ABCD and ODEs

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Abstract

We outline a relationship between conformal field theories and spectral problems of ordinary differential equations, and discuss its generalisation to models related to classical Lie algebras.

1 Introduction

The ODE/IM correspondence [1, 2, 3, 4] has established a link between two dimensional conformal field theory (CFT) and generalised spectral problems in ordinary differential and pseudo-differential equations. It is based on an equivalence between transfer matrix eigenvalues [5, 6] and Baxter Q-functions in integrable models (IMs), and spectral determinants [7, 8] of ordinary differential equations (ODEs).

In statistical mechanics, the transfer matrix and its largest eigenvalue – denoted by T in the following – are central objects. For example, consider the six-vertex model defined on a square lattice with N columns and N' rows; T can be written in terms of an auxiliary entire function Q through the so-called Baxter TQ relation. Up to an overall constant, Q is completely determined by the knowledge of the positions of its zeros, the Bethe roots, which are constrained by the Bethe ansatz equations (BAE). Subject to some qualitative information on the positions of the Bethe roots, easily deduced by studying systems with small size, the Bethe ansatz leads to a unique set of ground-state roots. In the $N' \to \infty$ limit the free energy per site f is simply related to T by

$$f \sim -\frac{1}{N} \ln T \ . \tag{1.1}$$

In [5, 6], Bazhanov, Lukyanov and Zamolodchikov showed how to adapt the same techniques directly to the conformal field theory (CFT) limit of the six-vertex model. In this setting, we consider the conformal field theory with Virasoro central charge $c = 1$ corresponding to the continuum limit of the six-vertex model, defined on an infinitely-long strip with twisted boundary conditions along the finite size direction. The largest transfer matrix eigenvalue T depends on three independent parameters: the (rescaled) spectral parameter ν, the anisotropy η and the twist ϕ. Defining E, M, l, ω, Ω through the following relations

$$E = e^{2\nu}, \quad \eta = \frac{\pi}{2}\frac{M}{M+1}, \quad \omega = e^{i\frac{\pi}{M+1}}, \quad \Omega = \omega^{2M}, \quad \phi = \frac{(2l+1)\pi}{2M+2} \tag{1.2}$$

the resulting TQ relation is

$$T(E,l,M)Q(E,l,M) = \omega^{-\frac{2l+1}{2}} Q(\Omega E,l,M) + \omega^{\frac{2l+1}{2}} Q(\Omega^{-1}E,l,M) \ . \tag{1.3}$$

The Baxter function Q for this largest eigenvalue is fixed by demanding entirety of both T and Q, and reality, positivity and 'extreme packing' for $l > -1/2$ of the set $\{E_i\}$ of zeros of Q. The BAE follow from the entirety of T and Q via

$$Q(E_i) = 0 \Rightarrow T(E_i)Q(E_i) = 0 \Rightarrow \frac{Q(\Omega E_i)}{Q(\Omega^{-1}E_i)} = -\omega^{2l+1} \ . \tag{1.4}$$

Surprisingly, equations (1.3) and (1.4) also emerge from an apparently unrelated context: the study of particular spectral problems for the following differential equation

$$\left(\left(\frac{d}{dx} - \frac{l}{x} \right) \left(\frac{d}{dx} + \frac{l}{x} \right) - x^{2M} + E \right) y(x, E, l) = 0 \ , \tag{1.5}$$

with x and E possibly complex. To see the emergence of (1.4) from (1.5), we start from the unique solution $\psi(x, E, l)$ of (1.5) on the punctured complex plane $x \in \mathbb{C} \setminus \{0\}$ which has the asymptotic

$$\psi \sim x^{-M/2} \exp(-\tfrac{1}{M+1} x^{M+1}) \ , \quad (M > 1) \tag{1.6}$$

as $|x| \to \infty$ in any closed sector contained in the sector $|\arg x| < \frac{3\pi}{2M+2}$. This solution is entire in E and x. From ψ we introduce a family of solutions to (1.5) using the 'Sibuya trick' (also known as 'Symanzik rescaling'):

$$\psi_k = \psi(\omega^k x, \ \Omega^k E, \ \ l) \ . \tag{1.7}$$

In (1.7), k takes integer values; any pair $\{\psi_k, \psi_{k+1}\}$ constitutes a basis of solutions to (1.5). An alternative way to characterize a solution to (1.5) is through its behaviour near the origin $x = 0$. The indicial equation is

$$(\lambda - 1 - l)(\lambda + l) = 0 \ , \tag{1.8}$$

and correspondingly we can define two (generally) independent solutions

$$\chi^+(x, E) = \chi(x, E, l) \sim x^{l+1} + O(x^{l+3}) \, , \tag{1.9}$$

and $\chi^-(x, E) = \chi(x, E, -l-1)$, which transform trivially under Symanzik rescaling as

$$\chi_k^+ = \chi^+(\omega^k x, \Omega^k E) = \omega^{(l+1)k} \chi^+(x, E) \ . \tag{1.10}$$

The trick is now to rewrite $\chi_0^+ = \chi^+(x, E)$ respectively in terms of the basis $\{\psi_0, \psi_1\}$ and $\{\psi_{-1}, \psi_0\}$:

$$2i\chi_0^+ \ = \ \omega^{-l-\frac{1}{2}} Q(\Omega E)\psi_0 - Q(E)\omega^{-\frac{1}{2}}\psi_1 \tag{1.11}$$

$$2i\chi_0^+ = 2i\omega^{l+1}\chi_{-1}^+ \ = \ \omega^{\frac{1}{2}} Q(E)\psi_{-1} - \omega^{l+\frac{1}{2}} Q(\Omega^{-1}E)\psi_0 \tag{1.12}$$

where the coefficients has been fixed by consistency among (1.11), (1.12) and (1.10) and

$$Q(E, l) = W[\psi_0, \chi_0^+] \ . \tag{1.13}$$

Here $W[f, g] = f\frac{dg}{dx} - g\frac{df}{dx}$ denotes the Wronskian of f and g. Taking the ratio (1.11)/(1.12) evaluated at a zero $E{=}E_i$ of Q leads immediately to the Bethe ansatz equations (1.4) without the need to introduce the TQ relation, though in this case it can be done very easily (see, for example the recent ODE/IM review article [4]). Correspondingly, χ becomes subdominant at $x \to \infty$ on the positive real axis: $\chi(x, E_i, l) \propto \psi(x, E_i, l)$. The motivation of dealing with χ, instead of ψ (1.6), is two-fold. Firstly, χ can be obtained by applying the powerful and numerically efficient iterative method proposed by Cheng many years ago [9] in the context of Regge pole theory, and applied to spectral problems of this sort in [10]. To this end we introduce the linear operator L, defined through its formal action

$$L[x^p] = \frac{x^{p+2}}{(p+l)(p-l-1)} \,. \tag{1.14}$$

So for any polynomial $\mathcal{P}(x)$ of x ,

$$\left(\frac{d}{dx} - \frac{l}{x}\right)\left(\frac{d}{dx} + \frac{l}{x}\right) L[\mathcal{P}(x)] = \mathcal{P}(x) \ , \tag{1.15}$$

and the basic differential equation (1.5), with the boundary conditions (1.9) at the origin, is equivalent to

$$\chi(x, E, l) = x^{l+1} + L\left[(x^{2M} - E)\chi(x, E, l)\right] \ . \tag{1.16}$$

Equation (1.16) is solvable by iteration and it allows the predictions of the ODE/IM correspondence to be checked with very high precision.

The initial results of [1, 2, 3] connected conformal field theories associated with the Lie algebra A_1 to (second-order) ordinary differential equations. The generalisation to A_{n-1}-models was established in [11, 12] but it was only recently [13] that the ODE/IM correspondence was generalised to the remaining classical Lie algebras B_n, C_n and D_n. Our attempts to derive generalised TQ relations from the proposed set of pseudo-differential equations were unsuccessful, but a series of well-motivated conjectures led us directly to the BAE, allowing us to establish the relationship between BAE and pseudo-differential equation parameters. Moreover, while the numerics to calculate the analogs of the functions ψ turned out to be very costly in CPU time, the generalisation of Cheng's method proved very efficient and allowed very high precision tests to be performed. This is our second main reason to deal with solutions defined through the behaviour about $x = 0$, rather than $x = \infty$.

2 Bethe ansatz for classical Lie algebras

For any classical Lie algebra $\mathfrak{g}$, conformal field theory Bethe ansatz equations depending on a set of $rank(\mathfrak{g})$ twist parameters γ=$\{\gamma_a\}$ can be written in a compact form as

$$\prod_{b=1}^{rank(\mathfrak{g})} \Omega^{B_{ab}\gamma_b} \frac{Q^{(b)}_{B_{ab}}(E^{(a)}_i, \gamma)}{Q^{(b)}_{-B_{ab}}(E^{(a)}_i, \gamma)} = -1 \,, \qquad i = 0, 1, 2, \dots \tag{2.1}$$

where $Q_k^{(a)}(E,\gamma) = Q^{(a)}(\Omega^k E,\gamma)$, and the numbers $E_i^{(a)}$ are the (in general complex) zeros of the functions $Q^{(a)}$. In (2.1) the indices a and b label the simple roots of the Lie algebra $\mathfrak{g}$, and

$$B_{ab} = \frac{(\alpha_a, \alpha_b)}{|\text{long roots}|^2} \ , \quad a, b = 1, 2, \ldots, rank(\mathfrak{g}) \tag{2.2}$$

where the α's are the simple roots of $\mathfrak{g}$. The constant $\Omega = \exp\left(i\frac{2\pi}{h^\vee \mu}\right)$ is a pure phase, μ is a positive real number and $h^\vee$ is the dual Coxeter number.

It turns out that the Bethe ansatz roots generally split into multiplets (strings) with approximately equal modulus $|E_i^{(a)}|$. The ground state of the model corresponds to a configuration of roots containing only multiplets with a common dimension $d_a = K/B_{aa}$; the model-dependent integer K corresponds to the degree of fusion (see for example [14]).

3 The pseudo-differential equations

To describe the pseudo-differential equations corresponding to the A_{n-1}, B_n, C_n and D_n simple Lie algebras we first introduce some notation. We need an n^{th}-order differential operator [12]

$$D_n(\mathbf{g}) = D(g_{n-1} - (n{-}1))\, D(g_{n-2} - (n{-}2)) \ \ldots\ D(g_1 - 1)\, D(g_0) \ , \tag{3.1}$$

$$D(g) = \left(\frac{d}{dx} - \frac{g}{x}\right) \ , \tag{3.2}$$

depending on n parameters

$$\mathbf{g}{=}\{g_{n-1}, \ldots, g_1, g_0\} \quad , \quad \mathbf{g}^\dagger{=}\{n-1-g_0, n-1-g_1, \ldots, n-1-g_{n-1}\} \ . \tag{3.3}$$

Also, we introduce an inverse differential operator $(d/dx)^{-1}$, generally defined through its formal action

$$\left(\frac{d}{dx}\right)^{-1} x^s = \frac{x^{s+1}}{s+1} \ , \tag{3.4}$$

and we replace the simple 'potential' $P(E,x) = (x^{2M} - E)$ of (1.5) with

$$P_K(E,x) = (x^{h^\vee M/K} - E)^K \ . \tag{3.5}$$

Using the notation of Appendix B in [13] the proposed pseudo-differential equations are reported below.

A_{n-1} models:

The A_{n-1} ordinary differential equations are

$$D_n(\mathbf{g}^\dagger)\chi^\dagger_{n-1}(x,E) = P_K(x,E)\chi^\dagger_{n-1}(x,E) \ , \tag{3.6}$$

with the constraint $\sum_{i=0}^{n-1} g_i = \frac{n(n-1)}{2}$ and the ordering $g_i < g_j < n-1,\ \forall\ i < j$. We introduce the alternative set of parameters $\gamma = \gamma(\mathbf{g}) = \{\gamma_a(\mathbf{g})\}$

$$\gamma_a = \frac{2K}{h^\vee M}\left(\sum_{i=0}^{a-1} g_i - \frac{a(h^\vee - 1)}{2}\right) . \tag{3.7}$$

The solution $\chi_{n-1}^\dagger(x, E)$ is specified by its $x \sim 0$ behaviour

$$\chi_{n-1}^\dagger \sim x^{n-1-g_0} + \text{subdominant terms}, \quad (x \to 0^+) . \tag{3.8}$$

In general, this function grows exponentially as x tends to infinity on the positive real axis. In Appendix B of [13], it was shown that the coefficient in front of the leading term, but for an irrelevant overall constant, is precisely the function $Q^{(1)}(E, \gamma)$ appearing in the Bethe Ansatz, that is

$$\chi_{n-1}^\dagger \sim Q^{(1)}(E, \gamma(\mathbf{g}))\ x^{(1-n)\frac{M}{2}} e^{\frac{x^{M+1}}{M+1}} + \text{subdominant terms}, \quad (x \to \infty) . \tag{3.9}$$

Therefore, the set of Bethe ansatz roots

$$\{E_i^{(1)}\} \leftrightarrow Q^{(1)}(E_i^{(1)}, \gamma) = 0 \tag{3.10}$$

coincide with the discrete set of E values in (3.6) such that

$$\chi_{n-1}^\dagger \sim o\left(x^{(1-n)\frac{M}{2}} e^{\frac{x^{M+1}}{M+1}}\right) , \quad (x \to \infty) . \tag{3.11}$$

This condition is equivalent to the requirement of absolute integrability of

$$\left(x^{(n-1)\frac{M}{2}} e^{-\frac{x^{M+1}}{M+1}}\right) \chi_{n-1}^\dagger(x, E) \tag{3.12}$$

on the interval $[0, \infty)$. It is important to stress that the boundary problem defined above for the function $\chi_{n-1}^\dagger$ (3.8) is in general different from the one discussed in Sections 3 and 4 in [13] involving $\psi(x, E)$. The latter function is instead a solution to the adjoint equation of (3.6) and characterised by recessive behaviour at infinity. Surprisingly, the two problems are spectrally equivalent and lead to identical sets of Bethe ansatz roots.

D_n models:

The D_n pseudo-differential equations are

$$D_n(\mathbf{g}^\dagger)\left(\frac{d}{dx}\right)^{-1} D_n(\mathbf{g})\chi_{2n-1}(x, E) = \sqrt{P_K(x, E)}\left(\frac{d}{dx}\right)\sqrt{P_K(x, E)}\,\chi_{2n-1}(x, E) . \tag{3.13}$$

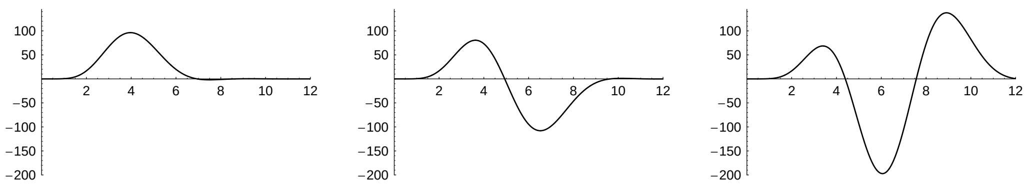

Figure 1: Lowest three functions $\Psi(x,E)$ for a D_4 pseudo-differential equation.

Fixing the ordering $g_i < g_j < h^\vee/2$, the $\mathbf{g} \leftrightarrow \gamma$ relationship is

$$\gamma_a = \frac{2K}{h^\vee M}\left(\sum_{i=0}^{a-1} g_i - \frac{a}{2}h^\vee\right) \ , \quad (a = 1, \ldots, n-2) \tag{3.14}$$

$$\gamma_{n-1} = \frac{K}{h^\vee M}\left(\sum_{i=0}^{n-1} g_i - \frac{n}{2}h^\vee\right), \ \gamma_n = \frac{K}{h^\vee M}\left(\sum_{i=0}^{n-2} g_i - g_{n-1} - \frac{n-2}{2}h^\vee\right) \ . \tag{3.15}$$

The solution is specified by requiring

$$\chi_{2n-1} \sim x^{h^\vee - g_0} + \text{subdominant terms}, \quad (x \to 0^+) \ , \tag{3.16}$$

$$\chi_{2n-1} \sim Q^{(1)}(E, \gamma(\mathbf{g}))\ x^{-h^\vee \frac{M}{2}} e^{\frac{x^{M+1}}{M+1}} + \text{subdominant terms}, \quad (x \to \infty). \tag{3.17}$$

Figure 1 illustrates $\Psi(x,E) = x^{h^\vee \frac{M}{2}} e^{-\frac{x^{M+1}}{M+1}} \chi_{2n-1}(x,E)$ for the first three eigenvalues of the D_4 pseudo-differential equation defined by $K{=}1, M = 1/3$ and $\mathbf{g}{=}(2.95{,}2.3{,}1.1{,}0.2)$.

B_n models:

The B_n ODEs are

$$D_n(\mathbf{g}^\dagger) D_n(\mathbf{g}) \chi^\dagger_{2n-1}(x,E) = \sqrt{P_K(x,E)}\left(\frac{d}{dx}\right)\sqrt{P_K(x,E)}\chi^\dagger_{2n-1}(x,E) \ . \tag{3.18}$$

With the ordering $g_i < g_j < h^\vee/2$, the $\mathbf{g} \leftrightarrow \gamma$ relation is

$$\gamma_a = \frac{2K}{h^\vee M}\left(\sum_{i=0}^{a-1} g_i - \frac{a}{2}h^\vee\right) \ . \tag{3.19}$$

The asymptotic behaviours about $x = 0$ and $x = \infty$ are respectively

$$\chi^\dagger_{2n-1} \sim x^{h^\vee - g_0} + \text{subdominant terms}, \quad (x \to 0^+) \ , \tag{3.20}$$

and

$$\chi^\dagger_{2n-1} \sim Q^{(1)}(E,\gamma(\mathbf{g}))\; x^{-h^\vee \frac{M}{2}} e^{\frac{x^{M+1}}{M+1}} + \text{subdominant terms}, \quad (x \to \infty)\ . \quad (3.21)$$

C_n models:

The pseudo-differential equations associated to the C_n systems are

$$D_n(\mathbf{g}^\dagger)\left(\frac{d}{dx}\right) D_n(\mathbf{g})\, \chi_{2n+1}(x,E) = P_K(x,E)\left(\frac{d}{dx}\right)^{-1} P_K(x,E)\, \chi_{2n+1}(x,E) \quad (3.22)$$

with the ordering $g_i < g_j < n$. The relation between the g's and the twist parameters in the BAE is

$$\gamma_a = \frac{2K}{h^\vee M}\left(\sum_{i=0}^{a-1} g_i - an\right),\ \gamma_n = \frac{K}{h^\vee M}\left(\sum_{i=0}^{n-1} g_i - n^2\right) \quad (3.23)$$

and

$$\chi^\dagger_{2n+1} \sim x^{2n-g_0} + \text{subdominant terms}, \quad (x \to 0^+)\ , \quad (3.24)$$

$$\chi^\dagger_{2n+1} \sim Q^{(1)}(E,\gamma) x^{-nM} e^{\frac{x^{M+1}}{M+1}} + \text{subdominant terms}, \quad (x \to \infty)\ . \quad (3.25)$$

Using a generalisation of Cheng's algorithm, the zeros of $Q^{(1)}(E,\gamma)$ can be found numerically and shown to match the appropriate Bethe ansatz roots [13].

In general, the 'spectrum' of a pseudo-differential equation may be either real or complex. In the A_{n-1}, B_n, D_n models with K=1*, the special choice $g_i = i$ leads to pseudo-differential equations with real spectra, a property which is expected to hold for a range of the parameters $\mathbf{g}$ (see, for example, [12]). The $K>1$ generalisation of the potential (3.5), proposed initially by Lukyanov for the A_1 models [15] but expected to work for all models, introduces a new feature. The eigenvalues corresponding to a K=2, 3 and K=4 case of the $SU(2)$ ODE are illustrated in figure 2. The interesting feature appears if we instead plot the logarithm of the eigenvalues as in figure 3. We see that the logarithm of the eigenvalues form 'strings', a well-known feature of integrable models. The string solutions approximately lie along lines in the complex plane, the deviations away from which can be calculated [13] using either WKB techniques, or by studying the asymptotics of the Bethe ansatz equations directly.

* The C_n spectrum is complex for any integer $K \geq 1$.

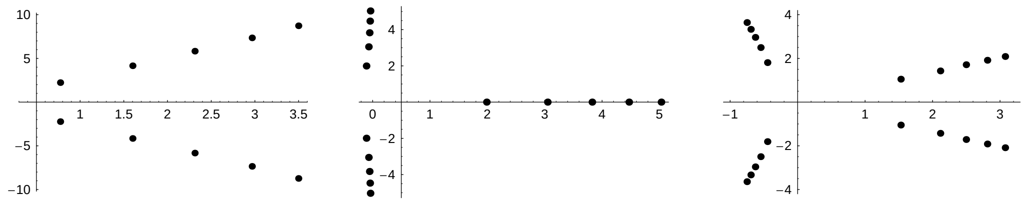

Figure 2: Complex E-plane: the eigenvalues for the $SU(2)$ model with $M = 3$, $g_0 = 0$ for $K = 2$, 3 and 4 respectively.

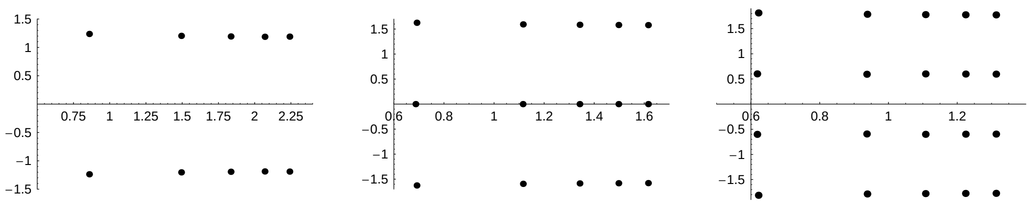

Figure 3: Complex ($\ln E$)-plane: two, three- and four-strings.

To end this section, we would like to comment briefly on the motivation behind the conjectured pseudo-differential equations of B_n, C_n and D_n type. Modulo the generalisation to $K>1$, the A_{n-1} type ODEs were derived in [12]. We began with the D_3 case since it coincides up to relabeling with A_3, implying that the D_3 function $Q^{(1)}(E, \gamma)$ coincides with the A_3 function $Q^{(2)}(E, \gamma)$. Fortunately, the latter is known [12] to encode the spectrum of a differential equation satisfied by the Wronskian of two solutions of the $Q^{(1)}$-related ODE. The generalisation to D_n models with larger n was then clear. Further supporting evidence came from a relationship between certain D_n lattice models and the sine-Gordon model, which appears as an $SU(2)$ problem. This relationship also extends to a set of B_n models, and leads naturally to the full B_n proposal. Finally, the C_n proposal arose from the B_n cases via a consideration of negative-dimension W-algebra dualities [16]. Numerical and analytical tests provided further evidence for the connection between these spectral problems and the Bethe ansatz equations for the classical Lie algebras.

4 Conclusions

The link between integrable models and the theory of ordinary differential equations is an exciting mathematical fact that has the potential to influence the future development of integrable models and conformal field theory, as well as some branches of classical and modern mathematics. Perhaps the most surprising aspect of the functions Q and T, only briefly discussed in this short note, is their variety of possible interpretations: transfer matrix eigenvalues of integrable lattice models in their CFT limit [5, 6], spectral determinants of Hermitian and PT-symmetric [17, 18] spectral problems (see for example [10]), g-functions of CFTs perturbed by relevant boundary operators [5, 19], and particular expectation values in the quantum problem of a Brownian particle [20]. Further, the (adjoint of the) operators (3.6), (3.13), (3.18) and (3.22) resemble in form the Miura-transformed Lax operators introduced by Drinfel'd and Sokolov in the context of generalised KdV equations, studied more recently in relation to the geometric Langlands correspondence [21, 22]. Clarifying this connection is an interesting open task. Here we finally observe that the proposed equations respect the well-known Lie algebras relations $D_2 \sim A_1 \oplus A_1$, $A_3 \sim D_3$, $B_1 \sim A_1$, $B_2 \sim C_2$. Also, at special values of the parameters the C_n equations are formally related to the D_n ones by the analytic continuation $n \to -n$, matching an interesting W-algebra duality discussed by Hornfeck in [16]:

$$\frac{(\widehat{D}_{-n})_K \times (\widehat{D}_{-n})_L}{(\widehat{D}_{-n})_{K+L}} \sim \frac{(\widehat{C}_n)_{-K/2} \times (\widehat{C}_n)_{-L/2}}{(\widehat{C}_n)_{-K/2-L/2}} \,. \tag{4.26}$$

The relationship between our equations and coset conformal field theories is another aspect worth investigation. We shall return to this point in a forthcoming publication.

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