Leibniz's Foundational Thought in 18th-Century Mathematical Debates

Theatrum naturae et artium – Leibniz und die Schauplätze der Aufklärung

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> *Il faut avouer que Leibnitz, tout grand méthaphysicien qu'il était, n'avait pas bien conçu la métaphysique du calcul differentiel.*
> *(Bailly,* Eloge de Leibnitz, *1768)*

Leibniz's effort in the foundation of mathematics were partly ignored, partly contested by his contemporaries, and were only partly able to originate subsequent strands of reflexion and controversy. The roots of these foundational problems in Leibniz's peculiar "philosophy of mathematics" and the presence of some known doctrines and theories of Leibniz's in the subsequent developments can be at least partially traced. But one must be aware, first of all, that Leibniz does not practice the study of the foundations of mathematics in our 19^{th}–20^{th} century sense.[1]

For Leibniz, mathematics are the result of the application of an intellectual activity, both of analysis and of demonstrative synthesis, to concepts that are built by common sense (that is, by the imagination in its function of a *sensorium commune*), and to entities that are built by fancy (that is, by imagination in its ordinary meaning). Space, lines, the continuum, are "imaginative" entities, that neither exist as such in the physical world, nor result from experience by intellectual abstraction. Numbers are concepts that arise from the inner perception of a multiplicity of structures that are common to the outer perceptions received by different senses. In both cases, there is a bundle of innate components at play. The ideas of mathematical concepts and entities are "virtually" innate: in our cognitive completion there are both the "seeds" of these ideas, and the ability through our ordinary faculties to fully develop those seeds as we experience the world. In our reason, the rules of rational proceeding are also innate, and we have an innate disposition to operate with symbols – be they words, signs, or images.

According to Leibniz, mathematical truths are analytic, that is, their validity hinges on their demonstrative connection of definitions, substitutions, and tautologies.[2]

1 I discussed this in Pasini, 2001. But see rather the Introduction, Beeley's essay and *passim* in Goethe et al., 2015.

2 It is well known how Leibniz interpreted the basic arithmetic operations: in the simplest and most familiar example, 2+2 = 4 is demonstrated using definitions and a substitution axiom: $2 =_{df} 1+1$, $3 =_{df} 2+1$, $4 =_{df} 3+1$; ergo 2+2 = 2+1+1 = 3+1, that is 4 *qed* (NE, IV, 7, § 10; A VI, 6, p. 413-14). It does not really matter how it was first figured out, whereas in Kant's treatment of 7+5 = 12, effecting the operation of counting is crucial to both performing and validity of any addition. German philosophical *Aufklärung* will not be impressed by Leibniz's analytic approach: see the anodine definition of addition in Wolff, 1732, Def. 26, § 61, and what follows. Sadly for our history, Bolzano is beyond any *möglicher Aufklärungsschauplatz*.

Such truths are *a priori* for Leibniz as for Wolff and Kant, although in a very different sense than Kant's: they exist *a priori* in the rational structure of the created world and in God's *regio idearum*; they also are virtually innate and are based on notions apprehended, as we have said, with a combination of experience and inner constitution.

Mathematics are amphibious, as for their possible foundation, between this psychological genesis and the purely demonstrative framework that Leibniz deems essential to such disciplines. From a purely mathematical point of view, the first domain can, so to say, be taken for granted. Demonstrative tools, techniques, formal rules of development, are instead a paramount preoccupation of Leibniz's[3]. For example, among the ideas that best characterize Leibniz's approach to mathematics, as to all demonstrative knowledge, is that of the progressive transformation of axioms into theorems, by deriving them from more fundamental axioms. He tried his hand at many such demonstrations. In a letter to Johann Bernoulli, with whom Leibniz often discusses such deep mathematical matters, we can read:

> "inveni, uno syllogismo comprehensam, innixo definitionis minoris et majoris et axiomati identico: *minus* enim definio, quod alterius (*majoris*) parti aequale est. Axioma autem identico quod adhibeo est: Unum quodque aequale sibi ipsi est, seu $a = a$. Hoc enim tanquam indemonstrabile sumo. Sic ergo argumentor in syllogismo primae figurae:
> Quidquid est aequale parti totius, id toto minus est, per *definitionem* majoris.
> Pars totius est aequalis partis totius, nempe sibi ipsi, per *axioma identicum*.
> Ergo pars totius toto minor est. Quod erat demonstrandum."[4]

Not many mathematicians, apart from Christian Wolff, would take up those ideas and approach. Looking up for the axiom in Wolff's works, we find in the *Ontology* a nearly identical demonstration, on the selfsame principles, and even with the same basic circularity:

> "§. 357. *Quaelibet pars totius est minor toto*. Quaelibet enim pars totius est sibimetipsi (§. 351), adeoque parti totius *per hypoth*. aequalis. Est igitur toto minor (§. 352). Demonstratio propositionis praesentis analyseos perfectae exemplum praebet, cum principia, quibus nititur, sint definitio et axioma veri nominis, propositio identica."[5]

This accord is not surprising, just like when they make use of similar concepts of substitution and identity. We must remember that Wolff had for some time direct access to Leibniz's ideas and texts, and they kept a lasting and lively correspondence. But even he would not offer such treatments in his properly mathematical works.

3 "Some decades ago I had the gall to open a lecture […] with the words: 'Leibniz knew what a proof is. Descartes did not'. I wish I were still an adolescent of the mind, and could allow myself to stroll today in the garden of unprotected aphorisms. Instead I have to say that Leibniz had a prescient grasp of the concept of proof we are now taught in elementary logic" (Hacking, 2015, p. 23–24; reference is made to Hacking, 1973).

4 GM 3, p. 321–22.

5 Wolff, 1730, I, III, 4, § 357, p. 278.

Yet again, this kind of inquiry has not the same foundational role it has for us. This is not to say that Leibniz does not consecrate systematic inquiries to themes that are undoubtedly connected with foundational problems. Leibniz addresses both general issues, like the validity of mathematics, and very particular issues, like imaginary quantities considered as useful, "well founded fictions", and at the same time as contradictory beings of which Nature is surprisingly able to make use. The latter example might allow us to distinguish, in turn, between investigations on mathematical grounds and investigations on a "foundational" grounds.[6]

Giving the proper form to mathematics, a form that will allow in turn to extend and perfect them; bringing to perfection the beginnning elements, that is, the "internal" foundations of mathematics; providing mathematics with a more "external" kind of foundation based on more general principles and reasons: these lines of research, with different approaches and not always increasing command, are pursued by Leibniz during his whole life. He repeatedly attempts, for instance, both the study of the starting propositions of the most fundamental mathematical domain (that is, in his view, geometry), and the study of general premises that on the metaphysical level can provide a foundation of mathematics.

To these intra- and extra-mathematical foundational domains he eventually adds a sort of middle ground, that he calls "metageometric":

> "quae nunc aggredimur tradere Geometriae interioris praecepta, Metageometrica compendio appellari possent, si nomina fingere ultra quam opus est nobis liberet. Nam Metaphysicum quiddam in ipsa Geometria tractant, quod mente magis quam imaginatione consequi licet."[7]

In his later years, actually, Leibniz devotes much effort to this "metaphysic" element that appears to be incorporated into mathematics. He has again, as in his youth, a quite creative attitude, and even profusely creates neologisms, very much à la Jungius, to describe the concepts he feels to need. In his *Initia rerum mathematicarum metaphysica* we find, for instance, "homogona" (correlated entities, such time and instant, or a figure and its perimeter, that can be considered as co-generated), or "prosultare" (to be brought into existence by sufficient conditions entailed by the existence of other entities, like the plane that is determined by three points) – key terms of a possible constructability of mathematical entities on a conceptual basis.[8] It is really a pity that most of this toil remained unknown to the 18th century.

Many themes, indeed, on which Leibniz exerted his mind, both privately and in exchanges, but concerning which he made public, at best, only some hints of his am-

6 An instance of the first kind is the 1712–13 dispute between Leibniz and Joh. Bernoulli, the latter claiming that log(-1) = log(√-1), that is 0, a real number. This will be tackled by Euler, who shows in 1749 that $\log(-1) = i(\pi+2n\pi)$, that is, a complex and multivalued result. But progress on a foundational ground will be made only with Lambert and Gauss, as it is well known, and Leibniz will not be, in this case, a source of direct inspiration. Foundation is analytic, and has nothing to do with the richness of creation.

7 Leibniz, 1875, p. 595.

8 GM 7, p. 20–21.

bitions rather than results, had no echo. Think of his work on the definition of the line, both straight and parallel[9]: it seems (to me, at least) that none of Leibniz's efforts are quoted in 18th-century works on, say, the parallel postulate.

Already at the end of the 19th century, Paul Stäckel, in his book on the history of parallel theory, would state that after the 1570s two centuries would pass before something interesting on this theme would be published in Germany: "denn die scharfsinnigen Bemerkungen, die Leibniz über die Grundlagen der Geometrie gemacht hatte, sind erst in diesem Jahrhundert aus seinem Nachlasse ans Licht gezogen worden"[10].

He quoted Abraham Gotthelf Kaestner's Preface to his *Anfangsgründe der Arithmetik und Geometrie*, which first appeared in 1757, where he declared to have worked for years on the problem: "Die Schwierigkeit, welche bei der Lehre von den Parallellinien sich findet, hat mich schon viele Jahre beschäftigt".[11] It must be added that in the whole piece Kaestner does not mention Leibniz. And the same Kaestner wrote a Preface to the first edition of Leibniz's *New Essays*,[12] where he nearly did not take into consideration any reasoning on the nature of mathematics that was there; maybe because debates about monads and Leibniz's theory of the continuum were still resonant[13], he was more interested, quite clearly, in the way that mathematics provided foundation for Leibniz's philosophy as he understood it, than vice versa.

The same happened, equally unfortunately, with most of his work on the axiomatic foundation of geometry and its extension to topological relations. I will not dwelve on Leibniz's *analysis situs*. Suffice it to remark that it also was mostly ignored by 18th-century mathematicians. This is testified by a note that Gauss wrote in 1833: "Von der *Geometria situs*, die Leibnitz ahnte und in die nur einem Paar Geometern (Euler und Vandermonde) einen schwachen Blick zu thun vergönnt war, wissen wir und haben wir nach anderthalbhundert Jahren noch nicht viel mehr wie nichts"[14]. Moreover, Schumacher would write to Gauss not much later, and not without surprise, that in Huygens' posthumous works there were letters by Leibniz about this theme:

> "Es sind zwei Bände posthuma von Huygen's herausgekommen, die seinen Briefwechsel mit Leibnitz und Hopital nebst allerhand dahin gehörenden Stücken enthalten. Das Interessanteste sind die erste Grundzüge zu Leibnitzen's *Geometria situs*, die man, soviel ich weiss, für ganz verloren hielt".[15]

9 On this see De Risi, 2016.

10 Stäckel, 1895, p. 139.

11 Ibidem.

12 Kaestner, 1765.

13 See Palaia, 1993; Pasini, 1994.

14 Gauss, 1877, p. 605. "Concerning the *Geometria situs*, foreseen by Leibnitz, and of which only a couple of geometers (Euler and Vandermonde) were allowed to catch a glimpse, we know and have obtained after a hundred and fifty years little more than nothing" (transl. in Nash, 1999, p. 362).

15 H.C. Schumacher to Gauss, 1834, in Gauss and Schumacher, 1860, vol. 2, p. 346–47; see also Huygens, 1833, p. 9.

The allusion is to Leibniz's letter to Huygens of 1679,[16] clearly the same quoted by Gauss in the cited fragment. Gauss already knew it – most likely from its publication in Samuel König's famous *Appel au public* (1752)[17], his defense against the pro-Maupertuis party in the controversy on the principle of least action. It was most likely Johann Heinrich Lambert's source too, a passage of whose constitutes a notable exception to 18th-century general unawareness. In the preface to his *Anlage zur Architectonic*, Lambert evoked Leibniz, seemingly seeking symbols for qualitative properties of geometrical entities:

> "*Leibnitz* scheint eigentlich Zeichen verlanget zu haben, die in Absicht auf das *Quale* eben den Dienst thun, den die Algeber mit ihren Zeichen in Absicht auf das *Quantum* thut. Es sollen also *Dinge* durch schickliche Zeichen an und für sich vorgestellet werden. Und dann verlanget man auch Zeichen für ihre *Verhältnissen*, *Verbindungen*, *Bestimmungen*, etc.".[18]

Analysis situs was conceived to be a specific branch of symbolic analysis. This brings us to the most foundational of all foundational ideas of Leibniz's, and the most fundamental among fundamental branches of logic and mathematics, *i. e.* his *Speciosa generalis*: the art of signs that includes the whole domain of symbolic analysis and would also allow, after the application of analysis, a universally understandable presentation in synthetic form of all resulting knowledge[19] :

> "Et cela fait encor juger, qu'il s'en faut beaucoup que l'Algebre soit l'art d'inventer, puisqu'elle même a besoin du secours d'un art plus general. Et l'on peut même dire que la Specieuse en general, c'est à dire, l'art des caracteres est un secours merveilleux parce qu'elle déchargé l'imagination".[20]

In place of the often recalled passion that Leibniz had for the *philosophia perennis*,[21] we meet here, indeed, a *perennis quaedam mathesis*. It begins with some prominent ancient mathematicians: "L'on ne doutera point, voyant l'Arithmetique de Diophante, et les livres Geometriques d'Apollonius et de Pappus, que les anciens n'en ayent eu quelque chose". Viète and Descartes have extended it to variable numbers and to

16 GM 2, p. 19: "je crois qu'il nous faut encore une autre Analyse proprement géométrique ou linéaire qui nous exprime directement situm, comme l'Algèbre exprime magnitudinem".

17 Gauss quotes the *Appel au public*, for instance, in his *Disquisitiones arithmeticae* (Gauss, 1801, p. 46 fn.). On its genesis and diffusion see Goldenbaum, 2015.

18 Lambert, 1771, vol. 1, Vorrede, p. XII.

19 On this point, Kant (*Principiorum primorum cognitionis metaphysicae nova dilucidatio*, 1755) likened Leibniz to an alchemist: "de hac arte, quam postquam Leibnizius inventam venditabat, eruditi omnes eodem cum tanto viro tumulo obrutam conquesti sunt, […] si, quod res est, aperte fateri fas est, vereor, ne, quod acutissimus Boerhaavius in Chemia alicubi de alchymistarum praestantissimis artificibus suspicatur, eos nempe […] velocitate quadam praevidendi ea pro factis narrasse, quae fieri posse, immo quae fieri debere colligebant, […] idem quoque viro incomparabili fato evenerit" (AW, I, p. 389–90).

20 NE, IV 17, § 13; A VI, 6, p. 488–89.

21 See among others Orio de Miguel, 1987.

algebraic analysis of curves. It includes now "le nouveau calcul des infinitesimales", the calculus (to which we shall get back), that generalizes it.[22]

Precisely the infinitesimal calculus provided one of the most fertile grounds for foundational controversy.[23] Foundational work is often, from a historical point of view, the symptom of a problem. If foundation is sought outside the perimeter of the science, this may indicate a lack of autonomy, maybe of epistemological autonomy. Infinitesimals raised concerns both from the outer and from the inner perspective[24]. Maybe because of this contrastive quality, they were the subject of public pronouncements by Leibniz that, differently from what we have seen above, would be noticed and continue to be discussed till the end of the 18th century.

Calculus has, according to what we can find in Leibniz's writings, a threefold foundation. To begin with, it works, and works exactly. There is much discussion, from Clüver to Nieuwentijt to Berkeley, on the role in the calculus of the introduction of errors subsequently elided. To such criticism, Leibniz's reply is constantly that even an angel could not detect the error, since it can be made smaller than any assignable quantity.[25] The famous mantra of French mathematicians who faced the headaches of the learning phase of calculus, *Allez en avant, la foi vous viendra* – "go forward, and faith will come to you", this half-military, half-priestly motto, with a tinge of pascalian wagering – was coined in a conversation between Alexis Fontaine and the Abbé Bossut in connection to these problems:

> "Qu'on me permette de citer à ce sujet un petit trait qui me regarde. Lorsque je commençais à étudier le livre du marquis de l'Hopital, j'avais de la peine à concevoir qu'on pût négliger absolument, sans erreur quelconque, une quantité infiniment petite, en comparaison d'une quantité finie. Je confiai mon embarras à un fameux géomètre [in the margin: *Fontaine*], qui me répondit: *Admettez les infiniment petits comme une hypothèse, étudiez la pratique du calcul, et la foi vous viendra*".[26]

22 NE, IV, 17, § 13; A VI, 6, p. 489.

23 On this much has been written, also by me. At present the best introduction to these controversies is Goldenbaum and Jesseph, 2008.

24 Oratorian Father Charles René Reyneau's *Analyse demontrée*, the calculus handbook on which noone else than d'Alembert would eventually study, suggested in total isolation that the calculus was a most "natural" branch of mathematics: "le principe de ce calcul est si naturel, que les premiers Geometres [i. e. Greek practitioners of exhaustion methods] l'ont fait servir quelques-unes de leurs demonstrations" (Reyneau, 1708, vol. 1, p. iv); new solutions, by means of infinitesimal analysis, of problems previously untreatable, "étoient tirée comme du fond de la nature, et des premiers et plus intimes principes du mouvement" (p. vii).

25 See for instance Leibniz to Clüver, in 1695: "je crois qu'on pourra accorder tout ce que vous dites, et ne laisser pas de se contenter des mesures de nostre façon puisqu'il est impossible, que l'erreur puisse jamais devenir notable, même à un ange" (A III, 6, p. 391); Clüver to Leibniz, in 1696: "Vous dites bien que l'erreur est si insensible que même un ange ne pourra pas comprendre la petitesse ou la difference, mais à quoy veut on soutenir et garder ces petits erreurs, lesquels enfin se peuvent engrossir de cette maniere, qu'ils deviennent tout à fait insupportables, sur tout quand on veut faire la dimension des corps solides" (A III, 6, p. 753; I corrected two errors in the text).

26 Bossut, 1802, vol. 2, pp. 141–42. The motto is also attributed to d'Alembert, who maybe simply

Alongside with this “don’t worry, be zappy” attitude, there is another line of defense, that one might dub “small is beautiful”. It is more clearly associated with foundation, and is first introduced in 1989, maybe as a reaction to Clüver’s criticism, in a writing on celestial mechanics: following a strategy he would develop against Nieuwentijt’s criticisms, he presents infinitesimals as incomparable quantities, that is, as quantities that do not respect the homogeneity that the fifth book of Euclid required for proportions: “Assumsi inter demonstrandum quantitates incomparabiliter parvas, verbi gratia differentiam duarum quantitatam communium ipsis quantitatibus incomparabilem”[27]. Along this line, scientific praxis offers an easy naturalistic interpretation of what corresponds to infinitesimals in physical reality:

> “si quis nolit adhibere infinite parvas, potest assumere tam parvas quam sufficere judicat, ut sint incomparabiles et errorem nullius momenti, imo dato minorem producant. Quemadmodum terra pro puncto, seu diameter terrae pro linea infinite parva habetur respectu coeli, sic demonstrari potest, si anguli latera habeant basin ipsis incomparabiliter minorem, angulum comprehensum fore recto incomparabiliter minorem, et differentiam laterum fore ipsis differentibus incomparabilem; item differentiam sinus totius, sinus complementi et secantis fore differentibus incomparabilem; item differentiam chordae, arcus et tangentis. Unde cum hae sint ipsae infinite parvae, erunt differentiae infinities infinite parvae, et sinus versus etiam erit infinities infinite parvus adeoque recto incomparabilia”.[28]

This can well correspond to the unlimited degrees of mathematical infinitude: “infiniti sunt gradus tam infinitorum quam infinite parvorum”.[29]

Leibniz would resort to this notion again in 1701, to counter Rolle’s and Gouye’s criticism and the polemics that had ensued in the Academy of sciences in Paris: “on n’a pas besoin de prendre l’infini icy à la rigeur”,[30] he wrote, just as in optics sunrays

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took it up. In Bertrand’s famous biography, the tale is told so: “On raconte qu’un jeune homme abordant le calcul différentiel y rencontrait des contradictions qui, s’il est mal enseigné, peuvent réellement s’y trouver. Il osa consulter d’Alembert […] La réponse est restée célèbre: “Allez en avant, la foi vous viendra”” (Bertrand, 1889, p. 56). He added pitilessly: “Ce mot brillant, mais dépourvu de toute vérité, explique assez bien les défauts de d’Alembert. Il se réserve d’éclairer chaque page par la lecture de la suivante; c’est ce qu’on appelle manquer de méthode” (p. 57). And it was not even his own *mot brillant*.

27 *Tentamen de motuum coelestium causis*, GM 6, p. 150–51. On it see Aiton, 1962; Aiton, 1964; Bertoloni Meli, 1993.

28 GM 6, p. 151. “If someone does not want to employ infinitely small quantities, he can take them to be as small as he judges sufficient to be incomparable, so that they produce an error of no importance and even smaller than any given [error]. Just as the Earth is taken for a point, or the diameter of the Earth for a line infinitely small with respect to the heavens, so it can be demonstrated that if the sides of an angle have a base incomparably less than them, the comprehended angle will be incomparably less than a rectilinear angle, and the difference between the sides will be incomparable with the sides themselves; also, the difference between the whole sine, the sine of the complement, and the secant will be incomparable to these differences” (transl. in Jesseph, 1998, p. 21).

29 Ibidem.

30 Leibniz, 1701, p. 270. See also A I, 20, p. 493–94.

are considered parallel by pretending that they come from an infinitely distant point. As for quantities:

> "quand il y a plusieurs degrés d'infini ou infiniment petit, c'est comme le globe de la terre est estimé un point à l'egard de la distance des fixes, et une boule que nous manions est encor un point en comparaison du semidiametre du globe de la terre, de sorte que la distance des fixes est comme un infini de l'infini par rapport au diametre de la boule".[31]

It was meant that, instead of the infinite and the infinitesimal, "on prend des quantités aussi grandes et aussi petites qu'il faut pour que l'erreur soit moindre que l'erreur donnée".[32] Thus, it would seem that incomparable quantities were introduced principally in order not only to naturalize infinitesimals, but also as a naturalized proxy for the combination of the overall correctness of calculus, and of the ultimate validity formula, that is, that the error will be smaller than any assignable quantity. This one was indeed an old idea of Leibniz's, who already in 1676 thought this would be the connection between exhaustion methods and infinitesimal methods: "differentia assumta etiam infinite parva minor fiat error".[33] Most likely he had first encountered the notion in Wallis, who used the phrases *minor quavis data* or *minor quavis assignabile* in his *Mechanics*, *Algebra*, and *Arithmetica infinitorum*.[34] Of course, in this way readers were left with a concept that was but fictional, or economical – infinitesimals are useful fictions, as Leibniz would write until the end of his life: "des fictions, mais de fictions utiles pour abréger et pour parler universellement".[35]

Leibniz wrote to Varignon that he was taking this stance to avoid that the calculus be meddled with metaphysical disputes: "mon dessein a esté de marquer, qu'on n'a point besoin de faire dependre l'analyse Mathematique des controverses metaphysiques, ny d'asseurer qu'il y a dans la nature des lignes infiniment petites à la rigueur".[36] Now the 18th century would instead be passionate about this issue. In fact, mention of metaphysics brings us to the famous idea of the "metaphysics of the calculus", also known as "metaphysics of infinitely small quantities", that brought about a century-long debate.

The first to give form to this idea was Jean-Pierre de Crousaz, a Swiss Calvinist pastor who taught mathematics at the university of Lausanne (then a theological school) between 1700 and 1724, and later at Groningen. He wrote successful books on disparate subjects, and in 1721 a *Commentaire sur l'Analyse des infiniment petits*, that dealt with l'Hôpital's book and articles:

31 Leibniz, 1701, p. 270–71.
32 Leibniz, 1701, p. 271.
33 A VI, 3, p. 434.
34 E.g. in the latter "Cum tamen crescente numero terminorum, excessus […] ita continuo minuatur, ut tandem quolibet assignabili minor evadat, (ut patet;) si in infinitum procedatur, prorsus evaniturus est" (Wallis, 1656, prop. XX, p. 16). Hobbes, in the *Marks of the Absurd Geometry […] of John Wallis*, rejected this as precisely such a mark (EW 7 p. 369).
35 D III, p. 500.
36 Leibniz to Varignon, February 2, 1702; GM 4, p. 91.

“Mon Commentaire est précédé de deux Discours: Dans le prémier, je m’applique á établir ce qu’il faut entendre par les Infiniment Petits, et à dissiper l’obscurité et l’équivoque de ce terme”; and one page later the formula is put in print: “Dans la Section IV^{e}, je reviens encore une fois à la Métaphysique des Infiniment Petits de divers genres”.[37] At page 1, we find the title: “Utilité de la Métaphysique des Infiniment Petits”.

This is not a quotation

Here “metaphysics” means nothing more (and nothing less, plainly) than a clarification of the kind and extent of the reality ascribed to infinitesimals. We shall spare ourselves those discussions on similarities between monads or simple beings and infinitely small quantities that were common coin in the already mentioned controversies on the doctrine of the monads around 1748, and are unfortunately still around in some piece of *Leibniz-Forschung*.

Crousaz had exposed the year before, in the second edition of a work on logic, a possible source for the metaphysics-of-the-calculus idea. He was discussing the importance of joining to the knowledge of symbolic notes, the intuitive knowledge of things themselves.[38] In a footnote he added this quotation:

> “‘Il est bon qu’une Métaphysique générale precede le Calcul, le dirige, et l’éclaire, mais ensuite c’est le Calcul qui donne la précision et les details’. *Hist. de l’Ac. des Sc.* 1706 p. 73”.[39]

The anonymous piece in the *Histoire de l’Académie des Sciences* to which Crousaz referred, titled *Sur le rapport des forces centrales à la pesanteur des corps*, was a synopsis made by Fontenelle, as the secretary of the Academy, of a *mémoire* by Varignon too long to fit in the acts: *Comparaison des forces centrales avec les pesanteurs absoluës des corps mûs de vitesses variées à discretion le long de telles courbes qu’on voudra*. The central point was: “Il doit donc y avoir une Equation algebrique de la force centrale, et de ces 4 principes dont elle dépend, et comme la pesanteur en est un, on aura par là le rapport de la force centrale à la pesanteur”. The passage Crousaz had in mind read so:

> “Mais pour avoir cette Equation, il faut que les idées metaphisiques que nous venons d’exposer soient exprimées geometriquement, et algebriquement. Il est bon qu’une Metaphisique génerale précede le calcul, le dirige, et l’éclaire, mais ensuite c’est le calcul qui donne la précision et les détails”.[40]

By the way, Varignon in his *mémoire* wrote nothing of this kind. It was an addition by Fontenelle, whose realistic approach to the mathematical infinite was shared by Crousaz:

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37 Crousaz, 1721, Préface, f. é 3*r*–3*v*. Crousaz’s work was sharply criticized by Johann Bernoulli in private and by Saurin publicly.

38 An idea that he explicitly connected with Leibniz’s warnings against the *cogitatio mere symbolica*: see Crousaz, 1721, Préface, f. é 4*r*.

39 Crousaz, 1720, vol. 2, p. 884.

40 Fontenelle, 1706, p. 73 in the Amsterdam edition (the one Crousaz had read), otherwise at p. 59.

"Il importe de se rendre très familières les idées des Infiniment Petits, et de se convaincre de la réalité de leur existence: On ne sauroit mieux faire que de suivre M. De Fontenelle, et de se promener avec lui dans ces différens ordres d'Infinis".[41]

Crousaz quoted various writings that had appeared in the *Histoire de l'Académie des sciences*. Some years later an enthusiastic tone would indeed be set by Fontenelle, first-person, in his exceedingly famous *Elemens de la géométrie de l'infini*, a work that celebrated the *hereuse révolution* of the use of the infinite in mathematics, and that most likely was at the origin of Voltaire's skepticism on it. Fontenelle wrote in the Preface, after having remembered Nieuwentijt's and Rolle's criticisms:

"Malgré tout cela, l'infini a triomphé, et s'est emparé de toutes les hautes spéculations des géomètres. Les infinis ou infiniment petits de tous les ordres sont aujourd'hui également établis; il n'y a plus deux partis dans l'académie; et si Leibnitz a chancelé, on se fie plus aux lumières qu'on tient de lui, qu'à son autorité même".[42]

Leibniz had wavered seriously: "Il semble qu'il se fût relâché jusqu'au point de réduire les infinis de différens ordres à n'être que des incomparables", i.e. the customary grain of sand, and this would but ruin "l'exactitude géométrique des calculs".[43]

Did any one, we might wonder at this point, take up Leibniz's natural analogies? We shall resort again to Christian Wolff, who, in his *Mathematisches Lexikon*, carefully paraphrased Leibniz's words of 1689 and 1701:

"*Quantitas infinite parva, Infinite parvum, infinitesima*, eine unendlich kleine Grösse. Wird in der Mathematick diejenige genennet, welche in Ansehung einer andern vor nichts zu halten ist, so, daß kein Fehler kan gezeiget werden, wenn man sie gleich hinweg läßt. Z. E. der halbe Diameter der Erde ist in Ansehung der Sonnen- und Sternen-Weite unendlich klein, denn wenn man ihn gleich in der Astronomie vor nichtss achtet, in Ansehung der ersten Bewegung, so kommt die Rechnung von dem Auf- und Untergang der Sterne richtig heraus".[44]

And in the umpteenth *Commentaire* that accompanied an edition of l'Hôpital's *Analyse*, the jesuit Aimé-Henri Paulian reproduced passages of the "Cours de mathématique de Wolf, Tom. I, pag. 418"[45] concerning the comparison of the infinites with grains of sand and so on. For sure, naturalization of geometrical notions was not on Fontenelle's horizon:

41 Crousaz, 1721, p. 166.

42 Fontenelle, 1727, f. b 1*r*.

43 Fontenelle, 1727, f. b 1*r*. "Ces différens ordres, dont l'ordre du fini est le premier et le plus bas, sont véritablement incomparables; c'est-à-dire, qu'une grandeur de l'un n'est rien par rapport à une grandeur de l'ordre supérieur, non dans le sens qu'un grain de sable ne seroit rien par rapport à un globe dont la distance du Soleil à Sirius seroit le rayon, mais dans un sens infiniment plus rigoureux" (f. 2*v*–3*r*).

44 Wolff, 1716, col. 1145.

45 Paulian, 1768, p. 258, note I.

> "La géométrie est toute intellectuelle, indépendante de la description actuelle, et de l'existence des figures dont elle découvre les propriétés. Tout ce qu'elle conçoit nécessaire est réel de la réalité qu'elle suppose dans son objet. L'infini qu'elle démontré est donc aussi réel que le fini".[46]

Clearly, Leibniz's more collected approach was not enough for Fontenelle or Crousaz. But Leibniz was no more a promeneur philosophique since his youth, and anyway he did not need *de se promener avec lui dans les différents ordres d'infinis*. Rather, when he told Varignon that he wanted to avoid metaphysical scuffles, he was thinking particularly of Fontenelle:

> "Entre nous je crois que Mons. de Fontenelle, qui a l'esprit galant et beau, en a voulu railler, lorsqu'il a dit qu'il vouloit faire des elemens metaphysiques de nostre calcul. Pour dire le vray, je ne suis pas trop persuadé moy même, qu'il faut considerer nos infinis et infiniment petits autrement que comme des choses ideales ou comme des fictions bien fondées".[47]

Nonetheless, it is to metaphysics that Leibniz turned in the following months and years, to make sure that the first solution, that was still the backbone of the second, would work. A combination of metaphysics and natural theology makes sure that there are not any infinitesimals in nature, that nonetheless

> "les règles de l'infini reussisent dans le fini, comme s'il y avait des infiniment petits metaphysiques, les règles de l'infini réussissent dans le fini comme s'il y avait des infiniment petits métaphysiques, quoyqu'on n'en ait point besoin; et que la matière ne parvienne jamais à des parcelles infiniment petites: c'est parce que tous se gouverne par raison".[48]

Although all continuity is ideal, reality behaves according to the law of continuity, or better, to some generalisations of the law, that sometimes Leibniz calls *lex iustitiae*: a law of the general order of things and of knowledge that at times surfaces in his mathematical writings. The various foundational strains were connected by the *Fondation du calcul infinitésimal par celuy de l'algèbre ordinaire*,[49] in which Leibniz tried to show that algebra already implies such relations between disappearing quantities as those that were made use of in his calculus when the differences that geometrically constitute the characteristic triangle were made to vanish, as it were, and their relations were nevertheless retained. In fact, vanishing quantities, not grains of sand, were the secret to this ultimate foundation, and metaphysical laws guaranteed the correctness of this passage.[50]

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46 Fontenelle, 1727, f. b 2*r*. Moreover, he separated metaphysical and geometrical infinity and maintained that objections based on the former kind would not apply to the latter (f. b 3*v*).

47 GM 4, p. 110.

48 GM 4, p. 93–94.

49 The strand of writings that developed from this approach were unknown to the public until a much later publication. See Goldenbaum and Jesseph, 2008, *passim*; Pasini, 1988.

50 In the vanished quantity, that is in an unextended point, the property sought after (a certain value) still exists because of continuity. Thus it is the opposite of our passage to the limit, on the

This vision of Leibniz's, that quite evidently has no connection to Fontenelle's or Crousaz's "metaphysics of the infinitely small", remains unknown or isolated. That expression, instead – the true catchword of the debate – shows an overwhelming power in occupying the whole horizon. This is easily shown by a couple of examples.

The author of our exergue, Jean Sylvain Bailly, was a person of importance: president of the Jeu de Paume, mayor of Paris from 1789 to 1791, astronomer. A composer of famous *Eulogies*, he won a prize from the Berlin Academy for his *Eloge de Leibniz* in the same year in which H. G. Justi's demolishing essay was crowned by the same institution in the prize competion on monads theory.[51] It was a second-hand piece of work, not at all technical. Only in the endnotes, he briefly tackles the issue of the metaphysical foundation of the infinitesimal calculus, with this unrestrained judgement:

> "Il faut avouer que Leibnitz, tout grand méthaphysicien qu'il était, n'avait pas bien conçu la métaphysique du calcul differentiel", since, while Newton conceived it as "la méthode de trouver les limites des rapports", Leibniz "fonde son calcul moins géométriquement" with infinitesimals. Bailly was really befuddled: "Il y a de plus dans son sisteme des infiniment petits, qui sont de même négligibles à l'égard de ceux du premier ordre; mais l'esprit se perd dans ces différens ordres d'infiniment petits"[52].

We find a similar attitude in Alexandre Savérien's *Histoire critique du calcul des infiniment-petits, contenant la métaphysique et la théorie de ce calcul*. It is remarkable that "metaphysics" appears only in the title, which could be the sign that this was really a mere catchword, as we have suggested, and the subtitle not more than a slogan. Instead, Savérien was strongly influenced by Locke, as we can see in this passage:

> "Les hommes [...] ont préféré dans tous les tems les idées abstraites aux objets sensibles. L'avenir les touche plus que le présent. Les substances spirituelles ou qu'on ne peut définir, l'intéressent davantage que les substances matérielles; et il sont affectés de l'Infini, dont ils n'ont aucune idée distincte, tandis qu'ils négligent de s'assurer du Fini, qu'il leur importe de connoître".[53]

He proposed a pragmatical approach: in the differential calculus,

> "c'est bien moins les quantités elles-mêmes, qui en sont l'objet, que le rapport de ces quantités: et ces rapport n'ont de valeurs réelles, que celles qu'on leurs assigne".[54]

But this would not remedy its lack of cognitive foundations, and Savérien's conclusion, "fort naturelle", was: "qu'il est dangereux de se trop livrer au Calcul".[55]

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existence of which continuity is in fact defined.

51 Justi, 1748.

52 Bailly, 1748, p. 56, n. 34.

53 Savérien, 1753, p. vi.

54 Savérien, 1753, p. xxix.

55 Savérien, 1753, p. xxxvi.

Is Fontenelle's enthusiasm, we might again wonder, countered only by psychologist refusal? Well, meanwhile some German mathematicians proposed a more drastic solution to the problem of infinitesimal quantities and their elision in the calculus. In Wolff's *Elementa analyseos mathematicae tam finitorum quam infinitorum*, which can be found in the first volume of his *Elementa matheseos universae*, we read:

> "Infinitesima itaque respectu ejus quantitatis, cui incomparabilis existit, pro nihilo habenda. [...] Hinc duae quantitates infinitesima differentes aequales sunt"[56].

This most drastic of all approaches is maybe also the most contrary to Leibniz's ideas on mathematics, although it is faithful to his denial of existence – neither real nor mathematical – to infinitely small quantities. Yet Varignon, in notes that were posthumously published as an *Eclaircissement sur l'analyse des infiniment petits*, already suggested this:

> "Toute quantité qui n'est augmentée ou diminuée que d'une partie infiniment petite par rapport à son tout, peut être prise pour la même qu'elle étoit avant ce changement: *mutatio indefinite parva, mutatio nulla*".[57]

And the same would be advocated by the harshest anti-Leibnizian and anti-Wolffian German mathematician of the time, Euler, in his *Institutiones Calculi differentialis*, where he proposed to define infinitesimals on the basis of the fundamental proposition $x+\mathrm{d}x = x$, that is, on their being "properly nothing":

> "§ 87. Cum igitur infinite parvum sit revera nihil, patet quantitatem finitam neque augeri neque diminui, si ad eam infinite parvum vel addamus vel ab ea subtrahamus. Sit a quantitas finita atque dx infinite parva, erit tam $a+dx$, quam a-dx, et generaliter $a \pm dx = 0$."[58]

In truth, with time, naive realism would be left to minor characters. For instance an abbé Paul Foucher, of Tours, wrote a *Géométrie métaphysique* in 1758, where he made use of infinitely small quantities in the elementar geometry. He would have on this a controversy with Louis Dupuy in the *Journal des Savants*. It is curious that the same Dupuis (as the secretary of the Académie des inscriptions et belles-lettres) would eventually read his Eulogy. During the controversy, Foucher would advocate the real existence of infinitesimal quantities after having proposed them as a hypothetical device:

> "Je parle selon mon opinion particuliere, lorsque je dis que les *infiniment petits* sont une vérité de fait constatée par la Physique et la Géométrie; et lorsque je les donne pour une simple hypothèse, c'est comme Auteur de la *Géométrie Métaphysique*, parce que dans cet ouvrage il ne me convenoit pas de prendre un autre ton."[59]

56 Wolff, 1715, §§ 3–4, p. 452.
57 Varignon, 1725, p. 2.
58 Euler, 1755, p. 80.
59 Foucher, 1759, p. 594.

The sponge that sucks up most of this foundational panorama – maybe the first in doing it – is d'Alembert. The resulting, quite composite approach is testified in d'Alembert's contributions for the *Encyclopédie*. In a section on "Differential calculus",[60] he pays homage to the usual formula:

> "Ce qu'il nous importe le plus de traiter ici, c'est la métaphysique du calcul différentiel".

This metaphysics is more important,

> "et peut-être plus difficile à développer que les regles mêmes de ce calcul".

d'Alembert rejects, just like Fontenelle and with the same words, Leibniz's naturalization of infinitesimal as incomparable quantities:

> "M. Leibnitz, embarrassé des objections qu'il sentoit qu'on pouvoit faire sur les quantités infiniment petites, telles que les considere le calcul différentiel, a mieux aimé réduire ses infiniment petits à n'être que des incomparables, ce qui ruineroit l'exactitude géométrique des calculs."

But then he takes, quite expectably, a stance nearly opposite to Fontenelle's. Newton, instead, with a "métaphysique […] très-exacte et très lumineuse", never considered differential calculus

> "comme le calcul des quantités infiniment petites, mais comme […] la méthode de trouver les limites des rapports".

This, in d'Alembert's view, leaves to infinitesimals a sheer economic function:

> "Quand une fois on [aura] bien comprise [la vraie métaphysique du calcul différentiel], on sentira que la supposition que l'on y fait de quantités infiniment petites, n'est que pour abréger et simplifier les raisonnemens."

So, with d'Alembert, we are brought directly back to an idea of Leibniz's, while purportedly opposing his views. It is time to bring to an end this exposition. And we may indeed come to it with the year 1795, when this cruel assessment was written by de Prony in a chronicle of Lagrange's analysis lessons:

> "On sait que, *mettant de côté la métaphysique vicieuse de Leibnitz*, l'un des inventeurs de ce calcul, les géomètres n'ont regardé comme rigoureuses que la méthode des fluxions et celle des limites, données par Newton".[61]

And it would be very good to end this history in 1795, in the full of the *tournant des Lumières*, also because only two years later Lazare Carnot was to publish the first edition of his *Réflexions sur la métaphysique du calcul infinitésimal*[62], which would

60 All the following quotations come from d'Alembert, 1754.

61 Gaspard, 1795, p. 208.

62 Carnot, 1797.

much impress various philosophers, but were but a reflex of these debates and represented, so to say, that kind of recapitulation that has the historical role of an epitaph.

> Just one short passage more. Prony adds that Lagrange has been able to "établir rigoureusement, non-seulement les procédés analytiques, mais encore l'application aux lignes et aux surfaces, du calcul différentiel et intégral, sans employer ni l'une ni l'autre des méthodes précédentes".[63]

This passage is not a quotation

Lagrange began his scientific career in Turin, and I am going to end with the *Miscellanea Taurinensia*, where his debut took place. Very interestingly, and characteristically, Lagrange avoided the subject: his contribution is very technical and cutting-edge. It was another prominent author in the *Miscellanea*, one who eventually would nearly become Pope, who turned up with it, in the second volume (the first French-titled one), in an article *De l'infini absolu consideré dans la grandeur, par le p. Gerdil barnabite*. Gerdil only apparently revived Fontenelle's criticism of Leibniz:

> "Leibnitz n'ignorant pas sans doute la force des preuves que la Géométrie même pouvoit fournir contre ces sortes de grandeur, reduisit ses infiniment petits à n'être que des incomparables, dans le même sens qu'un grain de sable seroit incomparable au globe de la Terre. Cette idée ne s'accorde guere à la vérité avec l'exactitude géométrique des calculs, mais elle fait voir du moins que Leibnitz étoit *bien éloigné d'admettre cette sorte d'infini*".[64]

Neither the "synthèse rigoureuse des anciens", nor modern sublime analysis, imply "la supposition de quantités infiniment petites": that is, and this exposes Gerdil's apologetic goals, "ne renferment rien qui tende à établir la réalité de l'infini absolu soit dans la quantité discrète, soit dans la quantité continue".[65] This is a sort of *Anti-Fontenelle*, with ultimate theist intention, just as Gerdil himself eventually will write an Anti-Rousseau. One wonders why he does not mention Berkeley – but this is not so important here. Instead, there is a real surprise waiting at p. 18 of this mémoire, where the reader finds a note written by Lagrange himself. It concerns the possibility to treat an asymptote as it were a tangent, "en faisant, pour ainsi dire, disparoître le point d'attouchement".[66] Lagrange adds:

> "Il en est ici comme dans la méthode des infiniment petits, où le calcul redresse aussi de lui même les fausses hipotéses que l'on y fait. On imagine par exemple qu'une courbe soit un poligone d'une infinité de petit côtés, dont chacun étant prolongé devienne une tangente à la courbe. Cette supposition est réellement fausse; car le petit côté prolongé ne peut jamais être autre chose qu'une véritable sécante: mais l'erreur est détruite par une autre erreur qu'on introduit dans le cal-

63 Ibidem.
64 Gerdil, 1761, p. 4.
65 Ibidem.
66 Lagrange, 1761, p. 18.

cul en y négligant comme nulles des quantités, qui selon la supposition ne sont qu'infiniment petites".[67]

The future proponent of the complete elimination of infinitesimal quantities by the purely analytic reduction of the calculus to a general theory of infinite series, concludes here that in the compensation of errors, that Berkeley had largely turned against mathematicians and their incredulity, and just in this, consists the metaphysics of Leibniz's calculus:

> "C'est en quoi consiste, ce me semble, la Métaphysique du calcul des infiniment petits, tel que l'a donné M. Leibnitz. La méthode de M. Newton est au contraire tout à fait rigoureuse soit dans les suppositions, soit dans les procédés du calcul".[68]

Under heaven, in these debates, was indeed a bit of chaos; and in this situation, it is maybe excellent to conclude with the opposite stance proposed by the incomparable Mme Du Chatelet, who lamented not the lack of a metaphysics for the calculus, but instead that "il nous manque un calcul pour la Métaphysique".[69]

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67 Ibidem.

68 Ibidem.

69 "Il nous manque un calcul pour la Métaphysique pareil à celui que l'on a trouvé pour la Géométrie, par le moyen duquel, avec l'aide de quelques données, on parvient à connoître des inconnues; peut-être quelque génie trouvera-t'il un jour ce calcul. Monsieur de Leibnits y a beaucoup pensé, il avoit sur cela des idées, qu'il n'a jamais par malheur communiquées à personne, mais quand même on le trouveroit, il y a apparence qu'il y a des inconnues dont on ne trouveroit jamais l'équation" (Du Châtelet, 1740, p. 13–14).

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