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## COMPLEMENTARY RESULTS FOR THE SPECTRAL ANALYSIS OF MATRICES IN GALERKIN METHODS WITH GB-SPLINES

**Abstract.** We collect some new results relative to the study of the spectral analysis of matrices arising in Galerkin methods based on generalized B-splines with high smoothness. We compute some estimates for their minimal eigenvalues, a bound for their condition number, and we devise some results on their spectral distribution.

### 1. Introduction

In this paper we collect some results relative to the study of the spectral analysis of matrices arising in Galerkin methods based on generalized B-splines with high smoothness, which has been considered in [1].

They are the generalization of the ones reported in [2] for the polynomial case, which have been studied also in the generalized context, although they do not appear in [1].

A Galerkin method can be a scheme to discretize, resulting in a linear system of algebraic equations, the following second order linear elliptic differential equation with constant coefficients and homogeneous Dirichlet boundary conditions:

$$(1) \quad \begin{cases} -u'' + \beta u' + \gamma u = f, & 0 < x < 1, \\ u(0) = 0, & u(1) = 0, \end{cases}$$

with  $f \in L_2((0, 1))$ ,  $\beta \in \mathbb{R}$ ,  $\gamma \geq 0$ ; a *stiffness matrix*  $A$  is constructed in the process.

The problem (1) refers to a one-dimensional setting, with constant coefficients and no geometry map. Although there are several other methods to solve such a differential problem, these one-dimensional results are crucial building blocks in the treatment of the multivariate problem; indeed, thanks to the tensor-product structure of the approximation spaces, the univariate results can be directly extended to the multivariate setting. This is very important, because the methods to solve certain partial differential equations in high dimension are by far less.

Furthermore, the use of GB-splines spaces as solution spaces can be advantageous for some practical applications: for example, they allow for an exact representation of polynomial curves, conic sections, helices and other profiles of salient interest in applications, which is important in the context of isogeometric analysis (IgA), where the same discretization and representation tools for the design and for the analysis are used. This has several advantages if applied to a CAD software, also in engineering.

For these reasons, we mainly focus on trigonometric and hyperbolic GB-splines, since they are the more interesting in practical applications: for example, ellipses (with the special case of circumferences) and helices can be exactly represented in terms of

trigonometric GB-splines, while hyperbolae and catenaries can be exactly represented in terms of hyperbolic GB-splines.

It is worthy to note that GB-splines are also plug-to-plug compatible with (polynomial) B-splines in IgA, since they are structurally similar; moreover, it is possible to select their section spaces according to a problem-oriented strategy, which takes into account the peculiar issues of the considered problem, both from a geometrical and analytical point of view. Thanks to these two aspects, GB-splines arose as a flexible and interesting tool in IgA approximation methods, by being used also in other fields like signal processing.

We are particularly interested in the spectral analysis of these matrices because iterative solvers like classical multigrid methods are not always effective in this context: they tend to lose accuracy while  $p$  is large, so we need an accurate description of their spectrum in order to construct solvers which are optimal and robust in  $p$ , see [3].

The paper is organized as follows. After in Section 2 we define the GB-splines and the matrices considered, in Section 3 some estimates for the minimal eigenvalues of some of these matrices are provided, while in Section 4 the condition number of the stiffness matrix is properly bounded. Section 5 depicts results about the spectral distribution of various matrices involved, while some concluding remarks are contained in Section 6.

## 2. GB-splines and matrices

We recall that  $N_{i,p}^{U,V}$  is the  $i$ -th GB-spline of degree  $p$  over the knot set

$$\{t_1, \dots, t_{n+2p+1}\} := \left\{ \underbrace{0, \dots, 0}_{p+1}, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, \underbrace{1, \dots, 1}_{p+1} \right\},$$

belonging to the generalized spline space of degree  $p$

$$\mathcal{S}_{n,p}^{U,V} := \left\{ s \in C^{p-1}([0, 1]) : s|_{\left[\frac{i}{n}, \frac{i+1}{n}\right]} \in \mathbb{P}_p^{U,V}, i = 0, \dots, n-1 \right\},$$

where  $\mathbb{P}_p^{U,V}$  is the *section space*

$$(2) \quad \mathbb{P}_p^{U,V} := \langle 1, x, \dots, x^{p-2}, U(x), V(x) \rangle, \quad x \in [0, 1],$$

with  $U, V \in C^{p-1}[0, 1]$  such that  $\{U^{(p-1)}, V^{(p-1)}\}$  is a Chebyshev system in  $[t_i, t_{i+1}]$ , i.e., any non-trivial element in the space  $\langle U^{(p-1)}, V^{(p-1)} \rangle$  has at most one zero in  $[t_i, t_{i+1}]$ ,  $i = p+1, \dots, p+n$ .

If  $\tilde{U}_i, \tilde{V}_i$  are the unique elements in  $\langle U^{(p-1)}, V^{(p-1)} \rangle$  satisfying

$$\tilde{U}_i(t_i) = 1, \tilde{U}_i(t_{i+1}) = 0, \quad \tilde{V}_i(t_i) = 0, \tilde{V}_i(t_{i+1}) = 1, \quad i = p+1, \dots, p+n,$$

then we can define  $N_{i,p}^{U,V} : [0, 1] \rightarrow \mathbb{R}$ ,  $i = 1, \dots, n+p$ , recursively as follows: for  $p = 1$ ,

$$N_{i,1}^{U,V}(x) := \begin{cases} \tilde{V}_i(x), & \text{if } x \in [t_i, t_{i+1}), \\ \tilde{U}_{i+1}(x), & \text{if } x \in [t_{i+1}, t_{i+2}), \\ 0, & \text{elsewhere,} \end{cases}$$

and for  $p \geq 2$ ,

$$N_{i,p}^{U,V}(x) := \delta_{i,p-1}^{U,V} \int_0^x N_{i,p-1}^{U,V}(s) ds - \delta_{i+1,p-1}^{U,V} \int_0^x N_{i+1,p-1}^{U,V}(s) ds,$$

where

$$\delta_{i,p}^{U,V} := \left( \int_0^1 N_{i,p}^{U,V}(s) ds \right)^{-1}.$$

Fractions with zero denominators are considered to be zero, and  $N_{i,p}^{U,V}(1) := \lim_{x \rightarrow 1^-} N_{i,p}^{U,V}(x)$ .

If  $U(x) = x^{p-1}$  and  $V(x) = x^p$ , this results in the classical polynomial B-splines, and we drop from the notation the dependence of  $U$  and  $V$ , by simply writing  $N_{i,p}(x)$ .

If  $U(x) = \cos(\alpha x)$  and  $V(x) = \sin(\alpha x)$ , with  $0 < \alpha < n\pi$ , we have the *trigonometric GB-splines*, for which we write also  $N_{i,p}^{\mathbb{T}\alpha}(x)$ .

If  $U(x) = \cosh(\alpha x)$  and  $V(x) = \sinh(\alpha x)$ , with  $0 < \alpha \in \mathbb{R}$ , we have the *hyperbolic* (or *exponential*) *GB-splines*, for which we write also  $N_{i,p}^{\mathbb{H}\alpha}(x)$ .

The notation  $N_{i,p}^{\mathbb{Q}\alpha}(x)$  is used when a statement holds for both  $N_{i,p}^{\mathbb{T}\alpha}(x)$  and  $N_{i,p}^{\mathbb{H}\alpha}(x)$ , but not necessarily for an arbitrary  $N_{i,p}^{U,V}(x)$ .

Since we are interested in having large values of  $n$ , or in the asymptotic behaviour while  $n \rightarrow \infty$ , we consider also trigonometric and hyperbolic spline spaces for which the *phase parameter*  $\alpha$  is proportional to  $n$ , rather than fixed.

Indeed,  $N_{i,p}^{\mathbb{Q}\alpha}(x) \rightarrow N_{i,p}(x)$  while  $n \rightarrow \infty$  if  $\alpha$  is fixed, so, without allowing  $\alpha$  to increase with  $n$  (which avoids this convergence), some advantages of the use of the GB-splines with respect to the polynomial B-splines are lost for large  $n$ .

Thus, we write  $N_{i,p}^{\mathbb{Q}\mu}(x)$ , where the case  $\mu = \alpha$  refers to a fixed phase parameter and it is known as *nested case*, while the case  $\mu = n\alpha$  refers to a phase parameter dependent of  $n$  with direct proportionality, and it is known as *non nested case*.

With GB-splines, the stiffness matrix  $A$  relative to the problem (1) can be written as  $A_{n,p}^{U,V}$ , which can be decomposed as

$$A_{n,p}^{U,V} = nK_{n,p}^{U,V} + \beta H_{n,p}^{U,V} + \frac{\gamma}{n} M_{n,p}^{U,V},$$

where

$$\begin{aligned} nK_{n,p}^{U,V} &:= \left[ \int_0^1 (N_{j+1,p}^{U,V})'(x) (N_{i+1,p}^{U,V})'(x) dx \right]_{i,j=1}^{n+p-2}, \\ H_{n,p}^{U,V} &:= \left[ \int_0^1 (N_{j+1,p}^{U,V})'(x) N_{i+1,p}^{U,V}(x) dx \right]_{i,j=1}^{n+p-2}, \\ \frac{1}{n} M_{n,p}^{U,V} &:= \left[ \int_0^1 N_{j+1,p}^{U,V}(x) N_{i+1,p}^{U,V}(x) dx \right]_{i,j=1}^{n+p-2}. \end{aligned}$$

While  $p \geq 2$ , for  $i = p+1, \dots, n$  we have  $N_{i,p}^{\mathbb{Q}\mu}(x) = \phi_p^{\mathbb{Q}\mu/n}(nx - i + p + 1)$  for certain functions  $\phi_p^{\mathbb{Q}\mu/n}$ , known as *cardinal GB-splines*. They are defined as follows: by

focusing again on (2), the space  $\mathbb{P}_p^{U,V}$ , we consider  $U, V \in C^{p-1}[0, p+1]$  such that  $\{U^{(p-1)}, V^{(p-1)}\}$  is a Chebyshev system in  $[0, 1]$ . By denoting with  $\tilde{U}, \tilde{V}$  the unique elements in the space  $\langle U^{(p-1)}, V^{(p-1)} \rangle$  satisfying

$$\tilde{U}(0) = 1, \tilde{U}(1) = 0, \quad \tilde{V}(0) = 0, \tilde{V}(1) = 1,$$

the (normalized) cardinal GB-spline of degree  $p \geq 1$  over the uniform knot set  $\{0, 1, \dots, p+1\}$  with sections in (2) is denoted by  $\phi_p^{U,V}$  and is defined recursively as follows: for  $p = 1$ ,

$$\phi_1^{U,V}(t) := \delta_1^{U,V} \begin{cases} \tilde{V}(t), & \text{if } t \in [0, 1), \\ \tilde{U}(t-1), & \text{if } t \in [1, 2), \\ 0, & \text{elsewhere,} \end{cases}$$

where  $\delta_1^{U,V}$  is a normalization factor given by

$$\delta_1^{U,V} := \left( \int_0^1 \tilde{V}(s) ds + \int_1^2 \tilde{U}(s-1) ds \right)^{-1}.$$

For  $p \geq 2$ ,

$$\phi_p^{U,V}(t) := \int_0^t (\phi_{p-1}^{U,V}(s) - \phi_{p-1}^{U,V}(s-1)) ds.$$

Note that, by already having  $N_{i,p}^{\mathbb{Q}\alpha}(x) \rightarrow N_{i,p}(x)$  for fixed  $\alpha$  and  $n \rightarrow \infty$ , it will also hold  $\phi_p^{\mathbb{Q}\alpha}(t) \rightarrow \phi_p(t)$  while  $\alpha \rightarrow 0$ .

Cardinal GB-splines are better suited than  $n$ -dependent GB-splines for studying some results on spectral distribution which will be depicted in Section 5.

### 3. Estimates for the minimal eigenvalues

In this section we provide estimates for the minimal eigenvalues of  $M_{n,p}^{\mathbb{Q}\mu}$  and  $K_{n,p}^{\mathbb{Q}\mu}$ . These estimates will be employed to obtain a lower bound for  $|\lambda_{\min}(A_{n,p}^{\mathbb{Q}\mu})|$ , where  $\lambda_{\min}(A_{n,p}^{\mathbb{Q}\mu})$  is an eigenvalue of  $A_{n,p}^{\mathbb{Q}\mu}$  with minimum modulus.

We begin by generalizing [2, Eq. (51)]. We remember that the result declares how, for every  $p \geq 1, n \geq 2$ , and  $\mathbf{x} = (x_1, \dots, x_{n+p-2}) \in \mathbb{R}^{n+p-2}$ , it holds

$$\underline{C}_p \frac{\|\mathbf{x}\|^2}{n} \leq \left\| \sum_{i=1}^{n+p-2} x_i N_{i+1,p} \right\|_{L_2([0,1])}^2 \leq \bar{C}_p \frac{\|\mathbf{x}\|^2}{n},$$

where the constants  $\underline{C}_p, \bar{C}_p > 0$  do not depend on  $n$  and  $\mathbf{x}$  (see also [9, Eq. (6.3) and Theorem 9.27]). It is possible to infer that, for every  $i = 2, \dots, n+p-1$ ,  $x \in \text{int}(\text{supp}(N_{i,p}(x)))$ , there hold, for proper positive constants  $\underline{\Gamma}_p^{\mathbb{Q}\alpha}, \bar{\Gamma}_p^{\mathbb{Q}\alpha}, \underline{\Gamma}_p^{\mathbb{Q}[0,\alpha]}, \bar{\Gamma}_p^{\mathbb{Q}[0,\alpha]}$ :

$$\underline{\Gamma}_p^{\mathbb{Q}\alpha} \leq \frac{N_{i,p}^{\mathbb{Q}\alpha}(x)}{N_{i,p}(x)} \leq \bar{\Gamma}_p^{\mathbb{Q}\alpha}, \quad \underline{\Gamma}_p^{\mathbb{Q}[0,\alpha]} \leq \frac{N_{i,p}^{\mathbb{Q}\alpha}(x)}{N_{i,p}(x)} \leq \bar{\Gamma}_p^{\mathbb{Q}[0,\alpha]},$$

where we use the notation  $\alpha^* = \frac{\alpha}{\lfloor \frac{\alpha}{\pi} \rfloor + 1}$  and

$$\begin{aligned} \underline{\Gamma}_p^{\mathbb{H}[0,\alpha]} &:= \inf_{\bar{\alpha} \in (0,\alpha]} \underline{\Gamma}_p^{\mathbb{H}\bar{\alpha}} \leq \inf_{n \in \mathbb{N}_0} \underline{\Gamma}_p^{\mathbb{H}\alpha/n}, & \underline{\Gamma}_p^{\mathbb{T}[0,\alpha]} &:= \inf_{\bar{\alpha} \in (0,\alpha^*]} \underline{\Gamma}_p^{\mathbb{T}\bar{\alpha}} \leq \inf_{n \geq \lfloor \frac{\alpha}{\pi} \rfloor + 1} \underline{\Gamma}_p^{\mathbb{T}\alpha/n}, \\ \overline{\Gamma}_p^{\mathbb{H}[0,\alpha]} &:= \sup_{\bar{\alpha} \in (0,\alpha]} \overline{\Gamma}_p^{\mathbb{H}\bar{\alpha}} \geq \sup_{n \in \mathbb{N}_0} \overline{\Gamma}_p^{\mathbb{H}\alpha/n}, & \overline{\Gamma}_p^{\mathbb{T}[0,\alpha]} &:= \sup_{\bar{\alpha} \in (0,\alpha^*]} \overline{\Gamma}_p^{\mathbb{T}\bar{\alpha}} \geq \sup_{n \geq \lfloor \frac{\alpha}{\pi} \rfloor + 1} \overline{\Gamma}_p^{\mathbb{T}\alpha/n}. \end{aligned}$$

Indeed,  $N_{i,p}^{\mathbb{Q}n\alpha}(x)$  and  $N_{i,p}(x)$  are zero in the same points, and the zeros in their supports (at the boundaries) have the same order; furthermore,  $N_{i,p}^{\mathbb{Q}n\alpha}(x) \rightarrow N_{i,p}(x)$  while  $n \rightarrow \infty$ .

As a consequence, with norms in  $L_2([0,1])$

$$\begin{aligned} \left\| \sum_{i=1}^{n+p-2} x_i \underline{\Gamma}_p^{\mathbb{Q}\alpha} N_{i+1,p} \right\|^2 &\leq \left\| \sum_{i=1}^{n+p-2} x_i N_{i+1,p}^{\mathbb{Q}n\alpha} \right\|^2 \leq \left\| \sum_{i=1}^{n+p-2} x_i \underline{\Gamma}_p^{\mathbb{Q}\alpha} N_{i+1,p} \right\|^2, \\ \left\| \sum_{i=1}^{n+p-2} x_i \underline{\Gamma}_p^{\mathbb{Q}[0,\alpha]} N_{i+1,p} \right\|^2 &\leq \left\| \sum_{i=1}^{n+p-2} x_i N_{i+1,p}^{\mathbb{Q}\alpha} \right\|^2 \leq \left\| \sum_{i=1}^{n+p-2} x_i \underline{\Gamma}_p^{\mathbb{Q}[0,\alpha]} N_{i+1,p} \right\|^2, \end{aligned}$$

and so, by setting  $\underline{\mathcal{C}}_p^{\mathbb{Q}\alpha} := (\underline{\Gamma}_p^{\mathbb{Q}\alpha})^2 \underline{\mathcal{C}}_p$ ,  $\overline{\mathcal{C}}_p^{\mathbb{Q}\alpha} := (\overline{\Gamma}_p^{\mathbb{Q}\alpha})^2 \overline{\mathcal{C}}_p$ ,  $\underline{\mathcal{C}}_p^{\mathbb{Q}[0,\alpha]} := (\underline{\Gamma}_p^{\mathbb{Q}[0,\alpha]})^2 \underline{\mathcal{C}}_p$ ,  $\overline{\mathcal{C}}_p^{\mathbb{Q}[0,\alpha]} := (\overline{\Gamma}_p^{\mathbb{Q}[0,\alpha]})^2 \overline{\mathcal{C}}_p$

$$(3) \quad \underline{\mathcal{C}}_p^{\mathbb{Q}\alpha} \frac{\|\mathbf{x}\|^2}{n} \leq \left\| \sum_{i=1}^{n+p-2} x_i N_{i+1,p}^{\mathbb{Q}n\alpha} \right\|^2 \leq \overline{\mathcal{C}}_p^{\mathbb{Q}\alpha} \frac{\|\mathbf{x}\|^2}{n},$$

$$(4) \quad \underline{\mathcal{C}}_p^{\mathbb{Q}[0,\alpha]} \frac{\|\mathbf{x}\|^2}{n} \leq \left\| \sum_{i=1}^{n+p-2} x_i N_{i+1,p}^{\mathbb{Q}\alpha} \right\|^2 \leq \overline{\mathcal{C}}_p^{\mathbb{Q}[0,\alpha]} \frac{\|\mathbf{x}\|^2}{n}.$$

We also recall the Poincaré inequality in the one-dimensional setting:

$$(5) \quad \|v\|_{L_2([0,1])} \leq \frac{1}{\pi} \|v'\|_{L_2([0,1])}, \quad \forall v \in H_0^1([0,1]),$$

where  $\frac{1}{\pi}$  is the best constant, see [4].

We can use (3)-(4) and (5) to prove the next theorem, by defining  $\underline{\mathcal{C}}_p^{\mathbb{Q}v} := \underline{\mathcal{C}}_p^{\mathbb{Q}\alpha}$  in the non-nested case,  $\underline{\mathcal{C}}_p^{\mathbb{Q}v} := \underline{\mathcal{C}}_p^{\mathbb{Q}[0,\alpha]}$  in the nested case.

**THEOREM 1.** *Let  $\underline{\mathcal{C}}_p^{\mathbb{Q}v} > 0$  be the constant in (3) or (4), then for all  $p \geq 2$  and  $n \geq 2$  the following properties hold.*

1.  $\lambda_{\min}(M_{n,p}^{\mathbb{Q}\mu}) \geq \underline{\mathcal{C}}_p^{\mathbb{Q}v}$ ,
2.  $K_{n,p}^{\mathbb{Q}\mu} \geq \frac{\pi^2}{n^2} M_{n,p}^{\mathbb{Q}\mu}$  and  $\lambda_{\min}(K_{n,p}^{\mathbb{Q}\mu}) \geq \frac{\pi^2 \underline{\mathcal{C}}_p^{\mathbb{Q}v}}{n^2}$ .

*Proof.* The proof is analogous to the one referring to the polynomial case, see [2, Theorem 8], by considering that  $M_{n,p}^{\mathbb{Q}\mu}$  and  $K_{n,p}^{\mathbb{Q}\mu}$  are still symmetric matrices, and having (3)-(5).  $\square$

REMARK 1. For every  $p \geq 2, n \geq 2$  and  $j = 1, \dots, n + p - 2$ , let  $\lambda_j(K_{n,p}^{\mathbb{Q}_\mu})$  be the  $j$ -th smallest eigenvalue of  $K_{n,p}^{\mathbb{Q}_\mu}$ , that is,  $\lambda_1(K_{n,p}^{\mathbb{Q}_\mu}) \leq \dots \leq \lambda_{n+p-2}(K_{n,p}^{\mathbb{Q}_\mu})$ . Then, we conjecture that for every  $p \geq 2$  and for each fixed  $j \geq 1$ ,

$$\lim_{n \rightarrow \infty} (n^2 \lambda_j(K_{n,p}^{\mathbb{Q}_\mu})) = j^2 \pi^2.$$

There is a motivation related to the connection between  $K_{n,p}^{\mathbb{Q}_\mu}$  and the problem (1), furthermore in the polynomial case it has been verified numerically for some values of  $p$  and  $j$ ; see [2, Remark 3] for details.

THEOREM 2. For all  $p \geq 2$  and all  $n \geq 2$ , let  $\lambda_{\min}(A_{n,p}^{\mathbb{Q}_\mu})$  be an eigenvalue of  $A_{n,p}^{\mathbb{Q}_\mu}$  with minimum modulus. Then,

$$(6) \quad |\lambda_{\min}(A_{n,p}^{\mathbb{Q}_\mu})| \geq \lambda_{\min}(\operatorname{Re}A_{n,p}^{\mathbb{Q}_\mu}) \geq \frac{\underline{C}_p^{\mathbb{Q}_v}(\pi^2 + \gamma)}{n},$$

with  $\underline{C}_p^{\mathbb{Q}_v} > 0$  being the same constant appearing in Theorem 1.

*Proof.* Let us preliminary recall that, having  $X^*$  the conjugate transposed of  $X$ , they hold:

$$\operatorname{Re}X := \frac{X + X^*}{2}, \quad \operatorname{Im}X := \frac{X - X^*}{2i}.$$

By the expression

$$A_{n,p}^{\mathbb{Q}_\mu} = nK_{n,p}^{\mathbb{Q}_\mu} + \beta H_{n,p}^{\mathbb{Q}_\mu} + \frac{\gamma}{n} M_{n,p}^{\mathbb{Q}_\mu},$$

and recalling that  $K_{n,p}^{\mathbb{Q}_\mu}, M_{n,p}^{\mathbb{Q}_\mu}$  are symmetric, while  $H_{n,p}^{\mathbb{Q}_\mu}$  is skew-symmetric, we infer that the real part of  $A_{n,p}^{\mathbb{Q}_\mu}$  is given by

$$\operatorname{Re}A_{n,p}^{\mathbb{Q}_\mu} = nK_{n,p}^{\mathbb{Q}_\mu} + \frac{\gamma}{n} M_{n,p}^{\mathbb{Q}_\mu}.$$

Therefore, by the minimax principle and by Theorem 1 we obtain

$$\lambda_{\min}(\operatorname{Re}A_{n,p}^{\mathbb{Q}_\mu}) \geq \lambda_{\min}(nK_{n,p}^{\mathbb{Q}_\mu}) + \lambda_{\min}\left(\frac{\gamma}{n} M_{n,p}^{\mathbb{Q}_\mu}\right) \geq n \frac{\pi^2 \underline{C}_p^{\mathbb{Q}_v}}{n^2} + \frac{\gamma}{n} \underline{C}_p^{\mathbb{Q}_v} = \frac{\underline{C}_p^{\mathbb{Q}_v}(\pi^2 + \gamma)}{n}.$$

The result is obtained by considering that  $|\lambda_{\min}(A_{n,p}^{\mathbb{Q}_\mu})| \geq \lambda_{\min}(\operatorname{Re}A_{n,p}^{\mathbb{Q}_\mu})$  because of the spectrum localization

$$\sigma(X) \subseteq [\lambda_{\min}(\operatorname{Re}X), \lambda_{\max}(\operatorname{Re}X)] \times [\lambda_{\min}(\operatorname{Im}X), \lambda_{\max}(\operatorname{Im}X)] \subset \mathbb{C}, \quad \forall X \in \mathbb{C}^{m \times m}.$$

□

The lower bound (6) remains bounded away from 0 for all  $\gamma \geq 0$  and, in particular, for the interesting value  $\gamma = 0$ .

The constant  $\underline{C}_p^{\mathbb{Q}_\nu}$  is theoretical, but with a table depicting  $|\lambda_{\min}(A_{n,p}^{\mathbb{H}_1})|$  for various values of  $n$  and  $p$ , where  $\beta = \gamma = 1$ , we can show its asymptotic proportionality with respect to  $\frac{1}{n}$ :

$p$	2				3			
$n$	10	20	40	80	10	20	40	80
$ \lambda_{\min}(A_{n,p}^{\mathbb{H}_1}) $	1.0856	0.55265	0.27757	0.13894	1.0723	0.55124	0.27741	0.13892
$\frac{ \lambda_{\min}(A_{n,p}^{\mathbb{H}_1}) }{ \lambda_{\min}(A_{\frac{n}{2},p}^{\mathbb{H}_1}) }$		0.50907	0.50225	0.50056		0.51407	0.50325	0.50078

#### 4. Conditioning

In this section we provide a bound for the condition number

$$\kappa_2(A_{n,p}^{\mathbb{Q}_\mu}) := \|A_{n,p}^{\mathbb{Q}_\mu}\|_2 \|(A_{n,p}^{\mathbb{Q}_\mu})^{-1}\|_2.$$

Let us start with the Fan-Hoffman theorem ([10]).

**THEOREM 3.** *Let  $X \in \mathbb{C}^{m \times m}$  and let:*

$$\|X\|_2 = s_1(X) \geq s_2(X) \geq \dots \geq s_m(X), \quad \lambda_1(\operatorname{Re}X) \geq \lambda_2(\operatorname{Re}X) \geq \dots \geq \lambda_m(\operatorname{Re}X)$$

*be the singular values of  $X$  and the eigenvalues of  $\operatorname{Re}X$ , respectively. Then*

$$s_j(X) \geq \lambda_j(\operatorname{Re}X), \quad \forall j = 1, \dots, m.$$

**THEOREM 4.** *For every  $p \geq 2$  there exists a constant  $\eta_p > 0$  such that*

$$\kappa_2(A_{n,p}^{\mathbb{Q}_\mu}) \leq \eta_p n^2, \quad \forall n \geq 2.$$

*Proof.* Fix  $p \geq 2$  and  $n \geq 2$ . Being either symmetric or skew-symmetric,  $K_{n,p}^{\mathbb{Q}_\mu}, H_{n,p}^{\mathbb{Q}_\mu}$  and  $M_{n,p}^{\mathbb{Q}_\mu}$  are normal matrices, and by applying [1, Lemma 7] we obtain for  $\|A_{n,p}^{\mathbb{Q}_\mu}\|_2$  the following bound, where  $\mathfrak{C}_{p,\alpha}$  is a constant dependent of  $p$  and  $\alpha$  (but not of  $n$ )

$$\begin{aligned} \|A_{n,p}^{\mathbb{Q}_\mu}\|_2 &= \left\| nK_{n,p}^{\mathbb{Q}_\mu} + \beta H_{n,p}^{\mathbb{Q}_\mu} + \frac{\gamma}{n} M_{n,p}^{\mathbb{Q}_\mu} \right\|_2 \leq \|nK_{n,p}^{\mathbb{Q}_\mu}\|_2 + |\beta| \|H_{n,p}^{\mathbb{Q}_\mu}\|_2 + \gamma \left\| \frac{1}{n} M_{n,p}^{\mathbb{Q}_\mu} \right\|_2 \\ &\leq \|nK_{n,p}^{\mathbb{Q}_\mu}\|_\infty + |\beta| \|H_{n,p}^{\mathbb{Q}_\mu}\|_\infty + \gamma \left\| \frac{1}{n} M_{n,p}^{\mathbb{Q}_\mu} \right\|_\infty \leq \mathfrak{C}_{p,\alpha} n + 2|\beta| + \frac{\gamma(p+1)}{n}. \end{aligned}$$

On the other hand, being  $s_{n+p-2}(A_{n,p}^{\mathbb{Q}_\mu})$  the minimum singular value of  $A_{n,p}^{\mathbb{Q}_\mu}$ , we have



$$\|(A_{n,p}^{\mathbb{Q}_\mu})^{-1}\|_2 = \frac{1}{s_{n+p-2}(A_{n,p}^{\mathbb{Q}_\mu})} \leq \frac{1}{\lambda_{\min}(\operatorname{Re}A_{n,p}^{\mathbb{Q}_\mu})} \leq \frac{n}{\underline{\mathcal{C}}_p^{\mathbb{Q}_v}(\pi^2 + \gamma)}.$$

so

$$\kappa_2(A_{n,p}^{\mathbb{Q}_\mu}) \leq \frac{\mathfrak{C}_{p,\alpha} n^2 + 2n|\beta| + \gamma(p+1)}{\underline{\mathcal{C}}_p^{\mathbb{Q}_v}(\pi^2 + \gamma)} \leq \frac{\mathfrak{C}_{p,\alpha} n^2 + n^2|\beta| + \gamma(p+1)(n^2/4)}{\underline{\mathcal{C}}_p^{\mathbb{Q}_v}(\pi^2 + \gamma)}.$$

that makes the theorem satisfied for  $\eta_p := \frac{1}{\underline{\mathcal{C}}_p^{\mathbb{Q}_v}(\pi^2 + \gamma)} [\mathfrak{C}_{p,\alpha} + |\beta| + \frac{\gamma(p+1)}{4}]$ .

□

Again, the constant  $\eta_p$  is theoretical, being dependent of  $\underline{\mathcal{C}}_p^{\mathbb{Q}_v}$ , but with a table depicting  $\kappa_2(A_{n,p}^{\mathbb{H}_1})$  for various values of  $n$  and  $p$ , where again  $\beta = \gamma = 1$ , we can show its asymptotic proportionality with respect to  $n^2$ :

$p$	2				3			
$n$	10	20	40	80	10	20	40	80
$ \kappa_2(A_{n,p}^{\mathbb{H}_1}) $	13.960	54.809	218.21	871.81	14.883	57.729	229.28	915.56
$\frac{ \kappa_2(A_{n,p}^{\mathbb{H}_1}) }{ \kappa_2(A_{\frac{n}{2},p}^{\mathbb{H}_1}) }$		3.9261	3.9813	3.9953		3.8789	3.9717	3.9932

## 5. More results on spectral distribution

We recall that, given a univariate function  $f : [-\pi, \pi] \rightarrow \mathbb{R}$  belonging to  $L_1([-\pi, \pi])$ , we can associate to  $f$  a family (sequence) of Hermitian Toeplitz matrices  $\{T_m(f)\}$  parameterized by the integer index  $m$  and defined for all  $m \geq 1$  in the following way:

$$T_m(f) := \begin{bmatrix} f_0 & f_{-1} & \cdots & \cdots & f_{-(m-1)} \\ f_1 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & f_{-1} \\ f_{m-1} & \cdots & \cdots & f_1 & f_0 \end{bmatrix} \in \mathbb{C}^{m \times m},$$

where

$$f_k := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-ik\theta} d\theta, \quad k \in \mathbb{Z},$$

are the Fourier coefficients of  $f$ ; the function  $f$  is called the generating function of  $T_m(f)$ . The following one is another important result regarding sequences of Toeplitz matrices.

**THEOREM 5.** *Let  $f \in L_1([-\pi, \pi])$  be a real-valued function, and let  $m_f := \operatorname{ess\,inf} f, M_f := \operatorname{ess\,sup} f$ , and suppose  $m_f < M_f$ . Then*

$$\lambda_{\min}(T_m(f)) \searrow m_f \text{ and } \lambda_{\max}(T_m(f)) \nearrow M_f \text{ as } m \rightarrow \infty.$$

Another result due to Parter [5] concerns the asymptotics of the  $j$ -th smallest eigenvalue  $\lambda_j(T_m(f))$ , for  $j$  fixed and  $m \rightarrow \infty$ .

**THEOREM 6.** *Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be continuous and  $2\pi$ -periodic. Let  $m_f := \min_{\theta \in \mathbb{R}} f(\theta) = f(\theta_{\min})$  and let  $\theta_{\min}$  be the unique point in  $(-\pi, \pi]$  such that  $f(\theta_{\min}) = m_f$ . Assume there exists  $s \geq 1$  such that  $f$  has  $2s$  continuous derivatives in  $(\theta_{\min} - \varepsilon, \theta_{\min} + \varepsilon)$  for some  $\varepsilon > 0$  and  $f^{(2s)}(\theta_{\min}) > 0$  is the first non-vanishing derivative of  $f$  at  $\theta_{\min}$ . Finally, for every  $m \geq 1$ , let  $\lambda_1(T_m(f)) \leq \dots \leq \lambda_m(T_m(f))$  be the eigenvalues of  $T_m(f)$  arranged in non-decreasing order. Then, for each fixed  $j \geq 1$ ,*

$$\lambda_j(T_m(f)) - m_f \sim_{m \rightarrow \infty} c_{s,j} \frac{f^{(2s)}(\theta_{\min})}{(2s)!} \frac{1}{m^{2s}},$$

where  $c_{s,j} > 0$  is a constant depending only on  $s$  and  $j$ .

**REMARK 2.** The constant  $c_{s,j}$  is the  $j$ -th smallest eigenvalue of the boundary value problem

$$\begin{cases} (-1)^s u^{(2s)}(x) = f(x), & \text{for } 0 < x < 1, \\ u(0) = u'(0) = \dots = u^{(s-1)}(0) = 0, u(1) = u'(1) = \dots = u^{(s-1)}(1) = 0, \end{cases}$$

see [5, p. 191]. Thus, we find that  $c_{1,j} = j^2 \pi^2$  for all  $j \geq 1$ .

If  $\dot{\phi}_p^{U,V}(t)$  is the first derivative of the cardinal GB-spline  $\phi_p^{U,V}(t)$  with respect to  $t$ , then we can define the functions

$$\begin{aligned} f_p^{U,V}(\theta) &= \sum_{k=-p}^p \left( \int_{\mathbb{R}} \dot{\phi}_p^{U,V}(t) \dot{\phi}_p^{U,V}(t-k) dt \right) \cos(k\theta), \\ h_p^{U,V}(\theta) &= \sum_{k=-p}^p \left( \int_{\mathbb{R}} \phi_p^{U,V}(t) \phi_p^{U,V}(t-k) dt \right) \cos(k\theta). \end{aligned}$$

and we denote as  $m_{h_p^{U,V}}$  the minimum of the function  $h_p^{U,V}(\theta)$  over  $[-\pi, \pi]$ , while  $M_{f_p^{U,V}}$  is the maximum of the function  $f_p^{U,V}(\theta)$  over  $[-\pi, \pi]$ .

Again, we omit  $U, V$  while referring to the polynomial case, and we use  $\mathbb{T}_\mu, \mathbb{H}_\mu, \mathbb{Q}_\mu$  while referring respectively to the trigonometric, the hyperbolic, and both cases.

This allows us to observe that, if  $n \geq 3p + 1$ , for  $k = 0, 1, \dots, p$  and  $i = 2p, \dots, n - p - 1$

$$\begin{aligned} (K_{n,p}^{\mathbb{Q}_\mu})_{i,i \pm k} &= \int_0^{p+1} \dot{\phi}_p^{\mathbb{Q}_\mu/n}(t \mp k) \dot{\phi}_p^{\mathbb{Q}_\mu/n}(t) dt, \\ (M_{n,p}^{\mathbb{Q}_\mu})_{i,i \pm k} &= \int_0^{p+1} \phi_p^{\mathbb{Q}_\mu/n}(t \mp k) \phi_p^{\mathbb{Q}_\mu/n}(t) dt. \end{aligned}$$

where, in every row, only at most  $2p + 1$  elements are nonzero, due to the compact support of the GB-splines:  $(K_{n,p}^{\mathbb{Q}\mu})_{i,i\pm k} = (M_{n,p}^{\mathbb{Q}\mu})_{i,i\pm k} = 0$  if  $k > p$ .

The matrices  $B_{n,p}^{\mathbb{Q}\mu}$  and  $C_{n,p}^{\mathbb{Q}\mu}$  are variants of respectively  $K_{n,p}^{\mathbb{Q}\mu}$  and  $M_{n,p}^{\mathbb{Q}\mu}$  in which the elements belonging to the first  $2p - 1$  rows and to the last  $2p - 1$  rows are set in a way such that the resulting matrices are  $(2p + 1)$ -band Toeplitz symmetric matrices, so:

$$\begin{aligned} (B_{n,p}^{\mathbb{Q}\mu})_{i,i\pm k} &= \int_0^{p+1} \phi_p^{\mathbb{Q}\mu/n}(t \mp k) \phi_p^{\mathbb{Q}\mu/n}(t) dt, \\ (C_{n,p}^{\mathbb{Q}\mu})_{i,i\pm k} &= \int_0^{p+1} \phi_p^{\mathbb{Q}\mu/n}(t \mp k) \phi_p^{\mathbb{Q}\mu/n}(t) dt. \end{aligned}$$

for  $k = 0, 1, \dots, p$  and  $i = 1, \dots, n + p - 2$ , provided that  $i - k \geq 1$  and  $i + k \leq n + p - 2$ . It can be noted that  $B_{n,p}^{\mathbb{Q}\mu} = T_{n+p-2}(f_p^{\mathbb{Q}\mu/n})$  and  $C_{n,p}^{\mathbb{Q}\mu} = T_{n+p-2}(h_p^{\mathbb{Q}\mu/n})$ ; furthermore, both  $f_p^{\mathbb{Q}\alpha}(\theta) \rightarrow f_p(\theta)$  and  $h_p^{\mathbb{Q}\alpha}(\theta) \rightarrow h_p(\theta)$  when  $\alpha \rightarrow 0$ , since  $\phi_p^{\mathbb{Q}\alpha}(t) \rightarrow \phi_p(t)$  while  $\alpha \rightarrow 0$ . In light of this, the following results are a consequence of Theorem 5.

**THEOREM 7.** *The following results hold:*

1.  $\lambda_{\min}(B_{n,p}^{\mathbb{Q}\alpha}) \searrow 0$  and  $\lambda_{\max}(B_{n,p}^{\mathbb{Q}\alpha}) \rightarrow M_{f_p}$  as  $n \rightarrow \infty$ ;
2. for each fixed  $j \geq 1$ ,

$$\lambda_j(B_{n,p}^{\mathbb{Q}\alpha}) \sim_{n \rightarrow \infty} \frac{j^2 \pi^2}{n^2},$$

where  $\lambda_1(B_{n,p}^{\mathbb{Q}\alpha}) \leq \dots \leq \lambda_{n+p-2}(B_{n,p}^{\mathbb{Q}\alpha})$  are the eigenvalues of  $B_{n,p}^{\mathbb{Q}\alpha}$  in non-decreasing order;

3.  $\lambda_{\min}(B_{n,p}^{\mathbb{Q}n\alpha}) \searrow 0$  and  $\lambda_{\max}(B_{n,p}^{\mathbb{Q}n\alpha}) \nearrow M_{f_p^{\mathbb{Q}\alpha}}$  as  $n \rightarrow \infty$ ;
4.  $\lambda_{\min}(C_{n,p}^{\mathbb{Q}\alpha}) \rightarrow m_{h_p}$  and  $\lambda_{\max}(C_{n,p}^{\mathbb{Q}\alpha}) \nearrow 1$  as  $n \rightarrow \infty$ ;
5.  $\lambda_{\min}(C_{n,p}^{\mathbb{Q}n\alpha}) \searrow m_{h_p^{\mathbb{Q}\alpha}}$  and  $\lambda_{\max}(C_{n,p}^{\mathbb{Q}n\alpha}) \nearrow 1$  as  $n \rightarrow \infty$ .

## 6. Conclusions

We saw some results relative to the study of the spectral analysis of matrices in Galerkin methods based on generalized B-splines with high smoothness. These results are "complementary" in the sense that they integrate the study of [1], which was more devoted to obtain the spectral distribution of the matrices involved as a main result of the topic, rather than to devise estimates for an eigenvalue, or bounds for the condition number.

In particular, a lower bound for  $|\lambda_{\min}(A_{n,p}^{\mathbb{Q}\mu})|$ , the modulus of the eigenvalue of the stiffness matrix  $A_{n,p}^{\mathbb{Q}\mu}$  with minimum modulus  $\lambda_{\min}(A_{n,p}^{\mathbb{Q}\mu})$ , has been seen to be proportional with respect to  $\frac{1}{n}$ , and a numerical example showed an actual asymptotic proportionality of  $|\lambda_{\min}(A_{n,p}^{\mathbb{Q}\mu})|$  itself, with respect to  $\frac{1}{n}$ , for large  $n$ .

Similarly, an upper bound for the condition number  $\kappa_2(A_{n,p}^{\mathbb{Q}_\mu})$  of the same matrix has been seen to be proportional with respect to  $n^2$ , and again a numerical example showed an actual asymptotic proportionality of  $\kappa_2(A_{n,p}^{\mathbb{Q}_\mu})$  itself, with respect to  $n^2$ , for large  $n$ .

As a completion, some matrices related to  $A_{n,p}^{\mathbb{Q}_\mu}$ , and in particular to  $K_{n,p}^{\mathbb{Q}_\mu}$  and  $M_{n,p}^{\mathbb{Q}_\mu}$ , that are respectively  $B_{n,p}^{\mathbb{Q}_\mu}$  and  $C_{n,p}^{\mathbb{Q}_\mu}$ , are investigated, the limits of their minimal and maximal eigenvalues for  $n \rightarrow \infty$  being determined, and for  $B_{n,p}^{\mathbb{Q}_\alpha}$  also the other eigenvalues.

All these results can be very useful in order to construct optimal and robust solvers, which do not lose accuracy if a large value of  $p$  is needed for geometry or analysis.

On the other hand, the similarity between (polynomial) B-splines and GB-splines, with a particular reference to trigonometric and hyperbolic GB-splines, is strengthened again by these results.

As possible generalizations and future perspectives, since the problem (1) refers to a setting with constant coefficients and no geometry map, we can consider the case in which the coefficients are not constant, or a geometry map is present, see [6] for the polynomial case.

Also, similar studies can be considered for matrices arising in *collocation* methods with GB-splines, too: for them, classical multigrid methods suffer of the same drawback with respect to  $p$ , so an accurate description of their spectrum is required again, see [7, 8] for the polynomial case.

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