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A MINIMAXMAX PROBLEM FOR IMPROVING THE TORSIONAL STABILITY OF RECTANGULAR PLATES

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ABSTRACT. We use a *gap function* in order to compare the torsional performances of different reinforced plates under the action of external forces. Then, we address a shape optimization problem, whose target is to minimize the torsional displacements of the plate: this leads us to set up a *minimaxmax* problem, which includes a new kind of worst-case optimization. Two kinds of reinforcements are considered: one aims at strengthening the plate, the other aims at weakening the action of the external forces. For both of them, we study the existence of optima within suitable classes of external forces and reinforcements. Our results are complemented with numerical experiments and with a number of open problems and conjectures.

1. INTRODUCTION

When pedestrians cross a footbridge or the wind hits a suspension bridge, the deck undergoes oscillations, which can be of three different kinds. The longitudinal oscillations, in the direction of the bridge, are usually harmless because bridges are planned to withstand them. The lateral oscillations, which move the deck horizontally away from its axis, may become dangerous, if the pedestrians walk synchronously; see the recent events at the London Millennium Bridge [1–3] and also earlier dramatic historical events [4, §1.2]. The torsional oscillations, which appear when the deck rotates around its main axis, are the most dangerous ones; they also appear in heavier structures such as suspension bridges; see [4, §1.3,1.4] for a survey.

Following [5], we view a bridge as a long narrow rectangular thin plate hinged at two opposite edges and free on the remaining two edges: this plate well describes decks of footbridges and suspension bridges, which, at the short edges, are supported by the ground. The corresponding Euler-Lagrange equation is given by a fourth order equation in a planar domain complemented by suitable boundary conditions; see (6). The solution of this equation represents the vertical displacement of the plate under the action of an external force. Note that the solution

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is continuous because for planar domains the energy space embeds into continuous functions. This does not occur in higher space dimensions or for lower order problems. The continuity of the solution is a crucial feature since it enables us to use the so-called *gap function* introduced in [6]. The gap function measures the difference of the vertical displacements on the two free edges of the plate and is therefore a measure of its torsional response. The number measuring the *maximal gap* is given by the maximum over the free edges of the gap function; see (7). Clearly, the maximal gap depends on the force through the Euler-Lagrange equation satisfied by the solution and one is led to seek the force which yields the largest torsional displacement. This gives a measure of the risk that the bridge collapses. In order to lower this risk, one may try different ways of reinforcing the deck.

Imagine that one has a certain amount of stiff material (e.g., steel) and has to decide where to place it within the plate in order to lower the maximal gap and, in turn, the torsional displacements. This material should occupy a proper open subset of the plate. In literature this kind of problem has been tackled in several ways; we refer to [7–9] for related problems on the torsion of a bar. Since this *shape optimization problem* is completely new, we choose two different strategies: we first assume that the stiff material reinforces a part of the plate, then we assume that it acts directly on the force and weakens it by a factor involving the characteristic function of the region occupied by the material and a constant measuring the strength of the stiff material. Reinforcing the plate means that we add the stiff material in critical parts of the plate in order to increase the energy necessary to bend it. Weakening the force means that we place some "aerodynamic damper" in order to reduce the action of the external force. These kinds of minimization problems naturally lead to homogenization [10], see also [9] for a stiffening problem for the torsion of a bar. Homogenization would lead to optimal designs with reinforcements scattered throughout the structure, namely designs impossible to implement for engineers. And since the design of the stiff structure should be usable for engineers, homogenization must be avoided and the class of admissible geometries for the reinforcements should be sufficiently small. In this respect, we mention the paper by Nazarov-Sweers-Slutskij [11], where only "macro" reinforcements are considered, although in a fairly different setting. The structural optimization problem that we tackle may be seen as the "dual problem" of the one considered in the seminal work by Michell [12], see also updated results in [13, Chapter 4]: our purpose is to determine the best performance of the stiff material by maintaining the cost whereas Michell aimed to determine the cheapest stiff material by maintaining the performance.

For both the two mentioned ways of introducing the reinforcement, our purpose is to optimize the maximal gap. We will introduce suitable classes, for both the force and the reinforcement, in which to set up the optimization problem. First we seek the "worst" forces for a given reinforcement. This number yields the maximal gap that may occur. Then, we seek the "best" reinforcements, which minimize the effect of the forces. We are then led to solve a *minimaxmax problem*. The existence of a maximal force and of a minimal reinforcement depends on how wide the classes are. In this paper, for the forces we mainly deal with the classes of Lebesgue

functions or of the dual of the energy space, while, concerning the reinforcements, we restrict our attention to simple designs, that may be appropriate for engineering applications: cross-type reinforcements, tiles of rectangular shapes, networks of bounded length, and general Lipschitz domains, see Definition 3.1.

This minimaxmax problem can also be seen as a *worst-case optimization problem*, since one is interested in minimizing the worst value of a functional among all possible designs. An extended presentation of worst-case optimization problems in structural mechanics can be found in [14]; see also [15] for a worst-case optimization problem of a compliance functional in the Lebesgue space.

This paper is organized as follows. In Section 2, we introduce rigorously the gap function with the minimaxmax problem. In Section 3, we identify suitable classes for which the minimaxmax problem admits a solution (i.e., worst forces with best reinforcements). In Section 4, we discuss symmetry properties of worst forces in the case of symmetric reinforcements. In Section 5, we investigate the worst force acting on a plate with no reinforcement. In Section 6, we analyze the effects of cross-type reinforcements, while, in Section 7, we consider more general polygonal-type reinforcements. In both cases we solve numerically the minimaxmax problem. Sections 8 to 11 are dedicated to the proofs of our results. Finally, Section 12 contains the conclusions on the work done.

2. Variational Setting and Reinforcements for the Plate

Up to scaling, in the following we may assume that the plate Ω has length π and width 2ℓ with $2\ell \ll \pi$ so that $\Omega =]0, \pi[\times] - \ell, \ell[\subset \mathbb{R}^2$. According to the Kirchhoff-Love theory [16, 17] (see also [18] for a modern presentation), the energy $\mathbb{E}$ of a vertical deformation u of the plate Ω subject to a load f may be computed through the functional

$$(1) \qquad \mathbb{E}(u) := \int_\Omega \left(\frac{(\Delta u)^2}{2} + (1 - \sigma)(u_{xy}^2 - u_{xx}u_{yy}) - fu \right) dxdy \, ,$$

where σ is the Poisson ratio and satisfies $0 < \sigma < 1$. This implies that the quadratic part of the energy $\mathbb{E}$ is positive. For the partially hinged plate under consideration, the functional $\mathbb{E}$ should be minimized in the space

$$H_*^2(\Omega) := \left\{ v \in H^2(\Omega) : \, v = 0 \text{ on } \{0, \pi\} \times] - \ell, \ell[\right\} ;$$

since Ω is a planar domain, one has the embedding $H^2(\Omega) \subset C^0(\overline{\Omega})$, and the condition on $\{0, \pi\} \times] - \ell, \ell[$ is satisfied pointwise. By [5, Lemma 4.1] we know that $H_*^2(\Omega)$ is a Hilbert space when endowed with the scalar product

$$(u, v)_{H_*^2} := \int_\Omega [\Delta u \Delta v + (1 - \sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})] \, dxdy$$

and associated norm $\|u\|_{H_*^2}^2 = (u, u)_{H_*^2}$, which is equivalent to the usual norm in $H^2(\Omega)$, that is, $\|u\|_{H^2}^2 = \|u\|_{L^2}^2 + \|D^2 u\|_{L^2}^2$. We also define $H_*^{-2}(\Omega)$ as the dual space of $H_*^2(\Omega)$ and we denote by $\langle \cdot, \cdot \rangle$ the corresponding duality. If $f \in L^1(\Omega)$ then the functional $\mathbb{E}$ is well-defined in $H_*^2(\Omega)$, while if $f \in H_*^{-2}(\Omega)$ we need to replace $\int_\Omega fu$ with $\langle f, u \rangle$.

Assume that the plate Ω is reinforced with a stiff material which occupies an open region $D \subset \Omega$ and that D belongs to a certain class $\mathcal{D}$, while f belongs to some space $\mathcal{F}$ of admissible forcing terms. We consider two possible ways of reinforcing the plate: either we stiffen the plate by increasing the cost of the bending energy, or we add an aerodynamic damper by weakening the force. This modifies the original energy (1) into the two following ways:

$$(2) \qquad \mathbb{E}_1(u) := \int_\Omega \left[(1 + d\chi_D) \left(\frac{(\Delta u)^2}{2} + (1-\sigma)(u_{xy}^2 - u_{xx}u_{yy}) \right) - fu \right] dxdy$$

and

$$(3) \qquad \mathbb{E}_2(u) := \int_\Omega \left[\frac{(\Delta u)^2}{2} + (1-\sigma)(u_{xy}^2 - u_{xx}u_{yy}) - \frac{f\,u}{1 + d\chi_D} \right] dxdy \,,$$

where χ_D is the characteristic function of D and $d > 0$ is the strength of the stiffening material. As for (1), the quadratic part of the functionals (2) and (3) are positive and should be minimized on the space $H_*^2(\Omega)$.

When dealing with $\mathbb{E}_1$, for any $D \subset \Omega$ open, we introduce the bilinear form

$$(4) \qquad (u,v)_D := \int_D \left[\Delta u \Delta v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx}) \right] dxdy$$

so that $(u,v)_\Omega = (u,v)_{H_*^2}$. Then, for all $f \in H_*^{-2}(\Omega)$ the minimizer of $\mathbb{E}_1$ satisfies the weak Euler-Lagrange equation

$$(5) \qquad (u_{f,D}, v)_{H_*^2} + d(u_{f,D}, v)_D = \langle f, v \rangle \qquad \forall v \in H_*^2(\Omega) \,,$$

which has no strong counterpart due to the lack of regularity of the term $(1+d\chi_D)$ that prevents an integration by parts.

On the other hand, due to the lack of regularity of the term $(1 + d\chi_D)$, $\mathbb{E}_2$ is not defined for all $f \in H_*^{-2}(\Omega)$, but it is well-defined for any $f \in L^1(\Omega)$; in this case the minimizer satisfies the equation $(u_{f,D}, v)_{H_*^2} = \int_\Omega \frac{fv}{1+d\chi_D}\,dxdy$, for all $v \in H_*^2(\Omega)$, which may also be written in its strong form:

$$(6) \qquad \begin{cases} (1 + d\chi_D)\Delta^2 u = f \,, & \text{in } \Omega \,, \\ u = u_{xx} = 0 \,, & \text{on } \{0, \pi\} \times]-\ell, \ell[\,, \\ u_{yy} + \sigma u_{xx} = u_{yyy} + (2-\sigma)u_{xxy} = 0 \,, & \text{on }]0, \pi[\times \{-\ell, \ell\} \,. \end{cases}$$

Since $0 < \sigma < 1$, both $\mathbb{E}_1$ and $\mathbb{E}_2$ admit a unique critical point in $H_*^2(\Omega)$, their absolute minimum. The minimizer may be different for $\mathbb{E}_1$ and $\mathbb{E}_2$ but we will denote both of them by $u_{f,D}$ since it will always be clear which functional we are dealing with. As we have just seen, the solution $u_{f,D}$ satisfies a weak Euler-Lagrange equation for $\mathbb{E}_1$ (but not a strong one) while it satisfies a strong Euler-Lagrange equation for $\mathbb{E}_2$ (and not a merely weak one).

Assume that some classes $\mathcal{F}$ and $\mathcal{D}$ of admissible f and D are given. Take $f \in \mathcal{F}$, $D \in \mathcal{D}$, and the minimizer $u_{f,D} \in H_*^2(\Omega) \subset C^0(\overline{\Omega})$ of $\mathbb{E}_1$ or $\mathbb{E}_2$, then compute its *gap function* with its *maximal gap*:

$$(7) \qquad \mathcal{G}_{f,D}(x) := u_{f,D}(x, \ell) - u_{f,D}(x, -\ell) \,, \qquad \mathcal{G}_{f,D}^\infty := \max_{x \in [0,\pi]} |\mathcal{G}_{f,D}(x)| \,.$$

In this way we have defined the map $\mathcal{G}_{f,D}^{\infty} \colon \mathcal{F} \times \mathcal{D} \to [0, \infty[$ with $(f, D) \mapsto \mathcal{G}_{f,D}^{\infty}$. Given $D \in \mathcal{D}$, we first seek the worst $f \in \mathcal{F}$ such that

$$(8) \qquad \mathcal{G}_D^{\infty} := \max_{f \in \mathcal{F}} \mathcal{G}_{f,D}^{\infty} = \max_{f \in \mathcal{F}} \max_{x \in [0,\pi]} |\mathcal{G}_{f,D}(x)|,$$

and then the best $D \in \mathcal{D}$ such that

$$(9) \qquad \mathcal{G}^{\infty} := \min_{D \in \mathcal{D}} \mathcal{G}_D^{\infty} = \min_{D \in \mathcal{D}} \max_{f \in \mathcal{F}} \max_{x \in [0,\pi]} |\mathcal{G}_{f,D}(x)|.$$

This is our *minimaxmax problem*. In the next sections we analyze some classes $\mathcal{F}$ and $\mathcal{D}$, where (8) and (9) admit a solution. Note that $\mathcal{G}^{\infty} = \mathcal{G}^{\infty}(\mathcal{F}, \mathcal{D})$ is monotone with respect to both the classes $\mathcal{F}$ and $\mathcal{D}$ but with opposite monotonicity.

3. Existence Results for the Minimaxmax Problem

We determine some classes $\mathcal{F}$ and $\mathcal{D}$ of admissible forces and reinforcements for which (8) and (9) admit a solution. The proofs are given in Section 8. We first show that $\mathcal{G}_D^{\infty}$, as in (8), is well-defined for some choices of the class $\mathcal{F}$.

Theorem 3.1. *For a given open set $D \subset \Omega$ and $p \in]1, +\infty]$, the maximization problems*

$$(10) \qquad \max \left\{ \mathcal{G}_{f,D}^{\infty} : f \in H_*^{-2}(\Omega) \ with \ \|f\|_{H_*^{-2}} = 1 \right\}, \quad (for \ \mathbb{E}_1),$$

$$(11) \qquad \max \left\{ \mathcal{G}_{f,D}^{\infty} : f \in L^p(\Omega) \ with \ \|f\|_{L^p} = 1 \right\}, \quad (for \ both \ \mathbb{E}_1 \ and \ \mathbb{E}_2),$$

admit a solution (in the considered space).

Then, we turn to problem (9). We introduce some classes $\mathcal{D}$ for which it is guaranteed the existence of a solution.

Definition 3.1 (Classes of admissible reinforcements)**.** *(a)* Cross-type reinforcements*: for $N, M \in \mathbb{N}$, $\mu \in]0, \pi/2N[$, $\varepsilon \in]0, \ell/M[$, $x_i \in [\mu, \pi - \mu]$ for $i = 1, \dots, N$ with $x_{i+1} - x_i > 2\mu$ for $i \leqslant N - 1$, and $y_j \in [-\ell + \varepsilon, \ell - \varepsilon]$ for $j = 1, \dots, M$ with $y_{j+1} - y_j > 2\varepsilon$ for $j \leqslant M - 1$, define*

$$\mathcal{C} := \left\{ D \subset \Omega : D = \left(\bigcup_{i=1}^{N}]x_i - \mu, x_i + \mu[\times] - \ell, \ell[\right) \cup \left(\bigcup_{j=1}^{M}]0, \pi[\times]y_j - \varepsilon, y_j + \varepsilon[\right) \right\}.$$

(b) Tiles of rectangular shapes*: for $N \in \mathbb{N}$ and $\varepsilon \in]0, \ell[$, define*

$$\mathcal{T} := \left\{ D \subset \Omega : D = \bigcup_{i=1}^{N} R^i, R^i \subset \Omega \ is \ an \ open \ rectangle \ with \ inradius \geqslant \varepsilon \right\}.$$

(c) Networks of bounded length*: for $\varepsilon \in]0, \ell[$ and $L > 0$, define*

$$\mathcal{N} := \left\{ D \subset \Omega : D = \Sigma^{\varepsilon} \ where \ \Sigma \subset \overline{\Omega} \ is \ closed, \ connected, \ \mathcal{H}^1(\Sigma) \leqslant L \right\}.$$

Here $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure of a set, and Σ^{ε} represents the ε-tubular neighborhood of Σ, namely the set of points in Ω at distance to Σ less than ε.

(d) Lipschitz trusses*: for $\varepsilon \in]0, \ell[$, define*

$$\mathcal{L} := \left\{ D \subset \Omega : D \ open \ with \ the \ inner \ \varepsilon\text{-cone property} \right\}.$$

We recall that by the inner ε-cone property we mean that at every point x of the boundary ∂D there is some truncated cone from x with an opening angle ε and radius ε inside D.

Notice that some of these classes are *monotone* with respect to set inclusion, namely $\mathcal{C} \subset \mathcal{T} \subset \mathcal{L}$, for suitable choices of the parameters N, ε, μ, L.

Theorem 3.2. *For a given $\kappa \in]0, 2\pi\ell[$ the minimization problem*

$$\tag{12} \min\{\mathcal{G}_D^\infty \,:\, D \in \mathcal{D} \text{ with } |D| = \kappa\},$$

admits a solution whenever the class $\mathcal{D}$ is one of those introduced in Definition 3.1 (with the parameters chosen so as to satisfy the area constraint).

4. Symmetric Framework for the Minimaxmax Problem

Whenever the class $\mathcal{D}$ of the minimaxmax problem (9) reduces to symmetric reinforcements, the class $\mathcal{F}$ can be reduced without changing the problem. We say that a set $D \subset \Omega$ is *symmetric with respect to the midline* (or, for short, symmetric) if $(x, y) \in D$ if and only if $(x, -y) \in D$, for all $(x, y) \in \Omega$. Then, we introduce the subspaces of even and odd functions with respect to y:

$$H_{\mathcal{E}}^2(\Omega) := \{u \in H_*^2(\Omega) : u(x, -y) = u(x, y) \,\forall (x, y) \in \Omega\},$$
$$H_{\mathcal{O}}^2(\Omega) := \{u \in H_*^2(\Omega) : u(x, -y) = -u(x, y) \,\forall (x, y) \in \Omega\}.$$

We first notice that

$$\tag{13} H_{\mathcal{E}}^2(\Omega) \perp H_{\mathcal{O}}^2(\Omega), \qquad H_*^2(\Omega) = H_{\mathcal{E}}^2(\Omega) \oplus H_{\mathcal{O}}^2(\Omega).$$

For all $u \in H_*^2(\Omega)$ we denote by $u^e \in H_{\mathcal{E}}^2(\Omega)$ and $u^o \in H_{\mathcal{O}}^2(\Omega)$ its components according to this decomposition, namely $u^e(x, y) = \frac{u(x,y)+u(x,-y)}{2}$ and $u^o(x, y) = \frac{u(x,y)-u(x,-y)}{2}$. The orthogonal projections $\mathcal{P}_{\mathcal{E}} : H_*^2(\Omega) \to H_{\mathcal{E}}^2(\Omega)$ and $\mathcal{P}_{\mathcal{O}} : H_*^2(\Omega) \to H_{\mathcal{O}}^2(\Omega)$ are defined onto these subspaces as $\mathcal{P}_{\mathcal{E}}u := u^e$ and $\mathcal{P}_{\mathcal{O}}u := u^o$, for every $u \in H_*^2(\Omega)$. Then, we define:

$$H_{\mathcal{E}}^{-2}(\Omega) := \{f \in H_*^{-2}(\Omega) : \langle f, v \rangle = 0 \,\forall v \in H_{\mathcal{O}}^2(\Omega)\},$$
$$H_{\mathcal{O}}^{-2}(\Omega) := \{f \in H_*^{-2}(\Omega) : \langle f, v \rangle = 0 \,\forall v \in H_{\mathcal{E}}^2(\Omega)\}.$$

In particular, $H_{\mathcal{O}}^2(\Omega) \subseteq \ker f$ for every $f \in H_{\mathcal{E}}^{-2}(\Omega)$ and $H_{\mathcal{E}}^2(\Omega) \subseteq \ker f$ for every $f \in H_{\mathcal{O}}^{-2}(\Omega)$. Moreover, $H_*^{-2}(\Omega) = H_{\mathcal{E}}^{-2}(\Omega) \oplus H_{\mathcal{O}}^{-2}(\Omega)$, that is for every $f \in H_*^{-2}(\Omega)$ there exists a unique couple $(f^e, f^o) \in H_{\mathcal{E}}^{-2}(\Omega) \times H_{\mathcal{O}}^{-2}(\Omega)$ such that $f = f^e + f^o$; with $f^e := f \circ \mathcal{P}_{\mathcal{E}}$ and $f^o := f \circ \mathcal{P}_{\mathcal{O}}$. As usual, we endow $H_*^{-2}(\Omega)$ with the norm $\|f\|_{H_*^{-2}} := \sup_{\|v\|_{H_*^2}=1} \langle f, v \rangle$, and we observe that

$$\tag{14} \|f\|_{H_*^{-2}} = \max\left\{\|f^e\|_{H_*^{-2}}, \|f^o\|_{H_*^{-2}}\right\} \qquad \forall f \in H_*^{-2}(\Omega).$$

The next result shows that if the reinforcement D is symmetric with respect to the midline then the worst forces f, whose existence is ensured by Theorem 3.1, can be sought in the class of odd distributions.

Theorem 4.1. *Assume that $D \subset \Omega$ is open and symmetric with respect to the x-axis. Then, (10) is equivalent to $\max\{\mathcal{G}_{f,D}^{\infty} : f \in H_{\mathcal{O}}^{-2}(\Omega), \|f\|_{H_*^{-2}} = 1\}$. Moreover, if $f \in H_{\mathcal{O}}^{-2}(\Omega)$ is such that $\|f\|_{H_*^{-2}} = 1$ and*

$$\mathcal{G}_{f,D}^{\infty} = \max \left\{ \mathcal{G}_{g,D}^{\infty} : g \in H_*^{-2}(\Omega) \text{ with } \|g\|_{H_*^{-2}} = 1 \right\},$$

then there exist infinitely many $g \in H_*^{-2}(\Omega)$ such that $\mathcal{P}_{\mathcal{E}}g \neq 0$, $\|g\|_{H_*^{-2}} = 1$, and $\mathcal{G}_{g,D}^{\infty} = \mathcal{G}_{f,D}^{\infty}$.

Theorem 4.1 states that, for a symmetric reinforcement D (possibly $D = \emptyset$ as for the free plate), the maximization of the gap function can be restricted to the class of odd distributions. But Theorem 4.1 *does not* state that *only* odd f attain the maximum. And indeed, $\mathcal{G}_{f,D}^{\infty}$ is not sensitive to the addition of some $\phi \in H_{\mathcal{E}}^{-2}(\Omega)$ to f, provided that the total norm is not exceeded. An interesting open problem is to determine whether there exists a unique $f \in H_{\mathcal{O}}^{-2}(\Omega)$ maximizing $\mathcal{G}_{f,D}^{\infty}$ (up to a sign change). We expect the answer to depend on D, in particular on possible additional symmetry properties of D. We prove Theorem 4.1 in Section 9.

Next, we have the following L^p-version of Theorem 4.1.

Theorem 4.2. *Assume that $D \subset \Omega$ is open and symmetric with respect to the x-axis and let $p \in]1, \infty]$. Then, problem (11) is equivalent to the maximization problem $\max \left\{ \mathcal{G}_{f,D}^{\infty} : f \in L^p(\Omega), f \text{ odd in } y, \|f\|_{L^p} = 1 \right\}$.*

If $1 < p < \infty$, then any maximizer is necessarily odd with respect to y.

If $p = \infty$ and an odd maximizer f satisfies $|f(x,y)| < 1$ on a subset of Ω of positive measure, then there exist infinitely many maximizers $g \in L^\infty(\Omega)$ such that $g^e \neq 0$ and $\|g\|_{L^\infty} = 1$.

Theorem 4.2 states that, for a symmetric reinforcement D, the maximization of the gap function can be restricted to the class of odd functions. Moreover, differently from Theorem 4.1, if $1 < p < \infty$ it says that *only* odd functions f attain the maximum. On the other hand, in the case $p = \infty$ oddness may fail, provided that there exists an odd maximizer satisfying the somewhat strange property stated in Theorem 4.2: the reason of this assumption will become clear in the proof of Theorem 4.2 given in Section 9.

5. Worst Cases on the Free Plate

In this section, we consider the free plate with no reinforcement ($D = \emptyset$) so that $\mathbb{E}_1$ and $\mathbb{E}_2$ coincide, and we study problem (10). For simplicity, for all $f \in H_*^{-2}(\Omega)$, we set $\mathcal{G}_f(x) = \mathcal{G}_{f,\emptyset}(x)$ and $\mathcal{G}_f^{\infty} = \mathcal{G}_{f,\emptyset}^{\infty}$.

Following the suggestion of Theorem 4.1, for any $z \in]0, \pi[$ we focus on the *odd distribution*

$$(15) \qquad T_z := \frac{\delta_{(z,\ell)} - \delta_{(z,-\ell)}}{2} \in H_{\mathcal{O}}^{-2}(\Omega),$$

where δ_P is the Dirac delta with mass concentrated at $P \in \overline{\Omega}$. Let $u_z \in H_*^2(\Omega)$ be the unique solution of the equation $(u_z, v)_{H_*^2} = \langle T_z, v \rangle$, for all $v \in H_*^2(\Omega)$. By the Riesz Theorem, this means that u_z is the representative of T_z and therefore (by taking $v = u_z$) $\|T_z\|_{H_*^{-2}}^2 = \|u_z\|_{H_*^2}^2 = \langle T_z, u_z \rangle = \mathcal{G}_{T_z}(z)/2$. This enables us to

normalize T_z and introduce the distribution $\overline{T}_z := \frac{\sqrt{2} T_z}{\sqrt{\mathcal{G}_{T_z}(z)}}$ such that $\|\overline{T}_z\|_{H_*^{-2}} = 1$. For any integer m, set

$$(16) \qquad \Upsilon_m := \frac{\sinh^2(m\ell)}{m^3 \left[(3+\sigma)\sinh(m\ell)\cosh(m\ell) + (1-\sigma)m\ell\right]} \,.$$

In Section 11 we prove the following result.

Proposition 5.1. *For all $x, z \in]0, \pi[$ we have*

$$\mathcal{G}_{T_z}(x) = \frac{4}{\pi(1-\sigma)} \sum_{m=1}^{\infty} \Upsilon_m \sin(mz) \sin(mx) \,, \qquad \mathcal{G}_{\overline{T}_z}(x) = \frac{\sqrt{2}\, \mathcal{G}_{T_z}(x)}{\sqrt{\mathcal{G}_{T_z}(z)}} \,.$$

Let us explain how Proposition 5.1 suggests a conjecture for the solution of (10) when $D = \emptyset$. Let Υ_m be as in (16) and consider the function Φ defined as $\Phi(x) := \sum_{m=1}^{\infty} \Upsilon_m \sin^2(mx)$, for every $x \in [0, \pi]$. Note that $\Phi(x) > 0$ for all $x \in]0, \pi[$ and

$$(17) \qquad \Phi(0) = \Phi(\pi) = 0 \,, \quad \Phi(\tfrac{\pi}{2}) = \sum_{k=0}^{\infty} \Upsilon_{2k+1} \,, \quad \Phi'(\tfrac{\pi}{2}) = 0 \,, \quad \Phi''(\tfrac{\pi}{2}) < 0 \,.$$

Some numerical computations and (17) suggest that Φ achieves its maximum at $x = \pi/2$:

$$(18) \qquad \Phi(\tfrac{\pi}{2}) > \Phi(x) \qquad \forall x \neq \tfrac{\pi}{2} \,.$$

Moreover, by Hölder's inequality, Proposition 5.1, and condition (18), for every $x, z \in]0, \pi[$

$$
\begin{aligned}
\left|\mathcal{G}_{\overline{T}_z}(x)\right| &= \frac{4\sqrt{2}}{\pi(1-\sigma)\sqrt{\mathcal{G}_{T_z}(z)}} \left|\sum_{m=1}^{\infty} \Upsilon_m \sin(mz) \sin(mx)\right| \\
&\leqslant \frac{4\sqrt{2}}{\pi(1-\sigma)\sqrt{\mathcal{G}_{T_z}(z)}} \sum_{m=1}^{\infty} \sqrt{\Upsilon_m}\, |\sin(mz)| \sqrt{\Upsilon_m}\, |\sin(mx)| \\
&\leqslant \frac{4\sqrt{2}}{\pi(1-\sigma)\sqrt{\mathcal{G}_{T_z}(z)}} \left(\sum_{m=1}^{\infty} \Upsilon_m \sin^2(mz)\right)^{\frac{1}{2}} \cdot \left(\sum_{m=1}^{\infty} \Upsilon_m \sin^2(mx)\right)^{\frac{1}{2}} \\
&= \frac{2\sqrt{2}}{\sqrt{\pi(1-\sigma)}}\, \Phi(x)^{\frac{1}{2}} \leqslant \frac{2\sqrt{2}}{\sqrt{\pi(1-\sigma)}}\, \Phi(\tfrac{\pi}{2})^{\frac{1}{2}} \,.
\end{aligned}
$$

Note that the above application of the Hölder inequality yields a strict inequality whenever $z \neq x$. Therefore, after taking the maximum over $[0, \pi]$ we deduce that $\mathcal{G}_{\overline{T}_z}^{\infty} < \frac{2\sqrt{2}}{\sqrt{\pi(1-\sigma)}} \Phi(\tfrac{\pi}{2})^{1/2}$ for every $z \neq \tfrac{\pi}{2}$ and that for $z = \tfrac{\pi}{2}$ the equality holds $\mathcal{G}_{\overline{T}_{\pi/2}}^{\infty} = \mathcal{G}_{\overline{T}_{\pi/2}}(\tfrac{\pi}{2}) = \frac{2\sqrt{2}}{\sqrt{\pi(1-\sigma)}} \Phi(\tfrac{\pi}{2})^{1/2}$. Hence, if (18) holds, then we would infer that

for all $z \in]0, \pi[$ we have $\mathcal{G}_{\overline{T}_z}^{\infty} \leqslant \mathcal{G}_{\overline{T}_{\pi/2}}^{\infty}$ with equality if and only if $z = \pi/2$.

This statement would prove that among all concentrated loads on the free edges of the plate Ω, the largest maximal gap is obtained when the load is concentrated (with opposite signs) at the midpoints $(\pi/2, \pm\ell)$. A numerical support of this

fact is provided by Table 1 below. The values collected there have been obtained using the software Mathematica, approximating the Fourier series for $\mathcal{G}_{\overline{T}_z}^\infty$ up to the 10,000-th term.

TABLE 1. Numerical values of $10^4 \times \mathcal{G}_{\overline{T}_z}^\infty$ and $10^4 \times \mathcal{G}_{T_z}^\infty$ (with $\ell = \pi/150$ and $\sigma = 0.2$).

z	$\frac{\pi}{20}$	$\frac{\pi}{18}$	$\frac{\pi}{16}$	$\frac{\pi}{14}$	$\frac{\pi}{12}$	$\frac{\pi}{10}$	$\frac{\pi}{8}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$10^4 \times \mathcal{G}_{\overline{T}_z}^\infty$	627.809	659.067	695.691	739.38	792.677	859.592	946.815	1066.21	1238.29	1429.87
$10^4 \times \mathcal{G}_{T_z}^\infty$	19.326	21.354	23.854	27.012	31.123	36.686	44.609	56.687	76.596	102.23

It is evident that the worst case is attained for $z = \frac{\pi}{2}$ and that the map $z \mapsto \mathcal{G}_{\overline{T}_z}^\infty$ is increasing on $[0, \pi/2]$ (note that it is symmetric with respect to $\pi/2$). For later use, we put in Table 1 also the values of $\mathcal{G}_{T_z}^\infty$.

6. Weakening the Force with Cross-Type Reinforcements

In this section, we minimize the energy $\mathbb{E}_2$ given in (3) finding the explicit solution and, in turn, the explicit gap function for particular choices of forces f and reinforcements D. We take symmetric cross-type reinforcements $D \in \mathcal{C}$ (see Definition 3.1) with one horizontal arm and $2N + 1$ vertical arms for some non-negative integer N. More precisely, fix $0 < \mu < \frac{(2N+1)\pi}{4(N+1)}$, $0 < \varepsilon < \ell$, (where the first condition prevents overlapping of vertical arms) and consider the set

$$(19) \qquad D_{\varepsilon,\mu}^N := \left( ]0, \pi[\times] - \varepsilon, \varepsilon[\right) \bigcup_{i=1}^{2N+1} \left(\left(\frac{\pi i}{2N+2} - \frac{\mu}{2N+1}, \frac{\pi i}{2N+2} + \frac{\mu}{2N+1} \right) \times] - \ell, \ell[\right) .$$

We will drop the subscripts in $D_{\varepsilon,\mu}^N$ in order to lighten the notation, writing them when needed to avoid confusion. To compare the effect of the reinforcements on the torsional instability, we are keeping the area of the set D^N fixed, indeed we have $|D^N| = 2\pi\varepsilon + 4\mu(\ell - \varepsilon)$ for any N. Furthermore, for $g \in L^2(]0, \pi[)$ and $\alpha > 0$ with $\alpha \notin \mathbb{N}$ (since this simplifies some computations), following the suggestion of Theorem 4.2, we consider the odd function

$$(20) \qquad f_\alpha(x, y) := R_\alpha \sinh(\alpha y) g(x)$$

with $R_\alpha := \frac{\alpha}{2C_g(\cosh(\alpha\ell) - 1)}$ and $C_g := \int_0^\pi |g(x)| \, dx$, so that $\|f_\alpha\|_{L^1} = 1$. We define

$$(21) \qquad \overline{\beta}_m := \frac{2\,\gamma_m\,\Upsilon_m}{C_g(1-\sigma)} \text{ and } \overline{\omega}_m := \frac{\gamma_m}{C_g} \frac{(1+\sigma)\sinh(m\ell)\cosh(m\ell) + (1-\sigma)m\ell}{(1-\sigma)m^2[(3+\sigma)\sinh(m\ell)\cosh(m\ell) + (1-\sigma)m\ell]} ,$$

where the coefficients Υ_m are as defined in (16), and

$$(22) \qquad \gamma_m := \frac{2}{\pi} \int_0^\pi g(x) \sin(mx) \, dx - \frac{2d}{\pi(1+d)} \sum_{i=1}^{2N+1} \int_{\frac{\pi i}{2N+2} - \frac{\mu}{2N+1}}^{\frac{\pi i}{2N+2} + \frac{\mu}{2N+1}} g(x) \sin(mx) \, dx .$$

Then, we obtain an explicit form for the gap function corresponding to problem (6) with $f = f_\alpha$ and $D = D^N$, and we analyze its asymptotic behavior as $\alpha \to +\infty$.

Theorem 6.1. *Let $\alpha > 0$ with $\alpha \notin \mathbb{N}$, let u_α be the unique solution of (6) with $f = f_\alpha$ and $D = D^N$, let $\mathcal{G}_\alpha$ be as in (7) with $u_{f,D} = u_\alpha$. As $\alpha \to +\infty$, $\mathcal{G}_\alpha(x) = \sum_{m=1}^{\infty} \beta_m(\alpha)\sin(mx)$ converges uniformly on $[0, \pi]$ to the function $\overline{\mathcal{G}}(x) := \sum_{m=1}^{\infty} \overline{\beta}_m \sin(mx)$, where the Fourier coefficients $\beta_m(\alpha)$ are so that $\beta_m(\alpha) = \overline{\beta}_m - \frac{\overline{\omega}_m}{\alpha} + o\left(\frac{1}{\alpha}\right)$, with $\overline{\beta}_m$ and $\overline{\omega}_m > 0$ given in (21).*

In Section 10 we prove Theorem 6.1. We derive the explicit value of $\beta_m(\alpha)$ in (57). Furthermore, we show that $\overline{\mathcal{G}}(x)$ is the gap function corresponding to a solution of the limit problem (58).

We exploit Theorem 6.1 to numerically solve the minimaxmax problem (9). More precisely, we fix $\ell = \pi/150$ and $\sigma = 0.2$ (two reasonable values for plates modeling the deck of a bridge, see [19]). Moreover, we take f_α in (20) with $g(x) = \sin(nx)$ for $n = 1, \ldots, 10$ and we call f^n its $H_*^{-2}(\Omega)$ limit as $\alpha \to +\infty$ (see Lemma 10.2) and $\overline{\mathcal{G}}_n$ the corresponding gap function. Then, we consider

$$(23) \qquad \mathcal{F} = \{f^1, ..., f^{10}\} \quad \text{and} \quad \mathcal{D} = \{D^0, ..., D^5\}.$$

The results are summarized in Table 2, in terms of the maximal gap $\overline{\mathcal{G}}_n^{\infty}$. The numerical values in Table 2 have been obtained using the software Mathematica, approximating the Fourier series for $\overline{\mathcal{G}}_n$ up to the 250-th term.

TABLE 2. Numerical values of $10^4 \times \overline{\mathcal{G}}_n^{\infty}$, with $\ell = \pi/150$, $\sigma = 0.2$, $d = 2$, $g(x) = \sin(nx)$, $D = D^N$, $\mu = 0.3$ (above) and $\mu = 0.5$ (below).

$10^4\times$	$\overline{\mathcal{G}}_1^{\infty}$	$\overline{\mathcal{G}}_2^{\infty}$	$\overline{\mathcal{G}}_3^{\infty}$	$\overline{\mathcal{G}}_4^{\infty}$	$\overline{\mathcal{G}}_5^{\infty}$	$\overline{\mathcal{G}}_6^{\infty}$	$\overline{\mathcal{G}}_7^{\infty}$	$\overline{\mathcal{G}}_8^{\infty}$	$\overline{\mathcal{G}}_9^{\infty}$	$\overline{\mathcal{G}}_{10}^{\infty}$
$\emptyset$	65.444	16.357	7.2665	4.0849	2.6123	1.8123	1.3300	1.0170	0.8023	0.6488
D^0	47.113	15.980	14.249	4.4296	11.422	2.6591	7.5961	2.0675	3.3673	1.6048
D^1	53.964	13.158	6.3585	4.0133	3.1797	2.9515	10.284	1.0582	9.9445	2.5730
D^2	55.292	13.979	5.9848	3.4987	2.1837	1.8092	1.4864	1.3153	1.2857	2.8377
D^3	55.839	13.892	6.2568	3.3920	2.2970	1.6152	1.2488	1.0158	0.8667	0.7611
D^4	56.135	14.080	6.2664	3.5181	2.1798	1.6029	1.1965	0.9264	0.7631	0.6461
D^5	56.320	14.050	6.2225	3.5437	2.2726	1.5216	1.1736	0.8864	0.7190	0.5998
$10^4\times$	$\overline{\mathcal{G}}_1^{\infty}$	$\overline{\mathcal{G}}_2^{\infty}$	$\overline{\mathcal{G}}_3^{\infty}$	$\overline{\mathcal{G}}_4^{\infty}$	$\overline{\mathcal{G}}_5^{\infty}$	$\overline{\mathcal{G}}_6^{\infty}$	$\overline{\mathcal{G}}_7^{\infty}$	$\overline{\mathcal{G}}_8^{\infty}$	$\overline{\mathcal{G}}_9^{\infty}$	$\overline{\mathcal{G}}_{10}^{\infty}$
$\emptyset$	65.444	16.357	7.2665	4.0849	2.6123	1.8123	1.3300	1.0170	0.8023	0.6488
D^0	37.707	14.748	16.541	5.3424	7.7463	3.7421	2.8388	1.8136	5.1181	1.1708
D^1	46.544	11.277	5.8180	4.0207	3.5078	3.6263	13.807	1.1909	12.949	3.1775
D^2	48.602	12.383	5.2331	3.0977	2.2697	1.7983	1.5803	1.4839	1.5789	3.7815
D^3	49.473	12.287	5.5878	2.9641	2.0589	1.4918	1.1993	1.0118	0.9058	0.8400
D^4	49.950	12.559	5.6012	3.1382	1.9056	1.4510	1.1105	0.8695	0.7391	0.6464
D^5	50.251	12.526	5.5421	3.1782	2.0384	1.3462	1.0587	0.8055	0.6678	0.5581

Several comments are in order. First we notice that, as expected from the statement of Theorem 6.1, the results do not depend on ε. Moreover, $\mu = 0.3$ means that the free edges of the plate are covered by the reinforcement on a percentage of 19% of their length, whereas $\mu = 0.5$ means that such a percentage is 31.8%.

It is worth noting that there is no monotonicity of $\overline{\mathcal{G}}_n^\infty$ with respect to either the number of branches, or to the frequency of $\sin(nx)$, nor to the reinforcement thickness μ. Also, we observe that each forcing term has its own "best truss" yielding a minimal maximal gap: the pattern is quite clear and it follows a descending diagonal in the two Tables 2. Basically, we see that D^{n-1} (i.e., the cross with $2n-1$ vertical arms) is the "best truss" for $g(x) = \sin(nx)$ and the reason is that the plate is reinforced in the points where g attains either a maximum or a minimum; we did not display all the related lines but the same pattern holds true until $N = 10$. In particular, if $n = 2$ we know that D^1 is the best reinforcement since there are parts of the truss under the two extremal points of $g(x) = \sin(2x)$, see the left-hand picture in Figure 1 where we depict the longitudinal behavior of $g(x) = \sin(2x)$ and the truss D^1 (black spots on the horizontal axis).
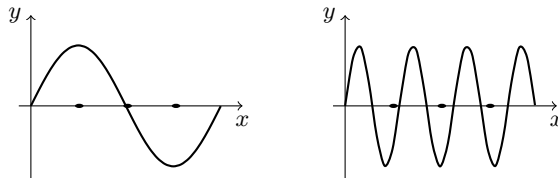

FIGURE 1. The forces $g(x) = \sin(2x)$ (left) and $g(x) = \sin(7x)$ (right) with the truss D^1.

We remark that some trusses aggravate the torsional instability, i.e., they increase the maximal gap $\overline{\mathcal{G}}_n^\infty$: this is due to a bad combination between the shape of the forcing term g and the location of D. For example, we observe that the reinforcement D^0 improves the performance when $g(x) = \sin(nx)$ with $n = 1, 2$, while for other values of n the torsional performance is worse than that of the unstiffened plate (with $D = \emptyset$). We also observe that there are some "anomalous values" of $\overline{\mathcal{G}}_n^\infty$, see e.g., the values corresponding to D^1 and $n = 7$ or $n = 9$: they are considerably larger than the other values in the same column and the reason is again that the place where D acts interacts badly with g. In particular, we notice that both $\sin(7x)$ and $\sin(9x)$ have the same sign in correspondence of $x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ that are the centers of the three vertical arms of D^1; in particular, $g(x) = \sin(7x) \implies g(\frac{\pi}{4}) = g(\frac{3\pi}{4}) = -\frac{1}{\sqrt{2}}$, $g(\frac{\pi}{2}) = -1$, see the right picture in Figure 1.

Next, we exploit Theorem 6.1 to solve analytically the maxmax problem (8) when D reduces to one horizontal bar (including the free plate). In general, maximizing a Fourier series is a tricky problem that can be solved only for particular choices of the coefficients, see e.g., [20]. This is why we focus on the set Γ of functions $g = g(x)$ satisfying one of the following:

- $g(x) = \sin(mx)$ with $m \in \mathbb{N}$.
- $g(x) = \sum_{m \geqslant N} \widetilde{\gamma}_m \sin(mx)$ with $\{\widetilde{\gamma}_m\}_{m \in \mathbb{N}} \subset \ell^2$ and $N \in \mathbb{N}$ large enough.
- $g(x) = \sin(mx) + \sin(3mx)$ with $m \in \mathbb{N}$.

- $g(x) = \sum_{m=1}^{N} \sin((2m-1)x)$ with $N \in \mathbb{N}$ sufficiently large.

Then, we define the class

$$\mathcal{F}_\Gamma := \left\{ f : f = \lim_{\alpha \to \infty} f_\alpha \text{ in } H_*^{-2}(\Omega) , \text{ with } f_\alpha \text{ as in (20) and } g \in \Gamma \right\}$$

and, in Section 10, we prove the following.

Theorem 6.2. *Let $\mathcal{F} = \mathcal{F}_\Gamma$ be as above and assume that $D =]0, \pi[\times] - \varepsilon, \varepsilon[$ for some $0 < \varepsilon < \ell$ (i.e., $\mu = 0$ in (19)). Then, the solution of the maxmax problem (8) is given by $[\lim_{\alpha \to \infty} R_\alpha \sinh(\alpha y)] \sin(x)$, where the limit is in $H_*^{-2}(\Omega)$.*

Theorem 6.2 states that the worst case as $\alpha \to \infty$ corresponds to the function $g(x) = \sin(x)$. However, $g(x) = \sin(x)$ seems not to be the worst case in general: to see this, compare the values of $\overline{\mathcal{G}}_1^\infty$ given in Table 2 with $\mathcal{G}_{T_{\pi/2}}^\infty$ given in Table 1.

7. Weakening Resonant Forces with Polygonal Reinforcements

In this section, we intend to study numerically the gap function (7) and the related minimaxmax problem (9) in the case the class $\mathcal{F}$ contains some "resonant-type" force f and the class $\mathcal{D}$ contains "not-so-nice" domains $D \in \mathcal{L}$ (see Definition 3.1). Hence, we minimize the energy (3).

Throughout this section, we fix $\ell = \pi/150$ and $\sigma = 0.2$ (two reasonable values for plates modeling the deck of a bridge, see [19]). Moreover, we assume that $\tanh(\sqrt{2}m\ell) > \sigma^2/(2-\sigma)^2\sqrt{2}m\ell$ so that $m \leqslant 2734$. Then, for any integer $m \leqslant 2734$, the first torsional eigenfunction $\overline{e}_m$ of Δ^2 with the boundary conditions in (30) having $m-1$ nodes in the x-direction and the corresponding eigenvalue ν_m are known; see [5]. Notice that $m^4 < \nu_m < \left(m^2 + \frac{\pi^2}{4\ell^2}\right)^2$. A detailed analysis of the variation of all the eigenvalues under domain deformations was performed in [21]. We aim to study the effect of a reinforcement D when the force f is at resonance, namely proportional to a torsional (odd) eigenfunction: we take $f = \overline{e}_m(x,y)$. For these functions f we then deal with problem (6) and we seek the best shape of the reinforcement D in order to lower the maximal gap $\mathcal{G}_D^\infty$. We numerically study problem (9) within classes of forces (with m from 1 to 5) and of reinforcements $\mathcal{D}$ of sets composed by two parallel strips, by triangles, by squares, and by hexagons as in Figure 2:

$$(24) \qquad \mathcal{F} = \{\overline{e}_1, ..., \overline{e}_5\} \quad \text{and} \quad \mathcal{D} = \{\text{Strips, Triangles, Squares, Hexagons}\}.$$

The black lines are the thick stiffening trusses D put below the plate and their total area is constant. More precisely, the first plate is reinforced by two parallel trusses of width $X = \frac{1046\pi}{750^2} \approx 0.00584$, thereby having a global area of $2\pi X$. The three remaining shapes all have two parallel trusses of width $W = \frac{\pi}{750}$ along the free edges of the plate for a total area of $2\pi W$, while the remaining area of $2\pi(X - W)$ is distributed in connecting transverse trusses which generate some polygons all along the plate, see again Figure 2. The triangular transverse truss is composed of 74 vertical segments having length $2\ell - 2W$ and 75 oblique segments having length $(\pi/75 - 2W)\sqrt{2}$, both having width 0.00287159 (approximately). The squared transverse truss is composed of 74 vertical segments having length
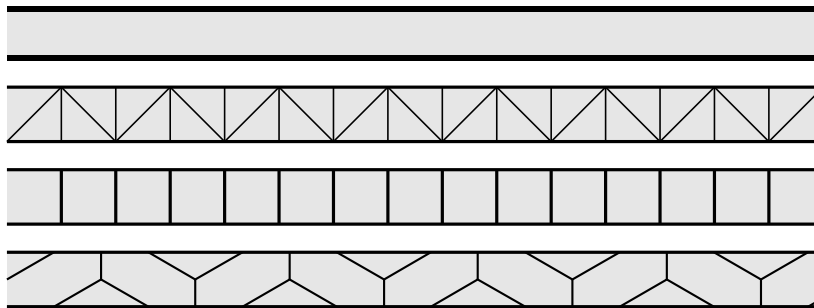
FIGURE 2. Qualitative patterns of the trusses in $\mathcal{D}$.

$2\ell - 2W$ and width W. Finally, the hexagonal transverse truss is composed of 17 Y-shaped components, alternating upwards and downwards, complemented by two segments at the opposite ends of the plate (playing the role of the oblique branches of Y), whose measures are $\ell - W$ for the length of the vertical legs and $Z = 0.0215211$ (approximated) for the width. These four designs, all belonging to the class $\mathcal{L}$ of Definition 3.1, have their own motivation. The first one is the most natural, putting reinforcements only on the two free edges. The triangular truss is the most frequently used by engineers. The third one is also natural, putting the simplest transverse connections between the free edges. Finally, a truss composed of regular hexagons was shown to have better bending performances in [22] where the "boundary effects" were neglected. In fact, what really counts is to have angles of size $2\pi/3$, as in *irrigation* or *traffic problems*, see [23–25]. Let us also mention that it has been known since the 19th century that soap bubbles reach an equilibrium on flat surfaces when the angles between three adjacent bubbles are always $2\pi/3$, see [26]. This angle has the peculiarity to "optimize the distances" and it is therefore interesting to measure its performance also in stiffening trusses. The numerical values for the maximal gap are reported in Table 3.

TABLE 3. Numerical values of $10^4 \times \mathcal{G}_{\overline{e}_m, D}^{\infty}$ for the different polygonal reinforcements D and resonant forces $\overline{e}_m$ (with $\ell = \pi/150$, $\sigma = 0.2$, and $d = 2$).

	$\overline{e}_1$	$\overline{e}_2$	$\overline{e}_3$	$\overline{e}_4$	$\overline{e}_5$
$\emptyset$	43.629	21.811	14.537	10.899	8.7147
Strips	25.448	6.3602	2.8255	1.5883	1.0157
Triangles	29.363	7.2105	3.2643	1.8409	1.1855
Squares	27.946	6.9846	3.1028	1.7442	1.1154
Hexagons	28.875	7.1787	3.2007	1.7919	1.1304

The class $\mathcal{D}$ that we have introduced here could be enlarged by considering also other geometries for D. Regarding the hexagonal design, we actually studied different positions of the intersections in the Y-shaped elements. The results

contained in Table 3 are given for elements where the intersections occur on the midline of the plate, hence with the vertical branch having length ℓ, while we performed computations also for cases where the vertical branch is longer or shorter than ℓ. Even though one might expect the gap functions to be monotone or to have a unique minimum point (with respect to the length of the vertical branch), this does not occur, the behavior being very specific depending on the particular resonant force $\overline{e}_m$ considered. In some cases, the maximal gap exits the range we saw in Table 3: for $\overline{e}_1$ and $\overline{e}_3$ the gap function is always bounded by that of the squares and that of the triangles, while for $\overline{e}_2$ the branch b of length $b = 2\ell/3$ produces a situation worse than the triangles, for $\overline{e}_4$ the cases $b = 4\ell/3, 11\ell/8$ perform better than the squares. For $\overline{e}_5$ the case $b = 23\ell/20$ performs worse than the triangles while the cases $b = 4\ell/3, 17\ell/20$ are better than the squares.

8. Proofs of the Existence Results

We first prove the continuity of the map defined in (7). We recall that in all the cases considered for the class $\mathcal{F}$, the weak* topology coincides with the weak topology, except when $\mathcal{F} = L^\infty(\Omega)$.

Proposition 8.1. *Let $\mathcal{F}$ be either $H_*^{-2}(\Omega)$ (for $\mathbb{E}_1$) or $L^p(\Omega)$ with $p \in\,]1, +\infty]$ (for both $\mathbb{E}_1$ and $\mathbb{E}_2$). Let also $\mathcal{D}$ be a class of open subdomains of Ω closed with respect to the L^1 topology. Then the map $\mathcal{G}_{f,D}^\infty : \mathcal{F} \times \mathcal{D} \to [0, \infty[$ with $(f, D) \mapsto \mathcal{G}_{f,D}^\infty$ is sequentially continuous when $\mathcal{F}$ is endowed with the weak* topology and $\mathcal{D}$ is endowed with the L^1 topology.*

Proof. Let $\{(f_n, D_n)\}_n \subset \mathcal{F} \times \mathcal{D}$ be such that $(f_n, D_n) \to (f, D)$ as $n \to +\infty$, hence $f_n \rightharpoonup^* f$ in $\mathcal{F}$ and $\chi_{D_n} \to \chi_D$ in L^1 as $n \to +\infty$. We denote by $u = u_{f,D}$ and $u_n = u_{f_n, D_n}$ the corresponding solutions of (5). Recalling (4), (5) with $f = f_n$ and $D = D_n$ reads

$$(25) \qquad (u_n, v)_{H_*^2} + d(u_n, v)_{D_n} = \langle f_n, v \rangle \qquad \forall v \in H_*^2(\Omega).$$

Since $f_n \rightharpoonup^* f$ in $\mathcal{F}$, the above equality with $v = u_n$ yields $\|u_n\|_{H_*^2} \leqslant C$ for some $C > 0$. In particular, $u_n \rightharpoonup \bar{u}$ up to a subsequence in $H_*^2(\Omega)$ for some $\bar{u} \in H_*^2(\Omega)$. Next, by adding and subtracting $d(u_n, v)_D$ in (25), we obtain that, for every $v \in H_*^2(\Omega)$

$$(26) \qquad (u_n, v)_{H_*^2} + d(u_n, v)_D + d(u_n, v)_{D_n \setminus D} - d(u_n, v)_{D \setminus D_n} = \langle f_n, v \rangle.$$

Since $\chi_{D_n} \to \chi_D$ in $L^1(\Omega)$ yields $|D_n \triangle D| \to 0$ as $n \to +\infty$, we deduce that $|(u_n, v)_{D_n \setminus D}| \leqslant C \|v\|_{H^2(D_n \setminus D)} = o(1)$ as $n \to +\infty$ and similarly $(u_n, v)_{D \setminus D_n} = o(1)$. By this, passing to the limit in (26), we conclude that $(\bar{u}, v)_{H_*^2} + d(\bar{u}, v)_D = \langle f, v \rangle$ for all $v \in H_*^2(\Omega)$; hence $\bar{u} \equiv u$. Furthermore, from the compactness of the embedding $H_*^2(\Omega) \subset C^0(\overline{\Omega})$, we obtain $u_n \to u$ in $C^0(\overline{\Omega})$. In terms of the gap functions, this means that $\mathcal{G}_{f_n, D_n}(x)$ converges uniformly to $\mathcal{G}_{f,D}(x)$ as $n \to +\infty$ over $[0, \pi]$. In particular, $\mathcal{G}_{f_n, D_n}^\infty \to \mathcal{G}_{f,D}^\infty$ as $n \to +\infty$. This concludes the proof. $\square$

Theorem 3.1. Fix $D \subset \Omega$. If $\{f_n\} \subset H_*^{-2}(\Omega)$ is a maximizing sequence for (10), since $\|f_n\|_{H_*^{-2}} = 1$, up to a subsequence, we have $f_n \rightharpoonup \overline{f}$ in $H_*^{-2}(\Omega)$. By Proposition 8.1, $\max \{\mathcal{G}_{f,D}^\infty : f \in H_*^{-2}(\Omega), \|f\|_{H_*^{-2}} = 1\} = \mathcal{G}_{\overline{f},D}^\infty$. Moreover, it must be $\|\overline{f}\|_{H_*^{-2}} = 1$. Otherwise, if $\|\overline{f}\|_{H_*^{-2}} < 1$, set $\widetilde{f} = \overline{f}/\|\overline{f}\|_{H_*^{-2}}$ and by linearity we get $\mathcal{G}_{\widetilde{f},D}^\infty = \mathcal{G}_{\overline{f},D}^\infty/\|\overline{f}\|_{H_*^{-2}} > \mathcal{G}_{\overline{f},D}^\infty$, a contradiction that proves the first part of Theorem 3.1.

Now, let $\{f_n\} \subset L^p(\Omega)$ be a maximizing sequence for (11) such that $\|f_n\|_{L^p} = 1$. Up to a subsequence and for some $\overline{f}$, we have $f_n \rightharpoonup \overline{f}$ in $L^p(\Omega)$ if $1 < p < \infty$ and $f_n \rightharpoonup^* \overline{f}$ in $L^\infty(\Omega)$. In particular, by lower semicontinuity of the norms with respect to these convergences, $\|\overline{f}\|_{L^p} \leqslant \|f_n\|_{L^p} = 1$. Moreover, by Proposition 8.1, we have $\max \{\mathcal{G}_{f,D}^\infty : f \in L^p(\Omega) \text{ with } \|f\|_{L^p} = 1\} = \mathcal{G}_{\overline{f},D}^\infty$. Finally, the proof that $\|\overline{f}\|_{L^p} = 1$ follows by arguing as above. $\qquad\qquad\square$

Theorem 3.2. Using the Direct Method of the Calculus of Variations, it is sufficient to find a topology for which the functional $D \mapsto \mathcal{G}_D^\infty$ defined in (10) is lower semicontinuous while the class of admissible sets $\mathcal{D}$ is compact. For this purpose we use the L^1-convergence of sets, namely the L^1-convergence of the characteristic functions associated to the sets. Indeed, by its definition (10) and the continuity proved in Proposition 8.1, it follows that the functional $\mathcal{G}^\infty$ is lower-semicontinuous with respect to the L^1-convergence of sets. Therefore, it remains to prove that the classes introduced in Definition 3.1 are compact with respect to this convergence: we do it for each class.

(a) Consider a sequence of crosses $\{D_n\}$ in $\mathcal{C}$: by the Bolzano-Weierstrass Theorem the sequences of points $\{x_n^i\}$ and $\{y_n^i\}$ converge, up to subsequences, to some $x^i \in [\mu, \pi - \mu]$, $i = 1, \ldots, N$, and some $y^j \in [-\ell + \varepsilon, \ell - \varepsilon]$, $j = 1, \ldots, M$, respectively. By the Lebesgue Dominated Convergence Theorem, it turns out that $|D_n \triangle D| \to 0$ as $n \to \infty$ where D is the cross

$$\Big(\bigcup_{i=1}^N (x^i - \mu, x^i + \mu \times] - \ell, \ell[\Big) \cup \Big(\bigcup_{j=1}^M]0, \pi[\times] y^j - \varepsilon, y^j + \varepsilon[\Big) ;$$

this means that $\chi_{D_n} \to \chi_D$ in $L^1(\Omega)$ as $n \to \infty$. Moreover, $|D| = \kappa$, thanks to the area constraint. Therefore, the class $\mathcal{C}$ with area constraint is compact with respect to the L^1-convergence of sets.

(b) To each rectangle $R \subset \Omega$ we associate its four vertices $V_1(R), \ldots, V_4(R)$ in such a way that $V_1(R)$ is the upper-right vertex (i.e., the one with largest y-coordinate in the case such a vertex is unique, otherwise the one with largest x-coordinate) and the remaining $V_i(R)$ are ordered clockwise. Consider a sequence of rectangles $\{R_n\}$ all having inradius at least ε: by the Bolzano-Weierstrass Theorem, up to extracting a subsequence (that we do not relabel), the sequence of vertices $\{V_1(R_n)\}$ converges to some point $V_1 \in \overline{\Omega}$. Up to extracting a further subsequence, the sequence of vertices $\{V_2(R_n)\}$ also converges to some point $V_2 \in \overline{\Omega}$. Repeating this argument for the remaining vertices, we infer that each of the four sequences of vertices $\{V_i(R_n)\}$ converges, up to subsequences, to some point $V_i \in \overline{\Omega}$ (for $i = 1, 2, 3, 4$). Let R be the open convex hull of the four

points $V_1, \ldots, V_4$; since, by construction, the scalar product of two consecutive sides is $(V_i(R_n)V_{i+1}(R_n), V_{i+1}(R_n)V_{i+2}(R_n)) = 0$ for $i = 1, 2, 3, 4$, where we set $V_5(R) := V_1(R)$ and $V_6(R) := V_2(R)$, passing to the limit as $n \to \infty$, and using the continuity of the scalar product it follows that $(V_i V_{i+1}, V_{i+1} V_{i+2}) = 0$ (for $i = 1, 2, 3, 4$, where $V_5 := V_1$ and $V_6 := V_2$). Moreover, since the distance between two consecutive vertices of R_n is larger than 2ε for all n, also the inradius of R is at least ε. This implies that the set R is an open rectangle having the distinct vertices $V_i(R) = V_i$ (for $i = 1, 2, 3, 4$). Moreover, by the Lebesgue Dominated Convergence Theorem, it turns out that $|R_n \triangle R| \to 0$ as $n \to \infty$; this means that $\chi_{R_n} \to \chi_R$ in $L^1(\Omega)$ as $n \to \infty$.

Then take a sequence of sets $D_n \in \mathcal{T}$ with $D_n = \cup_{i=1}^N R_n^i$. Using the argument above, up to subsequences, we have that $|R_n^i \triangle R^i| \to 0$ as $n \to \infty$ for some rectangles R^i all having inradius at least ε. Hence, $\chi_{R_n^i} \to \chi_{R^i}$ in $L^1(\Omega)$ for all $i = 1, ..., N$ and, in turn, $\chi_{D_n} \to \chi_D$ in $L^1(\Omega)$. The area constraint yields that $|D| = \kappa$. Therefore, the class $\mathcal{T}$ with area constraint is compact with respect to the L^1-convergence of sets.

(c) Let Σ_n be a sequence of closed connected sets with $\mathcal{H}^1(\Sigma_n) \leqslant L$. From the Blaschke Selection Theorem and the Gołab Theorem (see e.g. [27, Theorem 4.4.17]), up to a subsequence we know that $\Sigma_n \to \Sigma$ with respect to the Hausdorff distance, where Σ is a closed and connected set with $\mathcal{H}^1(\Sigma) \leqslant L$. Then the distance function to Σ_n converges to the distance function to Σ uniformly on $\overline{\Omega}$. This, with the fact that the Lebesgue measure of the set $\partial K^\varepsilon = \{x \in \Omega : \text{dist}_K = \varepsilon\}$ is zero, implies that K_n^ε converges in L^1 to K^ε (see [28]).

(d) Using again [28, Theorem 2.4.10] we obtain the compactness with respect to the L^1 convergence of the space $\mathcal{L}$ with area constraint.   □

9. Proofs of the Symmetry Results

Theorem 4.1. Let $f \in H_*^{-2}(\Omega)$ be such that $\|f\|_{H_*^{-2}} = 1$ and consider the solution $u_f \in H_*^2(\Omega)$ of (5). Since D is symmetric, following the decomposition (13) we may rewrite (5) as

$$(27) \quad (u_f^e, v^e)_{H_*^2} + (u_f^o, v^o)_{H_*^2} + d(u_f^e, v^e)_D + d(u_f^o, v^o)_D = \langle f^e, v^e \rangle + \langle f^o, v^o \rangle,$$

for all $v \in H_*^2(\Omega)$. Moreover, by (7), we have $\mathcal{G}_{f,D}(x) = u_f^o(x, \ell) - u_f^o(x, -\ell)$ and also that $\mathcal{G}_{f,D}^\infty = \max_{x \in [0,\pi]} |u_f^o(x, \ell) - u_f^o(x, -\ell)|$. In particular, if $f^o = 0$ then $u^o = 0$ and $\mathcal{G}_{f,D}^\infty = 0$ so that f cannot be a maximizer for $\mathcal{G}_{f,D}^\infty$. Hence, by (14), there exists $0 < \alpha \leqslant 1$ such that $\alpha = \|f^o\|_{H_*^{-2}} \leqslant \|f\|_{H_*^{-2}} = 1$. Consider now the problem $(w, v)_{H_*^2} + d(w, v)_D = \frac{1}{\alpha} \langle f^o, v \rangle$ for all $v \in H_*^2(\Omega)$. By linearity and by (27), its solution is $w = u^o/\alpha$, then $\mathcal{G}_{\frac{f^o}{\alpha},D}(x) = \frac{1}{\alpha} \mathcal{G}_{f,D}(x)$ and $\mathcal{G}_{\frac{f^o}{\alpha},D}^\infty = \frac{1}{\alpha} \mathcal{G}_{f,D}^\infty \geqslant \mathcal{G}_{f,D}^\infty$.

Hence, we have shown that for all $f \in H_*^{-2}(\Omega)$ such that $\|f\|_{H_*^{-2}} = 1$, there exists $g \in H_{\mathcal{O}}^{-2}(\Omega)$ such that $\|g\|_{H_*^{-2}} = 1$ ($g = f^o/\alpha$) and $\mathcal{G}_{g,D}^\infty \geqslant \mathcal{G}_{f,D}^\infty$. This proves the first part of Theorem 4.1.

The remaining part of Theorem 4.1 follows the inverse path. Let f be as in the statement and take any $\phi \in H_{\mathcal{E}}^{-2}(\Omega)$ such that $\|\phi\|_{H_*^{-2}} \leqslant 1$. Then, put $g = f + \phi$

so that $g^o = f$ and $g^e = \phi$. By (14) we have $\|g\|_{H_*^{-2}} = 1$. By slightly modifying the arguments above we see that $\mathcal{G}_{g,D}^\infty = \mathcal{G}_{f,D}^\infty$. $\qquad\square$

For the proof of Theorem 4.2 we need the following result.

Lemma 9.1. *Let* $1 \leqslant p \leqslant \infty$ *and* $a > 0$. *If* $\phi \in L^p(]-a,a[)$ *then*

$$\|\phi^o\|_{L^p} \leqslant \|\phi\|_{L^p}. \tag{28}$$

Moreover:

– if $p = 1$ *then the inequality in* (28) *is strict if and only if* $|\phi^o(x)| < |\phi^e(x)|$ *in a subset of* $]-a,a[$ *of positive measure;*

– if $1 < p < \infty$ *then the inequality in* (28) *is strict if and only if* ϕ *is not odd* $(\phi \not\equiv \phi^o)$;

– if $p = \infty$ *then the inequality in* (28) *is strict if and only if for any* $\{x_n\} \subset$ $]-a,a[$ *such that* $|\phi(x_n)| \to \|\phi\|_{L^\infty}$, *one has* $\liminf_n |\phi^e(x_n)| > 0$; *in particular, if* $\phi \in C^0[-a,a]$, *then the inequality is strict if and only if* $\phi^e(\overline{x}) \neq 0$ *in every point* $\overline{x}$ *where* $|\phi|$ *attains its maximum.*

Proof. Since $\phi^o(x) = \frac{\phi(x)-\phi(-x)}{2}$, the inequality (28) follows from the Minkowski inequality and the symmetry of $]-a,a[$.

If $p = 1$, then the Minkowski inequality, just used to obtain (28), reads

$$\int_{-a}^a |\phi^o(x)|\, dx = \frac{1}{2}\int_{-a}^a |\phi(x)-\phi(-x)|\, dx \leqslant \int_{-a}^a \tfrac{|\phi(x)|+|\phi(-x)|}{2}\, dx = \int_{-a}^a |\phi(x)|\, dx$$

so that it reduces to an equality if and only if

$$0 \geqslant \phi(x)\phi(-x) = \Big[\phi^e(x)+\phi^o(x)\Big]\Big[\phi^e(-x)+\phi^o(-x)\Big] = \phi^e(x)^2 - \phi^o(x)^2$$

for a.e. $x \in]-a,a[$. This means that $|\phi^e(x)| \leqslant |\phi^o(x)|$ for a.e. $x \in]-a,a[$. Since this is a necessary and sufficient condition, the statement for $p = 1$ is proved.

If $p \in]1,+\infty[$, the Minkowski inequality is itself obtained via an application of Hölder's inequality and equality holds if and only if the two involved functions are multiples of each other. In the present situation, this means that $\phi(x) = \alpha\phi(-x)$ for some $\alpha = \alpha(p) < 0$ and for a.e. $x \in]-a,a[$. The only possibility is that $\alpha = -1$, which means that $\phi = \phi^o$ and $\phi^e \equiv 0$. Since this is a necessary and sufficient condition, also the statement for $p > 1$ is proved.

If $p = \infty$, we claim that equality holds in (28) if and only if there exists $\{x_n\} \subset$ $]-a,a[$ such that $|\phi(x_n)| \to \|\phi\|_{L^\infty}$ and $\phi^e(x_n) \to 0$. Indeed, if such a sequence exists, then $|\phi^o(x_n)| = |\phi(x_n) - \phi^e(x_n)| \to \|\phi\|_{L^\infty}$ which proves that $\|\phi^o\|_{L^\infty} = \|\phi\|_{L^\infty}$. Conversely, if equality holds then there exists $\{x_n\} \subset]-a,a[$ such that $\phi^o(x_n) \to \|\phi\|_{L^\infty}$. This yields $\phi(x_n) - \phi^e(x_n) = \phi^o(x_n) \to \|\phi\|_{L^\infty}$ and $\phi^e(x_n) - \phi(-x_n) = -\phi^o(-x_n) = \phi^o(x_n) \to \|\phi\|_{L^\infty}$, which proves that $\phi^e(x_n) \to 0$ since otherwise one of $|\phi(\pm x_n)|$ would tend to exceed $\|\phi\|_{L^\infty}$. The claim is so proved and therefore the strict inequality occurs in the opposite situation: this proves the first statement.

In the case, where $\phi \in C^0([-a,a])$, the sequences just used to prove the statement may be replaced by their limits. $\qquad\square$

Theorem 4.2. For every $p \in]1, \infty]$, (28) combined with the argument in the proof of Theorem 4.1 yields that a maximizer f can be sought as an odd function.

If $1 < p < \infty$, by contradiction, let $f \in L^p(\Omega)$ such that $\|f\|_{L^p} = 1$ be a non-odd maximizer. Since $f \not\equiv f^o$, by Lemma 9.1 $\|f^o\|_{L^p} < \|f\|_{L^p} = 1$. Take now $\overline{f} = f^o / \|f^o\|_{L^p}$, recalling that f^e plays no role in the value of the gap function, we obtain $\|\overline{f}\|_{L^p} = 1$ and $\mathcal{G}^\infty_{\overline{f}, D} = \mathcal{G}^\infty_{f, D} / \|f^o\|_{L^p} > \mathcal{G}^\infty_{f, D}$, a contradiction.

If $p = \infty$, take an odd function $f \in L^\infty(\Omega)$ such that $\|f\|_{L^\infty} = 1$ and $\mathcal{G}^\infty_{f, D} = \max \left\{ \mathcal{G}^\infty_{\phi, D} \, : \, \phi \in L^\infty(\Omega) \text{ with } \|\phi\|_{L^\infty} = 1 \right\}$. If $|f(x, y)| < 1$ on a subset $\omega \subset \Omega$ of positive measure, take any even function h such that $h(x, y) \equiv 0$ in $\Omega \setminus \omega$ and $|h(x, y)| < 1 - |f(x, y)|$ in ω. Then, $g = f + h$ is not odd and satisfies $\|g\|_{L^\infty} = 1$, $g^e = h \neq 0$, and $\mathcal{G}^\infty_{g, D} = \mathcal{G}^\infty_{f, D}$ (by linearity since $\mathcal{G}^\infty_{h, D} = 0$). □

10. Proofs of Theorems 6.1 and 6.2

We prove Theorem 6.1 and Theorem 6.2 in several steps. Let $g \in L^2(]0, \pi[)$, $\alpha \geqslant 0$ with $\alpha \notin \mathbb{N}$ and

$$(29) \quad k_\alpha(x, y) = K_\alpha e^{\alpha y} g(x), \quad K_\alpha := \frac{\alpha}{2 C_g \sinh(\alpha \ell)} \text{ and } C_g := \int_0^\pi |g(x)| \, dx,$$

so that $\|k_\alpha\|_{L^1} = 1$. Let $h_\alpha := \frac{k_\alpha}{1 + d\chi_{D^N}}$; we first focus on the *auxiliary problem*

$$(30) \quad \begin{cases} \Delta^2 w = h_\alpha, & \text{in } \Omega, \\ w = w_{xx} = 0, & \text{on } \{0, \pi\} \times] - \ell, \ell[, \\ w_{yy} + \sigma w_{xx} = w_{yyy} + (2 - \sigma) w_{xxy} = 0, & \text{on }]0, \pi[\times \{-\ell, \ell\}. \end{cases}$$

Indeed, if w_α solves (30), then a multiple of its odd part

$$u_\alpha(x, y) := \frac{R_\alpha}{K_\alpha} w_\alpha^o(x, y) = \frac{\sinh(\alpha \ell)}{2(\cosh(\alpha \ell) - 1)} \left(w_\alpha(x, y) - w_\alpha(x, -y) \right)$$

solves problem (6) with $f = f_\alpha$ and $D = D^N$. Moreover, if $\mathcal{G}_\alpha(x)$ is the gap function corresponding to w_α, then $\frac{R_\alpha}{K_\alpha} \mathcal{G}_\alpha(x)$ is the gap function corresponding to u_α. Therefore, since $\frac{R_\alpha}{K_\alpha} = 1 + 2 e^{-\alpha \ell} + o(e^{-\alpha \ell})$ as $\alpha \to +\infty$, the limit of the gap function corresponding to u_α and the asymptotic behavior of the corresponding coefficients are exactly the same as those for w_α.

Now, we focus on the explicit solution of the auxiliary problem. We expand $g \in L^2(]0, \pi[)$ in a Fourier series

$$(31) \quad g(x) = \sum_{m=1}^\infty \widetilde{\gamma}_m \sin(mx), \quad \widetilde{\gamma}_m = \frac{2}{\pi} \int_0^\pi g(x) \sin(mx) \, dx.$$

Then, if we set

$$(32) \quad I^N := \bigcup_{i=1}^{2N+1} \left(\frac{\pi i}{2N + 2} - \frac{\mu}{2N + 1}, \frac{\pi i}{2N + 2} + \frac{\mu}{2N + 1} \right),$$

for every $x \in]0, \pi[$ and every $y \in]-\ell, -\varepsilon[\cup]\varepsilon, \ell[$, we have

$$(33) \qquad \frac{g(x)}{1+d\chi_{D^N}(x,y)} = \frac{g(x)}{1+d\chi_{I^N}(x)} = \sum_{m=1}^{\infty} \gamma_m \sin(mx) \,,$$

where the coefficients γ_m are as defined in (22), while if we set $\widehat{\gamma}_m = \frac{\widetilde{\gamma}_m}{1+d}$, we have $\frac{g(x)}{1+d\chi_{D^N}(x,y)} = \frac{g(x)}{1+d} = \sum_{m=1}^{\infty} \widehat{\gamma}_m \sin(mx)$ for all $x \in]0, \pi[$ and $y \in]-\varepsilon, \varepsilon[$.

In the sequel, we will need the following constants (only depending on m, α and ε):

$$F_1(\varepsilon) := \frac{\alpha(\alpha^2 - 3m^2)\sinh(m\varepsilon) + 2m^3\cosh(m\varepsilon)}{2m^3} + (\alpha^2 - m^2)\frac{m\sinh(m\varepsilon) - \alpha\cosh(m\varepsilon)}{2m^2}\varepsilon \,,$$

$$F_2(\varepsilon) := -\frac{\alpha(\alpha^2 - 3m^2)\cosh(m\varepsilon) + 2m^3\sinh(m\varepsilon)}{2m^3} - (\alpha^2 - m^2)\frac{m\cosh(m\varepsilon) - \alpha\sinh(m\varepsilon)}{2m^2}\varepsilon \,,$$

$$F_3(\varepsilon) := (\alpha^2 - m^2)\frac{\alpha\cosh(m\varepsilon) - m\sinh(m\varepsilon)}{2m^2} \,,$$

$$(34) \qquad F_4(\varepsilon) := -(\alpha^2 - m^2)\frac{\alpha\sinh(m\varepsilon) - m\cosh(m\varepsilon)}{2m^2} \,,$$

$$(35) \qquad \begin{aligned} F_1^{\pm}(\varepsilon) &:= F_1(\varepsilon) \pm e^{-2\alpha\varepsilon}F_1(-\varepsilon) \,, & F_2^{\pm}(\varepsilon) &:= F_2(\varepsilon) \pm e^{-2\alpha\varepsilon}F_2(-\varepsilon) \,, \\ F_3^{\pm}(\varepsilon) &:= F_3(\varepsilon) \pm e^{-2\alpha\varepsilon}F_3(-\varepsilon) \,, & F_4^{\pm}(\varepsilon) &:= F_4(\varepsilon) \pm e^{-2\alpha\varepsilon}F_4(-\varepsilon) \,, \end{aligned}$$

$$(36) \qquad a = a(m, \alpha, \varepsilon) := K_\alpha e^{\alpha\varepsilon}\frac{\gamma_m - \widehat{\gamma}_m}{(m^2 - \alpha^2)^2} \,,$$

$$\begin{aligned} G_1 &:= -\tfrac{a}{2}\big\{(1-\sigma)m^2\cosh(m\ell)F_1^+(\varepsilon) + m[2\cosh(m\ell) + (1-\sigma)m\ell\sinh(m\ell)]F_4^+(\varepsilon) \\ &\quad + (1-\sigma)m^2\sinh(m\ell)F_2^-(\varepsilon) + m[2\sinh(m\ell) + (1-\sigma)m\ell\cosh(m\ell)]F_3^-(\varepsilon)\big\} \\ G_2 &:= \tfrac{a}{2}\big\{(1-\sigma)m^3\cosh(m\ell)F_2^-(\varepsilon) - m^2[(1+\sigma)\cosh(m\ell) - (1-\sigma)m\ell\sinh(m\ell)]F_3^-(\varepsilon) \\ &\quad + (1-\sigma)m^3\sinh(m\ell)F_1^+(\varepsilon) - m^2[(1+\sigma)\sinh(m\ell) - (1-\sigma)m\ell\cosh(m\ell)]F_4^+(\varepsilon)\big\} \,, \\ G_3 &:= -\tfrac{a}{2}\big\{(1-\sigma)m^2\cosh(m\ell)F_1^-(\varepsilon) + m[2\cosh(m\ell) + (1-\sigma)m\ell\sinh(m\ell)]F_4^-(\varepsilon) \\ &\quad + (1-\sigma)m^2\sinh(m\ell)F_2^+(\varepsilon) + m[2\sinh(m\ell) + (1-\sigma)m\ell\cosh(m\ell)]F_3^+(\varepsilon)\big\} \,, \\ G_4 &:= \tfrac{a}{2}\big\{(1-\sigma)m^3\cosh(m\ell)F_2^+(\varepsilon) - m^2[(1+\sigma)\cosh(m\ell) - (1-\sigma)m\ell\sinh(m\ell)]F_3^+(\varepsilon) \\ (37)&\quad + (1-\sigma)m^3\sinh(m\ell)F_1^-(\varepsilon) - m^2[(1+\sigma)\sinh(m\ell) - (1-\sigma)m\ell\cosh(m\ell)]F_4^-(\varepsilon)\big\} \,. \end{aligned}$$

Then, we set

$$C_2 := \frac{m\cosh(m\ell)\left(K_\alpha\gamma_m\frac{\sigma m^2 - \alpha^2}{(m^2 - \alpha^2)^2}\sinh(\alpha\ell) + G_3\right) + \sinh(m\ell)\left(\alpha K_\alpha\gamma_m\frac{(2-\sigma)m^2 - \alpha^2}{(m^2 - \alpha^2)^2}\cosh(\alpha\ell) + G_4\right)}{m^2[(3+\sigma)\sinh(m\ell)\cosh(m\ell) + (1-\sigma)m\ell]} \,,$$

$$D_2 := \frac{m\sinh(m\ell)\left(K_\alpha\gamma_m\frac{\sigma m^2 - \alpha^2}{(m^2 - \alpha^2)^2}\cosh(\alpha\ell) + G_1\right) + \cosh(m\ell)\left(\alpha K_\alpha\gamma_m\frac{(2-\sigma)m^2 - \alpha^2}{(m^2 - \alpha^2)^2}\sinh(\alpha\ell) + G_2\right)}{m^2[(3+\sigma)\sinh(m\ell)\cosh(m\ell) - (1-\sigma)m\ell]} \,,$$

$$A_2 := \frac{D_2 m^2[(1+\sigma)\sinh(m\ell) - (1-\sigma)m\ell\cosh(m\ell)] - \alpha K_\alpha\gamma_m\frac{(2-\sigma)m^2 - \alpha^2}{(m^2 - \alpha^2)^2}\sinh(\alpha\ell) - G_2}{(1-\sigma)m^3\sinh(m\ell)} \,,$$

$$(38) \qquad B_2 := \frac{C_2 m^2[(1+\sigma)\cosh(m\ell) - (1-\sigma)m\ell\sinh(m\ell)] - \alpha K_\alpha\gamma_m\frac{(2-\sigma)m^2 - \alpha^2}{(m^2 - \alpha^2)^2}\cosh(\alpha\ell) - G_4}{(1-\sigma)m^3\cosh(m\ell)} \,,$$

and

$$(39) \qquad \begin{aligned} A_1 &:= A_2 + aF_1(\varepsilon) \,, & A_3 &:= A_2 + ae^{-2\alpha\varepsilon}F_1(-\varepsilon) \,, \\ B_1 &:= B_2 + aF_2(\varepsilon) \,, & B_3 &:= B_2 + ae^{-2\alpha\varepsilon}F_2(-\varepsilon) \,, \\ C_1 &:= C_2 + aF_3(\varepsilon) \,, & C_3 &:= C_2 + ae^{-2\alpha\varepsilon}F_3(-\varepsilon) \,, \\ D_1 &:= D_2 + aF_4(\varepsilon) \,, & D_3 &:= D_2 + ae^{-2\alpha\varepsilon}F_4(-\varepsilon) \,. \end{aligned}$$

The following statement allows us to determine the explicit solution of (30).

Proposition 10.1. *Assume that* $g \in L^2(]0, \pi[)$ *satisfies* (31). *For* $\alpha \geqslant 0$ *with* $\alpha \notin \mathbb{N}$, *let* K_α *be as in* (29). *Then, the unique solution of* (30) *is given by*

$$w_\alpha(x, y) = \begin{cases} w_1(x, y), & in \ ]0, \pi[\times]\varepsilon, \ell[, \\ w_2(x, y), & in \ ]0, \pi[\times]-\varepsilon, \varepsilon[, \\ w_3(x, y), & in \ ]0, \pi[\times]-\ell, -\varepsilon[. \end{cases}$$

with, for $i = 1, \ldots, 3$,

$$w_i(x, y) := \sum_{m=1}^{\infty} \left((A_i + C_i y) \cosh(my) + (B_i + D_i y) \sinh(my) + \frac{e^{\alpha y} K_\alpha \gamma_m^i}{(m^2 - \alpha^2)^2} \right) \sin(mx)$$

and the constants $A_i = A_i(m, \alpha, \varepsilon)$, $B_i = B_i(m, \alpha, \varepsilon)$, $C_i = C_i(m, \alpha, \varepsilon)$, *and* $D_i = D_i(m, \alpha, \varepsilon)$ *as defined in* (38) *and* (39), *while* $\gamma_m^1 = \gamma_m^3 = \gamma_m$ *and* $\gamma_m^2 = \widehat{\gamma}_m$.

Proof. In order to solve the problem, we split the domain Ω into three rectangles: $R_1 :=]0, \pi[\times]\varepsilon, \ell[$, $R_2 :=]0, \pi[\times]-\varepsilon, \varepsilon[$, $R_3 :=]0, \pi[\times]-\ell, -\varepsilon[$, so that we obtain

$$(40) \quad \begin{cases} \Delta^2 w_1 = K_\alpha e^{\alpha y} g(x)(1 + d\chi_{I^N}(x))^{-1}, & in \ R_1, \\ w_1 = (w_1)_{xx} = 0, & on \ \{0, \pi\} \times]\varepsilon, \ell[, \\ (w_1)_{yy} + \sigma(w_1)_{xx} = (w_1)_{yyy} + (2 - \sigma)(w_1)_{xxy} = 0, & on \ ]0, \pi[\times\{\ell\}, \end{cases}$$

$$(41) \quad \begin{cases} \Delta^2 w_2 = K_\alpha e^{\alpha y} g(x)(1 + d)^{-1}, & in \ R_2, \\ w_2 = (w_2)_{xx} = 0, & on \ \{0, \pi\} \times]-\varepsilon, \varepsilon[, \end{cases}$$

$$(42) \quad \begin{cases} \Delta^2 w_3 = K_\alpha e^{\alpha y} g(x)(1 + d\chi_{I^N}(x))^{-1}, & in \ R_3, \\ w_3 = (w_3)_{xx} = 0, & on \ \{0, \pi\} \times]-\ell, \varepsilon[, \\ (w_3)_{yy} + \sigma(w_3)_{xx} = (w_3)_{yyy} + (2 - \sigma)(w_3)_{xxy} = 0, & on \ ]0, \pi[\times\{-\ell\}, \end{cases}$$

where I^N is as defined in (32). We also have to add the junction conditions:

$$(43) \quad w_1 = w_2, \quad (w_1)_y = (w_2)_y, \quad (w_1)_{yy} = (w_2)_{yy}, \quad (w_1)_{yyy} = (w_2)_{yyy}, \quad in \ ]0, \pi[\times\{\varepsilon\}$$

$$(44) \quad w_2 = w_3, \quad (w_2)_y = (w_3)_y, \quad (w_2)_{yy} = (w_3)_{yy}, \quad (w_2)_{yyy} = (w_3)_{yyy}, \quad in \ ]0, \pi[\times\{-\varepsilon\}$$

Let $\phi \in H^4(]0, \pi[)$ be the unique solution of

$$(45) \quad \begin{cases} \phi''''(x) + 2\alpha^2 \phi''(x) + \alpha^4 \phi(x) = g(x)(1 + d\chi_{I^N}(x))^{-1}, & x \in]0, \pi[, \\ \phi(0) = \phi(\pi) = \phi''(0) = \phi''(\pi) = 0. \end{cases}$$

By (33), and recalling that $\alpha \notin \mathbb{N}$, ϕ may be written as

$$\phi(x) = \sum_{m=1}^{\infty} \frac{\gamma_m}{(m^2 - \alpha^2)^2} \sin(mx), \quad x \in]0, \pi[,$$

with the γ_m as defined in (22). Moreover, $\phi'' \in H^2(]0, \pi[)$ is given by

$$\phi''(x) = -\sum_{m=1}^{\infty} \frac{\gamma_m \, m^2}{(m^2 - \alpha^2)^2} \sin(mx), \quad x \in]0, \pi[,$$

and this series converges in $H^2(]0,\pi[)$ and, hence, uniformly. We will also need the constants

$$\zeta_m^1 := \gamma_m \frac{\sigma m^2 - \alpha^2}{(m^2 - \alpha^2)^2}, \qquad \zeta_m^2 := \gamma_m \frac{(2-\sigma)m^2 - \alpha^2}{(m^2 - \alpha^2)^2}.$$

Let us now restrict our attention to R_1. By the system (45), we have that $\Delta^2[K_\alpha e^{\alpha y}\phi(x)] = K_\alpha e^{\alpha y} g(x)(1 + d\chi_{I^N}(x))^{-1}$. Hence, if we introduce the auxiliary function $v_1(x,y) := w_1(x,y) - K_\alpha e^{\alpha y}\phi(x)$ with w_1 solving (40), we see that v_1 solves

$$(46) \quad \begin{cases} \Delta^2 v_1 = 0, & \text{in } R_1, \\ v_1 = (v_1)_{xx} = 0, & \text{on } \{0,\pi\}\times]\varepsilon,\ell[, \\ (v_1)_{yy} + \sigma(v_1)_{xx} = -K_\alpha e^{\alpha\ell}[\alpha^2\phi + \sigma\phi''], & \text{on }]0,\pi[\times\{\ell\}, \\ (v_1)_{yyy} + (2-\sigma)(v_1)_{xxy} = -K_\alpha \alpha e^{\alpha\ell}[\alpha^2\phi + (2-\sigma)\phi''], & \text{on }]0,\pi[\times\{\ell\}. \end{cases}$$

We seek solutions of (46) by separating variables, namely we seek functions $Y_m^1 = Y_m^1(y)$ such that $v_1(x,y) = \sum_{m=1}^{\infty} Y_m^1(y)\sin(mx)$ solves (46). Then

$$\Delta^2 v_1(x,y) = \sum_{m=1}^{\infty} [(Y_m^1)''''(y) - 2m^2(Y_m^1)''(y) + m^4(Y_m^1)(y)]\sin(mx),$$

and the equation in (46) yields

$$(47) \qquad (Y_m^1)''''(y) - 2m^2(Y_m^1)''(y) + m^4 Y_m^1(y) = 0, \quad \text{for } y \in]\varepsilon,\ell[.$$

The solutions of (47) are linear combinations of $\cosh(my)$, $\sinh(my)$, $y\cosh(my)$, $y\sinh(my)$, that is, $Y_m^1(y) = (A_1 + C_1 y)\cosh(my) + (B_1 + D_1 y)\sinh(my)$, where A_1, B_1, C_1, D_1 will be determined by imposing the boundary conditions in (43) and (46). By differentiating we obtain

$$(Y_m^1)'(y) = (A_1 m + D_1 + C_1 my)\sinh(my) + (B_1 m + C_1 + D_1 my)\cosh(my) + \sinh(my),$$

$$(Y_m^1)''(y) = (A_1 m^2 + 2D_1 m + C_1 m^2 y)\cosh(my) + (B_1 m^2 + 2C_1 m + D_1 m^2 y)\sinh(my),$$

$$(Y_m^1)'''(y) = (A_1 m^3 + 3D_1 m^2 + C_1 m^3 y)\sinh(my) + (B_1 m^3 + 3C_1 m^2 + D_1 m^3 y)\cosh(my).$$

The two boundary conditions on $]0,\pi[\times\{\varepsilon,\ell\}$, see (46), become respectively

$$\sum_{m=1}^{\infty} [(Y_m^1)''(\ell) - \sigma m^2 Y_m^1(\ell)]\sin(mx) = -K_\alpha e^{\alpha\ell}[\alpha^2\phi(x) + \sigma\phi''(x)],$$

$$\sum_{m=1}^{\infty} [(Y_m^1)'''(\ell) - (2-\sigma)m^2(Y_m^1)'(\ell)]\sin(mx) = -K_\alpha e^{\alpha\ell}[\alpha^2\phi(x) + (2-\sigma)\phi''(x)],$$

for all $x \in]0,\pi[$. Hence, from the Fourier expansion of ϕ, we deduce that $(Y_m^1)''(\ell) - \sigma m^2 Y_m^1(\ell) = K_\alpha e^{\alpha\ell}\zeta_m^1$, $(Y_m^1)'''(\ell) - (2-\sigma)m^2(Y_m^1)'(\ell) = K_\alpha e^{\alpha\ell}\alpha\zeta_m^2$. By plugging this information into the explicit form of the derivatives of Y_m^1 we find the system

$$(1-\sigma)m^2\cosh(m\ell)A_1 + m[2\cosh(m\ell) + (1-\sigma)m\ell\sinh(m\ell)]D_1 + (1-\sigma)m^2\sinh(m\ell)B_1$$
$$+m[2\sinh(m\ell) + (1-\sigma)m\ell\cosh(m\ell)]C_1 = K_\alpha e^{\alpha\ell}\zeta_m^1,$$
$$-(1-\sigma)m^3\cosh(m\ell)B_1 + m^2[(1+\sigma)\cosh(m\ell) - (1-\sigma)m\ell\sinh(m\ell)]C_1$$
$$-(1-\sigma)m^3\sinh(m\ell)A_1 + m^2[(1+\sigma)\sinh(m\ell) - (1-\sigma)m\ell\cosh(m\ell)]D_1 = K_\alpha e^{\alpha\ell}\alpha\zeta_m^2.$$

Similarly, for R_3 we introduce the function $v_3(x,y) := w_3(x,y) - K_\alpha e^{\alpha y}\phi(x)$, with w_3 solving (42), and we see that v_3 satisfies

(48)
$$\begin{cases} \Delta^2 v_3 = 0, & \text{in } R_3, \\ v_3 = (v_3)_{xx} = 0, & \text{on } \{0,\pi\}\times] - \ell, -\varepsilon[, \\ (v_3)_{yy} + \sigma(v_3)_{xx} = -K_\alpha e^{-\alpha\ell}[\alpha^2\phi + \sigma\phi''], & \text{on }]0,\pi[\times\{-\ell\}, \\ (v_3)_{yyy} + (2-\sigma)(v_3)_{xxy} = -K_\alpha \alpha e^{-\alpha\ell}[\alpha^2\phi + (2-\sigma)\phi''], & \text{on }]0,\pi[\times\{-\ell\}. \end{cases}$$

By separating variables, we seek functions $Y_m^3 = Y_m^3(y)$ so that the function $v_3(x,y) = \sum_{m=1}^{\infty} Y_m^3(y)\sin(mx)$ solves (48). Then

$$\Delta^2 v_3(x,y) = \sum_{m=1}^{\infty} [(Y_m^3)''''(y) - 2m^2(Y_m^3)''(yY_m^2) + m^4 Y_m^3(y)]\sin(mx)$$

and then $Y_m^3(y) = A_3\cosh(my) + B_3\sinh(my) + C_3 y\cosh(my) + D_3 y\sinh(my)$. As with R_1, we are then led to the system

$$(1-\sigma)m^2\cosh(m\ell)A_3 + m[2\cosh(m\ell) + (1-\sigma)m\ell\sinh(m\ell)]D_3$$
$$-(1-\sigma)m^2\sinh(m\ell)B_3 - m[2\sinh(m\ell) + (1-\sigma)m\ell\cosh(m\ell)]C_3 = K_\alpha e^{-\alpha\ell}\zeta_m^1,$$
$$-(1-\sigma)m^3\cosh(m\ell)B_3 + m^2[(1+\sigma)\cosh(m\ell) - (1-\sigma)m\ell\sinh(m\ell)]C_3$$
$$+(1-\sigma)m^3\sinh(m\ell)A_3 - m^2[(1+\sigma)\sinh(m\ell) - (1-\sigma)m\ell\cosh(m\ell)]D_3 = K_\alpha e^{-\alpha\ell}\alpha\zeta_m^2.$$

Finally, let $\widetilde{\gamma}_m$ be the Fourier coefficients of g, see (31), and $\widetilde{\phi} \in H^4(]0,\pi[)$ be defined as $\widetilde{\phi}(x) = \sum_{m=1}^{\infty}\frac{\widetilde{\gamma}_m}{(m^2-\alpha^2)^2}\sin(mx)$, for every $x \in]0,\pi[$. For R_2 we introduce the auxiliary function $v_2(x,y) := w_2(x,y) - K_\alpha e^{\alpha y}\frac{\widetilde{\phi}(x)}{1+d}$ with $\widetilde{\phi}$ as above and w_2 solving (41), and we see that v_2 satisfies

(49)
$$\begin{cases} \Delta^2 v_2 = 0 & \text{in } R_2 \\ v_2 = (v_2)_{xx} = 0 & \text{on } \{0,\pi\}\times] - \varepsilon, \varepsilon[. \end{cases}$$

We seek again solutions of (49) by separating variables, namely we seek functions $Y_m^2 = Y_m^2(y)$ such that $v_2(x,y) = \sum_{m=1}^{\infty} Y_m^2(y)\sin(mx)$ solves (49). Then

$$\Delta^2 v_2(x,y) = \sum_{m=1}^{\infty} [(Y_m^2)''''(y) - 2m^2(Y_m^2)''(y) + m^4 Y_m^2(y)]\sin(mx)$$

and then $Y_m^2(y) = A_2\cosh(my) + B_2\sinh(my) + C_2 y\cosh(my) + D_2 y\sinh(my)$.

In this case we have no boundary conditions to use as a constraint. Instead, we impose the junction conditions (43) and (44) by which we get the relations (39). Combining (39) with (48) and (49), we obtain a 4×4 system in the unknowns A_2, B_2, C_2, D_2 which decouples into the following 2×2 systems:

$$\begin{cases} (1-\sigma)m^2\cosh(m\ell)A_2 + m[2\cosh(m\ell) + (1-\sigma)m\ell\sinh(m\ell)]D_2 = K_\alpha\zeta_m^1\cosh(\alpha\ell) + G_1 \\ -(1-\sigma)m^3\sinh(m\ell)A_2 + m^2[(1+\sigma)\sinh(m\ell) - (1-\sigma)m\ell\cosh(m\ell)]D_2 = \alpha K_\alpha\zeta_m^2\sinh(\alpha\ell) + G_2 \end{cases}$$

and

$$\begin{cases} (1-\sigma)m^2\sinh(m\ell)B_2 + m[2\sinh(m\ell) + (1-\sigma)m\ell\cosh(m\ell)]C_2 = K_\alpha\zeta_m^1\sinh(\alpha\ell) + G_3 \\ -(1-\sigma)m^3\cosh(m\ell)B_2 + m^2[(1+\sigma)\cosh(m\ell) - (1-\sigma)m\ell\sinh(m\ell)]C_2 = \alpha K_\alpha\zeta_m^2\cosh(\alpha\ell) + G_4 \end{cases}$$

where the $G_i = G_i(m,\alpha,\varepsilon)$ are defined in (37). The solutions of the above systems are given in (38) and, combined with (39), allow us to write the explicit form of w_1, w_2 and w_3.

To complete the proof of Proposition 10.1 we show that the series defining w_α converges in the next lemma. $\qquad\square$

Lemma 10.1. *If $g \in L^2(]0, \pi[)$, then the series defining w_α in Proposition 10.1 converges uniformly in $\overline{\Omega}$ up to the second derivative.*

Proof. We start by studying the uniform convergence of the series which defines w_1. We have

$$|w_1| \leqslant \sum_{m=1}^\infty \left(\frac{e^{m\ell}}{2} (|A_1 + B_1| + \ell|C_1 + D_1|) + \frac{e^{-m\varepsilon}}{2} (|A_1 - B_1| + \ell|C_1 - D_1|) + \frac{e^{\alpha\ell} K_\alpha \gamma_m}{(m^2 - \alpha^2)^2} \right).$$

From the following relations

$$\frac{2m \tanh(m\ell)G_1}{a} + \frac{2G_2}{a} = \frac{1}{\cosh(m\ell)} [(1 - \sigma)m^3 F_2^- + 2m^2 F_3^- + m^3(1 - \sigma)\ell F_4^+)]$$
$$- (\sigma + 3)m^2 [F_3^- \cosh(m\ell) + F_4^+ \sinh(m\ell)],$$

$$\frac{2 \tanh(m\ell)G_4}{a} + \frac{2mG_3}{a} = \frac{1}{\cosh(m\ell)} [-(1 - \sigma)m^3 F_1^- - (1 - \sigma)m^3 \ell F_3^+ + m^2(1 + \sigma)F_4^-)]$$
$$- (\sigma + 3)m^2 [F_4^- \cosh(m\ell) + F_3^+ \sinh(m\ell)],$$

and after some lengthy calculations, as $m \to +\infty$, we get

$$(50) \qquad \begin{aligned} A_2 + B_2 &= -a(F_1(\varepsilon) + F_2(\varepsilon)) + \overline{K}_1 \frac{\gamma_m e^{-m\ell}}{m^3} + o\left(\frac{\gamma_m e^{-m\ell}}{m^3} \right), \\ C_2 + D_2 &= -a(F_3(\varepsilon) + F_4(\varepsilon)) + \overline{K}_2 \frac{\gamma_m e^{-m\ell}}{m^3} + o\left(\frac{\gamma_m e^{-m\ell}}{m^3} \right), \end{aligned}$$

for some $\overline{K}_1, \overline{K}_2 \in \mathbb{R} \setminus \{0\}$. Hence, as $m \to \infty$, we have

$$(51) \quad A_1 + B_1 = \overline{K}_1 \frac{\gamma_m e^{-m\ell}}{m^3} + o\left(\frac{\gamma_m e^{-m\ell}}{m^3} \right), \quad C_1 + D_1 = \overline{K}_2 \frac{\gamma_m e^{-m\ell}}{m^3} + o\left(\frac{\gamma_m e^{-m\ell}}{m^3} \right).$$

On the other hand, as $m \to +\infty$, we have

$$(52) \qquad \begin{aligned} A_2 - B_2 &= \overline{K}_3 (\gamma_m - \widehat{\gamma}_m) \frac{e^{-m\varepsilon}}{m^3} + o\left((\gamma_m - \widehat{\gamma}_m) \frac{e^{-m\varepsilon}}{m^3} \right), \\ C_2 - D_2 &= \overline{K}_4 (\gamma_m - \widehat{\gamma}_m) \frac{e^{-m\varepsilon}}{m^3} + o\left((\gamma_m - \widehat{\gamma}_m) \frac{e^{-m\varepsilon}}{m^3} \right), \end{aligned}$$

for some $\overline{K}_3, \overline{K}_4 \in \mathbb{R} \setminus \{0\}$. Hence, since as $m \to +\infty$ we have

$$a(F_1(\varepsilon) - F_2(\varepsilon)) = -K_\alpha \varepsilon e^{\alpha\varepsilon}(\gamma_m - \widehat{\gamma}_m) \frac{e^{m\varepsilon}}{2m^3} + o\left((\gamma_m - \widehat{\gamma}_m) \frac{e^{m\varepsilon}}{m^3} \right),$$

$$a(F_3(\varepsilon) - F_4(\varepsilon)) = K_\alpha e^{\alpha\varepsilon}(\gamma_m - \widehat{\gamma}_m) \frac{e^{m\varepsilon}}{2m^3} + o\left((\gamma_m - \widehat{\gamma}_m) \frac{e^{m\varepsilon}}{m^3} \right),$$

we conclude that

$$\begin{aligned} A_1 - B_1 &= -K_\alpha \varepsilon e^{\alpha\varepsilon}(\gamma_m - \widehat{\gamma}_m) \frac{e^{m\varepsilon}}{2m^3} + o\left((\gamma_m - \widehat{\gamma}_m) \frac{e^{m\varepsilon}}{m^3} \right), \\ C_1 - D_1 &= K_\alpha e^{\alpha\varepsilon}(\gamma_m - \widehat{\gamma}_m) \frac{e^{m\varepsilon}}{2m^3} + o\left((\gamma_m - \widehat{\gamma}_m) \frac{e^{m\varepsilon}}{m^3} \right). \end{aligned}$$

The above equalities, together with (51) and the summability of the coefficients $\widetilde{\gamma}_m$ and $\widehat{\gamma}_m$, prove the uniform convergence of the series defining w_1 up to the second derivative.

Next, we consider w_2. We estimate

$$|w_2| \leqslant \sum_{m=1}^{\infty} \left(\frac{e^{m\varepsilon}}{2} \left(|A_2 + B_2| + \varepsilon|C_2 + D_2|\right) + \frac{e^{m\varepsilon}}{2} \left(|A_2 - B_2| + \varepsilon|C_2 - D_2|\right) + \frac{e^{\alpha\varepsilon} K_\alpha \widehat{\gamma}_m}{(m^2 - \alpha^2)^2} \right).$$

Since as $m \to +\infty$ we have

$$a(F_1(\varepsilon) + F_2(\varepsilon)) = K_\alpha \varepsilon e^{\alpha\varepsilon}(\gamma_m - \widehat{\gamma}_m)\frac{e^{-m\varepsilon}}{2m^3} + o\left((\gamma_m - \widehat{\gamma}_m)\frac{e^{-m\varepsilon}}{m^3}\right),$$

$$a(F_3(\varepsilon) + F_4(\varepsilon)) = K_\alpha e^{\alpha\varepsilon}(\gamma_m - \widehat{\gamma}_m)\frac{e^{-m\varepsilon}}{2m^3} + o\left((\gamma_m - \widehat{\gamma}_m)\frac{e^{-m\varepsilon}}{m^3}\right),$$

from (50) we deduce that

$$A_2 + B_2 = -K_\alpha \varepsilon e^{\alpha\varepsilon}(\gamma_m - \widehat{\gamma}_m)\frac{e^{-m\varepsilon}}{2m^3} + o\left((\gamma_m - \widehat{\gamma}_m)\frac{e^{-m\varepsilon}}{m^3}\right),$$

$$C_2 + D_2 = -K_\alpha e^{\alpha\varepsilon}(\gamma_m - \widehat{\gamma}_m)\frac{e^{-m\varepsilon}}{2m^3} + o\left((\gamma_m - \widehat{\gamma}_m)\frac{e^{-m\varepsilon}}{m^3}\right).$$

This, jointly with (52) and the summability of the coefficients $\widetilde{\gamma}_m$, $\widehat{\gamma}_m$, proves the uniform convergence of the series defining w_2 up to the second derivative.

The computations for w_3 are similar to those for w_1 and we omit them. $\square$

Now, we focus on the limiting behavior of w_α as $\alpha \to +\infty$. We first determine the limit of h_α.

Lemma 10.2. *Let C_g be as in (29) and I^N be as in (32). As $\alpha \to +\infty$ we have that $h_\alpha \to \overline{h}$ in $H_*^{-2}(\Omega)$, where $\overline{h} \in H_*^{-2}(\Omega)$ is defined as follows*

$$\langle \overline{h}, v \rangle = \int_0^\pi \frac{g(x)}{C_g(1 + d\chi_{I^N}(x))} v(x, \ell)\, dx \qquad \forall v \in H_*^2(\Omega).$$

The proof is a consequence of an integration by parts, similar to that of Lemma 11.1 below and therefore we omit it. Next, we set

$$(53) \qquad \overline{w}(x, y) = \sum_{m=1}^{\infty} \left((\overline{A}_2 + \overline{C}_2 y) \cosh(my) + (\overline{B}_2 + \overline{D}_2 y) \sinh(my) \right) \sin(mx),$$

where $\overline{A}_2 = \overline{A}_2(m)$, $\overline{B}_2 = \overline{B}_2(m)$, $\overline{C}_2 = \overline{C}_2(m)$, $\overline{D}_2 = \overline{D}_2(m)$ are given by

$$\overline{A}_2 := \frac{\gamma_m \left((1 - \sigma)m\ell \sinh(m\ell) + 2\cosh(m\ell)\right)}{2C_g(1-\sigma)m^3 \left[(3 + \sigma)\sinh(m\ell)\cosh(m\ell) - (1 - \sigma)m\ell\right]},$$

$$\overline{B}_2 := \frac{\gamma_m \left((1 - \sigma)m\ell \cosh(m\ell) + 2\sinh(m\ell)\right)}{2C_g(1-\sigma)m^3 \left[(3 + \sigma)\sinh(m\ell)\cosh(m\ell) + (1 - \sigma)m\ell\right]},$$

$$\overline{C}_2 := -\frac{\gamma_m \sinh(m\ell)}{2C_g m^2 \left[(3 + \sigma)\sinh(m\ell)\cosh(m\ell) + (1 - \sigma)m\ell\right]},$$

$$\overline{D}_2 := -\frac{\gamma_m \cosh(m\ell)}{2C_g m^2 \left[(3 + \sigma)\sinh(m\ell)\cosh(m\ell) - (1 - \sigma)m\ell\right]},$$

with γ_m as in (22). The following lemma holds.

Lemma 10.3. *Let w_α be the unique solution of* (30), $\overline{w}$ *be as in* (53) *and $\overline{h}$ be as in Lemma 10.2. Then, as $\alpha \to +\infty$, we have that $w_\alpha(x,y) \to \overline{w}(x,y)$ in $H^2_*(\Omega)$ and $\overline{w}$ is the unique solution of the problem*

$$(54) \qquad (w, v)_{H^2_*} = \langle \overline{h}, v \rangle \qquad \forall v \in H^2_*(\Omega).$$

Proof. Let A_2, B_2, C_2, D_2 be as defined in (38), let $\overline{A}_2$, $\overline{B}_2$, $\overline{C}_2$, $\overline{D}_2$ be as defined in (53), we get $A_2 \to \overline{A}_2$, $B_2 \to \overline{B}_2$, $C_2 \to \overline{C}_2$, $D_2 \to \overline{D}_2$ as $\alpha \to +\infty$. Moreover, from (29), (34), and (36), as $\alpha \to +\infty$, we have

$$K_\alpha = \frac{\alpha e^{-\alpha \ell}}{C_g} + o(\alpha e^{-\alpha \ell}), \qquad a = \frac{e^{-\alpha(\ell - \varepsilon)}(\gamma_m - \widehat{\gamma}_m)}{C_g \alpha^3} + o\left(\frac{e^{-\alpha(\ell - \varepsilon)}}{\alpha^3}\right),$$

and

$$F_1(\varepsilon) = \frac{\sinh(m\varepsilon) - m\varepsilon \cosh(m\varepsilon)}{2m^3} \alpha^3 + o(\alpha^3), \quad F_3(\varepsilon) = \frac{\cosh(m\varepsilon)}{2m^2} \alpha^3 + o(\alpha^3),$$

$$F_2(\varepsilon) = \frac{m\varepsilon \sinh(m\varepsilon) - \cosh(m\varepsilon)}{2m^3} \alpha^3 + o(\alpha^3), \quad F_4(\varepsilon) = -\frac{\sinh(m\varepsilon)}{2m^2} \alpha^3 + o(\alpha^3).$$

Hence, recalling the definition of the constants A_i, B_i, C_i, D_i given in (38) and (39), for $i = 1, 2, 3$, we deduce that $A_i \to \overline{A}_2$, $B_i \to \overline{B}_2$, $C_i \to \overline{C}_2$, $D_i \to \overline{D}_2$, as $\alpha \to +\infty$. Therefore, by exploiting the summability of the coefficients in the series defining w_α, we conclude that $w_\alpha \to \overline{w}$ a.e. in Ω, as $\alpha \to +\infty$.

On the other hand, by Lemma 10.2, there holds $h_\alpha \to \overline{h}$ in $H^{-2}_*(\Omega)$ as $\alpha \to +\infty$. Let $w_{\overline{h}} \in H^2_*(\Omega)$ be the unique solution of (54), so that clearly $\|w_\alpha - w_{\overline{h}}\|_{H^2_*} \leqslant \|h_\alpha - \overline{h}\|_{H^{-2}_*}$, $w_\alpha \to w_{\overline{h}}$ in $H^2_*(\Omega)$ and uniformly in $\overline{\Omega}$. Hence, $\overline{w} \equiv w_{\overline{h}}$ and this concludes the proof of the lemma. $\qquad \square$

Proof of Theorem 6.1. By Proposition 10.1, the gap function corresponding to w_α is $\mathcal{G}_\alpha(x) = w_1(x, \ell) - w_3(x, -\ell) = \sum_{m=1}^\infty \xi_m(\alpha) \sin(mx)$ with the coefficients

$$\xi_m(\alpha) \quad := \quad (A_1 - A_3) \cosh(m\ell) + (B_1 + B_3) \sinh(m\ell) + (C_1 + C_3) \ell \cosh(m\ell)$$

$$(55) \qquad + (D_1 - D_3) \ell \sinh(m\ell) + \frac{\alpha \gamma_m}{C_g (m^2 - \alpha^2)^2},$$

while the gap function corresponding to $\overline{w}$ can be written as the function $\overline{\mathcal{G}}(x) = \overline{w}(x, \ell) - \overline{w}(x, -\ell) = \sum_{m=1}^\infty \overline{\beta}_m \sin(mx)$ with the coefficients

$$(56) \qquad \overline{\beta}_m \quad = \quad \left(2\overline{B}_2 \sinh(m\ell) + 2\overline{C}_2 \ell \cosh(m\ell)\right) \sin(mx) = \frac{2 \gamma_m \Upsilon_m}{C_g (1 - \sigma)},$$

where the coefficients Υ_m are as defined in (16). Since, by Lemma 10.3, the function w_α converges to $\overline{w}$ uniformly in $\overline{\Omega}$, we have

$$\max_{x \in [0, \pi]} |\mathcal{G}_\alpha(x) - \overline{\mathcal{G}}(x)| = \max_{x \in [0, \pi]} \left| \sum_{m=1}^\infty \xi_m(\alpha) \sin(mx) - \sum_{m=1}^\infty \overline{\beta}_m \sin(mx) \right|$$

$$\leqslant \max_{x \in [0, \pi]} |w_\alpha(x, \ell) - \overline{w}(x, \ell)| + \max_{x \in [0, \pi]} |w_\alpha(x, -\ell) - \overline{w}(x, -\ell)|,$$

and so the right-hand side converges to zero.

Next, we specify the asymptotic behavior of $\xi_m(\alpha)$. To this end, by Proposition 10.1, we note that, as $\alpha \to +\infty$, the following estimates hold

$$A_1 - A_3 = a(F_1(\varepsilon) - e^{-2\alpha\varepsilon}F_1(-\varepsilon)) = O(e^{-\alpha(\ell-\varepsilon)}) + O(e^{-\alpha(\ell+\varepsilon)}),$$

$$B_1 + B_3 = 2B_2 + a(F_2(\varepsilon) + e^{-2\alpha\varepsilon}F_2(-\varepsilon)) = 2B_2 + O(e^{-\alpha(\ell-\varepsilon)}) + O(e^{-\alpha(\ell+\varepsilon)}),$$

$$C_1 + C_3 = 2C_2 + a(F_3(\varepsilon) + e^{-2\alpha\varepsilon}F_3(-\varepsilon)) = 2C_2 + O(e^{-\alpha(\ell-\varepsilon)}) + O(e^{-\alpha(\ell+\varepsilon)}),$$

$$D_1 - D_3 = a(F_4(\varepsilon) - e^{-2\alpha\varepsilon}F_4(-\varepsilon)) = O(e^{-\alpha(\ell-\varepsilon)}) + O(e^{-\alpha(\ell+\varepsilon)}),$$

$$B_2 = \overline{B}_2 - \frac{\gamma_m\left((1+\sigma)\cosh(m\ell) - (1-\sigma)m\ell\sinh(m\ell)\right)}{2C_g(1-\sigma)m^2\left[(3+\sigma)\sinh(m\ell)\cosh(m\ell) + (1-\sigma)m\ell\right]}\frac{1}{\alpha} + o\left(\frac{1}{\alpha}\right),$$

$$C_2 = \overline{C}_2 - \frac{\gamma_m\cosh(m\ell)}{2C_g m\left[(3+\sigma)\sinh(m\ell)\cosh(m\ell) + (1-\sigma)m\ell\right]}\frac{1}{\alpha} + o\left(\frac{1}{\alpha}\right).$$

Recalling the definition of $\xi_m(\alpha)$, we conclude that, as $\alpha \to +\infty$,

$$
\begin{aligned}
\xi_m(\alpha) &= 2\overline{B}_2\sinh(m\ell) + 2\overline{C}_2\ell\cosh(m\ell) \\
&\quad - \frac{(1+\sigma)\sinh(m\ell)\cosh(m\ell) + (1-\sigma)m\ell}{C_g(1-\sigma)m^2[(3+\sigma)\sinh(m\ell)\cosh(m\ell) + (1-\sigma)m\ell]}\frac{\gamma_m}{\alpha} + o\left(\frac{1}{\alpha}\right) \\
&= \overline{\beta}_m - \frac{\overline{\omega}_m}{\alpha} + o\left(\frac{1}{\alpha}\right)
\end{aligned}
$$

with $\overline{\omega}_m$ as in (21).

In view of the discussion at the beginning of this section, the statement of Theorem 6.1 now follows simply by setting

$$(57) \qquad \beta_m(\alpha) := \frac{R_\alpha}{K_\alpha}\xi_m(\alpha)$$

with $\xi_m(\alpha)$ as given in (55). Notice that we still denote by $\mathcal{G}_\alpha(x)$ the gap functions corresponding to u_α. Furthermore, by Lemma 10.3, $u_\alpha(x,y) \to \overline{u}(x,y)$ in $H_*^2(\Omega)$ where $\overline{u}$ is the odd part of the unique solution of problem (54). Namely, $\overline{u}$ is the the unique solution of the problem

$$(58) \qquad (\overline{u}, v)_{H_*^2} = \langle \overline{h}^o, v \rangle \qquad \forall v \in H_*^2(\Omega)$$

where $\overline{h}^o$ is the odd part of $\overline{h}$ as defined in Lemma 10.2. $\qquad\qquad\square$

Proof of Theorem 6.2. Set $\mu = 0$ and $g(x) = \sin(nx)$ for some positive integer n, then the coefficients (22) become $\gamma_n = 1$ while $\gamma_m = 0$ if $m \neq n$. Furthermore, since $C_g = \int_0^\pi |\sin(nx)|\,dx = 2$, by Theorem 6.1 the corresponding gap function is

$$(59) \qquad \overline{\mathcal{G}}_n(x) = \overline{\Upsilon}_n\sin(nx),$$

where $\overline{\Upsilon}_n := \Upsilon_n/(1-\sigma)$ with the Υ_n as defined in (16). Hence, $\overline{\mathcal{G}}_n^\infty = \overline{\Upsilon}_n$. The thesis follows by showing that $\overline{\mathcal{G}}^\infty < \overline{\Upsilon}_1$ for every $g \in \Gamma$. Let us consider separately the four cases in the set Γ.

• Let $g(x) = \sin(mx)$ for some positive integer m. By (59) we know that $\overline{\mathcal{G}}_m^\infty = \overline{\Upsilon}_m$. Since $0 < \sigma < 1$, some lengthy computations reveal that $\frac{d\overline{\Upsilon}_m}{dm} < 0$, hence the map $m \mapsto \overline{\Upsilon}_m$ is strictly decreasing and $\max_m \overline{\mathcal{G}}_m^\infty = \overline{\Upsilon}_1$.

• For given $N \in \mathbb{N}$ and $\{\widetilde{\gamma}_m\}_{m\in\mathbb{N}} \subset \ell^2(\mathbb{N})$, let $g(x) = \sum_{m\geqslant N}\widetilde{\gamma}_m\sin(mx)$. Since $\sup_m |\widetilde{\gamma}_m| \leqslant 2C_g/\pi$ and $\sum_{m=1}^\infty \overline{\Upsilon}_m$ converges, by Theorem 6.1 we infer the existence of $N \in \mathbb{N}$ sufficiently large such that $\mathcal{G}^\infty \leqslant \frac{4}{\pi}\sum_{m\geqslant N}\overline{\Upsilon}_m < \overline{\Upsilon}_1$.

• Let m be a positive integer, and take $g(x) = \sin(mx) + \sin(3mx)$. By the prosthaphaeresis formulas, $g(x) = 4\sin(mx)\cos^2(mx)$ and we compute the value of C_g in (29). By putting $x_k = (k/m)\pi$ with $k = 0, ..., m$, for $m = 2n$ we have

$$C_g = \sum_{j=0}^{n-1} \left(\int_{x_{2j}}^{x_{2j+1}} \big(\sin(mx) + \sin(3mx) \big)\, dx - \int_{x_{2j+1}}^{x_{2j+2}} \big(\sin(mx) + \sin(3mx) \big)\, dx \right),$$

while for $m = 2n + 1$ we have

$$C_g = \sum_{j=0}^{n-1} \left[\int_{x_{2j}}^{x_{2j+1}} \big(\sin(mx) + \sin(3mx) \big)\, dx - \int_{x_{2j+1}}^{x_{2j+2}} \big(\sin(mx) + \sin(3mx) \big)\, dx \right]$$
$$+ \int_{x_{2n}}^{x_{2n+1}} \big(\sin(mx) + \sin(3mx) \big)\, dx.$$

In any case, we get that $C_g = 8/3$. Furthermore, by Theorem 6.1 we have

$$|\mathcal{G}(x)| = \frac{2}{C_g} | \left(\overline{\Upsilon}_m \sin(mx) + \overline{\Upsilon}_{3m} \sin(3mx) \right) | \leqslant \frac{3}{4} \left(\overline{\Upsilon}_m + \overline{\Upsilon}_{3m} \right) \leqslant \frac{3}{4} \left(\overline{\Upsilon}_1 + \overline{\Upsilon}_3 \right),$$

where in the last step we exploit the monotonicity of the map $n \mapsto \overline{\Upsilon}_n$. Finally, some lengthy computations reveal that $3F_3 < F_1$, hence $\mathcal{G}^\infty \leqslant \overline{\Upsilon}_1$ holds also for $g(x) = \sin(mx) + \sin(3mx)$.

• Take $g(x) = g_N(x) = \sum_{m=1}^N \sin((2m-1)x)$ for $N \in \mathbb{N}$ sufficiently large, to be fixed later. It is known that $g_N(x) = \sin^2(Nx)/\sin(x)$ for $x \in\,]0, \pi[$, see [20, p.73]. Hence,

$$C_{g_N} = \int_0^\pi |g_N(x)|\, dx = \int_0^\pi g_N(x)\, dx = 2 \sum_{m=1}^N \frac{1}{2m-1} \geqslant \log(2N - 1).$$

By this and Theorem 6.1 we deduce that

$$\mathcal{G}^\infty \leqslant \frac{2}{C_g} \sum_{m=1}^N \overline{\Upsilon}_m \leqslant \frac{2}{\log(2N-1)} \sum_{m=1}^N F_m(\ell) \leqslant \overline{\Upsilon}_1$$

for N sufficiently large, since the last summation converges when $N \to +\infty$. This concludes the proof of Theorem 6.2. □ □

11. Proof of Proposition 5.1

For $z \in\,]0, \pi[$, $0 < \eta < \min\{z, \pi - z\}$, and $\alpha > 0$, we take $g(x) = \chi_{[z-\eta,z+\eta]}(x)$ in (20) and we set $f_{\alpha,\eta}(x, y) := R_{\alpha,\eta} \sinh(\alpha y) \chi_{[z-\eta,z+\eta]}(x)$ with the constant $R_{\alpha,\eta} := \frac{\alpha}{4\eta(\cosh(\alpha\ell)-1)}$, so that $\|f_{\alpha,\eta}\|_{L^1} = 1$. Let us establish the first ingredient for the proof of Proposition 5.1.

Lemma 11.1. *Let $D = \emptyset$ and let $\mathcal{G}_{\alpha,\eta}$ and $\mathcal{G}_{T_z}$ (T_z as in (15)) be the gap functions corresponding to the solutions of (5) with $f = f_{\alpha,\eta}$ and $f = T_z$, respectively. As $(\alpha^{-1}, \eta) \to (0, 0)$ we have that $\mathcal{G}_{\alpha,\eta}(x) \to \mathcal{G}_{T_z}(x)$ uniformly on $[0, \pi]$.*

Proof. We first claim that $f_{\alpha,\eta} \to T_z$ in $H_*^{-2}(\Omega)$ as $(\alpha^{-1}, \eta) \to (0, 0)$, that is,

$$(60) \qquad \lim_{(\alpha^{-1},\eta)\to(0,0)} \int_\Omega f_{\alpha,\eta}(x, y)v(x, y)\, dxdy = \frac{v(z,\ell)-v(z,-\ell)}{2}, \quad \forall v \in H_*^2(\Omega).$$

Take $v \in H^2_*(\Omega)$ and compute

$$\int_\Omega f_{\alpha,\eta}(x,y)v(x,y)\,dxdy = R_{\alpha,\eta}\int_0^\pi \chi_{[z-\eta,z+\eta]}(x)\left(\int_{-\ell}^\ell \sinh(\alpha y)v(x,y)\,dy\right)dx$$

$$= \frac{1}{4\,\eta(\cosh(\alpha\ell)-1)}\int_{z-\eta}^{z+\eta}\left(\big[\cosh(\alpha y)v(x,y)\big]_{-\ell}^\ell - \int_{-\ell}^\ell \cosh(\alpha y)v_y(x,y)\,dy\right)dx$$

$$= \frac{\cosh(\alpha\ell)}{\cosh(\alpha\ell)-1}\frac{v(\xi_z,\ell)-v(\xi_z,-\ell)}{2} - \frac{1}{2(\cosh(\alpha\ell)-1)}\int_{-\ell}^\ell \cosh(\alpha y)v_y(\theta_z,y)\,dy,$$

for some $z-\eta < \xi_z, \theta_z < z+\eta$. Now we observe that, uniformly with respect to η, $\lim_{\alpha^{-1}\to 0}\int_{-\ell}^\ell \frac{\cosh(\alpha y)}{\cosh(\alpha\ell)-1}\,v_y(\theta_z,y)\,dy = 0$, by the Lebesgue Dominated Convergence Theorem and thus (60) follows. From this we infer that the corresponding solutions converge in $H^2_*(\Omega)$ and then the proof can be completed by arguing as in Proposition 8.1. □

In view of Lemma 11.1, the proof of Proposition 5.1 follows once we have proved the following statement.

Lemma 11.2. *Assume $z \in]0,\pi[$, $0 < \eta < \min\{z, \pi-z\}$ and $\alpha \geqslant 0$ with $\alpha \notin \mathbb{N}$. As $(\alpha^{-1},\eta) \to (0,0)$, the gap function $\mathcal{G}_{\alpha,\eta}(x)$ corresponding to the solution of (6) with $f = f_{\alpha,\eta}$ and $D = \emptyset$ converges to $\frac{4}{\pi(1-\sigma)}\sum_{m=1}^\infty \Upsilon_m \sin(mz)\sin(mx)$ uniformly on $[0,\pi]$, with the Υ_m as defined in (16).*

Proof. The explicit form of the gap function $\mathcal{G}_{\alpha,\eta}(x)$ follows from Theorem 6.1 by replacing f_α with $f_{\alpha,\eta}$ and D^N with $\emptyset$. Namely, we assume $\mu = \varepsilon = 0$ in (19) and $g(x) = \chi_{[z-\eta,z+\eta]}(x)$ in (20), hence $C_g = 2\eta$. With these specifications, by (39), we have $A_1 = A_3$, $B_1 = B_2 = B_3$, $C_1 = C_2 = C_3$, and $D_1 = D_3$, while $\gamma_m = \widetilde{\gamma}_m$ in (31). Hence, by (55) and (57), we it follows that the function $\mathcal{G}_{\alpha,\eta}(x) = \sum_{m=1}^\infty \beta_m(\alpha,\eta)\sin(mx)$ and

$$\beta_m(\alpha,\eta) = \frac{\sinh(\alpha\ell)}{\cosh(\alpha\ell)-1}\left(2B_2\sinh(m\ell) + 2C_2\ell\cosh(m\ell) + \frac{\alpha\widetilde{\gamma}_m}{2\eta(m^2-\alpha^2)^2}\right),$$

with B_2 and C_2 as in (38) and $\widetilde{\gamma}_m = \frac{4}{\pi m}\sin(mz)\sin(m\eta)$. By noting that $\frac{\widetilde{\gamma}_m(\eta)}{2\eta} = \frac{2\sin(mz)}{\pi} + o(\eta)$ as $\eta \to 0$ and exploiting (56), as $(\alpha^{-1},\eta) \to (0,0)$ we obtain

$$\beta_m(\alpha,\eta) = \big(1 + 2e^{-\alpha\ell} + o(e^{-\alpha\ell})\big)\left(\frac{2\sin(mz)}{\pi} + o(\eta)\right)\left(\frac{2\Upsilon_m}{1-\sigma} + o\left(\frac{1}{\alpha^2}\right)\right)$$

$$= \frac{4\Upsilon_m\sin(mz)}{\pi(1-\sigma)} + o(1),$$

with Υ_m as defined in (16). This completes the proof of the lemma. □

12. Conclusions, Perspectives and Open Problems

With possible applications to the deck of a footbridge or a suspension bridge, this paper deals with the problem of minimizing the torsional displacements of partially hinged reinforced plates. We showed that the gap function (7) is extremely useful to measure the torsional instability and that it gives hints on how to compare the torsional performances of different plates through the minimaxmax

problem (9), namely a robust shape optimization in the worst case setting. The demonstrated existence results prove that the problem is well-defined and, in some cases, it also allows to find properties of the worst force. On the other hand, some meaningful problems prove themselves to be very difficult to handle and optimal elements are hardly characterizable. This led us to provide some conjectures that we now motivate in detail.

There are other classes $\mathcal{D}$ where the minimum problem (9) admits a solution but the ones in Definition 3.1 appear particularly appropriate for engineering applications. A quite general class of admissible domains, where it is possible to define problem (9), is that of measurable sets with uniformly bounded De Giorgi perimeter and fixed area. In this setting, thanks to a compactness result for BV functions, the existence of a solution is still guaranteed. Therefore, we point out that it could be interesting to know if such a solution has enough regularity and geometrical properties to belong to one of those classes of Definition 3.1.

When $\mathcal{D} = \emptyset$, namely in the free plate case, in Proposition 5.1 we provide the explicit representation of the gap function when the load is concentrated on the boundary. The same statement seems out of reach for more general load, that is for odd distributions such as $\frac{\delta_{(z,w)} - \delta_{(z,-w)}}{2}$, with $z \in]0, \pi[$ and $w \in [0, \ell[$. Nevertheless, it is reasonable to expect that the gap function amplifies whenever $w \to \ell$ and z remains fixed. For this reason, we expect $\overline{T}_{\pi/2}$ to be the worst case among all possible normalized couples of odd concentrated loads. This leads to the following.

Conjecture 12.1. *When $D = \emptyset$, $\overline{T}_{\pi/2}$ and $-\overline{T}_{\pi/2}$ are the unique maximizers of the worst case problem* (10).

The worst case problem (10) may also be set up in different (smaller) classes of loads such as L^p-spaces and one has the maximization problem (11), see Theorem 3.1. We have no guess about the possible solutions of (11) when $p \in]1, \infty[$. We also suspect that there exists no maximizer for (11) in $L^1(\Omega)$; see Section 6. Moreover, we believe that the strange property stated in Theorem 4.2 for $p = \infty$ may not be fulfilled since we expect the following.

Conjecture 12.2. *Let $p = \infty$. For every $D \subset \Omega$, the unique maximizers of the problem* (11) *are the odd function $f(x,y) = y/|y|$, $y \neq 0$, and its opposite $-f$.*

Finally, as concerns the most ambitious goal to solve the minimaxmax problem (9), we conclude by stating two conjectures which are supported by numerical computations. More precisely, Table 2 in Section 6 suggests the following.

Conjecture 12.3. *Let $\mathcal{F}$ and $\mathcal{D}$ be as in (23). The optimum of the minimaxmax problem* (9) *is the couple (f^1, D^0).*

Table 3 in Section 7 shows that the least $\mathcal{G}_D^\infty$ is obtained for strips, then squares, hexagons, while the largest $\mathcal{G}_D^\infty$ is obtained for triangles. This is somehow surprising since squares are expected to be in between triangles and hexagons. Moreover, Table 3 suggests the following.

Conjecture 12.4. *Let $\mathcal{F}$ and $\mathcal{D}$ be as in (24). The optimum of the minimaxmax problem* (9) *is the couple $(\overline{e}_1, Strips)$.*

References

[1] Abrams, D.M., Eckhardt, B., McRobie, A., Ott, E., Strogatz, S.H.: Crowd synchrony on the Millennium Bridge. Nature, Brief Communications 438, 43-44 (2005)

[2] Macdonald, J.H.G.: Lateral excitation of bridges by balancing pedestrians. Proceedings of the Royal Society A, Mathematical, Physical & Engineering Sci. 465, 1055-1073 (2009)

[3] Sanderson, K.: Millennium bridge wobble explained. Nature (2008)

[4] Gazzola, F.: Mathematical models for suspension bridges. MS&A 15, Springer, Cham (2015)

[5] Ferrero, A., Gazzola, F.: A partially hinged rectangular plate as a model for suspension bridges. Disc. Cont. Dynam. Syst. A 35, 5879-5908 (2015)

[6] Berchio, E., Buoso, D., Gazzola, F.: A measure of the torsional performances of partially hinged rectangular plates. Integral methods in science and engineering 1, Theoretical techniques, 35-46 (2017)

[7] Chanillo, S., Kenig, C.E., To, T.: Regularity of the minimizers in the composite membrane problem in $\mathbb{R}^2$. J. Funct. Anal. 255, 2299-2320 (2008)

[8] Chanillo, S., Kenig, C.E.: Weak uniqueness and partial regularity for the composite membrane problem. J. Eur. Math. Soc. 10, 705-737 (2008)

[9] Kawohl, B., Stará, J., Wittum, G.: Analysis and numerical studies of a problem of shape design. Arch. Rat. Mech. Anal. 114, 349-363 (1991)

[10] Murat, F., Tartar, L.: Calculus of variations and homogenization. Topics in the Math. Modelling of Composite Materials 31, Progr. Nonlin. Diff. Eq. Appl. 139-173 (1997)

[11] Nazarov, S.A., Sweers, G.H., Slutskij, A.S.: Homogenization of a thin plate reinforced with periodic families of rigid rods. Sbornik Mathematics 202, 1127-1168 (2011)

[12] Michell, A.G.M.: The limits of economy of material in framed structures. Philosophical Magazine Series 6, Vol. 8, Issue 47, 589-597 (1904)

[13] Bendsøe, M.P., Sigmund, O.: Topology optimization. Springer, Berlin (2003)

[14] Allaire, G., Dapogny, C.: A linearized approach to worst-case design in parametric and geometric shape optimization. Math. Models Methods Appl. Sci. 24, 2199-2257 (2014)

[15] Cherkaev, A, Cherkaeva, E.: Principal compliance and robust optimal design. Journal of Elasticity 72, 71-98 (2003)

[16] Kirchhoff, G.R.:   Über das gleichgewicht und die bewegung einer elastischen scheibe. J. Reine Angew. Math. 40, 51-88 (1850)

[17] Love, A.E.H.: A treatise on the mathematical theory of elasticity. Cambridge Univ. Press, Cambridge (1927)

[18] Mansfield, E.H.: The bending and stretching of plates. Cambridge Univ. Press, Cambridge (2005)

[19] Berchio, E., Ferrero, A., Gazzola, F.: Structural instability of nonlinear plates modelling suspension bridges: mathematical answers to some long-standing questions. Nonlin. Anal. Real World Appl. 28, 91-125 (2016)

[20] Pólya, G., Szegő, G.: Problems and theorems in analysis, volume II. Springer 216 (1976)

[21] Berchio, E., Buoso, D., Gazzola, F.: On the variation of longitudinal and torsional frequencies in a partially hinged rectangular plate. ESAIM: COCV 24 (1), 63-87 (2018)

[22] Gazzola, F.: Hexagonal design for stiffening trusses. Ann. Mat. Pura Appl. 194, 87-108 (2015)

[23] Buttazzo, G., Oudet, E., Stepanov, E.: Optimal transportation problems with free Dirichlet regions. Variational methods for discontinuous structures, 41-65, Progr. Nonlinear Differential Equations Appl. 51, Birkhäuser, Basel (2002)

[24] Buttazzo, G., Stepanov, E.: Optimal transportation networks as free Dirichlet regions for the Monge-Kantorovich problem. Ann. Sc. Norm. Super. Pisa Cl. Sci. 2, 631-678 (2003)

[25] Buttazzo, G., Stepanov, E.: Minimization problems for average distance functionals. Calculus of variations: topics from the mathematical heritage of E. De Giorgi, 48-83, Quad. Mat. 14, Dept. Math., Seconda Univ. Napoli, Caserta (2004)

[26] Thomson, W.: On the division of space with minimum partitional area. Acta Math. 11, 121-134 (1887)

[27] Ambrosio, L., Tilli, P.: Analysis in metric spaces. Oxford University Press, Oxford (2004)

[28] Henrot, A., Pierre, M.: Variation et optimisation de formes. Springer, Berlin (2005)