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Pairs of nodal solutions for a Minkowski-curvature boundary value problem in a ball

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Abstract

By using a shooting technique, we prove that the quasilinear boundary value problem

$$\begin{cases} \operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) + \lambda q(|x|)|u|^{p-1}u = 0, & x \in \mathcal{B}, \\ u = 0, & x \in \partial\mathcal{B}, \end{cases}$$

where $\mathcal{B} \subset \mathbb{R}^N$ is a ball and $p > 1$, has more and more pairs of nodal solutions on growing of the parameter $\lambda > 0$. The radial Neumann problem and the periodic problem for the corresponding one-dimensional equation are considered, as well.

1 Introduction

In this paper, we study existence and multiplicity of radial solutions of the quasilinear equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) + f(|x|, u) = 0, \qquad x \in \mathcal{B}_R, \tag{1.1}$$

where $\mathcal{B}_R \subset \mathbb{R}^N$ is the ball of radius R, for $N \geq 2$.

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As well known, the above differential equation can be meant as a prescribed mean curvature equation in the Minkowski space [3, 14, 15]; in recent years, the solvability of the associated boundary value problems - even in the non-radial setting - has gained a lot of interest among researchers in the field of Nonlinear Analysis (see, for instance, [4, 5] and the references therein).

The model case for our investigation will be the Dirichlet boundary value problem

$$\begin{cases} \operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) + \lambda q(|x|)|u|^{p-1}u = 0, & x \in \mathcal{B}_R, \\ u = 0, & x \in \partial\mathcal{B}_R, \end{cases} \tag{1.2}$$

where $q : [0, R] \to \mathbb{R}$ is a continuous function, $p > 1$ and λ is a positive parameter. As shown in [6, 11], the role of λ is crucial for the existence of *positive* solutions of (1.2). In particular, it was proved therein that, when $q^+ \not\equiv 0$ and $\lambda > 0$ is large enough, two positive (and two negative) solutions appear, while in general nonexistence holds for $\lambda \to 0^+$ (see also [10] for a previous achievement in a one-dimensional setting). Later on, such a result has been extended to the genuine PDE setting in [5, 12].

The aim of this paper is to show that more and more *nodal* (i.e., sign-changing) solutions of (1.2) appear as well, on growing of the parameter λ. More precisely, as a corollary of our main result, we can state the following theorem.

Theorem 1.1. *Let $p > 1$ and let $q : [0, R] \to \mathbb{R}$ be a continuous function such that $q(r_0) > 0$ for some $r_0 \in [0, R]$. Then, for any integer $k \geq 1$, there exists $\lambda^*(k) > 0$ such that, for every $\lambda > \lambda^*(k)$, problem (1.2) has at least $4k$ nodal radial solutions.*

We anticipate that the above solutions will be distinguished according to their number of nodal regions; we refer to Theorem 3.1 for a precise description, in a more general setting.

The underlying idea of the high-multiplicity scheme in Theorem 1.1 can be traced back to a paper by Rabinowitz [18], dealing however with a semilinear second order equation like $u'' + q(x)g(u) = 0$, with $g(u)u < 0$ for $|u|$ large. Recently, this pattern has been shown to emerge in several different situations, both for ordinary [8] and partial differential equations [16], always requiring a *sublinear* behavior for $g(u)$ at infinity, i.e., $\limsup_{|u|\to\infty} g(u)/u \leq 0$. Theorem 1.1 can thus be seen as a further step in this line of research; it is remarkable, however, that the sublinearity of $g(u)$ at infinity is not needed, due to the peculiar properties of the Minkowski-curvature operator.

As for the proof of Theorem 1.1, we adopt a shooting approach for the equivalent ODE formulation

$$\begin{cases} (r^{N-1}\varphi(u'))' + \lambda r^{N-1}q(r)|u|^{p-1}u = 0 \\ u'(0) = 0, \ u(R) = 0, \end{cases}$$

that is, we consider the associated singular Cauchy problem

$$u'(0) = 0, \quad u(0) = \eta,$$

on varying of the initial datum $\eta \in \mathbb{R}$, looking for values $\eta \neq 0$ such that $u(R) = 0$. A careful study shows that small (i.e., $0 < |\eta| \to 0$) and large (i.e., $|\eta| > R$) solutions do not rotate around the origin in the phase-plane $(u, r^{N-1}\varphi(u'))$, while intermediate ones rotate more and more on growing of λ. As a consequence, similarly as in [8], a "double-gap" picture for the winding number is created and multiple *pairs* of solutions with precise nodal characterization can be provided.

It is worth noticing that the above argument also allows us to easily recover the existence of four one-signed radial solutions (two positive ones and two negative ones) already proved in [6, 11] with topological and variational techniques, respectively, see Remark 4.1. Incidentally, we mention that a shooting approach has been recently exploited to investigate the existence of radial ground-state solutions, as well [1, 2].

The plan of the paper is the following. In Section 2 we present a preliminary technical result, providing solutions rotating around the origin for a general planar Hamiltonian system. In Section 3 we state and prove our main result, dealing with a nonlinearity of the type $f(|x|, u) = \lambda q(|x|)g(u)$, under minimal assumptions on $g(u)$. We then show some numerical simulations in order to ease the reader's comprehension of the statement. Finally, Section 4 is devoted to several remarks about possible extensions and related results. In particular, we deal with the periodic problem for the one-dimensional counterpart of (1.1), namely

$$\left(\frac{u'}{\sqrt{1-(u')^2}}\right)' + f(t, u) = 0, \qquad t \in \mathbb{R},$$

often used to model the relativistic version of Newton's law (see [17] and its rich bibliography).

2 An auxiliary result

In this section, we are going to present an auxiliary result dealing with a planar Hamiltonian system of the type

$$\begin{cases} x' = X(t, y) \\ y' = -Y(t, x), \end{cases} \tag{2.1}$$

where $X, Y : I \times \mathbb{R} \to \mathbb{R}$ are locally Lipschitz continuous functions. Roughly speaking, we are going to prove that, whenever suitable (local) sign-conditions are assumed, the number of rotations around the origin of the planar path $(x(t), y(t))$ becomes arbitrarily large on growing of the width of the interval I.

To this end, we will write (whenever possible, namely for $x(t)^2 + y(t)^2 > 0$) solutions of (2.1) in (clockwise) polar coordinates as

$$x(t) = \rho(t)\cos\vartheta(t), \qquad y(t) = -\rho(t)\sin\vartheta(t), \tag{2.2}$$

with $\rho(t) > 0$. Notice that, of course, the angular coordinate $\vartheta(t)$ is defined up to integer multiples of 2π; however, the expression $\vartheta(t_2) - \vartheta(t_1)$, for any $t_1, t_2 \in I$, is uniquely determined, depending on the path $(x(t), y(t))$ only.

With this in mind, we state and prove the following result, which is a slight variant of [8, Lemma 3.2].

Proposition 2.1. *Let $a_i, b_i : (-\delta, \delta) \to \mathbb{R}$, with $i = 1, 2$, be locally Lipschitz continuous functions such that*

$$0 < a_1(s)s \le b_1(s)s \quad \text{and} \quad 0 < b_2(s)s \le a_2(s)s, \quad \text{for every } s \neq 0. \tag{2.3}$$

Then, for every positive integer j, there exist $\tau_j^* > 0$ and $\rho_j^* \in (0, \delta)$ such that, for every interval $I = [t_0, t_1] \subset \mathbb{R}$ with $t_1 - t_0 > \tau_j^*$ and for every locally Lipschitz continuous functions $X, Y : I \times \mathbb{R} \to \mathbb{R}$ satisfying

$$a_1(y)y \le X(t, y)y \le b_1(y)y, \quad \text{for every } t \in I,\ y \in (-\delta, \delta)$$

and

$$b_2(x)x \le Y(t, x)x \le a_2(x)x, \quad \text{for every } t \in I,\ x \in (-\delta, \delta),$$

it holds that any solution $(x(t), y(t))$ of (2.1), defined on I and satisfying $x(t_0)^2 + y(t_0)^2 = (\rho_j^*)^2$, fulfills $x(t)^2 + y(t)^2 > 0$ for every $t \in I$ and

$$\vartheta(t_1) - \vartheta(t_0) > j\pi.$$

Sketch of the proof. We are going to give a sketch of the proof, following the arguments in [8, Lemma 3.2]. The crucial point is to construct two spiraling planar curves controlling (from below and from above) the rotations of the planar path $(x(t), y(t))$; this can be done be gluing together pieces of level curves of suitable energy functions. Precisely, after having extended a_i and b_i to the whole real line in such a way that (2.3) still holds and that the primitives

$$A_i(s) = \int_0^s a_i(\xi)\, d\xi, \qquad B_i(s) = \int_0^s b_i(\xi)\, d\xi$$

are coercive, one defines the energies

$$\mathcal{E}_A(x, y) = A_1(y) + A_2(x), \qquad \mathcal{E}_B(x, y) = B_1(y) + B_2(x).$$

Straightforward computations show that, along a solution $(x(t), y(t))$ of (2.1), it holds

$$\frac{d}{dt}\mathcal{E}_A(x(t), y(t)) \ge 0, \ \text{ if } x(t)y(t) \ge 0, \qquad \frac{d}{dt}\mathcal{E}_A(x(t), y(t)) \le 0, \ \text{ if } x(t)y(t) \le 0$$

and

$$\frac{d}{dt}\mathcal{E}_B(x(t), y(t)) \ge 0, \ \text{ if } x(t)y(t) \le 0, \qquad \frac{d}{dt}\mathcal{E}_B(x(t), y(t)) \le 0, \ \text{ if } x(t)y(t) \ge 0.$$

Therefore, the aforementioned parameterized spirals

$$\mathbb{R} \ni \vartheta \mapsto (\rho_\pm(\vartheta)\cos\vartheta, -\rho_\pm(\vartheta)\sin\vartheta) \in \mathbb{R}^2 \setminus \{(0, 0)\}$$

can be obtained solving the differential equations

$$\frac{d\rho_\pm}{d\vartheta} = \rho_\pm \mathcal{M}_\pm(\rho_\pm(\vartheta)\cos\vartheta, -\rho_\pm(\vartheta)\sin\vartheta),$$

where

$$\mathcal{M}_-(x, y) = \begin{cases} \dfrac{b_1(y)x - b_2(x)y}{b_2(x)x + b_1(y)y}, & \text{if } xy \ge 0 \\[2ex] \dfrac{a_1(y)x - a_2(x)y}{a_2(x)x + a_1(y)y}, & \text{if } xy \le 0 \end{cases}$$

and

$$
\mathcal{M}_+(x,y) = \begin{cases} \dfrac{b_1(y)x - b_2(x)y}{b_2(x)x + b_1(y)y}, & \text{if } xy \le 0 \\[2mm] \dfrac{a_1(y)x - a_2(x)y}{a_2(x)x + a_1(y)y}, & \text{if } xy \ge 0. \end{cases}
$$

The remaining part of the proof follows exactly as in [8, Lemma 3.2]. □

3 Statement and proof of the main result

In this section, we state and prove our main result concerning the problem

$$
\begin{cases} \operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) + \lambda q(|x|)g(u) = 0, & x \in \mathcal{B}_R, \\ u = 0, & x \in \partial\mathcal{B}_R, \end{cases} \tag{3.1}
$$

where $\mathcal{B}_R \subset \mathbb{R}^N$ is the ball of radius R, for $N \geq 2$, and $\lambda > 0$.

Theorem 3.1. *Let $q : [0,R] \to \mathbb{R}$ be a continuous function such that $q(r_0) > 0$ for some $r_0 \in [0,R]$ and let $g : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function satisfying*

(g_0) *there exists $\delta > 0$ such that*

$$
g(u)u > 0 \quad \text{for every } u \in (-\delta,\delta) \setminus \{0\}
$$

and

$$
\lim_{u\to 0} \frac{g(u)}{u} = 0.
$$

Then, for any integer $k \geq 1$, there exists $\lambda^*(k) > 0$ such that, for every $\lambda > \lambda^*(k)$, problem (3.1) has at least $4k$ sign-changing radial solutions. More precisely, for every integer $j = 1,\dots k$ there exist four radial solutions $u^-_{l,j}$, $u^-_{s,j}$, $u^+_{s,j}$, $u^+_{l,j}$ of (3.1) satisfying

$$
u^-_{l,j}(0) < u^-_{s,j}(0) < 0 < u^+_{s,j}(0) < u^+_{l,j}(0)
$$

and having exactly $j+1$ nodal domains.

Theorem 1.1 in the Introduction is a direct consequence of this statement for $g(u) = |u|^{p-1}u$, which satisfies assumption (g_0) as long as $p > 1$. Notice however that in Theorem 3.1 only a *local* condition on $g(u)$ at zero is required. We also observe that the solutions found are distinguished through their nodal properties; precisely, for any $j = 1,\dots,k$, we find four solutions, two of them with positive value in the center of the ball (a small one $u^+_{s,j}$ and a large one $u^+_{l,j}$) and two with negative value therein (a small one $u^-_{s,j}$ and a large one $u^-_{l,j}$); see Section 3.2 for a more accurate discussion.

3.1 Proof of Theorem 3.1

Setting

$$\varphi(s) = \frac{s}{\sqrt{1-s^2}}, \qquad s \in (-1,1),$$

and writing $u(r) = u(|x|)$ with the usual abuse of notation, we convert the radial problem associated with (3.1) into

$$\begin{cases} (r^{N-1}\varphi(u'))' + \lambda r^{N-1} q(r) g(u) = 0 \\ u'(0) = 0, \ u(R) = 0. \end{cases} \tag{3.2}$$

Remark 3.1. Let us recall that a solution of (3.2) is meant as a function $u \in C^1([0,R])$ such that $|u'(r)| < 1$ for every $r \in [0,R]$, $r^{N-1}\varphi(u') \in C^1([0,R])$ and the differential equation in (3.2) is satisfied pointwise, together with the boundary conditions. Actually, by an easy regularity argument (see [11, Remark 3.3]) one can see that $\varphi(u') \in C^1([0,R])$, finally implying $u \in C^2([0,R])$.

As a first step, we are going to introduce an equivalent formulation of this problem, obtained from a suitable modification of both the differential operator and the nonlinear term (cf. [11, Proposition 2.1]). Precisely, on one hand we choose a locally Lipschitz continuous function $\tilde{f} : [0,R] \times \mathbb{R} \to \mathbb{R}$ such that

$$\tilde{f}(r,u) = \begin{cases} q(r)g(u) & \text{if } |u| \leq R \\ 0 & \text{if } |u| \geq R+1, \end{cases}$$

and we set $M = \sup_{r\in[0,R], u\in\mathbb{R}} |\tilde{f}(r,u)|$. On the other hand, we set $\gamma = \varphi^{-1}(\lambda M R) \in (0,1)$ and define the C^1-function $\tilde{\varphi} : \mathbb{R} \to \mathbb{R}$ as

$$\tilde{\varphi}(x) = \begin{cases} \varphi(x) & \text{if } |x| \leq \gamma \\ \varphi'(\gamma)(x+\gamma) - \varphi(\gamma) & \text{if } x < -\gamma \\ \varphi'(\gamma)(x-\gamma) + \varphi(\gamma) & \text{if } x > \gamma. \end{cases}$$

We then introduce the modified problem

$$\begin{cases} (r^{N-1}\tilde{\varphi}(u'))' + \lambda r^{N-1} \tilde{f}(r,u) = 0 \\ u'(0) = 0, \ u(R) = 0, \end{cases} \tag{3.3}$$

meaning its solutions as in Remark 3.1 (up to the requirement $|u'(r)| < 1$), and state the following lemma.

Lemma 3.1. *Let $u \in C^1([0,R])$; then, $u(r)$ is a solution of* (3.2) *if and only if $u(r)$ is a solution of* (3.3).

Proof. Let $u(r)$ be a solution of (3.3); integrating the equation and using $u'(0) = 0$, we obtain

$$r^{N-1}\tilde{\varphi}(u'(r)) = -\lambda \int_0^r s^{N-1} \tilde{f}(s,u(s))\, ds, \qquad \text{for every } r \in [0,R],$$

implying

$$|u'(r)| \leq \tilde{\varphi}^{-1}(\lambda M R) = \varphi^{-1}(\lambda M R) = \gamma < 1. \tag{3.4}$$

Hence, on one hand $\tilde{\varphi}(u'(r)) = \varphi(u'(r))$ for every $r \in [0, R]$; on the other hand, using $u(R) = 0$ we have that

$$|u(r)| \leq \int_0^R |u'(s)|\, ds \leq \gamma R < R, \qquad \text{for every } r \in [0, R], \tag{3.5}$$

so that $\tilde{f}(r, u(r)) = q(r)g(u(r))$, as well. Summing up, $u(r)$ solves (3.2).

The converse (which will not be used for our purposes) follows from similar arguments and we omit the proof. □

According to Lemma 3.1, from now on we deal with problem (3.3). Having in mind the shooting approach presented in the Introduction, we first state a global existence and continuous dependence result for the associated Cauchy problems.

Lemma 3.2. *For any $\eta \in \mathbb{R}$, the Cauchy problem*

$$\begin{cases} (r^{N-1}\tilde{\varphi}(u'))' + \lambda r^{N-1}\tilde{f}(r, u) = 0 \\ u'(0) = 0, \ u(0) = \eta, \end{cases} \tag{3.6}$$

has a unique solution $u(r; \eta)$ defined on $[0, R]$. Moreover, if $\eta_k \to \bar{\eta}$, then

$$u(r; \eta_k) \to u(r; \bar{\eta}), \qquad r^{N-1}\tilde{\varphi}(u'(r; \eta_k)) \to r^{N-1}\tilde{\varphi}(u'(r; \bar{\eta})), \tag{3.7}$$

uniformly on $[0, R]$.

Proof. We first observe that $u(r; \eta)$ is a solution of (3.6), defined on $[0, R]$, if and only if it is a fixed point of the operator $T : C([0, R]) \to C([0, R])$ defined by

$$Tu(r) = \eta - \int_0^r \tilde{\varphi}^{-1}\left(\frac{1}{s^{N-1}} \int_0^s \lambda \tau^{N-1}\tilde{f}(\tau, u(\tau))\, d\tau\right) ds;$$

incidentally, we explicitly notice that T is well-defined, despite the presence of the singular term $1/s^{N-1}$. We are going to establish existence and uniqueness of a fixed point by proving that a suitable iterate of T is a contraction with respect to the standard sup-norm $\|u\| = \sup_{r \in [0,R]} |u(r)|$. Precisely, denoting by $\mathrm{Lip}(\tilde{\varphi}^{-1})$ and $\mathrm{Lip}(\tilde{f})$ the Lipschitz constants of $\tilde{\varphi}^{-1}$ and $\tilde{f}$ respectively, and setting

$$L = \frac{\lambda}{N}\,\mathrm{Lip}(\tilde{\varphi}^{-1})\mathrm{Lip}(\tilde{f}),$$

we show by induction that, for every $k \in \mathbb{N}$ and for every u_1, $u_2 \in C([0, R])$, it holds

$$|T^k u_1(r) - T^k u_2(r)| \leq \frac{L^k r^{2k}}{k!}\|u_1 - u_2\|, \qquad \text{for every } r \in [0, R], \tag{3.8}$$

from which the thesis easily follows for k large enough.

For $k = 0$, the estimate (3.8) is trivial. Assuming it for $k \geq 0$, we have

$$\begin{aligned}
|T^{k+1}u_1(r) - T^{k+1}u_2(r)| &\leq \int_0^r \left| \tilde{\varphi}^{-1}\left(\frac{1}{s^{N-1}}\int_0^s \lambda\tau^{N-1}\tilde{f}(\tau, T^k u_1(\tau))\,d\tau\right) + \right.\\
&\quad \left. - \tilde{\varphi}^{-1}\left(\frac{1}{s^{N-1}}\int_0^s \lambda\tau^{N-1}\tilde{f}(\tau, T^k u_2(\tau))\,d\tau\right)\right| ds\\
&\leq \lambda \mathrm{Lip}(\tilde{\varphi}^{-1})\mathrm{Lip}(\tilde{f})\int_0^r \frac{1}{s^{N-1}}\int_0^s \tau^{N-1}|T^k u_1(\tau) - T^k u_2(\tau)|\,d\tau\,ds\\
&\leq \frac{\lambda^{k+1}\mathrm{Lip}(\tilde{\varphi}^{-1})^{k+1}\mathrm{Lip}(\tilde{f})^{k+1}}{N^k k!}\left(\int_0^r \frac{1}{s^{N-1}}\int_0^s \tau^{N+2k-1}\,d\tau\,ds\right)\|u_1 - u_2\|\\
&= \frac{\lambda^{k+1}\mathrm{Lip}(\tilde{\varphi}^{-1})^{k+1}\mathrm{Lip}(\tilde{f})^{k+1}r^{2k+2}}{N^k(N+2k)k!(2k+2)}\|u_1 - u_2\| \leq \frac{L^{k+1}r^{2(k+1)}}{(k+1)!}\|u_1 - u_2\|,
\end{aligned}$$

thus implying that (3.8) holds true.

The first convergence in (3.7) is a direct consequence of the above argument, while the second one now follows directly by integrating the differential equation in (3.6). $\square$

Henceforth, we set

$$v(r;\eta) = r^{N-1}\tilde{\varphi}(u'(r;\eta))$$

and we pass to (clockwise) polar-like coordinates, by writing

$$u(r;\eta) = \rho(r;\eta)\cos\theta(r;\eta), \quad v(r;\eta) = -\sqrt{\lambda}\rho(r;\eta)\sin\theta(r;\eta),$$

where $\theta(0;\eta) = 0$ for $\eta > 0$ and $\theta(0;\eta) = \pi$ for $\eta < 0$. Notice that this change of variables is admissible for $\eta \neq 0$, since

$$u(r;\eta)^2 + v(r;\eta)^2 > 0 \quad \text{for every } r \in [0, R],$$

this being a consequence of the uniqueness of the Cauchy problems (indeed, on $(0, R]$ the differential equation in (3.3) can be written as a first order planar system satisfying the assumption of Cauchy-Lipschitz theorem). Moreover, by standard results on path liftings, the continuity of the path $(u(r;\eta), v(r;\eta))$ with respect to η ensures that $\theta(r;\eta)$ depends continuously on $\eta \neq 0$, as well.

We also highlight the following property of the function $\theta(r;\eta)$, which will be used in the proofs.

Lemma 3.3. *For every r_1, r_2 such that $0 \leq r_1 \leq r_2 \leq R$, it holds that*

$$\theta(r_2;\eta) - \theta(r_1;\eta) > -\pi.$$

Proof. The result follows easily from the fact that

$$\begin{aligned}
\theta'(r;\eta) &= \sqrt{\lambda}\,\frac{u'(r;\eta)v(r;\eta) - v'(r;\eta)u(r;\eta)}{\lambda u(r;\eta)^2 + v(r;\eta)^2}\\
&= \sqrt{\lambda}\,\frac{r^{N-1}\tilde{\varphi}(u'(r;\eta))u'(r;\eta) - v'(r;\eta)u(r;\eta)}{\lambda u(r;\eta)^2 + v(r;\eta)^2},
\end{aligned}$$

whence $\theta'(r;\eta) \geq 0$ if $\theta(r;\eta) = \pi/2 + k\pi$, namely if $u(r;\eta) = 0$ and $v(r;\eta) \neq 0$ (see also [8, Lemma 3.1]). $\square$

From now on, we consider the case when $\eta > 0$ and show how to find the $2k$ solutions $u^+_{s,j}, u^+_{l,j}$ for $j = 1, \ldots, k$. A symmetric argument yields the other pairs of solutions.

According to the general strategy described in the Introduction, we first focus on the behavior of the small solutions; without loss of generality, we take the constant δ appearing in assumption (g_0) strictly less than R.

Lemma 3.4. *For every $\lambda > 0$, there exists $\eta_*(\lambda) \in (0, \delta)$ such that, for any $0 < \eta \leq \eta_*(\lambda)$, it holds that*

$$\theta(r; \eta) < \frac{1}{2}\pi, \qquad \text{for every } r \in [0, R].$$

Proof. Let us fix $\lambda > 0$ and choose $\varepsilon > 0$ such that

$$\sqrt{\varepsilon} < \frac{\pi}{2R}. \tag{3.9}$$

By assumption (g_0), there exists $\hat{\eta}(\lambda) \in (0, \delta)$ such that

$$\lambda \tilde{f}(r, u)u = \lambda q(r)g(u)u \leq \varepsilon u^2, \qquad \text{for every } r \in [0, R], \ |u| \leq \hat{\eta}(\lambda). \tag{3.10}$$

By Lemma 3.2, we finally find $\eta_*(\lambda) \in (0, \hat{\eta}(\lambda))$ such that, if $\eta \in [0, \eta_*(\lambda)]$, it holds

$$|u(r; \eta)| \leq \hat{\eta}(\lambda), \qquad \text{for every } r \in [0, R]. \tag{3.11}$$

With these positions, we are going to prove that the planar path $(u(r; \eta), v(r; \eta))$, for $r \in [0, R]$, cannot reach the negative v-semiaxis, which clearly implies the conclusion.

By contradiction, suppose that this is not the case. Then, there exists $[r_1, r_2] \subset [0, R]$ such that the path $r \mapsto (u(r), v(r)) = (u(r; \eta), v(r; \eta))$ satisfies

$$v(r_1) = 0 = u(r_2), \quad v(r) < 0 < u(r) \text{ for every } r \in (r_1, r_2). \tag{3.12}$$

Of course, also the path

$$r \mapsto \left(\sqrt{\varepsilon}u(r), \tilde{\varphi}(u'(r)) = \tfrac{v(r)}{r^{N-1}}\right)$$

satisfies (3.12); as a consequence, passing to clockwise polar coordinates as in (2.2), we find

$$\frac{1}{4} = \frac{\sqrt{\varepsilon}}{2\pi} \int_{r_1}^{r_2} \frac{u'(r)\tilde{\varphi}(u'(r)) - u(r)(\tilde{\varphi}(u'(r)))'}{\varepsilon u(r)^2 + \tilde{\varphi}(u'(r))^2} \, dr,$$

that is, in view of the differential equation in (3.3),

$$\frac{1}{4} = \frac{\sqrt{\varepsilon}}{2\pi} \int_{r_1}^{r_2} \frac{u'(r)\tilde{\varphi}(u'(r)) + \lambda \tilde{f}(r, u(r))u(r) + \frac{N-1}{r}\tilde{\varphi}(u'(r))u(r)}{\varepsilon u(r)^2 + \tilde{\varphi}(u'(r))^2} \, dr.$$

Recalling that $\tilde{\varphi}(u'(r))u(r) \leq 0$ for every $r \in [r_1, r_2]$ and using (3.10)-(3.11), together with the inequality $s\tilde{\varphi}(s) \leq \tilde{\varphi}(s)^2$, we finally obtain

$$\frac{1}{4} < \frac{\sqrt{\varepsilon}R}{2\pi},$$

contradicting (3.9). □

Second, we prove our key lemma; roughly speaking, we make larger solutions rotate as desired by enlarging the parameter λ.

Lemma 3.5. *For every integer $k \geq 1$, there exists $\lambda_k^* > 0$ such that for every $\lambda > \lambda_k^*$ there exists $\eta^*(\lambda) \in (0, R+1)$ such that*

$$\theta(R; \eta^*(\lambda)) > (k+1)\pi.$$

Proof. We first choose $0 < r_0^- < r_0^+ < R$ such that $q(r) \geq m > 0$ for $r \in [r_0^-, r_0^+]$ and state the following.

Claim. *There exist $\lambda_k^* > 0$ and $\Gamma_k \in (0, \delta)$ such that, for every $\lambda > \lambda_k^*$, every solution $(u(r), v(r)) = (\rho(r)\cos\theta(r), -\sqrt{\lambda}\rho(r)\sin\theta(r))$ of (3.3) satisfying $u(r_0^-)^2 + \frac{1}{\lambda}v(r_0^-)^2 = \Gamma_k^2$ fulfills*

$$\theta(r_0^+) - \theta(r_0^-) > (k+3)\pi.$$

Proof of the claim. We set

$$(x(t), y(t)) = \left(u\left(\frac{t}{\sqrt{\lambda}}\right), \frac{1}{\sqrt{\lambda}} v\left(\frac{t}{\sqrt{\lambda}}\right) \right),$$

for $t \in I_\lambda = [\sqrt{\lambda} r_0^-, \sqrt{\lambda} r_0^+]$. It is easily seen that $(x(t), y(t))$ solves the differential system

$$\begin{cases} x' = \dfrac{1}{\sqrt{\lambda}} \tilde{\varphi}^{-1}\left(\dfrac{\lambda^{(N/2)} y}{t^{N-1}}\right) =: X_\lambda(t, y) \\ y' = -\dfrac{t^{N-1}}{\lambda^{(N-1)/2}} \tilde{f}\left(\dfrac{t}{\sqrt{\lambda}}, x\right) =: -Y_\lambda(t, x). \end{cases}$$

In view of the choice of r_0^-, r_0^+, we have that, for every $t \in I_\lambda$ and $x \in (-\delta, \delta)$, it holds that

$$m(r_0^-)^{N-1} g(x)x \leq Y_\lambda(t, x)x \leq \|q^+\|_{L^\infty(0,R)} (r_0^+)^{N-1} g(x)x.$$

On the other hand, since

$$\frac{1}{\varphi'(\gamma)} s^2 \leq \tilde{\varphi}^{-1}(s)s \leq s^2, \qquad \text{for every } s \in \mathbb{R},$$

we find that, for every $t \in I_\lambda$ and $y \in (-\delta, \delta)$,

$$\frac{1}{\varphi'(\gamma)(r_0^+)^{N-1}} y^2 \leq X_\lambda(t, y)y \leq \frac{y^2}{(r_0^-)^{N-1}}.$$

Taking into account that, denoting by $\vartheta(t)$ the angular coordinate associated with the planar path $(x(t), y(t))$ as in (2.2), it holds that

$$\vartheta(\sqrt{\lambda} r_0^+) - \vartheta(\sqrt{\lambda} r_0^-) = \theta(r_0^+) - \theta(r_0^-),$$

Proposition 2.1 can be applied with the choice $j = k+3$, providing

$$\lambda_k^* = \left(\frac{\tau_{k+3}^*}{r_0^+ - r_0^-}\right)^2, \quad \Gamma_k = \rho_{k+3}^*.$$

□

We can now easily draw the conclusion: indeed, it holds that $u(r;0) \equiv 0$ and $u(r;R+1) \equiv R+1$, since $\tilde{f}(r,u) = 0$ for $|u| \geq R+1$. Accordingly, recalling that $\Gamma_k < \delta < R$, there exists $\eta^*(\lambda) \in (0, R+1)$ such that

$$u(r_0^-;\eta^*(\lambda))^2 + \frac{1}{\lambda} v(r_0^-;\eta^*(\lambda))^2 = \Gamma_k^2. \tag{3.13}$$

Since

$$\begin{aligned}\theta(R;\eta^*(\lambda)) = \theta(r_0^-;\eta^*(\lambda)) - \theta(0;\eta^*(\lambda)) + \theta(r_0^+;\eta^*(\lambda)) - \theta(r_0^-;\eta^*(\lambda)) \\ + \theta(R;\eta^*(\lambda)) - \theta(r_0^+;\eta^*(\lambda)),\end{aligned}$$

from the previous Claim and Lemma 3.3 the statement follows. $\square$

We are now in a position to conclude. Given $k \geq 1$, we fix $\lambda > \lambda_k^*$ (where λ_k^* is given by Lemma 3.5) and we consider the continuous function

$$(0, +\infty) \ni \eta \mapsto \theta(R;\eta) \in \mathbb{R}.$$

Of course, $\theta(R;R+1) = 0$; moreover, from Lemmas 3.4 and 3.5 we infer that $\eta_*(\lambda) < \eta^*(\lambda)$ and that

$$\theta(R;\eta_*(\lambda)) < \frac{\pi}{2} < (k+1)\pi < \theta(R;\eta^*(\lambda)).$$

Then, the Intermediate Value theorem gives, for any integer $j = 1, \dots, k$, two positive values $\eta_{s,j}^+, \eta_{l,j}^+$, with

$$\eta_*(\lambda) < \eta_{s,j}^+ < \eta^*(\lambda) < \eta_{l,j}^+,$$

such that

$$\theta(R;\eta_{s,j}^+) = \theta(R;\eta_{l,j}^+) = \left(\frac{1}{2} + j\right)\pi. \tag{3.14}$$

The solutions

$$u_{s,j}^+(r) = u(r;\eta_{s,j}^+), \qquad u_{l,j}^+(r) = u(r;\eta_{l,j}^+)$$

are therefore distinct solutions of (3.2); moreover, from (3.14), together with the fact that $\theta'(r) \geq 0$ whenever $u(r) = 0$ (compare with the proof of Lemma 3.3), it follows that $u_{s,j}^+(r)$ and $u_{l,j}^+(r)$ have exactly j zeros on $[0, R)$ (see also [8, Lemma 3.8]).

A similar argument works when $\eta < 0$, giving the other pair of solutions.

3.2 Some numerical simulations

To give a better insight into the statement of Theorem 1.1, we collect here below some numerical simulations obtained with MAPLE© software (see Figures 1, 2 and 3 below) for the equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 - |\nabla u|^2}}\right) + \lambda u^3 = 0, \qquad x \in \mathcal{B}_R, \tag{3.15}$$

choosing $N = 2$ and $R = 10$. The Dirichlet solutions shown are found for $\lambda = 5$; for completeness, we also depict the one-signed solutions already found in [6, 11] (see Remark 4.1).

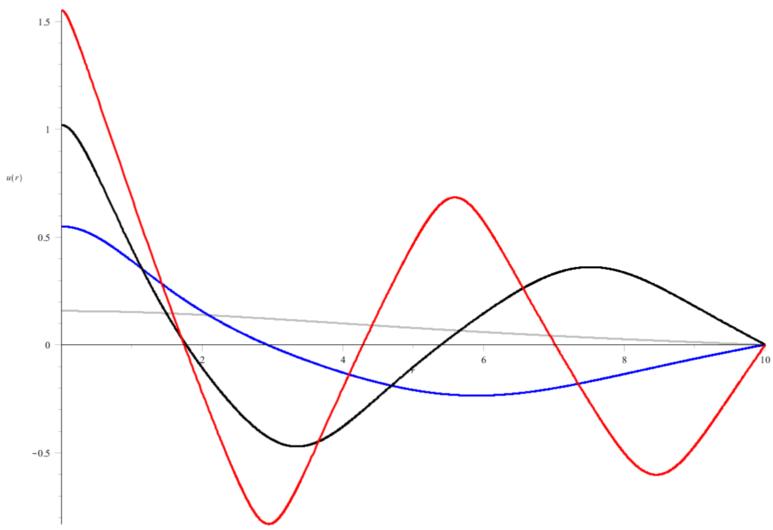

Figure 1: We plot the small solutions $u^+_{s,j}(r)$ of equation (3.15) for $j = 0, 1, 2, 3$ (in grey, blue, black, red, respectively). Notice that we also plotted the positive solution $u^+_{s,0}$, see Remark 4.1 in Section 4. Due to the oddness of the nonlinearity, here $u^-_{s,j}(r) = -u^+_{s,j}(r)$ for every $j = 0, 1, 2, 3$.

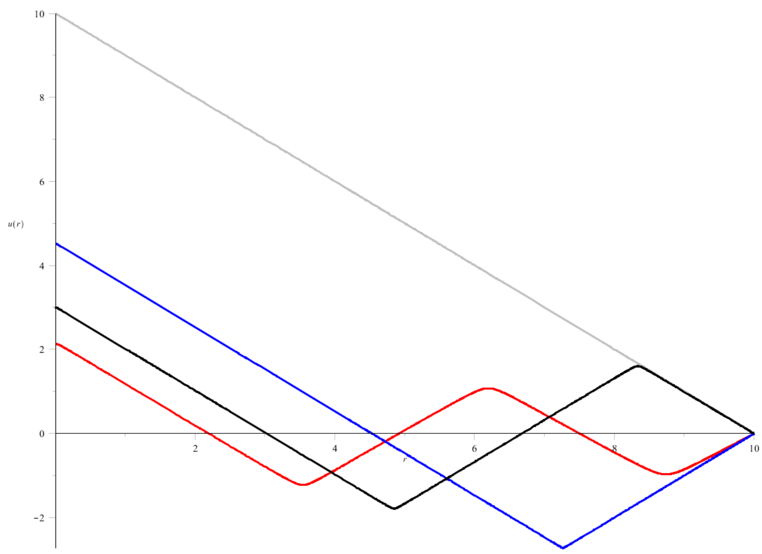

Figure 2: We plot the large solutions $u^+_{l,j}(r)$ of equation (3.15) for $j = 0, 1, 2, 3$ (in grey, blue, black, red, respectively). Again, we plotted the positive solution $u^+_{l,0}$, as well, and again $u^-_{l,j}(r) = -u^+_{l,j}(r)$.

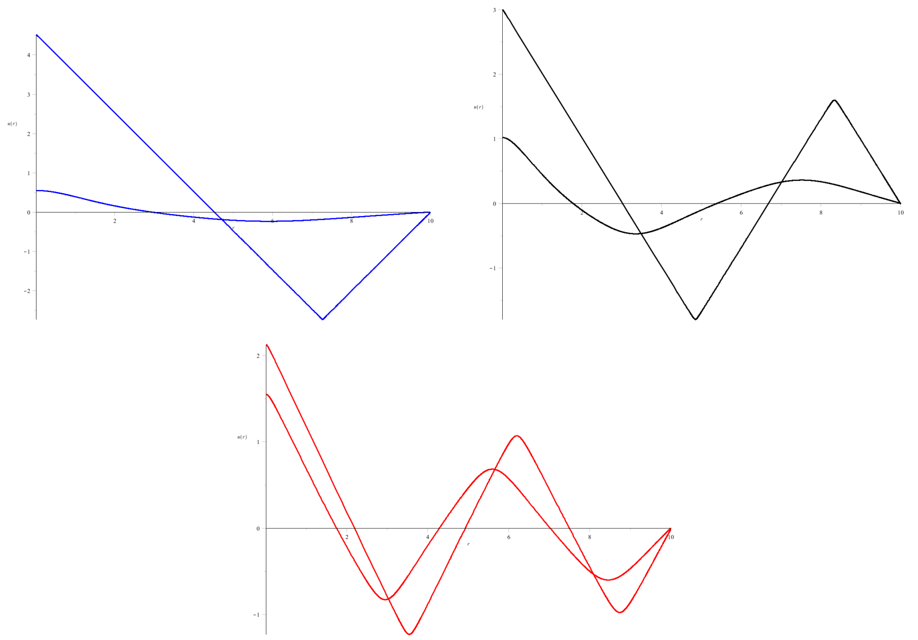

Figure 3: To better highlight the "multiple pairs" scheme, we group together the above displayed smaller and larger solutions of (3.15) having the same nodal properties: namely, $u^+_{s,1}$ and $u^+_{l,1}$ in blue, $u^+_{s,2}$ and $u^+_{l,2}$ in black, $u^+_{s,3}$ and $u^+_{l,3}$ in red.

The reader will certainly notice how the large solutions $u_{l,j}$ found above appear almost sharp-cornered, with slope approximately equal to ± 1; we remark that, in principle, this is not a drawback of numerics but rather a consequence of the peculiar properties of the Minkowski curvature operator, as an elementary singular perturbation analysis shows. For simplicity, we assume that $q(r) > 0$ for every $r \in [0, R]$ and $g(u)u > 0$ for every $u \neq 0$ and we discuss the case when $\{u_\lambda\}_\lambda$ is a family of *positive* solutions of (3.1). Passing to the radial formulation, it is immediate to see that u_λ is decreasing; moreover, using the arguments in the proof of Lemma 3.1, we obtain that $\{u_\lambda\}$ is bounded in the C^1-norm. By the sequential version of the Banach-Alaoglu theorem, there exists $u_\infty \in W^{1,\infty}$ such that, up to subsequences, $u_\lambda \to u_\infty$ in the weak*-topology of $W^{1,\infty}$. Notice that u_∞ is nonnegative and nonincreasing. Of course, it may be that $u_\infty \equiv 0$ (indeed, this is the case for the family of small positive solutions); however, assume that $u_\infty(r) > 0$ on $[0, \bar{r})$, for some $\bar{r} \in (0, R]$. Setting $v_\lambda(r) = r^{N-1}\varphi(u'_\lambda(r))$ and recalling that $v_\lambda(0) = 0$, we have

$$v_\lambda(r) = -\lambda \int_0^r q(s)g(u_\lambda(s))\,ds, \qquad \text{for every } r \in (0, R].$$

By Fatou Lemma, $v_\lambda(r) \to -\infty$ for any $r \in (0, R]$, so that $u'_\lambda(r) \to -1$ for any $r \in (0, R]$. By Lebesgue theorem, $u'_\lambda \to -1$ strongly in L^p for every $p \geq 1$, so that $u_\infty(r) = R - r$ (notice that, a posteriori, $\bar{r} = R$) and $u_\lambda \to u_\infty$ strongly in $W^{1,p}$ for every $p \geq 1$. Notice that this convergence is sharp: indeed, since $u_\lambda \in C^1$ and $u'_\lambda(0) = 0$, if the convergence were $W^{1,\infty}$ it should be $u'_\infty(0) = 0$ (while $u'_\infty(0) = -1$). The same argument works for negative solutions and, with some more care, is extendable to nodal solutions as well, showing that the graph of the singular limit $u_\infty(r)$ is a polygonal line with slopes -1 and 1.

4 Final remarks

We conclude the paper with some final remarks about possible extensions of Theorem 1.1 and related results.

Remark 4.1. One-signed solutions. The choice $j = 0$ in (3.14) is admissible, as well. In this way, the existence of four one-signed (two positive and two negative) radial solutions directly follows (compare with [6, 11]).

Remark 4.2. The Neumann problem. With similar arguments, we can deal as well with the Neumann problem

$$\begin{cases} \operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) + \lambda q(|x|)g(u) = 0, & x \in \mathcal{B}_R, \\ \partial_\nu u = 0, & x \in \partial\mathcal{B}_R. \end{cases}$$

Indeed, when passing to the ODE formulation, one is led to the Neumann problem

$$\begin{cases} (r^{N-1}\varphi(u'))' + \lambda r^{N-1}q(r)g(u) = 0 \\ u'(0) = 0,\ u'(R) = 0 \end{cases}$$

and it is possible to find solutions by imposing, instead of (3.14), the condition $\theta(R) = j\pi$, for any integer $j = 1, \ldots, k$. It has to be noticed, however, that an analogue of Lemma 3.1 can still be established only due to the fact that we deal with nodal solutions (ensuring the validity of (3.5)). Incidentally, we observe that in fact one-signed solutions of the Neumann problem typically do not exist (choose $q(r) \geq 0$ and integrate the equation).

Remark 4.3. Solutions on annuli. The same result as in Theorem 3.1 holds true if the ball $\mathcal{B}_R$ is replaced by the annular domain $\mathcal{A}(R_1, R_2) = \{x \in \mathbb{R}^N : R_1 < |x| < R_2\}$. Here, the radial formulation reads as the Dirichlet problem

$$\begin{cases} (r^{N-1}\varphi(u'))' + \lambda r^{N-1}q(r)g(u) = 0 \\ u(R_1) = 0,\ u(R_2) = 0 \end{cases} \tag{4.1}$$

and an analogue of Lemma 3.1 holds true also in this case, since both $u(r)$ and $u'(r)$ vanish at least once in $[R_1, R_2]$ (ensuring the validity of formulas (3.4) and (3.5)). One can thus set, for the truncated equivalent problem, the shooting scheme

$$u(R_1) = 0, \qquad u'(R_1) = \eta$$

and look for solutions satisfying $\theta(R_2; \eta) - \theta(R_1; \eta) = j\pi$, for $j = 1, \ldots, k$. The previous procedure of proof can still be followed, but unexpectedly some extra work is needed. Summarizing the main steps, it should be proved that:

- $\theta(R_2;\eta) - \theta(R_1;\eta) < \frac{\pi}{2}$ for $0 < |\eta|$ small enough, as can be shown using arguments on the lines of the ones in Lemma 3.4;

- $\theta(R_2;\eta) - \theta(R_1;\eta) < \frac{\pi}{2}$ for $|\eta|$ large enough, this being a consequence of the so-called elastic property (i.e., $u^2(r;\eta) + v^2(r;\eta) \to \infty$ uniformly when $|\eta| \to \infty$, see [19, Lemma 2] and notice that the differential equation in (4.1) is not anymore singular) and of the fact that $\tilde{f}(r,u) \equiv 0$ for $|u| \geq R+1$;

- $\theta(R_2;\eta^*(\lambda)) - \theta(R_1;\eta^*(\lambda)) > k\pi$ for $\lambda > \lambda_k^*$, this being provable as in Lemma 3.5 using the elastic property once more to prove that (3.13) holds true.

We refer the reader to [8] for a detailed proof following this scheme, even if in a different setting. Combining with the previous Remark 4.2, one deduces the validity of Theorem 3.1 for the Neumann problem on an annulus, as well.

Remark 4.4. The periodic problem. When $q : \mathbb{R} \to \mathbb{R}$ is a continuous and T-periodic function ($T > 0$), a version of Theorem 3.1 can also be stated for the T-periodic problem associated with the differential equation

$$\left(\frac{u'}{\sqrt{1-(u')^2}}\right)' + \lambda q(t) g(u) = 0.$$

In this case, instead of using the above shooting procedure, one has to apply the Poincaré-Birkhoff fixed point theorem. We briefly illustrate here below the steps of the proof, referring again the reader to [8]:

- we pass to the truncated problem $(\tilde{\varphi}(u'))' + \lambda\tilde{f}(t,u) = 0$; the analogue of Lemma 3.1 holds true since both $u(t)$ and $u'(t)$ vanish at least once in $[0,T]$ (see the corresponding discussions in Remarks 4.2 and 4.3);

- we write the truncated equation as the planar Hamiltonian system

  $$u' = \tilde{\varphi}^{-1}(v), \qquad v' = -\lambda\tilde{f}(t,u),$$

  and we denote by $(u(t;u_0,v_0), v(t;u_0,v_0))$ the solution satisfying the initial conditions $u(0) = u_0, v(0) = v_0$;

- we pass to polar-like coordinates

  $$\begin{cases} u(t;u_0,v_0) = \rho(t;u_0,v_0)\cos\theta(t;u_0,v_0) \\ v(t;u_0,v_0) = -\sqrt{\lambda}\rho(t;u_0,v_0)\sin\theta(t;u_0,v_0); \end{cases}$$

- we prove that $\theta(T;u_0,v_0) - \theta(0,u_0,v_0) < 2\pi$ both when $0 < u_0^2 + v_0^2$ is small enough and when it is large enough. This can be done again similarly as in the corresponding steps of Remark 4.3;

- we prove that there exists λ_k^* such that, for every $\lambda > \lambda_k^*$, there exists $\eta^*(\lambda)$ with $\theta(T;u_0,v_0) - \theta(0,u_0,v_0) > 2k\pi$ if $u_0^2 + v_0^2 = \eta^*(\lambda)^2$. This can be shown as in Lemma 3.5 (without loss of generality, we can assume $q(0) > 0$ so that the argument therein can be made slightly simpler).

The conclusion then follows from the Poincaré-Birkhoff fixed point theorem. We also mention that, combining the arguments in the present paper with the ones in [13, Section 3] it should be possible to prove the existence of pairs of *subharmonic* solutions (namely, mT-periodic solutions for $m > 1$) for the equation

$$\left(\frac{u'}{\sqrt{1-(u')^2}} \right)' + q(t)g(u) = 0.$$

In this case, the largeness of λ is replaced by the width of the periodicity interval (this requiring, however, that $q(t) > 0$ for every t). For the sake of briefness, we omit the details.

Remark 4.5. Some complementary situations. We finally observe that the superlinearity assumption in (g_0), namely $\lim_{u\to 0} g(u)/u = 0$, is used only to ensure the validity of Lemma 3.4. Accordingly, a "double gap" for the winding number can be established, thus producing solutions *in pairs*. If no asymptotic hypothesis at zero is required (keeping, however, the validity of the local sign assumption $g(u)u > 0$ for $u \neq 0$), then $2k$ solutions are still preserved for $\lambda > \lambda_k^*$, thanks to the gap between intermediate and large solutions. Of course, depending on the growth condition at zero, a more precise picture may be displayed. For instance, combining the arguments of the present paper with the ones in [9], it is possible to find, for every $\lambda > 0$ (and assuming $q(|x|) > 0$), infinitely many nodal radial solutions of the equation

$$\operatorname{div} \left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}} \right) + \lambda q(|x|)|u|^{p-1}u = 0, \qquad 0 < p < 1,$$

having any arbitrary integer number of nodal regions. For completeness, we also mention [7] for the case when $g(u)$ is linear at zero, in the context of the one-dimensional periodic problem.

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