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Abstract

We present the gravity dual to a class of three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories on a $U(1) \times U(1)$-invariant squashed three-sphere, with a non-trivial background gauge field. This is described by a supersymmetric solution of four-dimensional $\mathcal{N} = 2$ gauged supergravity with a non-trivial instanton for the graviphoton field. The particular gauge theory in turn determines the lift to a solution of eleven-dimensional supergravity. We compute the partition function for a class of Chern-Simons quiver gauge theories on both sides of the duality, in the large N limit, finding precise agreement for the functional dependence on the squashing parameter. This constitutes an exact check of the gauge/gravity correspondence in a non-conformally invariant setting.

Contents

1   Introduction and summary

There has been considerable interest recently in studying supersymmetric gauge theories on curved manifolds. Quite generally, supersymmetric field theories on compact curved backgrounds are particularly amenable to localization techniques, leading to vast simplifications in the exact computation of partition functions and other observables in strongly coupled field theories. The partition functions of $\mathcal{N} = 2$ gauge theories on S^4 and certain Wilson loops were computed in [1]. Using similar techniques, the partition functions of three-dimensional $\mathcal{N} = 3$ supersymmetric gauge theories on a round S^3 were first computed in [2], and subsequently generalized to $\mathcal{N} = 2$ gauge theories in [3, 4]. One can also consider curved manifolds other than round spheres. For

example, the superconformal indices of four-dimensional and three-dimensional field theories may be computed by putting the theories on $S^1 \times S^3$ [5, 6, 7] and $S^1 \times S^2$ [8, 9, 10, 11], respectively.

A more systematic analysis of the possible curved manifolds on which one can construct supersymmetric theories has been initiated in [12]. One particularly interesting possibility is that of deformed three-spheres, often referred to as *squashed* three-spheres. The partition functions of three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories on different squashed spheres were computed by Hama, Hosomichi and Lee (HHL) in [13]. An interesting deformation in [13] preserves a $U(1) \times U(1)$ isometry, and in the partition function leads to the appearance of the double sine function $s_b(z)$ [14], also referred to as the quantum dilogarithm function. This special function plays an important role in various contexts. For example, the double sine function and the $U(1)^2$-squashed sphere (which is a three-dimensional ellipsoid) are important ingredients in the AGT correspondence [15] and its 3d/3d version [16, 17]. The matching of partition functions also allows one to perform non-trivial tests of conjectured dualities between pairs of three-dimensional field theories [18, 19, 20, 21, 22, 23, 24, 25, 26].

Knowledge of the exact partition functions of three-dimensional Chern-Simons matter theories has also been key for some of the recent non-trivial tests of the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence. In [27] the free energy of the ABJM matrix model arising from localization on the round three-sphere was matched to the dual holographic free energy in the large N limit, in particular reproducing the famous $N^{3/2}$ gravity prediction from a purely field theoretic computation. This matching was extended in [28, 29] to examples with $\mathcal{N} = 3$ supersymmetry, and then subsequently to a large class of $\mathcal{N} = 2$ models in [30, 31, 32]. It is then natural to attempt to construct the gravity dual of $\mathcal{N} = 2$ Chern-Simons quiver theories on the $U(1)^2$-squashed three-sphere, and to compare the holographic free energy with the large N behaviour of the field theoretic free energy obtained from the HHL matrix integral. In this paper we will address this problem for a large class of $\mathcal{N} = 2$ Chern-Simons theories, finding exactly the same non-trivial dependence on the deformation parameter on the two sides.

Preview

In [13], HHL have shown that rigid $\mathcal{N} = 2$ supersymmetric Chern-Simons gauge theories can be put on a $U(1)^2$-invariant squashed three-sphere, by appropriately modifying the Lagrangian and supersymmetry transformations. In particular, the metric used in

[13] is, up to an irrelevant overall constant factor, given by

$$\mathrm{d}s_3^2 \;=\; f^2(\theta)\mathrm{d}\theta^2 + \cos^2\theta\,\mathrm{d}\varphi_1^2 + \frac{1}{b^4}\sin^2\theta\,\mathrm{d}\varphi_2^2 \;, \tag{1.1}$$

where the squashing parameter is $b^2 = \ell/\tilde{\ell}$, and the function $f^2(\theta)$ will be specified below. The spinor parameter χ entering the supersymmetry transformations obeys the modified Killing spinor equation

$$(\nabla_\alpha^{(3)} - \mathrm{i}A_\alpha^{(3)})\chi - \frac{\mathrm{i}}{2f(\theta)}\gamma_\alpha\chi \;=\; 0 \;, \tag{1.2}$$

where $\nabla_\alpha^{(3)}$, $\alpha = 1, 2, 3$, is the spinor covariant derivative constructed from the metric (1.1), γ_α generate $\mathrm{Cliff}(3,0)$, and

$$A^{(3)} \;=\; \frac{1}{2f(\theta)}\left(\mathrm{d}\varphi_1 - \frac{1}{b^2}\mathrm{d}\varphi_2\right) \tag{1.3}$$

is a background gauge field.[1] In [13] the squashed three-sphere (1.1) arises as the metric induced on the hypersurface

$$r_1^2 + b^4 r_2^2 \;=\; r^2 \tag{1.4}$$

in flat $\mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R}^2$ with metric

$$\mathrm{d}s_{\mathbb{R}^4}^2 \;=\; \mathrm{d}r_1^2 + \mathrm{d}r_2^2 + r_1^2\mathrm{d}\varphi_1^2 + r_2^2\mathrm{d}\varphi_2^2 \;. \tag{1.5}$$

Here one can take $r > 0$ to be any constant, although the metric in (1.1) is normalized so that $r = 1$. This leads to the particular function $f^2(\theta) = \sin^2\theta + \frac{1}{b^4}\cos^2\theta$, and by definition (1.1) is then the metric on an *ellipsoid*. However, notice that (1.1) is a non-singular metric on S^3 for *any* strictly positive (or negative) function $f(\theta)$ that approaches sufficiently smoothly $|f(\theta)| \to 1/b^2$ as $\theta \to 0$ and $|f(\theta)| \to 1$ as $\theta \to \frac{\pi}{2}$. This observation will be important in what follows.

The main result of [13] is that the partition function of a supersymmetric gauge theory in the background of (1.1) *and* the gauge field (1.3) can be computed using localization techniques, and reduces to a matrix integral generalizing that of [3, 4]. As we will describe in more detail in section 3.2, for a $U(N)^G$ Chern-Simons quiver gauge theory with Chern-Simons levels k_I, $I = 1, \ldots, G$, the partition function reads

$$Z_b \;=\; \frac{1}{N!^G}\int\left(\prod_{I=1}^{G}\prod_{i=1}^{N}\frac{\mathrm{d}\lambda_i^I}{2\pi}\right)\exp\left[-F_b\left(\lambda_i^I\right)\right] \;, \tag{1.6}$$

---

[1] Written here up to an irrelevant gauge transformation, $A^{(3)} = A_{\mathrm{HHL}}^{(3)} + \frac{1}{2}(\mathrm{d}\varphi_1 - \mathrm{d}\varphi_2)$.

where

$$F_b\left(\lambda_i^I\right) \;=\; -\frac{\mathrm{i}}{b^2}\sum_{I=1}^{G}\frac{k_I}{4\pi}\sum_{i=1}^{N}(\lambda_i^I)^2 - \sum_{I=1}^{G}\sum_{i<j}\left[\log\left(2\sinh\frac{\lambda_i^I-\lambda_j^I}{2}\right)\right. \tag{1.7}$$

$$\left.+\log\left(2\sinh\frac{\lambda_i^I-\lambda_j^I}{2b^2}\right)\right] - \sum_{I\to J}\sum_{i,j=1}^{N} s_b\left[\frac{\mathrm{i}Q}{2}(1-\Delta_{I,J}) - \frac{(\lambda_i^I-\lambda_j^J)}{2\pi b}\right] \;.$$

The first term in (1.7) comes from the classical Chern-Simons action, while the second term is a one-loop contribution from the gauge field multiplet. The final one-loop term in (1.7) contains a sum over bifundamental fields in the fundamental of the Ith gauge group factor and anti-fundamental of the Jth, of R-charge $\Delta_{I,J}$, and we have defined

$$Q \;\equiv\; b+\frac{1}{b} \;. \tag{1.8}$$

Importantly, it turns out that this result *does not* depend on the details of the function $f(\theta)$. In section 3.2 we will review the localization calculation of [13], emphasizing its independence of the precise choice of $f(\theta)$.

In this paper we will present the gravity dual to the set-up described above. In particular, we will discuss a 1/4 supersymmetric solution of $d=4$, $\mathcal{N}=2$ gauged supergravity (Einstein-Maxwell theory) that asymptotically approaches the metric (1.1) and gauge field (1.3), albeit with a function $f(\theta)$ that is different from that used in [13]. Indeed, while the HHL ellipsoid metric arises from the hypersurface (1.4) in *flat space*, instead our boundary three-metric arises from the *same* hypersurface equation (1.4), but now in *hyperbolic space* $\mathbb{H}^4$ (Euclidean AdS$_4$) with metric

$$\mathrm{d}s^2_{\mathbb{H}^4} \;=\; \frac{1}{r_1^2+r_2^2+1}\left[\mathrm{d}r_1^2+\mathrm{d}r_2^2+(r_2\mathrm{d}r_1-r_1\mathrm{d}r_2)^2\right] + r_1^2\mathrm{d}\varphi_1^2+r_2^2\mathrm{d}\varphi_2^2 \;. \tag{1.9}$$

More precisely, our three-metric arises from the limit $r\to\infty$ in (1.4), which leads to the particular function $f^2(\theta)=1/(b^4\cos^2\theta+\sin^2\theta)$ in (1.1). We may therefore refer to our particular squashed S^3 as a *hyperbolic ellipsoid*. Of course, by construction it arises as the conformal boundary of Euclidean AdS$_4$ (1.9), and thus unlike the HHL metric in reference [13] our squashed S^3 metric is conformal to the round metric on S^3. However, it will also be important to turn on an appropriate $U(1)$ instanton. This then uplifts to a supersymmetric solution of eleven-dimensional supergravity of the form $M_4\times Y_7$, where Y_7 is any Sasaki-Einstein seven-manifold, and the product is twisted [33, 34]. (It also uplifts [33] to a solution of the twisted, warped form $M_4\times_w N_7$, corresponding to M5-branes wrapping SLag three-cycles, although we will not use this in this paper.)

Moreover, the Killing spinor for the four-dimensional supergravity solution, restricted to the boundary, precisely solves the equation (1.2).

We will then compute the holographic free energy for this supergravity solution and compare it with the large N limit of the free energy of a large class of $\mathcal{N} = 2$ Chern-Simons quiver theories, obtained from (1.6). We will find exact agreement, and in particular in both cases we will show that the free energy satisfies

$$\mathcal{F}_b = \frac{Q^2}{4}\mathcal{F}_{b=1} \,, \tag{1.10}$$

where Q is given by (1.8). The $U(1)$ gauge field instanton breaks explicitly the symmetries of the conformal group $SO(2,3)$. Therefore, this constitutes an exact check of the gauge/gravity correspondence in a non-conformally invariant setting.

The rest of this paper is organized as follows. In section 2 we discuss the gravity side: we present the solution and compute its holographic free energy. In section 3 we discuss the field theory side: we review the computation of the partition function and extract the large N limit of the free energy. Section 4 concludes. In appendix A we discuss the solution in the more general context of Plebanski-Demianski solutions to Einstein-Maxwell theory. Appendices B and C contain some technical computational details for the Killing spinor and one-loop vector multiplet contribution to the partition function, respectively.

2 The gravity side

2.1 $d = 4$, $\mathcal{N} = 2$ gauged supergravity and $d = 11$ uplift

Our starting point is the action for the bosonic sector of $d = 4$, $\mathcal{N} = 2$ gauged supergravity [35]

$$S = -\frac{1}{16\pi G_4} \int \mathrm{d}^4x \sqrt{\det g_{\mu\nu}} \left(R + 6g^2 - F^2\right) \,. \tag{2.1}$$

Here R denotes the Ricci scalar of the four-dimensional metric $g_{\mu\nu}$, and the cosmological constant is given by $\Lambda = -3g^2$. The graviphoton is an Abelian gauge field A with field strength $F = \mathrm{d}A$, and we have denoted $F_{\mu\nu}F^{\mu\nu} = F^2$. We will mainly be working in Euclidean signature, and have denoted the four-dimensional Newton constant by G_4.

A solution to the equations of motion derived from (2.1), namely

$$R_{\mu\nu} = -3g^2 g_{\mu\nu} + 2\left(F_\mu{}^\rho F_{\nu\rho} - \tfrac{1}{4}F^2 g_{\mu\nu}\right) \,,$$
$$\mathrm{d} *_4 F = 0 \,, \tag{2.2}$$

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is supersymmetric if there is a non-trivial Dirac spinor ϵ satisfying the Killing spinor equation

$$\left[\nabla_\mu + \tfrac{1}{2}g\Gamma_\mu - \mathrm{i}gA_\mu + \tfrac{\mathrm{i}}{4}F_{\nu\rho}\Gamma^{\nu\rho}\Gamma_\mu\right]\epsilon = 0 . \tag{2.3}$$

Here Γ_μ, $\mu = 1, 2, 3, 4$, generate the Clifford algebra Cliff$(4, 0)$, so $\{\Gamma_\mu, \Gamma_\nu\} = 2g_{\mu\nu}$.

As shown in [33, 34], any such solution to $d = 4$, $\mathcal{N} = 2$ gauged supergravity uplifts to a supersymmetric solution of eleven-dimensional supergravity. More precisely, given any Sasaki-Einstein seven-manifold Y_7 with contact one-form η, transverse Kähler-Einstein metric $\mathrm{d}s_T^2$, and with the seven-dimensional metric normalized so that $R_{ij} = 6g_{ij}$, we write

$$\begin{aligned}
\mathrm{d}s_{11}^2 &= R^2 \left(\tfrac{1}{4}\mathrm{d}s_4^2 + \left(\eta + \tfrac{1}{2}A\right)^2 + \mathrm{d}s_T^2\right) , \\
G &= R^3 \left(\tfrac{3}{8}\mathrm{vol}_4 - \tfrac{1}{4} *_4 F \wedge \mathrm{d}\eta\right) .
\end{aligned} \tag{2.4}$$

Here $\mathrm{d}s_4^2$ is the four-dimensional gauged supergravity metric, with volume form vol_4, and we have set $g = 1$. The effective AdS$_4$ radius R is then determined by the quantization of the four-form flux G via

$$N = \frac{1}{(2\pi\ell_p)^6} \int_{Y_7} *_{11}G , \tag{2.5}$$

where ℓ_p is the eleven-dimensional Planck length, which leads to

$$R^6 = \frac{(2\pi\ell_p)^6 N}{6\mathrm{Vol}(Y_7)} . \tag{2.6}$$

The effective four-dimensional Newton constant is then

$$\frac{1}{16\pi G_4} = N^{3/2}\sqrt{\frac{\pi^2}{32 \cdot 27\,\mathrm{Vol}(Y_7)}} . \tag{2.7}$$

In fact it was more generally conjectured in [33] that given any $\mathcal{N} = 2$ warped AdS$_4 \times Y_7$ solution of eleven-dimensional supergravity there is a consistent Kaluza-Klein truncation on Y_7 to the above $d = 4$, $\mathcal{N} = 2$ gauged supergravity theory. Properties of such general solutions have recently been investigated in [36], and we expect the contact structure discussed there to play an important role in this truncation. In particular, it was shown in [36] that (2.7) remains true in this more general setting, provided one replaces the Riemannian volume by the contact volume.

2.2 Supergravity solution

We will be interested in the following supersymmetric solution to the above $d = 4$, $\mathcal{N} = 2$ gauged supergravity theory:

$$
\begin{aligned}
\mathrm{d}s_4^2 &= f_1^2(x,y)\mathrm{d}x^2 + f_2^2(x,y)\mathrm{d}y^2 + \frac{(\mathrm{d}\Psi + y^2\mathrm{d}\Phi)^2}{f_1^2(x,y)} + \frac{(\mathrm{d}\Psi + x^2\mathrm{d}\Phi)^2}{f_2^2(x,y)} \ , \\
A &= g(s^2 - 1)\frac{\mathrm{d}\Psi - xy\mathrm{d}\Phi}{2(y+x)} \ ,
\end{aligned}
\tag{2.8}
$$

where we have defined the functions

$$
f_1^2(x,y) \equiv \frac{y^2 - x^2}{g^2(x^2 - 1)(s^2 - x^2)} \ , \quad f_2^2(x,y) \equiv \frac{y^2 - x^2}{g^2(y^2 - 1)(y^2 - s^2)} \ . \tag{2.9}
$$

This arises as a special case of the class of Plebanski-Demianski solutions to Einstein-Maxwell theory [37], whose supersymmetry was investigated (in Lorentzian signature) in [38]. However, for our purposes it will be crucial to obtain the explicit form of the Killing spinor of this solution, in the context of $\mathcal{N} = 2$ gauged supergravity, and as far as we are aware this analysis is new.

The solution depends on one parameter s, which will take the values $s \in [1, \infty)$. In fact, as anticipated in the introduction, the metric in (2.8) is locally just the (Euclidean) AdS$_4$ metric, for any value of s, as is easily verified by checking that the Riemann curvature tensor obeys $R_{\mu\nu\rho\sigma} = -g^2(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$. The unusual coordinate system in (2.8) is inherited from its origin as a Plebanski-Demianski solution [37], as discussed further in appendix A. We shall also make use of the following coordinates:

$$
\frac{x^2 - 1}{s^2 - 1} \equiv \cos^2\theta \ , \qquad \Psi \equiv \frac{s\varphi_2 - \varphi_1}{g^2(s^2 - 1)} \ , \qquad \Phi \equiv \frac{s\varphi_1 - \varphi_2}{sg^2(s^2 - 1)} \ . \tag{2.10}
$$

Introducing also

$$
h^2(\theta) \equiv s^2\cos^2\theta + \sin^2\theta \ , \tag{2.11}
$$

the four-dimensional metric in (2.8) becomes

$$
\begin{aligned}
\mathrm{d}s_4^2 &= \frac{y^2 - h^2(\theta)}{g^2(y^2 - 1)(y^2 - s^2)}\mathrm{d}y^2 + \frac{y^2 - h^2(\theta)}{g^2 h^2(\theta)}\mathrm{d}\theta^2 + \frac{y^2 - 1}{g^2}\cos^2\theta\mathrm{d}\varphi_1^2 \\
&\quad + \frac{y^2 - s^2}{s^2 g^2}\sin^2\theta\mathrm{d}\varphi_2^2 \ .
\end{aligned}
\tag{2.12}
$$

Here the ranges of the coordinates are $y \in [s, \infty)$, $\theta \in [0, \frac{\pi}{2}]$, while φ_1 and φ_2 are periodic with period 2π. In particular this implies that $x \in [1, s]$ (when $s > 1$). This will be discussed further in section 2.3.

7

Introducing the orthonormal frame

$$e^1 = \frac{\mathrm{d}\Psi + y^2\mathrm{d}\Phi}{f_1} , \qquad e^3 = f_1\mathrm{d}x ,$$

$$e^2 = \frac{\mathrm{d}\Psi + x^2\mathrm{d}\Phi}{f_2} , \qquad e^4 = f_2\mathrm{d}y , \qquad (2.13)$$

the gauge field in (2.8) has field strength

$$F = \mathrm{d}A = \frac{g(s^2 - 1)}{2(y+x)^2}(e^{13} + e^{24}) . \qquad (2.14)$$

In particular, we see that F is anti-self-dual, $*_4 F = -F$, and hence that A is an instanton. We will see in section 2.6 that the action is indeed finite.

Notice that when $s = 1$ the gauge field strength is zero, and moreover the metric as presented in (2.12) is more obviously the metric on Euclidean AdS$_4$, since in this case $h^2(\theta) \equiv 1$ and

$$\mathrm{d}s_4^2 \,|_{s=1} = \frac{\mathrm{d}y^2}{g^2(y^2-1)} + \frac{y^2 - 1}{g^2}(\mathrm{d}\theta^2 + \cos^2\theta\mathrm{d}\varphi_1^2 + \sin^2\theta\mathrm{d}\varphi_2^2) . \qquad (2.15)$$

This describes Euclidean AdS$_4$ as a hyperbolic ball with boundary conformal to the round metric on S^3, the latter appearing in the round brackets. When $s > 1$ the metric (2.12) continues to be a smooth complete metric on AdS$_4$, but with y being a different choice of radial coordinate to that in (2.15), as we shall see in the subsection below. Of course, for $s > 1$ we are also turning on a non-trivial instanton in the graviphoton field, via (2.14).

2.3 Global structure

At large values of y the metric (2.12) is

$$\mathrm{d}s_4^2 = \frac{\mathrm{d}y^2}{g^2 y^2}\left[1 + O(\tfrac{1}{y^2})\right] + \frac{y^2}{g^2}\left[\mathrm{d}s_3^2 + O(\tfrac{1}{y^2})\right] , \qquad (2.16)$$

where

$$\mathrm{d}s_3^2 = \frac{\mathrm{d}\theta^2}{s^2\cos^2\theta + \sin^2\theta} + \cos^2\theta\mathrm{d}\varphi_1^2 + \frac{1}{s^2}\sin^2\theta\mathrm{d}\varphi_2^2 . \qquad (2.17)$$

Thus our Euclidean AdS$_4$ metric has as conformal boundary at $y = \infty$ the metric of a squashed S^3 of the form (1.1), where $s = b^2 \in [1, \infty)$ is the squashing parameter and $f^2(\theta) = 1/h^2(\theta) = 1/(s^2\cos^2\theta + \sin^2\theta)$. In particular, for $s = 1$ we recover the round metric on S^3, as already noted.

8

Returning to the full four-dimensional metric (2.12), it is immediate to see that we obtain a smooth induced metric on S^3 at *any* value of $y \in (s, \infty)$, and that this hence defines a smooth, but incomplete, metric on $\mathbb{R}_{>0} \times S^3$, where $y - s$ is a coordinate on $\mathbb{R}_{>0}$. It thus remains to examine what happens as y tends to s from above. Although one can examine this directly in the above coordinates, it is easier to see what is going on globally by changing coordinates again:

$$
\begin{aligned}
r_1^2 &\equiv (y^2 - 1) \cos^2 \theta \ , \\
r_2^2 &\equiv \frac{1}{s^2}(y^2 - s^2) \sin^2 \theta \ .
\end{aligned}
\tag{2.18}
$$

The metric (2.12) (multiplied by g^2) then becomes the Euclidean AdS$_4$ metric (1.9) presented in the introduction. The parameter s has disappeared, and we directly see the local equivalence to the Euclidean AdS$_4$ metric for all $s \in [1, \infty)$. Notice that for y large, (2.18) gives $y^2 \simeq r_1^2 + s^2 r_2^2$, as claimed in the introduction. Also notice that for $s > 1$, $y = s$ is simply the coordinate singularity $r_2 = 0$. The "centre" of AdS$_4$ in the coordinates (1.9) is $\{r_1 = r_2 = 0\}$, which is $\{y = s, \theta = \frac{\pi}{2}\}$.

It follows from this discussion that our metric (2.12) is simply the metric on the usual Euclidean AdS four-ball, but with a non-standard choice of radial coordinate y. In particular, this means that the conformal class of the induced metric on $y = \infty$ is that of a squashed S^3, with metric given by (2.17). In other words, we have a one-parameter family of "faces of AdS", to use the phrase coined in [39], given by choosing a radial coordinate that depends on s.

Of course, it will also be important for our application that the instanton in (2.8) depends on s. Notice that the field strength F in (2.8) is a non-singular globally defined two-form on our Euclidean AdS ball. Indeed, a short computation gives

$$
\begin{aligned}
F = \ & \frac{s^2 - 1}{2g(y + h(\theta))^2} \Big[\left(\cos^2 \theta \mathrm{d}\varphi_1 + s^{-1} \sin^2 \theta \mathrm{d}\varphi_2 \right) \wedge \mathrm{d}y \\
& - \left[(y^2 - 1)\mathrm{d}\varphi_1 - s^{-1}(y^2 - s^2)\mathrm{d}\varphi_2 \right] \sin \theta \cos \theta \wedge \frac{\mathrm{d}\theta}{h(\theta)} \Big] \ .
\end{aligned}
\tag{2.19}
$$

In particular, notice that $y + x$ is nowhere zero, since both y and $x = h(\theta)$ are strictly positive. On the other hand, there is a *self-dual* instanton for the same metric given by

$$
F_{\mathrm{SD}} = \frac{g(s^2 - 1)}{2(y - x)^2}(e^{13} - e^{24}) \ .
\tag{2.20}
$$

Compare this with (2.14). However, since $y = x$ on the locus $\{y = s, x = s\}$ (or equivalently $\{y = s, \theta = 0\}$), this self-dual instanton is singular on this locus.

9

2.4 Supersymmetry

In this subsection we discuss the supersymmetry of the solution. That the solution is supersymmetric is perhaps not surprising, given that we simply have an instanton on AdS_4 space. However, instantons work a little differently in AdS than on Ricci-flat manifolds, due to the cosmological constant, and the precise form of the Killing spinor on our background (particularly its asymptotic expansion) will be important in section 3. It is perhaps worth noting that the Killing spinor ϵ solving (2.3) is *not* (and even *a priori* could not be) one of the usual Killing spinors of AdS_4.

It will be convenient to choose the following representation of $\mathrm{Cliff}(4,0)$:

$$\hat{\Gamma}_a = \begin{pmatrix} 0 & \sigma_a \\ \sigma_a & 0 \end{pmatrix}, \qquad \hat{\Gamma}_4 = \begin{pmatrix} 0 & i\mathbb{I}_2 \\ -i\mathbb{I}_2 & 0 \end{pmatrix}, \qquad (2.21)$$

where σ_a, $a = 1, 2, 3$, denote the Pauli matrices, and hats denote tangent space quantities. Thus $\{\hat{\Gamma}_m, \hat{\Gamma}_n\} = 2\delta_{mn}$. In particular then $\hat{\Gamma}_5 \equiv \hat{\Gamma}_1 \hat{\Gamma}_2 \hat{\Gamma}_3 \hat{\Gamma}_4$ is given by

$$\hat{\Gamma}_5 = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}, \qquad (2.22)$$

and we may decompose the Dirac spinor ϵ into negative and positive chirality parts as

$$\epsilon = \begin{pmatrix} \epsilon^+ \\ \epsilon^- \end{pmatrix}. \qquad (2.23)$$

One can then substitute into the Killing spinor equation (2.3) using the orthonormal frame (2.13). In fact an immediate consequence of the integrability condition for (2.3) is the relation

$$2g F_{\mu\nu} \epsilon^+ = (\nabla^\rho F_{\mu\nu}) \gamma_\rho \epsilon^-, \qquad (2.24)$$

where we have defined $\hat{\gamma}_a = \sigma_a$, $a = 1, 2, 3$, $\hat{\gamma}_4 = i\mathbb{I}_2$, and made use of the Bianchi identity for F and that the metric is AdS_4. In the present case this may be rewritten

$$\epsilon^+ = -\frac{1}{g(y+x)} \left(\frac{i}{f_2} \mathbb{I}_2 + \frac{1}{f_1} \sigma_3 \right) \epsilon^-, \qquad (2.25)$$

allowing us to algebraically eliminate ϵ^+ in terms of ϵ^-. It is then straightforward, but somewhat tedious, to verify that

$$\epsilon^- = \sqrt{y+x} \begin{pmatrix} \lambda(x,y) \\ i\lambda^*(x,y) \end{pmatrix}, \qquad (2.26)$$

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where

$$\lambda(x,y) \equiv \left(\frac{\sqrt{(s^2 - x^2)(y^2 - 1)} - \mathrm{i}\sqrt{(x^2 - 1)(y^2 - s^2)}}{\sqrt{(s^2 - x^2)(y^2 - 1)} + \mathrm{i}\sqrt{(x^2 - 1)(y^2 - s^2)}} \right)^{1/2} , \qquad (2.27)$$

is the only solution to the Killing spinor equation (2.3), up to a constant of proportionality. Note in particular that the Killing spinor is, in the gauge where A takes the form in (2.8), independent of the angular coordinates Ψ and Φ (or equivalently φ_1 and φ_2).

The solution thus preserves $\mathcal{N} = 1$ supersymmetry, in the sense that it admits a single Dirac spinor ϵ solving (2.3). However, it will be important later that the charge conjugate spinor $\epsilon^c \equiv B\epsilon^*$ also satisfies the Killing spinor equation (2.3) but with A replaced by $-A$. Here B is the charge conjugation matrix

$$B \equiv \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix} , \qquad \varepsilon \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \qquad (2.28)$$

which satisfies the defining properties

$$B^{-1}\Gamma_\mu B = \Gamma_\mu^* , \qquad BB^* = -\mathbb{I}_4 . \qquad (2.29)$$

Thus it is clear that provided ϵ satisfies (2.3), then ϵ^c satisfies

$$\left[\nabla_\mu + \tfrac{1}{2}g\Gamma_\mu + \mathrm{i}gA_\mu - \tfrac{\mathrm{i}}{4}F_{\nu\rho}\Gamma^{\nu\rho}\Gamma_\mu \right] \epsilon^c = 0 . \qquad (2.30)$$

At large y it is straightforward to calculate the asymptotic expansion of the Killing spinor (2.26). Still working in the frame (2.13) one finds

$$\epsilon = \begin{pmatrix} -y^{1/2}\left[1 - \frac{h(\theta)}{2y} + O(\frac{1}{y^2}) \right] \mathrm{i}\chi \\ y^{1/2}\left[1 + \frac{h(\theta)}{2y} + O(\frac{1}{y^2}) \right] \chi \end{pmatrix} , \qquad (2.31)$$

where

$$\chi = \begin{pmatrix} \mathrm{i}e^{\mathrm{i}\theta} \\ e^{-\mathrm{i}\theta} \end{pmatrix} . \qquad (2.32)$$

The latter defines a spinor on the squashed three-sphere conformal boundary, and one finds that χ satisfies the following Killing spinor equation on the squashed sphere (2.17)

$$(\nabla_\alpha^{(3)} - \mathrm{i}gA_\alpha^{(3)})\chi + \frac{\mathrm{i}h(\theta)}{2}\gamma_\alpha\chi = 0 . \qquad (2.33)$$

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Here γ_α generate Cliff$(3,0)$, $\alpha = 1,2,3$, while $A^{(3)}$ denotes the asymptotic value of the gauge field in (2.8), namely

$$
\begin{aligned}
A^{(3)} &= -g(s^2 - 1)\frac{x}{2}\mathrm{d}\Phi \ , \\
&= -\frac{h(\theta)}{2g}\left(\mathrm{d}\varphi_1 - \frac{1}{s}\mathrm{d}\varphi_2\right) \ .
\end{aligned} \tag{2.34}
$$

Notice that (2.33), (2.34) are precisely of the form (1.2), (1.3) in the introduction, on identifying $f(\theta) = -1/h(\theta)$.

The Killing spinor in (2.32) is in the somewhat unusual frame

$$
\begin{aligned}
\hat{e}^1 &= \frac{1}{s}\cos\theta\sin\theta(s\mathrm{d}\varphi_1 - \mathrm{d}\varphi_2) \ , \quad \hat{e}^2 = \frac{1}{s}\left[\mathrm{d}\varphi_2 + \cos^2\theta(s\mathrm{d}\varphi_1 - \mathrm{d}\varphi_2)\right] \ , \\
\hat{e}^3 &= -\frac{\mathrm{d}\theta}{h(\theta)} \ ,
\end{aligned} \tag{2.35}
$$

inherited from (2.13). It is clearly more natural to define the following orthonormal frame for the squashed S^3

$$
\check{e}^1 = \cos\theta\,\mathrm{d}\varphi_1 \ , \qquad \check{e}^2 = \frac{1}{s}\sin\theta\,\mathrm{d}\varphi_2 \ , \qquad \check{e}^3 = -\frac{\mathrm{d}\theta}{h(\theta)} \ . \tag{2.36}
$$

In this frame, and with $\gamma_\alpha = \check{e}^a_\alpha\sigma_a$, one finds that the solution to (2.33) is

$$
\chi = e^{i\pi/4}\begin{pmatrix} e^{i\theta/2} \\ e^{-i\theta/2} \end{pmatrix} \ . \tag{2.37}
$$

This is of course related to (2.32) by a $U(1) \subset SU(2)$ rotation that covers the $SO(2) \subset SO(3)$ rotation relating the frame (2.36) to the corresponding frame given by (2.35). Notice that, in this frame, the Killing spinor (2.37) is independent of the squashing parameter s, and is identical (up to an irrelevant proportionality constant and the gauge transformation in footnote 1) to the Killing spinor $\bar{\epsilon}$ in section 2 of [13]. The Killing spinor ϵ of [13] coincides with the charge conjugate χ^c which satisfies the same Killing spinor equation but with $A^{(3)}$ replaced by $-A^{(3)}$.

2.5   AdS$_4$ with round S^3 boundary

As we argued in sections 2.2 and 2.3, our four-dimensional metric is in fact globally Euclidean anti de Sitter space. In this section we elaborate on this point, presenting the solution in more standard coordinates, and discussing the induced background gauge field and Killing spinors on the boundary.[2]

---

[2]This section was added in version 2 of the preprint in March 2012. We wish to thank Jerome Gauntlett and David Tong for discussions that prompted us to add this section.

A more standard coordinate system on AdS$_4$ is obtained by defining

$$q^2 \equiv r_1^2 + r_2^2 , \qquad \cos^2 \psi \equiv \frac{r_1^2}{r_1^2 + r_2^2} , \tag{2.38}$$

where r_1, r_2 were defined in (2.18). In these coordinates the metric (2.12) becomes

$$g^2 \mathrm{d}s_4^2 = \frac{\mathrm{d}q^2}{1 + q^2} + q^2 \left(\mathrm{d}\psi^2 + \cos^2 \psi \mathrm{d}\varphi_1^2 + \sin^2 \psi \mathrm{d}\varphi_2^2 \right) , \tag{2.39}$$

and the gauge field

$$A = \frac{-(1 + s\sqrt{1 + q^2})\mathrm{d}\varphi_1 + (s + \sqrt{1 + q^2})\mathrm{d}\varphi_2}{2g\sqrt{(1 + s\sqrt{1 + q^2})^2 + (1 - s^2)q^2 \cos^2 \psi}} . \tag{2.40}$$

In these coordinates the "squashing" parameter s manifestly parametrizes purely a deformation of the gauge field from pure gauge, corresponding to $s = 1$. A computation shows that the four-dimensional Killing spinor (2.26) is constructed from

$$\sqrt{y + x}\, \lambda(q, \psi) = \left(\frac{(s^2 - 1) + q^2(-\mathrm{i}\cos \psi + s \sin \psi)^2}{\sqrt{(1 - s\sqrt{1 + q^2})^2 + q^2(1 - s^2)\cos^2 \psi}} \right)^{1/2} , \tag{2.41}$$

and in particular this still depends non-trivially on s.

The metric $\mathrm{d}\tilde{s}_3^2$ and gauge field $\tilde{A}^{(3)}$ induced on the conformal boundary defined by $q = \infty$ are

$$\mathrm{d}\tilde{s}_3^2 = \mathrm{d}\psi^2 + \cos^2 \psi \mathrm{d}\varphi_1^2 + \sin^2 \psi \mathrm{d}\varphi_2^2 , \tag{2.42}$$

$$\tilde{A}^{(3)} = -\frac{\tilde{h}(\psi)}{2g} \left(\mathrm{d}\varphi_1 - \frac{1}{s}\mathrm{d}\varphi_2 \right) , \tag{2.43}$$

respectively, where

$$\tilde{h}^2(\psi) = \frac{s^2}{s^2 \sin^2 \psi + \cos^2 \psi} . \tag{2.44}$$

While the metric is precisely the round metric on the three-sphere, the gauge field is non-trivial and takes essentially the same form as in the original coordinates (2.34).

Since the change of coordinates (2.38) is globally smooth, it follows that the boundary metric (2.17) in the original $\theta, \varphi_1, \varphi_2$ coordinates must be in the same conformal class as the round three-sphere metric. One can confirm this by checking that the Cotton tensor of the metric (2.17) vanishes. More explicitly, the change of coordinates

$$\cos \psi = \frac{s \cos \theta}{h(\theta)} \tag{2.45}$$

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shows that the two metrics are related by a Weyl rescaling via

$$\frac{\mathrm{d}\theta^2}{h^2(\theta)} + \cos^2\theta \mathrm{d}\varphi_1^2 + \frac{1}{s^2}\sin^2\theta \mathrm{d}\varphi_2^2 = \frac{\tilde{h}^2(\psi)}{s^2}\left(\mathrm{d}\psi^2 + \cos^2\psi \mathrm{d}\varphi_1^2 + \sin^2\psi \mathrm{d}\varphi_2^2\right) , \quad (2.46)$$

while the gauge field correspondingly transforms as

$$A^{(3)} = -\frac{h(\theta)}{2g}\left(\mathrm{d}\varphi_1 - \frac{1}{s}\mathrm{d}\varphi_2\right) = -\frac{\tilde{h}(\psi)}{2g}\left(\mathrm{d}\varphi_1 - \frac{1}{s}\mathrm{d}\varphi_2\right) = \tilde{A}^{(3)} . \quad (2.47)$$

The Killing spinor on the boundary, which we will denote by $\tilde{\chi}$, may be extracted by expanding the four-dimensional spinor[3] determined from (2.41) in powers of $q^{1/2}$. We find that the three-dimensional spinor is

$$\tilde{\chi} = \begin{pmatrix} \sqrt{\mathrm{i}\cos\psi - s\sin\psi} \\ \sqrt{\mathrm{i}\cos\psi + s\sin\psi} \end{pmatrix} , \quad (2.48)$$

and it obeys the following equation[4]

$$(\tilde{\nabla}_\alpha - \mathrm{i}g\tilde{A}_\alpha^{(3)})\tilde{\chi} + \frac{\mathrm{i}\tilde{h}^2(\psi)}{2s}\tilde{\gamma}_\alpha\tilde{\chi} - \frac{1}{2}\partial_\psi \log \tilde{h}(\psi)\tilde{\gamma}_\alpha\sigma_3\tilde{\chi} = 0 , \quad (2.49)$$

where $\tilde{\nabla}_\alpha$ is the connection computed with the round metric in (2.42), related to the original one by $\tilde{\nabla}_\alpha = \nabla_\alpha - \frac{1}{2}\tilde{\gamma}_\alpha{}^\beta\partial_\beta \log \tilde{h}(\psi)$. Note that the spinor (2.48) depends on s, and it is therefore different from the standard Killing spinors on the round sphere, to which it reduces when $s = 1$. It may be worth comparing this construction with that in [13]: here we have a round metric, a gauge field, and a non-standard Killing spinor, whereas in [13] they have a squashed metric, a gauge field, and a standard Killing spinor.

Finally, defining a rescaled spinor χ as

$$\chi = \sqrt{\frac{\tilde{h}(\psi)}{s}}\tilde{\chi} , \quad (2.50)$$

and changing coordinates as in (2.45), the equation (2.49) becomes precisely the Killing spinor equation (2.33) obeyed by the original metric (2.17), gauge field (2.34), and spinor (2.37). We will briefly comment on the field theory implication of this in the concluding section.

---

[3]In order to do this, one has to note that the change of coordinates (2.38) induces a natural change of orthonormal frame adapted to the new radial coordinate q.

[4]We used the orthonormal frame defined by $\tilde{e}^1 = \cos\psi \mathrm{d}\varphi_1, \tilde{e}^2 = \sin\psi \mathrm{d}\varphi_2, \tilde{e}^3 = -\mathrm{d}\psi$.

2.6 The holographic free energy

In this section we derive an expression for the holographic free energy of the dual field theory by computing the holographically renormalized on-shell action for the gauged supergravity solution (2.8). This is a standard application of the prescriptions in the literature (see *e.g.* [40]), so the reader uninterested in the details may jump to the final formula for the free energy (2.62).

The total renormalized action comprises three types of term: the bulk on-shell action (2.1) is divergent and therefore one evaluates a regulated action, integrated up to a cut-off $y = r$. Then in general one needs to add boundary terms appropriate to the imposed boundary conditions, and counterterms that remove the divergent part and give a finite result in the limit $r \to \infty$. The general form is therefore

$$I = I_{\text{bulk}}^{\text{grav}} + I_{\text{bulk}}^F + I_{\text{bdry}}^{\text{grav}} + I_{\text{bdry}}^F + I_{\text{ct}}^{\text{grav}} + I_{\text{ct}}^F , \qquad (2.51)$$

where

$$I_{\text{bulk}}^{\text{grav}} + I_{\text{bulk}}^F = -\frac{1}{16\pi G_4} \int_{B_r} \mathrm{d}^4 x \sqrt{\det g_{\mu\nu}} \left(R[g_{\mu\nu}] + 6g^2 - F^2 \right) , \qquad (2.52)$$

$$I_{\text{bdry}}^{\text{grav}} = -\frac{1}{8\pi G_4} \int_{\partial B_r} \mathrm{d}^3 x \sqrt{\det \gamma_{\alpha\beta}} K , \qquad (2.53)$$

$$I_{\text{ct}}^{\text{grav}} = \frac{1}{8\pi G_4} \int_{\partial B_r} \mathrm{d}^3 x \sqrt{\det \gamma_{\alpha\beta}} \left(2g + \frac{1}{2g} R[\gamma_{\alpha\beta}] \right) , \qquad (2.54)$$

$$I_{\text{bdry}}^F = I_{\text{ct}}^F = 0 . \qquad (2.55)$$

Here (2.52) is simply the $d = 4$, $\mathcal{N} = 2$ gauged supergravity action (2.1) with which we started. We evaluate this on the solution (2.8), integrating over the ball B_r that is defined by taking $s \leq y \leq r$. The boundary integral (2.53) is the Gibbons-Hawking term, ensuring that the equations of motion (2.2) do indeed result from varying the action (2.52) with fixed boundary metric $\gamma_{\alpha\beta}$ on $\partial B_r \cong S^3$. Here K denotes the trace of the second fundamental form of this surface. Finally, (2.54) are the counterterms of reference [39]: the sum $I_{\text{bulk}}^{\text{grav}} + I_{\text{bdry}}^{\text{grav}}$ is divergent as we take the cut-off $r \to \infty$, and the counterterms precisely remove this divergence, giving a finite result for (2.51) as $r \to \infty$. $R[\gamma_{\alpha\beta}]$ of course denotes the Ricci scalar of the induced boundary metric in (2.54).

Let us now explain why the boundary term I_{bdry}^F for the gauge field A is not included in (2.51). The AdS/CFT duality requires specifying boundary conditions for fluctuating fields in the bulk. In the background of an asymptotically AdS$_4$ metric of the form

(2.16), we impose the following boundary condition for the gauge field A, in the gauge $A_y = 0$, as $y \to \infty$

$$A_\alpha = A_\alpha^{(3)} + \frac{1}{y} J_\alpha + O\left(\frac{1}{y^2}\right) . \qquad (2.56)$$

This amounts to saying that the gauge field is $O(1)$ to leading order as $y \to \infty$. Notice that our particular gauge field instanton in (2.8) satisfies (2.56). Assuming the boundary condition (2.56), the variation of the Maxwell action is then easily computed to be

$$\delta S_{\text{Maxwell}} = -\frac{1}{2} \int_{\text{bdry}} *_3 J \wedge \delta A^{(3)} . \qquad (2.57)$$

Thus holding $A^{(3)}$ fixed on the boundary leads to a well-defined variational problem for the Maxwell equations. In fact this is precisely the boundary condition we shall want, since we will be regarding $A^{(3)}$ in (2.34) as a fixed background gauge field in the next section. With this boundary condition we then do not need to add a boundary term for the variational problem. Notice from (2.56) that the Maxwell action is automatically finite, and there is no need for any counterterm for F.

It is now straightforward to compute (2.52) – (2.54), and take the limit $r \to \infty$ in (2.51). Let us quote the finite contributions. Using the Einstein equation, for the bulk gravity action we obtain

$$\begin{aligned}
I_{\text{bulk}}^{\text{grav}} &= \frac{3g^2}{8\pi G_4} \int \mathrm{d}^4 x \sqrt{\det g_{\mu\nu}} = \frac{3g^2}{8\pi G_4} \frac{(2\pi)^2}{g^4 s(s^2-1)} \int_1^s \mathrm{d}x \int_s^r \mathrm{d}y (y^2 - x^2) \\
&= \frac{\pi}{2G_4 g^2} + \text{divergent} ,
\end{aligned} \qquad (2.58)$$

where the divergent part will be precisely cancelled by the boundary terms. Curiously, we see that this result is independent of s and indeed it is exactly the same as that obtained for the round three-sphere. This might have been expected, since the bulk metric is just AdS$_4$. However, this expectation is certainly naive, and the result could have depended on s because of the particular slicing of AdS$_4$. While it would be interesting to investigate the role of a solution consisting of AdS$_4$ with squashed three-sphere boundary and no gauge field instanton, we will not pursue this presently. For the instanton action we compute

$$I_{\text{bulk}}^F = \frac{(2\pi)^2}{16\pi G_4} \frac{(s^2-1)}{sg^2} \int_1^s \mathrm{d}x \int_s^\infty \mathrm{d}y \frac{x-y}{(x+y)^3} = \frac{\pi}{8G_4 g^2} \frac{(s-1)^2}{s} , \qquad (2.59)$$

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which is finite as promised and vanishes correctly for $s = 1$. One can check that the terms $I_{\text{bdry}}^{\text{grav}} + I_{\text{ct}}^{\text{grav}}$ cancel the divergent part in (2.58) and do not contribute a finite part upon taking $r \to \infty$. Combining everything we obtain the finite result

$$I = \frac{\pi Q^2}{8g^2 G_4} \,, \tag{2.60}$$

where we have defined

$$Q \equiv \frac{s+1}{\sqrt{s}} = b + \frac{1}{b} \,, \qquad \text{where} \quad s \equiv b^2 \,. \tag{2.61}$$

We thus obtain the result for the round sphere, for which $s = 1$, multiplied by the factor $Q^2/4$. Note that clearly this result does not depend on the choice of coordinates, and thus in particular it applies also to the round sphere boundary metric (plus gauge field).

Finally, setting $g = 1$ in order to uplift to eleven-dimensional supergravity via (2.4), and using the Newton constant formula in (2.7), we obtain the gravitational free energy in the Euclidean quantum gravity approximation:

$$\mathcal{F}_b = I = N^{3/2} Q^2 \sqrt{\frac{\pi^6}{8 \cdot 27 \operatorname{Vol}(Y_7)}} = \frac{Q^2}{4} \mathcal{F}_{b=1} \,. \tag{2.62}$$

We shall reproduce this formula from a dual large N quantum field theory calculation in the next section.

3 The field theory side

3.1 Supersymmetric gauge theories on the $U(1)^2$-squashed S^3

In [13] the authors have constructed $\mathcal{N} = 2$ supersymmetric Lagrangians on a squashed three-sphere with metric (1.1), for gauge theories comprising Chern-Simons and Yang-Mills terms and matter fields in chiral multiplets. They have shown that the Lagrangians and supersymmetry variations may be appropriately modified if one includes a background gauge field A_α of the form (1.3), and the supersymmetry parameter[5] χ obeys the modified Killing spinor equation

$$(\nabla_\alpha - \mathrm{i} A_\alpha)\chi - \frac{\mathrm{i}}{2f(\theta)}\gamma_\alpha \chi = 0 \,. \tag{3.1}$$

---

[5]This was denoted ϵ in [13]; we hope this will not generate confusion.

Although the construction of [13] appears to require the existence of a "second" Killing spinor, denoted $\bar{\epsilon}$ there, in fact this is simply the charge conjugate χ^c, which in general satisfies the same Killing spinor equation (3.1) but with A_α replaced by $-A_\alpha$. In the following we will summarize the supersymmetric Lagrangians constructed by HHL, and their computation of the partition function using localization. For simplicity we will consider a single vector multiplet V and a single chiral multiplet Φ, transforming in the fundamental representation of the gauge group.

A 3d $\mathcal{N} = 2$ vector multiplet V consists of a gauge field $\mathscr{A}_\alpha$, a scalar field σ, a two-component Dirac spinor λ, and scalar field D, all transforming in the adjoint representation of the gauge group. The matter field Φ is a chiral multiplet, consisting of a complex scalar ϕ, a fermion ψ and an auxiliary scalar F, which we take here to be in the fundamental representation of the gauge group. This is assumed to have an arbitrary R-charge Δ. The $\mathcal{N} = 2$ Lagrangian constructed in [13] consists of three terms $S = S_{\mathrm{CS}} + S_{\mathrm{mat}} + S_{\mathrm{YM}}$, that we discuss in turn.[6] The Chern-Simons term is unchanged with respect to the expression in flat space and reads

$$S_{\mathrm{CS}} = \frac{k}{4\pi} \int \mathrm{Tr}\left[\mathscr{A} \wedge \mathrm{d}\mathscr{A} - \frac{2\mathrm{i}}{3}\mathscr{A} \wedge \mathscr{A} \wedge \mathscr{A} - *\mathbf{1}(\lambda^\dagger \lambda - 2D\sigma) \right] , \qquad (3.2)$$

where k is the integer Chern-Simons level. The matter Lagrangian reads

$$\begin{aligned}
S_{\mathrm{mat}} = \int \mathrm{d}^3x\sqrt{\det \gamma_{ij}}\bigg[& \mathscr{D}_\alpha\phi^\dagger\mathscr{D}^\alpha\phi + \phi^\dagger\sigma^2\phi + \mathrm{i}\phi^\dagger D\phi + F^\dagger F \\
& - \mathrm{i}\psi^\dagger\gamma^\alpha\mathscr{D}_\alpha\psi + \mathrm{i}\psi^\dagger\sigma\psi + \mathrm{i}\psi^\dagger\lambda\phi - \mathrm{i}\phi^\dagger\lambda^\dagger\psi \\
& + \mathrm{i}\phi^\dagger\frac{\sigma}{f}\phi + \frac{2\mathrm{i}(\Delta - 1)}{f}v^\alpha\mathscr{D}_\alpha\phi^\dagger\phi + \frac{\Delta(2\Delta - 3)}{2f^2}\phi^\dagger\phi + \frac{\Delta}{4}R\phi^\dagger\phi \\
& - \frac{1}{2f}\psi^\dagger\psi + \frac{\Delta - 1}{f}\psi^\dagger\gamma^\alpha v_\alpha\psi \bigg] .
\end{aligned} \qquad (3.3)$$

The first two lines reduce to the usual expressions in flat space (and $A_\alpha = 0$), while the last two lines are new terms necessary for supersymmetry in the curved background. Here R denotes the scalar curvature of the background metric, and v^α is the vector bilinear $v^\alpha \equiv \chi^\dagger\gamma^\alpha\chi$ constructed from the spinor χ, normalized so that $\chi^\dagger\chi = 1$, and satisfying $v^\alpha v_\alpha = 1$. The covariant derivative is defined as

$$\mathscr{D}_\alpha = \nabla_\alpha - \mathrm{i}[\mathscr{A}_\alpha, \cdot] - \mathrm{i}\Delta A_\alpha , \qquad (3.4)$$

---

[6]Instead of the conventions of [13], we will adopt a somewhat more standard notation.

where ∇_α is the metric covariant derivative, $\mathscr{A}_\alpha$ is the gauge field and A_α is the background $U(1)$ gauge field. Δ is the R-charge (or conformal dimension in the conformally invariant case) of the field on which $\mathscr{D}_\alpha$ acts. This Lagrangian is invariant under a set of supersymmetry variations, independently of the function f [13]; however we will not write these here. Notice that although in Euclidean signature one can have two independent supersymmetry parameters, denoted ϵ and η in [2], in the construction of [13] they are related: the second spinor is simply the charge conjugate of the first, as we have already noted. Finally, the Yang-Mills Lagrangian reads

$$
\begin{aligned}
S_{\mathrm{YM}} \;=\; & \frac{1}{g_{\mathrm{YM}}^2} \int \mathrm{d}^3 x \sqrt{\det \gamma_{ij}} \, \mathrm{Tr}\left[\frac{1}{4}\mathscr{F}_{\alpha\beta}\mathscr{F}^{\alpha\beta} + \frac{1}{2}\mathscr{D}_\alpha \sigma \mathscr{D}^\alpha \sigma + \frac{1}{2}\left(D + \frac{\sigma}{f} \right)^2 \right. \\
& \left. + \frac{\mathrm{i}}{2}\lambda^\dagger \gamma^\alpha \mathscr{D}_\alpha \lambda + \frac{\mathrm{i}}{2}\lambda^\dagger[\sigma, \lambda] - \frac{1}{4f}\lambda^\dagger \lambda \right] ,
\end{aligned}
\tag{3.5}
$$

where notice that the bosonic part is positive semi-definite, and hence the Yang-Mills Lagrangian acts as a regulator in the path integral. This will be important for the localization argument. For an Abelian gauge group there exists also a supersymmetric version of the FI parameter; however this is not relevant for the application in this paper.

3.2   Localization of the partition function

The supersymmetric Yang-Mills and matter Lagrangians above are in fact total supersymmetry variations with respect to the supersymmetry δ_{χ^c} generated by χ^c (of course one could just swap the definitions of χ^c and χ), and therefore they can be used for applying localization. In particular, we have that

$$
\begin{aligned}
\mathcal{L}_{\mathrm{YM}} \;&=\; \delta_{\chi^c}\left(\delta_\chi \mathrm{Tr}\left(\frac{1}{2}\lambda^\dagger \lambda - 2D\sigma \right) \right) , \\
\mathcal{L}_{\mathrm{mat}} \;&=\; \delta_{\chi^c}\left(\delta_\chi \mathrm{Tr}\left(\frac{1}{2}\psi^\dagger \psi - 2\mathrm{i}\phi^\dagger \sigma \phi \right) \right) .
\end{aligned}
\tag{3.6}
$$

Therefore both these terms may be included in the partition function multiplied by arbitrary parameters, so that the total (Euclidean) partition function of a Chern-Simons(-Yang-Mills)-matter theory may be written as

$$
Z \;=\; \int \mathcal{D}[\text{all fields}] \, \mathrm{e}^{-S_{\mathrm{CS}} - t S_{\mathrm{YM}} - (t+1)S_{\mathrm{mat}}} ,
\tag{3.7}
$$

and by the standard localization argument this is independent of the parameter t. The physical theories we are interested in correspond to the value $t = 0$, whereas in the limit

$t \to \infty$ all the contribution comes from the saddle-point, which is a supersymmetric configuration of fields in the curved background. This is characterized by all fields vanishing, except the scalar fields in the vector multiplet which satisfy

$$f D \;=\; -\sigma \;=\; \text{constant} . \tag{3.8}$$

Notice that σ is a matrix-valued constant field, while D is not constant and depends on f. However, we will see that this dependence will disappear completely from the final answer.

The partition function receives a classical contribution from the Chern-Simons action S_{CS} (3.2) evaluated on the solution (3.8), and a one-loop contribution from the Gaussian integral over quadratic fluctuations of all the fields (bosonic and fermionic) in $S_{\mathrm{mat}} + S_{\mathrm{YM}}$, around the classical solution (3.8). The key observation of the authors of [13] is that the bosonic and fermionic eigenmodes entering the one-loop determinants are paired by supersymmetry, and therefore their detailed form is irrelevant since they give cancelling contributions. One can thus circumvent a detailed computation of the spectrum of the relevant kinetic operators by identifying the few eigenmodes that do not pair, and therefore give a net contribution to the one-loop determinant.

Before describing the details, and our main aim of deriving (1.7), let us note that our key observation here is that essentially all the computations in section 5 of [13] go through independently of the specific functional form of $f(\theta)$ entering the metric (1.1). In fact one needs only that $f(\theta)$ enters the Killing spinor equation as in (1.2), and the gauge field as in (1.3), together with the boundary conditions $|f(\theta)| \to 1/b^2$ as $\theta \to 0$ and $|f(\theta)| \to 1$ as $\theta \to \frac{\pi}{2}$, which ensure regularity of the metric. Recall that for the particular ellipsoid metric in [13] one has $f(\theta) = \sqrt{\sin^2 \theta + \frac{1}{b^4} \cos^2 \theta}$, while our "hyperbolic ellipsoid" satisfies the same equations but with $f(\theta) = -1/h(\theta) = -1/\sqrt{b^4 \cos^2 \theta + \sin^2 \theta}$. Having emphasized this, we now briefly summarize the steps in section 5 of [13], and how these results then lead to the partition function given by (1.6), (1.7).

We consider first a chiral matter multiplet $\Phi = (\phi, \psi, F)$, which for simplicity we assume has unit charge under a single Abelian vector multiplet – the extension to arbitrary representations of a non-Abelian gauge group is straightforward, and we will write the result relevant for quiver theories at the end of the section. In this set-up, it is simple to verify that there is a pairing between eigenmodes of the scalar kinetic operator for ϕ and the spinor kinetic operator for ψ. More precisely, a scalar eigenmode with eigenvalue $\mu(\mu - 2i\sigma)$ is paired with two spinor eigenmodes with eigenvalues μ,

$2\mathrm{i}\sigma - \mu$. Here σ is the scalar in the vector multiplet under which Φ has unit charge, which is constant and satisfies (3.8). This pairing involves contractions or products with the Killing spinor χ, and the above statements then depend only on the Killing spinor equation (1.2) and the identity $v_\alpha \gamma^\alpha \chi = \chi$, but not on the specific expression for $f(\theta)$. The contributions of the paired modes to the partition function then precisely cancel, as is familiar in supersymmetric theories.

Thus we need only consider the modes that do *not* have a superpartner under the above pairing. The first such class of modes are spinor eigenmodes characterized by having zero inner product with the Killing spinor χ, so that the corresponding scalar in the would-be pairing is identically zero. One finds that the eigenvalues of such modes are

$$\mu = \mathrm{i}\sigma + m + nb^2 - \tfrac{1}{2}(\Delta - 2)(1 + b^2) \ , \tag{3.9}$$

where the eigenfunction has charge $(m, -n)$ under ∂_{φ_1}, ∂_{φ_2}, so $m, n \in \mathbb{Z}$. The dependence of the modes on the coordinate θ in turn depends on the function $f(\theta)$. However, the *normalizability* depends only on the *boundary conditions* of $f(\theta)$ at $\theta = 0$, $\theta = \pi/2$, and this is determined by regularity of the metric. The upshot is that the modes (3.9) are normalizable if and only if $m, n \geq 0$, precisely as in [13]. The second class of modes are where the two spinor eigenmodes associated to a given scalar are linearly dependent. In this case one finds the spectrum

$$\mu = \mathrm{i}\sigma - m - nb^2 - \tfrac{1}{2}\Delta(1 + b^2) \ , \tag{3.10}$$

where again normalizability requires $m, n \geq 0$. The first type of spinor modes (3.9) are left uncancelled by the scalar determinant, while the second type of spinor modes (3.10), while paired with a scalar, will then be double counted. Thus the first contribute to the numerator, while the second effectively contribute to the denominator in the one-loop determinant of the chiral multiplet, giving

$$Z_{\mathrm{one-loop}}^{\mathrm{mat}}(\sigma) = \frac{\det \Delta_\psi}{\det \Delta_\phi} = \prod_{m,n \geq 0} \frac{mb^{-1} + nb + \frac{Q}{2} + \mathrm{i}\frac{1}{b}\sigma + \frac{Q}{2}(1 - \Delta)}{mb^{-1} + nb + \frac{Q}{2} - \mathrm{i}\frac{1}{b}\sigma - \frac{Q}{2}(1 - \Delta)}$$
$$= s_b\big(\tfrac{\mathrm{i}Q}{2}(1 - \Delta) - \tfrac{1}{b}\sigma\big) \ , \tag{3.11}$$

where recall that $Q = b + 1/b$, and s_b is by definition the double sine function.

The analysis of the one-loop determinant of the vector multiplet $V = (\mathscr{A}_\alpha, \sigma, \lambda, D)$, for an arbitrary gauge group G, is very similar. In this case, after gauge fixing and

combining with the volume of the gauge group, only the transverse vector eigenmodes contribute to the one-loop determinant. In this case the transverse vector eigenmodes are paired with superpartner spinor eigenmodes, both of the same eigenvalue μ. The unpaired modes, which then contribute to the partition function, again fall into two classes. The first are spinor eigenmodes for the kinetic operator for λ that pair with identically zero vector eigenmodes. These have eigenvalues

$$\mu = m + nb^2 + i\alpha(\sigma) , \tag{3.12}$$

where α runs over the roots of G. Again normalizability requires $m, n \geq 0$, but not *both* zero, *i.e.* the mode $m = n = 0$ is *not* a normalizable unpaired spinor mode. The second are vector eigenmodes that pair with identically zero spinor eigenmodes. These also have eigenvalues (3.12), but now normalizability requires $m, n \leq -1$. The first class then contribute to the numerator, while the second contribute to the denominator in the one-loop determinant of the vector multiplet, giving a total contribution

$$
\begin{aligned}
Z_{\text{one-loop}}^{\text{vector}}(\sigma) &= \frac{\det \Delta_\lambda}{\det \Delta_{\mathscr{A}_\alpha^\perp}} \\
&= \prod_{\text{roots } \alpha} \frac{1}{i\alpha(\sigma)} \prod_{m,n \geq 0} \frac{m + nb^2 + i\alpha(\sigma)}{-m - 1 + (-n - 1)b^2 + i\alpha(\sigma)} \\
&= \prod_{\text{positive roots } \alpha} \frac{4 \sinh(\pi\alpha(\sigma)) \sinh(\pi b^{-2}\alpha(\sigma))}{\alpha(\sigma)^2} .
\end{aligned}
\tag{3.13}
$$

Notice here we have included the $m = n = 0$ mode in (3.12) in the numerator, but then explicitly divided by $i\alpha(\sigma)$ to remove it in the middle line of (3.13). The equality in the last line is explained in appendix C.

In fact we shall be interested only in the case where $G = U(N)$. In this case we may take the Cartan to be the diagonal $N \times N$ matrices, and write

$$\sigma = \left(\frac{\lambda_1}{2\pi}, \ldots, \frac{\lambda_N}{2\pi} \right) , \tag{3.14}$$

where $\frac{\lambda_i}{2\pi}$, $i = 1, \ldots, N$, are the eigenvalues of σ. Then the roots of G are labelled by integers $i \neq j$ with

$$\alpha_{ij}(\sigma) = \frac{\lambda_i - \lambda_j}{2\pi} , \tag{3.15}$$

with a choice of positive roots being $\{\alpha_{ij} \mid i < j\}$. Taking into account also the Vandermonde determinant (see appendix C), the one-loop vector multiplet determinant

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(3.13) then reduces to

$$\prod_{i<j} 4 \sinh \frac{\lambda_i - \lambda_j}{2} \sinh \frac{\lambda_i - \lambda_j}{2b^2} \, , \tag{3.16}$$

which is of the form presented in (1.7).

For a chiral multiplet Φ in a general representation $\mathcal{R}$ of the gauge group G, one should simply replace σ in (3.11) by $\rho(\sigma)$, and then take the product over weights ρ in a weight-space decomposition of $\mathcal{R}$. For the bifundamental representation of $U(N)_I \times U(N)_J$, this is

$$\rho_{ij}(\sigma) \;=\; \frac{\lambda_i^I - \lambda_j^J}{2\pi} \, , \tag{3.17}$$

which again directly leads to the form presented in (1.7).

Finally, the first term in (1.7) is the contribution from the classical Chern-Simons action, which upon localization reads

$$
\begin{aligned}
S_{\mathrm{CS}} &= \frac{\mathrm{i}k}{4\pi} \int_{S^3_{\mathrm{squashed}}} 2 \operatorname{Tr}(D\sigma) \\
&= -\frac{\mathrm{i}k}{4\pi} \int_{\theta=0}^{\pi/2} \int_{\varphi_1=0}^{2\pi} \int_{\varphi_2=0}^{2\pi} \sqrt{\det \gamma_{ij}} \; \mathrm{d}^3 x \frac{2}{f(\theta)} \operatorname{Tr} \sigma^2 \\
&= -\frac{\mathrm{i}k}{4\pi b^2} \sum_{i=1}^{N} \lambda_i^2 \, .
\end{aligned}
\tag{3.18}
$$

Here we have substituted $D = -\sigma/f$ (3.8), used the Riemannian measure $\sqrt{\det \gamma_{ij}} = \frac{1}{b^2} f(\theta) \sin\theta \cos\theta$ for the metric (1.1), so that $f(\theta)$ cancels in (3.18), and substituted $\operatorname{Tr} \sigma^2 = \sum_{i=1}^{N} \left(\frac{\lambda_i}{2\pi}\right)^2$. This completes our derivation of the partition function (1.6), (1.7).

3.3 Large N limit of the free energy

In this section we evaluate the partition function (1.6), for a large class of Chern-Simons quiver theories, in the "M-theory limit" in which the rank N is taken to infinity while the Chern-Simons levels k_I are held fixed. This is a relatively straightforward modification of the computation presented in [30, 31, 32], and so we shall be as brief as possible.[7]

---

[7]Very recently we note that a completely different method has been found for computing this M-theory limit [41].

As in [28], the idea is to compute the integral (1.6) in a saddle point approximation. Solutions to the saddle point equations may be viewed as zero force configurations between the eigenvalues λ_i^I, which interact via a potential. As the number of eigenvalues N for each gauge group tends to infinity, one has a continuum limit in which one can replace the sums over eigenvalues in (1.6) by integrals. In particular, one can then separate the interactions between eigenvalues into "long range forces", for which the interaction between eigenvalues is non-local, plus a local interaction. A key point, observed in [30], is that for an appropriate class of *non-chiral* Chern-Simons quiver theories, these long ranges forces automatically cancel. We begin by showing that this statement is unmodified for the corresponding supersymmetric theories on the squashed sphere, with $b \neq 1$.

The long range forces referred to above are related to the leading terms in an asymptotic expansion of the functions appearing in the integrand in (1.6). In particular, if we define

$$f_b(z) \equiv \log s_b(z) \ , \tag{3.19}$$

where $s_b(z)$ is the double sine function, then the long range forces are determined by

$$f_b^{\mathrm{asymp}}(z) \equiv \frac{\mathrm{i}\pi}{2} \left(z^2 + \frac{b^2 + b^{-2}}{12} \right) \mathrm{sign}\,(\mathrm{Re}\,z) \ . \tag{3.20}$$

Here $f_b(z) - f_b^{\mathrm{asymp}}(z)$ has the property that it tends to zero as $|\mathrm{Re}\,z| \to \infty$ [42, 43]. Similarly, we have

$$[\log \sinh z]^{\mathrm{asymp}} \equiv z\,\mathrm{sign}\,(\mathrm{Re}\,z) \ . \tag{3.21}$$

One then takes the continuum limit of (1.6), so that the sums become Riemann integrals

$$\frac{1}{N} \sum_{i=1}^{N} \longrightarrow \int_{x_{\min}}^{x_{\max}} \rho(x)\mathrm{d}x \ , \tag{3.22}$$

where we make the following ansatz for the eigenvalues [28]

$$\lambda^I(x) = N^\alpha x + \mathrm{i}y^I(x) \ , \tag{3.23}$$

with $\alpha > 0$. Note here that we have deformed the real eigenvalues in (1.6) into the complex plane in (3.23), as is often necessary when performing the saddle point method for evaluation of integrals, and that the function $\rho(x)$ describes the eigenvalue density.

In this limit, and substituting the functions $f_b(z)$ and $\sinh z$ by their asymptotic forms in (3.20), (3.21), we obtain the following long range contribution to F:

$$
\begin{aligned}
-F_{\text{asymp}} \; = \; & N^2 \int_{x_{\min}}^{x_{\max}} \rho(x) \mathrm{d}x \int_{x_{\min}}^{x_{\max}} \rho(x') \mathrm{d}x' \, \text{sign}(x - x') \bigg\{ \frac{Q}{4b} \sum_{I=1}^{G} \lambda^I(x) - \lambda^I(x') \\
& - \sum_{I \to J} \frac{Q}{4b}(1 - \Delta_{I,J}) \left[\lambda^I(x) - \lambda^J(x') \right] + \frac{i\pi}{2b^2} \left(\frac{\lambda^I(x) - \lambda^J(x')}{2\pi} \right)^2 \bigg\}
\end{aligned}
\tag{3.24}
$$

Here we have already used the fact that a constant inserted into the curly bracketed expression in (3.24) does not affect the integral, due to the skew symmetry under exchanging $x \leftrightarrow x'$. In fact this same symmetry may then be used to argue that the last quadratic term in (3.24) also contributes zero, provided that the quiver is *non-chiral*: that is, for every bifundamental field transforming as $I \to J$, there is an associated field transforming as $J \to I$. The terms quadratic in λ^I in (3.24) then cancel pairwise, and we may further simplify (3.24) to

$$
\begin{aligned}
-F_{\text{asymp}} \; = \; & \frac{QN^2}{2b} \int_{x_{\min}}^{x_{\max}} \rho(x) \mathrm{d}x \int_{x_{\min}}^{x_{\max}} \rho(x') \mathrm{d}x' \, \text{sign}(x - x') \bigg\{ \sum_{I=1}^{G} \lambda^I(x) \\
& - \frac{1}{2} \sum_{I \to J} (1 - \Delta_{I,J}) \left[\lambda^I(x) + \lambda^J(x) \right] \bigg\} .
\end{aligned}
\tag{3.25}
$$

The coefficient of $\lambda^I(x)$ in the integrand is then

$$
1 - \frac{1}{2} \sum_{\text{fixed } I \to J} (1 - \Delta_{I,J}) - \frac{1}{2} \sum_{\text{fixed } I \leftarrow J} (1 - \Delta_{J,I}) .
\tag{3.26}
$$

Thus provided this expression vanishes for each I, the long range contribution F_{asymp} is zero. As noted in [30], curiously (3.26) are in fact the beta function equations for the parent four-dimensional $\mathcal{N} = 1$ quiver gauge theory.

We thus now restrict to non-chiral Chern-Simons quiver gauge theories with an R-symmetry that satisfies (3.26). For such theories the long range forces between eigenvalues cancel, and it remains to compute the leading order contribution to the free energy in the M-theory limit. From (1.6) one easily computes

$$
F_{\text{classical}} \; = \; \frac{N^{1+\alpha}}{2\pi b^2} \int_{x_{\min}}^{x_{\max}} \rho(x) \mathrm{d}x \sum_{I=1}^{G} k_I x y^I(x) + o(N^{1+\alpha}) ,
\tag{3.27}
$$

so that the $b = 1$ result is simply rescaled by $1/b^2$. The one-loop contribution from each vector multiplet is

$$
F_{\text{gauge}} \; = \; \frac{\pi^2 b Q N^{2-\alpha}}{6} \int_{x_{\min}}^{x_{\max}} \rho(x)^2 \mathrm{d}x + o(N^{2-\alpha}) ,
\tag{3.28}
$$

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leading instead to a $bQ/2$ rescaling of the $b = 1$ result. Notice that in obtaining (3.28) we are effectively using the substitution

$$\log \sinh z - [\log \sinh z]^{\text{asymp}} \simeq -\frac{\pi^2}{6}\delta(\text{Re}\, z) \, , \qquad (3.29)$$

in (1.6) – a more detailed discussion of precisely how this delta function arises may be found around equation (3.33) of [30]. Finally, the one-loop matter contribution follows from the similar approximation (see also appendix A of [44])

$$f_b(z) - f_b(z)^{\text{asymp}} \simeq \frac{\pi}{3}\delta(\text{Re}\, z)\left[(\text{Im}\, z)^3 - \tfrac{1}{4}(b^2 + b^{-2})\text{Im}\, z\right] \, , \qquad (3.30)$$

which for a single bifundamental field $I \to J$ then gives

$$F_{I,J} = -\frac{2\pi^2 b N^{2-\alpha}}{3}\int_{x_{\min}}^{x_{\max}} \rho(x)^2 \mathrm{d}x\left[Y_{I,J}(x)^3 - \tfrac{1}{4}(b^2 + b^{-2})Y_{I,J}(x)\right] + o(N^{2-\alpha}) \, , (3.31)$$

where we have defined

$$Y_{I,J}(x) \equiv \frac{Q}{2}(1 - \Delta_{I,J}) - \frac{y^I(x) - y^J(x)}{2\pi b} \, . \qquad (3.32)$$

Now, the sum over G $U(N)$ vector multiplets gives G times the contribution (3.28). Using (3.26) we may then write

$$G = \sum_{I \to J}(1 - \Delta_{I,J}) \, , \qquad (3.33)$$

where the sum is over all bifundamental fields. Using the fact that the quiver is non-chiral, with each bifundamental $I \to J$ being paired with a corresponding bifundamental $J \to I$, the contributions from the one-loop vector and matter multiplets combine to give

$$F_{\text{one-loop}} = \frac{(bQ)^3\pi^2 N^{2-\alpha}}{2^3 \cdot 3b^2}\int_{x_{\min}}^{x_{\max}} \rho(x)^2 \mathrm{d}x \sum_{\text{pairs}\, I \leftrightarrow J}\frac{(2 - \Delta_{I,J}^+)}{2}\left\{\Delta_{I,J}^+(4 - \Delta_{I,J}^+)\right.$$
$$\left. -3\left[\frac{2\left(y^I(x) - y^J(x)\right)}{\pi bQ} + \Delta_{I,J}^-\right]^2\right\} + o(N^{2-\alpha}) \, , \qquad (3.34)$$

where we have defined

$$\Delta_{I,J}^\pm \equiv \Delta_{I,J} \pm \Delta_{J,I} \, , \qquad (3.35)$$

for each bifundamental pair.

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As in the $b = 1$ case, we thus see that in order for the classical and one-loop contributions in (3.27), (3.34) to be the same order in N, which in turn is necessary for a saddle point solution, we must take $\alpha = \frac{1}{2}$. Then making the change of variable

$$\hat{y}^I(x) \equiv \frac{2}{bQ}y^I(x) , \tag{3.36}$$

the leading order action obtained by combining the classical and one-loop terms is

$$F = N^{3/2}\left\{ \frac{bQ}{2b^2} \int_{x_{\min}}^{x_{\max}} \rho(x)\mathrm{d}x \left[\sum_{I=1}^{G} \frac{k_I}{2\pi}x\hat{y}^I(x) \right] + \frac{(bQ)^3}{2^3b^2} \frac{\pi^2}{3} \int_{x_{\min}}^{x_{\max}} \rho(x)^2\mathrm{d}x \right. \tag{3.37}$$

$$\left. \sum_{\text{pairs } I \leftrightarrow J} \frac{\left(2 - \Delta^+_{I,J}\right)}{2} \left\{ \Delta^+_{I,J}(4 - \Delta^+_{I,J}) - 3 \left[\frac{\hat{y}^I(x) - \hat{y}^J(x)}{\pi} + \Delta^-_{I,J} \right]^2 \right\} \right\} .$$

Setting $b = 1$ we precisely recover the results of [30, 31, 32]. For $b > 1$ we see that the classical contribution has effectively been scaled by $bQ/2b^2$, while the one-loop contribution has been scaled by $(bQ)^3/2^3b^2$, relative to the $b = 1$ result. Alternatively, we may view this as rescaling the *entire* action by the latter factor of $(bQ)^3/2^3b^2$, and in turn rescaling the *Chern-Simons couplings* k_I by $k_I \to (2/bQ)^2k_I$. Provided the Chern-Simons quiver theory is dual to M-theory on an $\mathrm{AdS}_4 \times Y_7$ background, then the free energy in the $b = 1$ case scales as $\sqrt{k}$ if one multiplies $k_I \to k \cdot k_I$, since the volume of Y_7 scales as $1/k$. Taking this into account, we see from (3.37) that the final result for the free energy, obtained by extremizing (3.37) and evaluating at the critical point, is given by

$$\mathcal{F}_b = F_{\text{critical}} = \frac{(bQ)^3}{2^3b^2} \cdot \frac{2}{bQ} \cdot \mathcal{F}_{b=1}$$

$$= \frac{Q^2}{4}\mathcal{F}_{b=1} . \tag{3.38}$$

The large N matching of the free energy on the round three-sphere, $\mathcal{F}_{b=1}$, with the holographic free energy computed in AdS_4 was first demonstrated in [27] for the ABJM model, and extended to larger classes of theories in [28, 29, 30, 31, 32]. Thus we precisely reproduce the dual gravity computation (2.62).

4   Discussion

In this paper we presented a class of supersymmetric solutions of eleven-dimensional supergravity, and conjectured that this is dual to supersymmetric $\mathcal{N} = 2$ gauge theories

on the background of a squashed three-sphere and a $U(1)$ gauge field, whose partition function may be computed using supersymmetric localization [13]. Indeed, although the restriction of our gravity solution to the three-dimensional boundary is slightly different to the background considered in [13], we have nevertheless argued that the localized partition functions for the two backgrounds are equal. Recall that in section 2.5 we showed that our particular squashed S^3 is related by a smooth Weyl transformation to the round S^3. This is particularly clear from the gravity dual description, where the two metrics simply arise from different slicings of AdS_4. However, what's not so clear is whether the localization and field theory partition function are invariant under Weyl rescalings, although we expect that this will be true. At least for large N, this would necessarily have to be true from the AdS/CFT correspondence. The possibility of obtaining the gravity dual of exactly the field theory background in [13], or for other choices of the function $f(\theta)$, remains an open problem.

As a non-trivial test of this correspondence we have successfully matched the holographic free energy to the large N behaviour of the field theoretic free energy, computed from the matrix model. On both sides the result takes the form of that of the round three-sphere result, multiplied by the factor $(Q/2)^2$ where $Q = b + 1/b$. One of the original motivations for studying supersymmetric gauge theories on the squashed S^3 in [13] was the relation, via the AGT correspondence [15], to Liouville or Toda theories with coupling b. Of course, a major difference here is that our N counts the number of M2-branes, while in the AGT correspondence it is M5-branes that appear.

This construction potentially has numerous generalizations. On the one hand one should explore the possibilities for curved backgrounds on which one can place rigid supersymmetric field theories, pursuing the work of [12]. On the other hand, it is then natural to attempt to construct gravity duals for each of these cases. Indeed, the relation between rigid and local supersymmetry is already clear from the results of [12]. We anticipate that immediate generalizations will arise from the class of Plebanski-Demianski solutions to four-dimensional $\mathcal{N} = 2$ gauged supergravity [38], or indeed from yet more general (Euclidean) supersymmetric solutions to this theory [45, 46, 47, 48]. For example, the gravity dual to the construction in [44] might be found within these classes. Another immediate extension is to embed (via a consistent truncation) our solution, and these generalizations, in the context of general $\mathcal{N} = 2$, $\mathrm{AdS}_4 \times Y_7$ solutions that the authors have investigated in [36]. It would also be natural to explore gauge/gravity dualities where the field theory lives on non-trivial curved backgrounds in dimensions other than three. In particular, we expect that this point of view should

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be useful for constructing supersymmetric gauge theories on deformed four-spheres or other curved four-manifolds.

Acknowledgments

A   Plebanski-Demianski origin of the solution

The Plebanski-Demianski solutions [37] are a large class of exact solutions to four-dimensional Einstein-Maxwell theory, *i.e.* they solve the equations of motion (2.2). In fact they are the *most general* such solutions of Petrov type D, and it is this property that allows one to solve the Einstein equations in closed form. Many well-known solutions, such as the Kerr-Newman solution describing a rotating, charged black hole, arise as particular limits.

Our starting point will be the form of the Plebanski-Demianski solutions essentially as presented in [38]. In Euclidean signature, the metric can be written

$$
\begin{aligned}
\mathrm{d}s_4^2 \;=\;& \frac{\mathcal{Q}(q)}{q^2 - p^2}(\mathrm{d}\tau + p^2\mathrm{d}\sigma)^2 + \frac{\mathcal{P}(p)}{p^2 - q^2}(\mathrm{d}\tau + q^2\mathrm{d}\sigma)^2 + \frac{q^2 - p^2}{\mathcal{Q}(q)}\mathrm{d}q^2 \\
& + \frac{p^2 - q^2}{\mathcal{P}(p)}\mathrm{d}p^2 \;,
\end{aligned}
\tag{A.1}
$$

where $\mathcal{P}(p)$ and $\mathcal{Q}(q)$ are quartic polynomials given by[8]

$$
\begin{aligned}
\mathcal{P}(p) \;&=\; g^2 p^4 - E p^2 + 2Np - P^2 + \alpha \;, \\
\mathcal{Q}(q) \;&=\; g^2 q^4 - E q^2 + 2Mq - Q^2 + \alpha \;.
\end{aligned}
\tag{A.2}
$$

---

[8]Note that the constant Q defined in this appendix is different from the parameter $Q = b + 1/b$ discussed in the main text. To obtain the metrics in the Euclideanized form presented here, one should take the solutions as presented in [38] and map $q \mapsto \mathrm{i}q$, $M \mapsto \mathrm{i}M$, $Q \mapsto \mathrm{i}Q$ (together with $\sigma \mapsto -\sigma$).

Here we have assumed a negative cosmological constant $\Lambda = -3g^2$, as in (2.2), and E, α, M, N, P and Q are arbitrary constants. The gauge field is

$$A = \frac{pP + qQ}{p^2 - q^2}\mathrm{d}\tau + pq\frac{qP + pQ}{p^2 - q^2}\mathrm{d}\sigma , \tag{A.3}$$

which thus depends only on the parameters P and Q. Moreover, one easily checks that when $P = \pm Q$ the gauge field A has self-dual/anti-self-dual curvature $F = \mathrm{d}A$ (depending on the choice of orientation), and that the metric (A.1) is Einstein.

In [38] the authors studied which of the Plebanski-Demianski solutions above are supersymmetric solutions to the $d = 4$, $\mathcal{N} = 2$ gauged supergravity described in section 2.1; that is, which admit non-trivial solutions to the Killing spinor equation (2.3). This leads to the following BPS equations for the parameters:

$$NQ + MP = 0 ,$$
$$\left[N^2 - M^2 - E(P^2 - Q^2)\right]^2 = 4g^2\alpha(P^2 - Q^2)^2 . \tag{A.4}$$

These arise from the BPS equations as presented in [38], on making the Euclidean change of variables described in the footnote above.

For applications to the AdS/CFT correspondence one is interested in solutions which have an asymptotic conformal boundary. It is then natural to assume that either p or q is the radial variable near this boundary, and without loss of generality we take this to be q. As $q \to \pm\infty$ the metric (A.1) tends to

$$g^2\mathrm{d}s_4^2 = \frac{\mathrm{d}q^2}{q^2} + q^2\mathrm{d}s_3^2 , \tag{A.5}$$

where the corrections are $O(1/q^2)$ relative to this metric, and the boundary three-metric is defined as

$$\frac{1}{g^2}\mathrm{d}s_3^2 = -\frac{\mathrm{d}p^2}{\mathcal{P}(p)} - \mathcal{P}(p)\mathrm{d}\sigma^2 + g^2(\mathrm{d}\tau + p^2\mathrm{d}\sigma)^2 . \tag{A.6}$$

In principle one could now carry out a systematic analysis of which solutions to the BPS equations (A.4) lead to a compact smooth boundary three-manifold of the form (A.6), with moreover a smooth interior metric (A.1).[9] However, motivated by the field theory analysis on the $U(1)^2$-squashed sphere in [13], we will content ourselves here by looking for a solution where the boundary three-metric (A.6) takes the form (1.1). We intend to return to the more general problem in future work.

---

[9]It is also important to ensure that the field strength F is everywhere non-singular.

We begin by noting that the polynomial $\mathcal{P}(p)$ in (A.2) may be written

$$\mathcal{P}(p) \;=\; g^2(p^2 - p_1^2)(p^2 - p_2^2) + 2Np \; . \tag{A.7}$$

This hence reduces to a simple quadratic in p^2 when $N = 0$. Assuming the latter, we may then introduce coordinates

$$\frac{p^2 - p_1^2}{p_2^2 - p_1^2} \;=\; \cos^2\theta \;, \qquad \frac{p_2^2 - p^2}{p_2^2 - p_1^2} \;=\; \sin^2\theta \;,$$

$$\sigma \;=\; \frac{1}{g^2(p_2^2 - p_1^2)}\left(\frac{1}{p_1}\varphi_1 - \frac{1}{p_2}\varphi_2\right) \;,$$

$$\tau \;=\; \frac{1}{g^2(p_2^2 - p_1^2)}\left(-p_1\varphi_1 + p_2\varphi_2\right) \;, \tag{A.8}$$

to obtain the boundary metric

$$\mathrm{d}s_3^2 \;=\; \frac{\mathrm{d}\theta^2}{p_2^2\cos^2\theta + p_1^2\sin^2\theta} + \frac{1}{p_1^2}\cos^2\theta\,\mathrm{d}\varphi_1^2 + \frac{1}{p_2^2}\sin^2\theta\,\mathrm{d}\varphi_2^2 \; . \tag{A.9}$$

Multiplying by p_1^2 and identifying $p_2/p_1 = s$ then precisely leads to our boundary metric (2.17). Notice that all we have assumed to obtain this result is $N = 0$.

Of course, we must then find a smooth filling of this boundary metric. Our four-dimensional metric and gauge field (2.8) arise from the solution

$$M \;=\; 0 \;, \qquad E^2 \;=\; 4g^2\alpha \;, \qquad P \;=\; -Q \tag{A.10}$$

of the BPS equations (A.4). The coordinates in (2.8) are obtained by making the additional rescalings

$$p \;=\; p_1 x \;, \qquad q \;=\; p_1 y \;, \qquad \tau \;=\; \frac{1}{p_1}\Psi \;, \qquad \sigma \;=\; \frac{1}{p_1^3}\Phi \; . \tag{A.11}$$

It is not difficult to see that (2.8), or equivalently (A.10), is the *only* regular solution of the BPS equations (A.4), although this involves analysing a number of subcases and we omit the details. Of course, in any case in principle one should show that (2.8) is the unique regular solution of the Einstein-Maxwell equations with appropriate boundary conditions, not just the unique solution within the supersymmetric Plebanski-Demianski class. This uniqueness question has been addressed in the mathematics literature for Einstein metrics – see, for example, [49, 50, 51] – but we are not aware of any detailed work on the problem in Einstein-Maxwell theory.

B   Supergravity Killing spinor

In this appendix we give some further details of the Killing spinor computation in section 2.4. It is straightforward to substitute the metric and gauge field (2.8) into the Killing spinor equation (2.3), using the orthonormal frame (2.13) and explicit basis of Cliff$(4,0)$ given in (2.21). In particular, one extracts the following y and x components of the Killing spinor equation:

$$\partial_y \epsilon^- + \frac{f_2}{f_1}\frac{1}{2(y-x)}\mathrm{i}\sigma_3\epsilon^- - \frac{gf_2}{2}\mathrm{i}\mathbb{1}_2\epsilon^+ = 0 , \tag{B.1}$$

$$\partial_y \epsilon^+ + \frac{f_2}{f_1}\frac{1}{2(y+x)}\mathrm{i}\sigma_3\epsilon^+ + \frac{gf_2}{2}\left(\mathrm{i}\mathbb{1}_2 + w\mathrm{i}\sigma_2\right)\epsilon^- = 0 , \tag{B.2}$$

$$\partial_x \epsilon^- + \frac{f_1}{f_2}\frac{1}{2(y-x)}\mathrm{i}\sigma_3\epsilon^- + \frac{gf_1}{2}\sigma_3\epsilon^+ = 0 , \tag{B.3}$$

$$\partial_x \epsilon^+ - \frac{f_1}{f_2}\frac{1}{2(y+x)}\mathrm{i}\sigma_3\epsilon^+ + \frac{gf_1}{2}\left(\sigma_3 + w\mathrm{i}\sigma_1\right)\epsilon^- = 0 . \tag{B.4}$$

Here we have defined the function

$$w(x,y) \equiv \frac{s^2 - 1}{(y+x)^2} , \tag{B.5}$$

so that the gauge field curvature is

$$F = \frac{gw}{2}(e^{13} + e^{24}) \tag{B.6}$$

in the frame (2.13). Using the algebraic relation (2.25), which recall follows from the integrability condition for the Killing spinor equation, we may eliminate ϵ^+ from (B.1) and (B.3), leading to

$$\left[\partial_y - \frac{1}{2(y+x)} + \frac{f_2}{f_1}\frac{y}{y^2 - x^2}\mathrm{i}\sigma_3\right]\epsilon^- = 0 , \tag{B.7}$$

$$\left[\partial_x - \frac{1}{2(y+x)} + \frac{f_1}{f_2}\frac{x}{y^2 - x^2}\mathrm{i}\sigma_3\right]\epsilon^- = 0 . \tag{B.8}$$

Since the Pauli matrix σ_3 is diagonal, equations (B.7), (B.8) lead to decoupled equations for the two components of ϵ^-. We thus write

$$\epsilon^- = \begin{pmatrix} \epsilon^-_+ \\ \epsilon^-_- \end{pmatrix} , \tag{B.9}$$

so that (B.7), (B.8) are equivalent to the four equations

$$\partial_y \epsilon^-_\pm + Y_\pm(x,y)\epsilon^-_\pm = 0 , \tag{B.10}$$

$$\partial_x \epsilon^-_\pm + X_\pm(x,y)\epsilon^-_\pm = 0 , \tag{B.11}$$

where we have defined

$$Y_\pm(x,y) \equiv -\frac{1}{2(y+x)} \pm \frac{f_2}{f_1}\frac{\mathrm{i}y}{y^2-x^2} \ , \tag{B.12}$$

$$X_\pm(x,y) \equiv -\frac{1}{2(y+x)} \pm \frac{f_1}{f_2}\frac{\mathrm{i}x}{y^2-x^2} \ . \tag{B.13}$$

The integrability condition for (B.10), (B.11) is $\partial_x Y_\pm(x,y) = \partial_y X_\pm(x,y)$, which is easily verified to hold. These are then first order linear homogeneous differential equations, which may be integrated to give

$$\epsilon_\pm^- = c_\pm \sqrt{y+x}\left(\frac{\sqrt{(s^2-x^2)(y^2-1)} \mp \mathrm{i}\sqrt{(x^2-1)(y^2-s^2)}}{\sqrt{(s^2-x^2)(y^2-1)} \pm \mathrm{i}\sqrt{(x^2-1)(y^2-s^2)}}\right)^{1/2} \ , \tag{B.14}$$

where $c_\pm$ are integration constants (*a priori* depending on the angular coordinates Ψ and Φ).

One can now substitute the solutions (B.14) into the remaining differential equations (B.2), (B.4), which one finds are satisfied if and only if

$$c_- = \mathrm{i}c_+ \ , \tag{B.15}$$

which leads to the form of the Killing spinor given in (2.26). Finally, from the Ψ and Φ components of the Killing spinor equation it is reasonably simple to extract $\partial_\Psi c_+ = \partial_\Phi c_+ = 0$, so that the spinor ϵ is independent of Ψ and Φ. A somewhat more lengthy calculation then confirms that the remaining components of the Killing spinor equation are all satisfied.

C   One-loop vector multiplet contribution

In this appendix, for completeness we explain how to show the equality between the middle and last lines of equation (3.13). We begin with a trick similar to that used in [2]: the eigenvalues of a matrix in the adjoint representation come in positive-negative pairs, so that (3.13) is even in σ. This implies that one can equivalently sum over only the *positive* roots in the middle line of (3.13), while at the same time multiplying the right-hand side by itself with $\sigma \mapsto -\sigma$, to obtain the same result. This leads to the equality

$$Z_{\text{one-loop}}^{\text{vector}}(\sigma) = \prod_{\text{positive roots}} \frac{1}{\alpha(\sigma)^2} \prod_{m,n\geq 0}\left[\frac{m+nb^2+\mathrm{i}\alpha(\sigma)}{m+1+(n+1)b^2-\mathrm{i}\alpha(\sigma)} \cdot \right.$$
$$\left. \frac{m+nb^2-\mathrm{i}\alpha(\sigma)}{m+1+(n+1)b^2+\mathrm{i}\alpha(\sigma)}\right] \ . \tag{C.1}$$

Next notice that (formally) all the numerator terms cancel against denominator terms in the product over all $m, n \geq 0$, *except* for the numerator contributions of $\{m = 0, n = 0\}$, $\{m = 0, n \geq 1\}$ and $\{m \geq 1, n = 0\}$, which are left uncancelled. The first of these cancels the $\alpha(\sigma)^2$ prefactor, and we immediately reduce to

$$Z^{\text{vector}}_{\text{one}-\text{loop}}(\sigma) \;\; = \;\; \prod_{\text{positive roots}} \prod_{n \geq 1} (n^2 + \alpha(\sigma)^2)(n^2 b^4 + \alpha(\sigma)^2) \,. \tag{C.2}$$

The above manipulations are somewhat formal, as this is clearly divergent. However, we may write

$$Z^{\text{vector}}_{\text{one}-\text{loop}}(\sigma) \;\; = \;\; \prod_{\text{positive roots}} \left(\prod_{n \geq 1} b^4 n^4 \right) \prod_{n \geq 1} \left(1 + \frac{\alpha(\sigma)^2}{n^2} \right) \left(1 + \frac{\alpha(\sigma)^2}{b^4 n^2} \right) \,, \tag{C.3}$$

and then use the product formula for $\sinh(\pi z)$:

$$\sinh(\pi z) \;\; = \;\; \pi z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{n^2} \right) \tag{C.4}$$

for the last product. Using the zeta function regularization, the divergent prefactor is (for $b \neq 0$)

$$\prod_{n \geq 1} b^4 n^4 \;\; \overset{\text{zeta reg}}{=} \;\; \frac{(2\pi)^2}{b^2} \,. \tag{C.5}$$

Putting everything together then gives the last line of (3.13). Notice we have corrected a factor of π^2 compared to the corresponding formula in the original reference [2].

Finally, we note that the denominator in the last line of (3.13) in fact cancels against the Vandermonde determinant when reducing the integral from the Lie algebra to its Cartan subalgebra. More precisely and specifically, the Haar measure for $U(N)$ is

$$d\mu \;\; = \;\; \prod_{i=1}^{N} d\sigma_i \, \Delta(\sigma)^2 \tag{C.6}$$

where σ_i denote the eigenvalues of σ (so $\sigma_i = \frac{\lambda_i}{2\pi}$) and $\Delta(\sigma)$ is the Vandermonde determinant

$$\Delta(\sigma) \;\; = \;\; \prod_{i<j} (\sigma_i - \sigma_j) \;\; = \;\; \prod_{\text{positive roots}} \alpha_{ij}(\sigma) \,. \tag{C.7}$$

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