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**QUADERNI DEL CORSO ESTIVO DI STATISTICA  
E CALCOLO DELLE PROBABILITÀ**

**N. 11**

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***EMPIRICAL PROCESSES:  
THEORY AND APPLICATIONS***

*Dalle lezioni del*

*“Corso Estivo di Statistica e Calcolo delle Probabilità”*

*Torignon (Aosta) - Luglio 2003*

*Jon A. Wellner, University of Washington*

*Moulinath Banerjee, University of Michigan*

*A cura di Sergio Venturini*

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