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Abstract

We prove gradient estimates for transition Markov semigroups (P_t) associated to SDEs driven by multiplicative Brownian noise having possibly unbounded C^1-coefficients, without requiring any monotonicity type condition. In particular, first derivatives of coefficients can grow polynomially and even exponentially. We establish pointwise estimates with weights for $D_x P_t \varphi$ of the form

$$\sqrt{t}\,|D_x P_t \varphi(x)| \le c\,(1 + |x|^k)\,\|\varphi\|_\infty, \quad t \in (0,1],\ \varphi \in C_b(\mathbb{R}^d),\ x \in \mathbb{R}^d.$$

To prove the result we use two main tools. First, we consider a Feynman–Kac semigroup with potential V related to the growth of the coefficients and of their derivatives for which we can use a Bismut-Elworthy-Li type formula. Second, we introduce a new regular approximation for the coefficients of the SDE. At the end of the paper we provide an example of SDE with additive noise and drift b having sublinear growth together with its derivative such that uniform estimates for $D_x P_t \varphi$ without weights do not hold.

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1 Introduction and main result

We consider the transition Markov semigroup (P_t) associated to the SDE

$$\begin{cases} dX(t) = b(X(t))dt + \sigma(X(t))dW(t), \\ X(0) = x \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

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where $W = (W(t))$ is an $\mathbb{R}^d$–valued standard Brownian motion defined on a fixed stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ (see, for instance, [Kr95] and [Ma08]) and $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to L(\mathbb{R}^d) = \mathbb{R}^d \otimes \mathbb{R}^d$ are C^1–functions. We write

$$\sigma(X(t))dW(t) = \sum_{i=1}^{d} \sigma_i(X(t))dW^i(t),$$

where $W^i = (W^i(t))$ are independent real Brownian motions, $i = 1, \ldots, d$. We have

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t, x))], \quad t \geq 0,\ x \in \mathbb{R}^d,\ \varphi \in C_b(\mathbb{R}^d). \tag{1.2}$$

We will prove pointwise gradient estimates such as

$$\sqrt{t}\,|D_x P_t\varphi(x)| \leq c\,w(x)\,\|\varphi\|_\infty, \quad t \in (0, 1],\ \varphi \in C_b(\mathbb{R}^d),\ x \in \mathbb{R}^d, \tag{1.3}$$

where $w(x)$ is a suitable weight related to the growth of b and σ and of their derivatives.

There are several recent papers on uniform gradient estimates without weights for non-degenerate diffusion semigroups which are of the form

$$\sqrt{t}\,|D_x P_t\varphi(x)| \leq c\,\|\varphi\|_\infty, \quad t \in (0, 1],\ \varphi \in C_b(\mathbb{R}^d),\ x \in \mathbb{R}^d. \tag{1.4}$$

Such papers deal with the case of unbounded coefficients and even with possibly irregular coefficients. We refer to [ElLi94], [Ce01], [BeFo04], [PrWa06] and [PoPr13] and the references therein. Uniform gradient estimates related to some non-linear parabolic equations are also considered in [PoPr13]. We also mention that $D_x P_T\varphi(x)$ is of interest in financial mathematics where it is called Greek: it represents the rate of change of the price of the derivative at time T with respect to the initial prices; see Chapter 6 in [Nu06].

Assuming regular unbounded coefficients and the existence of a Lyapunov function, a typical monotonicity condition one imposes to obtain (1.4) is

$$2\langle Db(x)h, h\rangle + \sum_{i=1}^{d} |D\sigma_i(x)h|^2 \leq C|h|^2, \quad x, h \in \mathbb{R}^d \tag{1.5}$$

(we denote by $|\cdot|$ and $\langle \cdot, \cdot \rangle$ the norm and the inner product in $\mathbb{R}^d$, $d \geq 1$). Recall that for the class of non-degenerate diffusions with b and σ globally bounded, estimates (1.4) holds without any control on the derivatives, see e.g. Section 5 in [St74].

In this paper we consider cases in which b and σ are regular, possibly unbounded but condition (1.5) does not hold in general. In such situations we can prove gradient estimates (1.3) with weights. Moreover, we provide a one-dimensional example of SDE with additive noise and drift b with sublinear growth together with its derivative such that uniform estimates (1.4) do not hold (see Section 6).

Using the notation $a(x) = \sigma(x)\sigma^*(x)$, where $\sigma^*(x)$ is the transposed matrix of $\sigma(x)$, we make the following assumptions.

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Hypothesis 1.1. *Let* $\gamma \in [1/2, 1]$, $M_0 > 0$ *and* $c_0 \geq 1$ *and consider*

$$f(t) = M_0 e^{c_0 \int_1^t s^{-\gamma} ds}, \quad t \geq 1. \tag{1.6}$$

(H1) There exists $L > 0$ *such that, for any* $x \in \mathbb{R}^d$,

$$2\langle b(x), x \rangle + Tr(a(x)) + \frac{8c_0 \langle a(x)x, x \rangle}{(1 + |x|^2)^\gamma} \leq L (1 + |x|^2)^\gamma.$$

(H2) We have, for any $x \in \mathbb{R}^d$, *for any* $h \in \mathbb{R}^d$, *with* $|h| = 1$,

$$2|Db(x)h| + \sum_{i=1}^d |D\sigma_i(x)h|^2 \leq f(1 + |x|^2). \tag{1.7}$$

(H3) $\sigma(x)$ *is invertible,* $x \in \mathbb{R}^d$. *Moreover there exists* $\nu > 0$ *such that*

$$\langle a(x)h, h \rangle \geq \nu |h|^2, \quad x, h \in \mathbb{R}^d. \tag{1.8}$$

The above function f allows for non-standard growth of the derivatives of b and σ (see also Remarks 1.3 and 1.4). For instance, when $\gamma = 1$ we have $f(t) = M_0 t^{c_0}$ and when $\gamma = 1/2$ we have $f(t) = M e^{2c_0 \sqrt{t}}$, $t \geq 1$. However there is a balance between the growth of the coefficients and the growth of its first derivatives.

Under the previous assumptions we prove gradient estimates (1.3) with weight

$$w(x) = [f(1 + |x|^2)]^2, \quad x \in \mathbb{R}^d.$$

Choosing $\gamma = 1$, we can allow $\sigma(x)$ to grow at most linearly and moreover $\langle b(x), x \rangle \leq C (1 + |x|^2)$, $x \in \mathbb{R}^d$. In this case, we can allow the following polynomial growth of the first derivatives:

$$2|Db(x)h| + \sum_{i=1}^d |D\sigma_i(x)h|^2 \leq M_0 (1 + |x|^2)^{c_0}, \quad x \in \mathbb{R}^d, \ h \in \mathbb{R}^d, \ |h| = 1,$$

and obtain $\sqrt{t} |D_x P_t \varphi(x)| \leq C (1 + |x|^2)^{2c_0} \|\varphi\|_\infty$, for any $\varphi \in C_b(\mathbb{R}^d)$. A simple one-dimensional example is

$$dX(t) = X(t) \sin(X^2(t)) dt + dW(t), \quad X(0) = x;$$

in this case $f(t) = 2t$ ($c_0 = 1$ and $\gamma = 1$). Other cases which we can consider are collected in Remark 1.3.

Existence and uniqueness of a strong solution $X(\cdot, x)$ to (1.1) is standard since (H1) implies the well-known non-explosion condition

$$2\langle b(x), x \rangle + Tr(a(x)) \leq L(1 + |x|^2), \quad x \in \mathbb{R}^d, \tag{1.9}$$

see e.g. the monographs [Kr95] and [Ma08]. We will also use hypothesis (H1) to get L^p-estimates for $f(|X(\cdot, x)|^2 + 1)$ (see Proposition 2.1 and Corollary 2.2).

Proving our estimates for $DP_t \varphi = D_x P_t \varphi$ requires some work because, due to the growth of the derivatives of b and σ, we cannot exploit the classical Bismut–Elworthy–Li formula, see [ElLi94]. Indeed under our assumptions we do not expect

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to have L^2-estimates for the derivative $D_xX(t,x)$ which appears in the classical Bismut–Elworthy–Li formula (cf. Lemma 4.2). Such L^2-estimates would lead to uniform gradient estimates which do not hold in general (cf. Section 6).

We prove weighted gradient estimates inspired by [DaDe03], [DaDe07] and [DaDe14], introducing a suitable potential V related to Hypothesis 1.1:

$$V(x) = f(1 + |x|^2), \quad x \in \mathbb{R}^d, \tag{1.10}$$

and studying the corresponding Feynman–Kac semigroup

$$S_t\varphi(x) = \mathbb{E}\Big[\varphi(X(t,x))\,e^{-\int_0^t V(X(s,x))\,ds}\Big], \quad t \geq 0,\ x \in \mathbb{R}^d. \tag{1.11}$$

More precisely, the Feynman–Kac semigroup we consider is a non-standard regular approximation of S_t (cf. Sections 3 and 4.2). We shall first prove gradient estimates for $\langle DS_t\varphi(x), h\rangle$ then we will return to $\langle DP_t\varphi(x), h\rangle$, using the identity

$$P_t\varphi(x) = S_t\varphi(x) + \int_0^t S_{t-s}(VP_s\varphi)(x)\,ds \tag{1.12}$$

which follows from the variation of constants formula, see Section 4.3. Indeed denoting by $\mathscr{L}$ and $\mathscr{K}$ the generators of P_t and S_t respectively, we have

$$\mathscr{L} = \mathscr{K} + V.$$

In order to estimate $\langle DS_t\varphi(x), h\rangle$ we deal with a probabilistic formula for the gradient of $S_t\varphi$ which holds when b, σ and V are Lipschitz C^1-functions (see [DaZa97]). This is why we introduce new regular approximations for b, σ and V in Section 3. Finally, we use a local regularity result as in [MPW02] and a localization argument to pass from Lipschitz estimates for the approximating SDEs with regular coefficients to Lipschitz estimates for the general SDE satisfying our assumptions (cf. Section 5). Our main result is the following:

Theorem 1.2. *Under Hypothesis 1.1, there exists $c = c(c_0, \gamma, M_0, L, d) > 0$ such that for any $\varphi \in C_b(\mathbb{R}^d)$, we have*

$$\sqrt{t\nu}\,|D_xP_t\varphi(x)| \leq c\,V^2(x)\|\varphi\|_\infty, \quad t \in (0,1],\ x \in \mathbb{R}^d. \tag{1.13}$$

Below, we make some comments.

Remark 1.3. (i) Choosing $\gamma = \frac{1}{2}$, we can consider the case when $a(x)$ is globally bounded and there exists $c > 0$ such that $\langle b(x), x\rangle \leq c$, for any $x \in \mathbb{R}^d$. In this case $f(t) = Me^{2c_0\sqrt{t}}$ and (H2) becomes

$$2|Db(x)h| + \sum_{i=1}^d |D\sigma_i(x)h|^2 \leq C\,e^{2c_0\sqrt{1+|x|^2}}, \quad x \in \mathbb{R}^d,$$

$h \in \mathbb{R}^d$, $|h| = 1$; our gradient estimates are

$$\sqrt{t\nu}\,|D_xP_t\varphi(x)| \leq c\,e^{4c_0\sqrt{1+|x|^2}}\,\|\varphi\|_\infty, \quad t \in (0,1],\ \varphi \in C_b(\mathbb{R}^d),\ x \in \mathbb{R}^d.$$

Remark 1.4. We stress that $V(x) = f(1 + |x|^2)$ is a Lyapunov function for the Kolmogorov operator $\mathscr{L}$ associated to our SDE. More precisely, by (H1) even $V^4(x)$ is a Lyapunov function. This fact will be used in Proposition 2.1 and Corollary 2.2. In our approach we will also use that (cf. (3.13))

$$|V'(x)| \leq C|V(x)|, \quad x \in \mathbb{R}^d.$$

We end the section with some notations. We denote by $|\cdot|$ and $\langle \cdot, \cdot \rangle$ or $\cdot$ the norm and the inner product in $\mathbb{R}^d$, $d \geq 1$. We indicate by $L(\mathbb{R}^d)$ the space of all $d \times d$-real matrices. If $A \in L(\mathbb{R}^d)$ then $Tr(A)$ indicates the trace of A. We use the Hilbert-Schmidt norm $\|\cdot\|$ on $L(\mathbb{R}^d)$. Moreover, $C_b(\mathbb{R}^d)$ is the space of all real continuous and bounded mappings $\varphi \colon \mathbb{R}^d \to \mathbb{R}$ endowed with the sup norm

$$\|\varphi\|_\infty = \sup_{x \in \mathbb{R}^d} |\varphi(x)|;$$

whereas $C_b^k(\mathbb{R}^d)$, $k \geq 1$, is the space of all real functions which are continuous and bounded together with their derivatives of order less or equal than k. Finally, $B_b(\mathbb{R}^d)$ is the space of all real, bounded and Borel mappings on $\mathbb{R}^d$.

We denote by C or c generic positive constants that might change from line to line and that may depend on c_0, γ, M_0, L and d if it is not specified.

2 Preliminary estimates for the solution $X(t, x)$

We first present estimates for $\mathbb{E}[V(|X(t, x))]$. Recall that in Theorem 4.1 of [Ma08] estimates for $\mathbb{E}(|X(t, x)|^{2m})$ are given. Here we are considering more general estimates (for instance, if $\gamma = 1/2$ we are establishing exponential type estimates because in this case $f(s) \sim e^{2c_0\sqrt{s}}$). Estimates as in [Ma08] are obtained by choosing $\gamma = 1$.

Note that condition (2.1) is weaker than (H1) which has $8c_0$ instead of $2c_0$.

Proposition 2.1. *Let $\gamma \in [1/2, 1]$ and f as in (1.6). Moreover suppose that*

$$2\langle b(x), x \rangle + Tr(a(x)) + \tfrac{2c_0}{(1+|x|^2)^\gamma}\langle a(x)x, x \rangle \leq L(1 + |x|^2)^\gamma, \quad x \in \mathbb{R}^d, \quad (2.1)$$

for some $L > 0$. Then the solution $X(t, x)$ verifies

$$\mathbb{E}[f(|X(t, x)|^2 + 1)] \leq e^{c_0 L t} f(|x|^2 + 1), \quad x \in \mathbb{R}^d, \ t \geq 0. \quad (2.2)$$

Proof. Recalling (1.10) in the sequel we set

$$g(t) = t^\gamma \text{ so that } f(t) = M_0 e^{c_0 \int_1^t \frac{1}{g(s)} ds}, \quad t \geq 1. \quad (2.3)$$

Let us fix $t > 0$. We apply Itô's formula setting $X(t, x) = X(t)$

$$V(X(t)) = V(x) + \int_0^t 2f'(1 + |X(s)|^2)X(s) \cdot b(X(s))ds$$

$$+ 2\int_0^t f'(1 + |X(s)|^2)\, X(s) \cdot \sigma(X(s))dW(s)$$

$$+ 2\int_0^t f''(1 + |X(s)|^2)\, |\sigma^*(X(s))X(s)|^2 ds$$

$$+ \int_0^t f'(1 + |X(s)|^2)\|\sigma(X(s))\|^2 ds.$$

Using that

$$f'(s) = \frac{c_0 f(s)}{g(s)}, \quad f''(s) \le f(s)\frac{c_0^2}{g^2(s)} = f'(s)\frac{c_0}{g(s)}, \quad s \ge 1,$$

we obtain

$$V(X(t)) \le V(x) + 2\int_0^t f'(1 + |X(s)|^2)\, X(s) \cdot \sigma(X(s))dW(s)$$

$$+ \left[\int_0^t f'(1 + |X(s)|^2)\left\{ 2X(s) \cdot b(X(s)) + \frac{2c_0}{g(1+|X(s)|^2)}\, |\sigma^*(X(s))X(s)|^2 \right. \right.$$

$$\left. \left. + \|\sigma(X(s))\|^2 \right\}ds\right]$$

$$\le V(x) + 2\int_0^t f'(1 + |X(s)|^2)\, X(s) \cdot \sigma(X(s))dW(s)$$

$$+ L\int_0^t f'(1 + |X(s)|^2)g(1 + |X(s)|^2)ds$$

$$\le V(x) + 2\int_0^t f'(1 + |X(s)|^2)\, X(s) \cdot \sigma(X(s))dW(s) + c_0 L\int_0^t V(X(s))ds.$$

Using the stopping times $\tau_n = \tau_n(x) = \inf\{t \ge 0 : |X(t)| \ge n\}$ and taking the expectation we get

$$\mathbb{E}[V(X(t \wedge \tau_n))] \le V(x) + c_0 L\int_0^t \mathbb{E}[V(X(s \wedge \tau_n))]ds.$$

By the Gronwall lemma we get an estimate for $\mathbb{E}[V(X(t \wedge \tau_n))]$; letting $n \to \infty$ (note that $\tau_n \uparrow \infty$ because condition (1.9) holds) we get the assertion. $\qquad\square$

Corollary 2.2. *Let $\gamma \in [1/2, 1]$ and f as in (1.6). Moreover suppose that*

$$2\langle b(x), x\rangle + Tr(a(x)) + \frac{8c_0}{(1+|x|^2)^\gamma}\langle a(x)x, x\rangle \le L(1 + |x|^2)^\gamma, \quad x \in \mathbb{R}^d. \quad (2.4)$$

Then the solution $X(t, x)$ verifies

$$\mathbb{E}[V^4(X(t, x))] = \mathbb{E}[f^4(|X(t, x)|^2 + 1)] \le e^{4c_0 Lt}\, V^4(x), \quad x \in \mathbb{R}^d, \ t \ge 0. \quad (2.5)$$

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Proof. Taking into account that (2.4) is like (2.3) with c_0 replaced by $4c_0$, we define $\tilde{f}(t) = M_0^4 e^{4c_0 \int_1^t \frac{1}{g(s)} ds}$, $t \geq 1$, and note that $\tilde{f}(t) = f^4(t)$. By the previous proposition we obtain easily the result. $\qquad\square$

Remark 2.3. The previous integral estimates (2.2) and (2.5) hold more generally with a function $f : [1, \infty) \to \mathbb{R}$ as in (2.3) and a corresponding C^1-function $g : [1, \infty) \to \mathbb{R}$ such that $1 \leq g(s) \leq s$, for $s \geq 1$.
In such case the assumptions must be changed replacing $(1+|x|^2)^\gamma$ with $g(1+|x|^2)$. For instance, assumption (2.4) becomes

$$2\langle b(x), x \rangle + Tr(a(x)) + \frac{8c_0}{g(1+|x|^2)}\langle a(x)x, x \rangle \leq Lg(1+|x|^2), \quad x \in \mathbb{R}^d. \quad (2.6)$$

One could consider the more general condition (2.6) instead of (H1) but then it is not clear how to obtain the results of Section 3 with $g(t)$ instead of t^γ (see in particular (3.6)).

3 Regular approximations with bounded derivatives for b, σ and V

Here we introduce Lipschitz C^1-approximations of b, σ and $V = f(1 + |\cdot|^2)$ which satisfy hypotheses (H1), (H2) and (H3), possibly replacing M_0 and L with cM_0 and cL for some $c > 0$ independent of n. Note that usual approximations for operators with unbounded coefficients do not work (see, in particular, [Lu98] and [Ce01]).

Let $\eta \in C_0^\infty(\mathbb{R}^d)$ be a function such that $0 \leq \eta(x) \leq 1$, $x \in \mathbb{R}^d$, and $\eta(x) = 1$ for $|x| \leq 1$. Moreover, $\eta(x) = 0$ for $|x| \geq 2$. Let us define the C^∞-mappings $\Phi_n : \mathbb{R}^d \to \mathbb{R}^d$,

$$\Phi_n(y) = \frac{y}{\sqrt{1 + \left(1 - \eta(\frac{y}{n})\right) \frac{|y|^2}{n^2}}}, \quad y \in \mathbb{R}^d, \ n \geq 1. \quad (3.1)$$

Note that $\Phi_n(y) = y$ if $|y| \leq n$. Moreover, $|\Phi_n(y)| \leq |y|$, if $n \leq |y| \leq 2n$ and $|\Phi_n(y)| \leq n$ if $|y| > 2n$. Hence, for any $y \in \mathbb{R}^d$, we get

$$|\Phi_n(y)| \leq |y| \wedge 2n, \quad n \geq 1. \quad (3.2)$$

In addition, for any $y, h \in \mathbb{R}^d$, when $n = 1$,

$$D\Phi_1(y)[h] = \frac{h}{\sqrt{1 + \left(1 - \eta(y)\right)|y|^2}}$$

$$- \frac{y}{2}\left(1 + \left(1 - \eta(y)\right)|y|^2\right)^{-3/2}\left[-\langle D\eta(y), h\rangle|y|^2 + 2\left(1 - \eta(y)\right)\langle y, h\rangle\right].$$

Hence, considering as before the cases $|y| \leq 1$, $1 < |y| \leq 2$ and $|y| > 2$, we find that there exists $c = c(\|D\eta\|_\infty) > 0$ such that

$$\sup_{y \in \mathbb{R}^d} |D\Phi_1(y)[h]| \leq c|h|, \quad h \in \mathbb{R}^d.$$

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Writing $\Phi_n(y) = n\Phi_1(\frac{y}{n})$ we find that, for any $n \geq 1$,

$$\sup_{y \in \mathbb{R}^d} |D\Phi_n(y)[h]| \leq c\,|h|, \quad h \in \mathbb{R}^d. \tag{3.3}$$

Recalling that $\|\Phi_n\|_\infty \leq 2n$, we consider the globally Lipschitz and bounded C^1-coefficients

$$b_n(x) = b(\Phi_n(x)), \quad \sigma_n(x) = \sigma(\Phi_n(x)), \ \ x \in \mathbb{R}^d, \ n \geq 1. \tag{3.4}$$

Note that, for any $n \geq 1$,

$$b(x) = b_n(x), \ \ \sigma_n(x) = \sigma(x), \ |x| \leq n, \quad x \in \mathbb{R}^d. \tag{3.5}$$

Each σ_n satisfies (H3) with the same ν. Concerning (H1) we clearly have for $|x| \leq n$:

$$2\langle b_n(x), x \rangle + \ Tr(a_n(x)) + \frac{8c_0\,\langle a_n(x)x,x \rangle}{(1+|x|^2)^\gamma} \leq L\,(1+|x|^2)^\gamma,$$

where $a_n(x) = \sigma_n(x)\sigma_n^*(x)$. Let us treat now the case $|x| > n$.

We can only consider the case when $\langle b_n(x), x \rangle > 0$. We find

$$2\langle b_n(x), x \rangle = 2\langle b(\Phi_n(x)), \Phi_n(x) \rangle \sqrt{1 + \left(1 - \eta(\tfrac{x}{n})\right)\frac{|x|^2}{n^2}}$$
$$\leq 2\langle b(\Phi_n(x)), \Phi_n(x) \rangle \left(1 + \frac{|x|}{n}\right),$$

and, similarly,

$$Tr(a_n(x)) + \frac{8c_0\,\langle a_n(x)x,x \rangle}{(1+|x|^2)^\gamma}$$
$$= Tr(a_n(x)) + 8c_0\,\langle a(\Phi_n(x))\Phi_n(x), \Phi_n(x) \rangle \left(1 + \left(1 - \eta(\tfrac{x}{n})\right)\frac{|x|^2}{n^2}\right)\frac{1}{(1+|x|^2)^\gamma}$$
$$\leq Tr(a(\Phi_n(x)) + 8c_0\langle a(\Phi_n(x))\Phi_n(x), \Phi_n(x) \rangle\,(1 + \frac{|x|^2}{n^2})\frac{1}{(1+|x|^2)^\gamma}$$
$$\leq Tr(a(\Phi_n(x))) + 8c_0\langle a(\Phi_n(x))\Phi_n(x), \Phi_n(x) \rangle\,(1 + \frac{|x|^2}{n^2})\frac{1}{|x|^{2\gamma}}$$
$$\leq \left(\frac{|x|^{2(1-\gamma)}}{n^{2(1-\gamma)}} + 1\right)Tr(a(\Phi_n(x))) + 8c_0\langle a(\Phi_n(x))\Phi_n(x), \Phi_n(x) \rangle\,\left(\frac{|x|^{2(1-\gamma)}}{n^{2(1-\gamma)}} + 1\right).$$

Hence, when $|x| > n$ we have, since $\gamma \in [1/2, 1]$,

$$2\langle b_n(x), x \rangle + Tr(a_n(x)) + \frac{8c_0\,\langle a_n(x)x,x \rangle}{(1+|x|^2)^\gamma}$$
$$\leq cL\,(1+n^{2\gamma})\frac{|x|}{n} + cL\,(1+n^{2\gamma})\frac{|x|^{2(1-\gamma)}}{n^{2(1-\gamma)}} \leq 2cL\,(|x|^{2-2\gamma} + n^{4\gamma-2}|x|^{2-2\gamma})$$
$$\leq 2cL\,(|x|^{2-2\gamma} + |x|^{2\gamma}) \leq 2cL\,(1+|x|^2)^\gamma,$$

since $|x|^{2(1-\gamma)} \leq |x|^{2\gamma}$ for $|x| \geq 1$, where $c > 0$ is independent of n.

Hence coefficients b_n and σ_n satisfy (H1) with L replaced by CL, for some constant $C > 0$ independent of n, i.e., we have:

$$2\langle b_n(x), x \rangle + Tr(a_n(x)) + \frac{8c_0\,\langle a_n(x)x,x \rangle}{(1+|x|^2)^\gamma} \leq CL\,(1+|x|^2)^\gamma. \tag{3.6}$$

Let us consider (H2). If $|x| < n$ we get: $Db_n(x) = Db(x)$, $D(\sigma_n)_i(x) = D\sigma_i(x)$. Let now $|x| \geq n$. We will use that if $y, k \in \mathbb{R}^d$, $k \neq 0$, then

$$2|Db(y)k| + \sum_{i=1}^{d} |D\sigma_i(y)k|^2 = 2|Db(y)\tfrac{k}{|k|}||k| + \sum_{i=1}^{d} |D\sigma_i(y)\tfrac{k}{|k|}|^2 |k|^2 \leq V(y)(1 + |k|^2).$$

We find, for $|x| \geq n$ and $|h| = 1$,

$$\sum_{i=1}^{d} |D(\sigma_n)_i(x)h|^2 + |Db_n(x)h|$$

$$= |Db\big(\Phi_n(x)\big)D\Phi_n(x)h| + \sum_{i=1}^{d} |D\sigma_i\big(\Phi_n(x)\big)D\Phi_n(x)h|^2 \leq c_1\, V(\Phi_n(x)), \quad (3.7)$$

with $c_1 > 0$ independent of n. Let us define:

$$V_n(x) = c_1 V(\Phi_n(x)), \quad x \in \mathbb{R}^d,\ n \geq 1. \qquad (3.8)$$

For any $x \in \mathbb{R}^d$, $n \geq 1$, for any $h \in \mathbb{R}^d$ with $|h| = 1$, we have

$$\sum_{i=1}^{d} |D(\sigma_n)_i(x)h|^2 + |Db_n(x)h| \leq V_n(x). \qquad (3.9)$$

Moreover, by (3.2) we get:

$$V_n(x) = c_1\, V(\Phi_n(x)) = c_1 f(1 + |\Phi_n(x)|^2) \leq c_1 V(x), \quad x \in \mathbb{R}^d,\ n \geq 1. \qquad (3.10)$$

We introduce the approximating SDEs:

$$\begin{cases} dX_n(t) = b_n(X_n(t))dt + \sigma_n(X_n(t))dW(t), \\ X_n(0) = x \in \mathbb{R}^d. \end{cases} \qquad (3.11)$$

Arguing as in Corollary 2.2 and using (3.6) and (3.10) we obtain

Proposition 3.1. *The solution $X_n(t, x)$ of (3.11) verifies, for any $x \in \mathbb{R}^d$,*

$$\mathbb{E}[V_n^4(X_n(t,x))] \leq c_1^4\, \mathbb{E}[V^4(X_n(t,x))] \leq Ce^{ct}\, V^4(x), \quad t \geq 0. \qquad (3.12)$$

Finally, since $\gamma \in [1/2, 1]$, we get the following useful estimates, for any $x \in \mathbb{R}^d$,

$$|V'(x)| = 2f'(1 + |x|^2)|x| = 2c_0 \tfrac{f(1+|x|^2)}{(1+|x|^2)^\gamma}|x| \leq 2c_0 \tfrac{V(x)}{\sqrt{1+|x|^2}}|x| \leq 2c_0 V(x) \qquad (3.13)$$

and

$$|V_n'(x)| \leq CV(x), \quad n \geq 1, \qquad (3.14)$$

with C independent of n.

4 Gradient estimates for SDEs with b_n and σ_n

In this section we prove the following crucial gradient type estimates for the transition semigroup (P_t^n) associated to the SDE (3.11).

Lemma 4.1. *There exists $c = c(c_0, M_0, L, \gamma, d) > 0$ such that, for any $\varphi \in C_b(\mathbb{R}^d)$, $x, h \in \mathbb{R}^d$, we have*

$$\sqrt{t\nu}\, |P_t^n \varphi(x + h) - P_t^n \varphi(x)| \le c\,|h|\, V^2(x)\|\varphi\|_\infty, \quad t \in (0, 1], \tag{4.1}$$

where $P_t^n \varphi(x) = \mathbb{E}[\varphi(X_n(t, x))]$.

4.1 Estimates for the derivative process $D_x X_n(t, x)$

We fix $n \ge 1$. Here we give an estimate for the derivative $D_x X_n(t, x)h$, which we denote by $\eta_n^h(t, x)$, $h \in \mathbb{R}^d$ ($X_n(t, x)$ is the solution to (3.11)). We also write

$$b_n'(x) = Db_n(x), \quad (\sigma_n)_i'(x) = D(\sigma_n)_i(x), \quad i = 1, \ldots, d, \quad x \in \mathbb{R}^d. \tag{4.2}$$

It is well known that $\eta_n^h(t, x)$ is a solution to the random equation

$$\begin{cases} \frac{d}{dt}\eta_n^h(t, x) = b_n'(X_n(t, x))\eta_n^h(t, x)dt + \sum_{i=1}^d (\sigma_n)_i'(X_n(t, x))\eta_n^h(t, x)dW^i(t), \\ \eta_n^h(0, x) = h \end{cases} \tag{4.3}$$

Lemma 4.2. *Using $V_n(x)$ defined in (3.8) the following estimate holds:*

$$\mathbb{E}\left[e^{-\int_0^t V_n(X_n(s, x))ds}|\eta_n^h(t, x)|^2\right] \le |h|^2, \quad t \ge 0, \quad x, h \in \mathbb{R}^d. \tag{4.4}$$

Proof. In the proof we write $X_n(t, x) = X_n(t)$, $\eta_n^h(t, x) = \eta_n(t)$ and introduce the process

$$Z_n(t) = e^{-\int_0^t V_n(X_n(s))ds}|\eta_n(t)|^2.$$

Since

$$d|\eta_n(t)|^2 = 2\langle \eta_n(t), d\eta_n(t)\rangle = 2\langle b_n'(X_n(t))\eta_n(t), \eta_n(t)\rangle dt$$

$$+ 2\sum_{i=1}^d \langle \eta_n(t), (\sigma_n)_i'(X_n(t))\eta_n(t)\rangle dW_t^i + \sum_{i=1}^d |(\sigma_n)_i'(X_n(t))\eta_n(t)|^2 dt$$

we get

$$dZ_n(t) = e^{-\int_0^t V_n(X(s))ds}\Big[\big(2\langle b_n'(X_n(t))\eta_n(t), \eta_n(t)\rangle$$

$$+ \sum_{i=1}^d |\sigma_i'(X_n(t))\eta_n(t)|^2 - V_n(X_n(t)|\eta_n(t)|^2)dt + 2\sum_{i=1}^d \langle \eta_n(t), (\sigma_n)_i'(X_n(t))\eta_n(t)\rangle dW_t^i\Big].$$

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Using (3.9) we get

$$dZ_n(t) \leq e^{-\int_0^t V_n(X(s))ds}[V_n(X_n(t)) - V_n(X_n(t))] |\eta_n(t)|^2 dt$$

$$+2\sum_{i=1}^{d} \langle \eta_n(t), (\sigma_n)'_i(X_n(t))\eta_n(t)\rangle dW^i(t) \Big]$$

and so, $\mathbb{P}$-a.s.,

$$Z_n(t) \leq |h|^2 + 2\sum_{i=1}^{d} \int_0^t \langle \eta_n(s), (\sigma_n)'_i(X_n(s))\eta_n(s)\rangle dW^i(s).$$

We find

$$\mathbb{E}[Z_n(t)] \leq |h|^2, \quad t \geq 0,$$

and the assertion holds. $\qquad\qquad\square$

4.2 Gradient estimates for the Feynman–Kac semigroup S_t^n

As we said in the introduction, we cannot estimate $D_x P_t^n \varphi$ for $\varphi \in C_b(\mathbb{R}^d)$ (uniformly in n) directly using the Bismut–Elworthy–Li formula (see [ElLi94]); it would be necessary to estimate $|\eta_x^h(t,x)|$ whereas we are only able to show (4.4). For this reason, we consider the potential $V_n(x)$, $x \in \mathbb{R}^d$, given in (3.8) and the Feynman–Kac semigroup S_t^n given by

$$S_t^n \varphi(x) = \mathbb{E}[\varphi(X_n(t,x)) e^{-\int_0^t V_n(X_n(s,x))\,ds}], \quad \varphi \in B_b(\mathbb{R}^d), \ t \geq 0, \ x \in \mathbb{R}^d.$$

We recall that the Bismut–Elworthy–Li formula generalises to S_t^n, thanks to the regularity of b_n, σ_n and V_n (see [DaZa97]). In fact for all $\varphi \in C_b(\mathbb{R}^d)$, setting

$$\beta_n(t) = \beta_n(t,x) = \int_0^t V_n(X_n(s,x))ds, \quad t \geq 0, \tag{4.5}$$

we have that $S_t^n \varphi$ is differentiable on $\mathbb{R}^d$, $t > 0$, and the following identity holds (we write $DV_n(x) = V_n'(x)$, $x \in \mathbb{R}^d$)

$$\langle DS_t^n \varphi(x), h\rangle = \tfrac{1}{t}\mathbb{E}\left[\varphi(X_n(t,x)) e^{-\beta_n(t)} \int_0^t \langle \sigma_n^{-1}(X_n(s,x))\,\eta_n^h(s,x), dW(s)\rangle\right]$$

$$-\mathbb{E}\left[\varphi(X_n(t,x)) e^{-\beta_n(t)} \int_0^t \left(1 - \tfrac{s}{t}\right) \langle V_n'(X_n(s,x)), \eta_n^h(s,x)\rangle \, ds\right]$$

$$=: I_1(\varphi, x, h, t, n) + I_2(\varphi, x, h, t, n) = I_1 + I_2, \quad t > 0, \ x, h \in \mathbb{R}^d. \tag{4.6}$$

Lemma 4.3. *Let* $\varphi \in C_b(\mathbb{R}^d)$, $t \in (0,1]$, $x \in \mathbb{R}^d$. *Then we have with* $C = C(c_0, \gamma, M_0, L, d) > 0$

$$|DS_t^n \varphi(x)| \leq \|\varphi\|_\infty \frac{C}{\sqrt{\nu t}} V(x). \tag{4.7}$$

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Proof. In the proof we write $X_n(t, x) = X_n(t)$, $\eta_n^h(t, x) = \eta_n(t)$ and consider the process $\beta_n(t)$ given in (4.5). We first consider I_1. We have

$$|I_1|^2 \leq \tfrac{1}{t^2} \left[\mathbb{E} \left(\varphi^2(X_n(t)) \right) \mathbb{E} \left(e^{-2\beta_n(t)} \left| \int_0^t \langle \sigma_n^{-1}(X_n(s))\eta_n(s), dW(s) \rangle \right|^2 \right) \right.$$

$$\leq \tfrac{1}{t^2} \left[\mathbb{E} \left(\varphi^2(X_n(t)) \right) \mathbb{E} \left(|z_n(t)|^2 \right), \right. \tag{4.8}$$

where

$$z_n(t) = e^{-\beta_n(t)} \int_0^t \langle \sigma_n^{-1}(X_n(s))\eta_n(s), dW(s) \rangle, \quad t \geq 0. \tag{4.9}$$

We now apply Itô's formula:

$$\begin{aligned} dz_n(t) &= -\beta_n{}'(t) e^{-\beta_n(t)} \int_0^t \langle \sigma_n^{-1}(X_n(s))\eta_n(s), dW(s) \rangle \, dt \\ &\quad + e^{-\beta_n(t)} \langle \sigma_n^{-1}(X_n(t))\eta_n(t), dW(t) \rangle \\ &= -\beta_n{}'(t) z_n(t) dt + e^{-\beta_n(t)} \langle \sigma_n^{-1}(X_n(t))\eta_n(t), dW(t) \rangle; \end{aligned}$$

we find

$$\begin{aligned} d(z_n(t)^2) &= 2z_n(t) \left(-\beta_n{}'(t) z_n(t) dt + e^{-\beta_n(t)} \langle \sigma_n^{-1}(X_n(t))\eta_n(t), dW(t) \rangle \right) \\ &\quad + e^{-2\beta_n(t)} |\sigma_n^{-1}(X_n(t))\eta_n(t)|^2 \, dt. \end{aligned}$$

Integrating, we find

$$\begin{aligned} |z_n(t)|^2 &= -2 \int_0^t |z(s)|^2 \beta_n{}'(s) \, ds + 2 \int_0^t z(s) e^{-\beta_n(s)} \langle \sigma_n^{-1}(X_n(s))\eta(s), dW(s) \rangle \\ &\quad + \int_0^t e^{-2\beta_n(s)} |\sigma_n^{-1}(X_n(s))\eta_n(s)|^2 ds. \end{aligned} \tag{4.10}$$

Neglecting the negative term in (4.10) and taking expectation, we find by Lemma 4.2 (recall that (H3) implies $|\sigma_n^{-1}(x)h|^2 \leq \tfrac{|h|^2}{\nu}$, $x, h \in \mathbb{R}^d$).

$$\begin{aligned} \mathbb{E}[|z_n(t)|^2] &\leq \mathbb{E} \left[\int_0^t e^{-2\beta_n(s)} |\sigma_n^{-1}(X_n(s))\eta_n(s)|^2 ds \right] \\ &= \int_0^t \mathbb{E} \left[e^{-2 \int_0^s V_n(X_n(r))dr} |\sigma_n^{-1}(X_n(s))\eta_n(s)|^2 \right] ds \leq \tfrac{c|h|^2 t}{\nu}, \quad t \geq 0. \end{aligned}$$

Coming back to (4.8) we obtain

$$|I_1(\varphi, x, h, t, n)|^2 \leq \tfrac{c|h|^2}{\nu t} \left[\mathbb{E} \left(\varphi^2(X_n(t, x)) \right) \right]. \tag{4.11}$$

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Now we treat $I_2 = I_2(\varphi, x, h, t, n)$.

$$I_2 = -\mathbb{E}\left[\varphi(X_n(t)) \, e^{-\beta_n(t)} \int_0^t \left(1 - \tfrac{s}{t}\right) \langle V_n'(X_n(s)), \eta_n(s) \rangle \, ds\right].$$

Using (3.14), we know that $|V_n'(X_n(s))| \leq CV(X_n(s))$; we deduce

$$|I_2| \leq c \, \|\varphi\|_\infty \Lambda(t), \tag{4.12}$$

where

$$\Lambda(t) = \mathbb{E}\left[e^{-\beta_n(t)} \int_0^t V(X_n(s)) \, |\eta_n(s)| \, ds\right]$$

$$\leq \int_0^t \mathbb{E}\left[V(X_n(s)) \, e^{-\beta_n(s)} \, |\eta_n(s)|\right] ds \tag{4.13}$$

$$\leq \left(\int_0^t \mathbb{E}\left[V^2(X_n(r))\right] dr\right)^{1/2} \left(\int_0^t \mathbb{E}\left[e^{-2\beta_n(s)} \, |\eta_n(s)|^2\right] ds\right)^{1/2}.$$

Now we use (3.12) and (4.4) to get

$$|I_2| \leq C \, t^{1/2} \, |h| \|\varphi\|_\infty V(x), \ \ t \in (0, 1]. \tag{4.14}$$

Finally, by (4.11) and (4.14), we get the assertion. $\qquad\square$

4.3 Proof of Lemma 4.1

We first remark that, for any $\varphi \in C_b(\mathbb{R}^d)$, $v(t, x) = S_t^n \varphi(x)$ solves

$$\begin{cases} \partial_t v(t, x) = \mathscr{L}_n v(t, x) - V_n(x) v(t, x), & t > 0, \\ v(0, x) = \varphi(x), \ \ x \in \mathbb{R}^d, \end{cases}$$

where $\mathscr{L}_n \phi(x) = \frac{1}{2} Tr(a_n(x) D^2 \phi(x)) + b_n(x) \cdot D\phi(x)$. Note that

$$\begin{cases} \partial_t u(t, x) = \mathscr{L}_n u(t, x) - V_n(x) u(t, x) + V_n(x) u(t, x), & t > 0, \\ u(0, x) = \varphi(x), \ \ x \in \mathbb{R}^d, \end{cases}$$

has the unique bounded classical solution $u(t, x) = P_t^n \varphi(x)$. It follows that

$$P_t^n \varphi(x) = S_t^n \varphi(x) + \int_0^t S_{t-s}^n (V_n \psi_s^n)(x) \, ds, \tag{4.15}$$

where $\psi_s^n(x) = P_s^n \varphi(x)$ is a regular bounded function. We deduce a formula for the directional derivative of $P_t^n \varphi$ along the direction h:

$$\langle D P_t^n \varphi(x), h \rangle = D_h P_t^n \varphi(x) = D_h S_t^n \varphi(x) + \int_0^t D_h S_{t-s}^n (V_n \psi_s^n)(x) \, ds, \tag{4.16}$$

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$t > 0$, $x, h \in \mathbb{R}^d$. We want to obtain an estimate for $D_h P_t^n \varphi$ as

$$|D_h P_t^n \varphi(x)| \leq C|h| \|\varphi\|_\infty \frac{1}{\sqrt{\nu t}} V^2(x), \ x, h \in \mathbb{R}^d, \ t \in (0, 1]. \qquad (4.17)$$

From (4.17) assertion (4.1) follows easily. We have already proved an estimate for $D_h S_t^n \varphi$ in Lemma 4.3 (note that there exists $C > 0$ such that $V(x) \leq C V^2(x)$, $x \in \mathbb{R}^d$).

As before we write $X_n(t, x) = X_n(t)$, $\eta_n^h(t, x) = \eta_n(t)$, $V' = DV$ and consider the process $\beta_n(t)$ given in (4.5).

We fix $t \in (0, 1]$, $s \in [0, t)$; setting $r = t - s$ we find

$$\langle DS_r^n (V_n \psi_s)(x), h \rangle$$
$$= \frac{1}{r} \mathbb{E}\left[V_n(X_n(r)) \psi_s^n(X_n(r)) \, e^{-\beta_n(r)} \int_0^r \langle \sigma_n^{-1}(X_n(p)) \, \eta_n(p), dW(p) \rangle \right]$$
$$- \mathbb{E}\left[V_n(X(r, x)) \psi_s^n(X_n(r)) \, e^{-\beta_n(r)} \int_0^r \left(1 - \tfrac{p}{r}\right) \langle V_n'(X_n(p)), \eta_n(p) \rangle \, dp \right]$$
$$=: I_1(\psi_s^n, x, h, r, n) + I_2(\psi_s^n, x, h, r, n) = I_1 + I_2,$$

for $x, h \in \mathbb{R}^d$ (cf. (4.6)). Since $\|\psi_s^n\|_\infty \leq \|\varphi\|_\infty$, $s \geq 0$, $n \geq 1$, we get arguing as in the proof of Lemma 4.3 (we are using similar notations)

$$|I_1|^2 \leq \|\varphi\|_\infty^2 \frac{1}{r^2} \left[\mathbb{E}\left(V_n^2(X_n(r)) \right) \mathbb{E}\left(e^{-2\beta_n(r)} \left| \int_0^r \langle \sigma_n^{-1}(X_n(p)) \eta_n(p), dW(p) \rangle \right|^2 \right) \right]$$

$$\leq \frac{2}{r} \frac{C|h|^2}{\nu} \|\varphi\|_\infty^2 \, \mathbb{E}\left[V_n^2(X_n(r)) \right].$$
$$(4.18)$$

Now we use (3.12) and obtain

$$|I_1|^2 \leq \frac{2}{r} \frac{c|h|^2}{\nu} \|\varphi\|_\infty^2 \, V^2(x). \qquad (4.19)$$

Now we treat I_2,

$$I_2 = -\mathbb{E}\left[V_n(X_n(r)) \psi_s^n(X_n(r)) \, e^{-\beta_n(r)} \int_0^r \left(1 - \tfrac{p}{r}\right) \langle V_n'(X_n(p)), \eta_n(p) \rangle \, dp \right].$$

We argue as in the estimates before (4.12):

$$|I_2| \leq C \|\varphi\|_\infty \Gamma_n(r), \qquad (4.20)$$

$$\Gamma_n(r) = \Gamma_n(r, x) = \mathbb{E}\left[V_n(X_n(r)) \int_0^r V_n(X_n(p)) e^{-\beta_n(p)} |\eta_n(p)| \, dp \right]. \qquad (4.21)$$

Using the generalized Hölder inequality, since $\frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1$, estimate (3.12) and Lemma 4.2 we can compute

$$\mathbb{E}\left[V_n(X_n(r)) \, V_n(X_n(p)) e^{-\beta_n(p)} |\eta_n(p)| \right]$$
$$\leq (\mathbb{E}\left[V_n^4(X_n(r)) \right])^{1/4} (\mathbb{E}\left[V_n^4(X_n(p)) \right])^{1/4} (\mathbb{E}\left[e^{-2\beta_n(p)} |\eta_n(p)|^2 \right])^{1/2}$$
$$\leq C|h| \, V^2(x), \ p \in [0, r].$$

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It follows that
$$\Gamma_n(r) \le C|h|\, V^2(x).$$

Collecting the previous estimates we obtain
$$|\langle DS_{t-s}(V_n\psi_s^n)(x), h\rangle| \le C|h|\|\varphi\|_\infty \Big(\tfrac{1}{\sqrt{\nu(t-s)}} + 1\Big)V^2(x).$$

We finally get, for $t \in [0,1]$, $x \in \mathbb{R}^d$,
$$\int_0^t |\langle DS_{t-s}^n(V_n\psi_s^n)(x), h\rangle| ds \le \tfrac{c}{\sqrt{\nu}}|h|\|\varphi\|_\infty V^2(x).$$

for some $c = c(L, d, M_0, \gamma, c_0) > 0$. Now (4.17) follows. Hence we have proved
$$|P_t^n\varphi(x+h) - P_t^n\varphi(x)| \le C|h|\|\varphi\|_\infty \tfrac{1}{\sqrt{\nu t}}V^2(x), \ x, h \in \mathbb{R}^d,\ t \in (0,1]. \quad (4.22)$$

The proof is complete.

5   Proof of Theorem 1.2

I Step.   Let us fix $\varphi \in C_b(\mathbb{R}^d)$. We first prove the Lipschitz estimate
$$|P_t\varphi(x+h) - P_t\varphi(x)| = |\mathbb{E}\big[\varphi(X(t, x+h)) - \varphi(X(t,x))\big]| \quad (5.1)$$
$$\le C|h|\|\varphi\|_\infty \tfrac{1}{\sqrt{\nu t}}V^2(x), \ x, h \in \mathbb{R}^d,\ t \in (0,1].$$

We deduce such estimate by a localization argument, passing to the limit in (4.22). To this purpose, let $x, h \in \mathbb{R}^d$ and set $X(t) = X(t, x+h)$, $Y(t) = X(t,x)$,
$$X_n(t) = X_n(t, x+h), \ Y_n(t) = X_n(t,x).$$

Note that the $\mathbb{R}^{2d}$-valued processes $Z(t) = (X(t), Y(t))$ and $Z_n(t) = (X_n(t), Y_n(t))$ are the unique strong solutions of the SDEs
$$\begin{cases} dZ(t) = \tilde{b}(Z(t))dt + \tilde{\sigma}(Z(t))dW(t), \\ Z(0) = (x+h, x) \in \mathbb{R}^{2d}, \end{cases}$$

$$\begin{cases} dZ_n(t) = \tilde{b}_n(Z_n(t))dt + \tilde{\sigma}_n(Z_n(t))dW(t), \\ Z_n(0) = (x+h, x) \in \mathbb{R}^{2d}, \end{cases}$$

where
$$\tilde{b}(x,y) = \tilde{b}(z) = \begin{pmatrix} b(x) \\ b(y) \end{pmatrix}, \quad \tilde{\sigma}(z) = \tilde{\sigma}(x,y) = \begin{pmatrix} \sigma(x) \\ \sigma(y) \end{pmatrix},$$

$$\tilde{b}_n(x,y) = \tilde{b}_n(z) = \begin{pmatrix} b_n(x) \\ b_n(y) \end{pmatrix}, \quad \tilde{\sigma}_n(z) = \tilde{\sigma}_n(x,y) = \begin{pmatrix} \sigma_n(x) \\ \sigma_n(y) \end{pmatrix}$$

(b_n and σ_n are the same as in Section 3). Let us consider the open set $A_n = B(0,n) \times B(0,n) \subset \mathbb{R}^{2d}$, $n \ge 1$, and consider the stopping times
$$\tau_n^Z = \tau_n^Z(x+h, x) = \inf\{t \ge 0 \ : \ Z(t) \notin A_n\} = \tau_n^X \wedge \tau_n^Y,$$

where $\tau_n^X = \tau_n^X(x+h) = \inf\{t \geq 0 \ : \ |X(t)| \geq n\}$; $\tau_n^Y = \tau_n^Y(x) = \inf\{t \geq 0 \ : \ |Y(t)| \geq n\}$. Moreover

$$\tau_n^{Z_n} = \tau_n^{Z_n}(x+h, x) = \inf\{t \geq 0 \ : \ Z_n(t) \notin A_n\}.$$

Since we know that

$$\tilde{b}_n(z) = \tilde{b}(z), \ \ \tilde{\sigma}_n(z) = \tilde{\sigma}(z), \ \ z \in A_n, \, n \geq 1,$$

and all the coefficients of the SDEs are locally Lipschitz, by a well-known localization principle we obtain that, for any $n \geq 1$, $\mathbb{P}$-a.s.,

$$\tau_n^Z = \tau_n^{Z_n}, \ \text{ and } \ Z(t \wedge \tau_n^Z) = Z_n(t \wedge \tau_n^{Z_n}), \ t \geq 0. \tag{5.2}$$

Let us fix $t > 0$. We show now that $Z_n(t)$ converges in law to $Z(t)$ (this fact could be also deduced by a general convergence result given in Theorem V.5 of [Kr95]; we provide a direct proof). Let $F(x, y)$ be a real continuous and bounded function on $\mathbb{R}^{2d}$. We have

$$|\mathbb{E}[F(Z(t)) - F(Z_n(t))]| = |\mathbb{E}[F(Z(t)) - F(Z_n(t))\, 1_{\{t \leq \tau_n^Z\}}]|$$
$$+ |\mathbb{E}[F(Z(t)) - F(Z_n(t))\, 1_{\{t > \tau_n^Z\}}]| \leq 2\|F\|_\infty\, \mathbb{P}(t > \tau_n^Z).$$

We have that $\mathbb{P}(t > \tau_n^Z) \leq \mathbb{P}(t > \tau_n^X) + \mathbb{P}(t > \tau_n^Y)$ and we know that

$$\lim_{n \to \infty} (\mathbb{P}(t > \tau_n^X) + \mathbb{P}(t > \tau_n^Y)) = 0$$

thanks to hypothesis (H1) which implies the non-explosion condition (1.9). Hence $|\mathbb{E}[F(Z(t)) - F(Z_n(t))]|$ tends to 0 as $n \to \infty$ and this shows the desired weak convergence.

Writing (4.22) with $F(u, v) = \varphi(u) - \varphi(v)$, $u, v \in \mathbb{R}^d$, we have:

$$|\mathbb{E}[F(Z_n(t))]| = |\mathbb{E}[\varphi(X_n(t, x+h)) - \varphi(X_n(t, x))]|$$
$$\leq C|h|\|\varphi\|_\infty \frac{1}{\sqrt{\nu t}} V^2(x).$$

By weak convergence, passing to the limit as $n \to \infty$ and get (5.1).

II Step. By Section 4 in [MPW02] we deduce in particular that $P_t\varphi \in C^2(\mathbb{R}^d)$, $t > 0$. To see this fact, let us fix $\varphi \in C_b(\mathbb{R}^d)$; recall that

$$P_t\varphi(x) = \lim_{n \to \infty} \mathbb{E}[\varphi(X(t, x))\, 1_{\{\tau_n^x > t\}}], \quad t > 0, \ x \in \mathbb{R}^d, \tag{5.3}$$

where, for $n \geq 1$, $\tau_n^x = \inf\{t \geq 0 \ : \ |X(t, x)| \geq n\}$. The previous formula holds because by (H1) we know that, $\mathbb{P}$-a.s., $\tau_n^x \to \infty$, as $n \to \infty$.

As in [MPW02] we can apply the classical interior Schauder estimates by A. Friedman to each $u_n(t, x) = \mathbb{E}[\varphi(X(t, x))\, 1_{\{\tau_n^x > t\}}]$; by Theorem 4.2 in [MPW02] we obtain that $P_t\varphi \in C_{loc}^{1+\alpha/2, 2+\alpha}((0, \infty) \times \mathbb{R}^d)$, for any $\alpha \in (0, 1)$.

Using that $P_t\varphi \in C^2(\mathbb{R}^d)$, $t > 0$, and our estimate (5.1) we easily obtain the gradient estimate (1.13). $\quad\square$

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Remark 5.1. (i) By a well-known argument, see Lemma 7.1.5 in [DaZa96], gradient estimates (1.13) hold also when $\varphi \in B_b(\mathbb{R}^d)$, i.e., we have

$$\sqrt{t\nu}\, |D_x P_t \varphi(x)| \leq c\, V^2(x)\|\varphi\|_\infty, \quad t \in (0,1], \ \varphi \in B_b(\mathbb{R}^d), \ x \in \mathbb{R}^d. \qquad (5.4)$$

(ii) Using the semigroup law, it is easy to check that, for any $t > 1$, we have (with $c = c(M_0, \gamma, L, c_0, d) > 0$)

$$\sqrt{\nu}|D_x P_t \varphi(x)| \leq c\, V^2(x)\|\varphi\|_\infty, \quad \varphi \in B_b(\mathbb{R}^d), \ x \in \mathbb{R}^d.$$

6 A counterexample to uniform gradient estimates

We mention that a one-dimensional counterexample to uniform gradient estimates is given in [BeFo04]. This concerns with diffusion semigroups having an invariant measure. In such example the drift term $b(x)$ grows faster than e^{x^4} for $x \to +\infty$ and the non-explosion is guaranteed by the existence of an invariant measure.

We consider transition Markov semigroups (P_t) associated to the one-dimensional SDEs
$$dX(t) = b(X(t))dt + \sqrt{2}\, dW(t), \quad X(0) = x \in \mathbb{R},$$

with $b \in C^1(\mathbb{R})$. We assume that there exists $\theta \in (0,1)$ and $k_0 > 0$ such that

$$\begin{aligned} &|b(x)| \leq k_0(1 + |x|^\theta); \\ &|b'(x)| \leq k_0(1 + |x|^{2\theta}), \quad x \in \mathbb{R}. \end{aligned} \qquad (6.1)$$

Before stating the next theorem we recall that if $b \in C^1(\mathbb{R})$ is also bounded then uniform gradient estimates hold. This is a special case of a result given in [St74].

Theorem 6.1. *Let us fix $\theta \in (0,1)$. There exists $b \in C^1(\mathbb{R})$ which satisfies (6.1) for some k_0 such that, for any $c > 0$, the inequality*

$$\|DP_t f\|_\infty \leq \tfrac{c}{\sqrt{t}}\|f\|_\infty \ \text{ for any } t \in (0,1], \ \text{ for any } f \in C_b(\mathbb{R}), \qquad (6.2)$$

does not hold, i.e., $\displaystyle \sup_{t \in (0,1]} \ \sup_{f \in C_b(\mathbb{R}),\, \|f\|_\infty \leq 1} \left(\sqrt{t}\,\|DP_t f\|_\infty\right) = \infty.$

Note that a drift b which satisfies (6.1) also verifies Hypothesis 1.1 with $\gamma = 1$ and $V(x) = k_0(1 + |x|^2)$ so that we can apply Theorem 1.2 to (P_t).

6.1 Preliminaries

The SDE is associated to

$$Lu(x) = u''(x) + b(x)u'(x), \quad x \in \mathbb{R}.$$

Under our assumptions b grows at most linearly and so $\mathbb{P}(\tau_N > t) \to 1$ as $N \to \infty$. where $\tau_N = \tau_N^x = \inf\{t \geq 0, \ : \ |X(t,x)| \geq N\}$, $N \geq 1$. Let now $g \in C_b(\mathbb{R})$.

If there exists $u \in C^2(\mathbb{R}) \cap C_b(\mathbb{R})$ such that

$$Lu(x) = g(x), \quad x \in \mathbb{R}.$$

then, by the Itô formula we know that, for any $N \geq 1$,

$$\mathbb{E}[u(X(t \wedge \tau_N^x, x))] = u(x) + \mathbb{E}\left[\int_0^{t \wedge \tau_N^x} g(X(s, x)) ds\right].$$

Passing to the limit as $N \to \infty$, we obtain

$$P_t u(x) = u(x) + \int_0^t P_s g(x) ds, \quad t \in [0, 1), \ x \in \mathbb{R}. \tag{6.3}$$

Assume by contradiction that uniform gradient estimates (6.2) hold. Then we would have

$$\|Du\|_\infty \leq \|DP_1 u\|_\infty + c \int_0^1 \frac{1}{\sqrt{s}} ds \, \|g\|_\infty \leq C(\|u\|_\infty + \|g\|_\infty) < \infty.$$

for some $C > 0$. Hence, uniform gradient estimates (6.2) do not hold if we show

Lemma 6.2. *Let us fix $\theta \in (0, 1)$. There exist $b \in C^1(\mathbb{R})$ which satisfies (6.1) for some $k_0 > 0$, a function $f \in C_b(\mathbb{R})$ and a bounded classical solution u to $Lu = f$ with unbounded derivative, i.e., $\sup_{x \in \mathbb{R}} |u'(x)| = \infty$.*

For the proof we need two elementary results about alternating series and improper Riemann integrals. The first one can be found in pages 637 and 638 of [Jo79].

Proposition 6.3. *Let $(a_k)_{k \geq 3}$ be a (strictly) decreasing sequence of positive numbers with limit 0 such that $(a_k - a_{k+1})_{k \geq 3}$ is decreasing too. Consider*

$$R_n = \sum_{k \geq n} (-1)^{k+1} a_k, \quad n \geq 3.$$

Then $R_n = (-1)^{n+1} |R_n|$ and

$$\frac{a_n}{2} \leq |R_n| \leq \frac{a_{n-1}}{2}, \quad n \geq 4. \tag{6.4}$$

Note that $a_k = \frac{1}{k^\gamma}$, $\gamma > 0$, verifies the previous conditions.

Under the assumptions of the previous result, by (6.4) we deduce that $|R_n|$ is decreasing and has limit 0. Hence, by the Leibnitz criterion,

$$\sum_{n \geq 4} R_n \text{ converges.} \tag{6.5}$$

We will use (6.5) in the sequel.

Proposition 6.4. *Let $a \in \mathbb{R}$ and let $g : [a, \infty) \to \mathbb{R}$ be a continuous function. Consider an increasing sequence $(z_n)_{n \geq n_0} \subset [a, \infty)$ such that $z_{n_0} = a$ and $\lim_{n \to \infty} z_n = \infty$. Then there exists $I = \lim_{x \to \infty} \int_a^x g(t) dt$ if the following two conditions are satisfied:*
  (i) $\sum_{n \geq n_0} \int_{z_n}^{z_{n+1}} g(t) dt = I$;
  (ii) $\lim_{n \to \infty} \sup_{x \in [z_n, z_{n+1}]} \left| \int_{z_n}^x g(t) dt \right| = 0$.

Proof. Let $\epsilon > 0$. There exists $N = N_\epsilon > n_0$ such that, for any $n \geq N$,

$$\Big| \sum_{k=n_0}^{n} \int_{z_k}^{z_{k+1}} g(t)dt - I \Big| = \Big| \int_a^{z_{n+1}} g(t)dt - I \Big| < \epsilon$$

and $\sup_{x \in [z_n, z_{n+1}]} \big| \int_{z_n}^{x} g(t)dt \big| < \epsilon$. Now if $x \geq z_{N+1}$ then $x \in [z_{N+k}, z_{N+k+1}]$ for some $k \geq 1$ and we find

$$\Big| \int_a^{x} g(t)dt - I \Big| \leq \Big| \int_a^{z_{N+k}} g(t)dt - I \Big| + \sup_{x \in [z_{N+k}, z_{N+k+1}]} \Big| \int_{z_{N+k}}^{x} g(t)dt \Big| < 2\epsilon.$$

$\square$

6.2 Proof of Lemma 6.2

Solutions u to $Lu = f$ are given by

$$u(x) = c_0 + \int_0^x e^{-B(t)} \Big(c_1 + \int_0^t f(s)e^{B(s)}ds \Big) dt, \quad x \in \mathbb{R},$$

where $B(t) = \int_0^t b(r)dr$. We set $c_0 = 0$. We will construct b satisfying (6.1) and a suitable changing sign function $f \in C_b(\mathbb{R})$ with the property that

$$\int_{-\infty}^{0} |f(t)|e^{B(t)}dt < \infty \quad \text{and}$$

$$J = \int_0^{\infty} f(t)e^{B(t)}dt \quad \text{is a convergent improper integral} \tag{6.6}$$

(the second integral will be only conditionally convergent since $f(t)e^{B(t)}$ will be not Lebesgue integrable on $\mathbb{R}_+$). Moreover we will require that

$$J = -c_1 \quad \text{where } c_1 = \int_{-\infty}^{0} f(t)e^{B(t)}dt. \tag{6.7}$$

Therefore

$$\int_{-\infty}^{+\infty} e^{B(s)} f(s)ds = 0.$$

Under the previous conditions we have:

$$u(x) = - \int_x^{0} e^{-B(t)} \Big(c_1 - \int_t^{0} f(s)e^{B(s)}ds \Big) dt$$

$$= - \int_x^{0} e^{-B(t)} \Big(\int_{-\infty}^{t} f(s)e^{B(s)}ds \Big) dt, \quad x < 0;$$

$$u(x) = - \int_0^{x} e^{-B(t)} \int_t^{\infty} f(s)e^{B(s)}dsdt, \quad x \geq 0, \tag{6.8}$$

where $\int_t^{\infty} f(s)e^{B(s)}ds = \lim_{M \to \infty} \int_t^{M} f(s)e^{B(s)}ds$ exists as an improper Riemann integral. Now we proceed in some steps.

I Step   To prove the lemma we need to find $b \in C^1(\mathbb{R}_+)$ satisfying (6.1) on $[0, \infty)$ and $f \in C_b(\mathbb{R}_+)$ such that condition (6.6) hold and moreover

$$f(0) = 0, \quad b(0) = 1, \quad b'(0) = 0. \tag{6.9}$$

We also need that the C^2-function u given in (6.8) verifies

$$\sup_{x \in \mathbb{R}_+} |u(x)| < \infty \quad \text{and} \tag{6.10}$$

$$\limsup_{x \to \infty} |u'(x)| = \infty. \tag{6.11}$$

Indeed once this is done, we can easily extend b and f to $(-\infty, 0]$ as follows:

$$b(x) = 1, \quad f(x) = kxe^x, \quad x < 0,$$

so that, for $x < 0$,

$$u(x) = -k \int_x^0 e^{-t}\Big(\int_{-\infty}^t se^{2s}ds\Big)dt = \tfrac{k}{4}\int_0^x e^{-t}(2te^{2t} - e^{2t})dt = \tfrac{k}{4}(2xe^x - 3e^x + 3),$$

where $k \in \mathbb{R}$ is such that $J = -\int_{-\infty}^0 f(r)e^{B(r)}dr = -k\int_{-\infty}^0 te^{2t}dt = k$ and so (6.7) holds.

Step 2   We define b on $[0, \infty)$.

We first consider a suitable positive C^2-function $l : [0, \infty) \to \mathbb{R}_+$ such that

$$l(x) = e^{-\int_0^x b(s)ds} = e^{-B(x)}, \quad x \geq 0;$$

it follows that $-\log(l(x)) = \int_0^x b(s)ds$ and we have

$$b(x) = -\frac{l'(x)}{l(x)}, \quad x \geq 0. \tag{6.12}$$

The function l is defined as follows.

We start with $\phi \in C_0^\infty(\mathbb{R})$ such that $\text{Supp}(\phi) \subset [-1, 1]$, $0 \leq \phi \leq 1$, $\phi(0) = 1$ and $\int_{\mathbb{R}} \phi(t)dt = 1$.

Let us introduce the following sequences of positive numbers (c_n), (δ_n) and (b_n)

$$c_n = n, \quad \delta_n = \tfrac{1}{n^{3\gamma}}, \quad b_n = n^{2\gamma} - 1, \quad n \geq 3, \tag{6.13}$$

with $\gamma = \tfrac{\theta}{5} \in (0, 1)$ (see condition (6.1)). The function l is given by

$$\begin{cases} l(x) = 1 + b_n\phi\big(\tfrac{x - c_n}{\delta_n}\big), & \text{if } x \in [c_n - \delta_n, c_n + \delta_n], \text{ for some } n \geq 3; \\[2mm] l(x) = 1, & \text{if } x \notin \bigcup_{n \geq 3}[c_n - \delta_n, c_n + \delta_n]. \end{cases} \tag{6.14}$$

We have $l(c_n) = 1 + b_n = n^{2\gamma}$, $l(c_n - \delta_n) = l(c_n + \delta_n) = 1$. In the sequel we will also use that

$$\int_{c_n - \delta_n}^{c_n + \delta_n} l(x)dx = 2\delta_n + \delta_n b_n = \delta_n(n^{2\gamma} - 1 + 2) = \delta_n + \tfrac{1}{n^\gamma}. \tag{6.15}$$

Note that b satisfies the first estimate in (6.1) since $|b(x)| = \frac{|l'(x)|}{|l(x)|} = 0$ if $x \notin \bigcup_{n \geq 3}[c_n - \delta_n, c_n + \delta_n]$ and if $x \in [c_n - \delta_n, c_n + \delta_n] = [n - \frac{1}{n^{3\gamma}}, n + \frac{1}{n^{3\gamma}}]$ then with $C = C(\phi) > 0$, since $l(x)| \geq 1$,

$$|b(x)| = \frac{|l'(x)|}{|l(x)|} \leq \|\phi'\|_\infty \frac{b_n}{\delta_n} = \|\phi'\|_\infty \frac{n^{2\gamma} - 1}{\delta_n} = \|\phi'\|_\infty (n^{5\gamma} - n^{3\gamma})$$
$$\leq Cn^{5\gamma} \leq C\left(n - \frac{1}{n^{3\gamma}}\right)^{5\gamma} + 1 \leq C(x^{5\gamma} + 1) \leq C(x^\theta + 1). \tag{6.16}$$

This shows the first estimate in (6.1) on $[0, \infty)$ (as required in Step 1). To prove that
$$|b'(x)| \leq c(1 + |x|^{2\theta}), \quad x \in \mathbb{R},$$
with $c = c(\phi) > 0$, we argue as in (6.16); for $x \in [c_n - \delta_n, c_n + \delta_n] = [n - \frac{1}{n^{3\gamma}}, n + \frac{1}{n^{3\gamma}}]$, we have with $C' = C'(\phi) > 0$:

$$|b'(x)| \leq \frac{b_n}{\delta_n^2}\|\phi''\|_\infty + \frac{b_n^2}{\delta_n^2}\|\phi'\|_\infty$$
$$\leq C'n^{10\gamma} \leq C'\left(n - \frac{1}{n^{3\gamma}}\right)^{10\gamma} + 1 \leq C'(x^{10\gamma} + 1) \leq C'(x^{2\theta} + 1).$$

This completes the proof of (6.1).

Step 3 *We define f on $[0, \infty)$.*

Let $a > 0$. We first consider the function $g(s) = 1 - \frac{|s|}{a}$, $s \in [-a, a]$. We note that
$$\int_{-a}^a dt \int_t^a g(s)ds = \int_{-a}^a g(s)(s + a)ds = a^2$$
and similarly, for any $z \in \mathbb{R}$,
$$\int_{z-a}^{z+a} dt \int_t^{z+a} \left(1 - \frac{|s-z|}{a}\right)ds = a^2. \tag{6.17}$$

Let us define, for $n \geq 3$,
$$a_n = \frac{1}{n^\gamma}, \quad x_n = n + \frac{1}{2},$$
where $\gamma = \frac{\theta}{5}$ as in (6.13). There exists an odd number $n_0 = n_0(\gamma) \geq 3$ such that
$$c_n - \delta_n = n - \frac{1}{n^{3\gamma}} < c_n + \delta_n < x_n - a_n = (n + \frac{1}{2}) - \frac{1}{n^\gamma} < x_n + a_n < c_{n+1} - \delta_{n+1}. \tag{6.18}$$
To this purpose it is enough to choose n_0 such that $\frac{1}{n_0^{3\gamma}} + \frac{1}{n_0^\gamma} < 1/2$. The function $f : [0, \infty) \to \mathbb{R}$ is defined as follows.

$$\begin{cases} f(x) = (-1)^{n+1}\left(1 - \frac{|x - x_n|}{a_n}\right), & \text{if } x \in [x_n - a_n, x_n + a_n], \text{ for some } n \geq n_0; \\ \\ f(x) = 0 & \text{if } x \notin \bigcup_{n \geq n_0}[x_n - a_n, x_n + a_n]. \end{cases}$$
$$\tag{6.19}$$

Note that, according to (6.14), since l is identically 1 on the support of f,
$$\frac{f(x)}{l(x)} = f(x), \quad x \geq 0. \tag{6.20}$$

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Moreover,

$$\lim_{n\to\infty} \sup_{x\in[x_n-a_n,x_n+a_n]} \left| \int_{x_n-a_n}^x f(s)ds \right| = 0.$$

Recall that $\int_{x_k-a_k}^{x_k+a_k} f(t)dt = (-1)^{k+1}a_k$. By Proposition 6.4 it follows that

$$\int_{x_n-a_n}^\infty f(t)dt = \sum_{k\geq n} \int_{x_k-a_k}^{x_k+a_k} f(t)dt = \sum_{k\geq n}(-1)^{k+1}a_k, \quad n\geq n_0. \tag{6.21}$$

Hence, in particular,

$$\int_0^\infty \tfrac{f(x)}{l(x)}dx = \int_{x_{n_0}-a_{n_0}}^\infty f(t)dt = \sum_{k\geq n_0}(-1)^{k+1}a_k$$

is a convergent integral and (6.6) holds. In the sequel we will also use that

$$\int_{x_n-a_n}^{x_n+a_n} dt \int_t^{x_n+a_n} f(s)ds = (-1)^{n+1}a_n^2, \quad n\geq n_0. \tag{6.22}$$

Step 4 We check that u given in (6.8) verifies (6.10) and (6.11).

Note that, for $x\geq c_{n_0-\delta_{n_0}}$,

$$u(x) = -\int_0^x l(t)dt \int_t^\infty \tfrac{f(s)}{l(s)}ds = C_0 - \int_{c_{n_0}-\delta_{n_0}}^x l(t)dt \int_t^\infty f(s)ds, \tag{6.23}$$

where $C_0 = -\int_0^{c_{n_0}-\delta_{n_0}} l(t)dt \int_t^\infty \tfrac{f(s)}{l(s)}ds$ thanks to (6.20).

Let us first check (6.11). We have, for $n\geq n_0$, taking into account (6.21) and (6.4),

$$|u'(c_n)| = \left| l(c_n) \int_{c_n}^\infty f(s)ds \right| = (b_n+1)\left| \int_{x_n-a_n}^\infty f(s)ds \right|$$

$$= n^{2\gamma}\left| \sum_{k\geq n}(-1)^{k+1}a_k \right| \geq n^{2\gamma}\tfrac{a_n}{2} = n^{2\gamma}\tfrac{1}{2n^\gamma} \to \infty,$$

as $n\to\infty$. This shows (6.11). The proof of (6.10) is more involved. By (6.23) it is enough to verify that there exists

$$\lim_{x\to\infty} \int_{c_{n_0}-\delta_{n_0}}^x l(t)dt \int_t^\infty f(s)ds = I \in \mathbb{R}. \tag{6.24}$$

Step 5 We check (6.24) by Proposition 6.4.

We first prove that

$$\sum_{n\geq n_0} \int_{c_n-\delta_n}^{c_{n+1}-\delta_{n+1}} l(t)dt \int_t^\infty f(s)ds \text{ is convergent.} \tag{6.25}$$

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We write, using (6.18),

$$\int_{c_n-\delta_n}^{c_{n+1}-\delta_{n+1}} l(t)dt \int_t^\infty f(s)ds \tag{6.26}$$

$$= \int_{c_n-\delta_n}^{c_n+\delta_n} l(t)dt \int_t^\infty f(s)ds + \int_{c_n+\delta_n}^{x_n-a_n} l(t)dt \int_t^\infty f(s)ds$$

$$+ \int_{x_n-a_n}^{x_n+a_n} l(t)dt \int_t^\infty f(s)ds + \int_{x_n+a_n}^{c_{n+1}-\delta_{n+1}} l(t)dt \int_t^\infty f(s)ds$$

$$= A_n + B_n + C_n + D_n.$$

To verify (6.25) it is enough to prove that $\sum_{n\geq n_0}(A_n + B_n)$ and $\sum_{n\geq n_0}(C_n + D_n)$ are convergent.

We have by (6.21) and (6.15), since $l = 1$ on $[c_n + \delta_n, x_n - a_n]$,

$$A_n + B_n = \int_{c_n-\delta_n}^{c_n+\delta_n} l(t)dt \int_{x_n-a_n}^\infty f(s)ds + \int_{c_n+\delta_n}^{x_n-a_n} l(t)dt \int_{x_n-a_n}^\infty f(s)ds$$

$$= \left[\int_{c_n-\delta_n}^{c_n+\delta_n} l(t)dt + \int_{c_n+\delta_n}^{x_n-a_n} dt \right] \sum_{k\geq n}(-1)^{k+1}a_k$$

$$= \left(\delta_n + \tfrac{1}{n^\gamma} + x_n - a_n - c_n - \delta_n \right) \sum_{k\geq n}(-1)^{k+1}a_k$$

$$= (x_n - c_n) \sum_{k\geq n}(-1)^{k+1}a_k = \tfrac{1}{2} \sum_{k\geq n}(-1)^{k+1}\tfrac{1}{k^\gamma}.$$

By Proposition 6.3 and the Leibnitz criterion (see (6.5)) we infer that

$$\sum_{n\geq n_0}(A_n + B_n) = \tfrac{1}{2} \sum_{n\geq n_0} \sum_{k\geq n}(-1)^{k+1}\tfrac{1}{k^\gamma} \quad \text{is convergent.}$$

Now, since $l = 1$ on $[x_n - a_n, c_{n+1} - \delta_{n+1}]$, recalling the definition of f we find

$$C_n + D_n = \int_{x_n-a_n}^{x_n+a_n} dt \int_t^\infty f(s)ds + \int_{x_n+a_n}^{c_{n+1}-\delta_{n+1}} dt \int_t^\infty f(s)ds$$

$$= \int_{x_n-a_n}^{x_n+a_n} dt \int_t^{x_n+a_n} f(s)ds + \int_{x_n-a_n}^{x_n+a_n} dt \int_{x_{n+1}-a_{n+1}}^\infty f(s)ds$$

$$+ \int_{x_n+a_n}^{c_{n+1}-\delta_{n+1}} dt \int_{x_{n+1}-a_{n+1}}^\infty f(s)ds.$$

Using (6.21) and (6.22) we get

$$C_n + D_n = (-1)^{n+1} a_n^2$$

$$+(2a_n + c_{n+1} - \delta_{n+1} - x_n - a_n) \sum_{k\geq n+1}(-1)^{k+1}\tfrac{1}{k^\gamma}$$

$$= (-1)^{n+1} a_n^2 + \tfrac{1}{2} \sum_{k\geq n+1}(-1)^{k+1}\tfrac{1}{k^\gamma} + (a_n - \delta_{n+1}) \sum_{k\geq n+1}(-1)^{k+1}\tfrac{1}{k^\gamma}.$$

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Applying again Proposition 6.3 and the Leibnitz criterion we obtain the convergence of $\sum_{n \geq n_0} C_n + D_n$ if we are able to show that the sequence

$$(a_n - \delta_{n+1}) = \left(\tfrac{1}{n^\gamma} - \tfrac{1}{(n+1)^{3\gamma}}\right) \text{ is definitively decreasing.}$$

This can be easily checked by looking at the function $g(s) = \left(\tfrac{1}{s^\gamma} - \tfrac{1}{(s+1)^{3\gamma}}\right)$; one proves that there exists $n_1 = n_1(\gamma) > n_0$ such that $(a_n - \delta_{n+1})_{n \geq n_1}$ is decreasing. This shows (6.25). Now let us define

$$I = \sum_{n \geq n_0} \int_{c_n - \delta_n}^{c_{n+1} - \delta_{n+1}} l(t) dt \int_t^\infty f(s) ds.$$

According to Proposition 6.4 to obtain (6.24), it remains to prove that

$$\Gamma_n = \sup_{x \in [c_n - \delta_n, c_{n+1} - \delta_{n+1}]} \left| \int_{c_n - \delta_n}^x l(t) dt \int_t^\infty f(s) ds \right| \to 0 \tag{6.27}$$

as $n \to \infty$. Using the same notations of (6.26) we find

$$\Gamma_n \leq 2 \sup_{x \in [c_n - \delta_n, c_n + \delta_n]} \int_{c_n - \delta_n}^x l(t) \left| \int_{x_n - a_n}^\infty f(s) ds \right| dt$$

$$+ 2 \sup_{x \in [c_n + \delta_n, x_n - a_n]} \int_{c_n + \delta_n}^x \left| \int_{x_n - a_n}^\infty f(s) ds \right| dt$$

$$+ 2 \sup_{x \in [x_n - a_n, x_n + a_n]} \left(\left| \int_{x_n - a_n}^x dt \int_t^{x_n + a_n} f(s) ds \right| \right.$$

$$+ \int_{x_n - a_n}^x \left| \int_{x_{n+1} - a_{n+1}}^\infty f(s) ds \right| dt \right)$$

$$+ 2 \sup_{x \in [x_n + a_n, c_{n+1} - \delta_{n+1}]} \int_{x_n + a_n}^x \left| \int_{x_{n+1} - a_{n+1}}^\infty f(s) ds \right| dt.$$

By the previous computations involving A_n, B_n, C_n and D_n, using also that

$$\left| \int_{x_n - a_n}^\infty f(s) ds \right| = \left| \sum_{k \geq n} (-1)^{k+1} a_k \right| \leq \tfrac{a_{n-1}}{2} = \tfrac{1}{2(n-1)^\gamma}, \quad n \geq n_0$$

(see (6.21) and Proposition 6.3) it is straightforward to prove (6.27). This shows that (6.24) holds and finishes the proof.

Remark 6.5. Concerning the drift term b of the previous result, it can also be checked that the derivative $b'(x)$ is unbounded from above and from below. Thus condition (1.5) does not hold for b.

Acnowledgement

References

[BeFo04] M. Bertoldi and S. Fornaro, *Gradient estimates in parabolic problems with unbounded coefficients,* Studia Math. 165 (2004), 221-254.

[Ce01] S. Cerrai, *Second order PDE's in finite and infinite dimension. A probabilistic approach,* Lecture Notes in Mathematics 1762, Springer-Verlag Berlin Heidelberg, 2001.

[DaDe03] G. Da Prato and A. Debussche, *Ergodicity for the 3D stochastic Navier–Stokes equations*, J. Math. Pures Appl. 82 (2003), 877–947.

[DaDe07] G. Da Prato and A. Debussche, *m–dissipativity of Kolmogorov operators corresponding to Burgers equations with space–time white noise,* Potential Anal 26 (2007), 31-55.

[DaDe14] G. Da Prato and A. Debussche, *Estimate for $P_t D$ for the stochastic Burgers equation.* Ann. Inst. Henri Poincare Probab. Stat. 52 (2016), 1248-1258.

[DaZa96] G. Da Prato and J. Zabczyk, *Ergodicity for Infinite Dimensional Systems,* London Mathematical Society Lecture Notes 229, Cambridge University Press, 1996.

[DaZa97] G. Da Prato and J. Zabczyk, *Differentiability of the Feynman–Kac semigroup and a control application,* Rend. Mat. Acc. Lincei. v. 8 (1997), 183–188.

[ElLi94] K.D. Elworthy and X. M. Li. *Formulae for the derivatives of heat semigroups,* J. Funct. Anal. 125 (1994), 252-286.

[Jo79] R. Johnsonbaugh, *Summing an alternating series,* Amer. Math. Monthly 86 (1979), 637-648.

[Kr95] N. V. Krylov, *Introduction to the theory of diffusion processes,* Translations of Mathematical Monographs 142, Amer. Math. Soc. Providence, 1995.

[Lu98] A. Lunardi, *Schauder theorems for linear elliptic and parabolic problems with unbounded coefficients in $\mathbb{R}^n$.* Studia Math. 128 (1998), 171-198.

[Ma08] X., Mao, *Stochastic differential equations and applications,* 2nd edition, Horwood Publishing Limited, Chichester, 2008.

[MPW02] G. Metafune, D. Pallara and M. Wacker, *Feller semigroups on $\mathbb{R}^N$,* Semigroup Forum 65 (2002), 159-205.

[Nu06] D. Nualart, *The Malliavin Calculus and Related Topics*, 2nd edition, Springer-Verlag Berlin Heidelberg, 2006.

[PoPr13] A. Porretta and E. Priola, *Global Lipschitz regularizing effects for linear and nonlinear parabolic equations,* J. Math. Pures Appl. (9) 100 (2013), 633-686.

[PrWa06] E. Priola and F. Y. Wang, *Gradient estimates for diffusion semigroups with singular coefficients,* J. Funct. Anal. 236 (2006), 244–264.

[St74] H. B. Stewart, *Generation of Analytic Semigroups by Strongly Elliptic Operators,* Trans. Amer. Math. Soc. 199 (1974), 141-162.