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On degeneracy loci of equivariant bi-vector fields on a smooth toric variety

Elena Martinengo

Abstract. *We study equivariant bi-vector fields on a toric variety. We prove that, on a smooth toric variety of dimension n, the locus where the rank of an equivariant bi-vector field is $\leq 2k$ is not empty and has at least a component of dimension $\geq 2k+1$, for all integers $k>0$ such that $2k<n$. The same is true also for $k=0$, if the toric variety is smooth and compact. While for the non compact case, the locus in question has to be assumed to be non empty.*

1. Introduction

Let X be a smooth toric variety associated to a fan $\Sigma \subset N \otimes_{\mathbb{Z}} \mathbb{R}$, where N is an n-dimensional lattice. Our aim is to study degeneracy loci of equivariant bi-vector fields on X. A bi-vector field π on X is a global section of the second exterior product of the holomorphic tangent bundle Θ_X. Let $\{z_1, \ldots, z_n\}$ be a system of local coordinates around a point $x \in X$, then a bi-vector field $\pi \in \Gamma(X, \bigwedge^2 \Theta_X)$ can be written locally as

$$\pi = \sum_{i,j=1}^{n} \pi^{ij}(z_1, \ldots, z_n) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j},$$

where $\Pi = \left(\pi^{ij}(z_1, \ldots, z_n)\right)_{ij}$ is an antisymmetric matrix of homolorphic functions defined locally around x. The rank of the bi-vector field π at the point x is the rank of the matrix Π calculated in x. The k-th degeneracy locus of π is

$$X_{\leq k} := \{x \in X \mid \mathrm{rk}_x \, \pi \leq k\}.$$

Since we are on a toric variety, it makes sense to consider bi-vector fields that are equivariant under the action of the torus. It is obvious that their rank is constant on the orbits of the action. We prove the following

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Main Theorem. *Let X be a smooth toric variety of dimension n and $\pi \in \Gamma(X, \bigwedge^2 \Theta_X)$ an equivariant bi-vector field on X. Consider an integer $k > 0$ such that $2k < n$. Then the degeneracy locus $X_{\leq 2k}$ is not empty and contains at least a component of dimension $\geq 2k+1$. Moreover, if the degeneracy locus $X_{\leq 0}$ is not empty, it contains at least a curve. If in addition X is compact, then $X_{\leq 0} \neq \emptyset$.*

Section 4 is dedicated to the proof of this result. The main ingredients are the following. In Section 3, we write the expression of an equivariant bi-vector field π on the open dense torus $(\mathbb{C}^*)^n \subset X$ and study the behaviour of it under the change of immersion $(\mathbb{C}^*)^n \hookrightarrow Y_\sigma$ of the torus on the affine toric subvariety $Y_\sigma \subset X$, when varying the maximal cone σ in the fan Σ. Looking at the affine charts, we are able to prove that, for all integers $k > 0$ such that $2k < n$, there exists a $(2k+1)$-dimensional orbit of the action of the torus on X contained in the degeneracy locus $X_{\leq 2k}$. For $k = 0$, the compactness of X is needed to prove with the same arguments the existence of a 1-dimensional orbit contained in $X_{\leq 0}$. While for X non compact, we have to assume $X_{\leq 0} \neq \emptyset$ to conclude.

The idea to investigate the degeneracy loci is not new. In [2], Bondal studied them for Poisson structures and formulated the following

Conjecture. *Let X be a connected Fano variety of dimension n and π a Poisson structure on it. Consider an integer $k \geq 0$ such that $2k < n$. If the degeneracy locus $X_{\leq 2k}$ is not empty, it contains a component of dimension $\geq 2k+1$.*

Recall that a Poisson structure on a smooth complex variety X is a bi-vector field $\pi \in \Gamma(X, \bigwedge^2 \Theta_X)$ such that the Poisson bracket

$$\{f, g\} := \pi \lrcorner (df \wedge dg)$$

satisfies the Jacobi identity on holomorphic functions.

Later in [1], Beauville conjectured the same property to be true for Poisson structures on smooth compact complex varieties. In [8], Polishchuk proved it for the maximal non trivial degeneracy locus in two cases: when X is the projective space and when π has maximal rank at the general point.

The interest of these authors for Poisson structures comes from different areas. One is the problem of classification of quadratic Poisson structures, i.e. Poisson brackets on the algebra of polynomials in n variables such that the brackets are quadratic forms. These structures arise as tangent spaces to noncommutative deformations of the polynomial algebra itself. On the other hand, Poisson manifolds, i.e. smooth complex varieties with a Poisson structure, provide a more flexible notion than the one of hyperkähler manifolds.

Bondal's conjecture on a toric variety was investigated by Gay in [6]. He proved it for invariant Poisson structures on a smooth toric variety and showed that, in this case, compactness is not necessary.

The next natural step is to study equivariant Poisson structures, or more generally equivariant bi-vector fields, on a toric variety, that is the object of our article.

2. Toric varieties

In this section we summarise some basic notions on toric variety, following mainly the book [3]. We also recall the classical references [4, 5, 7].

2.1. Notations

Let $\mathbb{C}^*$ be the multiplicative group of non zero elements of $\mathbb{C}$. A *complex algebraic torus* (or simply a *torus*) is an affine variety T isomorphic to the group $(\mathbb{C}^*)^n$ for some $n > 0$. There are two *lattices*, i.e. free abelian groups, of rank n associated to a torus $T = \operatorname{Spec} \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$:

- the lattice of characters of T:

$$\begin{aligned} M &= \{\chi \colon T \to \mathbb{C}^* \mid \chi \text{ group homomorphism}\} \\ &= \{\chi^{\underline{m}} \colon T \to \mathbb{C}^*,\ \chi^{\underline{m}}(t_1, \dots, t_n) = t_1^{m_1} \cdots t_n^{m_n} \mid \underline{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n\}\,; \end{aligned}$$

- the lattice of 1-parameter subgroups of T:

$$\begin{aligned} N &= \{\lambda \colon \mathbb{C}^* \to T \mid \lambda \text{ group homomorphism}\} \\ &= \left\{\lambda_{\underline{u}} \colon \mathbb{C}^* \to T,\ \lambda_{\underline{u}}(t) = (t^{u_1}, \dots, t^{u_n}) \mid \underline{u} = (u_1, \dots, u_n) \in \mathbb{Z}^n\right\}. \end{aligned}$$

There is a natural pairing $\langle -, - \rangle \colon M \times N \to \mathbb{Z}$, that associates to every $(\underline{m}, \underline{u})$ the integer of the character $\chi^{\underline{m}} \circ \lambda_{\underline{u}}$. It follows that M is the dual lattice of N and viceversa. Moreover, there is a canonical isomorphism $N \otimes_{\mathbb{Z}} \mathbb{C}^* \cong T$, given by $\underline{u} \otimes t \mapsto \lambda_{\underline{u}}(t)$, and the notation T_N is used to indicate the torus associated to the lattice N via this isomorphism. To fix notation: let $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$.

2.2. Toric variety associated to a fan

Definition 2.1. A *toric variety* is an irreducible variety X containing a torus $T \cong (\mathbb{C}^*)^n$ as a dense open subset, such that the action of T on itself extends to an action on X.

Given a cone, one can define an affine toric variety in the following way.

Definition 2.2. A *convex polyhedral cone* in $N_{\mathbb{R}}$ is a set of the form

$$\sigma = \operatorname{Cone}(S) := \left\{ \sum_{s \in S} \lambda_s s \mid \lambda_s \geq 0 \right\} \subset N_{\mathbb{R}}$$

where $S \subset N_{\mathbb{R}}$ is a finite set of *generators* of σ.

Note that a convex polyhedral cone is convex and it is a cone. In the following we refer to such cones simply with the name *polyhedral cones.*

Given a polyhedral cone σ, its *dual cone* is

$$\sigma^\vee := \{\underline{m} \in M_\mathbb{R} \mid \langle \underline{m}, \underline{u} \rangle \geq 0, \text{ for all } \underline{u} \in \sigma\},$$

$$\text{while} \quad \sigma^\perp := \{\underline{m} \in M_\mathbb{R} \mid \langle \underline{m}, \underline{u} \rangle = 0, \text{ for all } \underline{u} \in \sigma\},$$

that are both polyhedral cones. A *face* of a polyhedral cone σ is the intersection of the cone with an hyperplane $H_{\underline{m}} = \{\underline{u} \in N_\mathbb{R} \mid \langle \underline{m}, \underline{u} \rangle = 0\}$ for some $\underline{m} \in \sigma^\vee$. One writes $\tau \leq \sigma$ to indicate that τ is a face of σ. A face of dimension one is also called an *edge.*

Definition 2.3. Let σ be a polyhedral cone. It is called

- *strongly convex* if the origin is a face of σ;
- *rational* if $\sigma = \mathrm{Cone}(S)$, for some finite set $S \subset N$;

A strongly convex rational polyhedral cone σ has a canonical set of generators constructed as follow. Let ρ be an edge of σ. Since σ is strongly convex, ρ is a half line, and since σ is rational, the semigroup $\rho \cap N$ is generated by a unique element u_ρ, called a *ray generator.* The ray generators of the edges of σ are called the *minimal generators.*

Definition 2.4. A rational polyhedral cone σ is called *smooth* if the set of its minimal generators is a part of a $\mathbb{Z}$-basis of N.

The following theorem defines the affine toric variety associated to a cone.

Theorem 2.5. *Let $\sigma \subset N_\mathbb{R}$ be a rational polyhedral cone. The lattice points $S_\sigma := \sigma^\vee \cap M$ form a finitely generated semigroup. Let $\mathbb{C}[S_\sigma]$ be the group algebra associated to S_σ. Then*

$$Y_\sigma := \mathrm{Spec}\,(\mathbb{C}[S_\sigma])$$

is an affine toric variety. Moreover,

- $\dim Y_\sigma = \mathrm{rk}_\mathbb{Z} N \Leftrightarrow \sigma$ *is strongly convex,*
- Y_σ *is normal* $\Leftrightarrow \sigma$ *is strongly convex and rational,*
- Y_σ *is smooth* $\Leftrightarrow \sigma$ *is smooth.*

An important fact is that all affine normal toric varieties can be obtained in this way.

Example 2.6. The n-dimensional torus $(\mathbb{C}^*)^n \subset Y_\sigma$ is the subvariety associated to the 0-dimensional cone $\{0\}$. Indeed, $(\mathbb{C}^*)^n = \mathrm{Spec}\,\mathbb{C}[\chi^{\pm e_1^*}, \ldots, \chi^{\pm e_n^*}] = \mathrm{Spec}\,\mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$, where $\mathcal{E} := \{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{Z}^n$.

We will now associate a toric variety to a fan.

Definition 2.7. A *fan* $\Sigma := \{\sigma_i\}_{i\in I}$ is a finite collection of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ such that

- every face of a cone $\sigma \in \Sigma$ is a cone of Σ,
- for every $\sigma_1, \sigma_2 \in \Sigma$ the intersection $\sigma_1 \cap \sigma_2$ is a common face of σ_1 and σ_2.

Definition 2.8. A fan $\sigma \subset N_{\mathbb{R}}$ is called

- *smooth* if every cone of Σ is smooth;
- *complete* if its support $|\Sigma| = \bigcup_{\sigma\in\Sigma} \sigma$ is all $N_{\mathbb{R}}$.

As already seen, every cone σ defines an affine toric variety. Moreover, it turns out that the cones in the fan give the combinatorial data necessary to glue a collection of affine toric varieties together to yield an abstract toric variety denoted by $X := TV(\Sigma)$. The main result is the following

Theorem 2.9. *Let Σ be a fan in $N_{\mathbb{R}}$. The variety $X = TV(\Sigma)$ is a normal separated toric variety. Moreover,*

- *X is smooth $\Leftrightarrow$ Σ is smooth,*
- *X is compact in the classical topology $\Leftrightarrow$ Σ is complete.*

An important result is

Theorem 2.10. *Let X be a normal separated toric variety with torus T_N. Then there exists a fan $\Sigma \subset N_{\mathbb{R}}$ such that $X \cong TV(\Sigma)$.*

2.3. Orbit-cone correspondence

Let N be a n-dimensional lattice and M its dual lattice. Let $X = TV(\Sigma)$ be the toric variety associated to a fan $\Sigma \subset N_{\mathbb{R}}$. Let $\sigma \in \Sigma$ be a cone and $Y_\sigma = \operatorname{Spec}(\mathbb{C}[S_\sigma])$ be the affine toric subvariety associated to it. Since the semigroup S_σ is naturally M-graded, the torus $T_N = (\mathbb{C}^*)^n$ acts on Y_σ.

We can describe this action using coordinates as follows. Let $\{\underline{m}_1, \dots, \underline{m}_s\} \subset M \cong \mathbb{Z}^n$ be the characters that generate the semigroup $S_\sigma = \mathbb{C}[\chi^{\underline{m}_1}, \dots, \chi^{\underline{m}_s}]$, $\mathbb{C}[y_1, \dots, y_s] \to \mathbb{C}[\chi^{\underline{m}_1}, \dots, \chi^{\underline{m}_s}]$ the quotient map that induces the immersion $\iota\colon Y_\sigma \hookrightarrow \mathbb{A}^s$. Let $\underline{y} = (y_1, \dots, y_s) \in \iota(Y_\sigma) \subset \mathbb{A}^s$ and $\underline{t} = (t_1, \dots, t_n) \in (\mathbb{C}^*)^n$, then

$$(\mathbb{C}^*)^n \times Y_\sigma \to Y_\sigma, \quad \underline{t}\cdot\underline{y} := (\chi^{\underline{m}_1}(\underline{t})\cdot y_1, \dots, \chi^{\underline{m}_s}(\underline{t})\cdot y_s)\,.$$

To determine the orbits for this action, it is convenient to use the following intrinsic description of points of Y_σ. Each point $p \in Y_\sigma$ can be seen as a semigroups homomorphisms $\gamma_p\colon S_\sigma \to \mathbb{C}$, given by $\gamma_p(\underline{m}) = \chi^{\underline{m}}(p)$. The action of the torus

T_N on Y_σ translates into $t \cdot \gamma_p(\underline{m}) = \chi^{\underline{m}}(t) \cdot \chi^{\underline{m}}(p)$. For every cone $\sigma \in \Sigma$, we define the *distinguish point* of σ as the point in Y_σ associated to the homomorphism

$$\gamma_\sigma \colon S_\sigma \to \mathbb{C}, \quad \text{given by } \gamma_\sigma(\underline{m}) =: \begin{cases} 1 & \text{if } \underline{m} \in \sigma^\perp \cap M, \\ 0 & \quad \text{otherwise.} \end{cases}$$

For every cone $\sigma \in \Sigma$, consider

$$O(\sigma) := \{\gamma \colon S_\sigma \to \mathbb{C} \mid \gamma(\underline{m}) \neq 0 \text{ for } \underline{m} \in \sigma^\perp \cap M \text{ and } \gamma(\underline{m}) = 0 \text{ otherwise}\},$$

it is easy to see that it is invariant under the torus action, $\gamma_\sigma \in O(\sigma)$ and that $O(\sigma) = T_N \cdot \gamma_\sigma$. The main theorem is the following

Theorem 2.11. *Let $X = TV(\Sigma)$ be the toric variety associated to a fan Σ in $N_\mathbb{R}$. Then:*

- *There is a bijective correspondence*

$$\begin{aligned} \{\textit{cones } \sigma \textit{ in } \Sigma\} &\longleftrightarrow \{T_N - \textit{orbits in } X\}, \\ \sigma &\longleftrightarrow O(\sigma) \cong \operatorname{Hom}(\sigma^\perp \cap M, \mathbb{C}^*) \end{aligned}$$

- *Let $n = \dim N_\mathbb{R}$. For each cone $\sigma \in \Sigma$, $\dim O(\sigma) = n - \dim \sigma$.*

- *The affine open subset Y_σ is the union of orbits*

$$Y_\sigma = \bigcup_{\tau \leq \sigma} O(\tau).$$

- *$\tau \leq \sigma$ if and only if $O(\sigma) \subset \overline{O(\tau)}$ and $\overline{O(\tau)} = \bigcup_{\tau \leq \sigma} O(\sigma)$.*

To conclude the section, let's write explicitly the orbit $O(\tau)$ in Y_σ for a face τ of a smooth cone σ. Since σ is smooth, we can assume it is generated by the standard basis $\{e_1, \dots e_n\}$ of $\mathbb{Z}^n$. Take τ to be the face generated by $\{e_1, \dots, e_h\}$, with $1 \leq h \leq n$. By definition, the distinguish point γ_τ is the semigroup homomorphism

$$\gamma_\tau : S_\sigma \to \mathbb{C}, \qquad \gamma_\tau(e_i^*) =: \begin{cases} 0 & \text{for } \ 1 \leq i \leq h, \\ 1 & \text{for } \ h+1 \leq i \leq n. \end{cases}$$

Thus, the orbit is

$$\begin{aligned} O(\tau) &= T_N \cdot \gamma_\tau \\ &= \{(z_1, \dots, z_n) \in Y_\sigma \mid z_i = 0 \text{ for } 1 \leq i \leq h \text{ and } z_i \neq 0 \text{ for } h+1 \leq i \leq n\}. \end{aligned}$$

3. Degeneracy loci of equivariant bi-vector fields

3.1. Equivariant bi-vector fields

Fix a n-dimensional lattice N, its dual lattice M and a smooth fan $\Sigma \subset N_{\mathbb{R}}$. Let $X = TV(\Sigma)$ be the smooth toric variety associated to the fan Σ.

Our aim is to study *bi-vector fields* on the toric variety X, i.e. global sections of the bundle $\bigwedge^2 \Theta_X$, where Θ_X is the holomorphic tangent bundle of X. Let $\{z_1, \dots, z_n\}$ be a system of local coordinates around a point $x \in X$, then π can be written locally as

$$\pi = \sum_{i,j=1}^{n} \pi_{ij}(z_1, \dots, z_n) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j},$$

where $\Pi = (\pi_{ij}(z_1, \dots, z_n))_{ij}$ is an antisymmetric matrix of homolorphic functions defined around $x \in X$. The *rank* of the bi-vector field π at the point x is the rank of the matrix Π when calculated in x, we indicate it with $\mathrm{rk}_x\, \pi$.

In particular, we are interested in equivariant bi-vector fields.

Definition 3.1. A bi-vector field $\pi \in \Gamma(X, \bigwedge^2 \Theta_X)$ is *equivariant* under the torus action if there exists a character $\underline{\alpha} \in M$, such that

$$\pi(t \cdot p) = \chi^{\underline{\alpha}}(t) \cdot \pi(p), \quad \text{for all } p \in X \text{ and } t \in (\mathbb{C}^*)^n.$$

The character $\underline{\alpha}$ is called *multi-degree.* If $\underline{\alpha} = (1, \dots, 1)$, then the bi-vector field is *invariant.*

Lemma 3.2. *The rank of an equivariant bi-vector field π is constant on the orbit of the torus action.*

Proof. It is obvious, since the π is an equivariant form. $\square$

First we describe equivariant bi-vector fields on the open dense torus $(\mathbb{C}^*)^n \subset X$. Let $\underline{z} = (z_1, \dots, z_n)$ be a system of coordinates on $(\mathbb{C}^*)^n$ and Θ the holomorphic tangent bundle of $(\mathbb{C}^*)^n$.

Lemma 3.3. *With the above notations, a bi-vector field $\pi \in \Gamma((\mathbb{C}^*)^n, \bigwedge^2 \Theta)$ is equivariant of multi-degree $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$ if and only if it has the form*

$$\begin{aligned} \pi(\underline{z}) &= \sum_{i,j=1}^{n} a_{ij} z_1^{\alpha_1} \cdots z_i^{\alpha_i+1} \cdots z_j^{\alpha_j+1} \cdots z_n^{\alpha_n} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j} \\ &= \sum_{i,j=1}^{n} a_{ij} \cdot \underline{z}^{\underline{\alpha}} \cdot z_i \cdot z_j \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}, \end{aligned} \tag{3.1}$$

where $A = (a_{ij})_{ij}$ is an antisymmetric matrix with coefficients in $\mathbb{C}$ and $\underline{z}^{\underline{\alpha}} = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$.

Proof. First check such a π is equivariant. Recall that the action of the torus on itself is just the multiplication. For all $\underline{z}, \underline{t} \in (\mathbb{C}^*)^n$,

$$\begin{aligned}\pi(\underline{t} \cdot \underline{z}) &= \sum_{i,j=1}^{n} a_{ij} \cdot \underline{t}^{\underline{\alpha}} \underline{z}^{\underline{\alpha}} \cdot t_i z_i \cdot t_j z_j \frac{1}{t_i} \frac{\partial}{\partial z_i} \wedge \frac{1}{t_j} \frac{\partial}{\partial z_j} \\ &= \sum_{i,j=1}^{n} a_{ij} \cdot \underline{t}^{\underline{\alpha}} \underline{z}^{\underline{\alpha}} \cdot z_i \cdot z_j \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j} = \underline{t}^{\underline{\alpha}} \cdot \pi(\underline{z}).\end{aligned}$$

Moreover, the calculation shows that the only polynomials that are equivariant under the torus action are the monomials of a fixed multidegree $\underline{\alpha}$. $\square$

Next we study the behaviour of the form π under the change of immersion $(\mathbb{C}^*)^n \hookrightarrow Y_\sigma$ of the dense torus in the affine toric subvariety $Y_\sigma \subset X$, when varying the maximal cone σ in the fan Σ.

Let $X = TV(\Sigma)$ be a smooth toric variety and $\pi \in \Gamma(X, \bigwedge^2 \Theta_X)$ an equivariant bi-vector field on X. Let σ_0 and σ_1 be two maximal cones of Σ. Since Σ is a smooth fan, every maximal cone is generated by a $\mathbb{Z}$-basis of N. Without loss of generality, we can assume the standard basis $\mathcal{E} = \{e_1, \ldots, e_n\}$ of $\mathbb{Z}^n$ to be a system of generators of the cone σ_0 and the form π to be expressed on the toric affine subvariety Y_{σ_0} by (3.1), where $z_i := \chi^{e_i}$ are the coordinates on Y_{σ_0}. Let $\mathcal{V} := \{v_1, \ldots, v_n\}$ be a basis of $\mathbb{Z}^n$ that generates the cone σ_1 and $w_i := \chi^{v_i}$ the corresponding coordinates on Y_{σ_1}. The form π is described on Y_{σ_1} by the following

Lemma 3.4. *Let $X = TV(\Sigma)$ be a smooth toric variety and $\pi \in \Gamma(X, \bigwedge^2 \Theta_X)$ an equivariant bi-vector field on X. In the above notations, π can be expressed on Y_{σ_1} by*

$$\pi = \sum_{k,h=1}^{n} b_{kh} \cdot \underline{w}^{\underline{\beta}} \cdot w_k \cdot w_h \frac{\partial}{\partial w_k} \wedge \frac{\partial}{\partial w_h}, \tag{3.2}$$

with $\underline{w}^{\underline{\beta}} = w_1^{\beta_1} \cdots w_n^{\beta_n}$, $B = (b_{kh})_{kh} = S \cdot A \cdot S^t$ and $\underline{\beta} = R^t \cdot \underline{\alpha}$, where R is the matrix of base change from the system of generators $\mathcal{V}$ of σ_1 to the standard basis $\mathcal{E}$ that generates σ_0 and $S = R^{-1}$.

Proof. In the above notations, $R := (r_{ij})_{ij}$ is the matrix of base change from $\mathcal{V}$ to $\mathcal{E}$ and $S := R^{-1} = (s_{ij})_{ij}$ is the matrix of the base change from $\mathcal{E}$ to $\mathcal{V}$, i.e.

$$v_j = \sum_{i=1}^{n} r_{ij} e_i \quad \text{and} \quad e_j = \sum_{i=1}^{n} s_{ij} v_i.$$

The dual basis $\mathcal{V}^* := \{v_1^*, \ldots, v_n^*\}$ is a system of generators of the dual cone $\sigma_1^\vee$. The matrix of the base change from $\mathcal{V}^*$ to the dual of the standard basis $\mathcal{E}^*$ is $(R^{-1})^t$ and the matrix of the base change from $\mathcal{E}^*$ to $\mathcal{V}^*$ is R^t, i.e.

$$v_i^* = \sum_{j=1}^{n} s_{ij} e_j^* \quad \text{and} \quad e_i^* = \sum_{j=1}^{n} r_{ij} v_j^*.$$

The coordinates on the affine subvariety Y_{σ_1} are by definition $w_i := \chi^{v_i^*}$. Thus,

$$w_i := \chi^{v_i^*} = \chi^{\sum_{j=1}^n s_{ij} e_j^*} = \chi^{s_{i1}e_1^*} \cdot \ldots \cdot \chi^{s_{in}e_n^*} = z_1^{s_{i1}} \cdot \ldots \cdot z_n^{s_{in}} \quad \text{and}$$
$$z_i := \chi^{e_i^*} = \chi^{\sum_{j=1}^n r_{ij} v_j^*} = \chi^{r_{i1}v_1^*} \cdot \ldots \cdot \chi^{r_{in}v_n^*} = w_1^{r_{i1}} \cdot \ldots \cdot w_n^{r_{in}}.$$

Moreover

$$z_j \cdot \frac{\partial}{\partial z_j} = z_j \cdot \sum_{i=1}^n \frac{\partial w_i}{\partial z_j} \frac{\partial}{\partial w_i} = z_j \cdot \sum_{i=1}^n s_{ij} \cdot z_1^{s_{i1}} \cdots z_j^{s_{ij}-1} \cdots z_n^{s_{in}} \frac{\partial}{\partial w_i} = \sum_{i=1}^n s_{ij} w_i \frac{\partial}{\partial w_i}.$$

On the affine subvariety Y_{σ_1}, the espression of the bi-vector field π described in (3.1) becomes

$$\begin{aligned}
\pi &= \sum_{i,j=1}^n a_{ij} \cdot z_1^{\alpha_1} \cdots z_n^{\alpha_n} \cdot z_i \frac{\partial}{\partial z_i} \wedge z_j \frac{\partial}{\partial z_j} \\
&= \sum_{i,j=1}^n a_{ij} \cdot z_1^{\alpha_1} \cdots z_n^{\alpha_n} \cdot \left(\sum_{k=1}^n s_{ki} w_k \frac{\partial}{\partial w_k} \right) \wedge \left(\sum_{h=1}^n s_{hj} w_h \frac{\partial}{\partial w_h} \right) \\
&= \sum_{i,j=1}^n a_{ij} \, (w_1^{r_{11}} \cdots w_n^{r_{1n}})^{\alpha_1} \cdots (w_1^{r_{n1}} \cdots w_n^{r_{nn}})^{\alpha_n} \\
&\qquad \cdot \left(\sum_{k=1}^n s_{ki} w_k \frac{\partial}{\partial w_k} \right) \wedge \left(\sum_{h=1}^n s_{hj} w_h \frac{\partial}{\partial w_h} \right) \\
&= \sum_{k,h=1}^n \sum_{i,j=1}^n s_{ki} \cdot a_{ij} \cdot s_{hj} \cdot \underline{w}^{\underline{\beta}} \cdot w_k \frac{\partial}{\partial w_k} \wedge w_h \frac{\partial}{\partial w_h} \\
&= \sum_{k,h=1}^n b_{kh} \cdot \underline{w}^{\underline{\beta}} \cdot w_k \frac{\partial}{\partial w_k} \wedge w_h \frac{\partial}{\partial w_h},
\end{aligned}$$

where $B = (b_{kh})_{kh} = S \cdot A \cdot S^t$ and

$$\underline{\beta} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} = \begin{pmatrix} r_{11} & \ldots & r_{n1} \\ \vdots & \vdots & \vdots \\ r_{1n} & \ldots & r_{nn} \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = R^t \cdot \underline{\alpha}\,.$$

□

3.2. Degeneracy loci

In this subsection we leave the toric setting and consider a smooth complex variety X of dimension n with a bi-vector field $\pi \in \Gamma(X, \bigwedge^2 \Theta_X)$. There is a partition

$$X = \coprod_{s \in 2\mathbb{N}} X_s, \quad \text{where } X_s := \{x \in X \mid \mathrm{rk}_x \, \pi = s\},$$

where the sets X_s will not be subvarieties of X in general. While

$$X_{\leq s} := \{x \in X \mid \mathrm{rk}_x \pi \leq s\} = \coprod_{s \leq r} X_r$$

is a closed subvariety of X and is called the *s-th degeneracy locus* of π.

In [2], Bondal formulated the following

Conjecture 3.5. *Let X be a connected Fano variety and π a Poisson structure on it. Let $k \geq 0$ be an integer, such that $2k < n$. Then, if the subvariety $X_{\leq 2k}$ is not empty, it contains at least a component of dimension $\geq 2k+1$.*

We recall that, a *Fano variety* is a compact complex variety whose anticanonical bundle is ample.

A *Poisson structure* on a smooth complex variety X is a bi-vector field $\pi \in \Gamma(X, \bigwedge^2 \Theta_X)$ such that the Poisson bracket

$$\{f, g\} := \pi \lrcorner (df \wedge dg)$$

satisfies the Jacobi identity on holomorphic functions. An equivalent condition for a bi-vector field π to be a Poisson structure is to satisfy $[\pi, \pi]_{SN} = 0$, where $[-,-]_{SN}$ is the Schouten-Nijenhuis bracket, see [8]. More explicitly, let $\{z_1, \ldots, z_n\}$ be a system of local coordinates around a point $x \in X$ and

$$\pi = \sum_{i,j=1}^{n} \pi_{ij} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$$

be a bi-vector field. Then, $[\pi, \pi]_{SN} = 0$ if and only if

$$\sum_{i=1}^{n} \pi_{ij} \frac{\partial \pi_{hk}}{\partial z_i} + \pi_{ih} \frac{\partial \pi_{kj}}{\partial z_i} + \pi_{ik} \frac{\partial \pi_{jh}}{\partial z_i} = 0, \quad \text{for all } j, h, k. \tag{3.3}$$

In [1], Beauville stated the following more optimistic version of Bondal's conjecture.

Conjecture 3.6. *Let X be a smooth complex and compact variety and π a Poisson structure on it. Let $k \geq 0$ be an integer, such that $2k < n$. Then, if the subvariety $X_{\leq 2k}$ is not empty, it contains at least a component of dimension $\geq 2k+1$.*

As explained in [1], there are some arguments in favor of this conjecture.

Proposition 3.7. *Let X be a smooth compact complex variety and π a Poisson structure on it. Then,*

- *Every component of X_s has dimension $\geq s$.*
- *Let r be the generic rank of π. Assume that $c_1(X)^q \neq 0$ in $H^q(X, \Omega^q_X)$, where $q = \dim X - r + 1$. Then, the degeneracy locus $X \setminus X_r$ has a component of dimension $> r - 2$.*

- *Let X be a projective threefold. If X_0 is not empty, it contains a curve.*

In [8], Polishchuk proved Bondal's conjecture for the maximal nontrivial degeneracy locus in two cases: when X is the projective space and when the Poisson structure π has maximal possible rank at the general point.

In [6], Gay studied invariant Poissons structures on a smooth toric variety. In this setting he proved Bondal's conjecture and pointed out that compactness is not necessary.

We will study equivariant bi-vector fields on smooth toric varieties. Our result is a little stronger then the one conjectured by Bondal. For $k \neq 0$, we even prove the degeneracy loci to be not empty, while for $k = 0$ the non emptiness is true just for compact toric varieties. Even if we will consider all equivariant bi-vector fields on a toric variety X, we spell out when such a bi-vector field is a Poisson structure.

Lemma 3.8. *Let X be a toric variety and $\pi \in \Gamma(X, \bigwedge^2 \Theta_X)$ an equivariant bi-vector field described on the open dense torus $(\mathbb{C}^*)^n$ as in (3.1). It defines a Poisson structure on X if and only if*

$$\sum_{\substack{i=1 \\ i \neq j,h,k}}^{n} \alpha_i \left(a_{ij}a_{hk} + a_{ih}a_{kj} + a_{ik}a_{jh}\right) = 0 \quad \textit{for all } j \neq h \neq k.$$

Proof. By Lemma 3.4, the structure of π is invariant under change of affine chart. Thus, it is enough to prove the proposition on the open dense torus $(\mathbb{C}^*)^n$ on which π is described as in (3.1). Because of equation (3.3), π is a Poisson structure if and only if, for all j, h, k

$$\begin{aligned}
0 &= \sum_{i=1}^{n} \pi_{ij}\frac{\partial \pi_{hk}}{\partial z_i} + \pi_{ih}\frac{\partial \pi_{kj}}{\partial z_i} + \pi_{ik}\frac{\partial \pi_{jh}}{\partial z_i} \\
&= \sum_{i=1}^{n} a_{ij}\underline{z}^{\underline{\alpha}} z_i z_j \frac{\partial(a_{hk}\underline{z}^{\underline{\alpha}} z_h z_k)}{\partial z_i} + a_{ih}\underline{z}^{\underline{\alpha}} z_i z_h \frac{\partial(a_{kj}\underline{z}^{\underline{\alpha}} z_k z_j)}{\partial z_i} + a_{ik}\underline{z}^{\underline{\alpha}} z_i z_k \frac{\partial(a_{jh}\underline{z}^{\underline{\alpha}} z_j z_h)}{\partial z_i} \\
&= \underline{z}^{\underline{\alpha}}\underline{z}^{\underline{\alpha}} z_j z_h z_k \sum_{i=1}^{n} \alpha_i \left(a_{ij}a_{hk} + a_{ih}a_{kj} + a_{ik}a_{jh}\right).
\end{aligned}$$

Since the matrix $A = (a_{ij})_{ij}$ is antisymmetric, if two of the indexes i, j, h, k are equal the sum in the parenthesis vanishes and the statement is proved. $\square$

4. The main theorem

Let N be a n-dimensional lattice, M its dual lattice and Σ in $N_{\mathbb{R}}$. Let $X = TV(\Sigma)$ be the normal toric variety associated to the fan Σ. Recall that, Σ is smooth if and only if X is smooth and Σ is complete if and only if X is compact.

Main Theorem. *Let X be a smooth toric variety of dimension n and $\pi \in \Gamma(X, \bigwedge^2 \Theta_X)$ an equivariant bi-vector field on X. Consider an integer $k > 0$ such that $2k < n$. Then the degeneracy locus $X_{\leq 2k}$ is not empty and contains at least a component of dimension $\geq 2k+1$. Moreover, if the degeneracy locus $X_{\leq 0}$ is not empty, it contains at least a curve. If in addition X is compact, then $X_{\leq 0} \neq \emptyset$.*

To prove of the main theorem, we will use the following lemma, whose proof is just an easy calculation.

Lemma 4.1. *Let $\sigma \subset N_{\mathbb{R}}$ be a strongly convex polyhedral cone. Consider the hyperplane*

$$H_0 = \{(z_1, \dots, z_n) \mid a_1 z_1 + \dots + a_n z_n = 0\} \subset N_{\mathbb{R}}, \quad \text{with } a_i \in \mathbb{R}$$

and the semispaces

$$H_0^+ = \{(z_1, \dots, z_n) \mid a_1 z_1 + \dots + a_n z_n \geq 0\}$$
$$H_0^- = \{(z_1, \dots, z_n) \mid a_1 z_1 + \dots + a_n z_n \leq 0\}.$$

If $\sigma \cap H_0^- \neq \{0\}$, there exists at least one generating ray of σ contained in H_0^-. The same holds for H_0^+.

Proof of the Main Theorem. Let $X = TV(\Sigma)$ as above. Let Y_{σ_0} be the affine subvariety of X corresponding to a maximal cone σ_0 in the fan Σ. Since Σ is smooth, we can assume without loss of generality that σ_0 is generated by the standard basis $\{e_1, \dots, e_n\}$ of $\mathbb{Z}^n$. On Y_{σ_0} there are coordinates $z_i := \chi^{e_i}$ and the equivariant bi-vector field can be written as

$$\pi = \sum_{j>i=1}^{n} 2a_{ij} z_1^{\alpha_1} \cdots z_i^{\alpha_i+1} \cdots z_j^{\alpha_j+1} \cdots z_n^{\alpha_n} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j},$$

where $A = (a_{ij})_{ij}$ is an antisymmetric matrix with complex coefficients. Let $\Pi = (a_{ij} \cdot \underline{z}^{\underline{\alpha}} \cdot z_i \cdot z_j)_{ij}$ be the matrix whose rank defines the rank of the form π. Note that, if $a_{ij} \neq 0$ for some i and j, the exponents satisfy

$$\alpha_i, \alpha_j \geq -1, \quad \text{and} \quad \alpha_h \geq 0 \ \ \forall\, h \neq i, j.$$

First suppose $\alpha_i \geq 0$, for all $i \in \{1, \dots, n\}$. Let $k \geq 0$ such that $2k < n$. Consider the $(2k+1)$-dimensional orbit O defined on Y_{σ_0} by:

$$\{(z_1, \dots, z_n) \in Y_{\sigma_0} \mid z_i = 0 \text{ for } 1 \leq i \leq n-2k-1 \text{ and } z_i \neq 0 \text{ for } n-2k \leq i \leq n\}.$$

As observed at the end of Subsection 2.3, this orbit corresponds to the face τ of σ_0 generated by $\{e_1, \dots, e_{n-2k-1}\}$. The matrix Π calculated on a point of O has at least $n-2k-1$ zero rows and $n-2k-1$ zero columns, thus its rank is less or equal then $2k$ (since its rank is even). Thus, $O \subset X_{\leq 2k}$ is the component of dimension $2k+1$ we are looking for.

Suppose now there exist $h_1, \dots, h_s \in \{1, \dots, n\}$ such that $\alpha_{h_i} = -1$ for all $i \in \{1, \dots s\}$ and $\alpha_k \geq 0$ for all $k \in \{1, \dots, n\} \setminus \{h_1, \dots, h_s\}$. If $s = 1$, the above arguments prove the theorem. To be precise, one can choose any orbit O such that the coordinate $z_{h_1} \neq 0$. Recall that, if $a_{ij} \neq 0$ and $h \neq i, j$, the exponent $\alpha_h \geq 0$. Then, if $s > 2$, the matrix A is the zero matrix and there is nothing to prove. If $s = 2$, the matrix A has just two non-zero coefficients. To simplify the notation, assume $h_1 = 1$, $h_2 = 2$. Thus, $a_{12} = -a_{12} \neq 0$ are the only two non zero coefficients, the exponents are $\underline{\alpha} = (-1, -1, \alpha_3, \dots, \alpha_n)$ with $\alpha_i \geq 0$ for all $i \geq 3$, and on Y_{σ_0} the form π becomes

$$\pi = 2a_{12} z_3^{\alpha_3} \cdots z_n^{\alpha_n} \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2}. \tag{4.1}$$

Thus, its rank is ≤ 2 on the whole Y_{σ_0}. Let's analyse the degeneracy locus $X_{\leq 0}$. If there exists at least one $\alpha_i > 0$ for $i \geq 3$, on the $(n-1)$-dimensional orbit $\{(z_1, \dots, z_n) \in Y_{\sigma_0} \mid z_i = 0\}$ the form π has rank 0.

The remaining case is when π is expressed on Y_{σ_0} by

$$\pi = 2a_{12} \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2} \quad \text{with } a_{12} \neq 0. \tag{4.2}$$

In the compact case, we have to prove that there exists a subvariety of dimension ≥ 1 on which the form π has rank $= 0$. Consider the hyperplane $H_0 := \{(z_1, \dots, z_n) \mid z_1 + z_2 = 0\} \subset N_{\mathbb{R}}$. Recall that Σ is a fan, thus every cone is strongly convex, and it is complete. By Lemma 4.1, there exists a maximal cone $\sigma \in \Sigma$, such that at least one of its generating ray $\rho = (\rho_1, \dots, \rho_n)$ lies in the halfspace H_0^-. Let $r_j = (r_{1j}, \dots, r_{nj})$ for $j = 2, \dots, n$ the other rays generating σ. By lemma 3.4, on the affine chart Y_σ corresponding to the cone $\sigma \in \Sigma$ the form π becomes

$$\pi = \sum_{j>i=1}^{n} 2b_{ij} w_1^{\beta_1} \cdots w_i^{\beta_i+1} \cdots w_j^{\beta_j+1} \cdots w_n^{\beta_n} \frac{\partial}{\partial w_i} \wedge \frac{\partial}{\partial w_j}.$$

The coordinates w_i are the one on Y_σ corresponding rays $\{\rho, r_2, \dots, r_n\}$ and $\underline{\beta} = R^t \cdot \underline{\alpha}$, where R is the matrix of the base change between the basis given by the rays of σ and the canonical basis of $\mathbb{Z}^n$. Thus,

$$\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} = R^t \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \rho_1 & \cdots & \rho_n \\ r_{12} & \cdots & r_{n2} \\ \vdots & \cdots & \vdots \\ r_{1n} & \cdots & r_{nn} \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} -\rho_1 - \rho_2 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}.$$

Since $\rho = (\rho_1, \dots, \rho_n) \in H_0^-$, $\beta_1 = -\rho_1 - \rho_2 > 0$, and all the other exponent β_i are ≥ -1. On this chart, we can proved the existence of the required subvariety, as discussed above.

If X is not compact, i.e. the fan is not complete, the argument above does not hold. The degeneracy locus $X_{\leq 0}$ is not empty if and only if we can find an affine subvariety $Y_\sigma \subset X$ on which π has the form (4.1) with at least one $\alpha_i > 0$ and we can be proved the existence of the required subvariety, as discussed above. □

The proof shows quite clearly where compactness is needed in our theorem. Moreover, the hypothesis for π to be equivariant is necessary at least for the non compact case. Let's see some examples.

Example 4.2. Consider the toric variety $\mathbb{C}^n$. The degeneracy locus $X_{\leq 0}$ of any equivariant bi-vector field of the form

$$\pi = a\frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}, \quad \text{with } a \neq 0 \text{ and } i, j \in \{1, \dots, n\}$$

is empty, since the rank of π is two everywhere.

Example 4.3. This example is due to Gay [6]. On the toric variety $\mathbb{C}^3$, consider the bi-vector field

$$\pi = z_3\frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2} + z_2\frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_3} + 2z_1\frac{\partial}{\partial z_2} \wedge \frac{\partial}{\partial z_3}.$$

It is immediate to see that π is not equivariat. One can verify, it is a Poisson structure, i.e. it satisfies the condition $[\pi, \pi]_{SN} = 0$. The rank of π is the rank of the matrix

$$\begin{pmatrix} 0 & z_3 & z_2 \\ -z_3 & 0 & z_1 \\ -z_2 & -z_1 & 0 \end{pmatrix}.$$

Observe that, the degeneracy locus $X_{\leq 0} = \{0\}$.

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Dipartimento di Matematica "Giuseppe Peano", Università degli Studi di Torino, via Carlo Alberto 10, 10123 Torino, Italy.

elena.martinengo@unito.it