(FP_1) = \begin{pmatrix} 1 - r_a & r_a \\ 0 & 1 - r_p \end{pmatrix} \tag{18}$$

regardless of whether $FP_1 \in D_-$ or $FP_1 \in D_+$ (which depends on O_a). Eigenvalues of $J(FP_1)$ are $\nu_1(FP_1) = 1 - r_a$ and $\nu_2(FP_1) = 1 - r_p$. The corresponding eigenvectors are $v_1 = (1, 0)$ and $v_2 = (r_a/(r_a - r_p), 1)$. Clearly, whenever 305 $0 < r_a, r_p < 2$, the point FP_1 is asymptotically stable. Both eigenvalues are real and r_a, r_p are strictly positive. Thus, the only bifurcation due to that FP_1 can lose its stability is the flip bifurcation (at $r_a = 2$ or $r_p = 2$).

We remark, that the singularity arises when $r_a = r_p$. In this case there is only one eigenvector v_1 related to the eigenvalue ν_1 of the multiplicity 2. This implies that if the fixed point FP_1 is stable, namely, $r_a \in (0, 2)$, then every orbit attracted to FP_1 is asymptotically tangent to the line $P = 1$ in the neighborhood of FP_1.

5.2. FP_2

The fixed point $FP_2(A^-_{\mathrm{I},1}, 1)$ is always located inside D_+, that is, $1/A^-_{\mathrm{I},1} > O_a$. Indeed,

$$1 = P_{\mathrm{I}}(A^-_{\mathrm{I},1}) = \frac{\left(A^-_{\mathrm{I},1}\right)^2 - B_{\mathrm{I}} A^-_{\mathrm{I},1}}{A^-_{\mathrm{I},1} - 1} O_a > O_a A^-_{\mathrm{I},1} \quad \Leftrightarrow$$

$$\left(A^-_{\mathrm{I},1}\right)^2 - B_{\mathrm{I}} A^-_{\mathrm{I},1} < \left(A^-_{\mathrm{I},1}\right)^2 - A^-_{\mathrm{I},1} \quad \Leftrightarrow \quad -\frac{r_a}{b_a O_a} A^-_{\mathrm{I},1} < 0$$

and the latter inequality is always true (recall that $0 < A^-_{\mathrm{I},1} < 1$). The related Jacobian matrix is then computed as

$$J(FP_2) = \begin{pmatrix} J_{11} & J_{12} \\ 0 & 1 - r_p \end{pmatrix}, \tag{19}$$

where

$$\begin{aligned} J_{11} &= \big(b_a(1 - O_a) + r_a\big) A^-_{\mathrm{I},1} - b_a(1 - O_a) + r_a + 1, \\ J_{12} &= -b_a O_a \left(A^-_{\mathrm{I},1}\right)^3 + (b_a O_a + r_a) \left(A^-_{\mathrm{I},1}\right)^2 + b_a A^-_{\mathrm{I},1} - b_a. \end{aligned} \tag{20}$$

The eigenvalues of FP_2 are

$$\nu_1(FP_2) = J_{11}, \quad \nu_2(FP_2) = 1 - r_p. \tag{21}$$

The related eigenvectors are

$$v_1 = (1, 0), \quad v_2 = \left(\frac{J_{12}}{1 - r_p - J_{11}}, 1\right). \tag{22}$$

Both eigenvalues of FP_2 are real and the second is also strictly less than one. Hence, the only possible bifurcation in the direction v_2 is the flip bifurcation (at $r_p = 2$). It can be further shown that the other eigenvalue is always $\nu_1 = J_{11} >$

1. Hence, the point FP_2 is either the saddle or the unstable node. If it is the saddle, then it becomes the unstable node when $r_p = 2$ giving rise to a saddle 2-cycle with one point located above the line $P = 1$ and the other point below this line. Moreover, this flip bifurcation is the only local bifurcation that FP_2 can undergo.

5.3. $FP_{3,4}$

Let us show that the fixed points $FP_3(A^-_{\mathrm{II},1}, 1)$ and $FP_4(A^+_{\mathrm{II},1}, 1)$ are always located in D_-. Recall that these two points exist when either (A.8a) or (A.8b) holds. If (A.8a) is true, then $A^{\max}_{\mathrm{II}} > 0$ and

$$\left.\frac{\mathrm{d}P_{\mathrm{II}}}{\mathrm{d}A}\right|_{A=0} = O_a - \frac{r_a}{b_a} \quad \Rightarrow \quad 1 < \left.\frac{\mathrm{d}P_{\mathrm{II}}}{\mathrm{d}A}\right|_{A=0} < O_a.$$

The derivative $\mathrm{d}P_{\mathrm{II}}(A)/\mathrm{d}A$ clearly decreases to zero on the interval $[0, A^{\max}_{\mathrm{II}}]$ and then becomes negative on $(A^{\max}_{\mathrm{II}}, 1)$. It means that

$$P_{\mathrm{II}}(A) < O_a A \quad \text{for} \quad 0 < A < 1 \quad \Rightarrow \quad FP_{3,4} \in D_-.$$

On the other hand, if (A.8b) holds, then

$$\left.\frac{\mathrm{d}P_{\mathrm{II}}}{\mathrm{d}A}\right|_{A=0} = O_a - \frac{r_a}{b_a} < -2\sqrt{O_a} - 1 < 0.$$

This implies that

$$A^{\pm}_{\mathrm{II},1} < 0 \quad \Rightarrow \quad FP_{3,4} \in D_-.$$

The Jacobi matrix for FP_3 is

$$J(FP_3) = \begin{pmatrix} J_{11} & J_{12} \\ 0 & 1 - r_p \end{pmatrix}, \tag{23}$$

where

$$\begin{aligned} J_{11} &= -3b_a O_a \left(A^-_{\mathrm{II},1}\right)^2 + (4b_a O_a + 2b_a - 2r_a) A^-_{\mathrm{II},1} - 2b_a - b_a O_a + r_a + 1, \\ J_{12} &= b_a O_a \left(A^-_{\mathrm{II},1}\right)^3 + (r_a - b_a O_a)\left(A^-_{\mathrm{II},1}\right)^2 - b_a A^-_{\mathrm{II},1} + b_a. \end{aligned} \tag{24}$$

For obtaining similar expressions for FP_4 one has to replace $A^{-}_{\mathrm{II},1}$ with $A^{+}_{\mathrm{II},1}$ in (24). The eigenvalues of FP_3 (and similarly of FP_4) are

$$\nu_1(FP_3) = J_{11}, \quad \nu_2(FP_3) = 1 - r_p. \tag{25}$$

The related eigenvectors are

$$v_1 = (1,0), \quad v_2 = \left(\frac{J_{12}}{1 - r_p - J_{11}}, 1\right). \tag{26}$$

Let us check which bifurcations can appear in the direction v_1. For that we make certain transformations in the expression for J_{11}:

$$J_{11} - 1 = \left(B_{\mathrm{II}} + \frac{1}{O_a} - 2\right)\sqrt{\left(B_{\mathrm{II}} + \frac{1}{O_a}\right)^2 - \frac{4}{O_a}} - \left(B_{\mathrm{II}} + \frac{1}{O_a}\right)^2 - \frac{4}{O_a}.$$

The latter equals zero if

$$\left[\begin{array}{l} \left(B_{\mathrm{II}} + \dfrac{1}{O_a}\right)^2 - \dfrac{4}{O_a} = 0, \\[2ex] \left(B_{\mathrm{II}} + \dfrac{1}{O_a} - 2\right)^2 = \left(B_{\mathrm{II}} + \dfrac{1}{O_a}\right)^2 - \dfrac{4}{O_a}, \end{array}\right. \quad \Leftrightarrow \quad \left[\begin{array}{l} \dfrac{r_a}{b_a} = \left(1 \pm \sqrt{O_a}\right)^2, \\[2ex] \dfrac{r_a}{b_a O_a} = 0. \end{array}\right.$$

Notice that for $r_a/b_a = \left(1 - \sqrt{O_a}\right)^2$ with $0 < O_a < 1$ the branch $P = P^{\mathrm{L}}_{\mathrm{II}}(A)$ is tangent to the line $P = 1$, and hence, the points $FP_{3,4}$ do not exist. Consequently,

$$\nu_1(FP_3) = J_{11} = 1 \quad \Leftrightarrow \quad \left[\begin{array}{l} \left\{\begin{array}{l} \dfrac{r_a}{b_a} = \left(1 - \sqrt{O_a}\right)^2, \\[2ex] O_a > 1, \end{array}\right. \\[4ex] \dfrac{r_a}{b_a} = \left(1 + \sqrt{O_a}\right)^2. \end{array}\right. \tag{27}$$

When (27) holds, the point FP_3 (together with FP_4) appears due to the fold bifurcation. Moreover, for

$$\left\{\begin{array}{l} \dfrac{r_a}{b_a} < \left(1 - \sqrt{O_a}\right)^2, \\[2ex] O_a > 1 \end{array}\right. \quad \text{or} \quad \frac{r_a}{b_a} > \left(1 + \sqrt{O_a}\right)^2$$

the eigenvalues are

$$\nu_1(FP_3) < 1 \quad \text{and} \quad \nu_1(FP_4) > 1.$$

If additionally $r_p < 2$, then FP_3 is the stable node, while FP_4 is the saddle. Otherwise, FP_3 is the saddle and FP_4 is the unstable node. It can be also shown that there is always $\nu_1(FP_3) > -1$. Thus, FP_3 cannot undergo a flip bifurcation in the v_1 direction.

The second eigenvalue for both points is always $\nu_2 < 1$, and the only possible bifurcation in the direction v_2 is the flip bifurcation (at $r_p = 2$). Notice that this bifurcation occurs for both points simultaneously.

5.4. FP_5

As for the fixed point $FP_5(A_\mathrm{d}, A_\mathrm{d})$, it is located inside D_- (D_+) if $O_a > 1$ ($O_a < 1$). In both cases its Jacobi matrix has in general all four non-zero elements:

$$J^{\pm}(FP_5) = \begin{pmatrix} 1 - r_a \pm \frac{r_p b_a (O_a - 1)}{b_p (O_p - 1)} & r_a \mp \frac{r_p b_a (O_a - 1)}{b_p (O_p - 1)} \\ \frac{r_p^2}{b_p (O_p - 1)^2} & 1 + r_p + \frac{r_p^2 (O_p - 2)}{b_p (O_p - 1)^2} \end{pmatrix}. \tag{28}$$

The eigenvalues of $J^{\pm}(FP_5)$ may be complex numbers. It happens when

$$\left(2 - r_a \pm \frac{r_p b_a (O_a - 1)}{b_p (O_p - 1)} + r_p + \frac{r_p^2 (O_p - 2)}{b_p (O_p - 1)^2} \right)^2 - 4 \det J^{\pm}(FP_5) < 0. \tag{29}$$

In such a case it is possible for this point to undergo a Neimark-Sacker bifurcation. However, the left-hand side of (29) is too cumbersome to study analytically how different parameters influence its sign.

5.5. FP_i, $i = \overline{6, 11}$

The expressions for FP_i, $i = \overline{6, 11}$, are also too complicated to study their stability properties analytically.

6. Sample Dynamics

This section presents two examples of phase plane of the map Φ_μ for different parameter sets. Both examples show the complexity of the dynamics and, even when restricting the phase plane to values relevant for the application, coexistence of different attractors.

6.1. Example 1

Let us fix the parameter point μ_1 with $r_a = 0.03, r_p = 0.01, b_a = b_p = 0.1, O_a = 3, O_p = 1.5$. For such parameter values, the application target fixed point FP_1 is a stable node (see Sec. 5.1). Fig. 5a shows a phase plane of the map Φ_{μ_1}, where different colors correspond to attractors of different period or divergence. Namely, some orbits are attracted to a fixed point (light-blue region), some to an 8-cycle $\mathcal{O}_8$ (violet region), some converge to a 35-cycle $\mathcal{O}_{35}$ (orange region), while the others are divergent (gray region). The cycles $\mathcal{O}_8$ and $\mathcal{O}_{35}$ are located outside the feasible domain $D_{\mathcal{F}}$. Hence, the orbits having initial conditions inside the respective regions are nonfeasible and should be excluded from consideration in the applied context.

Let us consider the orbits convergent to the fixed point in more detail. We notice that for the mentioned parameter values there exist seven fixed points: FP_i, $i = 1, \ldots, 6$, and $i = 9$. All these fixed points, except for FP_5, belong to the feasible domain (to its interior or its boundary $\partial D_{\mathcal{F}}$). The points FP_1 and FP_3 are stable nodes, the points FP_2, FP_4, FP_5, and FP_9 are saddles, the point FP_6 is an unstable node. In Fig. 5b basins of attraction of FP_1 and FP_3 are shown by pink and brown colors, respectively, and some of their boundaries are marked by blue curves, which are stable sets of the four saddles.

The intersection of the basin of attraction of the application target point FP_1 and the feasible domain $D_{\mathcal{F}}$ is relatively small for the chosen parameter values. However, from the form of the immediate basin of FP_1 one can conclude that for the learning process to be effective, the initial value of the actual developmental level A must be sufficiently high regardless of the initial potential developmental level P. As has been already mentioned in Sec 2, evaluation of the current learner's knowledge level is a complicated task often requiring time and usage of multiple techniques. Therefore, in reality it can sometimes happen that the potential developmental level is estimated incorrectly and there is $P < A$. Though if initial A is large enough, the orbit eventually enters the feasible domain $D_{\mathcal{F}}$ converging to the desired point FP_1. In Fig. 5b two orbits with different initial conditions, one being outside and the other one located inside

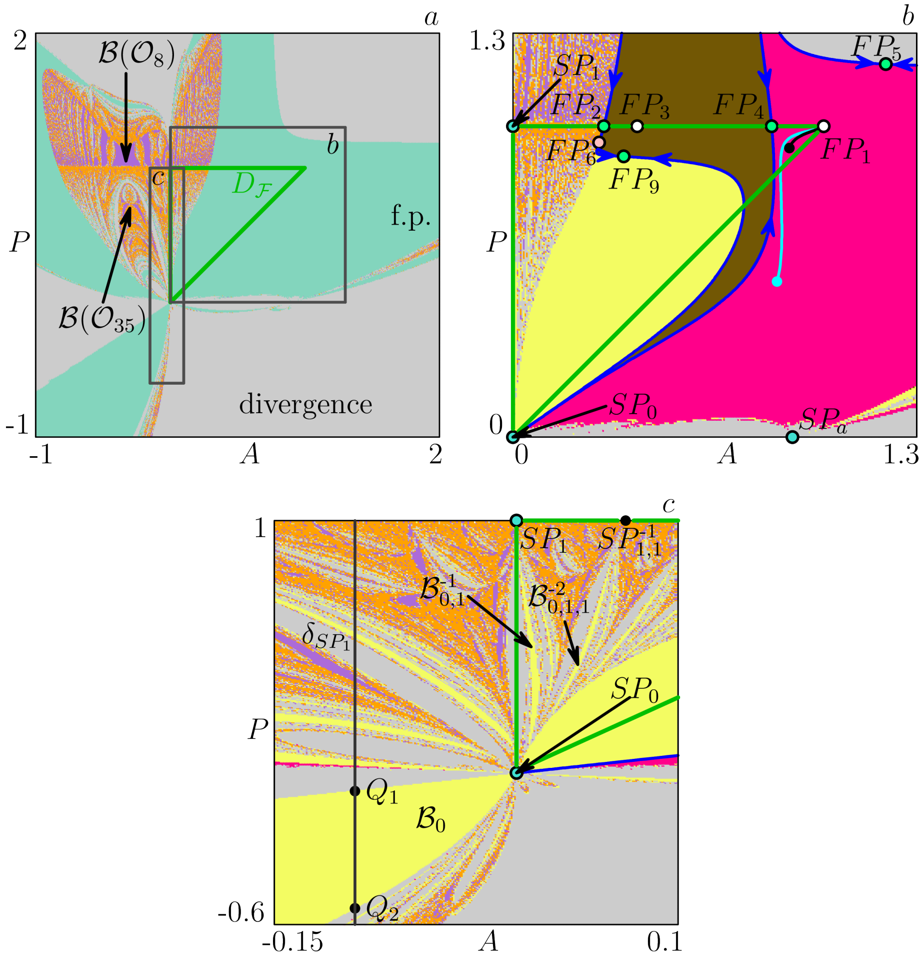

Figure 5: Phase space of Φ_{μ_1} revealing basins of different attractors. (a) Light-blue region corresponds to initial values whose orbits are attracted to a fixed point; violet region corresponds to the basin of $\mathcal{O}_8$; orange points constitute the basin of $\mathcal{O}_{35}$; gray color is related to divergent orbits. The rectangles mark the areas shown enlarged in the panels b and c. (b), (c) Basins of attraction of the stable nodes FP_1 (pink) and FP_3 (brown) and the focal point SP_0 (yellow). The other colors have the same meaning as in (a). Parameters are $r_a = 0.03, r_p = 0.01, b_a = 0.1, b_p = 0.1, O_a = 3, O_p = 1.5$.

$D_{\mathcal{F}}$, are shown by cyan and black lines, respectively.

As for the orbits whose initial points are located in the yellow region, they asymptotically approach the focal point SP_0. Recall from Sec. 3 that SP_0

belongs to its prefocal set δ_{SP_0}. Moreover, if coefficients ξ_1 and η_1 in Taylor series (6) are different from zero, the image of the respective arc $\gamma(\tau)$ intersects δ_{SP_0} exactly at SP_0 regardless of the slope $m = \eta_1/\xi_1$. And hence, SP_0 may play a role similar to that of an attracting fixed point. The basin of attraction of SP_0 contains elements characteristic for maps with denominator, as one can see in Fig. 5c. In particular, let us consider the part of this basin with three vertices in the points Q_1, Q_2 and SP_0, denoted as $\mathcal{B}_0$. The points Q_1 and Q_2 are the intersections of the respective basin boundaries with the prefocal set δ_{SP_1}, and hence, are both focalized into SP_1 by one of the inverses of Φ_{μ_1}. Due to this there exists a *crescent* between the two focal points, SP_0 and SP_1, denoted as $\mathcal{B}_{0,1}^{-1}$ in Fig. 5c, such that $\Phi_{\mu_1}(\mathcal{B}_{0,1}^{-1}) = \mathcal{B}_0$. Clearly there also exist an infinite sequence of preimages of $\mathcal{B}_{0,1}^{-1}$, each having a form of crescent between SP_0 and a respective preimage of SP_1. For instance, one can notice the region $\mathcal{B}_{0,1,1}^{-2}$ between SP_0 and $SP_{1,1}^{-1}$, where $\Phi_{\mu_1}(SP_{1,1}^{-1}) = SP_1$ and $\Phi_{\mu_1}(\mathcal{B}_{0,1,1}^{-2}) = \mathcal{B}_{0,1}^{-1}$.

For further details on characteristic basin structures occurring for maps with vanishing denominator see [11, 12, 13].

6.2. Example 2

In this example we fix the parameter point μ_2 with $r_a = 0.098$, $r_p = 0.09$, $b_a = b_p = 0.1$, $O_a = 0.2$, $O_p = 0.11$. All in all, there are seven fixed points: two stable nodes FP_1 and FP_5, four saddles FP_2, $FP_{7,8,9}$, and an unstable node FP_6. In addition, there are two non-periodic invariant sets. Fig. 6 shows basins of different attractors in the (A, P) phase plane. Blue points correspond to initial conditions whose orbits are attracted to FP_1, the basin of FP_5 (which is nonfeasible though) is plotted brown, orange region is related to the chaotic attractor $\mathcal{Q}$ located at the line $P = 1$, and the points colored pink have orbits ending up at the invariant closed curve Γ (shown violet). Grey region corresponds to divergence.

We remark further that the basin of FP_1 is separated from the others by the stable set of the saddle FP_2. Note that in comparison with the previous example, for the current parameter set the part of basin of FP_1 located inside

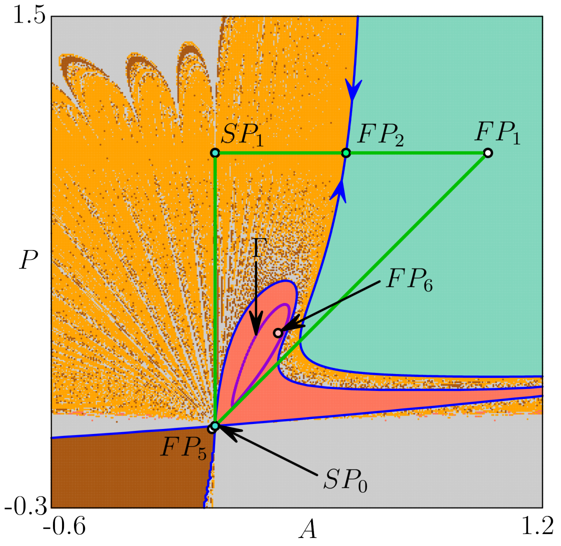

Figure 6: Phase space of Φ_{μ_2} revealing basins of four different attractors: the stable node FP_1 (light-blue), the stable node FP_5 (brown), the chaotic attractor $\mathcal{Q} \subset \{(A, P) \,:\, P = 1\}$ (orange), and the closed curve Γ (pink). Gray region is related to divergent orbits. Parameters are $r_a = 0.098, r_p = 0.09, b_a = 0.1, b_p = 0.1, O_a = 0.2, O_p = 0.11$.

the feasible domain $D_{\mathcal{F}}$ is essentially larger. However, the initial actual developmental level A again must not fall below a certain value in order to achieve the final educational goal $K = 1$. In case when the initial A is too small, or the original evaluation of the current learner's knowledge level is too far from the reality, that is, initial P is too far below the initial A, the learning is not effective. Indeed, such an orbit either eventually leaves the feasible domain $D_{\mathcal{F}}$ or is attracted to an invariant curve Γ. This curve Γ can be interpreted as a cyclic learning process in which the student achieving a certain developmental level gives up (for instance, gets bored of the subject) and gradually loses the skills acquired. At some point he/she starts fighting the educational goal anew, but eventually gives up again.

Note also that the focal points SP_0 and SP_1 are involved as well into formation of the basin structures, typical for maps with vanishing denominator, such as lobes and crescents. For example, the basin of $\mathcal{Q}$ consists of multiple lobes issuing from SP_0, forming a structure which resembles a fan centered at SP_0. And the parts of the basin of infinity (divergent orbits) located between these lobes have form of crescents.

Finally, the points $FP_{7,8,9}$ are located in the third quadrant of the plane and fall outside both, the feasible domain $D_{\mathcal{F}}$ and the area plotted in Fig. 6.

7. Conclusion

Models of education, such as the model described in this article, often imply processes of co-adaptation between a helper and a learner. That is, they imply a coupling of systems over time. The details of this coupling are described in the theoretical assumptions of the underlying model, such as a model of the zone of proximal development, a model of scaffolding, or one that combines both. In the current article, we have investigated a mathematical formalization of the latter type of model in the form of a 2D difference equation system. The resulting map is noninvertible, piecewise smooth and both its components assume the form of rational functions. This implies that in the phase space there exists a set of nondefinition, where at least one of denominators vanishes. It is not surprising that the map dynamics turns out to be rather complex and interesting.

In the current work we have made the first step in studying the mathematical model described and analyzed some of its properties. In particular, we have derived analytic expressions for finding fixed points of the map and obtained conditions for their stability. We have also determined focal points, at which at least one of the map components assumes the uncertainty zero over zero, and computed the related prefocal sets. Noteworthy, the focal point at the origin denoted SP_0 is rather peculiar, since its prefocal set coincides with the set of nondefinition. Moreover, there exist a family of smooth curves $\gamma_m(\tau)$ passing through SP_0 with a slope m, such that the image of $\gamma_m(\tau)$ intersects the related prefocal set δ_{SP_0} at the point SP_0 itself, regardless the value of m. This implies that SP_0 can play a role similar to that of a fixed point.

Finally, we have also examined the phase plane of the map for two different parameter sets. In both cases we have observed coexistence of several attractors, as well as complex basin structures having multiple lobes and crescents, which is a specific feature of maps with vanishing denominator. Another intriguing phenomenon has been revealed in the first example, where one of the attractors was the focal point at the origin.

It is important to note that the discovered structure and complexity [28] directly result from the map dynamics themselves. That is to say, the complexity is a genuine result of the nature of the processes that the model describes. Complexity [29] must not be added to the model, for instance, by invoking a host of additional variables, the dynamics of which are not controlled by the educational model as such and which thus serve as independent or error variables. The notion that such complexity is added from outside is quite typical of standard models in the educational sciences, for instance, regression models or structural equation models. Intuitively, or based on verbal reasoning alone, models that imply some sort of interaction between the participants in an educational process, are implicitly believed to be relatively simple, with the desired educational result, plus or minus random variation, as the standard outcome. However, if such models are expressed in the form of difference equation systems describing changes in their relevant variables, a thorough study of their map dynamics reveals their hidden intrinsic complexity. Recently, several works have discussed application of dynamical system approach to developmental processes and related contributions and challenges, see [16, 30, 31, 32, 33]. Our analysis supports the idea "that cognition and development take place, not in the head, but in the interactions between the mind and the environment" [31, p. 282] and provides a step to move away from the metatheoretical aspects of the dynamical system approach in developmental psychology (discussed in [34]) towards meeting the demands of those asking for quantitative rigor [31].

It goes without saying that the study of the map dynamics of a particular educational model is an investigation into the properties of the model, that is to say, an investigation into the range of possible observations one could make if the model provides a correct description of reality. But even if the model is correct, it still provides a rather radical idealization and simplification of that same reality. For instance, the model we have studied in this article is a deterministic model, but it is highly unlikely that real educational interactions of the type described by the model are indeed deterministic. It might be interesting to study how the map dynamics behave if it is subject to stochastic influences, perturbations or shocks from outside.

Finally, the model presented here is but one of a family of models of learning and development based on principles of co-adaptation between a developing or learning person and material or social environment that continuously adapts to this person's developing or learning needs. We have tried to formulate a model that is as general as possible, in terms of its underlying theoretical assumptions. Nevertheless, future research could focus on variations and further specifications of this general model, and investigate whether the resulting map dynamics have certain properties in common that are not only interesting from a mathematical and theoretical point of view, but that might also offer new insights into empirical data on learning and development.

References

[1] M. Koopmans, D. Stamovlasis, Introduction to education as a complex dynamical system, in: M. Koopmans, S. Dimitrios (Eds.), Complex Dynamical Systems in Education Concepts, Methods and Applications, Springer, New York, NY, 2016, pp. 1–7.

[2] P. van Geert, Nonlinear complex dynamical systems in developmental psychology, in: S. Guastello, M. Koopmans, D. Pincus (Eds.), Chaos and Complexity in Psychology: The Theory of Nonlinear Dynamical Systems, Cambridge University Press, Cambridge, UK, 2008, pp. 242–281.

[3] M. van Dijk, P. van Geert, K. Korecky Kröll, I. Maillochon, S. Laaha, W. U. Dressler, D. Bassano, Dynamic adaptation in child-adult language interaction, Language Learning 63 (2) (2013) 243–270.

[4] P. van Geert, Dynamic System of Development, Harvester Wheatsheaf, New York, NY, 1994.

[5] P. van Geert, A dynamic systems model of basic developmental mechanisms: Piaget, Vygotsky, and beyond, Psychological Review 105 (4) (1998) 634–677.

[6] C. H. Coombs, R. D. Dawes, A. Tversky, Mathematical Psychology, Prentice-Hall, Englewood Cliffs, NJ, 1970.

[7] P. van Geert, H. Steenbeek, Explaining after by before: Basic aspects of a dynamic systems approach to the study of development, Developmental Review 25 (2) (2005) 408–442.

[8] U. Merlone, P. van Geert, A dynamical model of proximal development: Multiple implementations, in: G. I. Bischi, A. Panchuk, D. Radi (Eds.), Qualitative Theory of Dynamical Systems, Tools and Applications for Economic Modelling, Springer International Publishing, Cham, 2016, pp. 305–324.

[9] L. S. Vygotsky, Mind in society: The development of higher psychological processes, (M. Cole, V. John-Steiner, S. Scribner and E. Souberman, Eds.) (A. R. Luria, M. Lopez-Morillas, M. Cole [with J. V. Wertsch], Trans.), Harvard University Press, Cambridge, MA, 1978, (Original manuscripts [ca. 1930–1934]).

[10] D. Wood, J. S. Bruner, G. Ross, The role of tutoring in problem solving, Journal of Child Psychology and Psychiatry 17 (2) (1976) 89–100. `doi:10.1111/j.1469-7610.1976.tb00381.x`.

[11] G. I. Bischi, L. Gardini, C. Mira, Plane maps with denominator. Part III: Nonsimple focal points and related bifurcations, International Journal of Bifurcation and Chaos 15 (2) (2005) 451–496.

[12] G. I. Bischi, L. Gardini, C. Mira, Plane maps with denominator. Part II: Noninvertible maps with simple focal points, International Journal of Bifurcation and Chaos 13 (8) (2003) 2253–2277.

[13] G. I. Bischi, L. Gardini, C. Mira, Plane maps with denominator. Part I: Some generic properties, International Journal of Bifurcation and Chaos 9 (1) (1999) 119–153.

[14] N. Pecora, F. Tramontana, Maps with vanishing denominator and their applications, Frontiers in Applied Mathematics and Statistics 2 (2016) 11. `doi:10.3389/fams.2016.00011`.
URL `https://www.frontiersin.org/article/10.3389/fams.2016.00011`

[15] F. Tramontana, Maps with vanishing denominator explained through applications in economics, Journal of Physics: Conference Series 692 (2016) 012006. `doi:10.1088/1742-6596/692/1/012006`.
URL `https://doi.org/10.1088/1742-6596/692/1/012006`

[16] T. Hollenstein, Twenty years of dynamic systems approaches to development: Significant contributions, challenges, and future directions, Child Development Perspectives 5 (4) (2011) 256–259. `doi:10.1111/j.1750-8606.2011.00210.x`.

[17] P. van Geert, Vygotskian dynamics of development, Human Development 37 (1994) 346–365. `doi:10.1159/000278280`.

[18] K. De Bot, W. Lowie, M. Verspoor, A dynamic systems theory approach to second language acquisition, Bilingualism: Language and Cognition 10 (1) (2007) 721. `doi:10.1017/S1366728906002732`.

[19] D. Tzuriel, S. Weiss, Cognitive modifiability as a function of mother-child mediated learning strategies, mothers' acceptance-rejection, and children's personality, Early Development and Parenting 7 (2) (1998) 79–99. `doi:10.1002/(SICI)1099-0917(199806)7:2<79::AID-EDP166>3.0.CO;2-#`.

[20] D. J. Sluss, A. J. Stremmel, A sociocultural investigation of the effects of peer interaction on play, Journal of Research in Childhood Education 18 (4) (2004) 293–305. `doi:10.1080/02568540409595042`.

[21] Z. T. Zhusubaliyev, E. Mosekilde, Bifurcations and chaos in piecewise-smooth dynamical systems, World Scientific, Singapore, 2003.

[22] M. di Bernardo, C. J. Budd, A. R. Champneys, P. Kowalczyk, Piecewise-smooth dynamical systems: Theory and applications, Springer, London, 2008.

[23] R. García-Ros, I. Talaya, F. Pérez-González, The process of identifying gifted children in elementary education: Teachers evaluations of creativity, School Psychology International 33 (6) (2012) 661–672. `doi:10.1177/0143034311421434`.

[24] R. F. De Haan, Identifying gifted children, The School Review 65 (1) (1957) 41–48. `doi:10.1086/442374`.

[25] B. D. Shaklee, Identification of young gifted students, Journal for the Education of the Gifted 15 (2) (1992) 134–144. `doi:10.1177/016235329201500203`.

[26] K. A. Johnston, B. K. Andersen, J. Davidge-Pitts, M. Ostensen-Saunders, Identifying student potential for ICT entrepreneurship using Myers-Briggs personality type indicators, Journal of Information Technology Education: Research 8 (1) (2009) 29–43.

[27] N. Schmitt, Development of rationale and measures of noncognitive college student potential, Educational Psychologist 47 (1) (2012) 18–29. `doi:10.1080/00461520.2011.610680`.

[28] D. P. Fromberg, The fractal dynamics of early childhood play development and nonlinear teaching and learning, in: M. Koopmans, S. Dimitrios (Eds.), Complex Dynamical Systems in Education Concepts, Methods and Applications, Springer, New York, NY, 2016, pp. 105–118.

[29] M. Koopmans, Perspectives on complexity, its definition and applications in the field, Complicity: An International Journal of Complexity and Education 1 (2017) 16–35.

[30] P. van Geert, The contribution of complex dynamic systems to development, Child Development Perspectives 5 (4) (2011) 273–278. `doi:10.1111/j.1750-8606.2011.00197.x`.

[31] M. Lewis, Dynamic systems approaches: Cool enough? Hot enough?, Child Development Perspectives 5 (4) (2011) 279–285. `doi:10.1111/j.1750-8606.2011.00190.x`.

[32] D. C. Witherington, T. E. Margett, How conceptually unified is the dynamic systems approach to the study of psychological development?, Child Development Perspectives 5 (4) (2011) 286–290. `doi:10.1111/j.1750-8606.2011.00211.x`.

[33] A. Fogel, Theoretical and applied dynamic systems research in developmental science, Child Development Perspectives 5 (4) (2011) 267–272. `doi:10.1111/j.1750-8606.2011.00174.x`.

[34] D. Witherington, The dynamic systems approach as metatheory for developmental psychology, Human Development 50 (2007) 127–153. `doi:10.1159/000100943`.

Appendix A. Determining fixed points as intersections of the two loci

Appendix A.1. Intersection of $f_1 = 0$ and $P \equiv 1$

- $P = A$ with $P \equiv 1$: First of all, there is always a fixed point $FP_1(1, 1)$, which is the desired target state from application viewpoint.
- $P = P_{\mathrm{I}}(A)$ with $P \equiv 1$: Solving

$$P_{\mathrm{I}}(A) = \frac{A^2 - B_{\mathrm{I}}A}{A - 1} O_a = 1, \tag{A.1}$$

where B_{I} is defined in (11), gives two solutions

$$A_{\mathrm{I},1}^{\pm} = \frac{1}{2}\left(B_{\mathrm{I}} + \frac{1}{O_a} \pm \sqrt{\left(B_{\mathrm{I}} + \frac{1}{O_a}\right)^2 - \frac{4}{O_a}}\right). \tag{A.2}$$

They are real whenever the discriminant Δ is not negative:

$$\Delta = \left(B_{\mathrm{I}} + \frac{1}{O_a}\right)^2 - \frac{4}{O_a} \geq 0.$$

Adding the term $\pm 4r_a/b_aO_a^2$ to the left-hand side of the last inequality gives

$$\begin{aligned} 1 + \frac{r_a^2}{b_a^2O_a^2} + \frac{1}{O_a^2} + \frac{2r_a}{b_aO_a} + \frac{2}{O_a} + \frac{2r_a}{b_aO_a^2} \pm \frac{4r_a}{b_aO_a^2} \\ = \left(1 + \frac{r_a}{b_aO_a} - \frac{1}{O_a}\right)^2 + \frac{4r_a}{b_aO_a^2} \geq 0. \end{aligned}$$

The latter always holds since $r_a > 0, b_a > 0$. Moreover, the inequality is always strict. It means that the two solutions $A_{\mathrm{I},1}^{\pm}$ are always real and

$$A_{\mathrm{I},1}^{-} < 1, \quad A_{\mathrm{I},1}^{+} > 1.$$

Clearly $A_{\mathrm{I},1}^{-}$ is the intersection point of $P = P_{\mathrm{I}}^{\mathrm{L}}(A)$ and $P = 1$, while $A_{\mathrm{I},1}^{+}$ is the intersection of $P = P_{\mathrm{I}}^{\mathrm{R}}(A)$ and $P = 1$. Hence, only $A_{\mathrm{I},1}^{-}$ is related to the fixed point, since only branch $P_{\mathrm{I}}^{\mathrm{L}}$ reduces $R_a(A, P)$ to zero. We additionally remark that $A_{\mathrm{I},1}^{-} > 0$ because $P = P_{\mathrm{I}}(A)$ is increasing and

$$P_{\mathrm{I}}(0) = 0, \quad \lim_{A\to 1-} P_{\mathrm{I}}(A) = \infty.$$

Let us denote

$$FP_2 = FP_2(A_{\mathrm{I},1}^{-}, 1). \tag{A.3}$$

Clearly, $FP_2 \in D_{\mathcal{F}}$, or more precisely, $FP_2 \in \partial D_{\mathcal{F}}$.

- $P = P_{\mathrm{II}}(A)$ with $P \equiv 1$: Similarly, from

$$P_{\mathrm{II}}(A) = \frac{A^2 - B_{\mathrm{II}}A}{A-1}O_a = 1, \tag{A.4}$$

where B_{II} is given in (11) two following solutions are obtained:

$$A^{\pm}_{\text{II},1} = \frac{1}{2}\left(B_{\text{II}} + \frac{1}{O_a} \pm \sqrt{\left(B_{\text{II}} + \frac{1}{O_a}\right)^2 - \frac{4}{O_a}}\right). \tag{A.5}$$

Let us denote

$$FP_3 = FP_3(A^{-}_{\text{II},1}, 1), \quad FP_4 = FP_4(A^{+}_{\text{II},1}, 1). \tag{A.6}$$

Again, the solutions $A^{\pm}_{\text{II},1}$ are real whenever the discriminant

$$\Delta = \left(B_{\text{II}} + \frac{1}{O_a}\right)^2 - \frac{4}{O_a} \geq 0,$$

but in contrast to the case of $P_{\text{I}}(A) = 1$, now the opposite inequality ($\Delta < 0$) is possible. This happens when

$$\left(1 - \sqrt{O_a}\right)^2 < \frac{r_a}{b_a} < \left(1 + \sqrt{O_a}\right)^2. \tag{A.7}$$

For the related parameter values both $A^{\pm}_{\text{II},1}$ are complex, and $FP_{3,4}$ do not exist. When Δ is positive, $A^{\pm}_{\text{II},1}$ are distinct real numbers. However, it does not immediately imply that the fixed points $FP_{3,4}$ exist. Indeed, recall that the expression (10b) defines two branches: $P^{\text{L}}_{\text{II}}(A)$ for $A < 1$ and $P^{\text{R}}_{\text{II}}(A)$ for $A > 1$, but the right branch P^{R}_{II} does not reduce $f_1(A, P)$ to zero. Formally, if $A^{\pm}_{\text{II},1} > 1$, then the points $FP_{3,4}$ are intersections of $P = P^{\text{R}}_{\text{II}}(A)$ and $P = 1$, but they are not fixed points of Φ_μ. In case where $A^{\pm}_{\text{II},1} < 1$ the fixed points $FP_{3,4}$ are intersections of $P = P^{\text{L}}_{\text{II}}(A)$ and $P = 1$.

To derive the region of parameter values for that the points $FP_{3,4}$ exist, we recall that $P_{\text{II}}(A)$ has a local maximum

$$\max_A P_{\text{II}}(A) = \left(\sqrt{\frac{r_a}{b_a}} - \sqrt{O_a}\right)^2 \overset{def}{=} P^{\max}_{\text{II}}$$

attained at $A^{\max}_{\text{II}}$ given in (12). Then we have to require that

(1) the opposite to (A.7) holds ($\Delta > 0$) and

(2) $P^{\max}_{\text{II}} > 1$.

The condition (1) is nothing else but

$$\frac{r_a}{b_a} < \left(1 - \sqrt{O_a}\right)^2 \quad \text{or} \quad \frac{r_a}{b_a} > \left(1 + \sqrt{O_a}\right)^2 .$$

The condition (2) is equivalent to

$$\left[\begin{array}{l} \sqrt{\dfrac{r_a}{b_a}} < \sqrt{O_a} - 1, \\ \\ \sqrt{\dfrac{r_a}{b_a}} > \sqrt{O_a} + 1 \end{array}\right. \quad \Leftrightarrow \quad \left[\begin{array}{l} \left\{\begin{array}{l} \dfrac{r_a}{b_a} < (\sqrt{O_a} - 1)^2, \\ \\ O_a > 1, \end{array}\right. \\ \dfrac{r_a}{b_a} > (\sqrt{O_a} + 1)^2. \end{array}\right.$$

Combining both conditions together implies

$$\left\{\begin{array}{l} \dfrac{r_a}{b_a} < \left(1 - \sqrt{O_a}\right)^2, \\ O_a > 1. \end{array}\right. \tag{A.8a}$$

or

$$\frac{r_a}{b_a} > \left(1 + \sqrt{O_a}\right)^2 \tag{A.8b}$$

Notice that if (A.8a) holds, the point of maximum $A_{\mathrm{II}}^{\max} > 0$, while for the parameters satisfying (A.8b) there is $A_{\mathrm{II}}^{\max} < 0$. In case of equality

$$\left\{\begin{array}{l} \dfrac{r_a}{b_a} = \left(1 - \sqrt{O_a}\right)^2, \\ O_a > 1. \end{array}\right. \quad \text{or} \quad \frac{r_a}{b_a} = \left(1 + \sqrt{O_a}\right)^2 \tag{A.9}$$

the curve $P = P_{\mathrm{II}}^{\mathrm{L}}(A)$ is tangent to the line $P = 1$, and the two fixed points coincide $FP_3 \equiv FP_4$. As shown in Sec. 5.3, this is exactly the condition for the fold bifurcation.

Appendix A.2. Intersection of $f_1 = 0$ and $P = P_-(A)$

- $P = A$ with $P = P_-(A)$: Solving

$$P_-(A) = \frac{1 + O_p A - \sqrt{(1 - O_p A)^2 - 4A\frac{r_p}{b_p}}}{2} = A \tag{A.10}$$

is equivalent to

$$1 + O_p A - 2A = \sqrt{(1 - O_p A)^2 - 4A\frac{r_p}{b_p}}.$$

This gives two solutions

$$A_0 = 0 \quad \text{and} \quad A_\mathrm{d} = 1 + \frac{r_p}{b_p(O_p - 1)}. \tag{A.11}$$

The solution A_0 (corresponding to the focal point SP_0) always exists, while A_d exists only provided that

$$\Delta|_{A_\mathrm{d}} = (1 - O_p A_\mathrm{d})^2 - 4A_\mathrm{d}\frac{r_p}{b_p} \geq 0, \tag{A.12}$$

$$1 + O_p A_\mathrm{d} - 2A_\mathrm{d} \geq 0. \tag{A.13}$$

The first inequality (A.12) can be rewritten as

$$\Delta|_{A_\mathrm{d}} = \left(\frac{O_p - 2}{O_p - 1} \cdot \frac{r_p}{b_p} + O_p - 1\right)^2 \geq 0,$$

which is always true. The second inequality (A.13) is equivalent to

$$\begin{cases} \dfrac{O_p - 2}{O_p - 1} < 0, \\[2ex] r_p \leq \dfrac{b_p(O_p - 1)^2}{2 - O_p} \end{cases} \quad \text{or} \quad \begin{cases} \dfrac{O_p - 2}{O_p - 1} > 0, \\[2ex] r_p \geq \dfrac{b_p(O_p - 1)^2}{2 - O_p} \end{cases} \quad \text{or} \quad O_p = 2. \tag{A.14}$$

Notice that if $O_p < 1$, the value A_d is the intersection point of $P = A$ and $P = P_-^\mathrm{L}(A)$, while if $O_p > 1$, it is the intersection point of $P = A$ and $P = P_-^\mathrm{R}(A)$. Finally,

$$\lim_{O_p \to 1-} A_\mathrm{d} = -\infty, \qquad \lim_{O_p \to 1+} A_\mathrm{d} = \infty.$$

Let us emphasize the particular case when the equality $r_p = \frac{b_p(O_p-1)^2}{2-O_p}$ holds. It immediately implies that $0 < O_p < 2$, since for $O_p \geq 2$ the value of r_p either falls outside the considered region for parameters or is infinite (for $O_p = 2$). Moreover,

1. for $0 < O_p < 1$ the solution of (A.10) is $A_\mathrm{d} = A_\mathrm{lim}^\mathrm{L}$ (defined in (15a)),
2. for $1 < O_p < 2$ the solution of (A.10) is $A_\mathrm{d} = A_\mathrm{lim}^\mathrm{R}$ (defined in (15b)),

Let us denote $FP_5 = FP_5(A_\mathrm{d}, A_\mathrm{d})$.

- $P = P_{\mathrm{I}}(A)$ with $P = P_-(A)$: The equality

  $$\begin{aligned} & P_{\mathrm{I}}(A) = P_-(A) \quad \Leftrightarrow \\ & 1 + O_pA - 2\left(-\frac{r_a}{b_a} - O_a + O_aA\right)\frac{A}{A-1} = \sqrt{(1-O_pA)^2 - 4A\frac{r_p}{b_p}} \end{aligned} \tag{A.15}$$

  immediately separates into $A = A_0 = 0$ and the cubic polynomial of A:

  $$a_1A^3 + a_2A^2 + a_3A + a_4 = 0 \tag{A.16}$$

  with

  $$\begin{aligned} a_1 &= O_a(O_a - O_p), \\ a_2 &= \frac{r_p}{b_p} + 2O_a(O_p - O_a) + O_p - O_a + \frac{r_a}{b_a}(O_p - 2O_a), \\ a_3 &= O_a(O_a - O_p) + 2(O_a - O_p) + \frac{r_a}{b_a}(2O_a - O_p) + \frac{r_a}{b_a}\left(1 + \frac{r_a}{b_a}\right) - 2\frac{r_p}{b_p}, \\ a_4 &= O_p - O_a + \frac{r_p}{b_p} - \frac{r_a}{b_a}. \end{aligned} \tag{A.17}$$

  The polynomial (A.16) with coefficients as in (A.17) always has three roots denoted as $A^1_{\mathrm{I,cub}}$, $A^2_{\mathrm{I,cub}}$, $A^3_{\mathrm{I,cub}}$. Among them there can be *at least one* real root and *at most three* real roots. Suppose that $A^1_{\mathrm{I,cub}}$ is always real. Although $A^i_{\mathrm{I,cub}}$, $i = 1, 2, 3$, can be obtained in explicit form by Cardano formulae (see Appendix B), the expressions are quite complicated, which hampers analytic investigation of the related fixed points.

  We also remark that for raising to the square both sides of (A.15) one has to guarantee that

  $$1 + O_pA - 2\left(-\frac{r_a}{b_a} - O_a + O_aA\right)\frac{A}{A-1} \geq 0. \tag{A.18}$$

  Thus, every $A^i_{\mathrm{I,cub}}$ also has to satisfy (A.18).

  Let us denote $FP_6 = FP_6(A^1_{\mathrm{I,cub}}, P^1_{\mathrm{I,cub}})$, $FP_7 = FP_7(A^2_{\mathrm{I,cub}}, P^2_{\mathrm{I,cub}})$, $FP_8 = FP_8(A^3_{\mathrm{I,cub}}, P^3_{\mathrm{I,cub}})$. The terms $P^i_{\mathrm{I,cub}}$, $i = 1, 2, 3$, are values of $P_{\mathrm{I}}(A)$ at the points $A^i_{\mathrm{I,cub}}$. Note that even if the cubic equation (A.16) always has at least one real root $A^1_{\mathrm{I,cub}}$, it does not imply that FP_6 always exists. Indeed, if $A^1_{\mathrm{I,cub}} > 1$, then the point $(A^1_{\mathrm{I,cub}}, P^1_{\mathrm{I,cub}})$ is the intersection of $P = P^{\mathrm{R}}_{\mathrm{I}}(A)$ and $P = P_-(A)$, and hence, it is not a fixed point of Φ_μ, since only branch $P^{\mathrm{L}}_{\mathrm{II}}$ reduces $R_a(A, P)$ to zero.

- $P = P_{\mathrm{II}}(A)$ with $P = P_-(A)$: Similarly to the previous case, the equality

$$\begin{aligned} &P_{\mathrm{II}}(A) = P_-(A) \quad \Leftrightarrow \\ &1 + O_p A - 2\left(\frac{r_a}{b_a} - O_a + O_a A\right)\frac{A}{A-1} = \sqrt{(1 - O_p A)^2 - 4A\frac{r_p}{b_p}} \end{aligned} \tag{A.19}$$

immediately separates into $A = A_0 = 0$ and the cubic polynomial of the form (A.16) but with coefficients slightly different from (A.17):

$$\begin{aligned} a_1 &= O_a(O_a - O_p), \\ a_2 &= \frac{r_p}{b_p} + 2O_a(O_p - O_a) + O_p - O_a + \frac{r_a}{b_a}(2O_a - O_p), \\ a_3 &= O_a(O_a - O_p) + 2(O_a - O_p) + \frac{r_a}{b_a}(O_p - 2O_a) + \frac{r_a}{b_a}\left(\frac{r_a}{b_a} - 1\right) - 2\frac{r_p}{b_p}, \\ a_4 &= O_p - O_a + \frac{r_p}{b_p} + \frac{r_a}{b_a}. \end{aligned} \tag{A.20}$$

The roots of the polynomial again can be obtained by Cardano formulae (see Appendix B) and are referred to as $A^i_{\mathrm{II,cub}}$, $i = 1, 2, 3$, with supposing that $A^1_{\mathrm{II,cub}}$ is always real. The related fixed points are denoted as $FP_9 = FP_9(A^1_{\mathrm{II,cub}}, P^1_{\mathrm{II,cub}})$, $FP_{10} = FP_{10}(A^2_{\mathrm{II,cub}}, P^2_{\mathrm{II,cub}})$, $FP_{11} = FP_{11}(A^3_{\mathrm{II,cub}}, P^3_{\mathrm{II,cub}})$.

Similarly to the previous case, every solution $A^i_{\mathrm{II,cub}}$, $i = 1, 2, 3$, of the cubic equation (A.16) with coefficients as in (A.20) has to satisfy the inequality

$$1 + O_p A - 2\left(\frac{r_a}{b_a} - O_a + O_a A\right)\frac{A}{A-1} \geq 0. \tag{A.21}$$

so that to guarantee validity of raising to square (A.19). Again the fixed point FP_9 exists provided that $A^1_{\mathrm{II,cub}} < 1$ by the same reason as for FP_6.

Appendix A.3. Intersection of $f_1 = 0$ *and* $P = P_+(A)$

- $P = A$ with $P = P_+(A)$: Solving

$$P_+(A) = \frac{1 + O_pA + \sqrt{(1 - O_pA)^2 - 4A\frac{r_p}{b_p}}}{2} = A$$

gives the only solution $A = A_\mathrm{d}$ defined in (A.11). Though A_d has to satisfy

$$2A_\mathrm{d} - 1 - O_pA_\mathrm{d} > 0, \tag{A.22}$$

which is different from (A.13). The inequality (A.22) is equivalent to

$$\begin{cases} \dfrac{2 - O_p}{b_p(O_p - 1)} < 0, \\ \\ r_p \leq \dfrac{b_p(O_p - 1)^2}{2 - O_p} \end{cases} \quad \text{or} \quad \begin{cases} \dfrac{2 - O_p}{b_p(O_p - 1)} > 0, \\ \\ r_p \geq \dfrac{b_p(O_p - 1)^2}{2 - O_p}. \end{cases} \tag{A.23}$$

Notice that the first inequalities in (A.23) have opposite signs to those of (A.14). This means that the two conditions (A.23) and (A.14) are in some sense complementary. Hence, the fixed point FP_5 exists for any parameter values, except for the set

$$\left\{\mu : r_p = \frac{b_p(O_p - 1)^2}{2 - O_p},\ O_p \geq 2\right\} \cup \{\mu : O_p = 1\}. \tag{A.24}$$

However, FP_5 is located on either $P = P_-(A)$ or $P = P_+(A)$, which depends on the parameters.

- $P = P_\mathrm{I}(A)$ with $P = P_+(A)$: The intersection points of $P_\mathrm{I}(A)$ with $P_+(A)$ are obtained from the cubic equation (A.16) with coefficients defined in (A.17) (the same equation as for the intersection of $P_\mathrm{I}(A)$ with $P_-(A)$). The only difference is that now every solution of (A.16) has to satisfy the inequality

$$1 + O_pA - 2\left(-\frac{r_a}{b_a} - O_a + O_aA\right)\frac{A}{A - 1} \leq 0 \tag{A.25}$$

(opposite sign to that in (A.18)). The same fixed points $FP_{6,7,8}$ are obtained. Thus, the points $FP_{6,7,8}$ are defined as intersections of $P_\mathrm{I}(A)$ with $P_-(A)$ if (A.18) holds or as intersections of $P_\mathrm{I}(A)$ with $P_+(A)$ if (A.25) is true.

- $P = P_{\mathrm{II}}(A)$ with $P = P_+(A)$: Similarly, equating $P_{\mathrm{II}}(A)$ to $P_+(A)$ implies the same cubic polynomial as equating $P_{\mathrm{II}}(A)$ to $P_-(A)$ giving the roots $A^i_{\mathrm{II,cub}}$, $i = 1, 2, 3$. However, now they have to satisfy inequality opposite to (A.21), that is,

  $$1 + O_p A - 2\left(\frac{r_a}{b_a} - O_a + O_a A\right)\frac{A}{A-1} \leq 0. \qquad \text{(A.26)}$$

  Consequently, depending on whether (A.21) or (A.26) holds, the fixed points $FP_{9,10,11}$ are intersections of $P_{\mathrm{II}}(A)$ with $P_-(A)$ or $P_{\mathrm{II}}(A)$ with $P_+(A)$, respectively.

Appendix B. Solving cubic equation: Cardano formulae

Reduce (A.16) to the canonical form

$$z^3 + pz + q = 0 \qquad \text{(B.1)}$$

with

$$p = \frac{3a_1a_3 - a_2^2}{3a_1^2}, \quad q = \frac{2a_2^3 - 9a_1a_2a_3 + 27a_1^2a_4}{27a_1^3}, \quad z = A + \frac{a_2}{3a_1}. \qquad \text{(B.2)}$$

Depending on the sign of the discriminant

$$\Delta = \frac{q^2}{4} + \frac{p^3}{27}$$

equation (B.1) can have different number of real roots and also complex conjugate roots.

If $\Delta < 0$, there are 3 real roots

$$z_i = 2\sqrt{-\frac{p}{3}}\cos\left(\frac{\phi + 2\pi(i-1)}{3}\right), \quad i = 1, 2, 3,$$

with

$$\begin{array}{lll} \phi = \arctan\left(-\frac{2}{q}\sqrt{-\Delta}\right) & \text{if} & q < 0, \\ \phi = \arctan\left(-\frac{2}{q}\sqrt{-\Delta}\right) + \pi & \text{if} & q > 0, \\ \phi = \frac{\pi}{2} & \text{if} & q = 0. \end{array}$$

If $\Delta > 0$ there is 1 real root and 2 complex conjugate ones

$$\begin{aligned}
z_1 &= -\sqrt[3]{\tfrac{q}{2} - \sqrt{\Delta}} - \sqrt[3]{\tfrac{q}{2} + \sqrt{\Delta}}, \\
z_2 &= \tfrac{1}{2}\left(\sqrt[3]{\tfrac{q}{2} - \sqrt{\Delta}} + \sqrt[3]{\tfrac{q}{2} + \sqrt{\Delta}}\right) + i\tfrac{\sqrt{3}}{2}\left(\sqrt[3]{\tfrac{q}{2} - \sqrt{\Delta}} - \sqrt[3]{\tfrac{q}{2} + \sqrt{\Delta}}\right), \\
z_3 &= \tfrac{1}{2}\left(\sqrt[3]{\tfrac{q}{2} - \sqrt{\Delta}} + \sqrt[3]{\tfrac{q}{2} + \sqrt{\Delta}}\right) - i\tfrac{\sqrt{3}}{2}\left(\sqrt[3]{\tfrac{q}{2} - \sqrt{\Delta}} - \sqrt[3]{\tfrac{q}{2} + \sqrt{\Delta}}\right).
\end{aligned}$$

If $\Delta = 0$ there are 2 real roots

$$z_1 = -2\sqrt[3]{\frac{q}{2}}, \quad z_2 = \sqrt[3]{\frac{q}{2}}.$$

The roots of the original equation (A.16) are obtained by

$$A_i = z_i - \frac{a_2}{3a_1}, \quad i = 1, 2, 3.$$