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Unmediated Communication in Games with Complete and Incomplete Information

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Abstract

In this paper we study the effects of adding unmediated communication to static, finite games of complete and incomplete information. We characterize \( S^U(G) \), the set of outcomes of a game \( G \), that are induced by sequential equilibria of cheap talk extensions. A cheap talk extension of \( G \) is an extensive-form game in which players communicate before playing \( G \). A reliable mediator is not available and players exchange private or public messages that do not affect directly their payoffs. We first show that if \( G \) is a game of complete information with five or more players and rational parameters, then \( S^U(G) \) coincides with the set of correlated equilibria of \( G \). Next, we demonstrate that if \( G \) is a game of incomplete information with at least five players, rational parameters and full support (i.e. all profiles of types have positive probability), then \( S^U(G) \) is equal to the set of communication equilibria of \( G \).

KEYWORDS: Communication, correlated equilibrium, communication equilibrium, sequential equilibrium, mechanism design, revelation principle.

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1 Introduction

Communication allows players to share information with one another and to coordinate their actions. By doing this, in many games players can attain outcomes that are not feasible otherwise. In this paper, we study the effects of adding communication to static, finite games of complete and incomplete information.

A system of communication specifies the rules according to which players communicate. Adding one to a game defines a new, extended game. In a game with communication players exchange messages before choosing their actions. The notions of correlated equilibrium (Aumann (1974)), communication equilibrium and the revelation principle (seeForges (1986) and Myerson (1982)) characterize the set of outcomes that players can achieve in games with communication. In particular, this set of outcomes coincides with the set of correlated equilibria if the game is one of complete information, and with the set of communication equilibria if the game is one of incomplete information. This is a very powerful and useful result. Although there are infinitely many ways in which players can exchange messages, it is very easy to characterize what they can achieve with communication. In fact, correlated and communication equilibria are simply defined by a number of linear inequalities. Therefore, the set of correlated or communication equilibria has a simple and tractable mathematical structure (it is a convex polyhedron). Consider, for example, a game of complete information, and suppose that we are interested in finding the highest payoff that a given player can achieve with communication. In principle, there are infinitely many games with communication to consider. However, given the characterization above, it is enough to find the best correlated equilibrium for that player. This simply involves maximizing a linear function over a convex polyhedron.

Since correlated and communication equilibria are important and commonly used solution concepts, it is worthwhile to examine carefully the conditions under which they can be applied. Forges (1986) and Myerson (1982) show that the set of correlated or commu-
nication equilibria coincides with the set of outcomes that players can achieve with communication under two critical assumptions. First, not only do players exchange messages with one another, but they can also communicate with a reliable and impartial mediator (*mediated* communication). Secondly, the *Nash equilibrium* concept is used to analyze the games with communication.

In many situations, however, the assumption that there is a reliable mediator is too strong, and players are restricted to exchange messages among themselves. In a bargaining setting, for example, the buyer and the seller usually undergo a number of rounds of direct conversations before reaching a final agreement, and no third party mediates their positions or filters their messages. Similarly, political leaders often communicate directly with their advisors. It is true that the mediator could be considered a machine that is programmed to perform a certain number of operations. Still, one has to assume that an impartial individual, different from the players, is available to program the machine. We cannot rely on the players, who would take advantage of the opportunity, and would program the machine in their best interest. When a mediator is not available, and players can only communicate with one another, we say that communication is direct or *unmediated*.

As far as the solution concept is concerned, Nash equilibrium is a legitimate candidate. However, a game with communication is an extensive-form game, and Nash equilibrium does not prevent players from using non credible threats and behaving irrationally in events that are reached with probability zero. In a game with communication, it is therefore appropriate to use a stronger solution concept, such as *sequential equilibrium* (Kreps and Wilson (1982)), which requires players to behave optimally on and off the equilibrium path.

The goal of this paper is to characterize the outcomes that players can achieve when an impartial mediator is not available and we adopt a solution concept stronger than Nash equilibrium. We further address the related question of whether it is possible to use correlated and communication equilibria when the assumptions made by Forges (1986) and Myerson (1982) are not satisfied.
Depending on the presence of a mediator and on the adopted solution concept, the existing literature on games with communication can be classified into four categories: (i) mediated communication, Nash equilibrium; (ii) mediated communication, sequential equilibrium; (iii) unmediated communication, Nash equilibrium; (iv) unmediated communication, sequential equilibrium. To illustrate our results in more details, and to describe how our findings contribute to the existing body of work, we now discuss these categories.

We have already mentioned that in case (i) the set of correlated or communication equilibria describes the outcomes that players can achieve with communication. To implement a correlated or communication equilibrium, a mediator usually performs lotteries and sends players private messages. However, Lehrer and Sorin (1997) show that correlated and communication equilibria can be implemented in communication systems in which the mediator behaves deterministically and sends only public messages (see also Lehrer (1996)).

What happens when communication is mediated and the solution concept is stronger than Nash equilibrium? For static games of complete information, the characterization of the outcomes attainable with communication in terms of correlated equilibria trivially extends to the case in which the sequential equilibrium concept is used.¹ A similar result can be derived for games of incomplete information with full support (i.e. games in which all profiles of players’ types have positive probability). In this case, the set of outcomes that players can achieve with mediated communication and the sequential equilibrium concept coincides with the set of communication equilibria. This result does not hold in games without full support. In Section 3, we show that there exist communication equilibria of games without full support that cannot be implemented. We also demonstrate the failure of the revelation principle in games that do not have full support.

There is an extensive literature on games with unmediated communication. Aumann

¹ However, Dhillon and Mertens (1996) show that the outcomes achievable with communication cannot be described by correlated equilibria when the perfect equilibrium concept is used to analyze the extended games with communication. They demonstrate the failure of the revelation principle in this context, and provide a characterization of the outcomes achievable with mediated communication only in two-player games.
and Hart (1999) use the Nash equilibrium concept and characterize the set of outcomes implementable for two-player games in which only one player has private information. Amitai (1996) generalizes Aumann and Hart’s (1999) results to two-player games in which both sides have private information. Urbano and Vila (1997, 2001) demonstrate that if the two players have bounded rationality and can solve only problems of limited computational complexity, then it is possible to implement all correlated equilibria and all communication equilibria without mediation. Results have also been obtained for specific games with two players, like double auctions (Matthews and Postlewaite (1989)) and the “battle of the sexes” (Banks and Calvert (1992)).

For games with four or more players, Barany (1992) and Forges (1990) provide a complete characterization of the effects of unmediated communication when the solution concept is Nash equilibrium. Barany (1992) considers static games of complete information and shows that any rational correlated equilibrium (i.e. any correlated equilibrium with rational components) can be implemented. Moreover, if a game also has rational parameters, then the set of outcomes attainable with unmediated communication coincides with the set of correlated equilibria (see Forges (1990)). Games of incomplete information are considered by Forges (1990). She demonstrates that if a game has four or more players and rational parameters, then the set of communication equilibria completely characterizes the outcomes achievable with unmediated communication.3

To implement a correlated equilibrium, Barany (1992) constructs a scheme of unmediated communication such that an action profile is chosen according to the correlated equilibrium distribution, and each player learns only her own action. Each message is sent by two players to a third one, and public verification of the past record is possible. To prevent

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3 Forges (1990) proposes a mechanism of communication in which players exchange messages at the ex-ante stage (before they learn their types) and at the interim stage (after each player learns her own type). However, Gerardi (2000) shows that in games with four or more players and rational parameters, it is possible to implement all communication equilibria even if communication takes place only at the interim stage.
unilateral deviations in the communication phase, Barany (1992) assumes that a player who receives two different messages stops the communication process and reports that cheating has occurred. Then messages are verified, and the deviator is identified and punished by the opponents who play a “minmax” strategy against her in the original game. If a receiver deviates and reports that cheating has occurred when it has not, then she is punished at her “minmax” level. Note that there are two situations in which Barany’s (1992) equilibrium fails to satisfy the sequential rationality criterion (Ben-Porath (1998, 2000)). First, the strategy profile that minmaxes a deviator is not necessarily a Nash equilibrium of the original game. Second, a receiver who observes a deviation may not have an incentive to report that cheating has occurred. These problems extend to Forges (1990), since she uses the communication scheme proposed by Barany (1992) to prove her result.

Ben-Porath (2000) uses the solution concept of sequential equilibrium. He provides sufficient conditions for a communication equilibrium to be implemented with unmediated communication in games with three or more players. Specifically, Ben-Porath (2000) shows that a communication equilibrium can be implemented provided that the game admits a Bayesian-Nash equilibrium in which the payoff of every type of each player is lower than the communication equilibrium payoff. The basic idea is that if a player deviates during the communication phase, then she is punished by her opponents, who play the Bayesian-Nash equilibrium. Using this approach one cannot implement, for example, a communication equilibrium in which the payoff of a player is lower than all Bayesian-Nash equilibrium payoffs: no punishment is available to prevent the player from deviating in the communication phase.

We propose a different system of communication that avoids this problem. We provide a complete characterization of the set of outcomes that can be implemented with unmediated communication in games with five or more players and rational parameters. Our solution concept in the extended games with communication is sequential equilibrium. We show that in games of complete information an outcome can be implemented with direct
communication if and only if it is a correlated equilibrium. For games with incomplete information and full support, we demonstrate that the set of outcomes achievable with unmediated communication coincides with the set of communication equilibria.

Our results provide support for the use of correlated and communication equilibria to describe the effects of communication. If a game has at least five players, rational parameters and full support, one can use the simple structure of the set of correlated or communication equilibria even if an impartial mediator is not available and players are sequentially rational.

The rest of the paper is organized as follows. In Section 2 we analyze games with complete information and in Section 3 we study games with incomplete information. In Section 4 we illustrate an application of our results in the context of mechanism design. Section 5 contains the proof of the main theorem and Section 6 concludes.

2 Games with Complete Information

In this section we restrict attention to games of complete information, and characterize the set of outcomes that players can achieve with unmediated communication.

Let $\Gamma = \langle P_1, \ldots, P_I, S_1, \ldots, S_I, u_1, \ldots, u_I \rangle$ be a finite normal-form game, where $P_1, \ldots, P_I$ are the players, $S_i$ is the set of actions available to player $P_i$, $S = \prod_{i=1}^I S_i$ is the set of action profiles, and $u_i : S \to \mathbb{R}$ is the payoff function of player $P_i$. We let $S_{-i} = \prod_{j\neq i} S_j$ denote the set of profiles of actions of players different from $P_i$. The set of probability distributions over $S$ is denoted by $\Delta(S)$.

We extend $\Gamma$ by allowing players to communicate before they choose their actions. A cheap talk extension of $\Gamma$ is an extensive-form game that consists of a communication phase and an action phase. During the communication phase, players exchange “cheap” messages, i.e. messages that do not affect directly their payoffs. Then, in the action phase, the original game $\Gamma$ is played.

To describe our results on unmediated communication, it is convenient first to recapitu-
late the more general case of mediated communication. We consider cheap talk extensions in which players exchange messages among themselves and communicate with an impartial mediator (see, for example, Myerson (1991)).

A strategy profile in a cheap talk extension of $\Gamma$ induces an outcome in $\Gamma$, i.e. a probability distribution over $S$. Let $N(\Gamma)$ denote the set of outcomes in $\Gamma$ induced by Nash equilibria of cheap talk extensions of $\Gamma$ (where mediated communication is allowed), and $S(\Gamma)$ denote the set of outcomes induced by sequential equilibria. In other words, a probability distribution $r \in \Delta(S)$ is in $N(\Gamma)$ (respectively, $S(\Gamma)$) if and only if there exists a cheap talk extension of $\Gamma$ and a Nash equilibrium (respectively, sequential equilibrium) of that extension that induces $r$.

To characterize the sets $N(\Gamma)$ and $S(\Gamma)$, we need to introduce the notion of correlated equilibrium. A probability distribution $r \in \Delta(S)$ is a correlated equilibrium of $\Gamma$ if and only if:

$$\sum_{s \in S} r(s) \left( u_i(s) - u_i(s_{-i}, \delta_i(s_i)) \right) \geq 0, \quad i = 1, \ldots, I, \quad \forall \delta_i : S_i \to S_i. \quad (1)$$

We let $C(\Gamma)$ denote the set of correlated equilibria of $\Gamma$. Being defined by finitely many linear inequalities, $C(\Gamma)$ is a convex polyhedron.

In the analysis below, we shall devote special attention to the class of correlated equilibria with rational components. We say that a correlated equilibrium $r$ is rational if for every action profile in $S$, the probability $r(s)$ is a rational number.

Any correlated equilibrium is in the set $S(\Gamma)$ (and therefore in $N(\Gamma)$). Let $r$ be a correlated equilibrium, and consider the following cheap talk extension of $\Gamma$. The mediator randomly selects an action profile in $S$ according to $r$, and informs each player $P_i$ only of the $i$th component of the chosen action profile. Then the players simultaneously choose their actions. Inequality (1) guarantees the existence of a sequential equilibrium in which each player follows the mediator’s recommendation. Clearly, the induced outcome in $\Gamma$ is the correlated equilibrium $r$.

The revelation principle for normal-form games states that any probability distribution
in $N(\Gamma)$ is a correlated equilibrium of $\Gamma$ (Forges (1986) and Myerson (1982)). Thus, it follows that both $N(\Gamma)$ and $S(\Gamma)$ coincide with $C(\Gamma)$, the set of correlated equilibria of $\Gamma$.

We now consider unmediated communication. We assume that a reliable mediator is not available, and consider cheap talk extensions defined as follow. We start by introducing \textit{plain} cheap talk extensions. Given a normal-form game $\Gamma$, a plain cheap talk extension of $\Gamma$ consists of a finite number of steps of communication, at the end of which $\Gamma$ is played. For each step, the plain cheap talk extension specifies the senders, that is, the players who are allowed to send a message. Further, for each sender, it specifies the receivers and the set of messages. Plain cheap talk extensions allow for simultaneous messages (if the sets of senders contain more than one player) and sequential messages (if the sets of senders are singleton). Further, we can have private messages, public messages, and intermediate situations in which a sender sends a message to a subset of her opponents.

We are ready to define a cheap talk extension with unmediated communication. In the first step, a number of players (possibly zero) sends \textit{public} messages. To each vector of messages the cheap talk extension associates a plain cheap talk extension.

In a plain cheap talk extension, the identity of senders and receivers and the set of messages of each step are predetermined and do not depend on previous messages. In contrast, in a cheap talk extension, the public messages sent in the first step may affect the set of messages and the identity of senders and receivers of subsequent steps. Of course, if the number of players sending public messages in the first step is zero, then the cheap talk extension is plain. In other words, plain cheap talk extensions are a special case of cheap talk extensions with unmediated communication.

Let $S^U(\Gamma)$ denote the set of outcomes of $\Gamma$ induced by a sequential equilibrium of some cheap talk extension with unmediated communication. Clearly, since mediated communication is more general than unmediated communication, $S^U(\Gamma)$ is included in $S(\Gamma)$ (and, therefore, in $N(\Gamma)$ and $C(\Gamma)$).

Our first result shows that, if there are at least five players, the set $S^U(\Gamma)$ contains all
rational correlated equilibria of $\Gamma$ (i.e. all correlated equilibria with rational components). When a correlated equilibrium $r$ belongs to $S^U(\Gamma)$, we say that $r$ can be implemented.

**Theorem 1** Let $\Gamma$ be a finite normal-form game with five or more players, and let $r$ be a rational correlated equilibrium of $\Gamma$. Then $r \in S^U(\Gamma)$.

A formal proof of Theorem 1 is in Section 5, where we construct $\bar{\Gamma}(r)$, a finite plain cheap talk extension of $\Gamma$ with unmediated communication, and a sequential equilibrium $\Psi(r)$ of $\bar{\Gamma}(r)$ that induces the distribution $r$ on $S$.

To implement a rational correlated equilibrium $r$, we propose a scheme of communication such that, if all players follow it, an action profile in $S$ is chosen according to the distribution $r$. Moreover, when the communication phase ends, player $P_i$, $i = 1, \ldots, I$, learns only which action in $S_i$ she has to play. Therefore, each player has an incentive to play the action that she learns, provided that her opponents behave likewise.

To give an intuition of our proof and explain how players learn their actions, let us consider the following simple example. There are three players, $P_1$, $P_2$, and $P_3$, and three action profiles, $s$, $s'$, and $s''$. Given an action profile, say $s$, let $pr_3(s)$ denote the action of $P_3$ in $s$. We assume $pr_3(s) = pr_3(s')$, and $pr_3(s) \neq pr_3(s'')$. The action profiles $s$ and $s'$ specify the same action for $P_3$, while $s''$ specifies a different action. Suppose we want the players to select an action profile at random, according to the uniform distribution. Moreover, $P_3$ has to learn her action in the chosen profile, while $P_1$ and $P_2$ do not have to learn anything.\(^4\) This can be accomplished in the following way. $P_1$ selects $\alpha$ and $\sigma$, two permutations on $\{s, s', s''\}$, at random, according to the uniform distribution, and independently of each other. $P_2$ selects $x$, an element of $\{s, s', s''\}$, at random, according to the uniform distribution. The chosen profile is $\sigma(x)$. Clearly, $\sigma(x)$ has a uniform distribution and is unknown to all players. We require $P_1$ to send the permutation $\alpha \sigma$ to

\(^4\)To keep the example as simple as possible, we require that only $P_3$ learns her action. Obviously, to implement a correlated equilibrium, each player has to learn her action in the chosen profile (see Section 5 for details).
Since $\alpha$ and $\sigma$ are independent of each other and uniformly distributed, $P_2$ does not learn anything about $\sigma$ (the conditional distribution of $\sigma$, given $\alpha \sigma$, is uniform). Then $P_2$ sends the element $\alpha \sigma (x)$ to $P_3$. Finally, $P_1$ sends $pr_3 \circ \alpha^{-1}$ (a mapping from $\{s, s', s''\}$ to $P_3$’s set of actions) to $P_3$. At this point $P_3$ computes $pr_3 \circ \alpha^{-1}(\alpha \sigma (x)) = pr_3 (\sigma (x))$ and learns her action in the chosen profile.

This system of communication, although simple, presents a serious problem: $P_1$ and $P_2$ might have an incentive to deviate from it. If, for example, $P_2$ sends $P_3$ a message different from $\alpha \sigma (x)$, $P_3$ will not learn the correct action. Clearly, in a game $P_2$ might prefer to induce $P_3$ not to learn the correct action.

We construct a more complex communication scheme that solves this problem and gives all players an incentive to follow it. We require that a message is sent by three different players to a fourth one. In equilibrium, all senders report the correct message. Moreover, we show that it is sequentially rational for a receiver to follow the message sent by the majority of the senders. This implies that a sender does not have profitable deviations during the communication stage. Even if she deviates, the receiver will receive the correct message from the other two senders.

Our communication scheme differs from the systems of communication proposed by Barany (1992) and Ben-Porath (1998, 2000) in other important aspects. In fact, the idea of using the majority rule to prevent unilateral deviations in the conversation phase cannot be applied to their communication schemes if there are only five players. Similarly to the example above, in Barany (1992) and Ben-Porath (1998, 2000), a combination of two random variables determines the action profile in $S$ that players play once communication is over (in our example the two variables are $\sigma$ and $x$). Clearly, it is crucial that a player does not know both random variables, otherwise she learns her opponents’ actions. However, this requirement cannot be satisfied if there are five players and every message is sent by three different players. In this case, at least one player must know both random variables. In our communication scheme, the action profile chosen depends on four random variables.
In this way, we are able to construct a communication scheme in which every message is sent by three players and no player learns her opponents’ actions.

This also explains our need for at least five players. A message goes from three players to a fourth one. This already requires four players. But if there are only four players all messages are public, and all players can learn the chosen action profile. At least another player is needed to generate private messages. Thus five is the smallest number of players that we need to use our communication scheme.

Finally, in contrast to Barany (1992) and Ben-Porath (1998), our system of communication does not require public verification of past messages. In the game that we present is Section 5, players can exchange oral messages.

Theorem 1 considers only rational correlated equilibria. An obvious question is whether players can implement correlated equilibria that involve probabilities which are irrational numbers. We now show that a correlated equilibrium with irrational components can be implemented provided that it can be expressed as a convex combination of rational correlated equilibria. Let $r_1, ..., r_K$ be $K$ rational correlated equilibria of a game, $\Gamma$, with five or more players. Consider the correlated equilibrium $r = \sum_{k=1}^{K} \rho_k r_k$, where $\rho_1, ..., \rho_K$ are positive numbers (rational or irrational) such that $\sum_{k=1}^{K} \rho_k = 1$. Theorem 1 guarantees that for every rational correlated equilibrium $r_k$, $k = 1, ..., K$, there exists a plain cheap talk extension $\bar{\Gamma}(r_k)$, and a sequential equilibrium $\Psi(r_k)$ of $\bar{\Gamma}(r_k)$ that induces the distribution $r_k$ on $S$.

To implement $r$, we let players perform a “jointly controlled lottery” in which every correlated equilibrium $r_k$ is selected with probability $\rho_k$. Specifically, consider the following cheap talk extension $\bar{\Gamma}(r)$. Players $P_1$ and $P_2$ simultaneously announce (to all players) two positive numbers in the unit interval. Let $\omega^i$ denote the number announced by player $P_i$,

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5 A jointly controlled lottery (Aumann, Maschler and Stearns (1968)) is a communication scheme that allows two or more players to randomly select an outcome. The scheme is immune against unilateral deviations in the sense that no player can, by her own decision, influence the probability distribution.
\( i = 1, 2 \). Let \( \omega = \chi (\omega^1, \omega^2) \), where
\[
\chi (\omega^1, \omega^2) := \begin{cases} 
\omega^1 + \omega^2 & \text{if } \omega^1 + \omega^2 \leq 1 \\
\omega^1 + \omega^2 - 1 & \text{if } \omega^1 + \omega^2 > 1
\end{cases}
\] (2)
and define \( \rho_0 \) to be equal to zero. If \( \omega \in \left(\sum_{j=0}^{k-1} \rho_j, \sum_{j=0}^{k} \rho_j\right) \), for some \( k = 1, ..., K \), then the game \( \bar{\Gamma} (r_k) \) is played.

Although Kreps and Wilson (1982) define sequential equilibria only for finite games, it is easy to extend their definition to \( \bar{\Gamma} (r) \). Notice that after \( P_1 \) and \( P_2 \) announce \( \omega^1 \) and \( \omega^2 \), respectively, a proper subgame is induced. We require that the equilibrium strategies and beliefs of \( \bar{\Gamma} (r) \), when restricted to a subgame \( \bar{\Gamma} (r_k) \), constitute a sequential equilibrium of \( \bar{\Gamma} (r_k) \). Moreover, player \( P_i, i = 1, 2 \), chooses the number \( \omega^i \) to maximize her expected payoff (given her strategy in the rest of the game and her opponents’ strategies).

Consider the following assessment. In the first step, both \( P_1 \) and \( P_2 \) randomly select a number in the interval \((0, 1]\), according to the uniform distribution. If the cheap talk extension \( \bar{\Gamma} (r_k) \) is selected, players play the sequential equilibrium \( \Psi (r_k) \). It is easy to check that this assessment constitutes a sequential equilibrium of the game \( \bar{\Gamma} (r) \). The only thing to note here is that, since \( \omega^2 \) is independent of \( \omega^1 \) and uniformly distributed, \( \omega \) is also independent of \( \omega^1 \) and uniformly distributed. This implies that \( P_1 \) is indifferent between all possible announcements in the first step (clearly, the same argument can be applied to \( P_2 \)). Therefore, we have proved that the correlated equilibrium \( r \) can be implemented.

For an important class of games, we are able to provide a complete characterization of the set of outcomes that can be implemented. We say that a normal-form game \( \Gamma \) is rational if all its parameters are rational numbers, i.e. if for every \( i = 1, ..., I \), and for every strategy profile \( s \) in \( S \), the payoff \( u_i (s) \) is a rational number. When \( \Gamma \) is rational, any correlated equilibrium can be expressed as a convex combination of rational correlated equilibria.\(^6\) Thus, any correlated equilibrium is in the set \( S^U (\Gamma) \). Since it is always the case that \( S^U (\Gamma) \) is contained in \( C (\Gamma) \), we conclude that the two sets are identical. The

\(^6\)If \( \Gamma \) is rational, the vertices of \( C (\Gamma) \) are rational correlated equilibria.
next corollary summarizes our results.

**Corollary 1** Let $\Gamma$ be a finite normal-form game with five or more players. Let $r$ be a convex combination of rational correlated equilibria of $\Gamma$. Then $r \in S^U(\Gamma)$. If $\Gamma$ is rational, $S^U(\Gamma) = C(\Gamma)$.

For any two-player game $\Gamma$, it is easy to show that $S^U(\Gamma)$ is the convex hull of the Nash equilibrium outcomes of $\Gamma$. Intuitively, when there are only two players, all messages are public, and therefore only lotteries over Nash equilibria can be implemented (see Aumann and Hart (1999)). Thus, in the class of games with rational parameters, a complete characterization of the set $S^U(\Gamma)$ is unavailable only when $\Gamma$ has three or four players. Obviously, our communication scheme cannot be applied if there are less than five players, but this does not rule out the possibility that all correlated equilibria can be implemented with some other communication scheme. We do not have any example of a correlated equilibrium in a game with three or four players that cannot be implemented. Barany (1992) provides the example of a rational correlated equilibrium in a three-player game that cannot be implemented (in Nash equilibrium) if the message spaces are finite. However, Forges (1990) shows that the correlated equilibrium considered by Barany (1992) is induced by a Nash equilibrium in a cheap talk extension where a continuum of messages is used. For a game $\Gamma$ with three or four players, a partial characterization of the set $S^U(\Gamma)$ is provided by Ben-Porath (2000), who gives sufficient conditions for a rational correlated equilibrium to be implemented.

We conclude this section with a comment on our definition of unmediated communication. The cheap talk extensions defined in this section are not the most general form of unmediated communication. One could think, for example, of situations in which a sender can choose the receivers of her message. However, Corollary 1 shows that restricting attention to our cheap talk extensions is without loss of generality in games with five or more players and rational parameters. For these games, allowing for more general forms of un-
mediated communication does not expand the set of outcomes that players can implement.

3 Games with Incomplete Information

We now consider games of incomplete information and show how unmediated communication allows players to expand the set of equilibrium outcomes.

Let $G = \langle P_1, ..., P_I, T_1, ..., T_I, S_1, ..., S_I, p, u_1, ..., u_I \rangle$ be a finite Bayesian game. Players are $P_1, ..., P_I$, and $S_i$ denotes the set of actions of $P_i$. As in Section 2, $S$ denotes the set of action profiles, $S_{-i}$ is the set of profiles of actions of players different from $P_i$, and $\Delta(S)$ denotes the set of probability distributions over $S$. In addition, $T_i$ is the set of types of $P_i$, and $T = \prod_{i=1}^I T_i$ is the set of type profiles. We let $T_{-i} = \prod_{j \neq i} T_j$ denote the set of profiles of types of players different from $P_i$. The payoffs of $P_i$ are described by $u_i : T \times S \to \mathbb{R}$. Finally, $p$ is a probability distribution over $T$.\footnote{For notational simplicity, we assume that beliefs in $G$ are consistent.} We say that game $G$ has full support if every profile of types occurs with positive probability, i.e. if $p(t) > 0$ for every $t$ in $T$. Games in which players’ types are independent constitute an obvious example of games with full support.

We proceed as in Section 2, and extend $G$ by introducing pre-play communication. Although we are interested in unmediated communication, it is convenient to start our analysis by considering first the case of mediated communication.

A strategy profile in a cheap talk extension of $G$ induces an outcome in $G$, i.e. a mapping from $T$ to $\Delta(S)$. As in the previous section, we let $N(G)$ denote the set of outcomes in $G$ induced by a Bayesian-Nash equilibrium of some cheap talk extension of $G$ (where mediated communication is allowed). Similarly, $S(G)$ denotes the set of outcomes in $G$ induced by a sequential equilibrium of some cheap talk extension of $G$.

The set $N(G)$ can be easily characterized in terms of communication equilibria. A
function \( q : T \to \Delta (S) \) is a communication equilibrium if and only if:

\[
\begin{align*}
\sum_{t_{-i} \in T_{-i}} \sum_{s \in S} p(t_{-i}|t_i) q(s|t_{-i}, t_i) u_i((t_{-i}, t_i) \neq s) & \geq \\
\sum_{t_{-i} \in T_{-i}} \sum_{s \in S} p(t_{-i}|t_i) q(s|t_{-i}, t_i') u_i((t_{-i}, t_i), (s_{-i}, \delta_i(s_i))) ,
\end{align*}
\]

(3)

We let \( CE(G) \) denote the set of communication equilibria of \( G \). \( CE(G) \) is defined by finitely many linear inequalities, and therefore is a convex polyhedron. We also say that a communication equilibrium \( q \) is rational if, for every action profile \( s \) in \( S \) and every type profile \( t \) in \( T \), the probability \( q(s|t) \) is a rational number.

To see that a communication equilibrium \( q \) belongs to the set \( N(G) \), consider the following cheap talk extension \( G^D(q) \), usually called the canonical game. First, each player \( P_i \) sends the mediator a message in \( T_i \). The mediator, after receiving a vector of messages \( t \), randomly selects an action profile in \( S \) according to the probability distribution \( q(\cdot|t) \) and informs each player \( P_i \) only of the \( i \)th component of the chosen profile. Finally, all players simultaneously choose their actions. It follows from inequality (3) that there exists a Bayesian-Nash equilibrium of \( G^D(q) \) in which every player reports her type truthfully to the mediator, and follows the mediator’s recommendation. The notion of communication equilibrium is a generalization of the notion of correlated equilibrium for games with incomplete information. In a correlated equilibrium, a mediator allows players to coordinate their actions. In a communication equilibrium, the mediator has two roles: she helps players coordinate their actions, and exchange their private information. The two notions of equilibria coincide in games with complete information.

On the other hand, it follows from the revelation principle that any outcome in \( N(G) \) is a communication equilibrium (see Forges (1986) and Myerson (1982)). Therefore, for any finite Bayesian game \( G \), the sets \( N(G) \) and \( CE(G) \) coincide.

The characterization of the set \( S(G) \) is, in general, less immediate. Clearly, for any game \( G \), \( S(G) \) is included in \( CE(G) \). Further, the set \( S(G) \) is non-empty (sequential equilibria exist in any extension of \( G \)) and convex (the mediator conducts a lottery among
different extensions and announces publicly the outcome of the lottery). To investigate whether any communication equilibrium of \( G \) is in \( S(G) \), we first need some definitions.

Given a communication equilibrium \( q \) of \( G \), consider the canonical game \( G^D(q) \). We say that \( q \) is regular if there is a sequential equilibrium of \( G^D(q) \) in which each player reports her type truthfully to the mediator, and obeys the mediator’s recommendation after being honest (a player may disobey the mediator’s recommendation if she did not report her type truthfully). Clearly, any regular communication equilibrium of \( G \) belongs to \( S(G) \).

When \( G \) has full support, it is easy to show that any communication equilibrium is regular. When all type profiles have positive probability, a player never learns that some other player lied to the mediator. This fact and inequality (3) imply that for a player who reports her type sincerely, it is sequentially rational to obey the mediator’s suggestion. Moreover, inequality (3) also guarantees that no player has an incentive to lie to the mediator. Thus, if \( G \) has full support, \( S(G) \) is equal to \( CE(G) \).

This result does not hold in games without full support. As we show in the next example, it is possible to construct a game \( G' \), without full support, such that the set \( S(G') \) is strictly included in the set of communication equilibria \( CE(G') \).

**Example 1** A Bayesian game \( G' \) with \( CE(G') \nsubseteq S(G') \).

\( G' \) is a two-player game. The set of types of players \( P_1 \) and \( P_2 \) are \( T_1 = \{t^1_1, t^2_1\} \) and \( T_2 = \{t^1_2, t^2_2\} \), respectively. \( P_1 \) has to choose an action from the set \( S_1 = \{s^1, s^2, s^3\} \), while \( P_2 \) does not choose an action. The probability distribution over the set of type profiles is:

\[
p(t^1_1, t^1_2) = p(t^1_1, t^2_2) = p(t^2_1, t^1_2) = \frac{1}{3}, \quad p(t^2_1, t^2_2) = 0.\]

Payoffs are described in Table 1, where, for each combination of type profile and action, we report the corresponding vector of payoffs (the first entry denotes \( P_1 \)'s payoff).

First, let us consider the game \( G' \) without communication. In any Bayesian-Nash equilibrium, type \( t^2_1 \) plays action \( s^2 \), and type \( t^1_1 \) chooses either \( s^1 \), or \( s^3 \), or a randomization between the two actions.
We now provide a complete characterization of the set $S(G^0)$. Consider a cheap talk extension of $G^0$. Since type $t_2^1$ of player $P_1$ knows that player $P_2$ has type $t_2^1$, sequential rationality implies that in every information set in which $t_2^1$ has to choose an action from $S_1$, she will play $s_2$.

Further, type $t_1^1$ of $P_1$ never chooses action $s_2$. In fact, independent of $P_2$’s type, action $s_2$ is dominated by actions $s_1$ and $s_3$. We now show that if $q$ belongs to $S(G^0)$, then $q(t_1^1, t_1^2) = q(t_2^1, t_2^2)$. Both types of $P_2$ prefer action $s_3$ to any other action when $P_1$ has type $t_1^1$. Suppose, by contradiction, that there exists a sequential equilibrium of a cheap talk extension that induces an outcome $q$, with $q(s_3|t_1^1, t_1^2) > q(s_3|t_1^1, t_2^2)$. Then type $t_2^2$ would have an incentive to deviate and mimic the behavior of type $t_1^1$ to increase the probability of action $s_3$. Similarly, suppose that a sequential equilibrium induces an outcome $q$ such that $q(s_3|t_1^1, t_2^2) < q(s_3|t_1^1, t_2^2)$. Type $t_1^1$ knows that, independent of her strategy, type $t_2^2$ of $P_1$ will play action $s_2$. Therefore $t_2^2$ has an incentive to mimic the behavior of $t_2^2$: if $P_1$ has type $t_2^1$, the probability of action $s_3$ will increase.

We conclude that $S(G^0)$ coincides with the set of equilibrium outcomes of $G'$. If the players are sequentially rational, communication cannot expand the set of equilibrium outcomes of $G'$. Given this, it is very easy to show that $S(G^0)$ is strictly included in the set of communication equilibria of $G'$. A communication equilibrium that does not belong to $S(G^0)$ is, for example, $q'$ defined by:

$$q'(t_1^1, t_1^2) = q'(t_2^1, t_2^2) = s_1^1, \quad q'(t_1^1, t_2^2) = s_3^3, \quad q'(t_2^1, t_1^2) = s_2^2,$$

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Table 1: Payoffs of the game $G'$
where we adopt the convention of writing, for example, \( q'(t^1_1, t^1_2) = s^1 \) to denote \( q'(s^1|t^1_1, t^1_2) = 1 \).

Although not all communication equilibria of \( G' \) belong to \( S(G') \), the revelation principle is still valid in the above example. Any element of \( S(G') \) is a regular communication equilibrium, and can be implemented with the canonical game. This result could suggest that for a Bayesian game \( G \), \( S(G) \) coincides with the set of regular communication equilibria. However, it turns out that this conjecture is incorrect. As the next example demonstrates, when a game \( G \) does not have full support, \( S(G) \) may contain communication equilibria that are not regular. In other words, the revelation principle does not hold in games without full support when the solution concept is sequential equilibrium.

**Example 2** The failure of the revelation principle.

Consider the following three-person game \( G'' \). The set of types of players \( P_1 \) and \( P_2 \) are

\[ T_1 = \{t^1_1, t^1_2\}, \quad \text{and} \quad T_2 = \{t^2_1, t^2_2\}, \]

respectively. \( P_3 \) does not have private information and is the only player to choose an action, from the set \( S_3 = \{s^1, s^2, s^3, s^4\} \). The probability distribution over the set of profiles of types is given by:

\[ p(t^1_1, t^2_1) = p(t^1_1, t^2_2) = p(t^2_1, t^1_2) = \frac{1}{3}, \quad p(t^2_1, t^2_2) = 0. \]

Finally, Table 2 describes the vector of payoffs for each pair of type profile and action (the first entry refers to \( P_1 \), the second one to \( P_2 \)).

The communication equilibrium of \( G'' \) that maximizes \( P_3 \)'s expected payoff is unique and equal to \( q'' \), where:

\[ q''(t^1_1, t^1_2) = s^1, \quad q''(t^1_1, t^2_2) = q''(t^2_1, t^1_2) = s^2, \quad q''(s^3|t^1_1, t^2_2) = q''(s^4|t^1_1, t^2_2) = \frac{1}{2}. \]

We now show that the communication equilibrium \( q'' \) is not regular. Consider the canonical game \( G''D(q'') \). \( P_3 \) has four information sets, one for each recommendation that the mediator can send. After receiving a recommendation, \( P_3 \) computes her beliefs over
the set \{\( (t_1^1, t_2^1) \), \( (t_1^2, t_2^2) \), \( (t_3^1, t_4^1) \) \}. In any consistent assessment of \( G''D(q'') \), \( P_3 \)'s beliefs after recommendation \( s^3 \) must coincide with her beliefs after \( s^4 \).\(^8\) Notice that it is sequentially rational for \( P_3 \) to choose action \( s^3 \) only if she assigns probability one to the profile of types \( (t_1^1, t_2^2) \). On the other hand, it is optimal for \( P_3 \) to play \( s^4 \) only if she assigns probability one to \( (t_1^2, t_2^1) \). But since \( P_3 \) has the same beliefs after \( s^3 \) and \( s^4 \), it cannot be sequentially rational to obey both recommendations. We conclude that \( q'' \) cannot be implemented with the canonical game \( G''D(q'') \).

However, \( q'' \) does belong to the set \( S(G'') \). Consider the following cheap talk extension \( \widehat{G}''(q'') \). Players \( P_1 \) and \( P_2 \) simultaneously report their messages to the mediator. Then the mediator recommends an action to \( P_3 \). Finally, \( P_3 \) chooses an action from \( S_3 \). The set of messages of \( P_i \), \( i = 1, 2 \), is \( \widehat{T}_i = \{t_1^i, t_2^i, t_3^i, t_4^i\} \), where \( t_3^i \) and \( t_4^i \) are two arbitrary messages. In Table 3, we report the mediator’s recommendation to \( P_3 \) for each vector of reports of \( P_1 \) and \( P_2 \).

The mediator adopts a deterministic behavior unless \( P_1 \) and \( P_2 \) report messages \( t_1^2 \) and \( t_2^2 \), respectively. In this case the mediator randomizes, with equal probability, between

---

\[^8\]The only thing that \( P_3 \) learns after receiving recommendations \( s^3 \) or \( s^4 \) is that the mediator received message \( t_1^i \) from \( P_1 \), and \( t_2^i \) from \( P_2 \). The mediator performs the lottery between \( s^3 \) and \( s^4 \) after \( P_1 \) and \( P_2 \) send their messages. Therefore, to prove the consistency of an assessment of \( G''D(q'') \), we cannot choose trembles for \( P_1 \) or \( P_2 \) that depend on the outcome of the mediator’s lottery. This, in turn, implies that \( P_3 \)'s beliefs after \( s^3 \) cannot differ from her beliefs after \( s^4 \).
Table 3: Mediator’s recommendation in $\hat{G}''(q'')$

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recommendations $s^3$ and $s^4$.

It is easy to show that the cheap talk extension $\hat{G}''(q'')$ admits a sequential equilibrium in which $P_1$ and $P_2$ “reveal their types truthfully” (i.e. $t_{ij}$ sends message $t_{ij}$, $i = 1, 2$, $j = 1, 2$), and $P_3$ obeys the mediator’s recommendation (clearly, this equilibrium induces $q''$). Intuitively, by introducing the new messages $t_3^i$ and $t_4^i$, $i = 1, 2$, we allow player $P_3$ to have different beliefs after recommendations $s^3$ and $s^4$ (recommendation $s^3$ can be induced by reports $(t_1^3, t_2^3)$, and $s^4$ by reports $(t_1^4, t_2^4)$). In this way, both obedience to $s^3$ and obedience to $s^4$ can be sequentially rational (see Gerardi (2001) for details).

A complete characterization of the set $S(G)$ when $G$ is a game without full support is still an open question and beyond the scope of this paper. In Gerardi (2001), we provide the solution for a special class of games. Specifically, we characterizes $S(G)$ when $G$ is a game in which one player is uninformed and has to choose an action, while all other players have private information but do not choose an action.

We now assume that an impartial mediator is not available and turn to unmediated communication. A cheap talk extension of $G$ is defined as follows. First, Nature selects a type profile $t$ according to the probability distribution $p$ and each player learns her own type. Then, in the communication phase, players exchange “cheap” messages as described in Section 2. Finally, in the action phase, players simultaneously choose their actions.

We denote by $S^U(G)$ the set of outcomes in a Bayesian game $G$ induced by a sequential equilibrium of some cheap talk extension with unmediated communication. Clearly, $S^U(G)$ is included in $CE(G)$.
We say that a communication equilibrium can be implemented if it belongs to $S^U(G)$.

We are ready to state our first result for games of incomplete information.

**Theorem 2** Let $G$ be a finite Bayesian game with five or more players, and let $q$ be a rational and regular communication equilibrium of $G$. Then $q \in S^U(G)$.

The proof of Theorem 2 (presented in Appendix B) consists of two steps. In the first one, we use a result due to Forges (1990). Consider a communication equilibrium (not necessarily rational) $q$ of a finite Bayesian game $G$ with at least four players. Forges (1990) constructs a cheap talk extension $G^F(q)$ in which, first, the players receive messages from a mediator, then exchange public and private messages, and, finally, choose their actions. Notice that in $G^F(q)$ the players do not send messages to the mediator, i.e. the mediator is a correlation device. Forges (1990) shows that $q$ is the outcome induced by a Bayesian-Nash equilibrium of $G^F(q)$. We demonstrate that if $q$ is regular, then the equilibrium in $G^F(q)$ that implements $q$ can be made sequential. However, communication in $G^F(q)$ is still mediated, since the mediator has to send messages to the players at the beginning of $G^F(q)$. In the second part of our proof, we use the fact that the communication equilibrium is rational and that there are at least five players. Under these assumptions, we show that the players, after learning their types, can use the communication scheme presented in Section 5 to generate the mediator’s messages. In other words, we construct a finite plain cheap talk extension $ar{G}(q)$ which starts with the communication scheme described in Section 5. At the end of it, the players exchange public and private messages as in $G^F(q)$ and, finally, choose their actions. We conclude our proof by showing that $ar{G}(q)$ admits a sequential equilibrium, $\Phi(q)$, that induces $q$.

Theorem 2 does not pertain to communication equilibria that are not regular or that have some irrational components. However, these equilibria can be implemented, provided that they can be expressed as a convex combination of rational and regular communication
equilibria. As in Section 2, we require the players to conduct a jointly controlled lottery among the cheap talk extensions in which the rational and regular communication equilibria are implemented.

Specifically, consider a game $G$ with at least five players. Let $q$ be a convex combination of $K$ regular and rational communication equilibria $q_1, ..., q_K$, with weights $\rho_1, ..., \rho_K$, respectively. It follows from Theorem 2 that for every $q_k (k = 1, ..., K)$ there exists a plain cheap talk extension $\tilde{G}(q_k)$, and a sequential equilibrium $\Phi(q_k)$ of $\tilde{G}(q_k)$ that induces $q_k$. We construct the following cheap talk extension $\tilde{G}(q)$. At the beginning of $\tilde{G}(q)$, players $P_1$ and $P_2$ simultaneously announce to all players two positive numbers in the unit interval ($P_i$ announces $\omega^i$, $i = 1, 2$). Let $\omega = \chi(\omega^1, \omega^2)$, where $\chi(\omega^1, \omega^2)$ is defined in equation (2), and let $\rho_0$ be equal to zero. If $\omega \in \left(\sum_{j=0}^{k-1} \rho_j, \sum_{j=0}^{k} \rho_j\right)$, for some $k = 1, ..., K$, then the cheap talk extension $\tilde{G}(q_k)$ is played.

The cheap talk extension $\tilde{G}(q)$ is not a finite game, since the first two players can announce any number in the interval $(0, 1]$. Moreover, after $P_1$ and $P_2$ report $\omega^1$ and $\omega^2$, respectively, a proper subgame is not induced, since the players have private information about their types. In general, consistent beliefs have not been defined in games with infinite strategy sets. Some authors have extended the notion of sequential equilibrium only to specific classes of infinite games. However, for our purposes, it is enough to consider assessments of $\tilde{G}(q)$ in which all types of $P_i$, $i = 1, 2$, select $\omega^i$ at random, according to the uniform distribution. We call these assessments simple. For a simple assessment, it is easy to define consistent beliefs. In this case, observing $\omega^1$ and $\omega^2$ does not provide any information about the types of $P_1$ and $P_2$. A simple assessment is a sequential equilibrium if the strategies and the beliefs, when restricted to a game $\tilde{G}(q_k)$, form a sequential equilibrium of $\tilde{G}(q_k)$. Moreover, for every type of $P_i$, $i = 1, 2$, selecting $\omega^i$ at random according to the uniform distribution, is optimal among all behavioral strategies.

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9 Notice that in games without full support, the set of regular communication equilibria need not be convex.

10 Manelli (1996), for example, considers signaling games.
Consider the simple assessment where the players play the sequential equilibrium $\Phi(q_k)$ if game $\bar{G}(q_k)$ is selected, i.e. if $\omega \in \left( \sum_{j=0}^{k-1} \rho_j, \sum_{j=0}^{k} \rho_j \right]$. The analysis at the end of Section 2 shows that for every type of $P_i$ ($i = 1, 2$), it is optimal to choose $\omega^i$ randomly according to the uniform distribution. Therefore, we conclude that $q$ can be implemented.

For some games, it is possible to provide a precise characterization of the set of communication equilibria that can be implemented. We say that a game has rational parameters if for every $i = 1, \ldots, I$, every action profile $s$ in $S$ and every profile of types $t$ in $T$, the payoffs $u_i(t, s)$ and the probability $p(t)$ are rational numbers. If $G$ has rational parameters, the vertices of $CE(G)$ are rational communication equilibria. Moreover the vertices of $CE(G)$ are regular communication equilibria if $G$ has full support (remember that in this case all communication equilibria are regular). Therefore, when $G$ has at least five players, rational parameters and full support, every communication equilibrium can be implemented. We can summarize our findings as follows.

**Corollary 2** Let $G$ be a finite Bayesian game with five or more players. Let $q$ be a convex combination of rational and regular communication equilibria of $G$. Then $q \in S^U(G)$. If $G$ has full support and rational parameters, $S^U(G) = CE(G)$.

It is an open question whether it is possible to implement communication equilibria that cannot be expressed as convex combinations of rational and regular communication equilibria. As Corollary 2 suggests, a complete characterization of the set of outcomes that can be implemented with unmediated communication is not available for games without full support or with irrational parameters or with less than five players. As has already been stated, for a game $G$ of incomplete information with at least three players, a partial characterization of $S^U(G)$ is provided by Ben-Porath (2000).
4 Mechanism Design with Imperfect Commitment

This section illustrates how our results on unmediated communication can be applied to solve a mechanism design problem. We consider a decision maker (or principal) who has to select an action, the payoff of which depends on the unknown state of the world. A number of experts (or agents) have private information about the state and are affected by the principal’s decision.

We model this environment as a Bayesian game, \( G_0 = (T_1, \ldots, T_{I-1}, S_I, p, u_1, \ldots, u_I) \). The principal, player \( I \), does not have private information and her set of actions is \( S_I \). Players \( 1, \ldots, I-1 \) are agents, and \( T_i \) denotes the set of types of agent \( i \). We let \( T = \prod_{i=1}^{I-1} T_i \) denote the set of the states of the world, and \( p \) is a probability distribution over \( T \). As usual, we use \( T_{-i} \) to denote the set of profiles of types of agents different from \( i \). Finally, the payoffs of each player \( i \) are described by the function \( u_i : T \times S_I \rightarrow \mathbb{R} \). We assume that \( G_0 \) is finite and has full support, rational parameters and at least four agents.

The decision maker faces the problem of finding a way to elicit as much information as possible from the experts. We start the analysis by considering the case where the principal can fully commit to a mechanism. In our environment, a mechanism is any function of the form \( \kappa : \Sigma_1 \times \cdots \times \Sigma_{I-1} \rightarrow \Delta (S_I) \), where \( \Sigma_i \) denotes the set of strategies available to agent \( i \).

For every profile of agents’ strategies, the mechanism specifies the probability distribution according to which the principal will select an action in \( S_I \). Direct-revelation mechanisms constitute a very important class of mechanisms. In a direct-revelation mechanism the set of strategies of each agent \( i \) coincides with \( T_i \), the set of her types.

A mechanism defines, in an obvious way, a game among the agents. As in the previous sections, our solution concept is sequential equilibrium. For any profile of agents’ strategies the principal can compute her expected payoff. The goal of the decision maker is to find the mechanism that gives her the highest sequential equilibrium payoff.\(^{11}\) The revelation

\(^{11}\) Actually, when full commitment to a mechanism is possible, the solution to the principal’s problem does not change if the solution concept is Bayesian-Nash equilibrium.
principle greatly simplifies this problem (see Dasgupta, Hammond and Maskin (1979), Harri
ds and Townsend (1981) and Myerson (1979), among others). According to this powerful
result, the decision maker can restrict attention to truthful equilibria\textsuperscript{12} of direct-revelation
mechanisms.

Let $q$ denote a mapping from the set of states $T$ to $\Delta(S_I)$, the set of probability distribu-
tions over $S_I$. The principal’s highest payoff is associated with the truthful equilibrium
of the direct-revelation mechanism $q^{FC}$, where $q^{FC}$ is a solution to the following problem:

$$\max_{q: T \to \Delta(S_I)} \sum_{t \in T} \sum_{s_I \in S_I} p(t) q(s_I | t) u_I(t, s_I),$$

subject to:

$$\sum_{t-i \in T-i} \sum_{s_I \in S_I} p(t-i | t_i) q(s_I | (t-i, t_i)) u_i((t-i, t_i), s_I) \geq \sum_{t-i \in T-i} \sum_{s_I \in S_I} p(t-i | t_i) q(s_I | (t-i, t_i')) u_i((t-i, t_i'), s_I),$$

$$i = 1, ..., I - 1, \quad \forall (t_i, t_i') \in T_i^2.$$  \hfill (5)

The decision maker recognizes that agents need incentives to reveal their information.
The constraints in inequality (5), usually called informational incentive constraints, guaran-
tee that it is rational for every agent to report her type truthfully to the principal, provided
that all the other agents behave likewise.

The game defined by the mechanism $q^{FC}$ may have other equilibria, in which the agents
do not report their types truthfully to the decision maker. When we use the revelation
principle, we implicitly assume that not only does the principal choose the mechanism, but
she can also designate a specific equilibrium of it. For example, the decision maker can
recommend (by a public announcement) all agents to play the truthful equilibrium in the
direct-revelation mechanism $q^{FC}$ (i.e. the truthful equilibrium becomes a focal point).

However, in many situations, a decision maker cannot fully commit to a mechanism.
Experts often express non-binding opinions and the principal has the power to choose
any action she desires. Therefore, we now model a situation of imperfect commitment.

\textsuperscript{12}As the name suggests, in a truthful equilibrium all agents honestly report their types.
Specifically, we assume that the decision maker can commit to a particular cheap talk extension of $G_0$, but she cannot commit to a strategy in the extended game. In other words, once the cheap talk extension is chosen, the principal becomes a player.\footnote{To give an example of what we mean by imperfect commitment, suppose that the principal chooses a cheap talk extension that contains an information set in which she has to send either message 0 or message 1. Full commitment to a specific behavioral strategy is not possible, but when the information set is reached the principal has only two options: she can say either 0 or 1 (she cannot send any other message or be silent).} Further, we assume that an impartial mediator is not available, and thus, we restrict attention to unmediated communication systems.

The problem of the decision maker is to find the cheap talk extension of $G_0$ with the highest sequential equilibrium payoff. Bester and Strausz (2001) and Wolinsky (1999) analyze a similar problem of mechanism design with imperfect commitment. Bester and Strausz (2001) consider the case where there is only one agent, and restrict attention to cheap talk extensions in which the agent sends a costless message, and the principal selects an action. They show that in order to maximize her expected utility, the decision maker chooses the game in which the set of messages coincides with the set of types of the agent. Wolinsky (1999) considers the case of multiple agents and allows the decision maker to partition the set of experts into different groups. Agents in the same group communicate among themselves and send a joint report to the principal who, in turn, chooses an action. Wolinsky (1999) shows that to elicit more information from the experts, the principal should allow partial communication among the agents, that is, the size of a group should be greater than one but smaller than the number of agents.

The results derived in the previous section allow us to find the optimal mechanism for the decision maker. Since $G_0$ has at least four agents (i.e. there are five or more players), rational parameters and full support, the set of outcomes that are induced by sequential equilibria in cheap talk extensions of $G_0$ coincides with the set of communication equilibria of $G_0$. So, we proceed in two steps. We first compute the communication equilibrium that maximizes the decision maker’s expected payoff. This simply requires maximization of the
expression (4) subject to the informational incentive constraint (5) and to the following constraint:

$$\sum_{t \in T} \sum_{s_I \in S_I} p(t) q(s_I | t) (u_I (t, s_I) - u_I (t, \delta_I (s_I))) \geq 0, \quad \forall \delta_I : S_I \rightarrow S_I.$$  \hspace{1cm} (6)

Constraint (6) is an obedience constraint, that provides the principal with the incentive to follow the mediator’s recommendation. Since $G_0$ has rational parameters, the maximization problem above admits a solution with rational components, which we denote by $q^*$. Moreover, the communication equilibrium $q^*$ is regular, since $G_0$ has full support.

The second step of our procedure concerns the implementation of $q^*$. To do this, the decision maker selects the cheap talk extension $\overline{G_0} (q^*)$ and recommends\(^{14}\) the sequential equilibrium $\Phi (q^*)$ (see Appendix B).

## 5 Proof of Theorem 1

In this section we prove that when a normal-form game $\Gamma$ has five or more players, any rational correlated equilibrium can be implemented. Given a rational correlated equilibrium $r$ of $\Gamma$, we construct a finite plain cheap talk extension $\overline{\Gamma} (r)$, and a sequential equilibrium $\Psi (r)$ of $\overline{\Gamma} (r)$ that induces the distribution $r$ on $S$.

We first illustrate how the players can generate the probability distribution $r$. Since $r$ is rational, there exists a positive integer $\tilde{m}$ (greater than one) and, for every $s$ in $S$, a non-negative integer $\tilde{m}_s$ such that $r(s) = \frac{\tilde{m}_s}{\tilde{m}+1}$. Define $X = \{1, \ldots, \tilde{m}, \tilde{m} + 1\}$, and let $\{X_s\}_{s \in S}$ be a partition of $X$ such that $|X_s| = \tilde{m}_s$ for every $s$ in $S$. For $i = 1, \ldots, I$, let the projection $pr_i : S \rightarrow S_i$ be defined by $pr_i (s) = s_i$ if $s = (s_1, \ldots, s_i, \ldots, s_I)$. We extend each projection $pr_i$ to $X$ as follows:

$$pr_i (x) = s_i \quad \text{if } x \in X_s \quad \text{and} \quad pr_i (s) = s_i.$$

\(^{14}\)The cheap talk extension $\overline{G_0} (q^*)$ admits multiple equilibria. Similarly to the case of full commitment, we assume that the decision maker can induce all agents to play the equilibrium she prefers.
It is easy to verify that if an element \( y \in X \) is randomly selected according to the uniform distribution, and if every player \( P_i \) chooses the action \( pr_i(y) \), then every action profile \( s \) in \( S \) is chosen with probability \( r(s) \).

Let \( \Lambda(X) \) denote the set of permutations on \( X \), and let \( m + 1 \) denote the cardinality of \( \Lambda(X) \). The sets \( X, \Lambda(X) \) and the projections \( pr_1, ..., pr_I \) are common knowledge among the players.

It is useful to divide the game \( \bar{\Gamma}(r) \) into several steps. For each step, we first illustrate the game and then present equilibrium behavioral strategies (hereafter simply called equilibrium strategies) and equilibrium beliefs. Then, we prove that the assessment \( \Psi(r) \) that we propose is sequentially rational. The proof that \( \Psi(r) \) is also consistent is relegated to Appendix A.

**Step 0. Random choices.**

In this step, which consists of several substeps, players “jointly select” random permutations on \( X \) and a random element of \( X \). We will explain later how the joint selections are made. We first list the random choices that players make. \( P_1, P_2, P_3, P_4, P_5, ..., P_I \) jointly select permutation \( \sigma \).\(^{15}\) \( P_2, P_3, P_4, P_5, P_6, ..., P_I \) choose two permutations, \( \tau \) and \( \beta_{41} \). \( P_2, P_3, P_4, P_5 \) choose permutation \( \beta_{31} \). \( P_1, P_3, P_4, P_5, P_6, ..., P_I \) select two permutations, \( \varphi \) and \( \beta_{42} \). \( P_1, P_3, P_4, P_5 \) choose permutation \( \beta_{32} \). \( P_1, P_2, P_3, P_5 \) select an element \( x \in X \). Finally, for \( i = 1, ..., 4 \), players in the set \( \{P_1, P_2, P_3, P_4\} \setminus \{P_i\} \) choose permutation \( \alpha_i \).

We let \( \mathcal{J}_i \) denote the set of random choices known to \( P_i \) at the end of Step 0. It is convenient to summarize Step 0 in Table 4, where for every player \( P_i \), we list the elements of the set \( \mathcal{J}_i \).

There is a separate substep for each random choice. Every choice is made according to the uniform distribution over the underlying probability space, and every choice is made independently of all others.

\(^{15}\)The proof we present is valid both for the cases \( I = 5 \) and \( I > 5 \). Clearly, any reference to player \( P_k \), with \( k > 5 \), is relevant only if there are more than five players.
Table 4: Random Choices of Step 0

We now describe how the players jointly select a random permutation, or an element of $X$. Consider, for example, the substep in which $P_1, P_2, P_3, P_4, P_6, ..., P_I$ have to select the random permutation $\sigma$. The two players with the lowest indices ($P_1$ and $P_2$ in this case) make two announcements simultaneously. Specifically, $P_i, i = 1, 2$, announces a permutation $\sigma^i \in \Lambda(X)$ to players in the set $\{P_1, P_2, P_3, P_4, P_6, ..., P_I\} \setminus \{P_i\}$. The chosen permutation will be $\sigma = \sigma^1\sigma^2$ (notice that $\sigma$ is common knowledge among $P_1, P_2, P_3, P_4, P_6, ..., P_I$). In equilibrium, $P_i$ selects a permutation $\sigma^i$ at random, according to the uniform distribution on $\Lambda(X)$. This implies that the random permutation $\sigma$ is uniformly distributed.

A similar procedure is used to make the remaining random choices.\(^\text{16}\) In every substep, the two players who have to make an announcement choose their messages randomly, according to the uniform distribution on the underlying probability space, independently of the messages that they have already sent or received.

Equilibrium beliefs are very simple. At the end of Step 0, a player either knows the realization of a given random variable, or believes that the random variable is uniformly distributed.

\(^\text{16}\)To select the random element $x$, $P_1$ and $P_2$ simultaneously announce $x^1 \in X$ and $x^2 \in X$, respectively. The chosen element is $x = x^1 + x^2$, where $+$ is mod $\tilde{m} + 1$.  

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**Sketch of $\bar{\Gamma}(r)$.**

To provide the reader with a better understanding of our construction, let us outline the rest of the game $\bar{\Gamma}(r)$ before presenting the next steps. The state $y := \sigma \tau \varphi(x)$ determines the action profile of the original game $\Gamma$ that players choose in the action phase (see below). In equilibrium, $y$ has a uniform distribution on $X$ since $\sigma$, $\tau$, $\varphi$ and $x$ are uniformly distributed. Notice that no player knows the realization of $y$ at the end of Step 0. This is crucial, since a player who is informed about $y$ knows the actions of the game $\Gamma$ that her opponents play when communication is over. However, Step $i$, $i = 1, ..., 4$, is carefully designed to make $P_i$ learn $\alpha_i(y)$ and nothing more ($P_5$ learns $\alpha_2(y)$ in Step 2, and for $k > 5$, $P_k$ learns $\alpha_4(y)$ in Step 4). The permutations $\alpha_1, ..., \alpha_4$ prevent a player from learning the state $y$. In Step 5 $P_i$, $i = 1, ..., 4$, learns the function $pr_i\alpha_i^{-1}$ ($P_5$ learns $pr_5\alpha_2^{-1}$, and for $k > 5$, $P_k$ learns $pr_k\alpha_4^{-1}$). So player $P_i$ learns her action $pr_i(y)$. Since the action profile $(pr_1(y), ..., pr_1(y))$ is chosen according to the correlated equilibrium distribution $r$, $P_i$ has exactly the same information as in the case in which she receives recommendation $pr_i(y)$ from a mediator who implements $r$. In the sixth and last step, every player $P_i$ chooses action $pr_i(y)$. Notice that $pr_i(y)$ is optimal for $P_i$ given her information and her opponents’ actions.

After Step 0, the game $\bar{\Gamma}(r)$ proceeds as follows.

**Step 1.** $P_2$, $P_3$ and $P_4$ send $P_1$ the permutation $\alpha_1\sigma\tau$.

Player $P_i$, $i = 2, 3, 4$, sends $P_1$ a permutation on $X$. The three senders report their messages simultaneously. Let $a$ be a permutation on $X$. The meaning of message $a$ in Step 1 is “The realization of the random permutation $\alpha_1\sigma\tau$ is $a$”. To simplify the presentation we say that in Step 1, $P_2$, $P_3$ and $P_4$ send $P_1$ the permutation $\alpha_1\sigma\tau$ (we adopt this terminology to describe the next steps).

The equilibrium strategies prescribe that each sender reports the realization of $\alpha_1\sigma\tau$ to $P_1$. For example, if the realizations of the random permutations $\alpha_1$, $\sigma$ and $\tau$ are $\hat{\alpha}_1$, $\hat{\sigma}$ and
\( \hat{\tau} \), respectively, in equilibrium \( P_2, P_3 \) and \( P_4 \) send \( P_1 \) the message \( \tilde{\alpha}_1 \tilde{\sigma} \tilde{\tau} \).\(^{17}\)

Before presenting equilibrium beliefs, let us clarify our terminology. As we will see, in Steps 1-5 of the game \( \tilde{\Gamma}(r) \), each message is sent by three players to a fourth one. We will often consider the message sent either by the majority of the senders, if a majority exists, or by the first sender in the description of the step (\( P_2 \) in Step 1), if the senders report three different messages. For expositional reasons, we refer to this message simply as the message sent by the majority of the senders. Moreover, we occasionally refer to the message sent by the majority as the message that the receiver “learns”. Finally, we let \( w_j \) denote a generic triple of messages sent in Step \( j \), and we use \( z_j^i \) to denote the message that \( P_i \) sends in Step \( j \).

We are now ready to describe equilibrium beliefs. At the end of Step 1, \( P_1 \) assigns probability one to the event that the realization of \( \alpha_1 \sigma \tau \) coincides with the message sent by the majority of the senders (we denote this message by \( \tilde{\alpha}_1 \tilde{\sigma} \tilde{\tau} \)). For any triple of messages \( w_1 \) we have:

\[
\Pr(\alpha_1 \sigma \tau = \tilde{\alpha}_1 \tilde{\sigma} \tilde{\tau}|w_1, J_1) = 1.
\]

This result is trivial if the three senders follow their equilibrium strategies and report the same message. To obtain the same result off the equilibrium path, in Appendix A we construct a sequence of completely mixed strategies that satisfies two requirements. First, \( P_2 \) is less likely to deviate than \( P_3 \) and \( P_4 \). Second, the probability that any two senders deviate converges to zero faster than the probability that only the third sender deviates. These two assumptions guarantee that in the limit (i.e. as the probability of deviating converges to zero) the majority of the senders report the true message with probability one.

In Appendix A, we also show that, conditional on \( P_1 \)’s information, both \( \alpha_1 \) and \( \tau \) are uniformly distributed over \( \Lambda(X) \). Intuitively, knowing the realization of \( \alpha_1 \sigma \tau \) does not make any realization of \( \tau \) more likely than the others. Every realization of \( \tau \) is made

\(^{17}\)We adopt the following notation: \( o \) for a random variable and \( \hat{o} \) for its realization.
compatible with the realization of $\alpha_1 \sigma \tau$ by one (and only one) realization of $\alpha_1$. Since $\alpha_1$ and $\tau$ are independent and uniformly distributed, the marginal distribution of $\tau$, given $\alpha_1 \sigma \tau$, is uniform. This, in turn, implies that the conditional distribution of the state $y = \sigma \tau \varphi(x)$ is uniform over $X$. In other words, $P_1$ does not learn anything new about $y$.

Our equilibrium beliefs do not say anything about $\beta_{31}$ and $\beta_{41}$, two random permutations unknown to $P_1$. To keep notation simple, we adopt the convention that the equilibrium beliefs specified in Step 0 hold when we are silent about a random variable. Thus, conditionally on $P_1$’s information, both $\beta_{31}$ and $\beta_{41}$ have a uniform distribution over $\Lambda(X)$.

Notice that $P_1$ knows (from Step 0) the permutation $\varphi$ and the element $x$. On the equilibrium path, she receives the realization of the permutation $\alpha_1 \sigma \tau$ (from all senders) and therefore learns $\alpha_1 \sigma \tau \varphi(x) = \alpha_1(y)$.

**Step 2.**

In this step, $P_5$ and $P_2$ learn $\alpha_2(y)$. Since $P_2$ does not know the permutation $\varphi$, she receives the permutation $\alpha_2 \sigma \tau \varphi$. This message should be sent by three senders. However, only two players, $P_3$ and $P_4$, know the realizations of $\sigma$, $\tau$ and $\varphi$. We divide Step 2 into two substeps: 2.1 and 2.2. In the first substep, $P_5$ (who knows $\tau$ and $\varphi$) receives the permutation $\alpha_2 \sigma$ from $P_1$, $P_3$ and $P_4$. Notice that $P_5$ does not know $\alpha_2$ and therefore does not learn anything about $\sigma$. In the second substep, $P_3$, $P_4$ and $P_5$ send the permutation $\alpha_2 \sigma \tau \varphi$ to $P_2$.

**Substep 2.1.** $P_1$, $P_3$ and $P_4$ send $P_5$ the permutation $\alpha_2 \sigma$. 

In equilibrium, $P_1$, $P_3$ and $P_4$ report the realization of $\alpha_2 \sigma$ to $P_5$. This means that $P_1$, $P_3$ and $P_4$ send the true message in any information set of Substep 2.1. As far as equilibrium beliefs are concerned, they are very similar to those derived in Step 1. Specifically, $P_5$ assigns probability one to the event that the majority of the senders sent the true message. We let $\overline{\alpha_2 \sigma}$ denote the permutation sent by the majority. For any triple of messages $w_{21}$ received by $P_5$, we have:

$$\Pr(\alpha_2 \sigma = \overline{\alpha_2 \sigma} | w_{21}, J_5) = 1.$$
Moreover, conditional on the information of $P_5$ in Substep 2.1, both $\alpha_2$ and $\sigma$ have a uniform distribution over $\Lambda(X)$.

$P_5$ knows the random permutations $\tau$ and $\varphi$, and the random element $x$. Since in equilibrium $P_5$ learns $\alpha_2 \sigma$, she can compute $\alpha_2 \sigma \tau \varphi(x) = \alpha_2 (y)$.

**Substep 2.2.** $P_3, P_4$ and $P_5$ send $P_2$ the permutation $\alpha_2 \sigma \tau \varphi$.

The equilibrium strategies prescribe that $P_3$ and $P_4$ report the realization of $\alpha_2 \sigma \tau \varphi$ to $P_2$. In equilibrium, $P_5$ sends $P_2$ the permutation $\alpha_2 \sigma \tau \varphi$.

As usual, we denote the message sent by the majority of the senders to $P_2$ by $\overline{\alpha_2 \sigma \tau \varphi}$. In Substep 2.2, $P_2$ knows $z_1^2$, the message that she sent in Step 1. For any message $z_1^2$ and for any triple of messages $w_{22}$ received in Substep 2.2, $P_2$’s beliefs are given by:

$$\Pr(\alpha_2 \sigma \tau \varphi = \overline{\alpha_2 \sigma \tau \varphi}, \overline{\alpha_2 \sigma} = \alpha_2 \sigma | w_{22}, z_1^2, J_2) = 1.$$  

With probability one, the realization of the random permutation $\alpha_2 \sigma \tau \varphi$ is the message sent by the majority of the senders. Further, independently of the triple of messages received, $P_2$ assigns probability one to the event that $P_5$ learnt the realization of $\alpha_2 \sigma$ in Substep 2.1. In Appendix A, we also show that, conditional on $P_2$’s information, both $\varphi$ and $\alpha_2$ are uniformly distributed over $\Lambda(X)$. This implies that the conditional distribution of the random state $y$ is uniform over $X$.

$P_2$ knows the realization of $x$, so she can compute $\overline{\alpha_2 \sigma \tau \varphi}(x)$. In equilibrium, $\overline{\alpha_2 \sigma \tau \varphi} = \alpha_2 \sigma \tau \varphi$, i.e. $P_2$ learns the realization of $\alpha_2 \sigma \tau \varphi$. Thus, at the end of Step 2, $P_2$ knows $\alpha_2 \sigma \tau \varphi(x) = \alpha_2 (y)$.

**Step 3.**

This step is designed to make $P_3$ learn the random element $\alpha_3(y)$. Since no player knows $y$, we need some preliminary steps in which the senders of the message to $P_3$ learn

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Note that from $P_5$’s point of view, $\overline{\alpha_2 \sigma}$ is a degenerate random permutation ($P_5$ knows the message that the majority of her senders report in Substep 2.1). In Substep 2.2, $P_2$ does not know the realization of $\overline{\alpha_2 \sigma}$. Technically speaking, $\overline{\alpha_2 \sigma}$ in Substep 2.1 is a different object from $\overline{\alpha_2 \sigma}$ in Substep 2.2. However, since the meaning of $\overline{\alpha_2 \sigma}$ in every step is clear and no ambiguity arises, we decide not to introduce further notation. We follow this convention in the rest of the section.
the necessary information. Specifically, in Substep 3.1, $P_5$ receives the permutation $\alpha_3\sigma$ from $P_1$, $P_2$ and $P_4$. Notice that $P_5$ does not know $\alpha_3$ and so she does not learn anything new about $\sigma$. In Substep 3.2, $P_2$, $P_4$ and $P_5$ send player $P_1$ the permutation $\beta_{31}\alpha_3\sigma\tau$. The permutation $\beta_{31}$ is unknown to $P_1$ and is used to prevent her from learning the realization of $\tau$. Then, in Substep 3.3, $P_2$ receives the permutation $\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi$ from $P_1$, $P_4$ and $P_5$. The fact that $P_2$ does not know $\beta_{32}$ implies that she does not get any information about the unknown permutation $\varphi$. Finally, in Substep 3.4, $P_1$, $P_2$ and $P_5$ send the element $\beta_{32}^{-1}\beta_{31}^{-1}(\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi(x)) = \alpha_3(y)$.

**Substep 3.1.** $P_1$, $P_2$ and $P_4$ send $P_5$ the permutation $\alpha_3\sigma$.

The equilibrium strategies prescribe that the three senders report the realization of $\alpha_3\sigma$. The message sent by the majority is denoted by $\overline{\alpha_3\sigma}$. At the end of Substep 3.1, $P_5$ knows the random choices in $I_5$, the message that she sent in Substep 2.2, $z_{22}^5$, and the triples of messages received in Substeps 2.1 and 3.1, $w_{21}$ and $w_{31}$. Given this information, $P_5$’s beliefs are given by:

$$
\Pr \left( \alpha_2\sigma = \overline{\alpha_2\sigma}, \alpha_3\sigma = \overline{\alpha_3\sigma} | w_{21}, w_{31}, z_{22}^5, I_5 \right) = 1.
$$

According to these beliefs, the majority of the senders report the true message in Substep 3.1. Further, the additional information that $P_5$ obtains in Substep 3.1 does not modify her beliefs from Substep 2.1. In Appendix A, we also demonstrate that in this step, $P_5$ does not learn anything about the state $y$. The conditional distribution of $\sigma$ is uniform.

**Substep 3.2.** $P_2$, $P_4$ and $P_5$ send $P_1$ the permutation $\beta_{31}\alpha_3\sigma\tau$.

In equilibrium, $P_2$ and $P_4$ send the realization of $\beta_{31}\alpha_3\sigma\tau$. $P_5$ sends the permutation $\overline{\beta_{31}\alpha_3\sigma\tau}$. We let $\overline{\beta_{31}\alpha_3\sigma\tau}$ denote the message sent by the majority of the senders to $P_1$. Equilibrium beliefs are given by:

$$
\Pr \left( \beta_{31}\alpha_3\sigma\tau = \overline{\beta_{31}\alpha_3\sigma\tau}, \alpha_3\sigma = \overline{\alpha_3\sigma}, \alpha_1\sigma\tau = \overline{\alpha_1\sigma\tau} | w_{32}, w_1, z_{31}^1, z_{21}^1, I_1 \right) = 1.
$$

First, with probability one, the message sent by the majority is the realization of the
random permutation $\beta_{31}\alpha_3\sigma\tau$. Second, $P_5$ learnt the realization of $\alpha_3\sigma$ in Substep 3.1 with probability one. Moreover, the additional information that $P_1$ gets in Substep 3.2 does not modify her beliefs from Step 1. Finally, the proof in Appendix A shows that $P_1$ does not update her beliefs about the marginal distributions of $\tau, \alpha_1$ and $\beta_{31}$ (conditional on $P_1$’s information, these random permutations are uniformly distributed).

**Substep 3.3.** $P_4, P_1$ and $P_5$ send $P_2$ the permutation $\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi$.

The equilibrium strategy of $P_4$ is to send $P_2$ the realization of the random permutation $\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi$ (notice that $P_4$ knows this permutation). Moreover, in equilibrium, $P_1$ reports the realization of $\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi$ and $P_5$ sends the realization of $\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi$.

Equilibrium beliefs are similar to those described in Substep 3.2. Specifically, $P_2$ assigns probability one to the event that the realization of the permutation $\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi$ coincides with $\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi$, the message sent by the majority of the senders. Moreover, with probability one, both $P_5$ and $P_1$ learnt the truth in Substeps 3.1 and 3.2. Finally, equilibrium beliefs derived in Substep 2.2 still hold. Formally, we have:

$$\Pr(\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi = \overline{\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi}, \beta_{31}\alpha_3\sigma\tau = \beta_{31}\alpha_3\sigma\tau, \overline{\alpha_3\sigma} = \alpha_3\sigma, \alpha_2\sigma\tau\varphi = \overline{\alpha_2\sigma\tau\varphi}, \alpha_2\sigma = \alpha_2\sigma|w_{33}, w_{22}, z_{32}, z_{31}, z_{11}, J_2) = 1.$$  

The proof that the assessment $\Psi(r)$ is consistent also shows that $P_2$ does not obtain any information about the state $y$. The conditional distribution of the permutation $\varphi$ is uniform.

**Substep 3.4.** $P_1, P_2$ and $P_5$ send $P_3$ the element $\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi(x)$.

$P_1, P_2$ and $P_5$ simultaneously send $P_1$ an element of the set $X$. In equilibrium, $P_1$ sends the message $\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi(x)$, $P_2$ reports the message $\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi(x)$ and $P_5$ chooses the element $\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi(x)$.

The message sent by the majority of the senders is denoted by $\overline{\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi(x)}$. $P_3$’s beliefs are given by:

$$\Pr(\overline{\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi(x)} = \beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi(x), \beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi = \beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi, \overline{\beta_{31}\alpha_3\sigma\tau} = \beta_{31}\alpha_3\sigma\tau, \overline{\alpha_3\sigma} = \alpha_3\sigma|w_{34}, z_1^3, z_{21}^3, z_{22}^3, J_3) = 1.$$
According to equilibrium beliefs, the message sent by the majority coincides with the realization of $\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi(x)$. Further, each sender knows the realization of the random permutation $\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi$. In Appendix A, we also show that, conditional on the information that $P_3$ has in Substep 3.4, $\alpha_3$ and $x$ are uniformly distributed on the underlying probability spaces. This, in turn, implies that the conditional distribution of the state $y$ is uniform (in other words, $P_3$ learns $\alpha_3(y)$ but nothing more).

**Step 4.**

We provide a brief description of this step, since it is almost identical to the previous one. The purpose of Step 4 is to let $P_4$ (and $P_6, ... P_I$, if $I > 5$) learn the realization of the random element $\alpha_4(y)$.

In Substep 4.1, $P_1$, $P_2$ and $P_3$ send $P_5$ the permutation $\alpha_4\sigma$. In Substep 4.2, $P_1$ receives the permutation $\beta_{41}\alpha_4\sigma\tau$ from $P_2$, $P_3$ and $P_5$. Then, in Substep 4.3, $P_3$, $P_1$ and $P_5$ report the permutation $\beta_{42}\beta_{41}\alpha_4\sigma\tau\varphi$ to $P_2$. In Substep 4.4, $P_1$, $P_2$ and $P_5$ send $P_4$ the element $\beta_{42}\beta_{41}\alpha_4\sigma\tau\varphi(x)$. $P_4$ knows the permutations $\beta_{41}$ and $\beta_{42}$, and so she learns $\alpha_4(y)$. As usual, $\beta_{42}\beta_{41}\alpha_4\sigma\tau\varphi(x)$ denotes the message sent by the majority of the senders to $P_4$ in Substep 4.4.

Step 4 ends here if there are exactly five players, otherwise it continues as follows. Consider $k = 6, ..., I$. In Substep 4. $(k-1)$, $P_1$, $P_2$ and $P_5$ send $P_k$ the element $\beta_{42}\beta_{41}\alpha_4\sigma\tau\varphi(x)$ \footnote{Then $P_k$ can use the permutations $\beta_{41}$ and $\beta_{42}$ to compute $\alpha_4(y)$.}. The message sent by the majority of the senders to $P_k$ is denoted by $\beta_{42}\beta_{41}\alpha_4\sigma\tau\varphi(x)^k$.

The equilibrium strategies of Step 4 are identical to the equilibrium strategies specified in Step 3. Every sender behaves sincerely, and uses the message sent by the majority of her senders when she does not know a random permutation.

According to equilibrium beliefs, the realization of the permutation that a player receives coincides with the message sent by the majority. Each receiver assigns probability one to the event that all her senders learnt the exact realization of the permutation that they report. Moreover, the information that a player obtains in Step 4 does not modify her
former beliefs. Finally, the conditional distribution of each random variable is uniform on the underlying probability space.

**Step 5.**

At this point of the game, every player knows an element of the set $X$. However, this element alone does not reveal any information about the state $y$. In this step every player $P_i$, $i = 1, \ldots, I$, learns which action in the set $S_i$ corresponds to $y$. We divide Step 5 into the following substeps.

**Substep 5.1.** $P_2$, $P_3$ and $P_4$ send $P_1$ the function $pr_1 \alpha_1^{-1}$.

In this step, $P_2$, $P_3$ and $P_4$ simultaneously send three messages to $P_1$. The (finite) set of feasible messages is denoted by $R_1$ and contains any mapping from $X$ to $S_1$ that can be generated by applying the function $pr_1$ to some permutation on $X$ (i.e. a message is feasible if and only if it can be expressed as $pr_1 \varsigma$, where $\varsigma$ is a permutation on $X$).

The equilibrium strategies prescribe that each sender sends $P_1$ the function $pr_1 \alpha_1^{-1}$. Note that all senders know the realization of the random permutation $\alpha_1$, and the projection $pr_1$ is common knowledge among the players.

As usual, according to equilibrium beliefs, the message sent by the majority (which we denote by $pr_1 \alpha_1^{-1}$) coincides with the realization of the random function $pr_1 \alpha_1^{-1}$. Moreover, $P_1$ does not modify her former beliefs. Formally, let $\mathcal{M}_1$ denote the sequence of messages sent and received by $P_1$ in Steps 1-5.1. We have:

$$
\Pr(pr_1 \alpha_1^{-1} = pr_1 \alpha_1^{-1}, \beta_{41} \alpha_4 \sigma \tau = \beta_{41} \alpha_4 \sigma \tau, \alpha_4 \sigma = \alpha_4 \sigma, \\
\beta_{31} \alpha_3 \sigma \tau = \beta_{31} \alpha_3 \sigma \tau, \alpha_3 \sigma = \alpha_3 \sigma, \alpha_1 \sigma \tau = \alpha_1 \sigma \tau | \mathcal{M}_1, J_1) = 1.
$$

Note that not all permutations on $X$ are compatible$^{20}$ with the message $pr_1 \alpha_1^{-1}$. Thus, at the end of this step, $P_1$ assigns probability zero to some realizations of $\alpha_1$. This, combined with the fact that $P_1$ knows the realization of $\alpha_1 \sigma \tau$ and $\sigma$, implies that some realizations of the random permutation $\tau$ have zero probability. Therefore, the conditional distribution

$^{20}$A permutation $\varsigma$ on $X$ is compatible with message $pr_1 \alpha_1^{-1}$ if $pr_1 \varsigma^{-1} = pr_1 \alpha_1^{-1}$. 
of the state $y$ cannot be uniform over $X$ anymore. We will come back to this point when we show that the assessment $\Psi (r)$ is sequentially rational.

The rest of Step 5 is similar to Substep 5.1. Specifically, consider $i = 2, 3, 4$. In Substep 5.i, players in the set $\{P_1, P_2, P_3, P_4\} \setminus \{P_i\}$ send $P_i$ the function $pr_i\alpha_i^{-1}$. In Substep 5.5, $P_1$, $P_3$ and $P_4$ send $P_5$ the function $pr_5\alpha_2^{-1}$. If there are more than five players, $P_1$, $P_2$ and $P_3$ send $P_k$ the function $pr_k\alpha_4^{-1}$ in Substep 5.k (where $k = 6, \ldots, I$). Sets of feasible messages, equilibrium strategies and equilibrium beliefs are similar to those described in Substep 5.1.

**Step 6.** The game $\Gamma$ is played.

In Step 6 all players simultaneously choose an action and then the game $\bar{\Gamma}(r)$ ends. $P_i$’s set of (pure) strategies in Step 6 is $S_i$, her set of actions in the original game.

Equilibrium strategies are formally described in Table 5. Roughly speaking, to choose an action, a player applies the projection function learned in Step 5 to the element of $X$ computed before Step 5. Consider, for example, $P_1$. In Step 1, she receives the permutation $\alpha_1\sigma\tau$ from the majority of the senders. $P_1$ knows (from Step 0) the realizations of $\phi$ and $x$ and computes the element $\alpha_1\sigma\tau\phi(x)$. In Substep 5.1, the majority of the senders send $P_1$ the function $pr_1\alpha_1^{-1}$. Her equilibrium strategy in Step 6 is to choose the action $pr_1\alpha_1^{-1}\alpha_1\sigma\tau\phi(x)$. The other players adopt similar strategies (see Table 5). Notice that the equilibrium strategy of a player in Step 6 does not depend on the messages that she sends in Steps 1-5.

**Sequential rationality.**

We now demonstrate that the assessment $\Psi (r)$ is sequentially rational. In particular, we restrict attention to player $P_1$ and show that, given her beliefs, she does not have profitable deviations. Our proof can be easily applied to any other player. Given the so-called “one-shot-deviation principle”, we only need to verify that deviations in a single information set are not profitable for $P_1$. 

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Consider an information set in Step 6. Given P₁'s beliefs, we may say that she knows\textsuperscript{21} the realization of \( \alpha_1 \sigma \tau \) and the realizations of the random variables in the set \( \mathcal{J}_1 \), where

\[
\mathcal{J}_1 = \{ \sigma, \varphi, x, \alpha_2, \alpha_3, \alpha_4, \beta_{32}, \beta_{42}, \beta_{34} \alpha_3 \sigma \tau, \beta_{41} \alpha_4 \sigma \tau, pr_1 \alpha_1^{-1} \}.
\]

Clearly, this implies that P₁ knows the realization of \( pr_1 (y) \), which we denote by \( s_1 \). Notice that \( y \) is independent of \( \mathcal{J}_1 \), since the random permutation \( \tau \) is independent of \( \mathcal{J}_1 \).

In any information set of Step 6, P₁ assigns probability one to the event that every player \( P_k, k = 2, ..., I \), chooses the action \( pr_k (y) \). We now compute the probability that P₁’s opponents play the action profile \( (s_2, ..., s_I) \) given her information. We have:

\[
\Pr \left( pr_2 (y) = s_2, ..., pr_I (y) = s_I | pr_1 (y) = s_1, \mathcal{J}_1, \alpha_1 \sigma \tau \right) = \frac{\Pr \left( pr_1 (y) = s_1, pr_2 (y) = s_2, ..., pr_I (y) = s_I, \mathcal{J}_1, \alpha_1 \sigma \tau \right)}{\Pr \left( pr_1 (y) = s_1, \mathcal{J}_1, \alpha_1 \sigma \tau \right)} = \frac{\Pr \left( pr_1 (y) = s_1, pr_2 (y) = s_2, ..., pr_I (y) = s_I \right)}{\Pr \left( pr_1 (y) = s_1, \mathcal{J}_1 \right)} = r \left( s_2, ..., s_I | s_1 \right),
\]

where \( r \) is the correlated equilibrium distribution that we want to implement. The third equality follows from \( y \) being independent of \( \mathcal{J}_1 \), and the last equality comes from the fact that \( y \) is uniformly distributed over \( X \). To explain the second equality, we need to

\textsuperscript{21}In fact, P₁ assigns probability one to the event that the messages sent by the majority coincide with the realizations of the corresponding permutations.

<table>
<thead>
<tr>
<th>Player</th>
<th>Equilibrium strategies</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_1 )</td>
<td>( pr_1 \alpha_1^{-1} \alpha_1 \sigma \tau \varphi (x) )</td>
</tr>
<tr>
<td>( P_2 )</td>
<td>( pr_2 \alpha_2^{-1} \alpha_2 \sigma \tau \varphi (x) )</td>
</tr>
<tr>
<td>( P_3 )</td>
<td>( pr_3 \alpha_3^{-1} \beta_{31} \beta_{32} \beta_{34} \alpha_3 \sigma \tau \varphi (x) )</td>
</tr>
<tr>
<td>( P_4 )</td>
<td>( pr_4 \alpha_4^{-1} \beta_{41} \beta_{42} \beta_{44} \beta_{41} \alpha_4 \sigma \tau \varphi (x) )</td>
</tr>
<tr>
<td>( P_5 )</td>
<td>( pr_5 \alpha_5^{-1} \beta_{2} \alpha_2 \sigma \tau \varphi (x) )</td>
</tr>
<tr>
<td>( P_6 )</td>
<td>( pr_6 \alpha_6^{-1} \beta_{41} \beta_{42} \beta_{44} \beta_{41} \alpha_4 \sigma \tau \varphi (x) )</td>
</tr>
<tr>
<td>( P_I )</td>
<td>( pr_I \alpha_I^{-1} \beta_{41} \beta_{42} \beta_{44} \beta_{41} \alpha_4 \sigma \tau \varphi (x) )</td>
</tr>
</tbody>
</table>

Table 5: Equilibrium Strategies in Step 6
introduce additional notation. Let $\bar{\varphi}$, $\tilde{x}$ and $\hat{pr}_1\alpha_1^{-1}$ denote the realizations of $\varphi$, $x$ and $pr_1\alpha_1^{-1}$ respectively. We say that the permutation $\alpha_1\sigma \tau$ on $X$ is compatible with $\tilde{\mathcal{I}}_1$ and $s_1$ if $\hat{pr}_1\alpha_1^{-1}\alpha_1\sigma \tau \bar{\varphi}(\tilde{x}) = s_1$. We let $Q$ denote the number of realizations of $\alpha_1\sigma \tau$ that are compatible with $\tilde{\mathcal{I}}_1$ and $s_1$. Then it is easy to show that for any compatible permutation $\alpha_1\sigma \tau$, the following holds:

$$
\Pr \left( \hat{pr}_1(y) = s_1, pr_2(y) = s_2, ..., pr_I(y) = s_I, \tilde{\mathcal{I}}_1, \alpha_1\sigma \tau = \alpha_1\sigma \tau \right) = \frac{1}{Q} \Pr \left( pr_1(y) = s_1, pr_2(y) = s_2, ..., pr_I(y) = s_I, \tilde{\mathcal{I}}_1 \right),
$$

and

$$
\Pr \left( pr_1(y) = s_1, \tilde{\mathcal{I}}_1, \alpha_1\sigma \tau = \alpha_1\sigma \tau \right) = \frac{1}{Q} \Pr \left( pr_1(y) = s_1, \tilde{\mathcal{I}}_1 \right).
$$

The two equalities above show that all realizations of $\alpha_1\sigma \tau$ compatible with $\tilde{\mathcal{I}}_1$ and $s_1$ are equally likely.

According to equation (7), $P_1$ has the same information as in the case where she receives recommendation $s_1$ from a reliable mediator who implements the correlated equilibrium $r$. Thus, the action $s_1$ maximizes $P_1$’s expected payoff. In other words, the equilibrium strategy of Step 6 is optimal for player $P_1$.

Consider now any $P_1$’s information set in Steps 1-5. Remember that $P_1$ assigns probability one to the event that all senders know the realization of the permutation that they report. Since every receiver follows the message sent by the majority of the senders, and since $P_1$’s action in Step 6 does not depend on her messages in Steps 1-5, we conclude that a deviation is not profitable.

Finally, we examine Step 0. Consider, for example, the substep in which $P_1$ and $P_2$ send two messages ($\sigma_1$ and $\sigma_2$, respectively) to determine the permutation $\sigma = \sigma_1\sigma_2$. Since $\sigma_2$ is uniformly distributed, the permutation $\sigma$ is independent of $\sigma_1$ and uniformly distributed. Any strategy is optimal in this substep for $P_1$, including the uniform randomization over the set $\Lambda(X)$. Clearly, a similar argument can be used to show that $P_1$ does not have profitable deviations in any other substep of Step 0. We conclude that the assessment $\Psi(r)$ is sequentially rational.
6 Conclusion

In this paper we characterize the outcomes of static games that sequentially rational players can implement with direct communication. We show that for a large class of games there is no difference between mediated and unmediated communication, in the sense that both forms of communication allow players to implement the same outcomes. Specifically, if a game has five or more players, rational parameters and full support, then the set of outcomes that can be implemented with unmediated communication coincides with the set of communication equilibria.

We use sequential equilibrium to analyze cheap talk extensions of a static game. However, our results provide a complete characterization of the effects of unmediated communication even for the case in which a solution concept weaker than sequential equilibrium, but stronger than Nash equilibrium (such as subgame-perfect equilibrium or perfect Bayesian equilibrium), is considered. In fact, by using a solution concept weaker than sequential equilibrium, one can always implement all the outcomes than we implement in this paper. If the solution concept is stronger than Nash equilibrium, only communication equilibria can be implemented. Thus, if a game satisfies our assumptions (five or more players, rational parameters and full support), the set of communication equilibria coincides with the set of outcomes that are induced by perfect Bayesian or subgame-perfect equilibria of cheap talk extensions with unmediated communication.

Future research is needed to obtain a complete characterization of the effects of direct communication in games that do not satisfy our assumptions. Moreover, we restrict our attention to static games. For extensive-form games, two important articles deal with mediated communication. Forges (1986) extends the notion of correlated and communication equilibria to multistage games. Myerson (1986) introduces a sequential rationality criterion in the context of multistage games with mediated communication. An interesting extension of this paper would be the analysis of unmediated communication in dynamic games.
Appendix A: Consistency of the Assessment $\Psi(r)$

In this appendix we show that the assessment $\Psi(r)$ presented in Section 5 is consistent. We construct a sequence of completely mixed strategies that converges to the equilibrium strategies. We compute beliefs along the sequence and show that, in the limit, they coincide with equilibrium beliefs. We proceed as in Section 5 and divide our analysis into different steps.

As far as Step 0 is concerned, we use the equilibrium strategies of Step 0 to construct a (constant) sequence of completely mixed strategies (in Step 0 all strategies are played with positive probability). This implies that, along the sequence, each random variable chosen in Step 0 has a uniform distribution on the underlying probability space. The assessment $\Psi(r)$ trivially satisfies consistency in Step 0.\(^{22}\)

Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence of positive numbers in the unit interval, converging to zero.

**Step 1.** $P_2$, $P_3$ and $P_4$ send $P_1$ the permutation $\alpha_1\sigma\tau$.

For any integer $n$, let $\varepsilon_{1,n} = \varepsilon_n^3$. We assume that in Step 1, $P_2$ follows her equilibrium strategy (and sends $P_1$ the realization of $\alpha_1\sigma\tau$) with probability $1 - \varepsilon_{1,n}^3$. To simplify the notation, we drop the subscript $n$ and say that $P_2$ reports the truth with probability $1 - \varepsilon_1^3$. $P_3$ and $P_4$ send the true message with probability $1 - \varepsilon_2^3$. All messages different from the realization of $\alpha_1\sigma\tau$ are equally likely. Suppose that the realization of the permutation $\alpha_1\sigma\tau$ is $\tilde{a}_1\tilde{\sigma}\tilde{\tau}$. Along the sequences of completely mixed strategies, $P_2$ sends the message $\tilde{a}_1\tilde{\sigma}\tilde{\tau}$ with probability $1 - \varepsilon_1^3$ and any other message in $\Lambda(X)$ with probability $\varepsilon_1^3$. $P_3$ and $P_4$ send the message $\tilde{a}_1\tilde{\sigma}\tilde{\tau}$ with probability $1 - \varepsilon_2^3$, and any other message with probability $\varepsilon_2^3$. We refer to these completely mixed strategies by saying that $P_2$ trembles from her equilibrium strategy with probability $\varepsilon_1^3$ and $P_3$ and $P_4$ tremble from their equilibrium strategies with probability $\varepsilon_1^3$. Trembles are independent across information sets.

We list the messages that a player receives in the order in which the senders appear in

\(^{22}\)Remember that at the end of Step 0, a player either knows the realization of a random variable or believes that the random variable is uniformly distributed.
the description of the step. For example, suppose that $a$, $b$ and $c$ are three permutations on $X$. In Step 1, the triple $(abc)$ means that the messages sent by $P_2$, $P_3$ and $P_4$ are $a$, $b$ and $c$, respectively. At the end of Step 1, $P_1$ faces one of the following five situations:

- $(aaa)$, which means that $P_2$, $P_3$ and $P_4$ sent the same message, $a$;
- $(aab)$, $(aba)$ and $(baa)$: in these three situations two players sent the same message, $a$, and one player sent a different message, $b$;
- $(abc)$: the senders reported three different messages.

We let $M = \{(aaa), (aab), (aba), (baa), (abc)\}$ denote the set of triples of messages in which $a$ is the message sent by the majority of the senders. Suppose that the realization of $\sigma$ is $\hat{\sigma}$ (note that $P_1$ knows $\hat{\sigma}$). Let $\hat{\tau}^1, \ldots, \hat{\tau}^{m+1}$ be $m + 1$ different permutations on $X$. Then for every $l = 1, \ldots, m + 1$, there exists a permutation $\hat{\alpha}^l_1$ in $\Lambda(X)$ such that $\hat{\alpha}^l_1 \hat{\sigma} \hat{\tau}^l = a$. Moreover, $l \neq l'$ implies $\hat{\alpha}^l_1 \neq \hat{\alpha}^{l'}_1$. We let $\Pr^n(\alpha_1 = \hat{\alpha}^l_1, \tau = \hat{\tau}^l|w_1, \sigma = \hat{\sigma})$ denote the probability that $\alpha_1 = \hat{\alpha}^l_1$ and $\tau = \hat{\tau}^l$ given that $P_1$ receives the triple of messages $w_1$, the realization of $\sigma$ is $\hat{\sigma}$ and the trembles of $P_2$, $P_3$ and $P_4$ are defined by $\varepsilon_{1,n}$.

We have to show that it satisfies consistency to assign probability one to the event that the message sent by the majority of the senders to $P_1$ is the realization of $\alpha_1 \sigma \tau$. Further, conditional on $P_1$’s information in Step 1, both $\alpha_1$ and $\tau$ are uniformly distributed over $\Lambda(X)$. To prove our claims it is enough to show that:

$$\lim_{n \to \infty} \Pr^n(\alpha_1 = \hat{\alpha}^l_1, \tau = \hat{\tau}^l|w_1, \sigma = \hat{\sigma}) = \frac{1}{m + 1},$$

for any triple of messages $w_1$ in $M$, and every $l = 1, \ldots, m + 1$.\footnote{Note that $P_1$ changes her beliefs about the joint distribution of $\alpha_1$ and $\tau$. After Step 0, $P_1$ believes that the pair of random permutations $(\alpha_1, \tau)$ is uniformly distributed on $\Lambda(X) \times \Lambda(X)$. On the other hand, at the end of Step 1, $P_1$ assigns probability $\frac{1}{m+1}$ only to the pairs of permutations $(\hat{\alpha}^l_1, \hat{\tau}^l)$, $l = 1, \ldots, m + 1$. However, this does not modify $P_1$’s beliefs about the marginal distribution of $\tau$ (which is uniform over $\Lambda(X)$).} So, let us first assume that
$P_1$ receives the triple of messages $(aaa)$ (this represents the situation that $P_5$ faces on the equilibrium path). By applying Bayes’ rule, we obtain:

$$\Pr^n\left(\alpha_1 = \hat{\alpha}_1^l, \tau = \hat{\tau}^l | (aaa), \sigma = \hat{\sigma}\right) = \frac{(1 - \varepsilon_1^3)(1 - \varepsilon_1^2)^2}{(m + 1) \left[ (1 - \varepsilon_1^3)(1 - \varepsilon_1^2)^2 + m \varepsilon_1^3 \right]}.$$ \hfill (8)

Suppose now that $P_1$ observes $(aab)$ or $(aba)$. In both cases we have:

$$\Pr^n\left(\alpha_1 = \hat{\alpha}_1^l, \tau = \hat{\tau}^l | (aab), \sigma = \hat{\sigma}\right) = \Pr^n\left(\alpha_1 = \hat{\alpha}_1^l, \tau = \hat{\tau}^l | (aba), \sigma = \hat{\sigma}\right) = \frac{(1 - \varepsilon_1^3)(1 - \varepsilon_1^2)^2 \varepsilon_1^3}{(m + 1) \left[ (1 - \varepsilon_1^3)(1 - \varepsilon_1^2)^2 + (m - 1) \varepsilon_1^3 \right]}.$$ \hfill (9)

When $P_1$ observes the messages $(baa)$ her beliefs are given by:

$$\Pr^n\left(\alpha_1 = \hat{\alpha}_1^l, \tau = \hat{\tau}^l | (baa), \sigma = \hat{\sigma}\right) = \frac{(1 - \varepsilon_1^2)^2 \varepsilon_1^3}{(m + 1) \left[ (1 - \varepsilon_1^2)^2 \varepsilon_1^3 + (1 - \varepsilon_1^3) \varepsilon_1^3 + (m - 1) \varepsilon_1^3 \right]}.$$ \hfill (10)

Finally, for the case $(abc)$, we have:

$$\Pr^n\left(\alpha_1 = \hat{\alpha}_1^l, \tau = \hat{\tau}^l | (abc), \sigma = \hat{\sigma}\right) = \frac{(1 - \varepsilon_1^3) \varepsilon_1^3}{(m + 1) \left[ (1 - \varepsilon_1^3) \varepsilon_1^3 + 2 (1 - \varepsilon_1^2) \varepsilon_1^3 + (m - 2) \varepsilon_1^3 \right]}.$$ \hfill (11)

As $n$ goes to infinity, the expressions in equations (8)-(11) converge to $\frac{1}{m+1}$.

To ease the presentation in the next steps, we introduce some notation. Consider a triple of messages $w$ in the set $M$. We let $g(\varepsilon, m; w)$ denote the value of the numerator of the expression above corresponding to the triple $w$, when $\varepsilon_1$ is equal to $\varepsilon$ and $m$ is equal to $m$. For example, $g(\varepsilon, m, (baa))$ is equal to $(1 - \varepsilon^2)^2 \varepsilon^3$. Further, we let $v(\varepsilon, m, w)$ denote the value of the denominator of the expression corresponding to the triple $w$ when $\varepsilon_1$ is equal to $\varepsilon$ and $m$ is equal to $m$. Note that for any triple of messages $w$ in $M$, we have $\lim_{\varepsilon \to 0} \frac{g(\varepsilon, m; w)}{v(\varepsilon, m; w)} = \frac{1}{m+1}$. 

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Step 2.

Substep 2.1. $P_1$, $P_3$ and $P_4$ send $P_5$ the permutation $\alpha_2\sigma$.

We assume that along the sequence of completely mixed strategies, $P_1$ trembles from her equilibrium strategy with probability $\varepsilon_1^2$ (that is, $P_1$ sends the realization of $\alpha_2\sigma$ with probability $1 - \varepsilon_1^2$ and any other message with probability $\varepsilon_1^2 / m$). $P_3$ and $P_4$ tremble from their equilibrium strategies with probability $\varepsilon_1^3$. The proof that the assessment $\Psi (r)$ satisfies consistency in Substep 2.1 is identical to the proof in Step 1, and is therefore omitted.

Substep 2.2. $P_3$, $P_4$ and $P_5$ send $P_2$ the permutation $\alpha_2\sigma\varphi$.

Let $\varepsilon_{1,n} = \varepsilon_{1,n}$. The probabilities of trembling are $\varepsilon_1^3$ for $P_3$ and $\varepsilon_1^2$ for $P_4$ and $P_5$.

The senders’ strategies in Step 2 do not depend on $z_2^2$, the message sent by $P_2$ in Step 1. Thus $P_2$’s beliefs are not a function of $z_2^2$.

Suppose that the message sent by the majority of the senders to $P_2$ is $a$, and the realizations of $\sigma$ and $\tau$ are $\hat{\sigma}$ and $\hat{\tau}$, respectively ($P_2$ knows these realizations). Let $\hat{\varphi}_1, ..., \hat{\varphi}_m$ be $m + 1$ different permutations on $X$. Then for every $l = 1, ..., m + 1$, we can find a permutation $\hat{\alpha}_l$ such that $\hat{\alpha}_l \hat{\sigma} \hat{\tau} \hat{\varphi}_l = a$. Moreover, $\hat{\alpha}_l \neq \hat{\alpha}_l'$ if $l$ differs from $l'$. To prove that the assessment $\Psi (r)$ satisfies consistency in Substep 2.2, it is enough to show that for any triple of messages $w_{22}$ in the set $M$, and for every $l = 1, ..., m + 1$, the following equality holds:

$$\lim_{n \to \infty} \Pr^n \left( \alpha_2 = \hat{\alpha}_l, \varphi = \hat{\varphi}_l, \alpha_2\sigma = \hat{\alpha}_l \hat{\sigma} \mid w_{22}, \sigma = \hat{\sigma}, \tau = \hat{\tau} \right) = \frac{1}{m + 1}. \quad (12)$$

The probability in equation (12) can be expressed as follows:

$$\Pr^n \left( \alpha_2 = \hat{\alpha}_l, \varphi = \hat{\varphi}_l, \alpha_2\sigma = \hat{\alpha}_l \hat{\sigma} \mid w_{22}, \sigma = \hat{\sigma}, \tau = \hat{\tau} \right) = \frac{\sum_{h, j, \tilde{\alpha} = 1}^{m + 1} \Pr^n \left( w_{22} \mid \alpha_2 = \hat{\alpha}_l, \varphi = \hat{\varphi}_l, \alpha_2\sigma = \hat{\alpha}_l \hat{\sigma}, \sigma = \hat{\sigma}, \tau = \hat{\tau} \right) \Pr^n \left( \alpha_2 = \hat{\alpha}_l \mid \alpha_2 = \hat{\alpha}_l, \sigma = \hat{\sigma} \right) \Pr^n \left( \alpha_2\sigma = \hat{\alpha}_l \hat{\sigma} \mid \alpha_2 = \hat{\alpha}_l, \sigma = \hat{\sigma} \right)}{\sum_{h, j, \tilde{\alpha} = 1}^{m + 1} \Pr^n \left( w_{22} \mid \alpha_2 = \hat{\alpha}_l, \varphi = \hat{\varphi}_l, \alpha_2\sigma = \hat{\alpha}_l \hat{\sigma}, \sigma = \hat{\sigma}, \tau = \hat{\tau} \right) \Pr^n \left( \alpha_2 = \hat{\alpha}_l \mid \alpha_2 = \hat{\alpha}_l, \sigma = \hat{\sigma} \right) \Pr^n \left( \alpha_2\sigma = \hat{\alpha}_l \hat{\sigma} \mid \alpha_2 = \hat{\alpha}_l, \sigma = \hat{\sigma} \right)}. \quad (13)$$

Suppose that the realization of $\alpha_2$ is $\hat{\alpha}_h^h$, $h = 1, ..., m + 1$. Given the sequences of completely mixed strategies that we have constructed in Step 0 and Substep 2.1, the probability
that the message sent by the majority of the senders to $P_5$ coincides with the permutation $\hat{\alpha}_2^h\hat{\sigma}$ (i.e. the probability that $P_5$ learns the true realization of $\alpha_2\sigma$) is given by:

$$\Pr^n\left(\alpha_2 = \hat{\alpha}_2^h, \sigma = \hat{\sigma} \mid \alpha_2 = \hat{\alpha}_2^h, \sigma = \hat{\sigma}\right) = 1 - \varepsilon_1^4 \left(\frac{1}{m} + 2\varepsilon_1 - \varepsilon_1^3 \left(1 + \frac{1}{m}\right)\right) := 1 - v(\varepsilon_1). \quad (14)$$

The probability that $P_5$ learnt some other permutation $\hat{\alpha}_2^k\hat{\sigma}$, different from the realization $\hat{\alpha}_2^h\hat{\sigma}$ is:

$$\Pr^n\left(\alpha_2 = \hat{\alpha}_2^k, \sigma = \hat{\sigma} \mid \alpha_2 = \hat{\alpha}_2^h, \sigma = \hat{\sigma}\right) = \frac{\varepsilon_1^4}{m} \left(\frac{1}{m} + 2\varepsilon_1 - \varepsilon_1^3 \left(1 + \frac{1}{m}\right)\right) := \frac{v(\varepsilon_1)}{m}. \quad (15)$$

Note that the probability that $P_5$ learnt a wrong state is an infinitesimal of the same order as $\varepsilon_1^4/m$. If $P_5$ did not learn the true realization of $\alpha_2\sigma$, then at least two senders deviated from their equilibrium strategies in Substep 2.1. The two most likely players to deviate are $P_3$ and $P_4$ and the probability that they send a pair of wrong messages is $\varepsilon_1^4/m$.

Substituting equations (14) and (15) into equation (13) yields:

$$\Pr^n\left(\alpha_2 = \hat{\alpha}_2^l, \varphi = \hat{\varphi}^l, \alpha_2\sigma = \hat{\alpha}_2^h\hat{\sigma} \mid w_{22}, \sigma = \hat{\sigma}, \tau = \hat{\tau}\right) = \frac{(1 - v(\varepsilon_1))g(\hat{\varepsilon}_1, m, w_{22})}{(1 - v(\varepsilon_1))g(\hat{\varepsilon}_1, m, w_{22}) + \frac{v(\varepsilon_1)}{m} \sum_{k \neq h} \Pr^n\left(w_{22} \mid \alpha_2 = \hat{\alpha}_2^h, \varphi = \hat{\varphi}^l, \alpha_2\sigma = \hat{\alpha}_2^k\hat{\sigma}, \sigma = \hat{\sigma}, \tau = \hat{\tau}\right)}.$$  

It is easy to verify that, for any triple of messages $w_{22}$, the above probability converges to $\frac{1}{m+1}$ as $n$ goes to infinity.

To give an intuition for this result, let us consider the different situations that $P_2$ can face in Substep 2.2. First of all, observing the triple $(aaa)$ is consistent with the fact that all senders in Substeps 2.1 and 2.2 played their equilibrium strategies. Therefore, with probability one, $P_5$ and $P_2$ received the true realizations of $\alpha_2\sigma$ and $\alpha_2\sigma\tau\varphi$, respectively.

We now consider the case where $P_2$ receives the triple $(aab)$. Although $P_3$’s message differs from the other two messages, $P_2$ believes that with probability one $P_5$ received the true realization of $\alpha_2\sigma$. Suppose that $P_3$ and $P_4$ sent the true message (the probability of
this event is one in the limit). There are three possible reasons why $P_5$ sent a different message: (i) $P_5$ received the true realization of $\alpha_2\sigma$ in Substep 2.1, but she deviated in Substep 2.2. (ii) $P_5$ received a wrong message in Substep 2.1, but followed her equilibrium strategy in Substep 2.2. (iii) $P_5$ received a wrong message in Substep 2.1, and then deviated. Note that only one deviation (by $P_5$) is needed for case (i). In case (ii) we need at least two deviations in Substep 2.1. Finally at least three deviations are required for case (iii). Since we assume that the probabilities of deviations in Substeps 2.1 and 2.2 are the same, in the limit we assign probability one to case (i), which requires the fewest deviations.

When $P_2$ receives $(aba)$, at least one player among $P_3$ and $P_4$ has deviated. A deviation by $P_4$ from her equilibrium strategy (in Substep 2.2) does not require any other deviation to justify the triple of messages $(aba)$ (in particular, a deviation by $P_4$ is consistent with the fact that $P_5$ learnt the true realization of $\alpha_2\sigma$). On the contrary, a deviation by $P_3$ requires at least another deviation (either $P_5$ learnt the truth and deviated, or she did not learn the truth). Again, since deviations occur with arbitrarily small probabilities, we assign probability one to the event that $P_4$ deviated in Substep 2.2.

We now turn to the triple of messages $(baa)$. We can justify this triple with a single deviation by $P_3$ in Substep 2.2. A deviation by $P_4$ requires at least another deviation. It is true that $P_3$ trembles with probability $\tilde{\varepsilon}_1^3$ in Substep 2.2, while $P_3$ trembles with probability $\tilde{\varepsilon}_1^2$, but any other deviation is independent and occurs at least with probability $\tilde{\varepsilon}_1^2$ or $\varepsilon_1^2$. Since $\tilde{\varepsilon}_1$ is equal to $\varepsilon_1$, we conclude that with probability one $P_3$ deviated in Substep 2.2 and $P_5$ learnt the truth in Substep 2.1.

Finally, we consider the case $(abc)$. Note that, independently of what $P_5$ learns in Substep 2.1, the probability that $P_2$ receives three different messages converges to zero faster than or at the same rate as $\tilde{\varepsilon}_1^2$. Since $P_3$ and $P_4$ have sent different messages, clearly, at least one of them has deviated from her equilibrium strategy in Substep 2.2 (in the limit we assign probability one to the event that the deviator is $P_4$, since she is more likely to tremble than $P_5$). As far as $P_5$ is concerned, it could be the case that she did not learn
the truth in Substep 2.1, which implies that at least two senders deviated in that substep. However, it could also be the case that $P_5$ did receive the true message in Substep 2.1, but then she deviated in Substep 2.2. We assign probability one to the last event, since it requires only one deviation.

**Step 3.**

**Substep 3.1.** $P_1$, $P_2$ and $P_4$ send $P_5$ the permutation $\alpha_3\sigma$.

We assume that $P_1$ trembles from her equilibrium strategy with probability $\varepsilon_3^1$, and that $P_2$ and $P_4$ tremble with probability $\varepsilon_2^2$. With this sequence of completely mixed strategies it is easy to show that, given any triple of messages $w_{31}$, in the limit, $P_5$ assigns probability one to the event that the realization of $\alpha_3\sigma$ is the message sent by the majority (see Step 1, above). This result does not change when we take into account $w_{21}$, the triple of messages received by $P_5$ in Substep 2.1, and regarding the permutation $\alpha_2\sigma$. This follows from the fact that the random permutations $\alpha_2, \alpha_3$ and $\sigma$ are independent of each other and uniformly distributed, and trembles are independent across information sets. For the same reason, in this step, $P_5$ does not update her beliefs from Substep 2.1. These results also imply that, conditional on $P_5$’s information, the marginal distribution of any of the three random permutations $\sigma, \alpha_2$ and $\alpha_3$, is uniform over $\Lambda(X)$.

**Substep 3.2.** $P_2$, $P_4$ and $P_5$ send $P_1$ the permutation $\beta_{31}\alpha_3\sigma\tau$.

For any integer $n$, let $\varepsilon_{2,n} = \varepsilon_n^2$ (notice that $\lim_{n\to\infty} \frac{\varepsilon_{1,n}}{\varepsilon_{2,n}} = 0$). $P_2$ trembles from her equilibrium strategy with probability $\varepsilon_3^1$, and $P_4$ and $P_5$ tremble with probability $\varepsilon_3^2$.

Note that $P_1$’s senders’ strategies do not depend on her message in Substep 2.1. Thus, $P_1$’s beliefs are independent of $z_{21}^1$. However, $P_1$’s beliefs do depend on $z_{31}^1$. The message that $P_5$ sends in Substep 3.2 is a function of the triple of messages that she receives in Substep 3.1. This triple, in turn, depends on $P_1$’s report, $z_{31}^1$.

Let $\tilde{\alpha}_3^1, \ldots, \tilde{\alpha}_3^{m+1}$ be $m + 1$ different permutations. We let $\tilde{\alpha}_3^1$ and $\tilde{\sigma}$ denote the realizations of $\alpha_3$ and $\sigma$, respectively ($\tilde{\alpha}_3^1$ and $\tilde{\sigma}$ are known to $P_1$). We also assume that the message sent by the majority of the senders to $P_1$ is the permutation $a$. To every
permutation, $\tilde{\tau}^l$, $l = 1, \ldots, m + 1$, we can assign a different permutation $\tilde{\beta}_{31}$ such that $\tilde{\beta}_{31} \tilde{\alpha}_3^{-1} \tilde{\tau}^l = a$.

We now show that for any triple of messages $w_{32}$ in the set $M$, for every $l = 1, \ldots, m + 1$, and for every message $z_{31}^1$, the following equality holds:

$$\lim_{n \to \infty} Pr^n \left( \beta_{31} = \tilde{\beta}_{31}^l, \tau = \tilde{\tau}^l, \tilde{\alpha}_3 \sigma = \tilde{\alpha}_3^{-1} \tilde{\tau} | w_{32}, z_{31}^1, \sigma = \tilde{\sigma}, \alpha_3 = \tilde{\alpha}_3^{-1} \right) = \frac{1}{m + 1}.$$  

This result, combined with the result of Step 1, and the fact that trembles are independent across information sets, and all random choices are independent of each other and uniformly distributed, implies that the assessment $\Psi(r)$ satisfies consistency in Substep 3.2.

Along the sequence of completely mixed strategies, we have:

$$\Pr^n \left( \beta_{31} = \tilde{\beta}_{31}^l, \tau = \tilde{\tau}^l, \tilde{\alpha}_3 \sigma = \tilde{\alpha}_3^{-1} \tilde{\tau} | w_{32}, z_{31}^1, \sigma = \tilde{\sigma}, \alpha_3 = \tilde{\alpha}_3^{-1} \right) = \frac{\sum_{h,j,k=1}^{m+1} Pr^n \left( w_{32} | \beta_{31} = \tilde{\beta}_{31}^h, \tau = \tilde{\tau}^j, \tilde{\alpha}_3 \sigma = \tilde{\alpha}_3^{-1} \tilde{\tau} \right) Pr^n \left( \tilde{\alpha}_3 \sigma = \tilde{\alpha}_3^{-1} \tilde{\tau} | \sigma = \tilde{\sigma}, \alpha_3 = \tilde{\alpha}_3^{-1} \right)}{\sum_{h,j,k=1}^{m+1} Pr^n \left( w_{32} | \beta_{31} = \tilde{\beta}_{31}^h, \tau = \tilde{\tau}^j, \tilde{\alpha}_3 \sigma = \tilde{\alpha}_3^{-1} \tilde{\tau} \right) Pr^n \left( \tilde{\alpha}_3 \sigma = \tilde{\alpha}_3^{-1} \tilde{\tau} | \sigma = \tilde{\sigma}, \alpha_3 = \tilde{\alpha}_3^{-1} \right)}.$$  

(16)

At this point we need to specify the action of $P_1$ in Substep 3.1. We first analyze the case where $P_1$ follows her equilibrium strategy, i.e. $z_{31}^1 = \tilde{\alpha}_3^{-1} \tilde{\tau}$. In this case, the probability that $P_5$ learns the realization of $\alpha_3 \sigma$ is given by:

$$\Pr^n \left( \tilde{\alpha}_3 \sigma = \tilde{\alpha}_3^{-1} \tilde{\tau} | \sigma = \tilde{\sigma}, \alpha_3 = \tilde{\alpha}_3^{-1}, z_{31}^1 = \tilde{\alpha}_3^{-1} \tilde{\tau} \right) = 1 - \frac{\varepsilon_4}{m}. \quad \text{(17)}$$

The probability that $P_5$ learns some other realization $\tilde{\alpha}_3^{-1} \tilde{\tau}$, $k \neq 1$, is:

$$\Pr^n \left( \tilde{\alpha}_3 \sigma = \tilde{\alpha}_3^{-1} \tilde{\tau} | \sigma = \tilde{\sigma}, \alpha_3 = \tilde{\alpha}_3^{-1}, z_{31}^1 = \tilde{\alpha}_3^{-1} \tilde{\tau} \right) = \frac{\varepsilon_4}{m^2}. \quad \text{(18)}$$

Since $P_1$ is the sender with the lowest probability of trembling, and she reports the truth, $P_5$ can learn a wrong state $\tilde{\alpha}_3^{-1} \tilde{\tau}$ if and only if both $P_2$ and $P_4$ send the message $\tilde{\alpha}_3^{-1} \tilde{\tau}$ (the probability of this event is $\frac{\varepsilon_4}{m^2}$).

---

24The permutations $\tilde{\tau}^1, \ldots, \tilde{\tau}^{m+1}$ were introduced in Step 1 of this section.
We substitute equations (17) and (18) into equation (16) and obtain:

\[
\Pr^n \left( \beta_{31} = \beta_{31}^l, \tau = \tau^l, \alpha_3 \sigma = \alpha_3^1 \sigma | w_{32}, z_{31} = \alpha_3^1 \sigma, \sigma = \hat{\sigma}, \alpha_3 = \hat{\alpha}_3^1 \right) = \\
\frac{1 - \frac{\varepsilon_2}{m}}{1 - \frac{\varepsilon_2}{m}} \cdot \Pr^n \left( w_{32} | \beta_{31} = \beta_{31}^l, \tau = \tau^l, \alpha_3 \sigma = \alpha_3^k \sigma, \sigma = \hat{\sigma}, \alpha_3 = \hat{\alpha}_3^k \right).
\]

It is easy to verify that, as \( n \) goes to infinity, the above probability converges to \( \frac{1}{m+1} \).

Consider the event that \( P_5 \) knows the permutation \( \alpha_3 \sigma \), and the message sent by the majority of the senders to \( P_1 \) coincides with the permutation \( \beta_{31} \alpha_3 \sigma \tau \). From the analysis of Substep 2.2 above, we know that this event has probability one if we are ignorant about \( P_1 \)'s action in Substep 3.1 and the trembles in Substeps 3.1 and 3.2 are equally likely. Now, since \( P_1 \) reports the true message to \( P_5 \), and deviations are less likely in Substep 3.1 than in Substep 3.2, we conclude that, a fortiori, \( P_5 \) is informed about the realization of \( \alpha_3 \sigma \), and that the message sent by the majority of the senders is the realization of \( \beta_{31} \alpha_3 \sigma \tau \).

We now assume that in Substep 3.1, \( P_1 \) deviates from her equilibrium strategy and sends the message \( \hat{\alpha}_3^k \hat{\sigma} \) to \( P_5 \), for some \( \hat{k} = 2, \ldots, m + 1 \). In this case, \( P_5 \) learns the truth if and only if both \( P_2 \) and \( P_4 \) report the realization of \( \alpha_3 \sigma \). This event occurs with probability:

\[
\Pr^n \left( \alpha_3 \sigma = \hat{\alpha}_3^k \hat{\sigma}, \sigma = \hat{\sigma}, \alpha_3 = \hat{\alpha}_3^1, z_{31} = \hat{\alpha}_3^k \hat{\sigma} \right) = \left( 1 - \varepsilon_1^2 \right)^2. \tag{19}
\]

The probability that \( P_5 \) learns the permutation sent by \( P_1 \) is:

\[
\Pr^n \left( \alpha_3 \sigma = \hat{\alpha}_3^k \hat{\sigma} | \sigma = \hat{\sigma}, \alpha_3 = \hat{\alpha}_3^1, z_{31} = \hat{\alpha}_3^k \hat{\sigma} \right) = \varepsilon_1^2 \left( 2 - \varepsilon_1^2 \left( \frac{m^2 + m - 1}{m^2} \right) \right). \tag{20}
\]

Clearly, \( P_5 \) is more likely to learn the permutation sent by \( P_1 \) than any other permutation \( \hat{\alpha}_3^k \hat{\sigma} \), for \( k \neq 1, \hat{k} \). In fact, in the last case we have:

\[
\Pr^n \left( \alpha_3 \sigma = \hat{\alpha}_3^k \hat{\sigma} | \sigma = \hat{\sigma}, \alpha_3 = \hat{\alpha}_3^1, z_{31} = \hat{\alpha}_3^k \hat{\sigma} \right) = \frac{\varepsilon_1^4}{m^2}. \tag{21}
\]

Substituting equations (19)-(21) into equation (16) yields:

50
\[
\Pr^n \left( \beta_{31} = \tilde{\beta}_{31}, \tau = \tilde{\tau}, \alpha_3 \sigma = \tilde{\alpha}_3 \hat{\sigma}, \sigma = \hat{\sigma}, \alpha_3 = \tilde{\alpha}_3 \right) = \frac{(1 - \varepsilon_1^2)^2 g(\varepsilon_2, m; w_{32})}{B_1},
\]
where:
\[
B_1 = (1 - \varepsilon_1^2)^2 v(\varepsilon_2, m; w_{32}) + \varepsilon_1^2 \left( 2 - \varepsilon_1^2 \left( \frac{m^2 + m - 1}{m} \right) \right)
\]
\[
\sum_{j \neq 1, k} \Pr^n \left( w_{32} | \beta_{31} = \tilde{\beta}_{31}, \tau = \tilde{\tau}, \alpha_3 \sigma = \tilde{\alpha}_3 \hat{\sigma}, \sigma = \hat{\sigma}, \alpha_3 = \tilde{\alpha}_3 \right)
\]
A sufficient condition for the above probability to converge to \( \frac{1}{m+1} \) is that \( \lim_{n \to \infty} \frac{\varepsilon_{1,n}}{\varepsilon_{2,n}} = 0 \). This condition is also necessary, at least in the cases where \( P_1 \) receives the triples (\( aab \)) and (\( abc \)). Suppose, for example, that \( P_2 \) and \( P_4 \) report the same message, \( a \), and that \( P_5 \) sends a different message. Consider a pair of permutations, \( \tilde{\beta}_{31}^l \) and \( \tilde{\tau}^l \), such that \( \tilde{\beta}_{31}^l \tilde{\alpha}_3 \hat{\sigma} \hat{\tau}^l = a \). Let us assume that \( P_5 \)'s message is \( b = \tilde{\beta}_{31}^l \tilde{\alpha}_3 \hat{\sigma} \hat{\tau}^l \). The following two situations are both compatible with the triple (\( aab \)). In the first scenario, we assume that \( P_5 \) learnt the true permutation \( \tilde{\alpha}_3 \hat{\sigma} \), but then deviated in Substep 3.2. The probability of this tremble is of the same order as \( \varepsilon_2^2 \). Suppose now that \( P_5 \) learnt the permutation \( \tilde{\alpha}_3 \hat{\sigma} \) and then she followed her equilibrium strategy. The probability of this event is an infinitesimal of the same order as \( \varepsilon_2^2 \). In Substep 3.2, \( P_1 \) can assign probability zero to the last event only if we assume that \( \varepsilon_{1,n} \) converges to zero faster than \( \varepsilon_{2,n} \), i.e. only if the trembles are much less likely in Substep 3.1 than in Substep 3.2.\(^{25}\)

**Substep 3.3.** \( P_4, P_1 \) and \( P_5 \) send \( P_2 \) the permutation \( \beta_{32} \beta_{31} \alpha_3 \sigma \tau \varphi \).

Let \( \varepsilon_{3,n} = \varepsilon_n \) for every \( n \), and note that \( \lim_{n \to \infty} \frac{\varepsilon_{1,n}}{\varepsilon_{3,n}} = 0 \) and \( \lim_{n \to \infty} \frac{\varepsilon_{2,n}}{\varepsilon_{3,n}} = 0 \). We assume that along the sequence of completely mixed strategies, \( P_4 \) trembles with probability \( \varepsilon_{3,n}^3 \), and \( P_1 \) and \( P_5 \) tremble with probability \( \varepsilon_{3,n}^2 \). Clearly, \( P_2 \)'s beliefs are a function of \( z_{31}^2 \) and \( z_{32}^2 \), the messages that she sends in Substeps 3.1 and 3.2, respectively. However, her beliefs do not depend on \( z_{1}^2 \), the message sent by \( P_2 \) in Step 1.

\(^{25}\)A similar problem arises when \( P_1 \) receives the triple of messages (\( abc \)).
As in the previous step, $\hat{\alpha}_3^1$ and $\hat{\sigma}$ denote the realizations of $\alpha_3$ and $\sigma$, respectively. In addition to them, $P_2$ knows the realizations of $\beta_{31}$ and $\tau$, which we denote by $\hat{\beta}_{31}$ and $\hat{\tau}$, respectively. Consider the $m + 1$ permutations $\hat{\phi}, \ldots, \hat{\phi}^{m+1}$ defined in Substep 2.2 of this section. For every $l = 1, \ldots, m + 1$ there exists a different permutation $\hat{\beta}_{32}^l$ such that

$$\hat{\beta}_{32}^l \hat{\beta}_{31} \hat{\alpha}_3^1 \hat{\sigma} \hat{\tau} \hat{\phi} = a$$

(we assume that $a$ is the message sent by the majority of the senders to $P_2$).

In order to demonstrate that the equilibrium beliefs of Substep 3.3 are consistent, it is enough to show that for any pair of messages $z_{31}^2$, $z_{32}^2$, for any triple of messages $w_{33}$ in $M$, and for every $l = 1, \ldots, m + 1$, the following equality is satisfied:

$$\lim_{n \to \infty} \Pr^n \left( \hat{\beta}_{32}^l \hat{\phi}, \hat{\beta}_{31} \hat{\alpha}_3 \hat{\sigma} \hat{\tau} = \hat{\beta}_{31} \hat{\alpha}_3 \hat{\sigma} \hat{\tau}, \hat{\alpha}_3 \hat{\sigma} = \hat{\alpha}_3 \hat{\sigma} | w_{33}, z_{31}^2, z_{32}^2, \hat{\beta}_{31}, \hat{\alpha}_3, \hat{\sigma}, \hat{\tau} \right) = \frac{1}{m + 1}.$$

The probability above can be expressed as:

$$\Pr^n \left( \hat{\beta}_{32}^l \hat{\phi}, \hat{\beta}_{31} \hat{\alpha}_3 \hat{\sigma} \hat{\tau} = \hat{\beta}_{31} \hat{\alpha}_3 \hat{\sigma} \hat{\tau}, \hat{\alpha}_3 \hat{\sigma} = \hat{\alpha}_3 \hat{\sigma} | w_{33}, z_{31}^2, z_{32}^2, \hat{\beta}_{31}, \hat{\alpha}_3, \hat{\sigma}, \hat{\tau} \right) = \frac{e(l, l, 1, 1)}{\sum_{h, j, \iota, k=1}^{m+1} e(h, j, \iota, k)}$$

where:

$$e(h, j, \iota, k) = \Pr^n \left( w_{33} | \hat{\beta}_{32}^h \hat{\phi}, \hat{\beta}_{31} \hat{\alpha}_3 \hat{\sigma} \hat{\tau} = \hat{\beta}_{31} \hat{\alpha}_3 \hat{\sigma} \hat{\tau}, \hat{\alpha}_3 \hat{\sigma} = \hat{\alpha}_3 \hat{\sigma} \right)$$

$$\Pr^n \left( \hat{\beta}_{31} \hat{\alpha}_3 \hat{\sigma} \hat{\tau} | z_{32}^2, \hat{\alpha}_3 \hat{\sigma} = \hat{\alpha}_3 \hat{\sigma}, \hat{\beta}_{31}, \hat{\alpha}_3, \hat{\sigma}, \hat{\tau} \right) \Pr^n \left( \hat{\alpha}_3 \hat{\sigma} = \hat{\alpha}_3 \hat{\sigma} | z_{32}^2, \hat{\alpha}_3 \hat{\sigma} \right)$$

Depending on whether $P_2$ reports the true messages in Substeps 3.1 and 3.2 or not, we have to consider four different cases. We start by assuming that $P_2$ follows her equilibrium strategies in both steps, i.e. $z_{31}^2 = \hat{\alpha}_3^1 \hat{\sigma}$ and $z_{32}^2 = \hat{\beta}_{31} \hat{\alpha}_3 \hat{\sigma} \hat{\tau}$. We first need to compute the probability that $P_3$ learns a given permutation. We have:

\footnote{We also use the permutations $\hat{\alpha}_3^l \ldots, \hat{\alpha}_3^{m+1}$, introduced in Substep 3.2 of this section.}
\[
Pr_n \left( \alpha_3 \sigma = \tilde{\alpha}_3^k \sigma | z_{32}^1 = \tilde{\alpha}_3 \sigma, \tilde{\alpha}_3 \sigma \right) = \begin{cases} 
1 - \varepsilon_1^5 & k = 1 \\
\frac{\varepsilon_1^5}{m} & k \neq 1 .
\end{cases} 
\] (23)

Suppose that \( P_3 \) learns the truth in Substep 3.1. The probability that \( P_1 \) learns a given permutation in Substep 3.2 is:

\[
Pr_n \left( \beta_{32} \tilde{\alpha}_3 \sigma \tau = \tilde{\beta}_{32} \tilde{\alpha}_3 \sigma \tilde{\tau} | z_{32}^2 = \tilde{\beta}_{32} \tilde{\alpha}_3 \sigma \tilde{\tau}, \tilde{\alpha}_3 \sigma = \tilde{\alpha}_3 \sigma, \tilde{\beta}_{31}, \tilde{\alpha}_3 \sigma, \tilde{\sigma}, \tilde{\tau} \right) = \begin{cases} 
1 - \frac{\varepsilon_1^4}{m} & \tau = 1 \\
\frac{\varepsilon_1^4}{m} & \tau \neq 1 .
\end{cases} 
\] (24)

Substituting equations (23) and (24) into equation (22) yields:

\[
Pr_n \left( \beta_{32} \tilde{\alpha}_3 \sigma \tau = \tilde{\beta}_{32} \tilde{\alpha}_3 \sigma \tilde{\tau}, \tilde{\alpha}_3 \sigma = \tilde{\alpha}_3 \sigma, \tilde{\beta}_{31}, \tilde{\alpha}_3 \sigma, \tilde{\sigma}, \tilde{\tau} \right) = \frac{(1 - \varepsilon_1^5) \left(1 - \frac{\varepsilon_1^4}{m}\right) g(\varepsilon_3, m; w_{33})}{B_2},
\]

where:

\[
B_2 = \left(1 - \varepsilon_1^5\right) \left(1 - \frac{\varepsilon_1^4}{m}\right) v(\varepsilon_3, m; w_{33}) + (1 - \varepsilon_1^5) \frac{\varepsilon_1^4}{m} \sum_{h,j, k \neq 1} Pr^n \left( w_{33} | \tilde{\beta}_{32}^h, \tilde{\varphi}^j, \beta_{31} \alpha_3 \sigma \tau = \tilde{\beta}_{32}^h \tilde{\alpha}_3^j \sigma \tilde{\tau}, \tilde{\alpha}_3 \sigma = \tilde{\alpha}_3 \sigma, \tilde{\beta}_{31}, \tilde{\alpha}_3 \sigma, \tilde{\sigma}, \tilde{\tau} \right) + \frac{\varepsilon_1^4}{m} \sum_{h,j, k \neq 1} Pr^n \left( w_{33} | \tilde{\beta}_{32}^h, \tilde{\varphi}^j, \beta_{31} \alpha_3 \sigma \tau = \tilde{\beta}_{32}^h \tilde{\alpha}_3^j \sigma \tilde{\tau}, \tilde{\alpha}_3 \sigma = \tilde{\alpha}_3 \sigma, \tilde{\beta}_{31}, \tilde{\alpha}_3 \sigma, \tilde{\sigma}, \tilde{\tau} \right) - \frac{\varepsilon_1^4}{m} \sum_{h,j, k \neq 1} Pr^n \left( \beta_{31} \alpha_3 \sigma \tau = \tilde{\beta}_{31} \tilde{\alpha}_3^k \sigma \tilde{\tau} | z_{32}^2 = \tilde{\beta}_{31} \tilde{\alpha}_3^k \sigma \tilde{\tau}, \tilde{\alpha}_3 \sigma = \tilde{\alpha}_3 \sigma, \tilde{\beta}_{31}, \tilde{\alpha}_3 \sigma, \tilde{\sigma}, \tilde{\tau} \right).
\]

Our assumptions regarding the trembles are sufficient to guarantee that the above probability converges to \( \frac{1}{m+1} \) as \( n \) goes to infinity.

We now consider the case where \( P_3 \) deviates in Substep 3.1 (she reports the message \( z_{31}^2 = \tilde{\alpha}_3^k \sigma \), for some \( \tilde{k} = 2, \ldots, m + 1 \)) and follows her equilibrium strategy in Substep 3.2. The probability that \( P_5 \) learns a given permutation in Substep 3.1 is given by:
\[
\Pr^n \left( \frac{\alpha_3 \sigma}{\alpha_3^k \sigma} = \alpha_3^k \sigma \mid z_{32}^1 = \alpha_3^k \sigma, \alpha_3^1 \sigma \right) = \begin{cases} 
(1 - \varepsilon_1^3) \left( 1 - \frac{\varepsilon_1^2}{m} \right) & k = 1 \\
\frac{\varepsilon_1^2}{m} \left( 1 + \varepsilon_1 - \frac{\varepsilon_1^2}{m} \right) & k = \tilde{k} \\
\frac{\varepsilon_1^3}{m} (1 - \frac{\varepsilon_1^3}{m}) & k \neq 1, \tilde{k}\end{cases} \tag{25}
\]

We now substitute equations (24) and (25) into equation (22). This gives us:

\[
\Pr^n \left( \beta_{32}^l \varphi', \beta_{31 \alpha_3 \sigma} = \beta_{31 \alpha_3 \sigma} \mid w_{33}, z_{31}^2 = \alpha_3^k \sigma, z_{32}^2 = \beta_{31 \alpha_3 \sigma} \mid \beta_{31 \alpha_3 \sigma}, \beta_{31}, \alpha_3^1 \right) \frac{1 - \varepsilon_1^3}{1 - \varepsilon_1^3} \frac{1 - \varepsilon_1^3}{1 - \varepsilon_1^3} g(\varepsilon_3, m; w_{33})
\]

where:

\[
B_3 = (1 - \varepsilon_1^3) \left( 1 - \frac{\varepsilon_1^2}{m} \right) v(\varepsilon_3, m; w_{33}) + \\
(1 - \varepsilon_1^3) \left( 1 - \frac{\varepsilon_1^2}{m} \right) \sum_{h,j, \ell} \Pr^n \left( w_{33} \mid \beta_{32}^h \varphi, \beta_{31 \alpha_3 \sigma} = \beta_{31 \alpha_3 \sigma} \mid w_{33}, z_{31}^2 = \alpha_3^k \sigma, z_{32}^2 = \beta_{31 \alpha_3 \sigma} \mid \beta_{31 \alpha_3 \sigma}, \beta_{31}, \alpha_3^1 \right) + \\
\frac{\varepsilon_1^3}{m} \left( 1 + \varepsilon_1 - \frac{\varepsilon_1^2}{m} \right) \sum_{h,j, \ell} \Pr^n \left( w_{33} \mid \beta_{32}^h \varphi, \beta_{31 \alpha_3 \sigma} = \beta_{31 \alpha_3 \sigma} \mid w_{33}, z_{31}^2 = \alpha_3^k \sigma, z_{32}^2 = \beta_{31 \alpha_3 \sigma} \mid \beta_{31 \alpha_3 \sigma}, \beta_{31}, \alpha_3^1 \right) + \\
\frac{\varepsilon_1^3}{m} \left( 1 - \frac{\varepsilon_1^3}{m} \right) \sum_{h,j, \ell} \Pr^n \left( w_{33} \mid \beta_{32}^h \varphi, \beta_{31 \alpha_3 \sigma} = \beta_{31 \alpha_3 \sigma} \mid w_{33}, z_{31}^2 = \alpha_3^k \sigma, z_{32}^2 = \beta_{31 \alpha_3 \sigma} \mid \beta_{31 \alpha_3 \sigma}, \beta_{31}, \alpha_3^1 \right) + \\
\Pr^n \left( \beta_{31 \alpha_3 \sigma} = \beta_{31 \alpha_3 \sigma} \mid z_{32}^2 = \beta_{31 \alpha_3 \sigma} \mid \alpha_3 \sigma = \alpha_3^k \sigma, \beta_{31}, \alpha_3^1 \right).
\]

Again, given our assumptions, we can conclude that the limit value of the above probability is \( \frac{1}{m+1} \).

We now assume that \( P_2 \) plays her equilibrium strategy in Substep 3.1 and deviates in Substep 3.2 (we assume that she sends message \( z_{32}^2 = \beta_{31 \alpha_3 \sigma} \varphi \tau \), for some \( \tilde{l} = 2, ..., m + 1 \)). Consider the case where \( P_3 \) learns the true realization of \( \alpha_3 \sigma \) (the probability of this event is given by equation (23)). Then the probability that \( P_1 \) learns a given permutation in Substep 3.2 is:
\[
\Pr^n \left( \beta_{31} \alpha_3 \sigma \tau = \beta_{31} \alpha_3 \tilde{\sigma} \tilde{\tau} \mid z_3^2 = \beta_{31} \tilde{\alpha}_3 \tilde{\sigma} \tilde{\tau}, \alpha_3 \sigma = \alpha_3^1 \tilde{\sigma}, \beta_{31}, \tilde{\alpha}_3, \tilde{\sigma}, \tilde{\tau} \right) = \\
\begin{cases} \\
(1 - \varepsilon_2^2)^2 & \text{if } \tau = 1 \\
\varepsilon_2^2 \left( 2 - \varepsilon_2^2 \left( \frac{m^2 + m - 1}{m^2} \right) \right) & \text{if } \tau \neq 1, \tilde{\tau} \\
\varepsilon_2^2 \left( \frac{m^2 + m - 1}{m^2} \right) & \text{if } \tau = \tilde{\tau}.
\end{cases}
\]

(26)

We substitute equations (23) and (26) into equation (22) and obtain:

\[
\Pr^n \left( \beta_{32}, \varphi, \beta_{31} \alpha_3 \sigma \tau = \beta_{31} \tilde{\alpha}_3 \tilde{\sigma} \tilde{\tau}, \alpha_3 \sigma = \alpha_3^1 \tilde{\sigma} \mid w_{33}, z_3^2 = \beta_{31} \tilde{\alpha}_3 \tilde{\sigma} \tilde{\tau}, \beta_{31}, \tilde{\alpha}_3, \tilde{\sigma}, \tilde{\tau} \right) = \\
\frac{(1 - \varepsilon_1^2)(1 - \varepsilon_2^2)^2 g(\varepsilon_3, m \mid w_{33})}{B_4},
\]

where:

\[
B_4 = (1 - \varepsilon_1^5)(1 - \varepsilon_2^3)^2 v(\varepsilon_3, m \mid w_{33}) + (1 - \varepsilon_1^5) \varepsilon_2^2 \left( 2 - \varepsilon_2^2 \left( \frac{m^2 + m - 1}{m^2} \right) \right) + \\
\sum_{h,j} \Pr^n \left( w_{33} \beta_{32}^h \varphi^j \beta_{31} \alpha_3 \sigma \tau = \beta_{31} \tilde{\alpha}_3 \tilde{\sigma} \tilde{\tau}, \alpha_3 \sigma = \alpha_3^1 \tilde{\sigma}, \beta_{31}, \tilde{\alpha}_3, \tilde{\sigma}, \tilde{\tau} \right) + \\
(1 - \varepsilon_1^5) \varepsilon_2^2 \sum_{k \neq 1, \tilde{\tau}} \Pr^n \left( w_{33} \beta_{32}^h \varphi^j, \beta_{31} \alpha_3 \sigma \tau = \beta_{31} \tilde{\alpha}_3 \tilde{\sigma} \tilde{\tau}, \alpha_3 \sigma = \alpha_3^1 \tilde{\sigma}, \beta_{31}, \tilde{\alpha}_3, \tilde{\sigma}, \tilde{\tau} \right) + \\
\frac{\varepsilon_1^2}{m} \sum_{h,j} \sum_{k \neq 1, \tilde{\tau}} \Pr^n \left( w_{33} \beta_{32}^h \varphi^j, \beta_{31} \alpha_3 \sigma \tau = \beta_{31} \tilde{\alpha}_3 \tilde{\sigma} \tilde{\tau}, \alpha_3 \sigma = \alpha_3^1 \tilde{\sigma}, \beta_{31}, \tilde{\alpha}_3, \tilde{\sigma}, \tilde{\tau} \right) \\
\Pr^n \left( \beta_{31} \alpha_3 \sigma \tau = \beta_{31} \tilde{\alpha}_3 \tilde{\sigma} \tilde{\tau} \mid z_3^2 = \beta_{31} \tilde{\alpha}_3 \tilde{\sigma} \tilde{\tau}, \alpha_3 \sigma = \alpha_3^1 \tilde{\sigma}, \beta_{31}, \tilde{\alpha}_3, \tilde{\sigma}, \tilde{\tau} \right).
\]

It is easy to check that our conditions regarding the trembles are sufficient for the above probability to be \( \frac{1}{m+1} \) in the limit. We now show that the condition \( \lim_{n \to \infty} \frac{\varepsilon_2}{\varepsilon_3, n} = 0 \) is also necessary to obtain the previous result. Suppose that \( P_2 \) receives message \( a \) from \( P_4 \) and \( P_5 \) and message \( b \) from \( P_1 \). Suppose that both \( P_1 \) and \( P_5 \) learnt the true realization of the permutation they received. Then the probability that \( P_2 \) receives the triple of messages\(^\text{27}\) \((aba)\) is an infinitesimal of the same order as \( \varepsilon_3^2 \). Now consider the scenario where \( P_3 \) learns the realization of \( \alpha_3 \sigma \), but \( P_1 \) learns the wrong permutation \( \beta_{31} \tilde{\alpha}_3 \tilde{\sigma} \tilde{\tau} \), i.e. the message sent by \( P_2 \). The probability of this event is an infinitesimal of the same order as \( \varepsilon_2^2 \). Suppose that

\(^{27}\)Remember that the first message in the triple refers to \( P_4 \), who is the least likely to tremble among the senders.
message \( b \) is equal to \( \hat{\beta}_{32}^l \hat{\beta}_{31} \hat{\alpha}_3 \hat{\sigma} \hat{\varphi}^l \), for some \( l = 1, \ldots, m + 1 \). In this case, \( P_2 \) receives the triple \((aba)\) only if every sender follows her equilibrium strategy in Substep 3.3 (in other words, no deviation is needed in Substep 3.3 in order to justify the triple \((aba)\)). In the limit, to assign probability zero to the latter event, we have to assume that \( \varepsilon_{2,n} \) converges to zero faster than \( \varepsilon_{3,n} \). A similar argument shows that the condition \( \lim_{n \to \infty} \frac{\varepsilon_{2,n}}{\varepsilon_{3,n}} = 0 \) is necessary for the case of \((abc)\), too.

Finally, we consider the case in which \( P_2 \) deviates in both substeps. We assume that \( P_2 \) sends message \( z_{32}^1 = \alpha_3^k \hat{\sigma} \) in Substep 3.1, and message \( z_{32}^2 = \hat{\beta}_{31} \hat{\alpha}_3 \hat{\sigma} \hat{\varphi} \) in Substep 3.2, where \( \hat{k} \) and \( \hat{\varphi} \) are different from one. As usual, we substitute equations (25) and (26) into equation (22). We obtain:

\[
\Pr^n \left( \hat{\beta}_{32}^l, \hat{\varphi}^l, \hat{\beta}_{31} \hat{\alpha}_3 \hat{\sigma} \hat{\varphi} \right) = \frac{\Pr^n \left( \hat{\beta}_{32}^l, \hat{\varphi}^l, \hat{\beta}_{31} \hat{\alpha}_3 \hat{\sigma} \hat{\varphi} \right)}{B_5},
\]

where:

\[
B_5 = (1 - \varepsilon_1^3) \left( 1 - \frac{\varepsilon_1^3}{m} \right) \left( 1 - \varepsilon_2^2 \right)^2 v(\varepsilon_3, m; w_{33}) + (1 - \varepsilon_1^3) \left( 1 - \frac{\varepsilon_1^3}{m} \right) \varepsilon_2^2 \left( 2 - \varepsilon_2^2 \left( \frac{m^2 + m - 1}{m^2} \right) \right)
\]

\[
\sum_{h,j} \Pr^n \left( w_{33} | \hat{\beta}_{32}^h, \hat{\varphi}^j, \hat{\beta}_{31} \hat{\alpha}_3 \hat{\sigma} \hat{\varphi} \right) = \hat{\beta}_{31} \hat{\alpha}_3 \hat{\sigma} \hat{\varphi} \right)
\]

\[
(1 - \varepsilon_1^3) \left( 1 - \frac{\varepsilon_1^3}{m} \right) \frac{\varepsilon_1^3}{m} \sum_{h,j} \sum_{i \neq 1, \hat{i}} \Pr^n \left( w_{33} | \hat{\beta}_{32}^h, \hat{\varphi}^j, \hat{\beta}_{31} \hat{\alpha}_3 \hat{\sigma} \hat{\varphi} \right) = \hat{\beta}_{31} \hat{\alpha}_3 \hat{\sigma} \hat{\varphi} \right)
\]

\[
\sum_{h,j,l} \Pr^n \left( w_{33} | \hat{\beta}_{32}^h, \hat{\varphi}^j, \hat{\beta}_{31} \hat{\alpha}_3 \hat{\sigma} \hat{\varphi} \right) = \hat{\beta}_{31} \hat{\alpha}_3 \hat{\sigma} \hat{\varphi} \right)
\]

\[
\frac{\varepsilon_1^3}{m} \sum_{h,j,l} \sum_{k \neq 1, \hat{k}} \Pr^n \left( w_{33} | \hat{\beta}_{32}^h, \hat{\varphi}^j, \hat{\beta}_{31} \hat{\alpha}_3 \hat{\sigma} \hat{\varphi} \right) = \hat{\beta}_{31} \hat{\alpha}_3 \hat{\sigma} \hat{\varphi} \right)
\]

Similarly to the previous cases, our assumptions regarding the trembles guarantee that in the limit, the above probability is \( \frac{1}{m+1} \). Here we show that the condition \( \lim_{n \to \infty} \frac{\varepsilon_{1,n}}{\varepsilon_{3,n}} = \)
0 is indeed necessary to obtain the previous result. Suppose that \( P_3 \) learns the permutation \( \tilde{\alpha}_3 \tilde{\kappa} \sigma \) sent by \( P_2 \) (the probability of this event is an infinitesimal of the same order as \( \varepsilon_1^2 \)). Then the probability that in Substep 3.2 \( P_1 \) learns the permutation \( \tilde{\beta}_{31} \tilde{\alpha}_3 \tilde{\tau} \sigma \) (sent by \( P_2 \) converges to one. Suppose now that \( \tilde{k} \) is different from \( \tilde{l} \). If in Substep 3.3 every sender follows her equilibrium strategy, \( P_2 \) receives three different messages. Remember that the probability of this event converges to zero at the same speed as \( \varepsilon_1^2 \). Now consider the event in which both \( P_1 \) and \( P_5 \) receive the true permutation. Then the probability that three different messages are sent in Substep 3.3 is an infinitesimal of the same order as \( \varepsilon_3^2 \). To assign probability zero to the first event (in the limit), we need to assume that 

\[
\lim_{n \to \infty} \frac{\varepsilon_{1,n}}{(\varepsilon_{a,n})^2} = 0.
\]

**Substep 3.4.** \( P_1, P_2 \) and \( P_5 \) send \( P_3 \) the element \( \beta_{32} \beta_{31} \alpha_3 \sigma \tau \varphi(x) \).

For any integer \( n \), let \( \varepsilon_{4,n} = \varepsilon_n \). We assume that \( P_1 \) trembles from her equilibrium strategy with probability \( \varepsilon_4^3 \). \( P_2 \) and \( P_5 \) tremble with probability \( \varepsilon_2^3 \).

With a slight abuse of notation, we let \( a \) denote an element of \( X \), and \( M \) denote the set of triples of messages in which \( a \) is the message sent by the majority. \( P_3 \) knows the permutations \( \beta_{32}, \beta_{31}, \sigma, \tau \) and \( \varphi \) (we denote their realizations by \( \tilde{\beta}_{32}, \tilde{\beta}_{31}, \tilde{\sigma}, \tilde{\tau} \) and \( \tilde{\varphi} \), respectively). Let \( \tilde{x}^1, \ldots, \tilde{x}^{\tilde{m}+1} \) denote \( \tilde{m} + 1 \) different elements of \( X \). For every element \( \tilde{x}^l \) in \( X \) there are \( \tilde{m}! \) different permutations\(^{28} \) \( \tilde{\alpha}_3^{(l)(\tilde{m}!)+1}, \ldots, \tilde{\alpha}_3^{(l+1)(\tilde{m}!)} \) such that 

\[
\tilde{\beta}_{32} \tilde{\beta}_{31} \tilde{\alpha}_3 \tilde{\tau} \tilde{\sigma} \tilde{\varphi} (\tilde{x}^l) = a, \text{ for every } l = l (\tilde{m}!) + 1, \ldots, (l + 1) (\tilde{m}!).
\]

Further, \( l \neq l' \) implies \( \tilde{\alpha}_3^j \neq \tilde{\alpha}_3^{j'} \), for every \( j = l (\tilde{m}!) + 1, \ldots, (l + 1) (\tilde{m})! \), and every \( j' = l' (\tilde{m}!) + 1, \ldots, (l' + 1) (\tilde{m}!) \).

\( P_3 \)'s beliefs do not depend on her actions in Steps 1 and 2.\(^{29} \) Thus, in order to prove that the assessment \( \Psi(r) \) satisfies consistency in Substep 3.4, it is sufficient to show that for every triple of messages \( w_{34} \) in \( M \), for every \( l = 1, \ldots, \tilde{m} + 1 \), and for every \( \tilde{l} = l (\tilde{m}!) + 1, \ldots, (l + 1) (\tilde{m}!) \), the following equality holds:

\(^{28}\) We relabel the indices of the permutations, if necessary.

\(^{29}\) The equilibrium strategies in Step 3 do not depend on \( P_3 \)'s messages in Steps 1 and 2.
Consider a step in which all the senders know the realization of the permutation that they have to send. Moreover, suppose that their trembles are defined by \( \varepsilon \). Then the probability that the receiver learns the truth is \( 1 - v(\varepsilon) \), and the probability that the receiver learns any other message is \( \frac{v(\varepsilon)}{m} \) (see equations (14) and (15)). Thus, we have:

\[
\Pr^n (\tilde{x}^t, \tilde{\alpha}_3^t, \beta_{32} \beta_{31} \alpha_3 \sigma \tau \varphi = \tilde{\beta}_{32} \tilde{\beta}_{31} \tilde{\alpha}_3^t \sigma \tilde{\tau} \tilde{\varphi}, \beta_{31} \alpha_3 \sigma \tau = \tilde{\beta}_{31} \tilde{\alpha}_3^t \sigma \tilde{\tau}, \overline{\tilde{\alpha}_3 \sigma} = \tilde{\alpha}_3^t \sigma \tilde{\tau} | w_{34}, \tilde{\beta}_{32}, \tilde{\beta}_{31}, \tilde{\sigma}, \tilde{\tau}, \tilde{\varphi}) = \frac{1}{m+1}.
\]

where:

\[
B_6 = (1 - v(\varepsilon_1)) (1 - v(\varepsilon_2)) (1 - v(\varepsilon_3)) \left( \frac{m+1}{m} \right) \left( \sum_{h=1}^{m+1} \Pr^n (w_{34} | \tilde{x}^h, \tilde{\alpha}_3^j, \beta_{32} \beta_{31} \tilde{\alpha}_3^j \sigma \tilde{\tau} \tilde{\varphi}, \beta_{31} \alpha_3 \sigma \tau = \tilde{\beta}_{31} \tilde{\alpha}_3^j \sigma \tilde{\tau}, \alpha_3 \sigma = \tilde{\alpha}_3^j \sigma | w_{34}, \tilde{\beta}_{32}, \tilde{\beta}_{31}, \tilde{\sigma}, \tilde{\tau}, \tilde{\varphi}) \right) + \\
\left( 1 - v(\varepsilon_1) \right) \left( \sum_{h=1}^{m+1} \Pr^n (w_{34} | \tilde{x}^h, \tilde{\alpha}_3^j, \beta_{32} \beta_{31} \tilde{\alpha}_3^j \sigma \tilde{\tau} \tilde{\varphi}, \beta_{31} \alpha_3 \sigma \tau = \tilde{\beta}_{31} \tilde{\alpha}_3^j \sigma \tilde{\tau}, \alpha_3 \sigma = \tilde{\alpha}_3^j \sigma | w_{34}, \tilde{\beta}_{32}, \tilde{\beta}_{31}, \tilde{\sigma}, \tilde{\tau}, \tilde{\varphi}) \right) + \\
\left( 1 - v(\varepsilon_2) \right) \left( \sum_{h=1}^{m+1} \Pr^n (w_{34} | \tilde{x}^h, \tilde{\alpha}_3^j, \beta_{32} \beta_{31} \tilde{\alpha}_3^j \sigma \tilde{\tau} \tilde{\varphi}, \beta_{31} \alpha_3 \sigma \tau = \tilde{\beta}_{31} \tilde{\alpha}_3^j \sigma \tilde{\tau}, \alpha_3 \sigma = \tilde{\alpha}_3^j \sigma | w_{34}, \tilde{\beta}_{32}, \tilde{\beta}_{31}, \tilde{\sigma}, \tilde{\tau}, \tilde{\varphi}) \right) + \\
(1 - v(\varepsilon_3)) \left( \sum_{h=1}^{m+1} \Pr^n (w_{34} | \tilde{x}^h, \tilde{\alpha}_3^j, \beta_{32} \beta_{31} \tilde{\alpha}_3^j \sigma \tilde{\tau} \tilde{\varphi}, \beta_{31} \alpha_3 \sigma \tau = \tilde{\beta}_{31} \tilde{\alpha}_3^j \sigma \tilde{\tau}, \alpha_3 \sigma = \tilde{\alpha}_3^j \sigma | w_{34}, \tilde{\beta}_{32}, \tilde{\beta}_{31}, \tilde{\sigma}, \tilde{\tau}, \tilde{\varphi}) \right)
\]

It is tedious but simple to check that as \( n \) goes to infinity, the above probability converges to \( \frac{1}{m+1} \).

To provide some intuition, suppose that \( P_3 \) receives the same message from two senders, and a different one from the third. Consider the two following scenarios. It is possible that all senders know the truth\(^{30}\) and only one of them deviated in Substep 3.4. Alternatively,

\(^{30}\)The probability of this event converges to one as \( n \) goes to infinity.
it is possible that some of the senders did not learn the truth. However, this implies that at least two players deviated in Substeps 3.1-3.3. Since deviations in Substep 3.4 are at least as likely as deviations in earlier steps, we conclude that, in the limit, the latter event has probability zero.

The case where $P_3$ receives three different messages can be analyzed in a similar way. If all senders know the truth, the probability of receiving three different messages is an infinitesimal of the same order as $\varepsilon^4_4$. If $P_1$ and/or $P_5$ did not learn the truth, then at least two players deviated in Substeps 3.1-3.2. Since $\varepsilon_{4,n}$ converges to zero faster than $\varepsilon_{1,n}$ and $\varepsilon_{2,n}$, we assign probability one to the event that both $P_1$ and $P_3$ know the truth. Given this, the probability that $P_2$ does not learn the truth in Substep 3.3 is an infinitesimal of the same order as $\varepsilon^4_4$. It is true that $\varepsilon_{3,n}$ is equal to $\varepsilon_{4,n}$, but note that $P_3$ cannot receive three different messages if both $P_1$ and $P_5$ follow their equilibrium strategies in Substep 3.4. At least one other deviation is needed. We conclude that all senders know the truth and two of them deviated in the last substep.

**Step 4.**

Consider the four sequences $\{\varepsilon_{j,n}\}_{n=1}^{\infty}$, $j = 1, \ldots, 4$, introduced in Step 3. In Substep 4.4, $j = 1, \ldots, 4$, the first sender in the description of the step trembles from her equilibrium strategy with probability $\varepsilon^3_{j,n}$. Each of the other two senders trembles with probability $\varepsilon^2_{j,n}$ (trembles are always independent across information sets). In each of the other steps (if any) $P_1$ trembles with probability $\varepsilon^3_4$, and $P_2$ and $P_5$ tremble with probability $\varepsilon^2_4$.

The analysis in Step 3 shows that, given the information acquired in Step 0 and Step 4, a receiver should assign probability one to the event that the majority of the senders report the true message and all of them learnt the realization of the corresponding permutation. Again, we use the fact that trembles are independent across information sets and all random choices are independent of each other and uniformly distributed to conclude that the above result does not depend on information from Steps 1-3. At the same time, the additional information in Step 4 does not modify the beliefs of earlier steps.

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Step 5.

Substep 5.1. $P_2$, $P_3$ and $P_4$ send $P_1$ the function $pr_1\alpha_1^{-1}$.

We assume that, along the sequence of mixed strategies, $P_2$ sends $P_1$ the realization of $pr_1\alpha_1^{-1}$ with probability $(1 - \varepsilon^3_n)$, and any other message with probability $\frac{\varepsilon^3_n}{|R_1| - 1}$, where $|R_1|$ is the number of feasible messages in Substep 5.1. $P_3$ and $P_4$ tremble with probability $\varepsilon^2_n$.

With another slight abuse of notation, let us assume that $a$ is the message sent by the majority of the senders to $P_1$ in Substep 5.1. Let $R^c$ denote the number of permutations compatible with $a$. We denote these permutations by $\hat{\alpha}_1^1, ..., \hat{\alpha}_1^{R^c}$ (again, we relabel the indices if necessary). Consider a triple of messages $w_{51}$ such that $a$ is the message sent by the majority. Given our sequence of mixed strategies, it is easy to verify that for any $l = 1, ..., R^c$, the following equality holds:

$$\lim_{n \to \infty} \Pr^n(a = \hat{\alpha}_1^l | w_{51}) = \frac{1}{R^c}.$$ 

This is sufficient to demonstrate that the assessment $\Psi(r)$ satisfies consistency in Substep 5.1.

Finally, notice that the equality above implies that, conditional on $P_1$’s information in Substep 5.1, the distribution of $\tau$ assigns probability $\frac{1}{R^c}$ only to the permutations $\hat{\tau}^1, ..., \hat{\tau}^{R^c}$ (the same result holds for the random distributions $\beta_{31}$ and $\beta_{41}$).

Since Substeps 5.2-5.I are identical to Substep 5.1, we conclude that the assessment $\Psi(r)$ is consistent.

Appendix B: Proof of Theorem 2

In this appendix we show how to implement a rational and regular communication equilibrium, $q$, of a Bayesian game $G$ with at least five players. As anticipated in Section 3, we proceed in two steps. First, we show that $q$ is the outcome induced by a sequential
equilibrium in a cheap talk extension, \( G^F(q) \), in which players use a correlation device to communicate. Then, we construct a plain cheap talk extension \( \bar{G}(q) \), and a sequential equilibrium of \( \bar{G}(q) \) that induces \( q \).

Remember that, given a communication equilibrium \( q \) of \( G \), the cheap talk extension \( G^D(q) \) is the following extensive-form game. In Step 1, players simultaneously send the mediator their messages (\( P_i \) sends a message in \( T_i \)). If the vector of reports is \( t \), the mediator randomly selects an action profile in \( S \) according to the probability distribution \( q(\cdot|t) \), and informs each player \( P_i \) only of the \( i \)th component of the chosen profile. Then, in Step 2, players simultaneously choose their actions (\( P_i \) chooses an action in \( S_i \)).

An information set of \( P_i \) in Step 2 is characterized by her type, \( t_i \), her report to the mediator in Step 1, \( t'_i \), and the mediator’s recommendation, \( s_i \). Obviously, the report \( t'_i \) and the recommendation \( s_i \) can be in the same information set only if there exist a profile of opponents’ types \( t_{-i} \), and a profile of opponents’ action \( s_{-i} \), such that \( q(s_i,s_{-i}|t'_i,t_{-i}) > 0 \).

We let \( L_i(t'_i) \) denote the set of recommendations that \( P_i \) can receive from the mediator after sending message \( t'_i \) in Step 1. Formally:

\[
L_i(t'_i) = \left\{ s_i \in S_i \mid \sum_{s_{-i} \in S_{-i}} \sum_{t_{-i} \in T_{-i}} q(s_i,s_{-i}|t'_i,t_{-i}) > 0 \right\}. \tag{27}
\]

Since \( q \) is regular, \( G^D(q) \) admits a sequential equilibrium \( \Phi^D(q) \) that satisfies the following two requirements. First, each player reports her true type to the mediator. Second, each player obeys the mediator’s recommendation after reporting her type truthfully to the mediator. We let \( \psi^D_i \) denote the equilibrium strategy of \( P_i \) in Step 2. For any information set \( (t_i,t'_i,s_i) \) in Step 2, \( \psi^D_i(t_i,t'_i,s_i) \) specifies a probability distribution over \( S_i \). The second requirement on \( \Phi^D(q) \) implies that for any type \( t_i \) in \( T_i \), and for any action \( s_i \) in \( L_i(t_i) \), \( \psi^D_i(t_i,t'_i,s_i) \) assigns probability one to \( s_i \). Clearly, \( \Phi^D(q) \) induces the outcome \( q \).

The assessment \( \Phi^D(q) \) also specifies the players’ beliefs. We denote \( P_i \)'s beliefs in Step 2 by \( \mu^D_i \). For each information set \( (t_i,t'_i,s_i) \), \( \mu^D_i \left( (t_j,t'_j,s_j)_{j \neq i} | t_i,t'_i,s_i \right) \) is the probability
that each opponent $P_j$ has type $t_j$, has reported $t'_j$ to the mediator and has received recommendation $s_j$. Notice that $\mu_i(t_i, t'_i, s_i)$ may be a probability distribution over a proper subset of $T_{-i}^2 \times S_{-i}$. However, $\mu_i(t_i, t'_i, s_i)$ can be easily extended to $T_{-i}^2 \times S_{-i}$ by assigning the value zero to those elements of $T_{-i}^2 \times S_{-i}$ that are not in the original domain of $\mu_i(t_i, t'_i, s_i)$. Without introducing further notation, hereafter we refer to $\mu_i(t_i, t'_i, s_i)$ as a probability distribution over $T_{-i}^2 \times S_{-i}$.

Finally, let us introduce the sequence of completely mixed strategies that is used to show that the assessment $\Phi^D(q)$ is consistent. Consider the $n$th element of the sequence, and let $t_i$ and $t'_i$ be two different types of player $P_i$. We let $\lambda_{i,n}(t'_i|t_i)$ denote the probability that $P_i$ sends the mediator message $t'_i$ when her type is $t_i$ (clearly, $\lim_{n \to \infty} \lambda_{i,n}(t'_i|t_i) = 0$). We denote by $\lambda_n$ the smallest probability that all players deviate from their equilibrium strategies in Step 1, that is:

$$
\lambda_n := \min_{(t_i, t'_i|t_i \neq t'_i)} \prod_{i=1}^I \lambda_{i,n}(t'_i|t_i). 
$$

We now turn to $G^F(q)$, the cheap talk extension proposed by Forges (1990). First, we need to describe how the mediator chooses her messages. The mediator randomly selects a function $f$ from $T$ to $S$ according to the following probability distribution:

$$
\Pr(f) = \Pr((f(t))_{t \in T}) = \prod_{t \in T} q(f(t)|t).
$$

Given two different type profiles, $t$ and $t'$, $f(t)$ is chosen independently of $f(t')$. Further, for each $i = 1, ..., I$, the mediator randomly selects a permutation $\gamma_i$ on $T_i$ according to the uniform distribution. Finally, for every $i = 1, ..., I$, the mediator also selects a permutation $\pi_i$ on $T \times S_i$ at random, according to the uniform distribution. $f, \gamma_1, ..., \gamma_I, \pi_1, ..., \pi_I$ are chosen independently of each other.

Given $f, \gamma_1, ..., \gamma_I$, let $f': T \to S$ be defined by:

\[\text{31} \text{Since } G^F(q) \text{ is analyzed in great detail in Forges (1990), we proceed quickly through its description.}\]
Let $f' (t_1, \ldots, t_I) = f (\gamma_1^{-1} (t_1), \ldots, \gamma_I^{-1} (t_I))$ for all $(t_1, \ldots, t_I) \in T$, and let $f'_i = T \to S_i$ be the projection of $f'$ on $S_i$. Further, let $\Upsilon$ be the identity mapping on $T$ and define $(\Upsilon \times f'_i) : T \to T \times S_i$ by $(\Upsilon \times f'_i) (t) = (t, f'_i (t))$. Then the mediator constructs the mapping $\zeta_i = \pi_i \circ (\Upsilon \times f'_i)$ from $T$ to $T \times S_i$, and the mapping $\theta_i = pr_{S_i} \circ \pi_i^{-1}$ from $T \times S_i$ to $S_i$ (i.e. $\theta_i$ is the projection on $S_i$ of $\pi_i^{-1}$). Clearly, $\theta_i \circ \zeta_i = f'_i$ and $\theta_i$ is independent of $f'_i$. Moreover, since the mapping $(\Upsilon \times f'_i)$ is injective, $\zeta_i$ is also independent of $f'_i$ (see Forges (1988, 1990)).

The extensive-form game $G^F (q)$ is as follows. The mediator reports $\gamma_i$, $\zeta_{i+1}$, $\zeta_{i+2}$, $\zeta_{i+3}$ (where $+$ is mod $I$) and $\theta_i$ to every player $P_i$. In Step 1, players simultaneously announce public messages ($P_i$ sends a message in $T_i$). Step 2 of $G^F (q)$ is divided into $I$ substeps. In Substep 2, player $P_i$ observes the messages sent by $P_{i-E_h}$ and has received the permutation $\zeta_i$ from the mediator. In equilibrium, $P_i$ announces the message $\gamma_i (t_i)$ in Step 1 (notice that this message is not a function of $\zeta_{i+1}$, $\zeta_{i+2}$, $\zeta_{i+3}$ or $\theta_i$). We also need to construct a sequence of completely mixed strategies. Suppose, as above, that $P_i$ has type $t_i$ and has received the permutation $\gamma_i$. Consider a type $t'_i$ different from $\gamma_i (t_i)$. In the $n$th element of the sequence of mixed strategies, the probability that $P_i$ reports message $t'_i$.

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32 In Forges (1990), messages of Step 2 may, but need not, be public. Here, we insist on private messages: in Substep 2, only $P_i$ observes the messages sent by $P_{i-E_3}$, $P_{i-E_2}$ and $P_{i-E_1}$. We will explain later why we need private messages in Step 2.
in Step 1 is equal to $\lambda_{i,n} (\gamma_i^{-1}(t'_i) | t_i)$. That is, the probability that $P_i$ (whose type is $t_i$) reports $t'_i$ in Step 1 of $G^F(q)$ coincides with the probability that she sends the mediator the message $\gamma_i^{-1}(t'_i)$ in the canonical game $G^D(q)$.

We now consider the equilibrium strategy of $P_i$ in Substep 2.$(i+h)$ (where $h = 1, 2, 3$). In addition to her type $t_i$, and to the mediator’s message $(\gamma_i, \zeta_{i+1}, \zeta_{i+2}, \zeta_{i+3}, \theta_i)$, $P_i$ knows the vector of reports in Step 1, say $t'$, and the messages that she has already sent and received in Step 2. The equilibrium strategy prescribes that $P_i$ sends message $\zeta_{i+h}(t')$ to $P_{i+h}$. Note that in equilibrium, in Step 2 each player receives the same message from the three senders.

As far as the sequence of completely mixed strategies is concerned, let $\{\epsilon_n\}_{n=1}^\infty$ be a sequence of positive numbers in the unit interval, converging to zero and satisfying the following condition A:

$$\lim_{n \to \infty} \frac{\epsilon_n}{\lambda_n} = 0, \quad (A)$$

where $\lambda_n$ is defined in equation (28). Consider the $n$th element of the sequence of mixed strategies. Consider $P_i$ in Substep 2.$(i+h)$ and suppose that $t'$ is the vector of reports in Step 1. We assume that with probability $(1 - \epsilon_n)$ $P_i$ follows the equilibrium strategy and sends $P_{i+h}$ message $\zeta_{i+h}(t')$. All other messages are equally likely. Condition (A) guarantees that the probability that a single player deviates in a substep of Step 2 converges to zero faster than the probability that all players deviate in Step 1.

Finally, we analyze Step 3. We let $d^j_i$ denote the message sent by $P_i$ to $P_j$ in Substep 2.$j$ (where $j = i+1, i+2, i+3$, and $+$ is mod $I$). An information set of $P_i$ in Step 3 is characterized by her type $t_i$, the mediator’s message $(\gamma_i, \zeta_{i+1}, \zeta_{i+2}, \zeta_{i+3}, \theta_i)$, the vector of reports in Step 1 $t'$, the three messages sent in Substeps 2.$(i+1)$ to 2.$(i+3)$ ($d^{i+1}_i, d^{i+2}_i, d^{i+3}_i$), and the three messages received in Substep 2.$i$ ($d^{i-1}_i, d^{i-2}_i, d^{i-3}_i$). We say that an information set in Step 3 is of class 1 if the two following conditions are satisfied. First, at least two senders sent the same message in Substep 2.$i$. We let $\mathcal{I}$ denote the message sent by the
majority of the senders. Second, the message $\overline{d}$ is such that $\theta_i(\overline{d})$ belongs to the set $L_i(\gamma_{i}^{-1}(t'_i))$, where $t'_i$ is the report of $P_i$ in Step 1 and the set $L_i(\cdot)$ is defined in equation (27). The second requirement implies that being in an information set of class 1 is not definitive evidence that the majority of the senders deviated from their equilibrium strategies in Substep $2.i$. In an information set of class 1, $P_i$ may think that the majority of her senders played the equilibrium strategy, even though this may require a deviation by some opponents in Step 1 (we will come back to this point shortly).

We let $\psi_i^F$ denote $P_i$'s equilibrium strategy in Step 3. To every information set of $P_i$ in Step 3, $\psi_i^F$ assigns a probability distribution on $S_i$. For any information set of class 1, $\psi_i^F$ satisfies:

$$\psi_i^F(t_i, \gamma_i, \zeta_{i+1}, \zeta_{i+2}, \zeta_{i+3}, \theta_i, t', d_i^{i+1}, d_i^{i+2}, d_i^{i+3}, d_{i-3}, d_{i-2}, d_{i-1}) = \psi_i^D(t_i, \gamma_{i}^{-1}(t'_i), \theta_i(\overline{d})),$$

where, remember, $\overline{d}$ denotes the message sent by the majority in Substep $2.i$, and $\psi_i^D$ is the equilibrium strategy of player $P_i$ in Step 2 of $G^D(q)$. That is, $P_i$ chooses the same strategy that she plays in $G^D(q)$ after sending the mediator the message $\gamma_{i}^{-1}(t'_i)$, and receiving the recommendation $\theta_i(\overline{d})$.

To define the equilibrium strategies in information sets that are not of class 1, we first need to analyze equilibrium beliefs. First of all notice that in any information set of Step 3, $P_i$ assigns probability one to the event that all her opponents are in information sets of class 1. In fact, messages in Step 2 are private, and the senders of Substep $2.j$ in equilibrium send $P_j$ the message $\zeta_j(t')$ (where $t'$ is the vector of reports in Step 1). It might be that $P_i$ is a sender of $P_j$ and $P_i$ deviated in Substep $2.j$. However, with probability one $P_j$ received the message $\zeta_j(t')$ from the other two senders.

Consider now any information set of class 1 of $P_i$. Condition (A) implies that $P_i$ assigns probability one to the event that the majority of her senders played their equilibrium strategies in Substep $2.i$. To give an intuition for this result, consider the following case.
Suppose that two senders reported the same message \( \vec{d} \) (with \( \theta_i \left( \vec{d} \right) \in L_i (\gamma_i^{-1} (t_i')) \)), and that the third sender sent a different message. Consider the \( n \)th element of the sequence of mixed strategies. The ratio between the probability that \( P_i \)'s information set is reached and the majority of the senders deviated, and the probability that \( P_i \)'s information set is reached and the majority played the equilibrium strategy is smaller than \( \epsilon_n / \left( (1 - \epsilon_n)^2 \lambda_n \right) \).

Condition (A) guarantees that in the limit (as \( n \) grows large), \( P_i \) assigns probability zero to the event that the majority of the senders deviated from their equilibrium strategies in Substep 2.\( i \). The same conclusion can be derived, \textit{a fortiori}, in an information set of class 1 in which all senders have reported the same message.

The above result, and the fact that the trembles in Step 1 are derived from the trembles of the equilibrium \( \Phi^D (q) \) of \( G^D (q) \), imply that in any information set of class 1, \( P_i \)'s (consistent) beliefs satisfy the following condition:

\[
\Pr((t_j, \gamma_j^{-1} (t_j') = \bar{t}_j, \theta_j (\zeta_j (t')) = s_j)_{j \neq i} | t_i, \gamma_i, \zeta_{i+1}, \zeta_{i+2}, \zeta_{i+3}, \theta_i, t', d_i^{i+1}, d_i^{i+2}, d_i^{i+3}, d_{i-3}^i, d_{i-2}^i, d_{i-1}^i) = \mu^D_i \left( (t_j, \bar{t}_j, s_j)_{j \neq i} | t_i, \gamma_i^{-1} (t_i'), \theta_i (\vec{d}) \right),
\]

for any vector \( (t_j, \bar{t}_j, s_j)_{j \neq i} \).

We now turn to analyze information sets that are not of class 1. Given the sequence of completely mixed strategies, \( P_i \) has some consistent beliefs (we will see that it does not matter what these beliefs are). However, as was said above, \( P_i \) assigns probability one to the event that each opponent \( P_j \) is in an information set of class 1. Since we have already specified equilibrium strategies in information sets of class 1, it is easy to compute equilibrium strategies in all other information sets of Step 3. \( P_i \) selects the strategy that maximizes her expected payoff, given her beliefs and her opponents’ strategies in information sets of class 1.

Note that in information sets of class 1 the equilibrium strategy of opponent \( P_j \) depends only on \( t_j \) (her type), \( \gamma_j^{-1} (t_j') \) (\( t_j' \) is \( P_j \)'s report in Step 1) and \( \theta_j (\vec{d}) \). Given all the information that \( P_i \) has in Step 3, her beliefs over the realizations of \( \left( t_j, \gamma_j^{-1} (t_j'), \theta_j (\vec{d}) \right) \) \( j \neq i \)
do not depend on the messages that she sent in Step 2. Therefore, equilibrium strategies in information sets that are not of class 1 may be selected so that they do not depend on the messages that a player sends in Step 2 (similarly to what happens in information sets of class 1). This concludes the description of the assessment $\Phi^F(q)$.

Checking that $\Phi^F(q)$ is indeed a sequential equilibrium is straightforward. Consider an information set of class 1 of $P_i$. She assigns probability one to the event that all her opponents are in information sets of class 1 (where the equilibrium strategies are defined by equation (29)). Since her beliefs satisfy equation (30), if $P_i$ has an incentive to deviate in an information set of class 1, then, given the assessment $\Phi^D(q)$, she would have an incentive to deviate from the strategy $\psi^D_i$ in some information set of Step 2 of $G^D(q)$, which is impossible. The strategies in information sets of Step 3 that are not in class 1 are optimal by construction. A deviation by a player in Step 2 does not have any impact on her action and on her opponents’ actions in Step 3. The optimality of the strategies of Step 1 comes from the fact that, given the assessment $\Phi^D(q)$, no player has an incentive to lie to the mediator in $G^D(q)$. Finally, beliefs of $\Phi^F(q)$ are consistent, since they are derived by applying Bayes’ rule to completely mixed strategies converging to the equilibrium strategies.

We now use the assumptions that $q$ is rational and that $G$ has five or more players, and show how unmediated communication can replace the mediator in $G^F(q)$. Let $\Xi_i$ denote the set of messages that $P_i$ can receive from the mediator in $G^F(q)$, and let $\xi$ denote the probability distribution over $\Xi = \prod_{i=1}^I \Xi_i$ according to which the mediator selects the random vector $\left(\gamma_i, (\zeta_{i+h})_{h=1}^3, \theta_i\right)_{i=1}^I$. If $q$ is rational, $\xi$ involves probabilities that are rational numbers. Therefore, there exists a finite set $\tilde{X}$, and functions $\tilde{\rho}_1 : \tilde{X} \to \Xi_i$ (for $i = 1, \ldots, I$), such that picking at random an element $\tilde{x} \in \tilde{X}$ according to the uniform distribution, and then computing $(\tilde{\rho}_1(\tilde{x}), \ldots, \tilde{\rho}_I(\tilde{x}))$, generates the probability distribution $\xi$ on $\Xi$. Note that by applying our communication scheme (Steps 0-5 in Section 5) to $\tilde{X}$, $\tilde{\rho}_1, \ldots, \tilde{\rho}_I$, players may randomly select a vector $\left(\gamma_i, (\zeta_{i+h})_{h=1}^3, \theta_i\right)_{i=1}^I$ according to the probability distribution $\xi$, in such a way that every player $P_i$ learns only $\left(\gamma_i, (\zeta_{i+h})_{h=1}^3, \theta_i\right)$.
We therefore combine our communication scheme and $G^F(q)$ to obtain $\bar{G}(q)$, a plain cheap talk extension. $\bar{G}(q)$ is an extensive form game that can be divided in nine steps. Steps 0-5 of $\bar{G}(q)$ coincide with Steps 0-5 of Section 5 applied to $\bar{X}$, $\bar{p}r_1, ..., \bar{p}r_I$. At the end of our communication protocol, Steps 1-3 of $G^F(q)$ are performed.

We now construct $\Phi(q)$, the sequential equilibrium of $\bar{G}(q)$ that induces $q$. In the first six steps of $\bar{G}(q)$ the equilibrium strategies of a player do not depend on her type. Specifically, every type of $P_i$ $(i = 1, ..., I)$ adopts the equilibrium strategies of $P_i$ in Steps 0-5 of Section 5.

To describe equilibrium strategies in the last three steps of $\bar{G}(q)$ we first need to discuss the information that a player has at the end of our communication scheme. Note that according to the terminology of Section 5, the message sent by the majority of the senders in a given step is a well-defined concept (if the senders send three different messages, the majority message is the message sent by the first player in the description of the step). $P_i$, $i = 1, ..., 4$, combines the random variables known in Step 0 with the majority messages in Step $i$ and in Substep 5.$i$ to construct a message in $\Xi_i$ (as described in Table 5). We refer to this message, as the message that $P_i$ learns after Step 5 of $\bar{G}(q)$. $P_5$ learns her message in $\Xi_5$ by using the majority messages received in Step 2 and in Substep 5.5 (and the random variables known in Step 0). If $G$ has more than five players, $P_k$ $(k = 6, ..., I)$ learns her message in $\Xi_k$ by using the majority messages received in Step 4 and in Substep 5.$k$ (and the random variables known in Step 0).

We are ready to define the equilibrium strategies of $\Phi(q)$ in the last three steps of $\bar{G}(q)$. Suppose that $P_i$ learns message $\left(\bar{\gamma}_i, (\bar{s}_{i+h})_{h=1}^{3}, \bar{\theta}_i\right)$. Then $P_i$ adopts the same behavioral strategies that $P_i$ chooses in equilibrium $\Phi^F(q)$ of $G^F(q)$ after receiving message $\left(\bar{\gamma}_i, (\bar{\zeta}_{i+h})_{h=1}^{3}, \bar{\theta}_i\right)$ from the mediator (note that messages of Steps 0-5 influence the equilibrium strategies of $P_i$ in the last three steps of $\bar{G}(q)$ only through the message that $P_i$ learns after Step 5).

To construct consistent beliefs, in Steps 0-5, for every type of $P_i$ we take the sequence
of completely mixed strategies of $P_i$ described in Appendix A.\textsuperscript{33} The mixed strategies of Steps 0-5 are therefore described by the converging sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ (see Appendix A). In Steps 6 and 7 of $\bar{G}(q)$, the sequences of completely mixed strategies are those used above to show consistency of the assessment $\Phi_F(q)$. Moreover, we assume that $\varepsilon_n$ converges to zero faster than the probability that all players deviate in Steps 6 and 7. Roughly speaking, this condition requires that deviations in Steps 0-5 are much less likely than deviations in Steps 6 and 7.

Note that if the majority of the senders follow their equilibrium strategies in Steps 0-5, then a random element $\tilde{x}$ of $\tilde{X}$ is selected according to the uniform distribution, and every player $P_i$ learns $\tilde{p}_{r,i}(\tilde{x})$ after Step 5. Our assumption above on the different order of infinitesimality of the deviations guarantees that at every information set in Steps 6-8, a player assigns probability one to the event that every player $P_j$ learnt $\tilde{p}_{r,j}(\tilde{x})$. This implies that the strategies specified by $\Phi(q)$ in Steps 6-8 satisfy the sequential rationality requirement. Sequential rationality of the assessment $\Phi(q)$ in the first five steps follows from our proof in Section 5. We conclude that $\Phi(q)$ is a sequential equilibrium of $\bar{G}(q)$ that induces $q$.

\textsuperscript{33}Note that at the end of Step 5, a player’s beliefs over her opponents’ types are described by the prior distribution $p$. In other words, in the first five steps a player does not learn anything new about her opponents’ types. This follows from the fact that in those steps, the equilibrium strategies and the sequences of mixed strategies are not functions of a player’s type.
References


