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The free energy of $\mathcal{N}=2$ supersymmetric AdS$_4$ solutions of M-theory

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Abstract

We show that general $\mathcal{N}=2$ supersymmetric AdS$_4$ solutions of M-theory with non-zero M2-brane charge admit a canonical contact structure. The free energy of the dual superconformal field theory on S^3 and the scaling dimensions of operators dual to supersymmetric wrapped M5-branes are expressed via AdS/CFT in terms of contact volumes. In particular, this leads to topological and localization formulae for the coefficient of $N^{3/2}$ in the free energy of such solutions.

1 Introduction

Tremendous progress has been achieved recently in understanding the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence, following the important results of [1, 2, 3]. In particular, for $\mathcal{N} \geq 2$ supersymmetry there is often good control on both sides of the correspondence. On the gravity side, the simplest setup is that of Freund-Rubin $\mathrm{AdS}_4 \times Y_7^{\mathrm{SE}}$ backgrounds of M-theory where Y_7^{SE} is a Sasaki-Einstein manifold[1], and deformations thereof. These are conjectured to be dual to the theory on a large number of multiple M2-branes placed at a Calabi-Yau four-fold singularity. Rather generally, these field theories are believed to be strongly coupled Chern-Simons-matter theories at a conformal fixed point.

While gravity computations are relatively amenable, obtaining results directly in the three-dimensional strongly coupled field theories has been prohibitively difficult until very recently. For this reason, non-trivial quantitative tests of the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence were not available. The situation has improved considerably with the results of [4, 5] (based on [6]), who showed that the partition function Z of $\mathcal{N} = 2$ supersymmetric field theories on the three-sphere can be reduced to more manageable *matrix integrals* using *localization* techniques. Moreover, in [4] it has been conjectured that at a conformal fixed point the *free energy*, defined as $\mathcal{F} = -\log|Z|$, is extremized as a function of all possible R-symmetries. If this is true, this quantity would then be analogous to the central charge a of four dimensional SCFTs. More generally, there are expectations that the free energy is a good measure of the number of degrees of freedom of three-dimensional field theories, even without supersymmetry.

In [7, 8, 9] the leading large N contribution to the free energy of Chern-Simons-matter theories on S^3 was computed for large classes of $\mathcal{N} = 2$ theories, and successfully matched to the gravity prediction in a class of Sasaki-Einstein geometries. This remarkable matching was first obtained in [10] for the ABJM theory and then in [11] for several $\mathcal{N} = 3$ examples. In this paper we will derive an expression for the (holographic) free energy $\mathcal{F}$, valid for a very general class of $\mathrm{AdS}_4 \times Y_7$ solutions dual to $\mathcal{N} = 2$ three-dimensional SCFTs. In fact, we will consider the most general class of M-theory AdS_4 solutions with non-zero M2-brane charge, finding very similar results to the type IIB AdS_5 geometries with non-zero D3-brane charge in [12]. We will prove the geometric formula

$$\mathcal{F} \;=\; N^{3/2}\sqrt{\frac{32\pi^6}{9\int_{Y_7}\sigma\wedge(\mathrm{d}\sigma)^3}}\ , \tag{1.1}$$

[1] Particular cases with $\mathcal{N} > 2$ include 3-Sasakian manifolds and orbifolds of the round seven-sphere.

where N is the quantized M2-brane charge and σ is a particular *contact form* on Y_7, that we will discuss.

We will also present a formula for the scaling dimension of BPS operators $\mathcal{O}_{\Sigma_5}$ dual to probe M5-branes wrapped on supersymmetric five-submanifolds $\Sigma_5 \subset Y_7$. In particular, the scaling dimension $\Delta(\mathcal{O}_{\Sigma_5})$ of these operators can be calculated from the contact volume of the five-submanifold Σ_5 as

$$\Delta(\mathcal{O}_{\Sigma_5}) \;\; = \;\; \pi N \left| \frac{\int_{\Sigma_5} \sigma \wedge (\mathrm{d}\sigma)^2}{\int_{Y_7} \sigma \wedge (\mathrm{d}\sigma)^3} \right| \, . \tag{1.2}$$

Both of these formulae are natural generalizations of those holding in the Sasaki-Einstein case, and are analogous to the results presented in [12].

The results of this paper stem from a systematic analysis of the geometry underlying general $\mathrm{AdS}_4 \times Y_7$ M-theory solutions preserving at least $\mathcal{N} = 2$ supersymmetry. In particular, we will identify a $\mathtt{u}(1)$ symmetry, generated by a Killing vector field ξ, which is the geometric counterpart to the $\mathtt{u}(1)$ R-symmetry of the dual $\mathcal{N} = 2$ superconformal field theory. In addition, we will demonstrate the existence of a *contact structure* on Y_7, that will play a key role in deriving our main results. These geometric objects are constructed from the Killing spinors preserved by the backgrounds [13, 12], and constitute a subset of a canonically defined $SU(2)$ structure on Y_7. In a subsequent work [14] we will present the necessary and sufficient conditions that this $SU(2)$ structure obeys in order to have an $\mathcal{N} = 2$ supersymmetric solution. In [14] we will also present more details of the computations that lead to the results discussed here.

2 Supersymmetric AdS_4 solutions of M-theory

Supersymmetric AdS_4 solutions of M-theory have been discussed before [15, 16, 17]; however, we will derive our results without recourse to the literature. In [14] we will present an analysis of the most general conditions for such solutions, in particular focusing on solutions preserving at least $\mathcal{N} = 2$ supersymmetry. In this section we summarize the Killing spinor equations that are used to derive many of the results presented in the remainder of the paper. We refer the reader to [14] for further details.

The bosonic fields of eleven-dimensional supergravity consist of a metric g_{11} and a three-form potential C with four-form field strength $G = \mathrm{d}C$. The signature of the metric is $(-, +, +, \ldots, +)$ and the action is

$$S \;\; = \;\; \frac{1}{2\kappa^2} \int R *_{11} \mathbf{1} - \frac{1}{2} G \wedge *_{11} G - \frac{1}{6} C \wedge G \wedge G \, , \tag{2.1}$$

where $2\kappa^2 = (2\pi)^8 \ell_p^9$ with ℓ_p the eleven-dimensional Planck length. We consider AdS_4 solutions of M-theory of the warped product form

$$\begin{aligned} g_{11} &= \mathrm{e}^{2\Delta} \left(g_{\mathrm{AdS}_4} + g_{Y_7} \right) , \\ G &= m \mathrm{vol}_4 + F . \end{aligned} \tag{2.2}$$

Here vol_4 denotes the Riemannian volume form on AdS_4, and without loss of generality[2] we take $\mathrm{Ric}_{\mathrm{AdS}_4} = -12 g_{\mathrm{AdS}_4}$. In order to preserve the $SO(3,2)$ invariance of AdS_4 we take Δ to be a function on the compact seven-manifold Y_7. F is the pull-back of a four-form on Y_7, and the Bianchi identity $\mathrm{d}G = 0$ requires that m is constant and F is closed.

In an orthonormal frame, the Clifford algebra Cliff(10, 1) is generated by gamma matrices Γ_A satisfying $\{\Gamma_A, \Gamma_B\} = 2\eta_{AB}$, where $A = 0, \ldots, 10$, and $\eta = \mathrm{diag}(-1, 1, \ldots, 1)$, and we choose a representation with $\Gamma_0 \cdots \Gamma_{10} = 1$. The Killing spinor equation is

$$\nabla_M \epsilon + \frac{1}{288} \left(\Gamma_M{}^{NPQR} - 8 \delta_M^N \Gamma^{PQR} \right) G_{NPQR}\, \epsilon = 0 , \tag{2.3}$$

where ϵ is a Majorana spinor and $M, N, \ldots$ are spacetime indices. We may decompose $\mathrm{Cliff}(10,1) \cong \mathrm{Cliff}(3,1) \otimes \mathrm{Cliff}(7,0)$ via

$$\Gamma_\alpha = \rho_\alpha \otimes 1 , \qquad \Gamma_{a+3} = \rho_5 \otimes \gamma_a , \tag{2.4}$$

where $\alpha, \beta = 0, 1, 2, 3$ and $a, b = 1, \ldots, 7$ are orthonormal frame indices for AdS_4 and Y_7 respectively, $\{\rho_\alpha, \rho_\beta\} = 2\eta_{\alpha\beta}$, $\{\gamma_a, \gamma_b\} = 2\delta_{ab}$, and we have defined $\rho_5 = \mathrm{i}\rho_0\rho_1\rho_2\rho_3$. Notice that our eleven-dimensional conventions imply that $\gamma_1 \cdots \gamma_7 = \mathrm{i}\, 1$.

The spinor ansatz preserving $\mathcal{N} = 1$ supersymmetry in AdS_4 is correspondingly

$$\epsilon = \psi^+ \otimes \mathrm{e}^{\Delta/2} \chi + (\psi^+)^c \otimes \mathrm{e}^{\Delta/2} \chi^c , \tag{2.5}$$

where ψ^+ is a positive chirality Killing spinor on AdS_4, so $\rho_5 \psi^+ = \psi^+$, satisfying

$$\nabla_\mu \psi^+ = \rho_\mu (\psi^+)^c . \tag{2.6}$$

The superscript c in (2.5) denotes charge conjugation in the relevant dimension, and the factor of $\mathrm{e}^{\Delta/2}$ is included for later convenience. Substituting (2.5) into the Killing

[2] The factor here is chosen to coincide with standard conventions in the case that Y_7 is a Sasaki-Einstein seven-manifold. For example, the AdS_4 metric in global coordinates then reads $g_{\mathrm{AdS}_4} = \frac{1}{4}(-\cosh^2\rho\, \mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \sinh^2\rho\, \mathrm{d}\Omega_2^2)$.

spinor equation (2.3) leads to the following algebraic and differential equations for the spinor field χ on Y_7:

$$\begin{aligned}\frac{1}{2}\gamma^n\partial_n\Delta\chi - \frac{im}{6}\mathrm{e}^{-3\Delta}\chi + \frac{1}{288}\mathrm{e}^{-3\Delta}F_{npqr}\gamma^{npqr}\chi + \chi^c &= 0\ ,\\ \nabla_m\chi + \frac{im}{4}\mathrm{e}^{-3\Delta}\gamma_m\chi - \frac{1}{24}\mathrm{e}^{-3\Delta}F_{mpqr}\gamma^{pqr}\chi - \gamma_m\chi^c &= 0\ .\end{aligned} \tag{2.7}$$

For a supergravity solution one must also solve the equations of motion resulting from (2.1), as well as the Bianchi identity $\mathrm{d}G = 0$.

Motivated by the discussion in the introduction, in this paper we will focus on $\mathcal{N} = 2$ supersymmetric AdS$_4$ solutions for which there are two independent solutions χ_1, χ_2 to (2.7). In particular, the general $\mathcal{N} = 2$ Killing spinor ansatz may be written as

$$\epsilon = \sum_{i=1,2}\psi_i^+ \otimes \mathrm{e}^{\Delta/2}\chi_i + (\psi_i^+)^c \otimes \mathrm{e}^{\Delta/2}\chi_i^c\ . \tag{2.8}$$

In this case there is a $\mathtt{u}(1)$ R-symmetry which rotates these spinors as a doublet. It is then convenient to introduce

$$\chi_\pm \equiv \frac{1}{\sqrt{2}}(\chi_1 \pm \mathrm{i}\chi_2)\ , \tag{2.9}$$

which will have charges ± 2 under the Abelian R-symmetry. For an $\mathcal{N} = 2$ solution one can show that the spinor equations (2.7) imply that, without loss of generality, one can normalize $\bar{\chi}_\pm\chi_\pm = 1$ [14]. We shall impose this normalization in what follows.

3 Contact structure

In this section we show that any $\mathcal{N} = 2$ supersymmetric AdS$_4$ solution with $m \neq 0$ admits a canonically defined contact structure. Moreover, the Reeb vector field ξ for this contact structure is also a Killing vector field which preserves all bosonic fields, and the spinors $\chi_\pm$ in (2.9) have charges ± 2 under ξ. We thus interpret ξ as the dual of the expected $\mathtt{u}(1)$ R-symmetry.

3.1 R-symmetry Killing vector

We begin by defining the one-form bilinear and its dual vector field

$$K \equiv \mathrm{i}\,\bar{\chi}_+^c\gamma_{(1)}\chi_-\ , \qquad \xi \equiv g_{Y_7}^{-1}(K,\cdot)\ , \tag{3.1}$$

where we denote $\gamma_{(n)} \equiv \frac{1}{n!}\gamma_{m_1\ldots m_n}\mathrm{d}y^{m_1}\wedge\ldots\wedge\mathrm{d}y^{m_n}$. *A priori* the one-form K in (3.1) is complex; however, one can show that the spinor equations (2.7) imply that $\mathrm{Im}\, K = 0$ so that K is real. It is then straightforward to show that K is a Killing one-form for the metric g_{Y_7} on Y_7, and hence that the dual vector field ξ is a Killing vector field. We note for future reference the square norm

$$\|\xi\|^2 \equiv g_{Y_7}(\xi,\xi) = |\bar{\chi}_+^c\chi_+|^2 + \frac{m^2}{36}\mathrm{e}^{-6\Delta} . \tag{3.2}$$

In particular when $m \neq 0$ we see that ξ is nowhere zero, and thus defines a one-dimensional foliation of Y_7.

The algebraic equation in (2.7) leads immediately to $\mathcal{L}_\xi\Delta = 0$, and using both equations in (2.7) one can show that

$$\mathrm{d}(\mathrm{e}^{3\Delta}\,\bar{\chi}_+^c\gamma_{(2)}\chi_-) = -\mathrm{i}\xi\lrcorner F . \tag{3.3}$$

It follows that

$$\mathcal{L}_\xi F = \mathrm{d}(\xi\lrcorner F) + \xi\lrcorner\mathrm{d}F = 0 , \tag{3.4}$$

provided the Bianchi identity $\mathrm{d}F = 0$ holds[3]. Thus ξ preserves all of the bosonic fields. One can also show that

$$\mathcal{L}_\xi\chi_\pm = \pm 2\mathrm{i}\,\chi_\pm , \tag{3.5}$$

so that $\chi_\pm$ have charges ± 2 under ξ. We thus identify ξ as the canonical vector field dual to the R-symmetry of the $\mathcal{N} = 2$ SCFT.

3.2 Contact form

Provided $m \neq 0$ we may define the real one-form bilinear

$$\sigma \equiv -\frac{6}{m}\mathrm{e}^{3\Delta}\,\bar{\chi}_+\gamma_{(1)}\chi_+ . \tag{3.6}$$

Using the spinor equations one can readily show that

$$\mathrm{d}\sigma = -\frac{12}{m}\mathrm{e}^{3\Delta}\,\mathrm{Re}\,\bar{\chi}_+^c\gamma_{(2)}\chi_- , \tag{3.7}$$

and an algebraic computation then leads to

$$\sigma\wedge(\mathrm{d}\sigma)^3 = \frac{2^7 3^4}{m^3}\mathrm{e}^{9\Delta}\mathrm{vol}_7 , \tag{3.8}$$

[3] In fact this is implied by supersymmetry, as we will show shortly – *cf.* equation (4.4).

where vol_7 denotes the Riemannian volume form of Y_7. It follows that $\sigma \wedge (\mathrm{d}\sigma)^3$ is a nowhere-zero top degree form on Y_7, and thus by definition σ is a contact form on Y_7. Again, straightforward algebraic computations lead to

$$\xi \lrcorner \sigma \;=\; 1 \;, \qquad \xi \lrcorner \mathrm{d}\sigma \;=\; 0 \;. \tag{3.9}$$

This implies that the Killing vector field ξ is also the unique Reeb vector field for the contact structure defined by σ.

4 Free energy on S^3

In this section we present a general supergravity formula for the free energy $\mathcal{F}$ of the dual $\mathcal{N} = 2$ SCFT on S^3. When $m \neq 0$, which is equivalent to a non-zero M2-brane charge of the AdS_4 background, this may be expressed in terms of the contact volume $\int_{Y_7} \sigma \wedge (\mathrm{d}\sigma)^3$ via (1.1).

4.1 Newton constant

The effective four-dimensional Newton constant G_4 is computed by dimensional reduction of eleven-dimensional supergravity on Y_7. More precisely, by definition $1/16\pi G_4$ is the coefficient of the four-dimensional Einstein-Hilbert term, in Einstein frame. A standard computation determines this to be

$$\frac{1}{16\pi G_4} \;=\; \frac{\pi \int_{Y_7} \mathrm{e}^{9\Delta} \mathrm{vol}_7}{2(2\pi \ell_p)^9} \;, \tag{4.1}$$

where recall that ℓ_p denotes the eleven-dimensional Planck length.

On the other hand, via the AdS/CFT correspondence G_4 also determines the free energy $\mathcal{F}$ of the dual CFT on S^3:

$$\mathcal{F} \;\equiv\; -\log |Z| \;=\; \frac{\pi}{2G_4} \;. \tag{4.2}$$

More precisely, the left hand side of (4.2) is minus the free energy of the unit radius AdS_4 computed in Euclidean quantum gravity, where Z is the gravitational partition function. The latter is regularized to give the finite result on the right hand side of (4.2) using the boundary counterterm subtraction method of [18]. Combining (4.1) and (4.2) leads to the supergravity formula

$$\mathcal{F} \;=\; \frac{4\pi^3 \int_{Y_7} \mathrm{e}^{9\Delta} \mathrm{vol}_7}{(2\pi \ell_p)^9} \;. \tag{4.3}$$

4.2 Flux quantization

Using the spinor equations (2.7) one can derive the general expression

$$mF \;=\; 6\,\mathrm{d}(\mathrm{e}^{6\Delta}\mathrm{Im}\,\bar{\chi}_+^c\gamma_{(3)}\chi_-)\ . \tag{4.4}$$

Thus provided $m \neq 0$ we see that F automatically obeys the Bianchi identity $\mathrm{d}F = 0$, and moreover F is in fact exact. There is thus no Dirac quantization condition for the four-form F when $m \neq 0$.[4]

On the other hand, the total M2-brane charge of the AdS_4 background is

$$N \;=\; -\frac{1}{(2\pi\ell_p)^6}\int_{Y_7} *_{11}G + \frac{1}{2}C\wedge G\ . \tag{4.5}$$

Dirac quantization requires this to be an integer. Equation (4.4) implies that $F = \mathrm{d}A$ where we may take the three-form potential A to be the globally defined form

$$A \;\equiv\; \frac{6}{m}\mathrm{e}^{6\Delta}\mathrm{Im}\,\bar{\chi}_+^c\gamma_{(3)}\chi_-\ . \tag{4.6}$$

Note that using (3.5) it immediately follows that this choice of gauge is invariant under ξ, that is,

$$\mathcal{L}_\xi A \;=\; 0\ . \tag{4.7}$$

Of course, one is free to add to A any closed three-form c, which will result in the same curvature F:

$$A \;\to\; A + \frac{1}{(2\pi\ell_p)^3}c\ . \tag{4.8}$$

If c is exact this is a gauge transformation of A and leads to a physically equivalent M-theory background. In fact more generally if c has integer periods then the transformation (4.8) is a large gauge transformation of A, again leading to an equivalent solution. It follows that only the cohomology class of c in the torus $H^3(Y_7;\mathbb{R})/H^3(Y_7;\mathbb{Z})$ is a physically meaningful parameter, and this corresponds to a marginal parameter in the dual CFT. In fact the free energy will be independent of this choice of c, which is why we have set $c = 0$ in (4.6). There is also the possibility of adding discrete torsion to A when $H^4_{\mathrm{torsion}}(Y_7;\mathbb{Z})$ is non-trivial, but we will not discuss this here.

Substituting our ansatz (2.2) into the general expression (4.5) leads to

$$N \;=\; \frac{1}{(2\pi\ell_p)^6}\int_{Y_7} m\mathrm{e}^{3\Delta}\mathrm{vol}_7 - \frac{1}{2}A\wedge F\ . \tag{4.9}$$

[4] This is certainly not the case for solutions with $m = 0$, as discussed in section 7.

Using (4.6) and the algebraic equation in (2.7) one can easily compute

$$N \;=\; \frac{1}{(2\pi\ell_p)^6}\frac{m^2}{2^5 3^2}\int_{Y_7}\sigma\wedge(\mathrm{d}\sigma)^3\ . \qquad (4.10)$$

Combining (4.10), (4.3) and (3.8) now leads straightforwardly to (1.1).

5 Scaling dimensions of BPS M5-branes

A probe M5-brane whose world-space is wrapped on a generalized calibrated five-submanifold $\Sigma_5 \subset Y_7$ and which moves along a geodesic in AdS$_4$ is expected to correspond to a BPS operator $\mathcal{O}_{\Sigma_5}$ in the dual three-dimensional SCFT. In particular, when Y_7 is a Sasaki-Einstein manifold, the scaling dimension of this operator can be calculated from the volume of the five-submanifold Σ_5 [19]. In this section we show that a simple generalization of this correspondence holds for the general $\mathcal{N}=2$ supersymmetric AdS$_4 \times Y_7$ solutions[5] treated in this paper, and in particular we prove the formula (1.2). The calculation is a simple adaptation of that presented in [22], and more details will appear in [14].

5.1 Generalized calibration

Given a Killing spinor ϵ of eleven-dimensional supergravity, it is simple to derive the BPS bound [23, 22]

$$\epsilon^\dagger\epsilon\, L_{\mathrm{DBI}}\,\mathrm{vol}_5 \;\geq\; \frac{1}{2}(\hat{k}\lrcorner H)\wedge H + \hat{\mu}\wedge H + \hat{\nu}\ . \qquad (5.1)$$

This bound is saturated if and only if $\mathcal{P}_-\epsilon = 0$, where $\mathcal{P}_- \equiv (1-\tilde{\Gamma})/2$ is the κ-symmetry projector, and corresponds to a probe M5-brane preserving supersymmetry. Here H is the three-form on the M5-brane, defined by $H = h + j^*C$ where h is closed and j^* denotes the pull-back to the M5-brane world-volume. The one-form $\hat{k}$, two-form $\hat{\mu}$ and five-form $\hat{\nu}$ denote the *pull-back* to Σ_5 of the differential forms [24] defined by the bilinears $k = \bar{\epsilon}\Gamma_{(1)}\epsilon$, $\mu = \bar{\epsilon}\Gamma_{(2)}\epsilon$, and $\nu = \bar{\epsilon}\Gamma_{(5)}\epsilon$, respectively, and vol$_5$ is the volume form on the world-space of the M5-brane. We have defined $\bar{\epsilon} \equiv \epsilon^\dagger\Gamma_0$ as usual.

We will use the static gauge embedding $\{\tau = \sigma^0, x^m = \sigma^m\}$, where τ is global time in AdS$_4$ and x^m, with $m = 1,\ldots,5$, are coordinates on Y_7. The Dirac-Born-Infeld Lagrangian is given by $L_{\mathrm{DBI}} = \sqrt{\det(\delta_m^{\ n} + H^{*n}_{m})}$, where the two-form $H^* \equiv *_5 H$ is

[5] Such supersymmetric M5-branes exist only for certain boundary conditions [20, 21], and our discussion here applies to these cases.

the world-space dual of H. Using the explicit form of the eleven-dimensional $\mathcal{N}=2$ Killing spinor (2.8) one can show that the bound (5.1) is saturated when $\rho=0$ (*i.e.* the M5-brane is at the centre of AdS_4) and

$$\frac{\mathrm{e}^{\Delta}}{2}L_{\mathrm{DBI}}\,\mathrm{vol}_5 \;=\; \frac{1}{2}(\hat{k}\lrcorner H)\wedge H+\hat{\mu}\wedge H+\hat{\nu}\ . \tag{5.2}$$

5.2 Energy of a BPS M5-brane

The energy density of an M5-brane can be computed by solving the Hamiltonian constraints, leading to

$$\mathcal{E} \;=\; P_\tau \;=\; T_{\mathrm{M5}}\left(\frac{\mathrm{e}^{\Delta}}{2}L_{\mathrm{DBI}}+\mathcal{C}_\tau\right)\ , \tag{5.3}$$

where $T_{\mathrm{M5}}=2\pi/(2\pi\ell_p)^6$ is the M5-brane tension and the contribution from the Wess-Zumino coupling is $\mathcal{C}_\tau\mathrm{vol}_5=\partial_\tau\lrcorner C_6-\frac{1}{2}(\partial_\tau\lrcorner C)\wedge(C-2H)$, with the potential C_6 defined through $\mathrm{d}C_6=*_{11}G+\frac{1}{2}C\wedge G$. However, from the explicit expression of C we presented earlier one can check that we have $\mathcal{C}_\tau=0$. The M5-brane energy is then given by

$$E_{\mathrm{M5}} \;=\; T_{\mathrm{M5}}\int_{\Sigma_5}\frac{\mathrm{e}^{\Delta}}{2}L_{\mathrm{DBI}}\,\mathrm{vol}_5 \;=\; T_{\mathrm{M5}}\int_{\Sigma_5}\frac{1}{4}(\xi\lrcorner H)\wedge H+\hat{\mu}\wedge H+\hat{\nu}\ , \tag{5.4}$$

where we used that the time-like Killing vector $k^\#$ dual to the one-form k is given by $k^\#=\partial_\tau+\frac{1}{2}\xi$. Let us briefly discuss this expression for the energy. With our gauge choice (4.6) for the three-form potential, in general we have $H=A+h$, where h is a closed three-form. If h is exact and invariant[6] under $k^\#$, namely $h=\mathrm{d}b$ with $\mathcal{L}_{k^\#}b=0$, then one can check that the integral does not depend on h. To see this, one has to recall that $\mathcal{L}_{k^\#}A=0$, use the results of [24], and apply Stokes' theorem repeatedly. If h is not exact, *a priori* it will contribute to the energy, and hence we expect the dimension of the dual operator to be affected. We leave an investigation of this interesting possibility for future work, and henceforth set $H=A$.

After some straightforward computations [14] the integrand in (5.4) can be evaluated in terms of the contact structure, and we get the remarkably simple result[7]

$$E_{\mathrm{M5}} \;=\; -T_{\mathrm{M5}}\frac{m^2}{2^6 3^2}\int_{\Sigma_5}\sigma\wedge(\mathrm{d}\sigma)^2\ . \tag{5.5}$$

Combining the latter with (4.10), and using the AdS/CFT dictionary $\Delta(\mathcal{O}_{\Sigma_5})=E_{\mathrm{M5}}$, leads straightforwardly to the formula (1.2) for the scaling dimension.

[6] One should obviously require that $\partial/\partial\tau$ and ξ generate symmetries of the M5-brane action.
[7] The sign arises from our choice of conventions, *cf.* [25].

6 Applications

As in [12], the formulae (1.1) and (1.2) have some immediate applications.

6.1 Topological and localization formulae

Let us suppose that the Reeb vector field ξ is quasi-regular, which means that all its orbits are closed and hence ξ integrates to a $U(1) = U(1)_R$ isometry of Y_7. Since (3.2) implies that ξ is nowhere zero, it follows that in this case Y_7 is the total space of a $U(1)$ principal orbifold bundle $\mathcal{L}$ over a six-dimensional orbifold $V \equiv Y_7/U(1)_R$; the latter is smooth precisely when $U(1)_R$ acts freely on Y_7. If we denote by v the canonically normalized generator of the $U(1)_R$ action, so that we may write $v = \partial/\partial\varphi$ where the coordinate φ has period 2π, then $\xi = kv$ for some constant $k > 0$, and the contact volume may be written

$$\frac{k^4}{(2\pi)^4}\int_{Y_7} \sigma \wedge (\mathrm{d}\sigma)^3 \;=\; \int_V c_1(\mathcal{L})^3 \;\in \mathbb{Q}\ . \tag{6.1}$$

Here we have used the general fact that the first Chern class $c_1(\mathcal{L})$ of a principal $U(1)$ orbifold bundle $\mathcal{L}$ over an orbifold V is a rational cohomology class. The constant k must also be rational, since one computes

$$\mathcal{L}_\xi \left(\bar{\chi}^c_\pm \chi_\pm\right) \;=\; \pm 4\mathrm{i}\, \bar{\chi}^c_\pm \chi_\pm\ , \tag{6.2}$$

which implies that $\bar{\chi}^c_\pm \chi_\pm$ has charge ± 4 under $U(1)_R$. On the other hand, $\bar{\chi}^c_\pm \chi_\pm$ must have an integer charge under v, in order to be single-valued in φ, implying that $4/k \in \mathbb{N}$.

The upshot is that for gravity solutions with a $U(1)_R$ isometry, the coefficient of $\pi N^{3/2}$ in the free energy (1.1) is the square root of a rational number, and that the latter has a topological interpretation as a Chern number. The corresponding result for supersymmetric AdS_5 solutions of Type IIB string theory in [12] is that the central charge a computed via supergravity is rational when one has a $U(1)_R$ isometry. In the dual $d = 4$, $\mathcal{N} = 1$ SCFT this is clear, since there is a well-known cubic expression for a in terms of R-charges with rational coefficients [26]. On the other hand, it is currently unclear, at least to the authors, why the coefficient of $\pi N^{3/2}$ in the free energy should be the square root of a rational number when one has rational R-charges in the $d = 3$, $\mathcal{N} = 2$ SCFT. We may thus regard this as a prediction of supergravity for field theory.

Also as in [12], we may write the contact volume in terms of a Duistermaat-Heckman integral on the cone X over Y_7

$$\int_{Y_7} \sigma \wedge (\mathrm{d}\sigma)^3 \quad = \quad \int_X \mathrm{e}^{-r^2/2} \, \frac{\omega^4}{4!} \ . \tag{6.3}$$

Here $r > 0$ is a coordinate on $\mathbb{R}_+$ in $X \cong \mathbb{R}_+ \times Y_7$, $\omega = \frac{1}{2}\mathrm{d}(r^2\sigma)$ is a symplectic form on X, and $r^2/2$ is a Hamiltonian function for the Reeb vector field ξ. The right hand side of (6.3) may then often be computed via localization. Roughly, this involves choosing an equivariant symplectic resolution of the singularity of X at $r = 0$. We refer to [12] and references therein for a more detailed discussion, especially in the case that X is symplectic toric. In practice, this is often a very useful method for computing the left hand side of (6.3) using only topological methods.

6.2 Massive deformations of $\mathrm{CY}_3 \times \mathbb{C}$

As a concrete example, in this section we briefly consider the supergravity solutions of [27]. The original solution in this paper is a warped $\mathrm{AdS}_4 \times \tilde{S}^7$ background with internal G-flux on the (squashed and stretched) seven-sphere $\tilde{S}^7$. This has a dual field theory interpretation as deforming the ABJM theory, dual to the round S^7 solution, by a mass deformation and flowing to the IR, with the warped AdS_4 solution of [27] describing the IR fixed point [28]. In fact more generally one can consider M2-branes probing the Calabi-Yau four-fold geometry $\mathrm{CY}_3 \times \mathbb{C}$, where CY_3 denotes *any* Calabi-Yau three-fold cone. One expects these to have dual field theory descriptions in which one can give a mass to a gauge-invariant scalar chiral primary operator, the latter being dual to a Kaluza-Klein mode arising from the holomorphic function z_0 on $\mathbb{C}$. This will trigger a renormalization group flow, whose end-point has a gravity dual described by a generalization of the warped $\tilde{S}^7$ solution of [27]. The latter is in fact then the special case $\mathrm{CY}_3 = \mathbb{C}^3$. Other special cases of such solutions, and their field theory duals, have been discussed recently in [29, 9]. We shall discuss the general case in more detail in [14].

Here we prove that these renormalization group flows are universal in the sense that the ratio of the free energies in the IR and UV is independent of the choice of three-fold CY_3. This was anticipated recently in [9]. A key point is that we do not need the generalization of the explicit supergravity solution in [27], but rather the universal formula

$$\frac{\mathcal{F}_{\mathrm{IR}}}{\mathcal{F}_{\mathrm{UV}}} \quad = \quad \sqrt{\frac{16}{27}} \tag{6.4}$$

in fact follows straightforwardly from the contact volume formula (1.1).

To see this, consider a general $\mathrm{CY}_3 \times \mathbb{C}$ Calabi-Yau four-fold, whose (generically singular) Sasaki-Einstein link Y_7^{SE} describes the UV background. This has at least a $\mathbb{C}^* \times \mathbb{C}^*$ symmetry, in which the first $\mathbb{C}^*$ acts on the CY_3, and under which the CY_3 Killing spinors have charge $\frac{1}{2}$, and the second $\mathbb{C}^*$ acts in the obvious way on the copy of $\mathbb{C}$. Let us denote the components of the Reeb vector field for the Calabi-Yau four-fold in this basis as (ξ_1, ξ_0). It is straightforward to see that the CY_4 Killing spinors have charge 2, as in equation (3.5), precisely when

$$\xi_1 + \xi_0 \;\; = \;\; 4 \, , \tag{6.5}$$

which is also equivalent to the holomorphic $(4,0)$-form $\Omega_{(4,0)} = \Omega_{(3,0)} \wedge \mathrm{d}z_0$ having charge 4. As shown in appendix B of [30], in general the contact volume is a function of the Reeb vector field. In our case the contact volume of Y_7 is given by the general formula

$$\mathrm{Vol}(Y_7)[\xi_1, \xi_0] \;\; = \;\; \frac{1}{\xi_0} \mathrm{Vol}(Y_5)[\xi_1] \, , \tag{6.6}$$

where Y_5 denotes the five-manifold link of CY_3. Then one easily shows that $\xi_1 = 3$ for a Sasaki-Einstein metric, so that (6.6) implies the relation $\mathrm{Vol}(Y_7^{\mathrm{SE}}) = \mathrm{Vol}(Y_5^{\mathrm{SE}})$ between Sasaki-Einstein volumes. Notice here that $\xi_0 = 1$ follows from (6.5), and that this indeed gives the expected scaling dimension $\Delta = \frac{1}{2}$ of a free chiral field.[8]

Let us now consider the IR solution corresponding to the mass deformation. Since the operator dual to z_0 is given a mass, the scaling dimension necessarily changes from $\Delta = \frac{1}{2}$ to $\Delta = 1$. Thus one expects $\xi_0 = 2$ in the supergravity solution, and one indeed sees that this is the case. The Killing spinors always have charge 2 (3.5); thus equation (6.5) still holds, and we conclude that $\xi_1 = 2$ for the mass-deformed background. Using this, together with the fact that the contact volume of Y_5 has homogeneous degree -3 [30] under ξ_1, we conclude from (6.6) that

$$\mathrm{Vol}(Y_7^{\mathrm{mass}}) \;\; = \;\; \frac{1}{2} \mathrm{Vol}(Y_5)[2] \;\; = \;\; \frac{1}{2} \cdot \left(\frac{2}{3}\right)^{-3} \mathrm{Vol}(Y_5)[3] \;\; = \;\; \frac{27}{16} \mathrm{Vol}(Y_7^{\mathrm{SE}}) \, . \tag{6.7}$$

Using (1.1) then leads directly to (6.4).

[8] Note there is a factor of $\frac{1}{2}$ in going from the geometric scaling dimension under $r\partial_r$ to the scaling dimension Δ in field theory, *cf.* equation (2.31) of [31].

7 Discussion

In this paper we have taken a first glimpse at the geometry characterizing general $\mathcal{N} = 2$ supersymmetric AdS_4 solutions of M-theory with non-zero M2-brane charge. In particular, we have shown that these admit a Killing vector ξ that realizes the $\mathtt{u}(1)$ R-symmetry of the dual field theories, and a contact one-form σ, in terms of which the free energy and the scaling dimensions of certain BPS operators can be written. The geometry of Y_7 is a natural generalization of Sasaki-Einstein geometry, precisely as was found in [12, 25] for general supersymmetric $AdS_5 \times Y_5$ solutions of type IIB supergravity. As an application of our results we have briefly discussed a class of solutions discovered in [27]. An investigation of new solutions is currently under way [14]. We should point out that there exist in the literature $\mathcal{N} = 2$ supersymmetric $AdS_4 \times Y_7$ backgrounds where the M2-brane charge vanishes. An example is the solution discussed in [32] (originally found in [33]), representing the near-horizon limit of M5-branes wrapped on a special Lagrangian submanifold inside a Calabi-Yau three-fold times $\mathbb{R}^2$. Our results do not apply to this solution, and in particular we expect that the free energy in this case will scale as N^3. However we leave the study of this class of solutions for the future.

In [14] the geometry of Y_7 will be explored in greater detail. In particular, we will show that this geometry is characterized by a local $SU(2)$ structure, which turns out to be strikingly similar to the $SU(2)$ structure characterizing supersymmetric $AdS_5 \times Y_6$ solutions of M-theory [34]. From the findings of this paper, and of [14], arise a number of interesting questions to address. In [25] the results of [13] and [12] have been elegantly reformulated in terms of generalized geometry of the cone $C(Y_5)$, where the metric and NS B-field are unified. Similarly, it is tempting to speculate that the geometry of the eight-dimenional cones $C(Y_7)$ will turn out to be an analogous kind of generalized geometry, that treats the metric and C-field on equal footing [35, 36, 37, 38]. We also anticipate a generalization of volume minimization of Sasaki-Einstein manifolds [39, 40], along the lines of [30]. Finally, it would be very interesting to understand whether a direct relationship between localization on the field theory side [6] and localization on the gravity side, discussed here and in [39, 40, 41, 12], can be established.

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References

[1] J. Bagger, N. Lambert, "Modeling Multiple M2's," Phys. Rev. **D75**, 045020 (2007). [hep-th/0611108].

[2] A. Gustavsson, "Algebraic structures on parallel M2-branes," Nucl. Phys. **B811** (2009) 66-76. [arXiv:0709.1260 [hep-th]].

[3] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, "N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals," JHEP **0810**, 091 (2008) [arXiv:0806.1218 [hep-th]].

[4] D. L. Jafferis, "The Exact Superconformal R-Symmetry Extremizes Z," [arXiv: 1012.3210 [hep-th]].

[5] N. Hama, K. Hosomichi, S. Lee, "Notes on SUSY Gauge Theories on Three-Sphere," JHEP **1103**, 127 (2011). [arXiv:1012.3512 [hep-th]].

[6] A. Kapustin, B. Willett and I. Yaakov, "Exact Results for Wilson Loops in Superconformal Chern-Simons Theories with Matter," JHEP **1003**, 089 (2010) [arXiv:0909.4559 [hep-th]].

[7] D. Martelli and J. Sparks, "The large N limit of quiver matrix models and Sasaki-Einstein manifolds," arXiv:1102.5289 [hep-th].

[8] S. Cheon, H. Kim and N. Kim, "Calculating the partition function of N=2 Gauge theories on S^3 and AdS/CFT correspondence," JHEP **1105**, 134 (2011) [arXiv:1102.5565 [hep-th]].

[9] D. L. Jafferis, I. R. Klebanov, S. S. Pufu and B. R. Safdi, "Towards the F-Theorem: N=2 Field Theories on the Three-Sphere," JHEP **1106**, 102 (2011) [arXiv:1103.1181 [hep-th]].

[10] N. Drukker, M. Marino and P. Putrov, "From weak to strong coupling in ABJM theory," arXiv:1007.3837 [hep-th].

[11] C. P. Herzog, I. R. Klebanov, S. S. Pufu and T. Tesileanu, "Multi-Matrix Models and Tri-Sasaki Einstein Spaces," Phys. Rev. D **83**, 046001 (2011) [arXiv:1011.5487 [hep-th]].

[12] M. Gabella, J. P. Gauntlett, E. Palti, J. Sparks and D. Waldram, "The Central charge of supersymmetric AdS_5 solutions of type IIB supergravity," Phys. Rev. Lett. **103**, 051601 (2009) [arXiv:0906.3686 [hep-th]].

[13] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, "Supersymmetric AdS_5 solutions of type IIB supergravity," Class. Quant. Grav. **23**, 4693 (2006) [arXiv:hep-th/0510125].

[14] M. Gabella, D. Martelli, A. Passias and J. Sparks, in progress.

[15] P. Kaste, R. Minasian, A. Tomasiello, "Supersymmetric M theory compactifications with fluxes on seven-manifolds and G-structures," JHEP **0307**, 004 (2003). [hep-th/0303127].

[16] A. Lukas and P. M. Saffin, "M theory compactification, fluxes and AdS_4," Phys. Rev. D **71**, 046005 (2005) [arXiv:hep-th/0403235].

[17] K. Behrndt, M. Cvetic and T. Liu, "Classification of supersymmetric flux vacua in M theory," Nucl. Phys. B **749**, 25 (2006) [arXiv:hep-th/0512032].

[18] R. Emparan, C. V. Johnson and R. C. Myers, "Surface terms as counterterms in the AdS/CFT correspondence," Phys. Rev. D **60**, 104001 (1999) [arXiv:hep-th/9903238].

[19] D. Berenstein, C. P. Herzog and I. R. Klebanov, "Baryon spectra and AdS/CFT correspondence," JHEP **0206**, 047 (2002) [arXiv:hep-th/0202150].

[20] I. R. Klebanov, S. S. Pufu and T. Tesileanu, "Membranes with Topological Charge and $\mathrm{AdS}_4/\mathrm{CFT}_3$ Correspondence," Phys. Rev. D **81**, 125011 (2010) [arXiv:1004.0413 [hep-th]].

[21] N. Benishti, D. Rodriguez-Gomez and J. Sparks, "Baryonic symmetries and M5 branes in the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence," JHEP **1007**, 024 (2010) [arXiv: 1004.2045 [hep-th]].

[22] D. Martelli and J. Sparks, "G-structures, fluxes and calibrations in M theory," Phys. Rev. D **68**, 085014 (2003) [arXiv:hep-th/0306225].

[23] O. Barwald, N. D. Lambert and P. C. West, "A Calibration bound for the M theory five-brane," Phys. Lett. B **463**, 33 (1999) [arXiv:hep-th/9907170].

[24] J. P. Gauntlett, S. Pakis, "The Geometry of $D = 11$ killing spinors," JHEP **0304**, 039 (2003). [hep-th/0212008].

[25] M. Gabella, J. P. Gauntlett, E. Palti, J. Sparks and D. Waldram, "AdS_5 Solutions of Type IIB Supergravity and Generalized Complex Geometry," Commun. Math. Phys. **299**, 365 (2010) [arXiv:0906.4109 [hep-th]].

[26] D. Anselmi, J. Erlich, D. Z. Freedman and A. A. Johansen, "Positivity constraints on anomalies in supersymmetric gauge theories," Phys. Rev. D **57**, 7570 (1998) [arXiv:hep-th/9711035].

[27] R. Corrado, K. Pilch and N. P. Warner, "An N=2 supersymmetric membrane flow," Nucl. Phys. B **629**, 74 (2002) [arXiv:hep-th/0107220].

[28] I. Klebanov, T. Klose and A. Murugan, "AdS_4/CFT_3 – Squashed, Stretched and Warped," JHEP **0903**, 140 (2009) [arXiv:0809.3773 [hep-th]].

[29] C. Ahn and K. Woo, "The Gauge Dual of A Warped Product of AdS_4 and A Squashed and Stretched Seven-Manifold," Class. Quant. Grav. **27**, 035009 (2010) [arXiv:0908.2546 [hep-th]].

[30] M. Gabella and J. Sparks, "Generalized Geometry in AdS/CFT and Volume Minimization," arXiv:1011.4296 [hep-th].

[31] J. P. Gauntlett, D. Martelli, J. Sparks and S. T. Yau, "Obstructions to the existence of Sasaki-Einstein metrics," Commun. Math. Phys. **273**, 803 (2007) [arXiv:hep-th/0607080].

[32] J. P. Gauntlett, N. Kim and D. Waldram, "M Five-branes wrapped on supersymmetric cycles," Phys. Rev. D **63**, 126001 (2001) [arXiv:hep-th/0012195].

[33] M. Pernici and E. Sezgin, "Spontaneous compactification of seven-dimensional supergravity theories," Class. Quant. Grav. **2**, 673 (1985).

[34] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, "Supersymmetric AdS_5 solutions of M theory," Class. Quant. Grav. **21**, 4335 (2004) [arXiv:hep-th/0402153].

[35] P. C. West, "E_{11} and M theory," Class. Quant. Grav. **18**, 4443 (2001) [arXiv:hep-th/0104081].

[36] C. M. Hull, "Generalised Geometry for M-Theory," JHEP **0707**, 079 (2007) [arXiv:hep-th/0701203].

[37] P. P. Pacheco and D. Waldram, "M-theory, exceptional generalised geometry and superpotentials," JHEP **0809**, 123 (2008) [arXiv:0804.1362 [hep-th]].

[38] D. S. Berman and M. J. Perry, "Generalized Geometry and M theory," JHEP **1106**, 074 (2011) [arXiv:1008.1763 [hep-th]].

[39] D. Martelli, J. Sparks and S. T. Yau, "The Geometric dual of a-maximisation for Toric Sasaki-Einstein manifolds," Commun. Math. Phys. **268**, 39 (2006) [arXiv:hep-th/0503183].

[40] D. Martelli, J. Sparks and S. T. Yau, "Sasaki-Einstein manifolds and volume minimisation," Commun. Math. Phys. **280**, 611 (2008) [arXiv:hep-th/0603021].

[41] K. M. Lee and H. U. Yee, "New $\mathrm{AdS}_4 \times X_7$ Geometries with N=6 in M Theory," JHEP **0703**, 012 (2007) [arXiv:hep-th/0605214].