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# A new method for helicity calculations <sup>★</sup>

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## Abstract

We propose a new helicity formalism based on the insertion in spinor lines of a complete set of states built up with appropriate unphysical spinors, which are eigenstates of  $\not{p}$ , where  $p$  is the momentum flowing in a fermion propagator. The corresponding eigenvalues can be real ( $p^2 > 0$ ) or imaginary ( $p^2 < 0$ ). The method is developed both for massless and massive fermions for which it turns out to be particularly fast. All relevant formulae are given.

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## 1. Introduction

In high energy collisions often many particles or partons widely separated in phase space are produced. The calculation of cross sections for these processes is made difficult by the large number of Feynman diagrams which appear in the perturbative expansion. This is due both to the complexity of non-abelian theories and to simple combinatorics, which generates more and more diagrams when the number of external particles grows.

If one computes unpolarized cross sections for a process described by many Feynman diagrams with the textbook method of considering the amplitude modulus squared and taking the traces, one can end up with a prohibitive number of traces to evaluate. The calculation becomes simpler if one uses the so-called helicity amplitudes techniques. In such an approach,

given an assigned helicity to external particles, one computes the contribution of every single diagram  $k$  as a complex number  $a_k$ , sums over all  $k$ 's and takes the modulus squared. To obtain unpolarized cross sections one simply sums the modulus squared for the various external helicities.

The use of helicity amplitude techniques in high energy physics dates back to Refs. [1,2]. Many different approaches have been developed [3–12], and even a brief survey of the vast literature on this subject goes far beyond our scope. Two among the most popular schemes are those of Refs. [7,10]. They can be used both with massless and massive particles. We have employed them in the past for several calculations. One of them is based [10] on the choice of a specific representation for the Dirac matrices. It is then possible to obtain explicit expressions for the spinors, for the matrices at the vertices and for the fermion propagators. The complex number corresponding to a given diagram is obtained multiplying these matrices. In the other one [7,12] all spinors for any physical momentum are defined in terms of a basic spinor for

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an auxiliary lightlike momentum. Decomposing the internal momenta in terms of the external ones, and using the fact that  $\sum_{\lambda} u\bar{u} = \not{p} + m$ ,  $\sum_{\lambda} v\bar{v} = \not{p} - m$ , all spinor lines are reduced to an algebraic combination of spinors products  $\bar{u}(p_1)u(p_2)$ . In order to use this method the polarization vectors of spin-1 particles have to be expressed as  $\bar{u}\gamma^{\mu}u$  currents.

In the following we describe an approach to helicity amplitudes which is based on the insertion in spinor lines of a complete set of states built up with unphysical spinors. By this we mean that for every internal spinor line (propagator) carrying momentum  $p$  in a Feynman diagram, we make use of the eigenstates of  $\not{p}$  with eigenvalues equal to the two determinations of  $\sqrt{p^2}$ . When  $p^2 < 0$ , as often is the case, the eigenvalues are imaginary. This generalization of the usual  $u$  and  $v$  type spinors allows us to treat in a unified way the case of spacelike and timelike momenta. This new formalism, which we have tested in several physical computations [13], combines the best features of Refs. [7,10], is highly flexible, has a modular structure and in our experience is faster than previous methods.

## 2. Spinors

The method we will describe makes use of generalized spinors  $u(p)$  which coincide with the usual ones when  $p^2 \geq 0$ , but are defined also for spacelike momenta. We discuss such a generalization in this section.

Spinors may be defined as eigenstates of  $\not{p}$ . This is equivalent to saying that they must satisfy the Dirac equations

$$\not{p}u(p) = +mu(p), \quad \not{p}v(p) = -mv(p). \quad (1)$$

Since  $\not{p}$  is not hermitian,  $m$  need not to be real, but it is constrained to satisfy  $m^2 = p^2$  as  $\not{p}^2 = p^2$ . When  $p^2 \leq 0$ , the two eigenvalues are imaginary and one can choose, for instance, to associate  $u(p)$  with the eigenvalue  $m$  whose imaginary part is positive. The remaining degeneracy is normally eliminated considering eigenstates of  $\gamma^5 \not{p}$ :

$$\begin{aligned} \gamma^5 \not{p}u(p, +) &= u(p, +), & \gamma^5 \not{p}u(p, -) &= -u(p, -), \\ \gamma^5 \not{p}v(p, +) &= v(p, +), & \gamma^5 \not{p}v(p, -) &= -v(p, -), \end{aligned} \quad (2)$$

where the polarization vector  $s$  has to satisfy the two conditions  $s \cdot p = 0$ ,  $s^2 = -1$ . The first one implies that  $\gamma^5 \not{p}$  commutes with  $\not{p}$ , the second that  $(\gamma^5 \not{p})^2 = 1$  and hence that its two eigenvalues are  $\pm 1$ .

One can easily construct an example of spinors satisfying Eqs. (1), (2) for any value of  $p^2$  with a straightforward generalization of the method of Ref. [7]. In such a method, one first defines spinors  $w(k_0, \lambda)$  for an auxiliary massless vector  $k_0$  satisfying

$$w(k_0, \lambda)\bar{w}(k_0, \lambda) = \frac{1 + \lambda\gamma_5}{2} \not{k}_0 \quad (3)$$

and with their relative phase fixed by

$$w(k_0, \lambda) = \lambda \not{k}_1 w(k_0, -\lambda), \quad (4)$$

with  $k_1$  a second auxiliary vector such that  $k_1^2 = -1$ ,  $k_0 \cdot k_1 = 0$ . Spinors for a four-momentum  $p$ , with  $m^2 = p^2$  are then obtained as

$$\begin{aligned} u(p, \lambda) &= \frac{\not{p} + m}{\sqrt{2} p \cdot k_0} w(k_0, -\lambda), \\ v(p, \lambda) &= \frac{\not{p} - m}{\sqrt{2} p \cdot k_0} w(k_0, -\lambda). \end{aligned} \quad (5)$$

One can readily check that  $u$  and  $v$  of Eq. (5) satisfy Eqs. (1), (2) also when  $p^2 \leq 0$  and  $m$  is imaginary.

Some care must be taken in defining in the general case the conjugate spinors  $\bar{u}(p, \lambda)$ ,  $\bar{v}(p, \lambda)$ . In fact the usual quantities  $\bar{u} = u^\dagger \gamma^0$ ,  $\bar{v} = v^\dagger \gamma^0$  do not satisfy normal Dirac equations

$$\bar{u}(p)\not{p} = +m\bar{u}(p), \quad \bar{v}(p)\not{p} = -m\bar{v}(p), \quad (6)$$

when  $m$  is imaginary. Taking the hermitian conjugate of Eqs. (1), one readily verifies on the contrary that  $\bar{u} = v^\dagger \gamma_0$ ,  $\bar{v} = u^\dagger \gamma_0$  do.

We define  $\bar{u}(p, \lambda)$ ,  $\bar{v}(p, \lambda)$  as the spinors satisfying both Eqs. (6) and

$$\begin{aligned} \bar{u}(p, +)\gamma^5 \not{p} &= \bar{u}(p, +), & \bar{u}(p, -)\gamma^5 \not{p} &= -\bar{u}(p, -), \\ \bar{v}(p, +)\gamma^5 \not{p} &= \bar{v}(p, +), & \bar{v}(p, -)\gamma^5 \not{p} &= -\bar{v}(p, -). \end{aligned} \quad (7)$$

With this definition the usual orthogonality relations among spinors are satisfied. As a consequence, choosing the normalization

$$\bar{u}(p, \lambda)u(p, \lambda) = 2m, \quad \bar{v}(p, \lambda)v(p, \lambda) = -2m, \tag{8}$$

the completeness relation

$$1 = \sum_{\lambda} \frac{u(p, \lambda)\bar{u}(p, \lambda) - v(p, \lambda)\bar{v}(p, \lambda)}{2m} \tag{9}$$

is verified for any value of  $p^2$ .

The explicit relation between  $u, v$  and  $\bar{u}, \bar{v}$  is

$$\bar{u}(p, \lambda) = u^\dagger(p, \lambda)\gamma^0, \quad \bar{v}(p, \lambda) = v^\dagger(p, \lambda)\gamma^0, \tag{10}$$

as usual when  $p^2 > 0$ . For  $p^2 < 0$  one has

$$\bar{u}(p, \lambda) = v^\dagger(p, \lambda)\gamma^0, \quad \bar{v}(p, \lambda) = u^\dagger(p, \lambda)\gamma^0, \tag{11}$$

if the components of the polarization vector  $s$  are real, and

$$\bar{u}(p, \lambda) = v^\dagger(p, -\lambda)\gamma^0, \quad \bar{v}(p, \lambda) = u^\dagger(p, -\lambda)\gamma^0, \tag{12}$$

if they become purely imaginary for imaginary  $m$ . This is precisely the case when one uses the spinors of Ref. [7]: the polarization vector is  $s^\mu = p^\mu/m - (m/p \cdot k_0)k_0^\mu$  and for a spacelike momentum  $p$  it is equal to the imaginary unit times a timelike vector. It is interesting to notice that for  $k_0^0 = \alpha(p^0 - |p|)/p^2$  and  $k_0 = \alpha\mathbf{p}(|p| - p^0)/|p|p^2$ , with  $\alpha$  an arbitrary factor, one recovers the usual helicity polarization vector  $s = (|p|^2, p_0\mathbf{p})/m|p|$ .

With the previous definitions, the spinors conjugate to  $u$  and  $v$  defined in Eq. (5) are given by the simple formulae

$$\begin{aligned} \bar{u}(p, \lambda) &= \bar{w}(k_0, -\lambda) \frac{\not{p} + m}{\sqrt{2 p \cdot k_0}}, \\ \bar{v}(p, \lambda) &= \bar{w}(k_0, -\lambda) \frac{\not{p} - m}{\sqrt{2 p \cdot k_0}} \end{aligned} \tag{13}$$

in all cases. At this point one may notice that there is an ambiguity in the definitions (5) and (13), since we have not specified which of the two determinations of the square root of  $p^2$  is to be used as  $m$  when  $p^2$  is negative. As we will show in the next sections, one can choose at will any of the two possibilities (for example

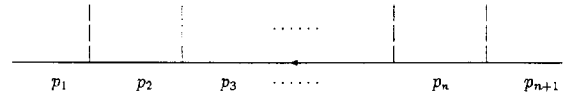


Fig. 1. Fermion line with  $n$  insertions.

$m = i\sqrt{|p^2|}$ ) since only  $m^2$  appears at the end of the computation of a complete Feynman diagram. The same applies to the determination of  $\sqrt{2 p \cdot k_0}$ .

### 3. Outline of the method. $T$ functions

The spinor part of every massive fermion line (Fig. 1) with  $n$  insertions has a generic expression of the following type,

$$\begin{aligned} T^{(n)} &= \bar{U}(p_1, \lambda_1) \chi_1(\not{p}_2 + \mu_2) \chi_2(\not{p}_3 + \mu_3) \cdots \\ &\quad \times (\not{p}_n + \mu_n) \chi_n U(p_{n+1}, \lambda_{n+1}), \end{aligned} \tag{14}$$

where  $\lambda_1$  and  $\lambda_{n+1}$  are the polarizations of the external fermions,  $p_1$  and  $p_{n+1}$  their momenta.  $p_2, \dots, p_n$  and  $\mu_2, \dots, \mu_n$  are the four-momenta and masses appearing in the fermion propagators.  $U(p, \lambda)$  ( $\bar{U}(p, \lambda)$ ) stands for either  $u(p, \lambda)$  ( $\bar{u}(p, \lambda)$ ) or  $v(p, \lambda)$  ( $\bar{v}(p, \lambda)$ ). The  $\chi$ 's are

$$\begin{aligned} \chi_i^S &\equiv \chi^S(c_{r_i}, c_{l_i}) \\ &= c_{r_i} \left( \frac{1 + \gamma_5}{2} \right) + c_{l_i} \left( \frac{1 - \gamma_5}{2} \right), \end{aligned} \tag{15}$$

when the insertion corresponds to a scalar (or pseudoscalar), or

$$\begin{aligned} \chi_i^V &\equiv \chi^V(\eta_i, c_{r_i}, c_{l_i}) \\ &= \eta_i \left[ c_{r_i} \left( \frac{1 + \gamma_5}{2} \right) + c_{l_i} \left( \frac{1 - \gamma_5}{2} \right) \right], \end{aligned} \tag{16}$$

when it corresponds to a vector particle whose ‘‘polarization’’ is  $\eta$ . Of course  $\eta$  can be the polarization vector of the external particle or the vector resulting from a complete subdiagram which is connected in the  $i$ th position to the fermion line.

Let us start considering the case in which there are only two insertions:

$$\begin{aligned} T^{(2)}(p_1; \eta_1, c_1; p_2; \eta_2, c_2; p_3) \\ = \bar{U}(p_1, \lambda_1) \chi_1(\not{p}_2 + \mu_2) \chi_2 U(p_3, \lambda_3). \end{aligned} \tag{17}$$

Here and in the following we indicate with  $c_i$  both the couplings  $c_r$ , and  $c_l$ . Obviously the vectors  $\eta$  only appear as an argument for vector insertions.

Even if  $m_2^2 \equiv p_2^2$  does not correspond to the mass of any physical particle, one can insert in Eq. (17), just before  $(\not{p}_2 + \not{\mu}_2)$ , a complete set of states in the form

$$1 = \sum_{\lambda} \frac{u(p_2, \lambda) \bar{u}(p_2, \lambda) - v(p_2, \lambda) \bar{v}(p_2, \lambda)}{2m_2} \quad (18)$$

and make use of the Dirac equations to get

$$\begin{aligned} T^{(2)} = & \frac{1}{2} \sum_{\lambda_2} [\bar{U}(p_1, \lambda_1) \chi_1 u(p_2, \lambda_2) \\ & \times \bar{u}(p_2, \lambda_2) \chi_2 U(p_3, \lambda_3) \times (1 + \mu_2/m_2) \\ & + \bar{U}(p_1, \lambda_1) \chi_1 v(p_2, \lambda_2) \times \bar{v}(p_2, \lambda_2) \chi_2 U(p_3, \lambda_3) \\ & \times (1 - \mu_2/m_2)]. \end{aligned} \quad (19)$$

This example can be generalized to any number of insertions and shows that the factors  $(\not{p}_i + \not{\mu}_i)$  can be easily eliminated, reducing all fermion lines essentially to products of  $T$  functions:

$$T_{\lambda_1 \lambda_2}(p_1; \eta, c; p_2) = \bar{U}(p_1, \lambda_1) \chi U(p_2, \lambda_2), \quad (20)$$

defined for any value of  $p_1^2$  and  $p_2^2$ .

It is convenient to use the spinors  $u(p, \lambda)$  and  $v(p, \lambda)$  defined in Eqs. (5), (13). With this choice, to which we will adhere from now on, the  $T$  functions (20) have a simple dependence on  $m_1$  and  $m_2$  and as a consequence the rules for constructing spinor lines out of them are simple. Every  $T$  function has in fact an expression of the following kind,

$$\begin{aligned} \tilde{T}_{\lambda_1 \lambda_2}(p_1; \eta, c; p_2) & \equiv \sqrt{p_1 \cdot k_0} \sqrt{p_2 \cdot k_0} T_{\lambda_1 \lambda_2}(p_1; \eta, c; p_2) \\ & = A_{\lambda_1 \lambda_2}(p_1; \eta, c; p_2) + M_1 B_{\lambda_1 \lambda_2}(p_1; \eta, c; p_2) \\ & + M_2 C_{\lambda_1 \lambda_2}(p_1; \eta, c; p_2) \\ & + M_1 M_2 D_{\lambda_1 \lambda_2}(p_1; \eta, c; p_2), \end{aligned} \quad (21)$$

where

$$\begin{aligned} M_i = & +m_i \quad \text{if } U(p_i, \lambda_i) = u(p_i, \lambda_i), \\ & -m_i \quad \text{if } U(p_i, \lambda_i) = v(p_i, \lambda_i). \end{aligned} \quad (22)$$

The functions  $A, B, C, D$  turn out to be independent of  $m_1$  and  $m_2$  and of the  $u$  or  $v$  nature of  $\bar{U}(p_1, \lambda_1)$

and  $U(p_2, \lambda_2)$ . This implies that the four  $T$  functions  $\bar{u}(p_1, \lambda_1) \chi u(p_2, \lambda_2)$ ,  $\bar{u}(p_1, \lambda_1) \chi v(p_2, \lambda_2)$ ,  $\bar{v}(p_1, \lambda_1) \chi u(p_2, \lambda_2)$  and  $\bar{v}(p_1, \lambda_1) \chi v(p_2, \lambda_2)$  can be computed simultaneously, speeding up the calculation considerably.

The functions  $A, B, C, D$  corresponding to a scalar and a vector insertion can be easily evaluated: using Eqs. (3), (5), (13), the spinor products can be written as

$$\begin{aligned} U(p_2, \lambda_2) \bar{U}(p_1, \lambda_1) = & \frac{1}{4\sqrt{2} p_1 \cdot k_0 \sqrt{2} p_2 \cdot k_0} \\ & \times (\not{p}_2 + M_2) \{ (1 + \lambda_1 \lambda_2) - (\lambda_1 + \lambda_2) \gamma^5 \\ & + \not{k}_1 [(\lambda_1 - \lambda_2) - (1 - \lambda_1 \lambda_2) \gamma^5] \} \not{k}_0 (\not{p}_1 + M_1). \end{aligned} \quad (23)$$

Multiplying to the right by  $\chi^S$  or  $\chi^V$  of Eqs. (15), (16), taking the trace and with the help of Eq. (21), one gets

$$\begin{aligned} A_{+-}^S = & c_l [k_0 \cdot p_1 k_1 \cdot p_2 - k_0 \cdot p_2 k_1 \cdot p_1 \\ & - i\epsilon(k_0, k_1, p_1, p_2)], \\ A_{-+}^S = & c_r [-k_0 \cdot p_1 k_1 \cdot p_2 + k_0 \cdot p_2 k_1 \cdot p_1 \\ & - i\epsilon(k_0, k_1, p_1, p_2)], \\ B_{++}^S = & c_r k_0 \cdot p_2, \quad B_{--}^S = c_l k_0 \cdot p_2, \\ C_{++}^S = & c_l k_0 \cdot p_1, \quad C_{--}^S = c_r k_0 \cdot p_1, \\ A_{++}^V = & c_r [-k_0 \cdot \eta p_1 \cdot p_2 + k_0 \cdot p_1 \eta \cdot p_2 \\ & + k_0 \cdot p_2 \eta \cdot p_1 + i\epsilon(k_0, \eta, p_1, p_2)], \\ A_{--}^V = & c_l [-k_0 \cdot \eta p_1 \cdot p_2 + k_0 \cdot p_1 \eta \cdot p_2 \\ & + k_0 \cdot p_2 \eta \cdot p_1 - i\epsilon(k_0, \eta, p_1, p_2)], \\ B_{+-}^V = & c_l [k_0 \cdot \eta k_1 \cdot p_2 - k_0 \cdot p_2 k_1 \cdot \eta \\ & - i\epsilon(k_0, k_1, \eta, p_2)], \\ B_{-+}^V = & c_r [-k_0 \cdot \eta k_1 \cdot p_2 + k_0 \cdot p_2 k_1 \cdot \eta \\ & - i\epsilon(k_0, k_1, \eta, p_2)], \\ C_{+-}^V = & c_r [-k_0 \cdot \eta k_1 \cdot p_1 + k_0 \cdot p_1 k_1 \cdot \eta \\ & + i\epsilon(k_0, k_1, \eta, p_1)], \\ C_{-+}^V = & c_l [k_0 \cdot \eta k_1 \cdot p_1 - k_0 \cdot p_1 k_1 \cdot \eta \\ & + i\epsilon(k_0, k_1, \eta, p_1)], \\ D_{++}^V = & c_l k_0 \cdot \eta, \quad D_{--}^V = c_r k_0 \cdot \eta. \end{aligned} \quad (24)$$

All functions  $A, B, C, D$  for a single insertion not reported in the preceding list are identically zero.

The function  $\epsilon$  is defined to be the determinant:

$$\epsilon(p, q, r, s) = \det \begin{vmatrix} p^0 & q^0 & r^0 & s^0 \\ p^1 & q^1 & r^1 & s^1 \\ p^2 & q^2 & r^2 & s^2 \\ p^3 & q^3 & r^3 & s^3 \end{vmatrix}. \quad (25)$$

#### 4. Spinor lines

We will show in this section how to compute recursively the functions

$$\begin{aligned} \tilde{T}^{(n)} &= T^{(n)} \sqrt{p_1 \cdot k_0} \sqrt{p_{n+1} \cdot k_0} \\ &(p_2 \cdot k_0) (p_3 \cdot k_0) \cdots (p_n \cdot k_0) \end{aligned} \quad (26)$$

from which the  $T^{(n)}$  themselves can then be immediately obtained at the end of the computation, dividing by the appropriate factors.

Let us denote with  $\tilde{T}, A, B, C, D$  the  $2 \times 2$  matrices whose elements are  $\tilde{T}_{\lambda_1 \lambda_2}, A_{\lambda_1 \lambda_2}, B_{\lambda_1 \lambda_2}, C_{\lambda_1 \lambda_2}, D_{\lambda_1 \lambda_2}$ . With this notation, making use of Eqs. (21) and (20), Eq. (19) reads

$$\begin{aligned} \tilde{T}^{(2)}(1, 2, 3) &= \frac{1}{2} \{ [A(1, 2) + M_1 B(1, 2) \\ &+ m_2 C(1, 2) + M_1 m_2 D(1, 2)] \\ &\times (1 + \mu_2/m_2) \times [A(2, 3) + m_2 B(2, 3) \\ &+ M_3 C(2, 3) + m_2 M_3 D(2, 3)] \\ &+ [A(1, 2) + M_1 B(1, 2) \\ &- m_2 C(1, 2) - M_1 m_2 D(1, 2)] \\ &\times (1 - \mu_2/m_2) \times [A(2, 3) - m_2 B(2, 3) \\ &+ M_3 C(2, 3) - m_2 M_3 D(2, 3)] \}, \end{aligned} \quad (27)$$

where we have used the shorthands  $(1, 2)$  and  $(1, 2, 3)$  for  $(p_1; \eta_1, c_1; p_2)$  and  $(p_1; \eta_1, c_1; p_2; \eta_2, c_2; p_3)$ , respectively.

Elementary algebra shows that  $\tilde{T}^{(2)}$  has again the same dependence on the external (possibly unphysical) masses as in (21):

$$\begin{aligned} \tilde{T}^{(2)}(1, 2, 3) &= A^{(2)}(1, 2, 3) + M_1 B^{(2)}(1, 2, 3) \\ &+ M_3 C^{(2)}(1, 2, 3) + M_1 M_3 D^{(2)}(1, 2, 3), \end{aligned} \quad (28)$$

with

$$\begin{aligned} A^{(2)}(1, 2, 3) &= A(1, 2) [A(2, 3) + \mu_2 B(2, 3)] \\ &+ C(1, 2) [\mu_2 A(2, 3) + p_2^2 B(2, 3)], \\ B^{(2)}(1, 2, 3) &= B(1, 2) [A(2, 3) + \mu_2 B(2, 3)] \\ &+ D(1, 2) [\mu_2 A(2, 3) + p_2^2 B(2, 3)], \\ C^{(2)}(1, 2, 3) &= A(1, 2) [C(2, 3) + \mu_2 D(2, 3)] \\ &+ C(1, 2) [\mu_2 C(2, 3) + p_2^2 D(2, 3)], \\ D^{(2)}(1, 2, 3) &= B(1, 2) [C(2, 3) + \mu_2 D(2, 3)] \\ &+ D(1, 2) [\mu_2 C(2, 3) + p_2^2 D(2, 3)]. \end{aligned} \quad (29)$$

This implies that  $A^{(2)}, B^{(2)}, C^{(2)}, D^{(2)}$  can be reinserted in an equation like Eq. (27) to give the  $\tilde{T}$  function  $\tilde{T}^{(3)}$  corresponding to a fermion line with three insertions, and so on. So one can generalize Eqs. (27)–(29) by induction: every  $\tilde{T}^{(i)}$  turns out to be of the form

$$\tilde{T}^{(i)} = A^{(i)} + M_1 B^{(i)} + M_{i+1} C^{(i)} + M_1 M_{i+1} D^{(i)} \quad (30)$$

and from the knowledge of  $A^{(i)}(1, \dots, i), B^{(i)}(1, \dots, i), C^{(i)}(1, \dots, i), D^{(i)}(1, \dots, i)$  and  $A^{(j)}(i+1, \dots, i+j), B^{(j)}(i+1, \dots, i+j), C^{(j)}(i+1, \dots, i+j), D^{(j)}(i+1, \dots, i+j)$ , one gets

$$\begin{aligned} A^{(i+j)} &= A^{(i)} [A^{(j)} + \mu_i B^{(j)}] \\ &+ C^{(i)} [\mu_i A^{(j)} + p_i^2 B^{(j)}], \\ B^{(i+j)} &= B^{(i)} [A^{(j)} + \mu_i B^{(j)}] \\ &+ D^{(i)} [\mu_i A^{(j)} + p_i^2 B^{(j)}], \\ C^{(i+j)} &= A^{(i)} [C^{(j)} + \mu_i D^{(j)}] \\ &+ C^{(i)} [\mu_i C^{(j)} + p_i^2 D^{(j)}], \\ D^{(i+j)} &= B^{(i)} [C^{(j)} + \mu_i D^{(j)}] \\ &+ D^{(i)} [\mu_i C^{(j)} + p_i^2 D^{(j)}], \end{aligned} \quad (31)$$

which build up  $\tilde{T}^{(i+j)}(1, \dots, i+j)$ . In Eq. (31)  $\mu_i$  and  $p_i$  are the mass and momentum of the propagator which connects the left ( $i$ ) insertions with the right ( $j$ ) ones.

It should by now be evident that the evaluation of any spinor line can be performed computing the  $A, B, C, D$  matrices relative to every single insertion and combining them with the help of Eq. (31) until one gets to the final  $T^{(n)}$ . It is important to point out that the unphysical masses  $m_i$  ( $i = 2, \dots, n$ ) do not appear in

Eqs. (29)–(31): only their squares  $p_i^2$  do. Therefore, as anticipated after Eq. (13), the identical final result is obtained using any of the two determinations of  $m_i = \sqrt{p_i^2}$ . The same conclusion can be drawn for  $\sqrt{p_i \cdot k_0}$  from Eq. (26) and the fact that the expressions (24) are independent of square roots.

## 5. $\tau$ matrices

It is convenient to cast the previous formulae in a matrix notation. We drop the superscripts ( $n$ ) when not necessary.

Every piece of a spinor line as well as every complete spinor line (for all fermion polarizations!) with  $n$  insertions is completely known when we know the matrix

$$\tau = \begin{pmatrix} A & C \\ B & D \end{pmatrix}. \quad (32)$$

The law of composition of two pieces of spinor line, connected by a fermion propagator with four-momentum  $p$  and mass  $\mu$ , whose matrices are

$$\tau_1 = \begin{pmatrix} A_1 & C_1 \\ B_1 & D_1 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} A_2 & C_2 \\ B_2 & D_2 \end{pmatrix}, \quad (33)$$

is simply (cf. Eq. (31))

$$\begin{pmatrix} A & C \\ B & D \end{pmatrix} = \begin{pmatrix} A_1 & C_1 \\ B_1 & D_1 \end{pmatrix} \begin{pmatrix} 1 & \mu \\ \mu & p^2 \end{pmatrix} \begin{pmatrix} A_2 & C_2 \\ B_2 & D_2 \end{pmatrix}. \quad (34)$$

We will sometimes indicate the above composition as follows,

$$\tau = \tau_1 \bullet \tau_2. \quad (35)$$

If we call  $\pi_i$  the matrix

$$\pi_i = \begin{pmatrix} 1 & \mu_i \\ \mu_i & p_i^2 \end{pmatrix} \quad (36)$$

corresponding to the propagator of four-momentum  $p_i$  and  $\tau_i$  the matrix associated with the  $i$ th insertion of Fig. 1, the  $\tau$  matrix of the whole spinor line can then be computed as follows,

$$\tau = \tau_1 \pi_2 \tau_2 \pi_3 \tau_3 \cdots \pi_{n-1} \tau_{n-1} \pi_n \tau_n. \quad (37)$$

Explicitly written as  $4 \times 4$  matrices,  $\tau$  and  $\pi_i$  are

$$\tau = \begin{pmatrix} A_{++} & A_{+-} & C_{++} & C_{+-} \\ A_{-+} & A_{--} & C_{-+} & C_{--} \\ B_{++} & B_{+-} & D_{++} & D_{+-} \\ B_{-+} & B_{--} & D_{-+} & D_{--} \end{pmatrix},$$

$$\pi_i = \begin{pmatrix} 1 & 0 & \mu_i & 0 \\ 0 & 1 & 0 & \mu_i \\ \mu_i & 0 & p_i^2 & 0 \\ 0 & \mu_i & 0 & p_i^2 \end{pmatrix}. \quad (38)$$

From the expressions of  $A, B, C, D$  given in Eq. (24), one can see that the  $\tau$ 's of a single vector or scalar insertion have the particular form

$$\tau^V = \begin{pmatrix} A_{++}^V & 0 & 0 & C_{+-}^V \\ 0 & A_{--}^V & C_{-+}^V & 0 \\ 0 & B_{+-}^V & D_{++}^V & 0 \\ B_{-+}^V & 0 & 0 & D_{--}^V \end{pmatrix},$$

$$\tau^S = \begin{pmatrix} 0 & A_{+-}^S & C_{++}^S & 0 \\ A_{-+}^S & 0 & 0 & C_{--}^S \\ B_{++}^S & 0 & 0 & 0 \\ 0 & B_{--}^S & 0 & 0 \end{pmatrix}. \quad (39)$$

When the insertion corresponds to a  $W$  boson, only  $A_{--}^V, C_{-+}^V, B_{+-}^V, D_{++}^V$  are different from zero. For practical computations we have implemented a set of routines which automatically write lines of Fortran code both for the expressions of the  $A, B, C, D$  functions and for their combination to form whole spinor lines. These routines of course avoid unnecessary and time consuming multiplications by the zeroes of the  $\tau$  matrices.

## 6. Massless spinor lines

When one has to deal with massless spinor lines, all the formulae given in Eq. (24) remain valid. But in this case the fact that all  $\mu_i$ 's as well as  $m_1$  and  $m_{n+1}$  are zero leads to significant simplifications. For example the rule (31) for combining pieces of spinor line is now

$$A^{(i+j)} = A^{(i)} A^{(j)} + p_i^2 C^{(i)} B^{(j)},$$

$$B^{(i+j)} = B^{(i)} A^{(j)} + p_i^2 D^{(i)} B^{(j)},$$

$$C^{(i+j)} = A^{(i)} C^{(j)} + p_i^2 C^{(i)} D^{(j)},$$

$$D^{(i+j)} = B^{(i)}C^{(j)} + p_i^2 D^{(i)}D^{(j)} \quad (40)$$

and the  $\pi_i$  matrices (37), (38) become diagonal.

Moreover, in order to compute the whole spinor line with  $n$  insertions it is not necessary to know the whole  $\tau^{(n)}$  matrix. It is clear from Eq. (30) that only  $A^{(n)}$  is needed. Therefore, if one for instance multiplies recursively the  $\tau$  matrices of the single insertions starting from the left, at every single step one needs to compute only  $A$  and  $C$ . In fact, from Eq. (40) with  $i = n - 1, j = 1$ , one sees that  $B^{(n-1)}$  and  $D^{(n-1)}$  are not needed to compute  $A^n$ . With  $i = n - 2, j = 1$  one verifies that  $B^{(n-2)}$  and  $D^{(n-2)}$  are not needed to compute  $A^{(n-1)}$  and  $C^{(n-1)}$ , and so on. Had one started from the right, only  $A$  and  $B$  would have had to be calculated for every product.

In most theories, like, e.g., the standard model, one has to consider only vector and axial-vector couplings to massless spinor lines. This means that one has to compute only  $\tau^V$  matrices (39) for every insertion. Combining two matrices of this type, one still gets a matrix whose only elements different from zero are on the two diagonals and therefore the  $\tau$  matrix of any piece of massless spinor line is “cross-diagonal” for these theories.

### 7. Combining spinor lines. A simple example

As a simple example of how to use in practice the above method, in this section we consider the process  $e^+e^- \rightarrow t\bar{t}g$ . For a more complicated application, we refer the reader to Ref. [13], where the calculation of the reaction  $e^+e^- \rightarrow W^+W^-b\bar{b}$  is presented in detail together with its phenomenological relevance for  $top$  and Higgs physics.

In the following, whenever in the argument of a  $\tau$  matrix or of one of its elements there will be an index  $\mu$  in place of a “polarization vector”  $\eta$ , we imply that the components  $\eta^\nu$  have been taken to be  $\eta^\nu = g^{\mu\nu}$ . In fact every  $\tau$  matrix satisfies the relation

$$\tau(p_1; \eta_1, \dots, \eta_i, \dots, p_n) = \eta_{i\mu} \tau(p_1; \eta_1, \dots, \mu, \dots, p_n) \quad (41)$$

and of course similar relations can be written for its  $A, B, C, D$  components. In case of just one insertion we have for example

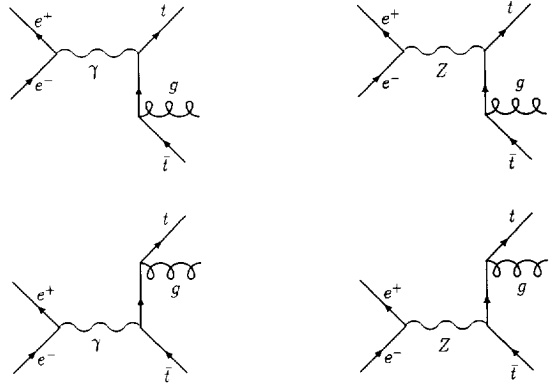


Fig. 2. Diagrams for the process  $e^+e^- \rightarrow t\bar{t}g$ .

$$A_{++}^V(p_1; \eta, c; p_2) = \eta_\mu A_{++}^V(p_1; \mu, c; p_2). \quad (42)$$

From Eq. (24) it is immediately seen that

$$A_{++}^V(p_1; \mu, c; p_2) = c_r(-p_1 \cdot p_2 k_0^\mu + k_0 \cdot p_1 p_2^\mu + k_0 \cdot p_2 p_1^\mu - i\epsilon_{\mu\nu\rho\sigma} k_0^\nu p_1^\rho p_2^\sigma), \quad (43)$$

with  $\epsilon_{0123} = 1$ .

The process  $e^+e^- \rightarrow t\bar{t}g$  is described by the diagrams in Fig. 2. In this case all insertions on spinor lines correspond to vector particles. We will indicate the four-momenta of the particles with their names, so that  $s = (e^+ + e^-)^2$ . We will also denote with  $c^{fV}$  the couplings of a generic fermion  $f$  to a vector particle  $V$ .

One has to choose  $k_0$  and  $k_1$  and the form of the polarization vectors  $\eta_i^s$  for the gluon in terms of his momentum. One can for instance use  $k_0 = (1, 1, 0, 0)$ ,  $k_1 = (0, 0, 1, 0)$  and the real polarization vectors proposed in Ref. [10] or normal helicity eigenvectors. One then starts computing with Eqs. (24)  $A_{jj}(e^+; \mu, c^{e\gamma}; e^-)$  and  $A_{jj}(e^+; \mu, c^{eZ}; e^-)$ , with  $j$  ( $= +, -$ ) the polarization of the electron. With them one constructs the “polarization” vectors

$$\eta_{j\mu}^\gamma = \frac{A_{jj}(e^+; \mu, c^{e\gamma}; e^-)}{s}, \quad \eta_{j\mu}^Z = \frac{A_{jj}(e^+; \mu, c^{eZ}; e^-)}{s - m_Z^2 + i\Gamma_Z m_Z}. \quad (44)$$

Using again Eqs. (24), it is easy to compute the tau matrices relative to the insertions in the upper (u) and

lower (d) part of the line:

$$\begin{aligned}
 \tau_\gamma^u[j] &= \tau(t; \eta_j^\gamma, c'^\gamma; -\bar{t} - g), \\
 \tau_Z^u[j] &= \tau(t; \eta_j^Z, c'^Z; -\bar{t} - g), \\
 \tau_g^u[i] &= \tau(t; \eta_i^g, c'^g; t + g), \\
 \tau_\gamma^d[j] &= \tau(t + g; \eta_j^\gamma, c'^\gamma; \bar{t}), \\
 \tau_Z^d[j] &= \tau(t + g; \eta_j^Z, c'^Z; \bar{t}), \\
 \tau_g^d[i] &= \tau(-\bar{t} - g; \eta_i^g, c'^g; \bar{t})
 \end{aligned} \quad (45)$$

and then the sum of  $\tau_\gamma$  and  $\tau_Z$  matrices

$$\begin{aligned}
 \tau_{\gamma Z}^u[j] &= \tau_\gamma^u[j] + \tau_Z^u[j], \\
 \tau_{\gamma Z}^d[j] &= \tau_\gamma^d[j] + \tau_Z^d[j].
 \end{aligned} \quad (46)$$

The final  $\tau$  matrices are obtained just composing with the law (29), (34)  $\tau_{\gamma Z}^u[j]$  with  $\tau_g^u[i]$  and  $\tau_g^d[i]$  with  $\tau_{\gamma Z}^d[j]$ :

$$\begin{aligned}
 \tau_1[i, j] &= \tau_{\gamma Z}^u[j] \bullet \tau_g^d[i], \\
 \tau_2[i, j] &= \tau_g^u[i] \bullet \tau_{\gamma Z}^d[j].
 \end{aligned} \quad (47)$$

From  $\tau[i, j]$  the polarized amplitude is then obtained with the help of Eq. (30). Indicating with  $l$  the polarization of the top and with  $m$  that of the antitop, one has

$$\begin{aligned}
 \widetilde{T}_1[i, j, l, m] &= A_{1lm}[i, j] + m_t B_{1lm}[i, j] \\
 &\quad - m_t C_{1lm}[i, j] - m_t^2 D_{1lm}[i, j], \\
 \widetilde{T}_2[i, j, l, m] &= A_{2lm}[i, j] + m_t B_{2lm}[i, j] \\
 &\quad - m_t C_{2lm}[i, j] - m_t^2 D_{2lm}[i, j]
 \end{aligned} \quad (48)$$

and the amplitude

$$\begin{aligned}
 \text{Amp}[i, j, l, m] &= \frac{1}{\sqrt{e^+ \cdot k_0} \sqrt{e^- \cdot k_0} \sqrt{t \cdot k_0} \sqrt{\bar{t} \cdot k_0}} \\
 &\quad \times \left( \frac{\widetilde{T}_1[i, j, l, m]}{(-\bar{t} - g) \cdot k_0 ((-\bar{t} - g)^2 - m_t^2 + i\Gamma_t m_t)} \right. \\
 &\quad \left. + \frac{\widetilde{T}_2[i, j, l, m]}{(t + g) \cdot k_0 ((t + g)^2 - m_t^2 + i\Gamma_t m_t)} \right). \quad (49)
 \end{aligned}$$

## 8. Comments and conclusions

The method we have demonstrated has been tested in several computations of physical amplitudes, both

unpolarized and partially or fully polarized. Our results have always been in perfect agreement with those obtained in other formalisms.

Even if we use for external fermions the same spinors as in Ref. [7], we do not also have to use the polarizations they suggest for vector particles. In effect, especially for massive vector particles, we often use the real polarization vectors suggested in Ref. [10]. If one defines  $k_0 = (1, 0, 0, -1)$ ,  $k_1 = (0, 1, 0, 0)$ , the massless spinors of Ref. [7] coincide with those of Ref. [10]. This implies that with this choice of  $k_0$  and  $k_1$  and the polarization for vector particles of Ref. [10] our results must agree, in the limit of spinor masses going to zero, with those obtained with the method of Ref. [10] for every single diagram and every polarization of external particles. This has been checked for several examples and it may be used as a valid test of the correctness of the results.

We believe that our method has some advantages with respect to those in Refs. [7,10]. In Ref. [10] it is proposed, using the particular form of the chiral representation for  $\gamma$  matrices, to decompose the calculation of every spinor line in a sum of terms of the type  $\chi^\dagger \phi_1, \dots, \phi_n \chi$ , where  $\phi_\pm = a_0 \pm \sigma \cdot a$  and  $a_\mu$  are either momenta of the propagators or ‘‘polarization’’ vectors. When all  $a_\mu$ ’s are real (and hence only when all ‘‘polarization’’ vectors correspond to external particles and not to other pieces of diagram inserted on the fermion line), the  $\phi_\pm$ ’s are written as  $(a_0 \mp |a|)\chi_+(a)\chi_+^\dagger(a) + (a_0 \pm |a|)\chi_-(a)\chi_-^\dagger(a)$ , where  $\chi_\pm(a)$  are two-component spinor eigenstates of the helicity  $\sigma \cdot a/|a|$ . In this way the computation of spinor lines is reduced to spinorial products [15]. This expansion of the  $\phi_\pm$ ’s already used eigenstates depending on the four-momenta of the propagators, as we do. There are however important differences: the decomposition of the  $\phi$ ’s is performed also for vectors corresponding to external particles helicities, while we decompose only the propagators. We have the possibility of using complex polarization vectors, which have been instrumental in obtaining extremely simple expressions for massless helicity amplitudes. Flexibility in the choice of polarization vectors is a desirable feature also when massive fermions are involved. In Ref. [10] the complete set is constructed with two-component spinors which are eigenstates of the he-



licity, while our complete sets are formed with four-component spinor eigenstates of  $a_\mu \gamma^\mu$ . It is just the fact that we employ four-component spinors, that we do not separate the various terms of the spinor lines, and that we use eigenstates of  $4 \times 4$  matrices  $a_\mu \gamma^\mu$  when  $a$  corresponds to the four-momenta of the spinor propagators, that allows us to treat the massive case in a compact and economical way. In such a fashion we can easily combine different diagrams with common intermediate fermion legs and treat the insertion of parts of the diagram on a spinor line as simply as the insertion of an external particle with complex polarization vector. A similar modularity can be achieved in the formalism of Ref. [10] when one multiplies spinors and matrices, without using spinor products. Compared with this latter approach, our method has only one  $\tau$  matrix to compute for each insertion, instead of one matrix for each vertex and one for each propagator. Moreover, we can freely choose the auxiliary vectors  $k_0$  and  $k_1$  which can be useful both in simplifying the expressions generated in intermediate stages and as a test of the final result.

In comparison with the method of Refs. [7,12], our formalism is much more compact. It avoids the proliferation of terms generated by the expansion of the momenta flowing in fermion propagators in terms of the external momenta. Even with respect to the more efficient method suggested in Ref. [8] for treating internal propagators, we have a smaller number of terms. The relationship between the matrix elements of  $u$ -type and  $v$ -type spinors (21), (22), (30) leads to simpler expressions than the introduction of additional auxiliary vectors, especially when long fermion lines are present and the insertions do not all correspond to external particles. As a consequence, it is much easier in our formalism to keep track of partial results and to set up recursive schemes of evaluation which compute and store for later use subdiagrams of increasing size and complexity. Our method is also more flexible in the choice of the polarization vectors for external vector particles and one can avoid the extra integration needed to obtain the correct sum over polarization

of the formalism of Refs. [7,12]. It allows one to directly compute cross sections with polarized  $W$ 's and  $Z$ 's for any desired polarization.

In conclusion, we have shown that it is possible to insert in spinor lines completeness relations in order to diagonalize the operators  $\not{p}$  of the propagators of momentum  $p$ . This is particularly convenient when one uses the spinors [7]. We have presented the formalism necessary to perform actual calculations, both in the massless and in the massive case.

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