POLITECNICO DI TORINO

UNIVERSITÀ DEGLI STUDI DI TORINO

# POLITECNICO & UNIVERSITÀ DI TORINO

# Global microlocal analysis on $\mathbb{R}^d$ with applications to hyperbolic partial differential equations and modulation spaces

**Submitted by**: Ahmed Abdeljawad

**Supervisor**: Prof. Sandro Coriasco (University of Turin)

**Coordinator**: Prof. Riccardo Adami (Polytechnic of Turin)
Prof. Anna Maria Fino (University of Turin)
**Academic years**: 2015-2018
**Scientific disciplinary sector**: MAT/05 - Mathematical Analysis

A thesis submitted for the degree of Doctor of Philosophy

Turin 2019

# Acknowledgment

The writing of a dissertation in a distant land, far away from home, can be a lonely and isolating experience, yet it is obviously not possible without the personal and practical support of numerous people.

First and foremost, I voice my praise, respect and sincere appreciation to my supervisor Prof. Sandro Coriasco. With his enthusiasm, his inspiration, and his great efforts to explain things clearly and simply, he made my PhD studies a great learning process. His careful advices and the uncountable conversations I have had with him enriched myself.

I thank the Polytechnic and the University of Turin for the opportunity to carry on my studies, in particular the research group of Microlocal and Time-Frequency Analysis, that gave me the possibility to participate to many useful conferences and seminaries.

I owe a debt of gratitude to Prof. Joachim Toft (Linneaus University), for introducing me the subject of time-frequency analysis on modulation spaces, for his kindness, patience, and the devoting of time and energies to stimulating discussions.

Towards my co-authors Alessia Ascanelli, Marco Cappiello, Sandro Coriasco, and Joachim Toft, I feel deep gratitude.

To all of my dear friends and colleagues in Växjö, I'd like to express my deepest gratitude for their hospitality and friendliness, during my visits.

Also I want to express my warmest thanks to all who contributed to the success of this thesis in any way.

I am indebted to my parents for their patience and encouragement throughout all these years. Last but not the least, I would like to thank my sisters and my brothers, and all my friends in Italy, Sweden and in Tunisia, for supporting me spiritually all the time.

To my second half Maryam and she knows why!

*Ahmed Abdeljawad*

# Abstract

This thesis treats different aspects of microlocal and time-frequency analysis, with particular emphasis on techniques involving multi-products of Fourier integral operators and one-parameter group properties for pseudo-differential operators.
In the first part, we study a class of hyperbolic Cauchy problems, associated with linear operators and systems with polynomially bounded coefficients, variable multiplicities and involutive characteristics, globally defined on $\mathbb{R}^d$. We prove well-posedness in Sobolev-Kato spaces, with loss of smoothness and decay at infinity. We also obtain results about propagation of singularities, in terms of wave-front sets describing the evolution of both smoothness and decay singularities of temperate distributions.
In the second part, we deduce lifting property for modulation spaces and construct explicit isomorpisms between them. To prove such results, we study one-parameter group properties for pseudo-differential operators with symbols in some Gevrey-Hörmander classes. Furthermore, we focus on some classes of pseudo-differential operators with symbols admitting anisotropic exponential growth at infinity. We deduce algebraic and invariance properties of these classes. Moreover, we prove mapping properties for these operators on Gelfand-Shilov spaces of type $\mathscr{S}$ and modulation spaces.

# Contents

# Introduction

One of the fundamental goals of classical analysis is a thorough study of functions near a point, that is, locally. It is also well known that the decay properties of the Fourier transform of a distribution, that is, its representation in the frequency domain, are connected to its smoothness. Starting from this observation, microlocal techniques were developed, in the second half of the twentieth century, as part of the study of linear partial differential equations, to obtain informations on the local behavior of the solutions. The term microlocal implies localization not only close to a point $x_0$ in the configuration space $X$, typically a (smooth) manifold, but also in a neighbourhood of the covariable (or frequency variable) $\xi_0$, that is, close to points $(x_0, \xi_0)$ of the cotangent space (of an open subset) of $X$. Loosely speaking, microlocal analysis is analysis near points and directions, that is, in the *phase space*, based on Fourier, and other type of, transforms. Many basic ideas date back to the original works by Hörmander [79], Kohn and Nirenberg [85], and Maslov [91], in which they generalized existing notions from analysis to investigate distributions and their singularities. A wide range of even more comprehensive and careful treatments of this subjects are now available, in particular, related to the concept of wave-front set of a distribution and to various functional spaces.

This dissertation consists of two parts, each one focused on related, but independent, topics and applications of the microlocal analysis techniques, with $\mathbb{R}^d$ chosen as configuration space. The various objects of interests will be anyway globally defined, that is, carrying informations, for instance, on their "behaviour at infinity" (e.g., decay, regularity with respect to certain functional or distributional spaces on the whole $\mathbb{R}^d$, etc.).

Part I deals with a class of hyperbolic partial differential equations and the corresponding families of Fourier integral operators giving their solutions.

Part II is devoted to study classes of pseudo-differential operators on the so-called modulation and Gelfand-Shilov functional and distributional spaces.

To introduce the contents of Part I, we begin by recalling the definition of Fourier integral operators on $\mathbb{R}^d$. Namely, they are linear maps, initially defined on $\mathscr{S}$, that, in their simplest form, can be written as

$$\left(\mathrm{Op}_\varphi(a)f\right)(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\varphi(x,\xi)} a(x,\xi)\widehat{f}(\xi)\, d\xi, \quad f \in \mathscr{S}(\mathbb{R}^d), \qquad (0.1)$$

where $a(x,\xi)$ is the *amplitude* or *symbol*, $\varphi(x,\xi)$ is the *phase function*, and the Fourier transform $\widehat{f}$ of $f$ is defined[1] in (0.10). In the case of the elementary phase function $\varphi(x,\xi) = \langle x,\xi\rangle$, the Fourier integral operator (0.1) are (left-quantized) pseudo-differential operator.

A most widely used class of amplitudes is the one introduced by Hörmander in [81], the so called $S^m_{\varrho,\delta}(\mathbb{R}^{2d}) = S^m_{\varrho,\delta}$ class, that consists of functions $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ satisfying, for $m \in \mathbb{R}$, $\varrho, \delta \in [0,1]$, $\delta \leqslant \varrho$,

$$|\partial^\alpha_\xi \partial^\beta_\xi a(x,\xi)| \leqslant C_{\alpha\beta}(1+|\xi|)^{m-\varrho|\alpha|+\delta|\beta|}, \quad x,\xi \in \mathbb{R}^d,$$

for suitable $C_{\alpha\beta} > 0$, $\alpha, \beta \in \mathbb{Z}^d_+$. In the classical theory, the phase function $\varphi \in C^\infty(\mathbb{R}^d \times (\mathbb{R}^d \backslash 0))$ is homogeneous of degree 1 in the frequency variable $\xi$. Often it is also assumed to satisfy the *non-degeneracy condition*, that is, the mixed Hessian matrix $[\frac{\partial^2\varphi}{\partial x_j \partial \xi_k}]$ has non-vanishing determinant. This is an important assumption when dealing with boundedness of the operators (0.1) on $L^2$ or $H^s$ and other functional spaces.

The study of these operators, which are intimately connected to the theory of linear partial differential operators, has a long history. In [81], Hörmander credits the original local notion of Fourier integral operators to Lax in the paper [87], where the objective was the study of the singularities of hyperbolic differential equations (see also Maslov [91]). There is a huge number of results and applications concerning regularity, boundedness and compositions (of Fourier integral operators and pseudo-differential operators). We refer the reader to [42, 43, 51, 76, 77, 81, 86, 107] and the references quoted therein for a wider overview of the existing literature.

In particular, we will be concerned with the application of Fourier integral operators to the study of hyperbolic type equations. Also this related literature is quite large, see again the sources quoted above and their list of references. More precisely, for some $T > 0$, we will consider the Cauchy problem

$$\begin{cases} Lu(t,s) = f(t) & (t,s) \in [0,T], s \leqslant t \\ (D^k_t)u(s,s) = g_k & k = 0,\ldots,m-1, \end{cases} \qquad (0.2)$$

[1] Notice the presence of the normalization factor $(2\pi)^{-\frac{d}{2}}$ in the definition of $\widehat{f}$.

with $f$ and $g_k$, $k = 0, \ldots, m-1$, chosen in suitable functional spaces,

$$\begin{aligned} L \equiv L(t, D_t; x, D_x) &= D_t^m + \sum_{j=1}^{m} P_j(t; x, D_x) D_t^{m-j} \\ &= D_t^m + \sum_{j=1}^{m} \sum_{|\alpha| \leqslant j} c_{j\alpha}(t; x) D_x^\alpha D_t^{m-j}, \end{aligned} \tag{0.3}$$

$m \in \mathbb{N}$, and the differential operators $P_j(t; x, D_x)$, $j = 1, \ldots, m$, given by

$$P_j(t; x, D_x) = \sum_{|\alpha| \leqslant j} c_{j\alpha}(t; x) D_x^\alpha,$$

having polynomially bounded coefficients, namely,

$$|\partial_t^k \partial_x^\beta c_{j\alpha}(t; x)| \lesssim \langle x \rangle^{j-|\beta|}, \quad \beta \in \mathbb{Z}_+^d, x \in \mathbb{R}^d, t \in [0, T]. \tag{0.4}$$

$j = 1, \ldots, m$, $\alpha \in \mathbb{Z}_+^d$, $|\alpha| \leqslant j$. Denoting by $L_m = \sigma_p(L)$ the principal symbol of $L$, that is

$$[\sigma_p(L)](t, \tau; x, \xi) = \tau^m + \sum_{j=1}^{m} \left[ \sum_{|\alpha|=j} \widetilde{c}_{j\alpha}(t; x) \xi^\alpha \right] \tau^{m-j}$$

where $\widetilde{c}_{j\alpha}$, the principal part of $c_{j,\alpha}$, satisfies (0.4), see Chapter 4 for the precise definition. We assume $L$ to be hyperbolic, that is

$$L_m(t, \tau; x, \xi) = \prod_{j=1}^{m} (\tau - \tau_j(t; x, \xi)),$$

with real-valued, smooth *characteristics roots* $\tau_j$, $j = 1, \ldots, m$.

Such global problems on the whole of $\mathbb{R}^d$ have been considered by Cordes (cf. [41]), mostly in the case of strictly hyperbolic operators, that is, when the characteristic roots are all distinct and satisfy a "separation condition" at infinity, see Chapter 4. While the classical setting recalled above is appropriate for the study of the local behaviour of solutions to problems of the type (0.2), it does not allow to obtain informations about their behaviour at infinity. A better suited environment for this aim is provided by the so-called SG symbols, introduced independently by Cordes and Parenti [95]. The calculus on $\mathbb{R}^d$ has been extended to a class of non-compact manifolds (the so-called SG-manifolds), including the manifolds with finitely many ends, by Schrohe [106]. Namely, a symbol $a(x, \xi)$ belongs to the SG symbol class $S^{m,\mu}(\mathbb{R}^{2d}) = S^{m,\mu}$, $m, \mu \in \mathbb{R}$, if $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and

$$|D_x^\alpha D_\xi^\beta a(x, \xi)| \leqslant C_{\alpha\beta} \langle x \rangle^{m-|\alpha|} \langle \xi \rangle^{\mu-|\beta|}, x, \xi \in \mathbb{R}^d,$$

for suitable constants $C_{\alpha\beta} > 0$, $\alpha, \beta \in \mathbb{Z}_+^d$. Operators $L$ of the type (0.3), where the operators $P_j$ turn out to be defined by means of parameter-dependent symbols of order $(m-j, m-j)$, $j = 1, \ldots, m$, indeed appear as local representations, on (unbounded) coordinate patches, of natural differential operators on manifolds with ends (see, e.g., the corresponding example in [47]).

The systematic study of (0.2) through Fourier integral operators techniques was performed in [43] in the case of operators $L$ with constant multiplicities (see Definition 4.2), based on the calculus of the so-called SG Fourier integral operators developed in [42]. Therein, the Fourier integral operators (0.1) have symbol $a \in S^{m,\mu}$, while the phase function $\varphi$ satisfies $\varphi \in S^{1,1}$ and

$$c\langle\xi\rangle \leqslant \langle\varphi'_x(x,\xi)\rangle \leqslant C\langle\xi\rangle, \quad c\langle x\rangle \leqslant \langle\varphi'_\xi(x,\xi)\rangle \leqslant C\langle x\rangle, \quad x, \xi \in \mathbb{R}^d,$$

for suitable $c, C > 0$. Notice that no homogeneity assumption on $\varphi$ is required.

Compared with [43], we will focus here on the more general situation of involutive operators. That is, those hyperbolic operators $L$ whose characteristic roots fulfill the next main Assumption A.

**Assumption A.** *Let the characteristic roots* $\tau_j \in C^\infty([0,T]; S^{1,1}(\mathbb{R}^{2d}))$, $j = 1, \ldots, m$, *be an involutive family. Explicitly, they are real-valued, and, for any* $j, k = 1, \ldots, m$, *there exist real symbols* $b_{j,k}$ *and* $d_{j,k} \in C^\infty([0,T]; S^{0,0}(\mathbb{R}^{2d}))$ *such that the* Poisson bracket $\{\tau - \tau_j, \tau - \tau_k\}$ *satisfy*

$$\begin{aligned}\{\tau - \tau_j, \tau - \tau_k\} &= \partial_t \tau_j - \partial_t \tau_k + \tau'_{j,\xi} \cdot \tau'_{k,x} - \tau'_{j,x} \cdot \tau'_{k,\xi} \\ &= b_{j,k}(\tau_j - \tau_k) + d_{j,k}\end{aligned}$$

*holds true on* $[0,T] \times \mathbb{R}^d \times \mathbb{R}^d$.

Assumption A allows to prove the existence of a representation of the fundamental solution of (0.2), involving finitely many SG Fourier integral operators, related to the characteristic roots of $L$, analogously to the results in [86, 94, 110] in the classical Hörmander symbols, local setting. Such well-posedness result in the naturally associated scale of functional spaces, the so-called Sobolev-Kato spaces $H^{r,\varrho}$, $r, \varrho \in \mathbb{R}$, further generalizes the quoted results, both in the local as well as in the global setting. To achieve this result, a careful analysis of algebraic properties of the involved operators is needed. Namely, completing the work in [8], commutative properties of multi-products of SG Fourier integral operators are proved under the Assumption A for the symbols generating their phase functions, so extending the similar results in [86, 94, 110] to the SG case. Moreover, a result for the propagation of singularities for the Cauchy problem (0.2) is obtained, taking advantage of the structure of its fundamental solution operator, in terms of the global wave-front sets studied in [46–48]. A future application will concern the study of stochastic versions of the Cauchy problem (0.2) for a SG-involutive operator, in the spirit of [10]. The results proved in Part I are contained in the preprint [1].

To summarize the contents of Part II, we first recall some basic notions of time-frequency analysis. A time-frequency representation transforms a function $f$ on $\mathbb{R}^d$ into a function on the time-frequency space $\mathbb{R}^d \times \mathbb{R}^d$. The goal is to obtain a description of $f$, that is, local both in time and in frequency. The standard time-frequency representations, such as the short-time Fourier transform and its various modifications known as Wigner distribution, radar ambiguity function, Gabor transform, all encode time-frequency information.

A main ingredient of time-frequency analysis are the so-called modulation spaces. They were introduced in [55] by Feichtinger, to measure the time-frequency concentration of a function or distribution on the time-frequency space. Nowadays they have become popular among mathematicians and engineers in view of their numerous applications in signal processing [58, 59], pseudo-differential and Fourier integral operators [33, 34, 38, 98, 99, 109, 111, 112, 118, 120, 122, 123, 125, 126] and quantum mechanics [39, 65]. They are interesting also because, by appropriate choices of the elements entering their definition, they coincide with many "standard" functional spaces, like $L^p$, $H^s_p$, etc. Since many mapping properties are known for pseudo-differential and Fourier integral operators acting on such spaces, it is useful to establish homeomorphisms (lifts) between modulation spaces and other functional spaces. In particular, if such homeomorphisms can be expressed terms of pseudo-differential operators, the corresponding calculi can then provide further mapping properties among the original modulation spaces themselves.

More precisely, the topological vector spaces $V_1$ and $V_2$ are said to possess lifting property if there exists a "convenient" homeomorphism (that is, a lifting) between them. For example, for any weight $\omega$ on $\mathbb{R}^d$, $p \in (0, \infty]$ and $s \in \mathbb{R}$ the mappings $f \mapsto \omega \cdot f$ and $f \mapsto (1-\Delta)^{s/2} f$ are homeomorphisms from the (weighted) Lebesgue space $L^p_{(\omega)}$ and the Sobolev space $H^s_p$, respectively, into $L^p = H^0_p$, with inverses $f \mapsto \omega^{-1} \cdot f$ and $f \mapsto (1-\Delta)^{-s/2} f$, respectively. (Cf. [80] and Part II, Chapter 5 for notations.) Hence, these spaces possess lifting properties.

It is sometimes relatively simple to deduce lifting properties between (quasi-)Banach spaces of functions and distributions, if the definition of their norms only differs by a multiplicative weight on the involved distributions, or on their Fourier transforms, which is the case in the above example. A more complicated situation appear when there are some kind of interactions between multiplication and differentiation in the definition of the involved vector spaces. This is a typical situation for many functional spaces in microlocal and time-frequency analysis, since multiplications on the Fourier transform side are linked to differentiations of the involved elements. An interesting example where such interactions occur concerns the extended family of Sobolev spaces, introduced by Bony and Chemin in [14] (see also [89]). More precisely, let $\omega, \omega_0$ be suitable weight functions and $g$ a suitable Riemannian metric, which are defined on the phase space $W \simeq T^*\mathbb{R}^d \simeq \mathbb{R}^{2d}$. Bony and Chemin introduced in [14] the generalised Sobolev space $H(\omega, g)$ which fits the Hörmander-Weyl calculus well, in the sense that $H(1, g) = L^2$, and if $a$ belongs to the Hörmander class $S(\omega_0, g)$, then the Weyl operator $\operatorname{Op}^w(a)$ with symbol $a$ is continuous from $H(\omega_0\omega, g)$ to $H(\omega, g)$ (see Chapter 1 below for notation about the pseudo-differential operators and their Weyl quantization $\operatorname{Op}^w(a)$). Moreover, they deduced group algebras, from which it follows that to each such weight $\omega_0$, there exist symbols $a$ and $b$ such that

$$\operatorname{Op}^w(a) \circ \operatorname{Op}^w(b) = \operatorname{Op}^w(b) \circ \operatorname{Op}^w(a) = I, a \in S(\omega_0, g), b \in S(1/\omega_0, g), \quad (0.5)$$

where $I$ is the identity operator on $\mathscr{S}'$. In particular, by the continuity properties of $\operatorname{Op}^w(a)$ it follows that $H(\omega_0\omega, g)$ and $H(\omega, g)$ possess lifting properties with the homeomorphism $\operatorname{Op}^w(a)$, and $\operatorname{Op}^w(b)$ as its inverse.

The existence of $a$ and $b$ in (0.5) is a consequence of solution properties of the evolution equation

$$(\partial_t a)(t, \,\cdot\,) = (b + \log \vartheta)\# a(t, \,\cdot\,), a(0, \,\cdot\,) = a_0 \in S(\omega, g), \;\; \vartheta \in S(\vartheta, g), \quad (0.6)$$

which involve the Weyl product $\#$ and a fixed element $b \in S(1, g)$. It is proved that (0.6) has a unique solution $a(t, \,\cdot\,)$ which belongs to $S(\omega\vartheta^t, g)$ (cf. [14, Theorem 6.4] or [89, Theorem 2.6.15]). The existence of $a$ and $b$ in (0.5) will follow by choosing $\omega = a_0 = 1$, $t = 1$ and $\vartheta = \omega_0$.

An important class of operators in quantum mechanics and time-frequency analysis concerns Toeplitz, or localisation operators. The main issue in [72, 73] is to show that the Toeplitz operator $\operatorname{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0\omega)}$ into $M^{p,q}_{(\omega)}$ for suitable $\omega_0$. The assumptions on $\omega_0$ in [72] is that it should be polynomially moderate and satisfies $\omega_0 \in S^{(\omega_0)}$. In [73] such assumptions have been relaxed (see the quoted paper for details), but here we work under further different hypotheses.

One of the main results in Part II, which is similar to [72, Theorem 0.1], can be stated as follows (see Chapter 5 below for the notation).

**Theorem 0.1.** *Let* $s \geqslant 1$, $\omega, \omega_0 \in \mathscr{P}^0_{E,s}(\mathbb{R}^{2d})$, $p, q \in (0, \infty]$ *and let* $\phi \in \mathcal{S}_s(\mathbb{R}^d)$. *Then the Toeplitz operator* $\operatorname{Tp}_\phi(\omega_0)$ *is an isomorphism from* $M^{p,q}_{(\omega)}(\mathbb{R}^{2d})$ *onto* $M^{p,q}_{(\omega/\omega_0)}(\mathbb{R}^{2d})$.

We notice that, in contrast to [72, 73], such lifting properties also hold for modulation spaces which may fail to be Banach spaces, since $p$ and $q$

in Theorem 0.1 are allowed to be smaller than 1. Moreover, differently from [73], we do not impose in Theorem 0.1 and in its related results that $\omega_0$ should be radial in each phase shift (cf. e. g. [73, Theorem 4.3]). Our lifting results then extend those proved in [72, 73]. They are contained in the preprint [3].

Another important family of functional and distributional spaces are the so-called Gelfand-Shilov spaces of type $\mathscr{S}$. They have been introduced in the book [64] by Gelfand and Shilov, as an alternative functional setting to the Schwartz space $\mathscr{S}(\mathbb{R}^d)$ of smooth and rapidly decreasing functions, for Fourier analysis and for the study of partial differential equations. Namely, fixed $s > 0, \sigma > 0$, the space $\mathcal{S}_s^\sigma(\mathbb{R}^d) = \mathcal{S}_s^\sigma$ can be defined as the space of all functions $f \in C^\infty$ satisfying an estimate of the form

$$\sup_{\alpha,\beta\in\mathbb{Z}_+^d} \sup_{x\in\mathbb{R}^d} \frac{|x^\beta \partial^\alpha f(x)|}{h^{|\alpha+\beta|}\alpha!^\sigma \beta!^s} < \infty \tag{0.7}$$

for some constant $h > 0$, or the equivalent condition

$$\sup_{\alpha\in\mathbb{Z}_+^d} \sup_{x\in\mathbb{R}^d} \frac{|e^{r|x|^{\frac{1}{s}}} \partial^\alpha f(x)|}{h^{|\alpha|}\alpha!^\sigma} < \infty \tag{0.8}$$

for some constants $h, r > 0$. For $\sigma > 1$, $\mathcal{S}_s^\sigma$ represents a natural global counterpart of the Gevrey class $G^\sigma$ but, in addition, the condition (0.8) encodes a precise description of the behavior at infinity of $f$. Together with $\mathcal{S}_s^\sigma$ one can also consider the space $\Sigma_s^\sigma$, which has been defined in [97] by requiring (0.7) (respectively (0.8)) to hold for every $h > 0$ (respectively for every $h, r > 0$). The duals of $\mathcal{S}_s^\sigma$ and $\Sigma_s^\sigma(\mathbb{R}^d)$ and further generalizations of these spaces have been then introduced in the spirit of Komatsu theory of ultradistributions, see [28, 97].

After their appearance, Gelfand-Shilov spaces have been recognized as a natural functional setting for pseudo-differential and Fourier integral operators, due to their nice behavior under Fourier transformation, and applied in the study of several classes of partial differential equations, see e. g. [7,17–22].

According to the condition on the decay at infinity of the elements of $\mathcal{S}_s^\sigma$ and $\Sigma_s^\sigma$, we can define on these spaces pseudo-differential operators with symbols admitting an exponential growth at infinity. These operators are commonly known as operators *of infinite order* and they have been studied in [15] in the analytic class and in [26, 84, 131] in the Gevrey spaces where the symbol has an exponential growth only with respect to $\xi$ and applied to the Cauchy problem for hyperbolic and Schrödinger equations in Gevrey classes, see [26, 27, 32, 83]. Parallel results have been obtained in Gelfand-Shilov spaces for symbols admitting exponential growth both in $x$ and $\xi$, see [17, 18, 21, 22, 25, 100].

The above results concern the non-quasi-analytic isotropic case $s = \sigma > 1$. In [24], the authors consider the more general case $s = \sigma > 0$, which is interesting in particular in connection with Shubin-type pseudo-differential operators, cf. [19, 23]. We further generalize the results of [24] to the case when $s > 0$ and $\sigma > 0$ may be different from each other. Thus the symbols we consider may have different rates of exponential growth and anisotropic Gevrey-type regularity in $x$ and $\xi$. More precisely, the symbols satisfy the conditions

$$\sup_{\alpha,\beta\in\mathbb{Z}_+^d}\ \sup_{x,\xi\in\mathbb{R}^d} \frac{|e^{-r(|x|^{\frac{1}{s}}+|\xi|^{\frac{1}{\sigma}})}\partial_x^\alpha\partial_\xi^\beta a(x,\xi)|}{h^{|\alpha+\beta|}\alpha!^\sigma\beta!^s} < \infty \tag{0.9}$$

for suitable restrictions on the constants $h, r > 0$ (cf. (0.8)). We prove that if $h > 0$, and (0.9) holds true for every $r > 0$, then the pseudo-differential operator $\operatorname{Op}(a)$ is continuous on $\mathcal{S}_s^\sigma$ and on $(\mathcal{S}_s^\sigma)'$. If instead $r > 0$, and (0.9) holds true for every $h > 0$, then we prove that $\operatorname{Op}(a)$ is continuous on $\Sigma_s^\sigma$ and on $(\Sigma_s^\sigma)'$ (cf. Theorems 10.8 and 10.14). We also prove that pseudo-differential operators with symbols satisfying such conditions form algebras (cf. Theorems 10.17 and 10.18), and that our span of pseudo-differential operators is invariant under the choice of representation (cf. Theorem 10.6).

An important ingredient in the analysis which is used to prove such properties concerns characterizations of the symbols above in terms of suitable estimates of their short-time Fourier transforms. Such characterizations are deduced in Chapter 9. All these results come from the preprint [2].

Finally, in Chapter 11, we consider pseudo-differential operators, where the symbols are of infinite orders, possess suitable Gevrey regularities, and are allowed to grow sub-exponentially together with all their derivatives. Our purpose is to extend boundedness results, in [120], of the pseudo-differential operators when acting on modulation spaces.

Similar investigations were performed in [126] in the case $s = \sigma$ (i. e. the isotropic case). Therefore, the results in Chapter 11 are more general in the sense of the anisotropicity of the considered symbol classes. Moreover, we use different techniques compared to [126]. These results are collected in the preprint [4].

# Notation

Let $\mathbb{R}^d$ be the usual Euclidean space given by

$$\mathbb{R}^d = \{(x_1, x_2, \ldots, x_d) : x_j \in \mathbb{R}\}.$$

We denote points in $\mathbb{R}^d$ by $x, y, \xi, \eta$, etc. Let $x = (x_1, x_2, \ldots, x_d)$ and $y = (y_1, y_2, \ldots, y_d)$ be any two points in $\mathbb{R}^d$. The inner product $\langle x, y\rangle$ of $x$ and $y$ is defined by

$$\langle x, y\rangle = \sum_{j=1}^{d} x_j y_j,$$

and the norm $|x|$ of $x$ is defined by

$$|x| = \left(\sum_{j=1}^{d} x_j^2\right)^{\frac{1}{2}} = \langle x, x\rangle^{\frac{1}{2}}.$$

The so-called Japanese bracket of $x$ is $\langle x\rangle = (1 + |x|^2)^{1/2}$ when $x \in \mathbb{R}^d$.

On $\mathbb{R}^d$, the simplest differential operators are $\dfrac{\partial}{\partial x_j} = \partial_j$, $j = 1, 2, \ldots, d$.

As usual, the operator $D_{x_j}$, given by $D_{x_j} = -i\dfrac{\partial}{\partial x_j}$, where $i^2 = -1$, is sometimes more convenient, especially when dealing with formulae involving Fourier transform.

In what follows we write $f(\theta) \lesssim g(\theta)$, $\theta \in \Omega \subseteq \mathbb{R}^d$, if there is a constant $c > 0$ such that $|f(\theta)| \leqslant c|g(\theta)|$ for all $\theta \in \Omega$. Moreover, if $f(\theta) \lesssim g(\theta)$ and $g(\theta) \lesssim f(\theta)$ for all $\theta \in \Omega$, we write that $f \asymp g$.

We will make use of multi-indices, which will keep the notation (relatively) short. Given $\mathbb{N} = \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, a multi-index is a vector $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_+^d$. For $\alpha \in \mathbb{Z}_+^d$, we define the length of $\alpha$ as $|\alpha| = \alpha_1 + \cdots + \alpha_d$ and its factorial as $\alpha! = \alpha_1! \ldots \alpha_d!$. Moreover, for $x \in \mathbb{R}^d$ and $\alpha \in \mathbb{Z}_+^d$ we define $x^\alpha = x_1^{\alpha_1} \ldots x_d^{\alpha_d}$, with the usual abuse of notation $x_j^0 \equiv 1$.

If $\alpha$, $\beta \in \mathbb{Z}_+^d$, we write $\beta \leqslant \alpha$ if and only if

$$\beta_j \leqslant \alpha_j, \text{ for all } j = 1, \ldots, d.$$

The difference $(\alpha - \beta)$ is the multi-index $(\alpha_1 - \beta_1, \alpha_2 - \beta_2, \ldots, \alpha_d - \beta_d)$ whenever $\beta \leqslant \alpha$. We also set

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$$

if $\beta \leqslant \alpha$ and $\binom{\alpha}{\beta} = 0$ otherwise. It is easy to check that, for $\alpha \leqslant \beta$,

$$\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_d}{\beta_d},$$

with the usual binomial coefficients on the right-hand side.

Any polynomial $p : \mathbb{R}^d \to \mathbb{C}$ of degree $m \in \mathbb{Z}_+$ can then be written as

$$p(x) = \sum_{|\alpha| \leqslant m} c_\alpha x^\alpha, \; c_\alpha \in \mathbb{C}.$$

$\mathscr{S}(\mathbb{R}^d)$ is the (Fréchet) space of infinitely differentiable functions $u$ "rapidly decreasing at infinity". Explicitly $u \in \mathscr{S}(\mathbb{R}^d)$ if $u \in C^\infty(\mathbb{R}^d)$ and

$$\sup_x |x^\alpha \partial^\beta u(x)| < \infty, \; \alpha, \beta \in \mathbb{N}^d.$$

The space $\mathscr{S}'(\mathbb{R}^d)$ is the space of tempered distributions, that is, of all the continuous linear functionals from $\mathscr{S}(\mathbb{R}^d)$ to $\mathbb{C}$.

For $u \in \mathscr{S}(\mathbb{R}^d)$, we denote by $\mathscr{F}u$ or $\widehat{u}$ the Fourier transform of $u$, given by

$$\widehat{u}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-i\langle x,\xi\rangle} u(x)\, dx \equiv \int_{\mathbb{R}^d} e^{-i\langle x,\xi\rangle} u(x)\, đx, \tag{0.10}$$

where $đx = (2\pi)^{-\frac{d}{2}} dx$.

We denote by $I$ the identity operator, while $I_d$ denotes the $d \times d$ identity matrix.

# Part I

# Involutive weakly hyperbolic Cauchy problems on $\mathbb{R}^d$ for operators with polynomially bounded coefficients

# Chapter 1

# Pseudo-differential and Fourier integral operators of SG type

Pseudo-differential operators generalize differential and singular integral operators. Classes of operators globally defined on $\mathbb{R}^d$ where studied, e. g., by Shubin [107], Helffer and Robert [76,77] and others. We here focus on the so-called SG classes of symbols and operators. The investigation of $SG$ pseudo-differential operators goes back to the works of Parenti [95] and Cordes [41]. $SG$ pseudo-differential operators are defined by mean of Symbols of Global type. They are also called by Schulze [105] "pseudo-differential operators with conical exit at infinity". In the terminology introduced by Melrose [92], they are also known as "scattering operators". The calculus of Fourier integral operators originally developed by Coriasco in [42, 43] is based on the class of $SG$ symbols.

Standard references about these topics are, e. g., Duistermaat [51], Hörmander [81], Cordes [41], Grigis-Sjöstrand [66], Kumano-go [86], Shubin [107], Treves [130] and Wong [132]. For an introduction to the main properties of the theory of pseudo-differential operators with symbols in $S^m_{1,0}$, see, e. g., Saint-Raymond [104] and Abels [5]. The present introductory chapter is mainly based on [1, 8, 41, 52, 86, 107], from which we took most of the materials.

Here we will recall some properties of $SG$ pseudo-differential operators.

## 1.1 Calculus for symbols of $SG$ type

First of all, we present some basic material about the $SG$ calculus. We begin with the definition of the symbol class which we are interested in.

**Definition 1.1.** *The class $S^{m,\mu}(\mathbb{R}^{2d})$ of SG symbols of order $m, \mu \in \mathbb{R}$,* *is given by all the functions* $a \in C^\infty(\mathbb{R}^{2d})$ *such that, for any multiindices* $\alpha, \beta \in \mathbb{Z}_+^d$, *there exist constants* $C_{\alpha\beta} > 0$ *such that*

$$|D_\xi^\alpha D_x^\beta a(x,\xi)| \leqslant C_{\alpha,\beta}\, \langle x\rangle^{m-|\beta|}\langle\xi\rangle^{\mu-|\alpha|} \tag{1.1}$$

*hold true, for all* $x, \xi \in \mathbb{R}^d$. *For* $a \in S^{m,\mu}(\mathbb{R}^{2d})$ , $m, \mu \in \mathbb{R}$, *we define the semi-norms* $\|\!|a|\!\|_l^{m,\mu}$ *by*

$$\|\!|a|\!\|_l^{m,\mu} = \max_{|\alpha+\beta|\leqslant l} \sup_{x,\xi\in\mathbb{R}^d} \langle x\rangle^{-m+|\beta|}\langle\xi\rangle^{-\mu+|\alpha|}|D_\xi^\alpha D_x^\beta a(x,\xi)|, \tag{1.2}$$

*where* $l \in \mathbb{Z}_+$.

The quantities (1.2) define a Fréchet topology on $S^{m,\mu}$. Moreover, we let

$$S^{\infty,\infty}(\mathbb{R}^{2d}) = \bigcup_{m,\mu\in\mathbb{R}} S^{m,\mu}(\mathbb{R}^{2d}), \qquad S^{-\infty,-\infty}(\mathbb{R}^{2d}) = \bigcap_{m,\mu\in\mathbb{R}} S^{m,\mu}(\mathbb{R}^{2d}).$$

The functions $a \in S^{m,\mu}(\mathbb{R}^{2d})$ can be $(\nu \times s)$-matrix-valued. In such case, the estimate (1.1) must be valid for each entry of the matrix. Very often, in the sequel we will omit the base spaces $\mathbb{R}^d$, $\mathbb{R}^{2d}$ from the notation, and we write $S^{m,\mu}$, $\mathscr{S}$, $\mathscr{S}'$, etc. The next technical lemma is useful when dealing with compositions of $SG$ operators.

**Lemma 1.2.** *Let* $f \in S^{m,\mu}(\mathbb{R}^{2d})$, $m, \mu \in \mathbb{R}$, *and* $g$ *vector-valued in* $\mathbb{R}^d$ *such that* $g \in S^{0,1}(\mathbb{R}^{2d})$ *and* $\langle g(x,\xi)\rangle \asymp \langle\xi\rangle$. *Then* $f(x, g(x,\xi))$ *belongs to* $S^{m,\mu}(\mathbb{R}^{2d})$.

The proof of Lemma 1.2 can be found in [44], and can of course be extended to the other composition cases, namely, $h(x,\xi)$ vector valued in $\mathbb{R}^d$ such that it belongs to $S^{1,0}(\mathbb{R}^{2d})$ and $\langle h(x,\xi)\rangle \asymp \langle x\rangle$, implying that $f(h(x,\xi),\xi)$ belongs to $S^{m,\mu}$.

We now recall definition and properties of the pseudo-differential operators $a(x,D) = \mathrm{Op}(a)$, $a \in S^{m,\mu}$.

**Definition 1.3.** *Let* $a \in \mathscr{S}(\mathbb{R}^{2d})$, *and* $t \in \mathbb{R}$ *be fixed. Then, the pseudo-differential operator* $\mathrm{Op}_t(a)$ *is the linear and continuous operator on* $\mathscr{S}(\mathbb{R}^d)$ *defined by the formula*

$$(\mathrm{Op}_t(a)u)(x) = (2\pi)^{-d} \iint_{\mathbb{R}^{2d}} e^{i\langle x-y,\xi\rangle} a((1-t)x+ty,\xi)u(y)\,dyd\xi$$

*(cf. Chapter XVIII in [80]). For general* $a \in \mathscr{S}'(\mathbb{R}^{2d})$, *the pseudo-differential operator* $\mathrm{Op}_t(a)$ *is defined as the continuous operator from* $\mathscr{S}(\mathbb{R}^d)$ *to* $\mathscr{S}'(\mathbb{R}^d)$ *with distribution kernel*

$$K_{t,a}(x,y) = (2\pi)^{-\frac{d}{2}}(\mathscr{F}_2^{-1}a)((1-t)x+ty, x-y).$$

Focusing on the case where $a \in S^{m,\mu}$, if $t = 0$, then $\mathrm{Op}_t(a)$ is the so-called Kohn-Nirenberg representation $\mathrm{Op}(a) = a(x, D)$, also known as the left-quantization, represented as follows

$$(\mathrm{Op}(a)u)(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} a(x,\xi)\widehat{u}(\xi)\, d\xi, \quad u \in \mathscr{S}, \tag{1.3}$$

with $\widehat{u}$ is the Fourier transform of $u \in \mathscr{S}$, given by (0.10).

Moreover, if $t = 1/2$, then $\mathrm{Op}_t(a)$ is the so-called Weyl quantization $\mathrm{Op}^w(a) = a^w(x, D)$ represented as

$$(\mathrm{Op}^w(a)u)(x) = (2\pi)^{-d} \iint_{\mathbb{R}^{2d}} e^{i\langle x-y,\xi\rangle} a\left((x+y)/2, \xi\right) u(y)\, dy d\xi, \quad u \in \mathscr{S}.$$

**Theorem 1.4.** *The operators in* (1.3) *form a graded algebra with respect to composition, that is, for $m_j, \mu_j \in \mathbb{R}$, $j = 1, 2$, we have*

$$\mathrm{Op}(S^{m_1,\mu_1}(\mathbb{R}^{2d})) \circ \mathrm{Op}(S^{m_2,\mu_2}(\mathbb{R}^{2d})) \subseteq \mathrm{Op}(S^{m_1+m_2,\mu_1+\mu_2}(\mathbb{R}^{2d})).$$

*The symbol $c \in S^{m_1+m_2,\mu_1+\mu_2}$ of the composed operator* $\mathrm{Op}(a) \circ \mathrm{Op}(b)$, *where $a \in S^{m_1,\mu_1}$, $b \in S^{m_2,\mu_2}$, admits the asymptotic expansion*

$$c(x,\xi) \asymp \sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha a(x,\xi)\, D_x^\alpha b(x,\xi).$$

**Remark 1.5.** *Theorem 1.4 implies that the symbol $c$ equals $a \cdot b$ modulo $S^{m_1+m_2-1,\mu_1+\mu_2-1}(\mathbb{R}^{2d})$.*

The residual elements of the calculus are operators with symbols in the space

$$S^{-\infty,-\infty}(\mathbb{R}^{2d}) = \bigcap_{(m,\mu)\in\mathbb{R}^2} S^{m,\mu}(\mathbb{R}^{2d}) = \mathscr{S}(\mathbb{R}^{2d}),$$

that is, those having kernel in $\mathscr{S}(\mathbb{R}^{2d})$, continuously mapping $\mathscr{S}'(\mathbb{R}^d)$ to $\mathscr{S}(\mathbb{R}^d)$.

**Definition 1.6.** *An operator $A = \mathrm{Op}(a)$ is called* elliptic *(or $S^{m,\mu}$-*elliptic*) if $a \in S^{m,\mu}(\mathbb{R}^{2d})$ and there exists $R \geqslant 0$ such that*

$$C\langle x\rangle^m \langle\xi\rangle^\mu \leqslant |a(x,\xi)|, \qquad |x| + |\xi| \geqslant R, \tag{1.4}$$

*for some constant $C > 0$.*

**Theorem 1.7.** *An elliptic SG operator $A \in \mathrm{Op}(S^{m,\mu}(\mathbb{R}^{2d}))$ admits a parametrix $P \in \mathrm{Op}(S^{-m,-\mu})$ such that*

$$PA = I + K_1, \quad AP = I + K_2,$$

*for suitable $K_1, K_2 \in \mathrm{Op}(S^{-\infty,-\infty})$, where $I$ denotes the identity operator.*

It is a well-known fact that $SG$-operators give rise to linear continuous mappings from $\mathscr{S}$ to itself, extendable as linear continuous mappings from $\mathscr{S}'$ to itself.

**Proposition 1.8.** $\mathrm{Op}(S^{m,\mu}(\mathbb{R}^{2d}))$ *act continuously between the so-called Sobolev-Kato (or weighted Sobolev) space, that is from* $H^{s,\sigma}$ *to* $H^{s-m,\sigma-\mu}$, *where* $H^{r,\varrho}$, $r,\varrho\in\mathbb{R}$, *is defined as*

$$H^{r,\varrho}=\left\{u\in\mathscr{S}'\colon \|u\|_{r,\varrho}=\|\langle . \rangle^{r}\langle D\rangle^{\varrho}u\|_{L^2}<\infty\right\}.$$

Next we introduce parameter-dependent symbols, where the parameters give rise to bounded families in $S^{m,\mu}$.

**Definition 1.9.** *Let* $\Omega\subseteq\mathbb{R}^N$, *for* $N\geqslant 1$. *We write* $f\in C^k\left(\Omega,S^{m,\mu}(\mathbb{R}^{2d})\right)$, *with* $m$, $\mu\in\mathbb{R}$ *and* $k\in\mathbb{Z}_+$ *or* $k=\infty$, *if*

*(i)* $f=f(\omega;x,\xi)$, $\omega\in\Omega$, $x$, $\xi\in\mathbb{R}^d$.

*(ii) For any* $\omega\in\Omega$, $\partial_\omega^\alpha f(\omega)\in S^{m,\mu}(\mathbb{R}^{2d})$, *for all* $\alpha\in\mathbb{Z}_+^N$, $|\alpha|\leqslant k$.

*(iii)* $\{\partial_\omega^\alpha f(\omega)\}_{\omega\in\Omega}$ *is bounded in* $S^{m,\mu}(\mathbb{R}^{2d})$, *for all* $\alpha\in\mathbb{Z}_+^N$, $|\alpha|\leqslant k$.

**Lemma 1.10.** *Let* $\Omega\subseteq\mathbb{R}^N$, *for* $N\geqslant 1$, $a\in C^k\left(\Omega,S^{m,\mu}(\mathbb{R}^{2d})\right)$ *and* $h\in C^k\left(\Omega,S^{0,0}(\mathbb{R}^{2d})\otimes\mathbb{R}^N\right)$ *such that* $k\in\mathbb{Z}_+$ *or* $k=\infty$. *Assume also that for any* $\omega\in\Omega$, $x,\xi\in\mathbb{R}^d$, *the function* $h(\omega;x,\xi)$ *takes value in* $\Omega$. *Then, setting* $b(\omega)=a(h(\omega))$, *that is,* $b(\omega;x,\xi)=a(h(\omega;x,\xi));x,\xi)$, *we find* $b\in C^k\left(\Omega,S^{m,\mu}(\mathbb{R}^{2d})\right)$.

*Proof.* (i) The fact that $b\in C^k\left(\Omega_\omega,C^\infty(\mathbb{R}^d_x\times\mathbb{R}^r_\xi)\otimes\mathbb{R}^N\right)$ is an immediate consequence of hypotheses and definitions.

(ii) For any fixed $\omega\in\Omega$, $b(\omega)\in S^{m,\mu}$. Indeed, by Faá di Bruno formula, see [13], for $\sum|\gamma_j|+|\gamma|=|\alpha|$ and $\sum|\delta_j|+|\delta|=|\beta|$, we get

$$\begin{aligned}\left|\partial_x^\alpha\partial_\xi^\beta b(\omega;x,\xi)\right|&\lesssim\sum\left|\left(\partial_x^\gamma\partial_\xi^\delta\partial_\omega^\kappa a\right)(h(\omega;x,\xi);x,\xi)\prod_j\partial_x^{\gamma_j}\partial_\xi^{\delta_j}h(\omega;x,\xi)\right|\\&\lesssim\sum\langle x\rangle^{m-|\gamma|}\langle\xi\rangle^{\mu-\delta}\left(\prod_j\langle x\rangle^{-|\gamma_j|}\langle\xi\rangle^{-\delta_j}\right)=\langle x\rangle^{m-|\alpha|}\langle\xi\rangle^{\mu-\beta}.\end{aligned}$$

Notice that the same argument shows that $(\partial_\omega^\alpha a)(h(\omega;x,\xi);x,\xi)\in S^{m,\mu}$ for any $\alpha\in\mathbb{Z}_+^N$, $|\alpha|\leqslant k$ and all $\omega\in\Omega$.

(iii) By Faá di Bruno formula, for any $\alpha\in\mathbb{Z}_+^N$, such that $|\alpha|\leqslant k$, $(\partial_\omega^\alpha b)(\omega)$ is in the span of

$$H\equiv\left\{(\partial_\omega^\kappa a)(h(\omega))\prod_j(\partial_\omega^{\kappa_j}h)(\omega)\right\}\subset S^{m,\mu}$$

and is bounded, since that is true for all the elements of $H$. □

## 1.2 Multi-product of SG pseudo-differential operators

This section is devoted to the proof of the next Theorem 1.11, dealing with the composition (or multi-product) of $(M+1)$ *SG* pseudo-differential operators, where $M \geqslant 1$. This extends the results of Theorem 1.4 to any arbitrary number of factors.

**Theorem 1.11.** *Let $M \geqslant 1$, $p_j \in S^{m_j,\mu_j}(\mathbb{R}^{2d})$, $m_j, \mu_j \in \mathbb{R}$, $j = 1, \ldots, M+1$. Consider the multi-product*

$$Q_{M+1} = P_1 \cdots P_{M+1} \tag{1.5}$$

*of the operators $P_j = \mathrm{Op}(p_j)$, $j = 1, \ldots, M+1$, and denote by $q_{M+1} \in S^{m,\mu}(\mathbb{R}^{2d})$, $m = m_1 + \cdots + m_{M+1}$, $\mu = \mu_1 + \cdots + \mu_{M+1}$, the symbol of $Q_{M+1}$. Then, if each factor $p_j$ belongs to a bounded subset $U_j \subset S^{m_j,\mu_j}(\mathbb{R}^{2d})$ for $j = 1, \ldots, M+1$, it follows that $q_{M+1}$ belongs to a bounded* [1] *subset $U \subset S^{m,\mu}(\mathbb{R}^{2d})$.*

We split the proof of Theorem 1.11 into two steps. The first one consists in obtaining an expression for $q_{M+1}$ as an oscillatory integral. The second one deals with the boundedness claim on $q_{M+1}$ in $S^{m,\mu}(\mathbb{R}^{2d})$, based on the expression obtained in the first step.

**Lemma 1.12.** *Under the hypotheses of Theorem 1.11, $q_{M+1}$ can be written as an oscillatory integral, namely,*

$$q_{M+1}(x,\xi) = \iint\limits_{\mathbb{R}^{dM}\times\mathbb{R}^{dM}} e^{-i\psi(y,\eta)} \prod_{j=1}^{M+1} p_j(x + y_{j-1}, \xi + \eta_j)\, đy đ\eta, \tag{1.6}$$

*where $y_0 = \eta_{M+1} = 0 \in \mathbb{R}^d$, $y = (y_1, \ldots, y_M)$, $\eta = (\eta_1, \ldots, \eta_M)$, $y, \eta \in \mathbb{R}^{dM}$,*

$$\psi(y,\eta) = \sum_{j=1}^{M} \langle y_j, \eta_j - \eta_{j+1}\rangle = \sum_{j=1}^{M} \langle y_j - y_{j-1}, \eta_j\rangle,$$

*and*

$$\begin{aligned} đy đ\eta &= đy_1 \ldots đy_M đ\eta_1 \ldots đ\eta_M \\ &\equiv (2\pi)^{-\frac{d}{2}} dy_1 \ldots (2\pi)^{-\frac{d}{2}} dy_M (2\pi)^{-\frac{d}{2}} d\eta_1 \ldots (2\pi)^{-\frac{d}{2}} d\eta_M. \end{aligned}$$

**Remark 1.13.** *We recall that the space of amplitudes of order $\tau \in \mathbb{R}$ on $\mathbb{R}^N$ for $N \in \mathbb{Z}_+$, denoted by $\mathscr{A}^\tau(\mathbb{R}^N)$, is defined as*

$$\mathscr{A}^\tau(\mathbb{R}^N) = \{a \in C^\infty(\mathbb{R}^N) : \text{ for any } \alpha \in \mathbb{Z}_+^N \text{ there exists } C_\alpha > 0 : |\partial_z^\alpha a(z)| \leqslant C_\alpha \langle z\rangle^\tau\}.$$

[1] With respect to the corresponding Fréchet topologies, loosely speaking $q_{M+1}$ depends continuously on $p_j$, $j = 1, \ldots, M+1$.

*Moreover, we write*

$$|a|_k^\tau := \max_{|\alpha|\leqslant k} \sup_{z\in\mathbb{R}^N} \langle z\rangle^{-\tau}|\partial_z^\alpha a(z)|, \quad k \in \mathbb{Z}_+,$$

*for the associated sequence of (monotonically increasing) semi-norms, see [104].*

*Proof of Lemma 1.12.* We first show that formally (1.6) holds true, then we check that the right-hand side can be regarded as an oscillatory integral, in the sense described, e.g., in [104].

We will prove (1.6) by induction. First, we show that it holds true for $M = 1$. With $u \in \mathscr{S}$, in view of the general theory of $SG$ pseudo-differential operators, we find, formally,

$$\begin{aligned}(P_1P_2u)(x) &= [\mathrm{Op}(q_2)u](x)\\ &= \int_{\mathbb{R}^d} e^{i\langle x,\eta\rangle} p_1(x,\eta) \int_{\mathbb{R}^d} e^{-i\langle y,\eta\rangle} \int_{\mathbb{R}^d} e^{i\langle y,\xi\rangle} p_2(y,\xi)\widehat{u}(\xi) đ\xi\, đyđ\eta\\ &= \iiint\limits_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} e^{i(\langle x,\eta\rangle - \langle y,\eta\rangle + \langle y,\xi\rangle)} p_1(x,\eta)p_2(y,\xi)\widehat{u}(\xi) đ\xi\, đyđ\eta\end{aligned}$$

$$= \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle}\left[\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} e^{i(\langle x,\eta\rangle - \langle y,\eta\rangle + \langle y,\xi\rangle - \langle x,\xi\rangle)} p_1(x,\eta)p_2(y,\xi)\, đyđ\eta\right]\widehat{u}(\xi)\, đ\xi$$

$$\begin{aligned}\Rightarrow q_2(x,\xi) &= \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} e^{i\langle x-y,\eta-\xi\rangle} p_1(x,\eta)p_2(y,\xi)\, đyđ\eta\\ &= \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} e^{-i\langle y_1,\eta_1\rangle} p_1(x,\xi+\eta_1)p_2(x+y_1,\xi)\, đy_1đ\eta_1,\end{aligned} \tag{1.7}$$

which is (1.6) with $M = 1$. The final expression of $q_2$ in (1.7) actually holds true, in view of the general theory of oscillatory integrals, which allows to exchange the order of integration and linear change of variables. In fact, $\langle y_1, \eta_1\rangle$ is nondegenerate[2].

It only remains to prove that $a_{x,\xi}(y_1, \eta_1) = p_1(x, \xi + \eta_1)p_2(x + y_1, \xi)$ is an amplitude of some order with respect to $(y_1, \eta_1)$. By Peetre's inequality

[2]The Hessian is indead the identity matrix. Alternatively, as observed in [104], the identity

$$\langle y_1, \eta_1\rangle = \frac{1}{4}\left(|y_1 + \eta_1|^2 - |y_1 - \eta_1|^2\right) \tag{1.8}$$

explicitly shows its $d$ positive and $d$ negative eigenvalues.

(see, e.g., Lemma 1.18 of [104]), with $\alpha, \beta \in \mathbb{Z}^d_+$,

$$\begin{aligned}\left|\partial^\alpha_{\eta_1}\partial^\beta_{y_1}\left[p_1(x,\xi+\eta_1)p_2(x+y_1,\xi)\right]\right| &\\ \leqslant C_{\alpha\beta}\langle x\rangle^{m_1}\langle \xi+\eta_1\rangle^{\mu_1-|\alpha|}\langle x+y_1\rangle^{m_2-|\beta|}\langle\xi\rangle^{\mu_2} &\\ \leqslant C_{\alpha\beta}\langle x\rangle^{m_1}\langle \xi+\eta_1\rangle^{\mu_1}\langle x+y_1\rangle^{m_2}\langle\xi\rangle^{\mu_2} &\\ \leqslant C_{\alpha\beta}\langle x\rangle^{m_1+m_2}\langle\xi\rangle^{\mu_1+\mu_2}\langle y_1\rangle^{|m_2|}\langle\eta_1\rangle^{|\mu_1|} &\\ \leqslant C_{\alpha\beta}\langle x\rangle^{m}\langle\xi\rangle^{\mu}\langle y_1,\eta_1\rangle^{\tilde{m}+\tilde{\mu}}, &\quad (1.9)\end{aligned}$$

where $m = m_1 + m_2$, $\mu = \mu_1 + \mu_2$, $\tilde{m} = |m_1| + |m_2|$, $\tilde{\mu} = |\mu_1| + |\mu_2|$, $\langle y_1, \eta_1\rangle^2 = 1+|y_1|^2+|\eta_1|^2$. It follows that $a_{x,\xi}(y_1,\eta_1)$ is, for any $(x,\xi)\in\mathbb{R}^{2d}$, an amplitude with respect to $(y_1,\eta_1)$, of order $(\tilde{m}+\tilde{\mu})$. Hence, (1.7) is a well defined oscillatory integral. From the general theory of oscillatory integrals, see [104], denoting by $[t] = \max\{k\in\mathbb{Z}\colon k\leqslant t\}$ the integer part of $t\in\mathbb{R}$, we have

$$\left|\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} e^{-i\langle y_1,\eta_1\rangle}a_{x,\xi}(y_1,\eta_1)\,đy_1đ\eta_1\right| \leqslant C\,|a_{x,\xi}|^{\tilde{m}+\tilde{\mu}}_{[2d+\tilde{m}+\tilde{\mu}+1]} \leqslant \tilde{C}\,\langle x\rangle^{m_1+m_2}\langle\xi\rangle^{\mu_1+\mu_2}, \tag{1.10}$$

which implies, for $\alpha,\beta,\gamma,\delta\in\mathbb{Z}^d_+$, $\gamma\leqslant\alpha$, $\delta\leqslant\beta$, and $C_{\alpha\beta\gamma\delta} = \binom{\alpha}{\gamma}\cdot\binom{\beta}{\delta}$,

$$\begin{aligned}&|\partial^\alpha_x\partial^\beta_\xi q_2(x,\xi)| = \\ &= \left|\sum_{\gamma\leqslant\alpha,\delta\leqslant\beta} C_{\alpha\beta\gamma\delta}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} e^{-i\langle y_1,\eta_1\rangle}(\partial^\gamma_x\partial^\delta_\xi p_1)(x,\xi+\eta_1)(\partial^{\alpha-\gamma}_x\partial^{\beta-\delta}_\xi p_2)(x+y_1,\xi)\,đy_1đ\eta_1\right| \\ &\leqslant C_{\alpha\beta}\sum_{\gamma\leqslant\alpha,\delta\leqslant\beta}\langle x\rangle^{m_1-|\gamma|+m_2-|\alpha-\gamma|}\langle\xi\rangle^{\mu_1-|\delta|+\mu_2-|\beta-\delta|} \leqslant \widetilde{C}_{\alpha\beta}\langle x\rangle^{m-|\alpha|}\langle\xi\rangle^{\mu-|\beta|},\end{aligned} \tag{1.11}$$

that is, $q_2\in S^{m,\mu}$, as stated, and the desired claim holds true for $M=1$.

Assume now that (1.6) holds true for $M\geqslant 1$. Let $u\in\mathscr{S}$, and write

$$\begin{aligned}[Q_{M+2}u](x) &= [\mathrm{Op}(q_{M+2})u](x) \\ &= [((P_1P_2\cdots P_{M+1})P_{M+2})u](x) = [(Q_{M+1}P_{M+2})u](x),\end{aligned}$$

with $Q_{M+1} = \mathrm{Op}(q_{M+1})$ and $q_{M+1}\in S^{m,\mu}$ given by (1.6) and (1.5), respectively. Using (1.7), with $q_{M+1}$ and $p_{M+2}$ in place of $p_1$ and $p_2$, respectively, arguing as above we find

$$
\begin{aligned}
&q_{M+2}(x,\xi)\\
&= \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} e^{-i\langle y_{M+1},\eta_{M+1}\rangle} q_{M+1}(x,\xi+\eta_{M+1}) p_{M+2}(x+y_{M+1},\xi)\, đy_{M+1}đ\eta_{M+1},
\end{aligned}
\tag{1.12}
$$

then, in view of (1.11), $q_{M+2} \in S^{(m_1+\cdots+m_{M+1})+m_{M+2},(\mu_1+\cdots+\mu_{M+1})+\mu_{M+2}}$. Moreover, with

$$
\begin{cases}
\psi(y,\eta) &= \langle y_1,\eta_1\rangle & \text{for } M=1\\
\psi(y,\eta) &= \sum\limits_{j=1}^{M-1}\langle y_j,\eta_j-\eta_{j+1}\rangle + \langle y_M,\eta_M\rangle & \text{for } M\geqslant 2,
\end{cases}
$$

inserting (1.6) into (1.12), in view of the inductive hypothesis, it follows that

$$
\begin{aligned}
&q_{M+2}(x,\xi) =\\
&\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} e^{-i\langle y_{M+1},\eta_{M+1}\rangle} q_{M+1}(x,\xi+\eta_{M+1}) p_{M+2}(x+y_{M+1},\xi)\, đy_{M+1}đ\eta_{M+1}\\
&\qquad = \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} e^{-i\langle y_{M+1},\eta_{M+1}\rangle} p_{M+2}(x+y_{M+1},\xi)\times\\
&\qquad\Bigg[\iint\limits_{\mathbb{R}^{dM}\times\mathbb{R}^{dM}} e^{-i\psi(y,\eta)} \left(\prod_{j=1}^{M} p_j(x+y_{j-1},\xi+\eta_j+\eta_{M+1})\right)\\
&\qquad\qquad p_{M+1}(x+y_M,\xi+\eta_{M+1})\, đyđ\eta\Bigg]\, đy_{M+1}đ\eta_{M+1}.
\end{aligned}
$$

After the change of variables $\eta_j = \widetilde{\eta}_j - \eta_{M+1}$, $j=1,\dots,M$, and writing $\widetilde{\eta}_{M+1} = \eta_{M+1}$, $\widetilde{\eta}_{M+2} = 0$, $\widetilde{y}_j = y_j$, $j = 1,\dots,M+1$, $\widetilde{y}_0 = 0$, $\widetilde{y} = (\widetilde{y}_1,\dots,\widetilde{y}_{M+1})$, $\widetilde{\eta} = (\widetilde{\eta}_1,\dots,\widetilde{\eta}_{M+1})$, we get

$$
q_{M+2}(x,\xi) = \iint\limits_{\mathbb{R}^{d(M+1)}\times\mathbb{R}^{d(M+1)}} e^{-i\widetilde{\psi}(\widetilde{y},\widetilde{\eta})} \prod_{j=1}^{M+2} p_j(x+\widetilde{y}_{j-1},\xi+\widetilde{\eta}_j)\, d\widetilde{y}đ\widetilde{\eta}, \tag{1.13}
$$

where:

- if $M=1$,

$$
\begin{aligned}
\widetilde{\psi}(\widetilde{y},\widetilde{\eta}) &= \langle y_1,\eta_1\rangle + \langle y_2,\eta_2\rangle = \langle \widetilde{y}_1,\widetilde{\eta}_1-\widetilde{\eta}_2\rangle + \langle \widetilde{y}_2,\widetilde{\eta}_2\rangle\\
&= \langle \widetilde{y}_1,\widetilde{\eta}_1-\widetilde{\eta}_2\rangle + \langle \widetilde{y}_2,\widetilde{\eta}_2-\widetilde{\eta}_3\rangle = \sum_{j=1}^{M+1}\langle \widetilde{y}_j,\widetilde{\eta}_j-\widetilde{\eta}_{j+1}\rangle
\end{aligned}
$$

- if $M \geqslant 2$,

$$\begin{aligned}
\widetilde{\psi}(\widetilde{y},\widetilde{\eta}) &= \sum_{j=1}^{M-1}\langle y_j,\eta_j-\eta_{j+1}\rangle+\langle y_M,\eta_M\rangle+\langle y_{M+1},\eta_{M+1}\rangle \\
&= \sum_{j=1}^{M-1}\langle \widetilde{y}_j,\widetilde{\eta}_j-\widetilde{\eta}_{j+1}\rangle+\langle \widetilde{y}_M,\widetilde{\eta}_M-\widetilde{\eta}_{M+1}\rangle+\langle \widetilde{y}_{M+1},\widetilde{\eta}_{M+1}\rangle \\
&= \sum_{j=1}^{M}\langle \widetilde{y}_j,\widetilde{\eta}_j-\widetilde{\eta}_{j+1}\rangle+\langle \widetilde{y}_{M+1},\widetilde{\eta}_{M+1}-\widetilde{\eta}_{M+2}\rangle=\sum_{j=1}^{M+1}\langle \widetilde{y}_j,\widetilde{\eta}_j-\widetilde{\eta}_{j+1}\rangle.
\end{aligned}$$

That is, (1.13) is (1.6) with $M+1$ in place of $M$, as desired.

From the general theory of oscillatory integrals, it follows that all the computations in the induction step, namely, linear changes of variables and exchange of integration order, are justified. In fact, the phase function $\psi$ is nondegenerate, and $a^{M+1}_{x,\xi}(y,\eta)=\prod_{j=1}^{M+1}p_j(x+y_{j-1},\xi+\eta_j)$ is an amplitude of order $|m_1|+\cdots+|m_{M+1}|+|\mu_1|+\cdots+|\mu_{M+1}|$ with respect to $(y,\eta)$, for any $M\geqslant 1$, $(x,\xi)\in\mathbb{R}^{2d}$.

$M=1$ is already proved. For $M\geqslant 2$, with arbitrary $\alpha_1,\ldots,\alpha_M$, $\beta_1,\ldots,\beta_M\in\mathbb{Z}^d_+$, setting $\alpha_{M+1}=\beta_0=0\in\mathbb{Z}^d_+$, a similar arguments show that

$$\begin{aligned}
&\left|\partial^{\alpha_1}_{\eta_1}\ldots\partial^{\alpha_M}_{\eta_M}\partial^{\beta_1}_{y_1}\ldots\partial^{\beta_M}_{y_M}\prod_{j=1}^{M+1}p_j(x+y_{j-1},\xi+\eta_j)\right| \\
&\qquad\leqslant C^{\alpha_1\ldots\alpha_M}_{\beta_1\ldots\beta_M}\prod_{j=1}^{M+1}\left|(\partial^{\beta_{j-1}}_x\partial^{\alpha_j}_\xi p_j)(x+y_{j-1},\xi+\eta_j)\right| \\
&\qquad\leqslant \widetilde{C}^{\alpha_1\ldots\alpha_M}_{\beta_1\ldots\beta_M}\prod_{j=1}^{M+1}\langle x+y_{j-1}\rangle^{m_j-|\beta_{j-1}|}\langle\xi+\eta_j\rangle^{\mu_j-|\alpha_j|} \\
&\qquad\leqslant \widetilde{C}^{\alpha_1\ldots\alpha_M}_{\beta_1\ldots\beta_M}\prod_{j=1}^{M+1}\langle x+y_{j-1}\rangle^{m_j}\langle\xi+\eta_j\rangle^{\mu_j} \\
&\qquad\leqslant \widetilde{\widetilde{C}}^{\alpha_1\ldots\alpha_M}_{\beta_1\ldots\beta_M}\prod_{j=1}^{M+1}\langle x\rangle^{m_j}\langle y_{j-1}\rangle^{|m_j|}\langle\xi\rangle^{\mu_j}\langle\eta_j\rangle^{|\mu_j|} \\
&\qquad\leqslant \left(\widetilde{\widetilde{C}}^{\alpha_1\ldots\alpha_M}_{\beta_1\ldots\beta_M}\langle x\rangle^m\langle\xi\rangle^\mu\right)\langle y,\eta\rangle^{\widetilde{m}+\widetilde{\mu}},
\end{aligned} \tag{1.14}$$

where we used Peetre's inequality, such that

$$\begin{aligned}
m&=m_1+\cdots+m_{M+1}, & \mu&=\mu_1+\cdots+\mu_{M+1}, \\
\widetilde{m}&=|m_1|+\cdots+|m_{M+1}|, & \widetilde{\mu}&=|\mu_1|+\cdots+|\mu_{M+1}|.
\end{aligned}$$

It only remains to prove that $\psi(y,\eta)$ is a nondegenerate phase function for any $M$. By means of (1.8), we already proved it for $M = 1$. For $M \geqslant 2$, we proceed in a similar way, rewriting $\psi$ as a linear combination of squared norms of suitable vectors, obtained through invertible linear maps applied to $y$ and $\eta$. Again, this shows that $\psi$ has $dM$ positive and $dM$ negative eigenvalues. Indeed, observe that, for $M \geqslant 2$,

$$\psi(y,\eta) = \sum_{j=1}^{M}\langle y_j, \eta_j - \eta_{j+1}\rangle = \sum_{j=1}^{M-1} \left(\langle y_j, \eta_j - \eta_{j+1}\rangle + \langle y_M, \eta_M\rangle\right)$$

$$= \frac{1}{4}\sum_{j=1}^{M-1} \left(|y_j + \eta_j - \eta_{j+1}|^2 - |y_j - \eta_j + \eta_{j+1}|^2 + |y_M + \eta_M|^2 - |y_M - \eta_M|^2\right)$$

$$= \frac{1}{4}\left[\sum_{j=1}^{M}(|y_j + \zeta_j|^2 - |y_j - \zeta_j|^2)\right]_{\substack{\zeta_j = \eta_j - \eta_{j+1}, j = \dots, M-1,\\ \zeta_M = \eta_M}}$$

The proof is complete. □

**Remark 1.14.** *Recalling the first inequality in* (1.10)*, which holds true for any amplitude in any dimension and any nondegenerate phase function, the last part of the proof of Lemma 1.12 shows also that it is possible to prove directly* $q_{M+1} \in S^{m,\mu}(\mathbb{R}^{2d})$*. In fact, it is enough to extend* (1.11)*, as it is possible, to a product of* $M+1$ *factors, and use* (1.14)*.*

*Proof of Theorem 1.11.* In view of Lemma 1.12, we can write the symbol $q_{M+1}(x,\xi)$ of the multi-product $Q_{M+1}$ in the form (1.6). Taking into account Remark 1.14, the proof of Lemma 1.12 also shows that the semi-norms of $q_{M+1}$ in $S^{m,\mu}(\mathbb{R}^{2d})$ depend continuously on the semi-norms of $p_j$ in $S^{m_j,\mu_j}(\mathbb{R}^{2d})$, $j = 1, \dots, M+1$ (see, in particular, (1.9) and (1.11)). This implies the claimed boundedness result. □

## 1.3 Fourier integral operators of SG type

Here we give a short summary of the main properties of the class of Fourier integral operators we will be dealing with. In particular, we recall their compositions with the *SG* pseudo-differential operators, and the compositions between Type I and Type II operators.

The Fourier integral operators defined, for $u \in \mathscr{S}(\mathbb{R}^d)$, as

$$u \mapsto (\mathrm{Op}_\varphi(a)u)(x) = (2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{i\varphi(x,\xi)}a(x,\xi)\hat{u}(\xi)\,d\xi, \tag{1.15}$$

and

$$u \mapsto (\mathrm{Op}^*_\varphi(b)u)(x) = (2\pi)^{-d}\iint_{\mathbb{R}^{2d}} e^{i(\langle x,\xi\rangle - \varphi(y,\xi))}\overline{b(y,\xi)}u(y)\,dyd\xi, \tag{1.16}$$

with suitable choices of phase function $\varphi$ and symbols $a$ and $b$, are often called Fourier operators of type I and II, respectively.

The operators $\mathrm{Op}_\varphi(a)$ and $\mathrm{Op}^*_\varphi(a)$ are called *SG* Fourier integral operators of type I and type II, respectively, when $a \in S^{m,\mu}$, and $\varphi$ satisfies the requirements of the next Definition 1.15. Note that a type II operator satisfies $\mathrm{Op}^*_\varphi(a) = \mathrm{Op}_\varphi(a)^*$, that is, it is the formal $L^2$-adjoint of the type I operator $\mathrm{Op}_\varphi(a)$.

**Definition 1.15** (*SG* phase functions)**.** *A real-valued function $\varphi \in C^\infty(\mathbb{R}^{2d})$ belongs to the class $\mathcal{P}$ of SG phase functions if it satisfies the following conditions:*

*(1)* $\varphi \in S^{1,1}(\mathbb{R}^{2d})$.

*(2)* $\langle \varphi'_x(x,\xi)\rangle \asymp \langle \xi \rangle$ *as* $|(x,\xi)| \to \infty$.

*(3)* $\langle \varphi'_\xi(x,\xi)\rangle \asymp \langle x \rangle$ *as* $|(x,\xi)| \to \infty$.

The *SG* Fourier integral operators of type I and type II, $\mathrm{Op}_\varphi(a)$ and $\mathrm{Op}^*_\varphi(b)$, are defined as in (1.15) and (1.16), respectively, with $\varphi \in \mathcal{P}$ and $a, b \in S^{m,\mu}$. Notice that we do not assume any homogeneity hypothesis on the phase function $\varphi$. The next Theorem 1.16 treats compositions between *SG* pseudo-differential operators and *SG* Fourier integral operators. It was originally proved in [42], see also [47, 49, 50].

**Theorem 1.16.** *Let $\varphi \in \mathcal{P}$ and assume $p \in S^{t,\tau}(\mathbb{R}^{2d})$, $a, b \in S^{m,\mu}(\mathbb{R}^{2d})$. Then,*

$$\mathrm{Op}(p) \circ \mathrm{Op}_\varphi(a) = \mathrm{Op}_\varphi(c_1 + r_1) = \mathrm{Op}_\varphi(c_1) \mod \mathrm{Op}(S^{-\infty,-\infty}(\mathbb{R}^{2d})),$$

$$\mathrm{Op}(p) \circ \mathrm{Op}^*_\varphi(b) = \mathrm{Op}^*_\varphi(c_2 + r_2) = \mathrm{Op}^*_\varphi(c_2) \mod \mathrm{Op}(S^{-\infty,-\infty}(\mathbb{R}^{2d})),$$

$$\mathrm{Op}_\varphi(a) \circ \mathrm{Op}(p) = \mathrm{Op}_\varphi(c_3 + r_3) = \mathrm{Op}_\varphi(c_3) \mod \mathrm{Op}(S^{-\infty,-\infty}(\mathbb{R}^{2d})),$$

$$\mathrm{Op}^*_\varphi(b) \circ \mathrm{Op}(p) = \mathrm{Op}^*_\varphi(c_4 + r_4) = \mathrm{Op}^*_\varphi(c_4) \mod \mathrm{Op}(S^{-\infty,-\infty}(\mathbb{R}^{2d})),$$

*for some $c_j \in S^{m+t,\mu+\tau}(\mathbb{R}^{2d})$, $r_j \in S^{-\infty,-\infty}(\mathbb{R}^{2d})$, $j = 1, \ldots, 4$.*

In order to obtain the composition of *SG* Fourier integral operators of type I and type II, some more hypotheses are needed, leading to the definition of the classes $\mathcal{P}_r$ and $\mathcal{P}_r(\tau)$ of regular SG phase functions, cf. [86].

**Definition 1.17** (Regular *SG* phase function)**.** *Let $\tau \in [0,1)$ and $r > 0$. A function $\varphi \in \mathcal{P}$ belongs to the class $\mathcal{P}_r(\tau)$ if it satisfies the following conditions:*

*(1)* $|\det(\varphi''_{x\xi})(x,\xi)| \geqslant r$, *for any* $x, \xi \in \mathbb{R}^d$.

*(2) The function* $J(x,\xi) := \varphi(x,\xi) - \langle x,\xi\rangle$ *is such that*

$$\sup_{\substack{x,\xi\in\mathbb{R}^d\\ |\alpha+\beta|\leqslant 2}} \frac{|D_\xi^\alpha D_x^\beta J(x,\xi)|}{\langle x\rangle^{1-|\beta|}\langle\xi\rangle^{1-|\alpha|}} \leqslant \tau. \tag{1.17}$$

*If only condition (1) holds, we write* $\varphi \in \mathcal{P}_r$.

**Remark 1.18.** *Notice that if condition* (1.17) *holds true for any* $\alpha$, $\beta \in \mathbb{Z}_+^d$, *then* $J(x,\xi)/\tau$ *is bounded with constant* 1 *in* $S^{1,1}(\mathbb{R}^{2d})$. *Notice also that condition (1) in Definition 1.17 is authomatically fulfilled when condition (2) holds true for a sufficiently small* $\tau \in [0,1)$.

For $\ell \in \mathbb{N}$, we also introduce the semi-norms

$$\|J\|_{2,\ell} := \sum_{2\leqslant|\alpha+\beta|\leqslant 2+\ell} \sup_{(x,\xi)\in\mathbb{R}^{2d}} \frac{|D_\xi^\alpha D_x^\beta J(x,\xi)|}{\langle x\rangle^{1-|\beta|}\langle\xi\rangle^{1-|\alpha|}},$$

and

$$\|J\|_\ell := \sup_{\substack{x,\xi\in\mathbb{R}^d\\ |\alpha+\beta|\leqslant 1}} \frac{|D_\xi^\alpha D_x^\beta J(x,\xi)|}{\langle x\rangle^{1-|\beta|}\langle\xi\rangle^{1-|\alpha|}} + \|J\|_{2,\ell}.$$

Notice that $\varphi \in \mathcal{P}_r(\tau)$ means that (1) of Definition 1.17 and $\|J\|_0 \leqslant \tau$ hold, and then we define the following subclass of the class of regular *SG* phase functions.

**Definition 1.19.** *Let* $\tau \in [0,1)$, $r > 0$, $\ell \geqslant 0$. *A function* $\varphi$ *belongs to the class* $\mathcal{P}_r(\tau,\ell)$ *if* $\varphi \in \mathcal{P}_r(\tau)$ *and* $\|J\|_\ell \leqslant \tau$ *for the corresponding function* $J$.

Theorem 1.20 below shows that the composition of SG Fourier integral operators of type I and type II with the same regular *SG* phase functions is a SG pseudo-differential operator cf. [43].

**Theorem 1.20.** *Let* $\varphi \in \mathcal{P}_r$ *and assume* $a \in S^{m,\mu}(\mathbb{R}^{2d})$, $b \in S^{t,\tau}(\mathbb{R}^{2d})$. *Then,*

$$\mathrm{Op}_\varphi(a) \circ \mathrm{Op}_\varphi^*(b) = \mathrm{Op}(c_5 + r_5) = \mathrm{Op}(c_5) \bmod \mathrm{Op}(S^{-\infty,-\infty}(\mathbb{R}^{2d})),$$

$$\mathrm{Op}_\varphi^*(b) \circ \mathrm{Op}_\varphi(a) = \mathrm{Op}(c_6 + r_6) = \mathrm{Op}(c_6) \bmod \mathrm{Op}(S^{-\infty,-\infty}(\mathbb{R}^{2d})),$$

*for some* $c_j \in S^{m+t,\mu+\tau}(\mathbb{R}^{2d})$, $r_j \in S^{-\infty,-\infty}(\mathbb{R}^{2d})$, $j = 5, 6$.

Furthermore, asymptotic formulate can be given for $c_j$, $j = 1, \ldots, 6$, in terms of $\varphi$, $p$, $a$ and $b$, see [42]. Finally, when $a \in S^{m,\mu}$ is elliptic and $\varphi \in \mathcal{P}_r$, the corresponding *SG* Fourier integral operators admit a parametrix, that is, there exist $b_1, b_2 \in S^{-m,-\mu}$ such that

$$\mathrm{Op}_\varphi(a) \circ \mathrm{Op}_\varphi^*(b_1) = \mathrm{Op}_\varphi^*(b_1) \circ \mathrm{Op}_\varphi(a) = I \bmod \mathrm{Op}(S^{-\infty,-\infty}),$$

$$\mathrm{Op}_\varphi^*(a) \circ \mathrm{Op}_\varphi(b_2) = \mathrm{Op}_\varphi(b_2) \circ \mathrm{Op}_\varphi^*(a) = I \bmod \mathrm{Op}(S^{-\infty,-\infty}),$$

where $I$ is the identity operator, see again [42, 47, 50].

**Theorem 1.21.** *Let* $\varphi \in \mathcal{P}_r$ *and* $a \in S^{m,\mu}(\mathbb{R}^{2d})$, $m, \mu \in \mathbb{R}$. *Then, for any* $r, \varrho \in \mathbb{R}$, $\mathrm{Op}_\varphi(a)$ *and* $\mathrm{Op}^*_\varphi(a)$ *continuously map* $H^{r,\varrho}(\mathbb{R}^d)$ *to* $H^{r-m,\varrho-\mu}(\mathbb{R}^d)$.

## 1.4 Multi-products of $SG$ phase functions and regular $SG$ Fourier integral operators

Here we recall a few results from [8] concerning the multi-products of $SG$ phase function and Fourier integral operators of type I. Let us consider a sequence $\{\varphi_j\}_{j\in\mathbb{N}}$ of regular $SG$ phase functions $\varphi_j(x,\xi) \in \mathcal{P}_r(\tau_j)$, $j \in \mathbb{N}$, with

$$\sum_{j=1}^{\infty} \tau_j =: \tau_0 < 1/4. \tag{1.18}$$

By Definition 1.17 and assumption (1.18), the sequence $\{J_k(x,\xi)/\tau_k\}_{k\geqslant 1}$ bounded in $S^{1,1}(\mathbb{R}^{2d})$ and for every $\ell \in \mathbb{N}$ there exists a constant $c_\ell > 0$ such that

$$\|J_k\|_{2,\ell} \leqslant c_\ell \tau_k \quad \text{and} \quad \sum_{k=1}^{\infty} \|J_k\|_{2,\ell} \leqslant c_\ell \tau_0. \tag{1.19}$$

We set $\overline{\tau}_M = \sum_{j=1}^{M} \tau_j$. With a fixed integer $M \geqslant 1$, we denote, for $x \equiv x_0$ and $\xi \equiv \xi_{M+1}$,

$$\begin{aligned}
&(X,\Xi) = (x_0, x_1, \ldots, x_M, \xi_1, \ldots, \xi_M, \xi_{M+1}) := (x, T, \Theta, \xi),\\
&(T,\Theta) = (x_1, \ldots, x_M, \xi_1, \ldots, \xi_M),
\end{aligned}$$

and define the function of $2(M+1)d$ real variables

$$\psi(X,\Xi) := \sum_{j=1}^{M} \left(\varphi_j(x_{j-1},\xi_j) - \langle x_j, \xi_j\rangle\right) + \varphi_{M+1}(x_M, \xi_{M+1}).$$

For every fixed $(x,\xi) \in \mathbb{R}^{2d}$, the critical points $(Y,N) = (Y,N)(x,\xi)$ of the function of $2Md$ variables $\widetilde{\psi}(T,\Theta) = \psi(x,T,\Theta,\xi)$ are the solutions to the system

$$\begin{cases}
\psi'_{\xi_j}(X,\Xi) = \varphi'_{j,\xi}(x_{j-1},\xi_j) - x_j = 0 & j = 1,\ldots,M,\\
\psi'_{x_j}(X,\Xi) = \varphi'_{j+1,x}(x_j,\xi_{j+1}) - \xi_j = 0 & j = 1,\ldots,M,
\end{cases} \tag{1.20}$$

in the unknowns $(T,\Theta)$. That is $(Y,N) = (Y_1,\ldots,Y_M,N_1,\ldots,N_M)(x,\xi)$ satisfies, if $M = 1$,

$$\begin{cases}
Y_1(x,\xi) = \varphi'_{1,\xi}(x, N_1(x,\xi))\\
N_1(x,\xi) = \varphi'_{2,x}(Y_1(x,\xi),\xi),
\end{cases} \tag{1.21}$$

or, if $M \geqslant 2$,

$$\begin{cases} Y_1(x,\xi) = \varphi'_{1,\xi}(x, N_1(x,\xi)) \\ Y_j(x,\xi) = \varphi'_{j,\xi}(Y_{j-1}(x,\xi), N_j(x,\xi)), & j = 2,\dots,M \\ N_j(x,\xi) = \varphi'_{j+1,x}(Y_j(x,\xi), N_{j+1}(x,\xi)), & j = 1,\dots,M-1 \\ N_M(x,\xi) = \varphi'_{M+1,x}(Y_M(x,\xi),\xi). \end{cases} \tag{1.22}$$

In the sequel we will only refer to the system (1.22), tacitly meaning (1.21) when $M = 1$.

**Definition 1.22** (Multi-product of *SG* phase functions)**.** *If, for every fixed* $(x,\xi) \in \mathbb{R}^{2d}$*, the system* (1.22) *admits a unique solution* $(Y,N) = (Y,N)(x,\xi)$*, we define*

$$\phi(x,\xi) = (\varphi_1 \sharp \cdots \sharp \varphi_{M+1})(x,\xi) := \psi(x, Y(x,\xi), N(x,\xi), \xi). \tag{1.23}$$

*The function $\phi$ is called multi-product of the SG phase functions $\varphi_1,\dots,\varphi_{M+1}$.*

The following properties of the multi-product of *SG* phase functions can be found in [8].

**Proposition 1.23.** *Under the assumptions* (1.17) *and* (1.18)*, the system* (1.22) *admits a unique solution* $(Y,N)$*, satisfying*

$$\{(Y_j - Y_{j-1})/\tau_j\}_{j\in\mathbb{N}} \text{ is bounded in } S^{1,0}(\mathbb{R}^{2d}),$$
$$\{(N_j - N_{j+1})/\tau_{j+1}\}_{j\in\mathbb{N}} \text{ is bounded in } S^{0,1}(\mathbb{R}^{2d}).$$

**Proposition 1.24.** *Under the assumptions* (1.17) *and* (1.18)*, the multi-product $\phi(x,\xi)$ in Definition 1.22 is well-defined for every $M \geqslant 1$ and has the following properties:*

*(1) There exists $k \geqslant 1$ such that $\phi(x,\xi) = (\varphi_1 \sharp \cdots \sharp \varphi_{M+1})(x,\xi) \in \mathcal{P}_r(k\bar{\tau}_{M+1})$ and, setting*

$$J_{M+1}(x,\xi) := (\varphi_1 \sharp \cdots \sharp \varphi_{M+1})(x,\xi) - \langle x,\xi\rangle,$$

*the sequence $\{J_{M+1}/\bar{\tau}_{M+1}\}_{M\geqslant 1}$ is bounded in $S^{1,1}(\mathbb{R}^{2d})$.*

*(2) The following relations hold:*

$$\begin{cases} \phi'_x(x,\xi) = \varphi'_{1,x}(x, N_1(x,\xi)) \\ \phi'_\xi(x,\xi) = \varphi'_{M+1,\xi}(Y_M(x,\xi),\xi), \end{cases}$$

*where $(Y,N)$ are the critical points* (1.22)*.*

*(3) The associative law holds true:*

$$\varphi_1 \sharp (\varphi_2 \sharp \cdots \sharp \varphi_{M+1}) = (\varphi_1 \sharp \cdots \sharp \varphi_M) \sharp \varphi_{M+1}.$$

(4) *For any* $\ell \geqslant 0$ *there exist* $0 < \tau^* < 1/4$ *and* $c^* \geqslant 1$ *such that, if* $\varphi_j \in \mathcal{P}_r(\tau_j, \ell)$ *for all* $j$ *and* $\tau_0 \leqslant \tau^*$, *then* $\phi \in \mathcal{P}_r(c^* \bar{\tau}_{M+1}, \ell)$.

Passing to regular $SG$ Fourier integral operators, one can prove the following algebra properties (cf. [8]).

**Theorem 1.25.** *Let* $\varphi_j \in \mathcal{P}_r(\tau_j)$, $j = 1, 2, \ldots, M$, $M \geqslant 2$, *be such that* $\tau_1 + \cdots + \tau_M \leqslant \tau \leqslant \frac{1}{4}$ *for some sufficiently small* $\tau > 0$, *and set*

$$
\begin{aligned}
\Phi_0(x, \xi) &= \langle x, \xi \rangle, \\
\Phi_1 &= \varphi_1, \\
\Phi_j &= \varphi_1 \sharp \cdots \sharp \varphi_j, \ j = 2, \ldots, M, \\
\Phi_{M,j} &= \varphi_j \sharp \varphi_{j+1} \sharp \cdots \sharp \varphi_M, j = 1, \ldots, M-1, \\
\Phi_{M,M} &= \varphi_M, \\
\Phi_{M,M+1}(x, \xi) &= \langle x, \xi \rangle.
\end{aligned}
$$

*Assume also* $a_j \in S^{m_j, \mu_j}(\mathbb{R}^{2d})$, *and set* $A_j = \operatorname{Op}_{\varphi_j}(a_j)$, $j = 1, \ldots, M$. *Then, the following properties hold true:*

(1) *Given* $q_j, q_{M,j} \in S^{0,0}(\mathbb{R}^{2d})$, $j = 1, \ldots, M$, *such that*

$$
\operatorname{Op}^*_{\Phi_j}(q_j) \circ \operatorname{Op}_{\Phi_j}(1) = I, \quad \operatorname{Op}^*_{\Phi_{M,j}}(1) \circ \operatorname{Op}_{\Phi_{M,j}}(q_{M,j}) = I,
$$

*set* $Q^*_j = \operatorname{Op}^*_{\Phi_j}(q_j)$, $Q_{M,j} = \operatorname{Op}_{\Phi_{M,j}}(q_{M,j})$, *and*

$$
R_j = \operatorname{Op}_{\Phi_{j-1}}(1) \circ A_j \circ Q^*_j, \quad R_{M,j} = Q_{M,j} \circ A_j \circ \operatorname{Op}^*_{\Phi_{M,j+1}}(1), \quad j = 1, \ldots, M.
$$

*Then,* $R_j, R_{M,j} \in \operatorname{Op}(S^{0,0}(\mathbb{R}^{2d}))$, $j = 1, \ldots, M$, *and*

$$
A = A_1 \circ \cdots \circ A_M = R_1 \circ \cdots \circ R_M \circ \operatorname{Op}_{\Phi_M}(1) = \operatorname{Op}^*_{\Phi_{M,1}}(1) \circ R_{M,1} \circ \cdots \circ R_{M,M}.
$$

(2) *There exists* $a \in S^{m,\mu}(\mathbb{R}^{2d})$, $m = m_1 + \cdots + m_M$, $\mu = \mu_1 + \cdots + \mu_M$ *such that, setting* $\phi = \varphi_1 \sharp \cdots \sharp \varphi_M$,

$$
A = A_1 \circ \cdots \circ A_M = \operatorname{Op}_\phi(a).
$$

(3) *For any* $l \in \mathbb{Z}_+$ *there exist* $l' \in \mathbb{Z}_+$, $C_l > 0$ *such that*

$$
|||a|||_l^{m,\mu} \leqslant C_l \prod_{j=1}^{M} |||a_j|||_{l'}^{m_j, \mu_j}.
$$

## 1.5 Eikonal equations and Hamilton-Jacobi systems in $SG$ classes

Given a real-valued symbol $a \in C([0,T]; S^{\epsilon,1}(\mathbb{R}^{2d}))$ with $\epsilon \in [0,1]$, consider the so-called *eikonal equation*

$$\begin{cases} \partial_t \varphi(t,s;x,\xi) & = a(t;x,\varphi'_x(t,s;x,\xi)), \quad t \in [0,T_0], \\ \varphi(s,s;x,\xi) & = \langle x,\xi \rangle, \quad\quad\quad\quad\quad\quad\quad s \in [0,T_0], \end{cases} \tag{1.24}$$

with $0 < T_0 \leqslant T$.

**Remark 1.26.** *Note that the eikonal equation* (1.24) *appears in the so-called geometric optics approach to the solution of* $\mathcal{L}u = f$, $u(0) = u_0$ *for the hyperbolic operator*

$$\mathcal{L} = D_t - a(t;x,D_x) \qquad \text{on } [0,T],$$

*where* $D_t = -i\partial_t$.

We now focus on the Hamilton-Jacobi system corresponding to the real-valued Hamiltonian $a \in C([0,T]; S^{1,1}(\mathbb{R}^{2d}))$, namely,

$$\begin{cases} \partial_t q(t,s;y,\eta) & = -a'_\xi\left(t; q(t,s;y,\eta), p(t,s;y,\eta)\right), \\ \partial_t p(t,s;y,\eta) & = \quad a'_x(t; q(t,s;y,\eta), p(t,s;y,\eta)), \end{cases} \tag{1.25}$$

where $t, s \in [0,T]$, $T > 0$, and the Cauchy data

$$\begin{cases} q(s,s;y,\eta) = y, \\ p(s,s;y,\eta) = \eta. \end{cases} \tag{1.26}$$

We recall how the solution of (1.25), (1.26) is related to solution of (1.24) in the $SG$ context. We mainly refer to known results from [41, Ch. 6] and [43].

**Proposition 1.27.** *Let* $a \in C\left([0,T]; S^{1,1}(\mathbb{R}^{2d})\right)$ *be real-valued. Then, the solutions* $q(t,s;y,\eta)$ *and* $p(t,s;y,\eta)$ *of the Hamilton-Jacobi system* (1.25) *with the Cauchy data* (1.26) *satisfy*

$$\langle q(t,s;y,\eta) \rangle \asymp \langle y \rangle, \qquad \langle p(t,s;y,\eta) \rangle \asymp \langle \eta \rangle. \tag{1.27}$$

**Proposition 1.28.** *Under the same hypotheses of Proposition 1.27, the maximal solution of the Hamilton-Jacobi system* (1.25) *with the Cauchy data* (1.26) *is defined on the whole product of intervals* $[0,T] \times [0,T]$.

**Proposition 1.29.** *The solution* $(q,p)$ *of the Hamilton-Jacobi system* (1.25) *with* $a \in C^\infty([0,T]; S^{1,1}(\mathbb{R}^{2d}))$ *real-valued, and the Cauchy data* (1.26), *satisfies* *(i)* *q belongs to* $C^\infty([0,T]^2; S^{1,0}(\mathbb{R}^{2d}))$.

*(ii)* *p belongs to* $C^\infty([0,T]^2; S^{0,1}(\mathbb{R}^{2d}))$.

**Lemma 1.30.** *The following statements hold true:*

*(i)* *Let* $a \in C([0,T]; S^{1,1}(\mathbb{R}^{2d}))$ *be real-valued. Then, the solution* $(q,p)(t,s; y,\eta)$ *of the Hamilton-Jacobi system* (1.25) *with Cauchy data* (1.26) *satisfies, for a sufficently small* $T_0 \in (0,T]$ *and a fixed* $t$ *such that* $0 \leqslant s,t \leqslant T_0,\ s \neq t$,

$$\begin{cases} (q(t,s;y,\eta)-y)/(t-s) & \text{is bounded in} \quad S^{1,0}(\mathbb{R}^{2d}) \\ (p(t,s;y,\eta)-\eta)/(t-s) & \text{is bounded in} \quad S^{0,1}(\mathbb{R}^{2d}) \end{cases} \tag{1.28}$$

*and*

$$\begin{cases} q(t,s;y,\eta), q(t,s;y,\eta)-y \in C^1(I(T_0); S^{1,0}(\mathbb{R}^{2d})), \\ p(t,s;y,\eta), p(t,s;y,\eta)-\eta \in C^1(I(T_0); S^{0,1}(\mathbb{R}^{2d})), \end{cases} \tag{1.29}$$

*where, for* $T > 0$, $I(T) = \{(t,s) \colon 0 \leqslant t,s \leqslant T\}$.

*(ii)* *Furthermore, if, additionally, a belongs to* $C^\infty([0,T]; S^{1,1}(\mathbb{R}^{2d}))$, *then,* $q(t,s;y,\eta) - y \in C^\infty(I(T_0); S^{1,0}(\mathbb{R}^{2d}))$ *and* $p(t,s;y,\eta) - \eta \in C^\infty(I(T_0); S^{0,1}(\mathbb{R}^{2d}))$.

The proof of Lemma 1.30 combines techniques and results similar to those used in [41], [43] and [86]. For the sake of completeness we prove it.

*Proof.* From the Hamilton-Jacobi system (1.25) one can use the fact that

$$\partial_t q(t,s;y,\eta) = -a'_\xi(t; q(t,s;y,\eta), p(t,s;y,\eta)),$$

where $a(t;x,\xi)$ belongs to $S^{1,1}(\mathbb{R}^{2d})$ for all $t \in [0,T]$. Then it follows that there exists a constant $C > 0$ such that

$$|a'_\xi(t; q(t,s;y,\eta), p(t,s;y,\eta))| \leqslant C\langle q(t,s;y,\eta)\rangle, \text{ for any } t,s \in I(T_0),\ y,\eta \in \mathbb{R}^d.$$

Using that together with the initial condition (1.26) we can write

$$q(t,s;y,\eta) - y = \int_s^t (\partial_t q)(\tau,s;y,\eta)\, d\tau,$$

then we get, for a suitable constant $C > 0$,

$$|q(t,s;y,\eta) - y| \leqslant \int_s^t \langle q(\tau,s;y,\eta)\rangle\, d\tau \leqslant C|t-s|\langle y\rangle,$$

where we also use (1.27).

Similarly, we prove that

$$|p(t,s;y,\eta)-\eta| \leqslant \tilde{C}|t-s|\langle\eta\rangle,$$

for some $\tilde{C}>0$.

Now following [86], let us fix

$$(Q_1,P_1)=\left(\frac{\partial q}{\partial y}(t,s;y,\eta),\frac{\partial p}{\partial y}(t,s;y,\eta)\right)$$

and

$$(Q_2,P_2)=\left(\frac{\partial q}{\partial \eta}(t,s;y,\eta),\frac{\partial p}{\partial \eta}(t,s;y,\eta)\right),$$

that is we consider the case when $|\alpha+\beta|=1$. Then, we can write the derivative of the matrix constructed by the columns $(Q_1,P_1)$ and $(Q_2,P_2)$ as

$$\frac{d}{dt}\begin{pmatrix} Q_1 & Q_2 \\ P_1 & P_2 \end{pmatrix}=\begin{pmatrix} -a''_{x,\xi} & -a''_{\xi,\xi} \\ a''_{x,x} & a''_{x,\xi} \end{pmatrix}\begin{pmatrix} Q_1 & Q_2 \\ P_1 & P_2 \end{pmatrix}, \tag{1.30}$$

with Cauchy data at $t=s$

$$\begin{pmatrix} Q_1 & Q_2 \\ P_1 & P_2 \end{pmatrix}_{|t=s}=\begin{pmatrix} I_d & 0 \\ 0 & I_d \end{pmatrix},\ \text{with } I_d \text{ the identity matrix of size } d. \tag{1.31}$$

Then, integrating the left hand side of (1.30) between $s$ and $t$, using its Cauchy data (1.31), we get the following block matrix

$$\begin{pmatrix} \frac{\partial q}{\partial y}(t,s;y,\eta)-I_d & \frac{\partial q}{\partial \eta}(t,s;y,\eta) \\ \frac{\partial p}{\partial y}(t,s;y,\eta) & \frac{\partial p}{\partial \eta}(t,s;y,\eta)-I_d \end{pmatrix}.$$

Using the fact that

$$\frac{\partial q}{\partial y}(t,s;y,\eta)-I_d=\frac{\partial}{\partial y}\left(\int_s^t \partial_t q(\tau,s;y,\eta)\,d\tau\right),$$

by an induction argument on the derivatives, the results hold true.

By (1.25), Proposition 1.29 and (1.30), together with the Cauchy data (1.31), one can get the next estimate

$$\left\|\int_s^t \frac{\partial}{\partial y}(-a'_\xi(\tau,q(\tau,s;y,\eta),p(\tau,s;y,\eta))d\tau\right\| =$$

$$\left\|\int_s^t \frac{\partial q}{\partial y}(\tau,s)\cdot(-a''_{x,\xi})(\tau,q(\tau,s),p(\tau,s))+\frac{\partial p}{\partial y}(\tau,s)\cdot(-a''_{\xi,\xi})(\tau,q(\tau,s),p(\tau,s))\,d\tau\right\|$$

$$\leqslant C_1|t-s|,$$

with $C_1 > 0$. It follows

$$\left\| \frac{\partial q}{\partial y}(t,s;y,\eta) - I_d \right\| \leqslant C_1|t-s|,$$

and

$$\langle y\rangle^{-1}\langle\eta\rangle \left\| \frac{\partial q}{\partial \eta}(t,s;y,\eta) \right\| \leqslant C_2|t-s|.$$

Using the same argument as before and Proposition 1.27 we conclude that

$$\begin{aligned}\left\| \frac{\partial q}{\partial \eta}(t,s;y,\eta) \right\| &= \left\| \int_s^t \frac{\partial q}{\partial \eta}(-a''_{x,\xi}) + \frac{\partial p}{\partial \eta}(-a''_{\xi,\xi})\, d\tau \right\| \\ &\leqslant \tilde{C}_2|t-s| \left( \langle y\rangle\langle\eta\rangle^{-1} + \langle p\rangle\langle q\rangle^{-1} \right) \leqslant C_2|t-s|\langle y\rangle\langle\eta\rangle^{-1}.\end{aligned}$$

Same steps as before give as the following inequalities

$$\left\| \frac{\partial p}{\partial \eta}(t,s) - I_d \right\| \leqslant C_1|t-s|,$$
$$\left\| \frac{\partial p}{\partial y}(t,s) \right\| \leqslant C_2|t-s|\langle y\rangle^{-1}\langle\eta\rangle.$$

Thus, there exists a constant $C_3 > 0$ such that

$$\left\| \frac{\partial q}{\partial y} - I_d \right\| + \langle y\rangle^{-1}\langle\eta\rangle \left\| \frac{\partial q}{\partial \eta}(t,s) \right\| \leqslant C_3|t-s|,$$
$$\langle y\rangle\langle\eta\rangle^{-1} \left\| \frac{\partial p}{\partial y}(t,s) \right\| + \left\| \frac{\partial p}{\partial \eta}(t,s) - I_d \right\| \leqslant C_3|t-s|.$$

So far, we proved that our statement is true for $|\alpha+\beta| = 1$. Now, assume that it holds true up to $|\alpha+\beta| = r$ with $r \geqslant 2$. By induction, one can conclude. In fact for $|\alpha+\beta| \geqslant 2$, we get $\partial_y^\beta \partial_\eta^\alpha q_{|t=s} = \partial_y^\beta \partial_\eta^\alpha p_{|t=s} = 0$, then we have

$$\partial_y^\beta \partial_\eta^\alpha q(t,s) = \int_s^t \partial_y^\beta \partial_\eta^\alpha \left( -a'_\xi(\tau, q(\tau,s), p(\tau,s)) \right) d\tau,$$

where the derivatives $\partial_y^\beta \partial_\eta^\alpha \Big( -a'_\xi(\tau, q(\tau,s), p(\tau,s)) \Big)$ in the span of terms of the type

$$\Big( \partial_x^\sigma \partial_\xi^\gamma \left(-a'_\xi\right)(\tau, q(\tau,s), p(\tau,s)) \Big) \prod_{j=1}^{|\sigma|} \partial_y^{\beta_j} \partial_\eta^{\alpha_j} q(\tau,s) \prod_{i=1}^{|\gamma|} \partial_y^{\beta'_i} \partial_\eta^{\alpha'_i} p(\tau,s), \quad (1.32)$$

where $|\alpha_j + \beta_j| \geqslant 1$, $|\alpha'_j + \beta'_j| \geqslant 1$, $\sum \alpha_j + \sum \alpha'_i = \alpha$, $\sum \beta_j + \sum \beta'_i = \beta$. The tensors $\partial_x^\sigma \partial_\xi^\gamma \Big(-a'_\xi\Big)$ in (1.32) and vectors $\partial_y^{\beta_j} \partial_\eta^{\alpha_j} q(\tau,s)$, $\partial_y^{\beta'_i} \partial_\eta^{\alpha'_i} p(\tau,s)$ are to be contracted in arbitrary order, $q$ with $\partial_x$, $p$ with $\partial_\xi$.

Differentiating (1.32) with respect to $y$ or $\eta$, such that to order of the differentiation is equal to 1 and using our assumption for $|\alpha+\beta| = r \geqslant 2$, we conclude the proof of (1.28), the argument is similar to the one used to show the second part of (1.28), i.e. the boundedness of $\partial_y^\beta \partial_\eta^\alpha (p-\eta)/(t-s)$ for $|\alpha+\beta| \geqslant 2$.

Now, we prove (1.29). Indeed, for any $(t,s) \in I(T_0)$, if we write

$$\begin{aligned} \partial_s q(t,s) &= a'_\xi(s,y,\eta) - \int_s^t \{\partial_s q(\tau,s)(a''_{x,\xi}) + \partial_s p(\tau,s)(a''_{\xi,\xi})\} d\tau, \\ \partial_s p(t,s) &= -a'_x(s,y,\eta) + \int_s^t \{\partial_s q(\tau,s)(a''_{x,x}) + \partial_s p(\tau,s)(a''_{x,\xi})\} d\tau, \end{aligned} \tag{1.33}$$

from the Hamilton-Jacobi system (1.25) and (1.33), one can write $\partial_y^\beta \partial_\eta^\alpha \big(\partial_s q(t,s)\big)$ for $|\alpha+\beta| = 1$ as

$$\begin{aligned} \partial_y^\beta \partial_\eta^\alpha \left(\partial_s q(t,s)\right) = \partial_x^\beta \partial_\xi^\alpha a'_\xi(s,y,\eta) + \int_s^t & \partial_y^\beta \partial_\eta^\alpha \partial_s q(\tau,s) a''_{x,\xi} + \partial_s q \partial_y^\beta \partial_\eta^\alpha q \frac{\partial}{\partial x} a''_{x,\xi} \\ & + \partial_s q(\tau,s) \partial_y^\beta \partial_\eta^\alpha p(\tau,s) \frac{\partial}{\partial \xi} a''_{x,\xi} + \partial_y^\beta \partial_\eta^\alpha \partial_s p(\tau,s) a''_{\xi,\xi} \\ & + \partial_s \partial_y^\beta \partial_\eta^\alpha q(\tau,s) \frac{\partial}{\partial x} a''_{\xi,\xi} + \partial_s p \partial_y^\beta \partial_\eta^\alpha p(\tau,s) \frac{\partial}{\partial \xi} a''_{\xi,\xi}\, d\tau. \end{aligned}$$

Then, by Proposition 1.27 and Proposition 1.29, we can conclude

$$\left| \partial_y^\beta \partial_\eta^\alpha \left(\partial_s q(t,s)\right) \right| \leqslant C_{1_{\alpha,\beta}} \langle y \rangle^{1-|\beta|} \langle \eta \rangle^{-|\alpha|} + C_{2_{\alpha,\beta}} |t-s| \langle y \rangle^{1-|\beta|} \langle \eta \rangle^{-|\alpha|},$$

where $C_{1_{\alpha,\beta}} C_{2_{\alpha,\beta}} > 0$. This implies that (1.29) holds true for $|\alpha+\beta| = 1$, then by iteration we conclude the proof for any order of $\alpha$ and $\beta$. Similar proof holds for the proof of the second part of (1.29), and (ii) holds by induction after the use of (i) and (1.29). □

Now, we observe that there exists a constant $T_1 \in (0, T_0]$ such that $q(t,s;y,\eta)$ is invertible with respect to $y$ for any $(t,s) \in I(T_1)$ and any $\eta \in \mathbb{R}^d$. Indeed, this holds by continuity and the fact that

$$q(s,s;y,\eta) = y \Rightarrow \frac{\partial q}{\partial y}(s,s;y,\eta) = I_d.$$

We denote the inverse function by $\overline{q}$, that is

$$y = \overline{q}(t,s) = \overline{q}(t,s;x,\eta) \Leftrightarrow x = q(t,s;y,\eta),$$

which exists on $I(T_1)$. Moreover, $\overline{q} \in C^\infty(I(T_1); S^{1,0}(\mathbb{R}^{2d}))$, cf. [41, 43].

Observe now that, in view of Lemma 1.30, $\|\frac{\partial q}{\partial y} - I_d\| \to 0$ when $t \to s$, uniformly on $I(T_1)$. Then, one can deduce the following result, which is an extension to the analogous ones that can be found e.g., in [41, 86].

**Lemma 1.31.** *Assume that* $a \in C([0,T]; S^{1,1}(\mathbb{R}^{2d}))$ *is real-valued, and let* $T_1 \in (0, T_0]$, $\epsilon_1 \in (0,1]$ *be constants such that on* $I(T_1)$ *we have*

$$\left\| \frac{\partial q}{\partial y} - I_d \right\| \leqslant 1 - \epsilon_1. \tag{1.34}$$

*Then, the mapping* $x = q(t,s;y,\xi) : \mathbb{R}^d_y \ni y \longmapsto x \in \mathbb{R}^d_x$ *with* $(t,s,\xi)$ *understood as parameter, has the inverse function* $y = \overline{q}(t,s;x,\xi)$ *satisfying*

$$\begin{cases} \overline{q}(t,s;x,\xi) - x \text{ belongs to } C^1(I(T_1); S^{1,0}(\mathbb{R}^{2d})), \\ \{(\overline{q}(t,s;x,\xi) - x)/|t-s|\} \text{ is bounded in } S^{1,0}(\mathbb{R}^{2d}), 0 \leqslant s,t \leqslant T_1; s \neq t. \end{cases} \tag{1.35}$$

*Moreover, if, additionally,* $a \in C^\infty([0,T]; S^{1,1}(\mathbb{R}^{2d}))$*, we also have* $\overline{q}(t,s;x,\xi)$, $(\overline{q}(t,s;x,\xi) - x) \in C^\infty(I(T_1); S^{1,0}(\mathbb{R}^{2d}))$.

*Proof.* Setting $F(x,y) = x + y - q(t,s;y,\xi)$, it follows $F(x,\cdot) : \mathbb{R}^d \ni y \longmapsto F(x,y) \in \mathbb{R}^d$ is a contracting map for any $x \in \mathbb{R}^d$, by (1.34). Then, the inverse $y = \overline{q}(t,s;x,\xi)$ in uniquely determined as the fixed point and it has the same regularity as $q$. The boundedness of the inverse function claimed in (1.35) is an immediate consequence of (1.28) and (1.29). All other claims follow by similar arguments. □

**Proposition 1.32.** *Let* $a \in C([0,T]; S^{1,1}(\mathbb{R}^{2d}))$ *be real-valued,* $q(t,s;y,\eta)$, $p(t,s;y,\eta)$ *and* $\overline{q}(t,s;x,\xi)$ *be the symbols constructed in the previous Lemmas 1.30 and 1.31. We define* $u(t,s;y,\eta)$ *by*

$$\begin{aligned} u(t,s;y,\eta) = \langle y,\eta\rangle + \int_s^t \Big( & a(\tau; q(\tau,s;y,\eta), p(\tau,s;y,\eta)) \\ & - \langle a'_\xi(\tau; q(\tau,s;y,\eta), p(\tau,s;y,\eta)), p(\tau,s;y,\eta)\rangle \Big)\, d\tau, \end{aligned} \tag{1.36}$$

*and*

$$\varphi(t,s;x,\xi) = u(t,s;\overline{q}(t,s;x,\xi),\xi). \tag{1.37}$$

*Then,* $\varphi(t,s;x,\xi)$ *is a solution of the eikonal equation* (1.24) *and satisfies*

$$\varphi'_\xi(t,s;x,\xi) = \overline{q}(t,s;x,\xi), \tag{1.38}$$

$$\varphi'_x(t,s;x,\xi) = p(t,s;\overline{q}(t,s;x,\xi),\xi), \tag{1.39}$$

$$\partial_s\varphi(t,s;x,\xi) = -a(s;\varphi'_\xi(t,s;x,\xi),\xi), \tag{1.40}$$

$$\langle \varphi'_x(t,s;x,\xi)\rangle \asymp \langle\xi\rangle \text{ and } \langle\varphi'_\xi(t,s;x,\xi)\rangle \asymp \langle x\rangle. \tag{1.41}$$

*Moreover, for any* $l \geqslant 0$ *there exists a constant* $c_l \geqslant 1$ *and* $T_l \in (0, T_1]$ *such that* $c_l T_l < 1$, $\varphi(t,s;x,\xi)$ *belongs to* $\mathcal{P}_r(c_l|t-s|, l)$ *and* $\{J(t,s)/|t-s|\}$ *is bounded in* $S^{1,1}(\mathbb{R}^{2d})$ *for* $0 \leqslant t,s \leqslant T_l \leqslant T_1$, $t \neq s$, *where* $J(t,s;x,\xi) = \varphi(t,s;x,\xi) - \langle x,\xi\rangle$.

*Proof.* The function $\varphi$ defined in (1.37) is the solution of (1.24) with its Cauchy data. In fact using the definition of $x$ and the Cauchy data of the Hamilton-Jacobi system (1.26), we get

$$q(s,s;y,\xi)=y\Leftrightarrow \overline{q}(s,s;x,\xi)=x$$

which implies that $\varphi(s,s;x,\xi)=\langle x,\xi\rangle$ is the desired Cauchy data for the eikonal equation (1.24). Now, we have to check that $\varphi$ satisfies,

$$\partial_t\varphi(t,s;x,\xi)=a(t;x,\varphi'_x(t,s;x,\xi)). \tag{1.42}$$

Following [43], let $u$ be defined as in (1.36). Then, (1.37) is equivalent to

$$\varphi(t,s;q(t,s;y,\xi),\xi)=u(t,s;y,\xi). \tag{1.43}$$

Let

$$\psi(t,s;y,\xi)=u'_y(t,s;y,\xi)-\frac{\partial q}{\partial y}(t,s;y,\xi)\cdot p(t,s;y,\xi),$$

where the last part of the right hand side is a product between a matrix and a vector. Hence

$$\psi(s,s;y,\xi)=\xi-I_d\cdot\xi=0.$$

Recalling that $\varphi$ solves (1.42) and $(q,p)$ is solution of the Hamilton-Jacobi system and (1.39) holds true. First, we observe that

$$\begin{aligned}\frac{\partial u}{\partial t}(t,s;y,\xi)&=(\partial_t\varphi)(t,s;q(t,s;y,\xi))+\varphi'_x(t,s;q(t,s;y,\xi))\partial_t q(t,s;y,\xi)\\&=\partial_t\varphi(t,s;q(t,s;y,\xi))+p(t,s;y,\xi)\cdot(-a'_\xi)(t,q(t,s;y,\xi),p(t,s;y,\xi)).\end{aligned}$$

Moreover,

$$\begin{aligned}\partial_t\Psi&=\partial_y(\partial_t u)-\partial_t(\frac{\partial q}{\partial y}\cdot p)\\&=\frac{\partial}{\partial y}\big(a(t,q(t,s;y,\xi),p(t,s;y,\xi))-\langle a'_\xi(t,q(t,s;y,\xi),p(t,s;y,\xi)),p\rangle\big)\\&\qquad-\frac{\partial\left(\partial_t q\right)}{\partial y}\cdot p-\frac{\partial q}{\partial y}\cdot\partial_t p\\&=\frac{\partial q}{\partial y}\cdot a'_x+a'_\xi\cdot\frac{\partial p}{\partial y}+\frac{\partial\left(a'_\xi\right)}{\partial y}\cdot p-a'_\xi\cdot\frac{\partial p}{\partial y}-\frac{\partial\left(a'_\xi\right)}{\partial y}\cdot p-\frac{\partial q}{\partial y}\cdot a'_x\\&=0.\end{aligned}$$

Therefore, we have $\psi(t,s;y,\xi)=0$ for all $(t,s;y,\xi)\in I(T_1)\times\mathbb{R}^{2d}$, which implies that

$$u'_y(t,s;y,\xi)=\frac{\partial q}{\partial y}(t,s;y,\xi)\cdot p(t,s;y,\xi). \tag{1.44}$$

Moreover, by (1.43) we have

$$u'_y(t,s;y,\xi) = \frac{\partial q}{\partial y}(t,s;y,\xi)\cdot\varphi'_x(t,s;q(t,s;y,\xi),\xi), \tag{1.45}$$

and by (1.44), (1.45) and the fact that $q$ is invertible in $I(T_1)$, it follows that

$$p(t,s;y,\xi) = \varphi'_x(t,s;q(t,s;y,\xi),\xi). \tag{1.46}$$

Differentiating (1.43) with respect to $t$, and inserting (1.25), (1.36) and (1.46), it follows that $\varphi$ indeed satisfies (1.24).

Equality (1.46) implies (1.39), then in order to prove (1.38) we use (1.36) and modify the function $\psi$ as follows:

$$\psi(t,s;y,\xi) = u'_\xi(t,s;y,\xi) - \frac{\partial q}{\partial \xi}(t,s;y,\xi)\cdot p(t,s;y,\xi).$$

Obviously $\psi(s,s;y,\xi) = y$, and derivative with respect to $t$ shows that $\psi$ is constant. Then

$$u'_\xi(t,s;y,\xi) - \frac{\partial q}{\partial \xi}(t,s;y,\xi)\cdot p(t,s;y,\xi) = y.$$

Moreover, by the equality (1.39) and the derivative of (1.43) with respect to $\xi$, we get (1.38).

We can show that (1.40) holds true, after showing the independence of $\partial_s\varphi$ from $t$. Indeed using (1.25) and (1.39), we get

$$\begin{aligned}
&\partial_t\,(\partial_s\varphi)(t,s;q(t,s),\xi)\\
&\qquad= (\partial_t\partial_s\varphi)\,(t,s;q(t,s),\xi) + \big\langle \big(\partial_s\varphi'_x\big)\,(t,s;q(t,s),\xi), \partial_t q(t,s)\big\rangle\\
&\qquad= (\partial_t\partial_s\varphi)\,(t,s;q(t,s),\xi) - \big\langle \big(\partial_s\varphi'_x\big)\,(t,s;q(t,s),\xi), a'_\xi(t;q(t,s),p(t,s))\big\rangle\\
&\qquad= \partial_s\Big(\partial_t\varphi(t,s;z,\xi) - a(t;z,\varphi'_x(t,s;z,\xi)\Big)\Big|_{z=q(t,s;y,\xi)}\\
&\qquad= 0.
\end{aligned}$$

The last equality follows by (1.24). Moreover, (1.24), (1.38) and the independence of $(\partial_s\varphi)$ from the variable $t$, together with the observation $x = q(t,s;y,\xi)\big|_{t=s} = y$, imply

$$\begin{aligned}
\partial_s\varphi(t,s;x,\xi) &= (\partial_s\varphi)\,(s,s;q(s,s),\xi) = (\partial_s\varphi)\,(s,s;y,\xi)\\
&= \{\partial_t[\varphi(t,t;y,\xi)] - (\partial_t\varphi)\,(t,t;y,\xi)\}\,\big|_{t=s}\\
&= \big\{\partial_t(\langle y,\xi\rangle) - a(s;y,\varphi'_x(s,s;y,\xi))\big\}\\
&= -a(s;y,\xi) = -a(s;\varphi'_\xi(t,s;x,\xi),\xi),
\end{aligned}$$

which concludes the proof of (1.40).

With the aim of determining the class of the function $\varphi$ cited in (1.37), the next lemma is an adapted version from Lemma 2.3 in [43].

Using Lemma 1.2 and recalling that $a \in C([0,T];S^{1,1})$ is real-valued, $q \in C^\infty(I(T_1);S^{1,0})$, $p \in C^\infty(I(T_1);S^{0,1}), \overline{q} \in C^\infty(I(T_1);S^{1,0})$, as defined above, we conclude that $\varphi$ belongs to $C^1(I(T_1);S^{1,1})$ and it is a real-valued function. Concerning the regularity condition, we have $\varphi''_{x,\xi}\big|_{t=s} = I_d$. Then by a continuity argument, one can possibly decreasing $T_1$, obtain (1.47) for all $t,s \in I(T_1)$, $x,\xi \in \mathbb{R}^d$

$$1/2 \leqslant \det\left(\varphi''_{x,\xi}\right) \leqslant 3/2. \tag{1.47}$$

In view of Proposition 1.27, (1.38) and (1.39); we get the equivalences (1.41). Now, for $0 \leqslant c_0|t-s| < 1$, we show the following:

$$\sup_{\substack{x,\xi\in\mathbb{R}^d\\ |\alpha+\beta|\leqslant 2}} \frac{\left|D_\xi^\alpha D_x^\beta J(t,s;x,\xi)\right|}{\langle x\rangle^{1-|\beta|}\langle\xi\rangle^{1-|\alpha|}} \leqslant c_0|t-s|. \tag{1.48}$$

Here, as usual, we can restrict $I(T_1)$ to obtain $c_0|t-s| < 1$, so that, by (1.37) and (1.36), we get

$$J(t,s;x,\xi) = \int_s^t a(\tau;x,p(\tau;s;\overline{q}(\tau,s),\xi)) - \langle a'_\xi(\tau;x,p(\tau,s;\overline{q}(\tau,s),\xi)), p(\tau,s;\overline{q}(\tau,s),\xi)\rangle\, d\tau. \tag{1.49}$$

Then, observing that

$$|D_\xi^\alpha D_x^\beta J(t,s;x,\xi)| = \Big|\int_s^t \left(D_\xi^\alpha D_x^\beta a - \sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \beta_1+\beta_2=\beta}} \frac{\alpha!}{\alpha_1!\alpha_2!}\frac{\beta!}{\beta_1!\beta_2!}\langle D_\xi^{\alpha_1}D_x^{\beta_1}a'_\xi, D_\xi^{\alpha_2}D_x^{\beta_2}p\rangle\right) d\tau\Big|,$$

and for $|\alpha+\beta| \leqslant 2$, Lemma 1.2 and Proposition 1.27 imply that the integrand is a *SG* symbol of order $(1,1)$. This implies the desired estimate (1.48). Therefore, $\varphi(t,s) \in \mathcal{P}_r(c_0|t-s|)$ where $c_0$ depends on the semi-norms of $a$. Next we show that $\varphi(t,s) \in \mathcal{P}_r(c_l|t-s|,l)$. In fact, using (1.49) in addition both $a$ and $\langle a'_\xi, p\rangle$ are in $S^{1,1}$, thus imply that for any $l \geqslant 0$ there exists a constant $c_l \geqslant 1$ and $T_l \in [0,T_1]$ such that for $|\alpha+\beta| \leqslant 2+l$, the next estimate holds for $(t,s) \in I(T_l)$

$$\frac{\left|D_\xi^\alpha D_x^\beta J(t,s;x,\xi)\right|}{\langle x\rangle^{1-|\beta|}\langle\xi\rangle^{1-|\alpha|}} \leqslant c_l|t-s|.$$

The above estimate confirms also that $J(t,s)/|t-s|$ bounded in $S^{1,1}$. □

As a simple realization of a sequence of phase functions satisfying (1.17) and (1.18), we recall the following example, see [8] and [86].

**Example 1.33.** *Let $\varphi(t,s;x,\xi)$ be the solution of the eikonal equation* (1.24) *with $\varepsilon = 1$. Choose the partition $\Delta_{M+1}(T_1) \equiv \Delta_{T_1}$ of the interval $[s,t]$, $0 \leqslant s \leqslant t \leqslant T_1$, given by*

$$s = t_{M+1} \leqslant t_M \leqslant \cdots \leqslant t_1 \leqslant t_0 = t,$$

*and define the sequence of phase functions*

$$\chi_j(x,\xi) = \begin{cases} \varphi(t_{j-1},t_j;x,\xi) & 1 \leqslant j \leqslant M+1, \\ \langle x,\xi \rangle & j \geqslant M+2. \end{cases}$$

*From Proposition 1.32 we know that $\chi_j \in \mathcal{P}_r(\tau_j)$ with $\tau_j = c_0(t_{j-1} - t_j)$ for $1 \leqslant j \leqslant M+1$ and with $\tau_j = 0$ for $j \geqslant M+2$. Condition* (1.18) *is fulfilled by the choice of a small enough positive constant $T_1$, since*

$$\sum_{j=1}^{\infty} \tau_j = \sum_{j=1}^{M+1} c_0(t_{j-1} - t_j) = c_0(t-s) \leqslant c_0 T_1 < \frac{1}{4}$$

*if $T_1 < (4c_0)^{-1}$. Moreover, from Proposition 1.32, we know that $\|J_j\|_{2,0} \leqslant c_0|t_j - t_{j-1}| = \tau_j$ for all $1 \leqslant j \leqslant M+1$ and $J_j = 0$ for $j \geqslant M+2$, so each one of the $J_j$ satisfies* (1.17).

**Corollary 1.34.** *Let $a \in C([0,T];S^{1,1}(\mathbb{R}^{2d}))$ be real-valued, and let $q$, $p$ and $\overline{q}$ be the symbols constructed in Lemma 1.30 and Lemma 1.31, respectively. Then, $J \in C^1(I(T_1);S^{1,1}(\mathbb{R}^{2d}))$. Moreover, if, additionally, $a \in C^\infty([0,T];S^{1,1}(\mathbb{R}^{2d}))$, we find $J \in C^\infty(I(T_1);S^{1,1}(\mathbb{R}^{2d}))$.*

## 1.6 Classical symbols of SG type

In the last chapter of Part I we will focus on the subclass of symbols and operators which are SG-classical, that is, $a \in S^{m,\mu}_{\mathrm{cl}}(\mathbb{R}^d) \subset S^{m,\mu}(\mathbb{R}^d)$. In this section we summarize some of their main properties, using materials coming from [11] (see, e. g., [52] for additional details and proofs).

**Definition 1.35.** *i) A symbol $a(x,\xi)$ belongs to the class $SG^{m,\mu}_{\mathrm{cl}(\xi)}(\mathbb{R}^d)$ if there exist $a_{m-i,\cdot}(x,\xi) \in \widetilde{\mathscr{H}}^{m-i}_{\xi}(\mathbb{R}^d)$, $i = 0,1,\ldots$, homogeneous functions of order $m-i$ with respect to the variable $\xi$, smooth with respect to the variable $x$, such that, for a 0-excision function $\omega$,*

$$a(x,\xi) - \sum_{i=0}^{N-1} \omega(\xi)\, a_{m-i,\cdot}(x,\xi) \in S^{m-N,\mu}(\mathbb{R}^d), \quad N = 1,2,\ldots;$$

*ii)* *A symbol* $a(x,\xi)$ *belongs to the class* $SG^{m,\mu}_{\mathrm{cl}(x)}(\mathbb{R}^d)$ *if there exist* $a_{\cdot,\mu-k}(x,\xi) \in \widetilde{\mathscr{H}}^{\mu-k}_x(\mathbb{R}^d)$, $k = 0, \ldots,$ *homogeneous functions of order* $\mu - k$ *with respect to the variable* $x$, *smooth with respect to the variable* $\xi$, *such that, for a* 0*-excision function* $\omega$,

$$a(x,\xi) - \sum_{k=0}^{N-1} \omega(x)\, a_{\cdot,\mu-k}(x,\xi) \in S^{m,\mu-N}(\mathbb{R}^d), \quad N = 1, 2, \ldots$$

**Definition 1.36.** *A symbol* $a(x,\xi)$ *is SG-classical, and we write* $a \in S^{m,\mu}_{\mathrm{cl}(x,\xi)}(\mathbb{R}^d) = S^{m,\mu}_{\mathrm{cl}}(\mathbb{R}^d) = S^{m,\mu}_{\mathrm{cl}}$, *if*

*i)* *there exist* $a_{m-j,\cdot}(x,\xi) \in \widetilde{\mathscr{H}}^{m-j}_\xi(\mathbb{R}^d)$ *such that, for a* 0*-excision function* $\omega$, $\omega(\xi)\, a_{m-j,\cdot}(x,\xi) \in S^{m-j,\mu}_{\mathrm{cl}(x)}(\mathbb{R}^d)$ *and*

$$a(x,\xi) - \sum_{j=0}^{N-1} \omega(\xi)\, a_{m-j,\cdot}(x,\xi) \in S^{m-N,\mu}(\mathbb{R}^d), \quad N = 1, 2, \ldots;$$

*ii)* *there exist* $a_{\cdot,\mu-k}(x,\xi) \in \widetilde{\mathscr{H}}^{\mu-k}_x(\mathbb{R}^d)$ *such that, for a* 0*-excision function* $\omega$, $\omega(x)\, a_{\cdot,\mu-k}(x,\xi) \in S^{m,\mu-k}_{\mathrm{cl}(\xi)}(\mathbb{R}^d)$ *and*

$$a(x,\xi) - \sum_{k=0}^{N-1} \omega(x)\, a_{\cdot,\mu-k} \in S^{m,\mu-N}(\mathbb{R}^d), \quad N = 1, 2, \ldots.$$

*We set* $L^{m,\mu}_{\mathrm{cl}(x,\xi)}(\mathbb{R}^d) = L^{m,\mu}_{\mathrm{cl}}(\mathbb{R}^d) = \mathrm{Op}(S^{m,\mu}_{\mathrm{cl}}(\mathbb{R}^{2d}))$.

The next two results are especially useful when dealing with $SG$-classical symbols.

**Theorem 1.37.** *Let* $a_k \in S^{m-k,\mu-k}_{\mathrm{cl}}(\mathbb{R}^{2d})$, $k = 0, 1, \ldots,$ *be a sequence of SG-classical symbols and* $a \asymp \sum_{k=0}^{\infty} a_k$ *its asymptotic sum in the general SG-calculus. Then,* $a \in S^{m,\mu}_{\mathrm{cl}}(\mathbb{R}^{2d})$.

**Theorem 1.38.** *Let* $\mathbb{B}^d = \{x \in \mathbb{R}^d : |x| \leqslant 1\}$ *and let* $\chi$ *be a diffeomorphism from the interior of* $\mathbb{B}^d$ *to* $\mathbb{R}^d$ *such that*

$$\chi(x) = \frac{x}{|x|(1-|x|)} \quad \text{for} \quad |x| > 2/3.$$

*Choosing a smooth function* $[x]$ *on* $\mathbb{R}^d$ *such that* $1 - [x] \neq 0$ *for all* $x$ *in the interior of* $\mathbb{B}^d$ *and* $|x| > 2/3 \Rightarrow [x] = |x|$, *for any* $a \in SG^{m,\mu}_{\mathrm{cl}}(\mathbb{R}^{2d})$ *denote by* $(D^{\mathbf{m}}a)(y,\eta)$, $\mathbf{m} = (m,\mu)$, *the function*

$$b(y,\eta) = (1 - [\eta])^{m_1}(1 - [y])^{m_2} a(\chi(y), \chi(\eta)).$$

*Then,* $D^{\mathbf{m}}$ *extends to a homeomorphism from* $S^{m,\mu}_{\mathrm{cl}}(\mathbb{R}^{2d})$ *to* $C^\infty(\mathbb{B}^d \times \mathbb{B}^d)$.

Note that the definition of SG-classical symbol implies a condition of compatibility for the terms of the expansions with respect to $x$ and $\xi$. In fact, defining $\sigma_\psi^{m-j}$ and $\sigma_e^{\mu-i}$ on $S^{m,\mu}_{\mathrm{cl}(\xi)}$ and $S^{m,\mu}_{\mathrm{cl}(x)}$, respectively, as

$$\begin{aligned}\sigma_\psi^{m-j}(a)(x,\xi) &= a_{m-j,\cdot}(x,\xi), \quad j=0,1,\ldots,\\ \sigma_e^{\mu-i}(a)(x,\xi) &= a_{\cdot,\mu-i}(x,\xi), \quad i=0,1,\ldots,\end{aligned}$$

it is possible to prove that for the homogeneous components $a_{m-j,\mu-i} = \sigma_{\psi e}^{m-j,\mu-i}(a)$ one finds

$$\begin{aligned}a_{m-j,\mu-i} = \sigma_\psi^{m-j}(\sigma_e^{\mu-i}(a)) = \sigma_e^{\mu-i}(\sigma_\psi^{m-j}(a)),&\\ j=0,1,\ldots,\ i=0,1,\ldots.&\end{aligned}$$

Moreover, the algebra property of $SG$-operators and Theorem 1.37 imply that the composition of two $SG$-classical operators is still classical. For $A = \mathrm{Op}(a) \in L^{m,\mu}_{\mathrm{cl}}$ the triple $\sigma(A) = (\sigma_\psi(A), \sigma_e(A), \sigma_{\psi e}(A)) = (a_{m,\cdot}\,,\, a_{\cdot,\mu}\,,\, a_{m,\mu})$ is called the *principal symbol of* $A$. This definition keeps the usual multiplicative behaviour, that is, for any $A \in L^{r,\rho}_{\mathrm{cl}}$, $B \in L^{s,\sigma}_{\mathrm{cl}}$, $r,\rho,s,\sigma \in \mathbb{R}$, $\sigma(AB) = \sigma(A)\,\sigma(B)$, with componentwise product in the right-hand side. We also set

$$\begin{aligned}\sigma_p\,(A)\,(x,\xi) = {}& \sigma_p\,(a)\,(x,\xi) =\\ = {}& a_{\mathbf{m}}(x,\xi) = \omega(\xi)a_{m,\cdot}(x,\xi) + \omega(x)(a_{\cdot,\mu}(x,\xi) - \omega(\xi)a_{m,\mu}(x,\xi))\end{aligned}$$

for a 0-excision function $\omega$. Theorem 1.39 below allows to express the ellipticity of $SG$-classical operators in terms of their principal symbol.

**Theorem 1.39.** *An operator $A \in L^{m,\mu}_{\mathrm{cl}}(\mathbb{R}^{2d})$ is elliptic if and only if each element of the triple $\sigma(A)$ is everywhere non-vanishing on its domain of definition.*

**Remark 1.40.** *The composition results in the previous Section 1.3 have classical counterparts. Namely, when all the involved starting elements are SG-classical, the resulting objects (multi-products, amplitudes, etc.) are SG-classical as well.*

# Chapter 2

# Commutative law for multi-products of SG phase functions

In this chapter our aim is to prove, under suitable hypotheses, the commutative law for multi-product of regular *SG* phase functions. Through this result, we further expand the theory of *SG* Fourier integral operators. In particular we will be able to apply it to obtain the solution of Cauchy problems for weakly hyperbolic linear differential operators, with polynomially bounded coefficients, and involutive characteristics. Notice that roots of constant multiplicities are always involutive, the converse is not true in general see, e. g., [9, 41]. An example of operator with involutive roots having variable multiplicities can be found, e. g., in [10, 94].

More precisely, we focus on the $\sharp$-product of regular *SG* phase functions obtained as solutions to eikonal equations. Namely, let $[\varphi_j(t,s)](x,\xi) = \varphi_j(t,s;x,\xi)$ be the phase functions defined by the eikonal equations (1.24), with $\varphi_j$ in place of $\varphi$ and $a_j$ in place of $a$, where $a_j \in C([0,T];S^{1,1})$, $a_j$ real-valued, $j \in \mathbb{N}$. Moreover, let $I_{\varphi_j}(t,s) = I_{\varphi_j(t,s)} = \mathrm{Op}_{\varphi_j(t,s)}(1)$ be the *SG* Fourier integral operator with phase function $\varphi_j(t,s)$ and symbol identically equal to 1.

Assume that $\{a_j\}_{j\in\mathbb{N}}$ is bounded in $C([0,T];S^{1,1})$. Then, by Proposition 1.32, there exists a constant $c$, independent of $j$, such that

$$\varphi_j(t,s) \in \mathcal{P}_r(c|t-s|),\ j \in \mathbb{N}.$$

**Definition 2.1.** *We make a choice of* $T_1$*, once and for all, such that*

$$cT_1 \leqslant \tau_0 \tag{2.1}$$

*for the constant* $\tau_0$ *defined in* (1.18)*. Moreover, for convenience below, we* *define, for* $M \in \mathbb{Z}_+$,

$$\begin{aligned} \boldsymbol{t}_{M+1} &= (t_0, \dots, t_{M+1}) \in \Delta(T_1), \\ \boldsymbol{t}_{M+1,j}(\tau) &= (t_0, \dots, t_{j-1}, \tau, t_{j+1}, \dots, t_{M+1}), \end{aligned} \tag{2.2}$$

*where* $\Delta(T_1) = \Delta_{M+1}(T_1) = \{(t_0, \dots, t_{M+1})\colon 0 \leqslant t_{M+1} \leqslant t_M \leqslant \cdots \leqslant t_0 \leqslant T_1\}$.

## 2.1 Parameter-dependent multi-products of regular SG phase functions

Let $M \geqslant 1$ be a fixed integer and $a_j \in C([0,T]; S^{1,1})$, $j = 1, \dots, M+1$. Then, trivially, $\{a_j\}_{j=1}^{M+1}$ is bounded in $C([0,T]; S^{1,1})$ and we have the following well-defined multi-product

$$\phi(\mathbf{t}_{M+1}; x, \xi) = [\varphi_1(t_0, t_1) \sharp \varphi_2(t_1, t_2) \sharp \dots \sharp \varphi_{M+1}(t_M, t_{M+1})](x, \xi), \tag{2.3}$$

where we set $t_0 = t$, $t_{M+1} = s$, for $\mathbf{t}_{M+1} \in \Delta(T_1)$ from (2.2). Explicitly, $\phi$ is defined as in (1.23), by means of the critical points $(Y, N) = (Y, N)(\mathbf{t}_{M+1}; x, \xi)$, obtained, when $M \geqslant 2$, as solutions of the system

$$\begin{cases} Y_1(\mathbf{t}_{M+1}; x, \xi) &= \varphi'_{1,\xi}(t_0, t_1; x, N_1(\mathbf{t}_{M+1}; x, \xi)) \\ Y_j(\mathbf{t}_{M+1}; x, \xi) &= \varphi'_{j,\xi}(t_{j-1}, t_j; Y_{j-1}(\mathbf{t}_{M+1}; x, \xi), N_j(\mathbf{t}_{M+1}; x, \xi)), \\ & \qquad (j = 2, \dots, M) \\ N_j(\mathbf{t}_{M+1}; x, \xi) &= \varphi'_{j+1,x}(t_j, t_{j+1}; Y_j(\mathbf{t}_{M+1}; x, \xi), N_{j+1}(\mathbf{t}_{M+1}; x, \xi)), \\ & \qquad (j = 1, \dots, M-1) \\ N_M(\mathbf{t}_{M+1}; x, \xi) &= \varphi'_{M+1,x}(t_M, t_{M+1}; Y_M(\mathbf{t}_{M+1}; x, \xi), \xi), \end{cases} \tag{2.4}$$

namely,

$$\begin{aligned} \phi(\mathbf{t}_{M+1}; x, \xi) := \sum_{j=1}^{M} \Big[ & \varphi_j(t_{j-1}, t_j; Y_{j-1}(\mathbf{t}_{M+1}; x, \xi), N_j(\mathbf{t}_{M+1}; x, \xi)) \\ - \langle Y_j(\mathbf{t}_{M+1}; x, \xi), N_j(\mathbf{t}_{M+1}; x, \xi) \rangle \Big] & + \varphi_{M+1}(t_M, t_{M+1}; Y_M(\mathbf{t}_{M+1}; x, \xi), \xi). \end{aligned} \tag{2.5}$$

Next, we give some properties of the multi-product $\phi$. The next Propositions 2.2 and 2.3, Corollary 2.4 and Proposition 2.5 are extension of analogous results from [8] to the parameter-dependent case needed here.

**Proposition 2.2.** *Let* $\phi$ *be the multiproduct* (2.5)*, with real-valued* $a_j \in C([0,T]; S^{1,1}(\mathbb{R}^{2d}))$, $j = 1, \dots, M+1$*. Then, the following properties hold true.*

*(i)*

$$\begin{cases} \partial_{t_0}\phi(\boldsymbol{t}_{M+1};x,\xi) &= a_1(t_0;x,\phi'_x(\boldsymbol{t}_{M+1};x,\xi)) \\ \partial_{t_j}\phi(\boldsymbol{t}_{M+1};x,\xi) &= a_{j+1}(t_j;Y_j(\boldsymbol{t}_{M+1};x,\xi),N_j(\boldsymbol{t}_{M+1};x,\xi)) \\ & \quad - a_j(t_j;Y_j(\boldsymbol{t}_{M+1};x,\xi),N_j(\boldsymbol{t}_{M+1};x,\xi)), \\ & \qquad (j=1,\dots,M)) \\ \partial_{t_{M+1}}\phi(\boldsymbol{t}_{M+1};x,\xi) &= -a_{M+1}(t_{M+1},\phi'_\xi(\boldsymbol{t}_{M+1};x,\xi),\xi). \end{cases} \tag{2.6}$$

*(ii) For any $\varphi_j$ solution to the eikonal equation associated with the Hamiltonian $a_j$, we have*

$$\begin{cases} \varphi_i(t,s)\sharp\varphi_j(s,s) &= \varphi_i(t,s) \\ \varphi_i(s,s)\sharp\varphi_j(t,s) &= \varphi_j(t,s), \end{cases}$$

*for all $i,j=1,\dots,M+1$.*

*Proof.* The claim (i) comes from the fact that $\phi$ is defined by (2.5), that $\varphi_j$ is the solution of the eikonal equation (1.24) related to $a_j$ and satisfies the properties of Proposition 1.32, and that $(Y,N)$ satisfy (1.22). Moreover, Example 3.3 of [8] shows that (ii) holds true. □

**Proposition 2.3.** *Let $\{a_j\}_{j\in\mathbb{N}}$ be a family of parameter-dependent, real-valued symbols, bounded in $C^\infty([0,T];S^{1,1}(\mathbb{R}^{2d}))$, and $(Y,N)$ be the solution of (2.4). Then, for $\gamma_k\in\mathbb{Z}_+$, $k=0,1\dots,M+1$, the following properties hold true.*

*(i)*

$$\begin{cases} \{\partial_{t_0}^{\gamma_0}\dots\partial_{t_{M+1}}^{\gamma_{M+1}}(Y_j-Y_{j-1})\}_{j\in\mathbb{N}} \text{ is bounded in } S^{1,0}(\mathbb{R}^{2d}), \\ \{\partial_{t_0}^{\gamma_0}\dots\partial_{t_{M+1}}^{\gamma_{M+1}}(N_j-N_{j+1})\}_{j\in\mathbb{N}} \text{ is bounded in } S^{0,1}(\mathbb{R}^{2d}). \end{cases} \tag{2.7}$$

*(ii) For $J_{M+1}(\boldsymbol{t}_{M+1};x,\xi)=\phi(\boldsymbol{t}_{M+1};x,\xi)-\langle x,\xi\rangle$, we have*

$$\{\partial_{t_0}^{\gamma_0}\dots\partial_{t_{M+1}}^{\gamma_{M+1}}J_{M+1}\} \text{ is bounded in } S^{1,1}(\mathbb{R}^{2d}).$$

*Proof.* From the fact that $y_j(\mathbf{t}_{M+1};x,\xi) = Y_j - Y_{j-1} = J'_{j,\xi}(Y_{j-1},N_j)$, $\eta_j(\mathbf{t}_{M+1};x,\xi) = N_j - N_{j+1} = J'_{j+1,x}(Y_j,N_{j+1})$ (cf. [8, Lemma 3.5]) and that $a_j$ belongs to $C^\infty([0,T];S^{1,1})$, for $j\geqslant 1$, then $\partial_t^{\gamma_0}\partial_s^{\gamma_{M+1}}J_j(t,s;x,\xi)$ belongs to $S^{1,1}$ as stated in Corollary 1.34. From this and Theorem 1.23, we get (i) for $|\gamma|=\gamma_0+\dots+\gamma_{M+1}=0$, and Proposition 1.24 implies (ii), for $|\gamma|=0$. We now proceed by induction on $\gamma$.

**Step** $|\gamma|=1$. We need to check (i) for the first order derivatives with respect to the $t_k$ variables where $k=0,\dots,M+1$. Let us start with the

$t_0 = t$ derivatives. To this aim, we switch from the system (1.22) in the unknown $(Y, N)$ to the equivalent system (2.9) in the unknown $(\bar{Y}, \bar{N}) = (y_1, \ldots, y_M, \eta_1, \ldots, \eta_M) \in \mathbb{R}^{2Md}$ as follows. Define

$$\begin{cases} z^0 & := 0 \\ z^j & := \sum_{k=1}^j y_k\,, \quad j = 1, \ldots, M \\ \zeta^j & := \sum_{k=j}^M \eta_k\,, \quad j = 1, \ldots, M \\ \zeta^{M+1} & := 0, \end{cases} \tag{2.8}$$

and consider the system

$$\begin{cases} y_k = J'_{k,\xi}(x + z^{k-1}, \xi + \zeta^k), & k = 1, \ldots, M \\ \eta_k = J'_{k+1,x}(x + z^k, \xi + \zeta^{j+1}), & k = 1, \ldots, M. \end{cases} \tag{2.9}$$

Then, we have

$$\begin{cases} \partial_t y_k & = \partial_t J'_{k,\xi}(x + z^{k-1}, \xi + \zeta^k) + J''_{k,\xi x}(x + z^{k-1}, \xi + \zeta^k)\partial_t z^{k-1} \\ & \qquad + J''_{k,\xi\xi}(x + z^{k-1}, \xi + \zeta^k)\partial_t \zeta^k, \\ \partial_t \eta_k & = \partial_t J'_{k+1,x}(x + z^k, \xi + \zeta^{k+1}) + J''_{k+1,xx}(x + z^k, \xi + \zeta^{k+1})\partial_t z^k \\ & \qquad + J''_{k+1,x\xi}(x + z^k, \xi + \zeta^{k+1})\partial_t \zeta^{k+1}. \end{cases} \tag{2.10}$$

In view of (1.17) and Corollary 1.34, we obtain

$$\begin{aligned} &\langle x\rangle^{-1} \cdot \|\partial_t y_k\| + \langle \xi\rangle^{-1} \cdot \|\partial_t \eta_k\| \\ &\leqslant \tau_k \cdot \langle x\rangle^{-1} \cdot \Big\{\langle x + z^{k-1}\rangle + \|\partial_t z^{k-1}\| + \langle x + z^{k-1}\rangle\langle \xi + \zeta^k\rangle^{-1}\|\partial_t \zeta^k\|\Big\} \\ &+ \tau_{k+1} \cdot \langle \xi\rangle^{-1} \cdot \Big\{\langle \xi + \zeta^{k+1}\rangle + \langle x + z^k\rangle^{-1}\langle \xi + \zeta^{k+1}\rangle\|\partial_t z^k\| + \|\partial_t \zeta^{k+1}\|\Big\}. \end{aligned}$$

From [8, Theorem 3.6] we have

$$\frac{2}{3}\langle x\rangle \leqslant \langle x + z^{k-1}\rangle \leqslant \frac{4}{3}\langle x\rangle \text{ and } \frac{2}{3}\langle \xi\rangle \leqslant \langle \xi + \zeta^k\rangle \leqslant \frac{4}{3}\langle \xi\rangle,$$

then we get

$$\begin{aligned} \langle x\rangle^{-1} \cdot \|\partial_t y_k\| + \langle \xi\rangle^{-1} \cdot \|\partial_t \eta_k\| &\leqslant \tau_k \cdot \langle x\rangle^{-1} \cdot \Big\{\frac{4}{3}\langle x\rangle + \|\partial_t z^{k-1}\| + 2\langle x\rangle\langle \xi\rangle^{-1}\|\partial_t \zeta^k\|\Big\} \\ &+ \tau_{k+1} \cdot \langle \xi\rangle^{-1} \cdot \Big\{\frac{4}{3}\langle \xi\rangle + 2\langle x\rangle^{-1}\langle \xi\rangle\|\partial_t z^k\| + \|\partial_t \zeta^{k+1}\|\Big\} \\ &\leqslant \tau_k \cdot \Big\{\frac{4}{3} + \sum_{k=1}^M \langle x\rangle^{-1}\|\partial_t y_k\| + 2\langle \xi\rangle^{-1} \sum_{k=1}^M \|\partial_t \eta_k\|\Big\} \\ &+ \tau_{k+1} \cdot \Big\{\frac{4}{3} + 2\sum_{k=1}^M \langle x\rangle^{-1}\|\partial_t y_k\| + \sum_{k=1}^M \langle \xi\rangle^{-1}\|\partial_t \eta_k\|\Big\}, \end{aligned}$$

where in the last inequality, we have used (2.8). Summing for $k = 1, \dots, M$, we get, for any $x, \xi \in \mathbb{R}^d$,

$$\sum_{k=1}^{M} \Big( \langle x \rangle^{-1} \cdot \|\partial_t y_k\| + \langle \xi \rangle^{-1} \cdot \|\partial_t \eta_k\| \Big)$$

$$\leqslant \bar{\tau}_M \left\{ \frac{4}{3} + \sum_{k=1}^{M} \langle x \rangle^{-1} \|\partial_t y_k\| + 2\langle \xi \rangle^{-1} \sum_{k=1}^{M} \|\partial_t \eta_k\| \right\}$$

$$+ \bar{\tau}_{M+1} \left\{ \frac{4}{3} + 2 \sum_{k=1}^{M} \langle x \rangle^{-1} \|\partial_t y_k\| + \sum_{k=1}^{M} \langle \xi \rangle^{-1} \|\partial_t \eta_k\| \right\}$$

$$\leqslant 3\bar{\tau}_{M+1} \left\{ 1 + \sum_{k=1}^{M} \Big( \langle x \rangle^{-1} \cdot \|\partial_t y_k\| + \langle \xi \rangle^{-1} \cdot \|\partial_t \eta_k\| \Big) \right\}.$$

The last inequality implies that

$$\sum_{k=1}^{M} \Big( \langle x \rangle^{-1} \cdot \|\partial_t y_k\| + \langle \xi \rangle^{-1} \cdot \|\partial_t \eta_k\| \Big) \leqslant \frac{3\tau_0}{1 - 3\tau_0} < 3,$$

and this hold true due to (1.18). Substituting the above estimate in (2.10), we obtain

$$\|\partial_t y_k\| \leqslant \|J_k\|_{2,0} \left\{ \frac{4}{3} \langle x \rangle + \|\partial_t z^{k-1}\| + 2\langle x \rangle \langle \xi \rangle^{-1} \|\partial_t \zeta^k\| \right\}$$

$$\leqslant 2\|J_k\|_{2,0} \langle x \rangle \left\{ 1 + \sum_{k=1}^{M} \Big( \langle x \rangle^{-1} \cdot \|\partial_t y_k\| + \langle \xi \rangle^{-1} \cdot \|\partial_t \eta_k\| \Big) \right\}$$

$$\leqslant c_0 \|J_k\|_{2,0} \langle x \rangle.$$

With similar computation, we obtain

$$\|\partial_t \eta_k\| \leqslant \|J_{k+1}\|_{2,0} \Big\{ \langle \xi + \zeta^{k+1} \rangle + \langle x + z^k \rangle^{-1} \langle \xi + \zeta^{k+1} \rangle \|\partial_t z^k\| + \|\partial_t \zeta^{k+1}\| \Big\}$$

$$\leqslant \|J_{k+1}\|_{2,0} \left\{ \frac{4}{3} \langle \xi \rangle + 2\langle x \rangle^{-1} \langle \xi \rangle \|\partial_t z^k\| + \|\partial_t \zeta^{k+1}\| \right\}$$

$$\leqslant 2\|J_{k+1}\|_{2,0} \langle \xi \rangle \Big\{ 1 + \langle x \rangle^{-1} \|\partial_t z^k\| + \langle \xi \rangle^{-1} \|\partial_t \zeta^{k+1}\| \Big\}$$

$$\leqslant 2\|J_{k+1}\|_{2,0} \langle \xi \rangle \left\{ 1 + \sum_{k=1}^{M} \Big( \langle x \rangle^{-1} \cdot \|\partial_t y_k\| + \langle \xi \rangle^{-1} \cdot \|\partial_t \eta_k\| \Big) \right\}$$

$$\leqslant C_0 \|J_{k+1}\|_{2,0} \langle \xi \rangle.$$

Similarly, for any $x, \xi \in \mathbb{R}^d$ and , $\nu = 1, \dots, M+1$, we have

$$\|\partial_{t_\nu} y_k(x,\xi)\| \leqslant C_\nu \|J_k\|_{2,0}\langle x\rangle, \qquad \|\partial_{t_\nu} \eta_k(x,\xi)\| \leqslant C_\nu \|J_{k+1}\|_{2,0}\langle \xi\rangle.$$

This conclude the step $|\gamma| = 1$.

**Inductive step.** We assume that our construction holds true for $|\gamma| = N$, and prove the estimate for $|\gamma| = N+1$. Using (1.17), we obtain

$$\begin{cases} \left|\partial_{t_0}^{\gamma_0} \dots \partial_{t_{M+1}}^{\gamma_{M+1}} y_k\right| \leqslant c\langle x\rangle, \\ \left|\partial_{t_0}^{\gamma_0} \dots \partial_{t_{M+1}}^{\gamma_{M+1}} \eta_k\right| \leqslant c\langle \xi\rangle, \end{cases} \tag{2.11}$$

for $|\gamma| = N$ and $k = 1, \dots, M$. We need (2.11) for the boundedness of $y_k$ and $\eta_k$ with respect to $\partial_{t_0}^{\gamma_0} \dots \partial_{t_{M+1}}^{\gamma_{M+1}}$ where $|\gamma| = N+1$. Thus we compute the $|\gamma|$ derivative of $\partial_{t_\nu} y_k$ where $|\gamma| = N$, $\nu = 0, \dots, M+1$ and $k = 1, \dots, M$.

$$\partial_{t_0}^{\gamma_0} \dots \partial_{t_{M+1}}^{\gamma_{M+1}} \{\partial_{t_\nu} y_k\} = \partial_{t_0}^{\gamma_0} \dots \partial_{t_{M+1}}^{\gamma_{M+1}} \Big\{\partial_{t_\nu} J'_{k,\xi}(x + z^{k-1}, \xi + \zeta^k)$$
$$+ J''_{k,\xi x}(x + z^{k-1}, \xi + \zeta^k)\partial_{t_\nu} z^{k-1} + J''_{k,\xi\xi}(x + z^{k-1}, \xi + \zeta^k)\partial_{t_\nu}\zeta^k\Big\}. \tag{2.12}$$

We use Faá di Bruno formula to obtain estimates on (2.12) i. e., obtaining the $\gamma$ derivatives of $\partial_{t_\nu} y_k$ with respect to $\mathbf{t}_{M+1}$. Hence, we get the following formula for the derivative of $H_k$ where $H_k \in \{\partial_{t_\nu} J'_{k,\xi}, J''_{k,\xi x}, J''_{k,\xi\xi}\}$, with respect to $\partial_{t_0}^{\gamma_0} \dots \partial_{t_{M+1}}^{\gamma_{M+1}}$, where $|\gamma| = N$:

$$\partial_{t_{k-1}}^{\gamma_{k-1}} \partial_{t_k}^{\gamma_k} H_k + \partial_x^s \partial_\xi^r H_k \prod^s \partial_{t_0}^{\sigma_0} \dots \partial_{t_{M+1}}^{\sigma_{M+1}} z^{k-1} \prod^r \partial_{t_0}^{\rho_0} \dots \partial_{t_{M+1}}^{\rho_{M+1}} \zeta^k.$$

Applying (2.11), we obtain

$$|\partial_{t_0}^{\gamma_0} \dots \partial_{t_{M+1}}^{\gamma_{M+1}} \left(\partial_{t_\nu} J'_{k,\xi}(x + z^{k-1}(\mathbf{t}_{M+1})), \xi + \zeta^k(\mathbf{t}_{M+1}))\right)| \leqslant \|J_k\|_{2,0}\langle x\rangle$$
$$+ \sum_{\substack{\sigma_1+\dots+\sigma_r+\rho_1+\dots+\rho_q=\gamma \\ \sigma_i\neq 0;\, \rho_i\neq 0}} C_{q,r,\gamma} \|J_k\|_{2,q+r} \langle\xi\rangle^{-q}\langle x\rangle^{1-r} \cdot \underbrace{\langle\xi\rangle \dots \langle\xi\rangle}_{q\,times} \underbrace{\langle x\rangle \dots \langle x\rangle}_{r\,times}$$
$$\leqslant C_\gamma \|J_k\|_{2,|\gamma|}\langle x\rangle,$$

moreover

$$|\partial_{t_0}^{\gamma_0} \dots \partial_{t_{M+1}}^{\gamma_{M+1}} \left(J''_{k,\xi x}(x + z^{k-1}(\mathbf{t}_{M+1}), \xi + \zeta^k(\mathbf{t}_{M+1}))\right)| \leqslant C_\gamma \|J_k\|_{2,|\gamma|},$$

and

$$|\partial_{t_0}^{\gamma_0} \dots \partial_{t_{M+1}}^{\gamma_{M+1}} \left(J''_{k,\xi\xi}(x + z^{k-1}(\mathbf{t}_{M+1}), \xi + \zeta^k(\mathbf{t}_{M+1}))\right)|$$
$$\leqslant C_\gamma \|J_k\|_{2,|\gamma|}\langle x\rangle\langle\xi\rangle^{-1}.$$

Thus, substituting these last estimates in (2.12) and using (2.8), we get

$$|\partial_{t_0}^{\gamma_0}\dots\partial_{t_{M+1}}^{\gamma_{M+1}}\{\partial_{t_\nu}y_k\}| \leqslant C'_\gamma\|J_k\|_{2,|\gamma|}\langle x\rangle$$

$$+\|J_k\|_{2,0}\sum_{k=1}^{M}\left(|\partial_{t_0}^{\gamma_0}\dots\partial_{t_{M+1}}^{\gamma_{M+1}}\{\partial_{t_\nu}y_k\}|+2\langle x\rangle\langle\xi\rangle^{-1}|\partial_{t_0}^{\gamma_0}\dots\partial_{t_{M+1}}^{\gamma_{M+1}}\{\partial_{t_\nu}\eta_k\}|\right). \tag{2.13}$$

By similar computation, we can show that

$$|\partial_{t_0}^{\gamma_0}\dots\partial_{t_{M+1}}^{\gamma_{M+1}}\{\partial_{t_\nu}\eta_k\}| \leqslant \tilde{C}'_\gamma\|J_{k+1}\|_{2,|\gamma|}\langle\xi\rangle+\|J_{k+1}\|_{2,0}\langle x\rangle^{-1}\langle\xi\rangle\cdot$$

$$\cdot\sum_{k=1}^{M}\left(2|\partial_{t_0}^{\gamma_0}\dots\partial_{t_{M+1}}^{\gamma_{M+1}}\{\partial_{t_\nu}y_k\}|+\langle x\rangle\langle\xi\rangle^{-1}|\partial_{t_0}^{\gamma_0}\dots\partial_{t_{M+1}}^{\gamma_{M+1}}\{\partial_{t_\nu}\eta_k\}|\right). \tag{2.14}$$

Summing up (2.13) and (2.14) for $k=1,\dots,M$, we get

$$\sum_{\nu=1}^{M}\left(|\partial_{t_0}^{\gamma_0}\dots\partial_{t_{M+1}}^{\gamma_{M+1}}\{\partial_{t_\nu}y_k\}|+\langle x\rangle\langle\xi\rangle^{-1}|\partial_{t_0}^{\gamma_0}\dots\partial_{t_{M+1}}^{\gamma_{M+1}}\{\partial_{t_\nu}\eta_k\}|\right)$$

$$\leqslant 3\left(\sum_{k=1}^{M}\|J_{k+1}\|_{2,0}\right)\cdot$$

$$\cdot\sum_{\nu=1}^{M}\left(|\partial_{t_0}^{\gamma_0}\dots\partial_{t_{M+1}}^{\gamma_{M+1}}\{\partial_{t_\nu}y_k\}|+\langle x\rangle\langle\xi\rangle^{-1}|\partial_{t_0}^{\gamma_0}\dots\partial_{t_{M+1}}^{\gamma_{M+1}}\{\partial_{t_\nu}\eta_k\}|\right)$$

$$+\bar{C}_\gamma\left(\sum_{\nu=1}^{M}\|J_k\|_{2,|\gamma|}+\sum_{k=1}^{M}\|J_{k+1}\|_{2,|\gamma|}\right)\langle x\rangle$$

$$\leqslant 3c_0\tau_0\sum_{\nu=1}^{M}\left(|\partial_{t_0}^{\gamma_0}\dots\partial_{t_{M+1}}^{\gamma_{M+1}}\{\partial_{t_\nu}y_k\}|+\langle x\rangle\langle\xi\rangle^{-1}|\partial_{t_0}^{\gamma_0}\dots\partial_{t_{M+1}}^{\gamma_{M+1}}\{\partial_{t_\nu}\eta_k\}|\right)$$

$$+2c_{|\gamma|}\tau_0\langle x\rangle,$$

where $c_0$ and $c_{|\gamma|}$ are the constants defined in (1.19). Using the fact that $c_0=1$, we get

$$\sum_{\nu=1}^{M}\left(|\partial_{t_0}^{\gamma_0}\dots\partial_{t_{M+1}}^{\gamma_{M+1}}\{\partial_{t_\nu}y_k\}|+\langle x\rangle\langle\xi\rangle^{-1}|\partial_{t_0}^{\gamma_0}\dots\partial_{t_{M+1}}^{\gamma_{M+1}}\{\partial_{t_\nu}\eta_k\}|\right)$$

$$\leqslant \bar{C}'_\gamma\frac{\tau_0}{1-3\tau_0}\langle x\rangle<\bar{C}'_\gamma\langle x\rangle. \tag{2.15}$$

The last inequality holds true thanks to the choice of $\tau_0$ in (1.18). Substituting (2.15) in (2.13) and (2.14) we finally obtain

$$|\partial_{t_0}^{\gamma_0}\dots\partial_{t_{M+1}}^{\gamma_{M+1}}\{\partial_{t_\nu}y_k\}| \leqslant C_\gamma\|J_k\|_{2,|\gamma|}\langle x\rangle, \tag{2.16}$$

$$|\partial_{t_0}^{\gamma_0}\dots\partial_{t_{M+1}}^{\gamma_{M+1}}\{\partial_{t_\nu}\eta_k\}| \leqslant C_\gamma\|J_{k+1}\|_{2,|\gamma|}\langle\xi\rangle. \tag{2.17}$$

The proof of (i) is complete since (2.16) and (2.17) are the desired estimates for $|\gamma| = N+1$.

(ii) follows from the boundedness of the Hamiltonian $\{a_j\}$ using (i) and (2.6). $\square$

**Corollary 2.4.** *Under the hypotheses of Proposition 2.3, we have, for some* $T_1 \in (0, T]$ *as in Proposition 1.32, and* $j = 0, \ldots, M+1$,

$$\begin{cases} Y_j \text{ belongs to } C^\infty\left(\Delta(T_1); S^{1,0}(\mathbb{R}^{2d})\right), \\ N_j \text{ belongs to } C^\infty\big(\Delta(T_1); S^{0,1}(\mathbb{R}^{2d})\big). \end{cases}$$

*Proof.* The proof follows by induction on $j$. For $j = 1$, $Y_1(\mathbf{t}_{M+1}; x, \xi) = Y_1(\mathbf{t}_{M+1}; x, \xi) - x + x$ observing that $x \in S^{1,0}$ and for any $\alpha, \beta \in \mathbb{Z}_+^d$ we have

$$\left|\partial_x^\beta \partial_\xi^\alpha Y_1\right| \leqslant \left|\partial_x^\beta \partial_\xi^\alpha (Y_1 - x)\right| + \left|\partial_x^\beta \partial_\xi^\alpha x\right| \leqslant C_{\alpha\beta} \langle x\rangle^{1-|\beta|} \langle \xi\rangle^{-|\alpha|},$$

since, by Proposition 2.3, $\{Y_j - Y_{j-1}\}_{j=1}^M$ is bounded in $C^\infty(\Delta(T_1); S^{1,0})$. Now, assume that the statement holds true up to $j = M-1$. Then, $Y_M \in S^{1,0}$, in view of to the fact that $Y_M = (Y_M - Y_{M-1}) + Y_{M-1}$, the inductive hypothesis $Y_{M-1} \in C^\infty(\Delta(T_1); S^{1,0})$ and (2.7). The same argument implies that $N_j \in C^\infty(\Delta(T_1); S^{0,1})$. $\square$

For any $\{a_j\} \subset C^\infty([0,T]; S^{1,1})$, we consider the solution $(q_j, p_j)(t, s; y, \eta)$ of the Hamilton-Jacobi system (1.25), with the Hamiltonian $a_j$ in place of $a$. We define the trajectory $(\tilde{q}_j, \tilde{p}_j)(\mathbf{t}_{j-1}, \sigma; y, \eta)$, for $(y, \eta) \in \mathbb{R}^d \times \mathbb{R}^d$, $\mathbf{t}_{j-1} \in \Delta(T_1)$, $\sigma \in [t_j, t_{j-1}]$, $j \in \mathbb{N}$, by

$$\begin{cases} (\tilde{q}_1, \tilde{p}_1)(t_0, \sigma; y, \eta) & = (q_1, p_1)(\sigma, t_0; y, \eta), \qquad t_1 \leqslant \sigma \leqslant t_0, \\ (\tilde{q}_j, \tilde{p}_j)(\mathbf{t}_{j-1}, \sigma; y, \eta) & = (q_j, p_j)(\sigma, t_{j-1}; (\tilde{q}_{j-1}, \tilde{p}_{j-1})(\mathbf{t}_{j-1}; y, \eta)), \\ & \qquad\qquad t_j \leqslant \sigma \leqslant t_{j-1}, j \geqslant 2. \end{cases} \tag{2.18}$$

**Proposition 2.5.** *Let* $(Y, N) = (Y, N)(\boldsymbol{t}_{M+1}; x, \xi)$ *be the solution of* (2.4) *under the hypotheses of Proposition 2.3. Then, we have*

$$\begin{cases} q_1(t_1, t_0; x, \phi'_x(\boldsymbol{t}_{M+1}; x, \xi)) = Y_1(\boldsymbol{t}_{M+1}; x, \xi) \\ p_1(t_1, t_0; x, \phi'_x(\boldsymbol{t}_{M+1}; x, \xi)) = N_1(\boldsymbol{t}_{M+1}; x, \xi), \end{cases} \tag{2.19}$$

$$\begin{cases} q_j(t_j, t_{j-1}; Y_{j-1}(\boldsymbol{t}_{M+1}; x, \xi), N_{j-1}(\boldsymbol{t}_{M+1}; x, \xi)) & = Y_j(\boldsymbol{t}_{M+1}; x, \xi), \\ p_j(t_j, t_{j-1}; Y_{j-1}(\boldsymbol{t}_{M+1}; x, \xi), N_{j-1}(\boldsymbol{t}_{M+1}; x, \xi)) & = N_j(\boldsymbol{t}_{M+1}; x, \xi), \\ & \quad (2 \leqslant j \leqslant M), \end{cases} \tag{2.20}$$

*and, for any* $j \leqslant M$,

$$(\tilde{q}_j, \tilde{p}_j)(\boldsymbol{t}_j; x, \phi'_x(\boldsymbol{t}_{M+1}; x, \xi)) = (Y_j, N_j)(\boldsymbol{t}_{M+1}; x, \xi). \tag{2.20$_j$}$$

*Proof.* Arguing as in [86, Chapter 10, Section 4], taking into account that $(q,p)$ is a solution to the Hamilton-Jacobi system (1.25), from the fact that the equation $x = q(t,s;y,\xi)$ has the unique solution $y = \bar{q}(t,s;x,\xi)$, for $0 \leqslant s \leqslant t \leqslant T$, $x, \xi \in \mathbb{R}^d$, using (1.38) and (1.39), see Proposition 1.32, with $\varphi_j$ in place of $\varphi$, we get the following equalities for any $j \geqslant 1$:

$$\begin{cases} q_j(t,s;\varphi'_{j,\xi}(t,s;x,\xi),\xi) = x \\ p_j(t,s;\varphi'_{j,\xi}(t,s;x,\xi),\xi) = \varphi'_{j,x}(t,s;x,\xi). \end{cases}$$

Thus, using the uniqueness of the solution to the Hamilton-Jacobi system (1.25) for $a = a_j$, we get

$$\begin{cases} q_j(s,t;x,\varphi'_{j,x}(t,s;x,\xi)) = \varphi'_{j,\xi}(t,s;x,\xi), \\ p_j(s,t;x,\varphi'_{j,x}(t,s;x,\xi)) = \xi. \end{cases} \tag{2.21$_j$}$$

From Proposition 1.24, recalling (1.22), it follows

$$\phi'_x(\mathbf{t}_{M+1};x,\xi) = \varphi'_{1,x}(t_0,t_1;x,N_1(\mathbf{t}_{M+1};x,\xi)).$$

Using this with (1.20) and (2.21$_j$) with $j=1$, we obtain

$$\begin{cases} q_1(t_1,t_0;x,\phi'_x(\mathbf{t}_{M+1};x,\xi)) & = q_1(t_1,t_0;x,\varphi'_{1,x}(t_0,t_1;x,N_1(\mathbf{t}_{M+1};x,\xi))) \\ & = \varphi'_{1,\xi}(t_0,t_1;x,N_1(\mathbf{t}_{M+1};x,\xi)) \\ & = Y_1(\mathbf{t}_{M+1};x,\xi) \\ p_1(t_1,t_0;x,\phi'_x(\mathbf{t}_{M+1};x,\xi)) & = p_1(t_1,t_0;x,\varphi'_{1,x}(t_0,t_1;x,N_1(\mathbf{t}_{M+1};x,\xi))) \\ & = N_1(\mathbf{t}_{M+1};x,\xi), \end{cases}$$

which is (2.19). In view of (2.4)

$$\varphi'_{j,x}(t_{j-1},t_j;Y_{j-1}(\mathbf{t}_{M+1};x,\xi),N_j(\mathbf{t}_{M+1};x,\xi)) = N_{j-1}(\mathbf{t}_{M+1};x,\xi),\ j \geqslant 2.$$

Using (2.21$_j$), we find

$$\begin{aligned} & q_j(t_j,t_{j-1};Y_{j-1}(\mathbf{t}_{M+1};x,\xi),N_{j-1}(\mathbf{t}_{M+1};x,\xi)) \\ & = q_j(t_j,t_{j-1};Y_{j-1}(\mathbf{t}_{M+1};x,\xi),\varphi'_{j,x}(t_{j-1},t_j;Y_{j-1}(\mathbf{t}_{M+1};x,\xi),N_j(\mathbf{t}_{M+1};x,\xi))) \\ & \qquad = \varphi'_{j,\xi}(t_{j-1},t_j;Y_{j-1}(\mathbf{t}_{M+1};x,\xi),N_j(\mathbf{t}_{M+1};x,\xi)) = Y_j(\mathbf{t}_{M+1};x,\xi), \end{aligned}$$

and

$$\begin{aligned} & p_j(t_j,t_{j-1};Y_{j-1}(\mathbf{t}_{M+1};x,\xi),N_{j-1}(\mathbf{t}_{M+1};x,\xi)) \\ & = p_j(t_j,t_{j-1};Y_{j-1}(\mathbf{t}_{M+1};x,\xi),\varphi'_{j,x}(t_{j-1},t_j;Y_{j-1}(\mathbf{t}_{M+1};x,\xi),N_j(\mathbf{t}_{M+1};x,\xi))) \\ & \qquad = N_j(\mathbf{t}_{M+1};x,\xi), \end{aligned}$$

which give (2.20). The proof of $(2.20_j)$ is obtained by an inductive argument, based on (2.20). □

Notice that, from the Propositions 2.3 and 2.5, we obtain also the following result.

**Proposition 2.6.** *Let $\{a_j\}_{j\in\mathbb{N}}$ be a family of parameter-dependent, real-valued symbols, bounded in $C^\infty([0,T];S^{1,1}(\mathbb{R}^{2d}))$. Then we have, for the trajectory $(\tilde{q}_j,\tilde{p}_j)(\boldsymbol{t}_{j-1},\sigma;y,\eta)$ defined in (2.18),*

$$\begin{cases} \{\partial_{t_0}^{\gamma_0}\dots\partial_{t_{j-1}}^{\gamma_{j-1}}\partial_\sigma^{\gamma_j}\tilde{q}_j\}_{j\in\mathbb{N},\,\boldsymbol{t}_{j-1}\in\Delta(T_1),\,\sigma\in[t_j,t_{j-1}]} \text{ is bounded in } S^{1,0}(\mathbb{R}^{2d}),\\ \{\partial_{t_0}^{\gamma_0}\dots\partial_{t_{j-1}}^{\gamma_{j-1}}\partial_\sigma^{\gamma_j}\tilde{p}_j\}_{j\in\mathbb{N},\,\boldsymbol{t}_{j-1}\in\Delta(T_1),\,\sigma\in[t_j,t_{j-1}]} \text{ is bounded in } S^{0,1}(\mathbb{R}^{2d}),\end{cases}$$

*where $\gamma_k\in\mathbb{Z}_+$ for $0\leqslant k\leqslant j$.*

## 2.2 An auxiliary equation

Consider the following quasi-linear partial differential equation

$$\begin{cases} \partial_{t_{j-1}}\Upsilon(\mathbf{t}_{M+1})-L(\Upsilon(\mathbf{t}_{M+1}),\mathbf{t}_{M+1})\cdot\Upsilon'_x(\mathbf{t}_{M+1})-H(\mathbf{t}_{M+1,j}(\Upsilon(\mathbf{t}_{M+1})))=0,\\ \Upsilon_{|t_{j-1}=t_j}=t_{j+1}\end{cases} \tag{2.22}$$

where, for $s\in\mathbb{R}$, $\mathbf{t}_{M+1,j}(s)$ is defined in (2.2), $\Upsilon(\mathbf{t}_{M+1})=\Upsilon(\mathbf{t}_{M+1};x,\xi)\in C^\infty(\Delta(T_1);S^{0,0})$, and $L(\tau,\mathbf{t}_{M+1})=L(\tau,\mathbf{t}_{M+1};x,\xi)$ is a vector-valued family of symbols of order $(1,0)$ such that $L\in C^\infty([t_{j+1},t_{j-1}]\times\Delta(T_1),S^{1,0})$, where $T_1$ is the same as in (2.1). For the sake of brevity, in (2.22) we have written $L(\Upsilon(\mathbf{t}_{M+1}),\mathbf{t}_{M+1})$ in place of $L(\Upsilon(\mathbf{t}_{M+1};x,\xi),\mathbf{t}_{M+1};x,\xi)$ and similarly, $H(\mathbf{t}_{M+1,j}(\Upsilon(\mathbf{t}_{M+1})))$ in place of $H(\mathbf{t}_{M+1,j}(\Upsilon(\mathbf{t}_{M+1};x,\xi));x,\xi)$.

We also assume that

$$H(\mathbf{t}_{M+1,j}(\tau))=H(\mathbf{t}_{M+1,j}(\tau);x,\xi)\in C^\infty(\Delta(T_1);S^{0,0}),$$

is such that

$$H(\mathbf{t}_{M+1};x,\xi)>0,\quad H(\mathbf{t}_{M+1};x,\xi)|_{t_j=t_{j-1}}\equiv 1,$$

for any $\mathbf{t}_{M+1}\in\Delta(T_1)$, $(x,\xi)\in\mathbb{R}^{2d}$.

The following Lemma 2.7 (cf. [110]) is the key result to prove the main Theorem 2.10. In fact, it gives the solution of the characteristics system

$$\begin{cases} \partial_{t_{j-1}}R(\mathbf{t}_{M+1})=-L(K(\mathbf{t}_{M+1}),\mathbf{t}_{M+1};R(\mathbf{t}_{M+1}),\xi)\\ \partial_{t_{j-1}}K(\mathbf{t}_{M+1})=\quad H(\mathbf{t}_{M+1,j}(K(\mathbf{t}_{M+1}));R(\mathbf{t}_{M+1}),\xi)\\ R_{|t_{j-1}=t_j}=y,\quad K_{|t_{j-1}=t_j}=t_{j+1},\end{cases} \tag{2.23}$$

which then easily provides the solution to the quasi-linear equation (2.22). The latter, in turn, is useful to simplify the computations in the proof of Theorem 2.10.

**Lemma 2.7.** *There exists a constant* $T_2 \in (0, T_1]$, $T_1$ *from Definition 2.1, such that* (2.23) *admits a unique solution* $(R, K) = (R, K)(\boldsymbol{t}_{M+1}; y, \xi) \in C^\infty(\Delta(T_2); (S^{1,0}(\mathbb{R}^{2d}) \otimes \mathbb{R}^d) \times S^{0,0}(\mathbb{R}^{2d}))$, $t_{j-1} \in [t_j, T_2]$, *which satisfies, for any* $\boldsymbol{t}_{M+1} \in \Delta(T_2)$, $(y, \xi) \in \mathbb{R}^{2d}$,

$$\left\| \frac{\partial R}{\partial y}(\boldsymbol{t}_{M+1}; y, \xi) - I \right\| \leqslant C(t_{j-1} - t_{j+1}), \tag{2.24}$$

*for a suitable constant* $C > 0$ *independent of* $M$, *and*

$$\begin{cases} t_{j+1} \leqslant K(\boldsymbol{t}_{M+1}; y, \xi) \leqslant t_{j-1} \\ K_{|t_{j-1}=t_j} = t_{j+1}. \end{cases} \tag{2.25}$$

*Proof.* First, we notice that, as a consequence of Lemma 1.10, the compositions in the right-hand side of (2.23) are well-defined. Moreover, they produce symbols of order $(1, 0)$ and $(0, 0)$, respectively, provided that $(R, K)$ belongs to $C^\infty(\Delta(T_2); S^{1,0} \times S^{0,0})$ and $K(\mathbf{t}_{M+1}) \in [t_{j+1}, t_{j-1}]$ for any $\mathbf{t}_{M+1} \in \Delta(T_2)$.

We focus only on the variables $(t_{j-1}, t_j, t_{j+1}; y, \xi)$, since all the others here play the role of (fixed) parameters, on which the solution clearly depends smoothly. We then omit them in the next computations. We will also write, to shorten some of the formulae, $(R, K)(s) = (R, K)(\mathbf{t}_{M+1,j-1}(s); y, \xi)$, $s \in [t_j, T_2]$, $T_2 \in (0, T_1]$ sufficiently small, to be determined below, $\mathbf{t}_{M+1} \in \Delta(T_2)$, $(y, \xi) \in \mathbb{R}^{2d}$.

We rewrite (2.23) in integral form, namely

$$\begin{cases} R(s) = y - \displaystyle\int_{t_j}^{s} L(K(\sigma); \sigma, t_j, t_{j+1}; R(\sigma), \xi)\, d\sigma \\ K(s) = t_{j+1} + \displaystyle\int_{t_j}^{s} H(\sigma, K(\sigma), t_{j+1}; R(\sigma), \xi)\, d\sigma, \end{cases} \tag{2.26}$$

$s \in [t_j, T_2]$, $\mathbf{t}_{M+1} \in \Delta(T_2)$, $(y, \xi) \in \mathbb{R}^{2d}$, and solve (2.26) by the customary Picard method of successive approximations. That is, we define the sequences

$$\begin{cases} R_{l+1}(s) = y - \displaystyle\int_{t_j}^{s} L(K_l(\sigma); \sigma, t_j, t_{j+1}; R_l(\sigma), \xi)\, d\sigma \\ K_{l+1}(s) = t_{j+1} + \displaystyle\int_{t_j}^{s} H(\sigma, K_l(\sigma), t_{j+1}; R_l(\sigma), \xi)\, d\sigma, \end{cases} \tag{2.27}$$

for $l = 1, 2, \ldots$, $s \in [t_j, T_2]$, $\mathbf{t}_{M+1} \in \Delta(T_2)$, $(y, \xi) \in \mathbb{R}^{2d}$, with

$$R_0(s) = y, \qquad K_0(s) = s - t_j + t_{j+1}.$$

We start by showing that $\{R_l\}_l$ and $\{K_l\}_l$ are bounded in $C^\infty(\Delta(T_2); S^{1,0})$ and $C^\infty(\Delta(T_2); S^{0,0})$, respectively. Also we aim at showing that (2.24) and (2.25) hold true for $R_l$ and $K_l$ in place of $R$ and $K$, respectively, for any $l \in \mathbb{Z}_+$, uniformly with respect to $l, j, M$. This follows by induction.

Namely, notice that all the stated properties are true for $l = 0$. Indeed, it is clear that $R_0 \in C^\infty(\Delta(T_2); S^{1,0})$ and $K_0 \in C^\infty(\Delta(T_2); S^{0,0})$, with semi-norms bounded by $\max\{1, 2T_2\}$. (2.24) with $R_0$ in place of $R$ is trivial, while (2.25) with $K_0$ in place of $K$ follows immediately, by inserting $s = t_{j-1}$ in $K_0(s)$ and recalling that $\mathbf{t}_{M+1} \in \Delta(T_2) \Rightarrow t_{j-1} - t_j + t_{j+1} \in [t_{j+1}, t_{j-1}]$.

Assume now that (2.24) and (2.25) hold true for $(R_\ell, K_\ell)$ for all the values of the index $\ell$ up to $l \geqslant 0$. We then find, by the same composition argument mentioned above, $R_{l+1} \in C^\infty(\Delta(T_2); S^{1,0})$ and $K_{l+1} \in C^\infty(\Delta(T_2); S^{0,0})$, with semi-norms uniformly bounded with respect to $l$, since they depend only on the semi-norms of $L$, $H$, $R_l$, $K_l$, and $T_2$. It follows that (2.24) holds true also for $R_{l+1}$ in place of $R$, since

$$\begin{aligned}\left\| \frac{\partial R_{l+1}}{\partial y}(\mathbf{t}_{M+1}; y, \xi) - I \right\| &= \left\| \int_{t_j}^{t_{j-1}} \frac{\partial}{\partial y} \left[ L(K_l(\sigma); \sigma, t_j, t_{j+1}; R_l(\sigma), \xi) \right] d\sigma \right\| \\ &\leqslant C(t_{j-1} - t_j) \leqslant C(t_{j-1} - t_{j+1}),\end{aligned}$$

for a suitable constant $C > 0$ independent of $l$. By the definition of $K_{l+1}$, for $l \geqslant 0$, clearly we get

$$K_{l+1}(\mathbf{t}_{M+1}; y, \xi)|_{t_{j-1}=t_j} = t_{j+1}.$$

It is also immediate that the hypotheses and the definition of $K_{l+1}$, $l \geqslant 0$, imply that $K_{l+1}(\mathbf{t}_{M+1}; y, \xi)$ is, for any fixed $(y, \xi) \in \mathbb{R}^{2d}$,

$$0 \leqslant t_{M+1} \leqslant \cdots t_{j+1} \leqslant t_{j-1} \leqslant \cdots t_0 \leqslant T_2 \leqslant T_1,$$

a monotonically decreasing function with respect to $t_j \in [t_{j+1}, t_{j-1}]$. (2.25) with $K_{l+1}$ in place of $K$, $l \geqslant 0$, follows by such property and the hypotheses on $H$. Observing that, for any $l \geqslant 0$, $\mathbf{t}_{M+1} \in \Delta(T_2)$, $(y, \xi) \in \mathbb{R}^{2d}$, $s \in [t_{j+1}, t_{j-1}]$,

$$K_{l+1}(\mathbf{t}_{M+1,j-1}(s); y, \xi)|_{t_j=t_{j+1}} = s, \quad K_{l+1}(\mathbf{t}_{M+1,j-1}(s); y, \xi)|_{t_j=s} = t_{j+1}, \tag{2.28}$$

which, in particular, also shows

$$K_{l+1}(\mathbf{t}_{M+1}; y, \xi)|_{t_j=t_{j+1}} = t_{j-1}, \quad K_{l+1}(\mathbf{t}_{M+1}; y, \xi)|_{t_j=t_{j-1}} = t_{j+1}. \tag{2.29}$$

Notice that (2.29), together with the monotonicity property of $K_{l+1}(\mathbf{t}_{M+1}; y, \xi)$ with respect to $t_j \in [t_{j+1}, t_{j-1}]$, $l \geqslant 0$, $\mathbf{t}_{M+1} \in \Delta(T_2)$, $(y, \xi) \in \mathbb{R}^{2d}$, complete the proof of (2.25) with $K_{l+1}$ in place of $K$ and the argument. Then, it just remains to prove (2.28).

We again proceed by induction. (2.28) is manifestly true for $K_0$. Assume then that it holds true for $K_\ell$ for all the values of the index $\ell$ up to $l \geqslant 0$. We find, in view of the hypotheses on $H$ and the inductive hypothesis,

$$\begin{aligned} &K_{l+1}(\mathbf{t}_{M+1,j-1}(s); y,\xi)|_{t_j=t_{j+1}} \\ &\qquad = t_{j+1} + \int_{t_{j+1}}^{s} H(\sigma, K_l(\mathbf{t}_{M+1,j-1}(\sigma), t_{j+1}, t_{j+1}; y,\xi), R_l(\sigma),\xi)\, d\sigma \\ &\qquad = t_{j+1} + \int_{t_{j+1}}^{s} H(\sigma,\sigma, t_{j+1}; R_l(\sigma),\xi)\, d\sigma = t_{j+1} + \int_{t_{j+1}}^{s} d\sigma = s, \end{aligned}$$

$$\begin{aligned} &K_{l+1}(\mathbf{t}_{M+1,j-1}(s); y,\xi)|_{t_j=s} \\ &\qquad = t_{j+1} + \int_{s}^{s} H(\sigma, K_l(\mathbf{t}_{M+1,j-1}(\sigma), s, t_{j+1}; y,\xi), R_l(\sigma),\xi)\, d\sigma = t_{j+1}, \end{aligned}$$

which completes the proof of (2.28).

In order to show that $\{R_l\}$ and $\{K_l\}$ converge, we employ Taylor formula with respect to the variable $t_{j-1}$. For an arbitrary $N \in \mathbb{Z}_+$ we can write

$$\begin{aligned} K_{l+1}(t_{j-1}) - K_l(t_{j-1}) &= \sum_{k<N} \frac{\left(\left(\partial^k_{t_{j-1}} K_{l+1}\right)(t_j) - \left(\partial^k_{t_{j-1}} K_l\right)(t_j)\right)(t_{j-1}-t_j)^k}{k!} \\ &+ \frac{1}{N!}\int_{t_j}^{t_{j-1}} (t_{j-1}-\sigma)^N \left(\left(\partial^{N+1}_{t_{j-1}} K_{l+1}\right)(\sigma) - \left(\partial^{N+1}_{t_{j-1}} K_l\right)(\sigma)\right) d\sigma \end{aligned} \tag{2.30}$$

and

$$\begin{aligned} R_{l+1}(t_{j-1}) - R_l(t_{j-1}) &= \sum_{k<N} \frac{\left(\left(\partial^k_{t_{j-1}} R_{l+1}\right)(t_j) - \left(\partial^k_{t_{j-1}} R_l\right)(t_j)\right)(t_{j-1}-t_j)^k}{k!} \\ &+ \frac{1}{N!}\int_{t_j}^{t_{j-1}} (t_{j-1}-\sigma)^N \left(\left(\partial^{N+1}_{t_{j-1}} R_{l+1}\right)(\sigma) - \left(\partial^{N+1}_{t_{j-1}} R_l\right)(\sigma)\right) d\sigma, \end{aligned} \tag{2.31}$$

respectively. The summations in the above equalities (2.30) and (2.31) are actually identically vanishing. To prove this assertion, we proceed by induction on $N$. Indeed, the claim trivially holds true for $N = 1$, where we immediately see

$$K_{l+1}(t_j) - K_l(t_j) = 0, \quad \text{and } R_{l+1}(t_j) - R_l(t_j) = 0,$$

in view of (2.27), which implies, for any $l \geqslant 0$, $K_l(t_j) = t_{j+1}$ and $R_l(t_j) = y$. Also, when $N = 2$ we find

$$\partial_{t_{j-1}} K_{l+1}(t_j) - \partial_{t_{j-1}} K_l(t_j) = \partial_{t_{j-1}} R_{l+1}(t_j) - \partial_{t_{j-1}} R_l(t_j) = 0,$$

since, for any $\ell \geqslant 0$,

$$\left(\partial_{t_{j-1}} K_{\ell+1}\right)(s) = H(s, K_\ell(s), t_{j+1}; R_\ell(s), \xi),$$

and

$$\left(\partial_{t_{j-1}} R_{\ell+1}\right)(s) = -L(K_\ell(s), s, t_j, t_{j+1}; R_\ell(s), \xi).$$

Now, assume that the claim holds true at the step $N+1$, $N \geqslant 1$. By Faá di Bruno formula for the derivatives of composed functions, one can show that

$$\partial_{t_{j-1}}^{N+1} R_{l+1}(t_j) = -\partial_{t_{j-1}}^{N}\left[L(K_l(\cdot); \cdot, t_j, t_{j+1}; R_l(\cdot), \xi)\right](t_j)$$

is in the span of

$$-\left(\partial_\tau^{j_1} L\right)\left(\prod_{\ell=1}^{j_1} \partial_{t_{j-1}}^{j_{1\ell}} K_l\right)\left(\partial_{t_{j-1}}^{j_2} L\right)\left(\frac{\partial^\alpha L}{\partial x^\alpha}\right)\left(\prod_{i=1}^{|\alpha|} (\partial_{t_{j-1}}^{j_{\alpha i}} R_l)^{\beta_i}\right)(t_j), \qquad (2.32)$$

where $j_1 + j_2 + |\alpha| = N$, $\sum \beta_i = |\alpha|$.

Similarly,

$$\partial_{t_{j-1}}^{N+1} K_{l+1}(t_j) = \partial_{t_{j-1}}^{N}\left[H(\cdot, K_l(\cdot), t_{j+1}; R_l(\cdot), \xi)\right](t_j)$$

is in the span of

$$\left(\partial_{t_{j-1}}^{j_1} H\right)\left(\partial_{t_j}^{j_2} H\right)\left(\prod_{\ell=1}^{j_2} \partial_{t_{j-1}}^{j_{2\ell}} K_l\right)\left(\frac{\partial^\alpha H}{\partial x^\alpha}\right)\left(\prod_{i=1}^{|\alpha|} (\partial_{t_{j-1}}^{j_{\alpha i}} R_l)^{\beta_i}\right)(t_j), \qquad (2.33)$$

where $j_1 + j_2 + |\alpha| = N$, $\sum \beta_i = |\alpha|$. Using (2.32) and (2.33), and the fact that the coefficients in the expressions of the derivatives under examination are independent of $l$, we conclude that

$$\begin{aligned}&\partial_{t_{j-1}}^{N+1} K_{l+1}(t_j) - \partial_{t_{j-1}}^{N+1} K_l(t_j) \\ &= \sum H_{j_1,j_2,\alpha}\left(\prod_{\ell=1}^{\beta} \partial_{t_{j-1}}^{\beta_\ell} K_l \prod_{i=1}^{\delta} \partial_{t_{j-1}}^{\delta_i} R_l - \prod_{\ell=1}^{\beta} \partial_{t_{j-1}}^{\beta_\ell} K_{l-1} \prod_{i=1}^{\delta} \partial_{t_{j-1}}^{\delta_i} R_{l-1}\right)(t_j),\end{aligned}$$

where

$$H_{j_1,j_2,\alpha} = C_{j_1,j_2,\alpha}\left(\partial_{t_{j-1}}^{j_1} H\right)\left(\partial_{t_j}^{j_2} H\right)\left(\frac{\partial^\alpha H}{\partial x^\alpha}\right)(t_j),$$

and $C_{j_1,j_2,\alpha}$ is a suitable constant. The right-hand side of the above formula is identically zero, in view of the recurrence assumption on both the sequences, that is, for $k < N+1$ we have

$$\partial_{t_{j-1}}^{k} K_{l+1}(t_j) - \partial_{t_{j-1}}^{k} K_l(t_j) = 0, \text{ and } \partial_{t_{j-1}}^{k} R_{l+1}(t_j) - \partial_{t_{j-1}}^{k} R_l(t_j) = 0.$$

Then, the expression can be rewritten as a linear combination of products involving only such differences. The argument for $\left(\partial_{t_{j-1}}^{N+1} R_{l+1}(t_j) - \partial_{t_{j-1}}^{N+1} R_l(t_j)\right)$ is completely similar. This proves the claim on the summations in (2.30) and (2.31).

Now, using standard inequalities for the remainder, together with the fact that $\{R_l\}_l$ is bounded in $C^\infty(\Delta(T_2); S^{1,0})$, while $\{K_l\}_l$ is bounded in $C^\infty(\Delta(T_2); S^{0,0})$, from (2.30) and (2.31) we get, for any $\alpha, \beta \in \mathbb{Z}_+$,

$$\sup_{(y,\xi)\in\mathbb{R}^{2d}} \left| \partial_y^\alpha \partial_\xi^\beta \left(K_{l+1} - K_l\right)(\mathbf{t}_{M+1,j-1}(t_{j-1}); y, \xi) \langle y \rangle^{|\alpha|} \langle \xi \rangle^{|\beta|} \right| \leqslant C_{\alpha\beta} \frac{(t_{j-1} - t_j)^{N+1}}{(N+1)!}, \quad (2.34)$$

with $C_{\alpha\beta}$ independent of $j$ and $N$. Similarly, we get

$$\sup_{(y,\xi)\in\mathbb{R}^{2d}} \left| \partial_y^\alpha \partial_\xi^\beta \left(R_{l+1} - R_l\right)(\mathbf{t}_{M+1,j-1}(t_{j-1}); y, \xi) \langle y \rangle^{-1+|\alpha|} \langle \xi \rangle^{|\beta|} \right| \leqslant \widetilde{C}_{\alpha\beta} \frac{(t_{j-1} - t_j)^{N+1}}{(N+1)!}, \quad (2.35)$$

where $\widetilde{C}_{\alpha\beta}$ independent of $j$, $N$.

Writing $l$ in place of $N$ in the right-hand side of (2.34) and (2.35), it easily follows that $(R_l, K_l)$ converges, for $l \to +\infty$, to a unique fixed point $(R, K)$, which satisfies the stated symbol estimates. Since, as we showed above, the properties (2.24) and (2.25) hold true for $(R_l, K_l)$ in place of $(R, K)$, $l \geqslant 0$, uniformly with respect to $M, j, l$, they also hold true for the limit $(R, K)$. The proof is complete. □

The next Corollary 2.8 is a standard result in the theory of Cauchy problems for quasi-linear partial differential equations of the form (2.22), see, e. g., [53]. Its proof is based on the hypotheses on $L$ and $H$, and the properties of the solution of (2.23).

**Corollary 2.8.** *Under the same hypotheses of Lemma 2.7, denoting by* $\bar{R}(\boldsymbol{t}_{M+1}; x, \xi)$ *the solution of the equation*

$$R(\boldsymbol{t}_{M+1}; y, \xi) = x, \quad \boldsymbol{t}_{M+1} \in \Delta(T_2), x, \xi \in \mathbb{R}^d,$$

*the function*

$$\Upsilon(\boldsymbol{t}_{M+1}; x, \xi) = K(\boldsymbol{t}_{M+1}; \bar{R}(\boldsymbol{t}_{M+1}; x, \xi), \xi)$$

*solves the Cauchy problem* (2.22) *for* $x, \xi \in \mathbb{R}^d$, $\boldsymbol{t}_{M+1} \in \Delta(T_2)$, *for a sufficiently small* $T_2 \in (0, T_1]$.

**Remark 2.9.** *Notice that* (2.24) *implies that, for all* $y, \xi \in \mathbb{R}^d$, $\boldsymbol{t}_{M+1} \in \Delta(T_2)$, $T_2 \in (0, T_1]$ *suitably small, the Jacobian matrix* $\frac{\partial R}{\partial y}(\boldsymbol{t}_{M+1}; y, \xi)$ *belongs to a suitably small open neighbourhood of the identity matrix, so it is invertible, with norm in an interval of the form* $[1 - \varepsilon, 1 + \varepsilon]$*, for positive, arbitrarily small* $\varepsilon$*. A standard argument in the SG symbol theory (see, e.g., [41, 42, 44]), shows that* $\bar{R} \in C^\infty(\Delta(T_2), S^{1,0}(\mathbb{R}^{2d}) \otimes \mathbb{R}^d)$ *and that*

$$\langle R(\boldsymbol{t}_{M+1}; y, \xi)\rangle \asymp \langle y \rangle, \quad \langle \bar{R}(\boldsymbol{t}_{M+1}; x, \xi)\rangle \asymp \langle x \rangle,$$

*with constants independent of* $\boldsymbol{t}_{M+1} \in \Delta(T_2)$, $\xi \in \mathbb{R}^d$*. This also implies that* $\Upsilon$ *satisfies* $t_{j+1} \leqslant \Upsilon(\boldsymbol{t}_{M+1}; x, \xi) \leqslant t_{j-1}$, $\boldsymbol{t}_{M+1} \in \Delta(T_2)$, $x, \xi \in \mathbb{R}^d$ *and* $\Upsilon \in C^\infty(\Delta(T_2); S^{0,0}(\mathbb{R}^{2d}))$*.*

## 2.3 Commutative law for multi-products of SG phase functions given by solutions of eikonal equations

Let $\{a_j\}_{j \in \mathbb{N}}$ be a bounded family of parameter-dependent, real-valued symbols in $C^\infty([0,T]; S^{1,1})$ and let $\{\varphi_j\}_{j \in \mathbb{N}}$ be the corresponding family of phase functions in $\mathcal{P}_r(c|t-s|)$, obtained as solutions to the eikonal equations associated with $a_j$, $j \in \mathbb{N}$. In the aforementioned multi-product (2.3), we commute $\varphi_j$ and $\varphi_{j+1}$, defining a new multi-product $\phi_j$, namely

$$\begin{aligned}\phi_j(\mathbf{t}_{M+1}; x, \xi) = (\varphi_1(t_0, t_1) \sharp \varphi_2(t_1, t_2) \sharp \dots \sharp \varphi_{j-1}(t_{j-2}, t_{j-1}) \sharp \\ \sharp \varphi_{j+1}(t_{j-1}, t_j) \sharp \varphi_j(t_j, t_{j+1}) \sharp \\ \sharp \varphi_{j+2}(t_{j+1}, t_{j+2}) \sharp \dots \sharp \varphi_{M+1}(t_M, t_{M+1}))\,(x, \xi),\end{aligned} \tag{2.36}$$

where $\mathbf{t}_{M+1} = (t_0, t_1, \dots, \dots t_{M+1}) \in \Delta(T_1)$.

**Assumption I** (Involutiveness of symbol families)**.** *Given the family of parameter-dependent, real-valued symbols* $\{a_j\}_{j \in \mathbb{N}} \subset C^\infty([0,T]; S^{1,1}(\mathbb{R}^{2d}))$*, there exist families of parameter-dependent, real-valued symbols* $\{b_{j,k}\}_{j,k \in \mathbb{N}}$ *and* $\{d_{j,k}\}_{j,k \in \mathbb{N}}$*, such that* $b_{j,k}, d_{j,k} \in C^\infty([0,T]; S^{0,0}(\mathbb{R}^{2d}))$, $j, k \in \mathbb{N}$*, and the Poisson brackets*

$$\begin{aligned}\{\tau - a_j(t; x, \xi), \tau - a_k(t; x, \xi)\} := \partial_t a_j(t; x, \xi) - \partial_t a_k(t; x, \xi) \\ + a'_{j,\xi}(t; x, \xi) \cdot a'_{k,x}(t; x, \xi) - a'_{j,x}(t; x, \xi) \cdot a'_{k,\xi}(t; x, \xi)\end{aligned}$$

*satisfy*

$$\{\tau - a_j(t; x, \xi), \tau - a_k(t; x, \xi)\} = b_{j,k}(t; x, \xi) \cdot (a_j - a_k)(t; x, \xi) + d_{j,k}(t; x, \xi),$$

*for all* $j, k \in \mathbb{N}$, $t \in [0, T]$, $x, \xi \in \mathbb{R}^d$*.*

We can now state our first main theorem.

**Theorem 2.10.** *Let $\{a_j\}_{j\in\mathbb{N}}$ be a family of parameter-dependent, real-valued symbols, bounded in $C^\infty\left([0,T];S^{1,1}(\mathbb{R}^{2d})\right)$, and let $\varphi_j \in \mathcal{P}_r(c|t-s|)$, for some $c>0$, be the phase functions obtained as solutions to (1.24) with $a_j$ in place of $a$, $j\in\mathbb{N}$. Consider $\phi(\boldsymbol{t}_{M+1})$ and $\phi_j(\boldsymbol{t}_{M+1})$ defined in (2.3) and (2.36), respectively, for any $M\geqslant 2$ and $j\leqslant M$. Then, Assumption I implies that there exists $T'\in(0,T_2]$, independent of $M$, such that we can find a symbol family $Z_j\in C^\infty(\Delta(T');S^{0,0}(\mathbb{R}^{2d}))$ satisfying, for all $\boldsymbol{t}_{M+1}\in\Delta(T')$, $x,\xi\in\mathbb{R}^d$,*

$$\begin{aligned}&t_{j+1}\leqslant Z_j(\boldsymbol{t}_{M+1};x,\xi)\leqslant t_{j-1},\\&Z_j|_{t_j=t_{j-1}}=t_{j+1},\ \text{and}\ Z_j|_{t_j=t_{j+1}}=t_{j-1}.\end{aligned}\tag{2.37}$$

*Moreover, we have*

$$\begin{aligned}&\phi_j(\boldsymbol{t}_{M+1};x,\xi)\\&=\phi(\boldsymbol{t}_{M+1,j}(Z_j(\boldsymbol{t}_{M+1};x,\xi));x,\xi)+\Psi_j(\boldsymbol{t}_{M+1};x,\xi),\,\boldsymbol{t}_{M+1}\in\Delta(T'),x,\xi\in\mathbb{R}^d\end{aligned}$$

*where $\Psi_j\in C^\infty(\Delta(T');S^{0,0}(\mathbb{R}^{2d}))$ satisfies*

$$\Psi_j\equiv 0 \quad \text{if} \quad d_{j,j+1}\equiv 0 \text{ in Assumption I.}$$

*Proof.* We show that the argument originally given in [110] extends to the SG setting, in view of Lemma 2.7 above. Let $\{(Y_1,\dots,Y_M,N_1,\dots,N_M)\}(\mathbf{t}_{M+1};x,\xi)$ be the solution of the critical point system

$$\begin{cases}x_j &= \varphi'_{j,\xi}(t_{j-1},t_j;x_{j-1},\xi_j)\\ \xi_j &= \varphi'_{j+1,x}(t_j,t_{j+1};x_j,\xi_{j+1}),\end{cases}$$

such that $x_j,\xi_j\in\mathbb{R}^d$, $x_0=x$, and $\xi_{M+1}=\xi$ (cf. [8,86]).

In view of (2.36), let $\left(\tilde{Y}_1,\dots,\tilde{Y}_M,\tilde{N}_1,\dots,\tilde{N}_M\right)(\mathbf{t}_{M+1};x,\xi)$ be the solution to the critical points problem, for the phase functions in modified order, namely,

$$\begin{cases}x_k &= \varphi'_{k,\xi}(t_{k-1},t_k;x_{k-1},\xi_k) \qquad \text{if } k\in\{1,\dots,j-1,j+2,\dots,M\}\\ x_j &= \varphi'_{j+1,\xi}(t_{j-1},t_j;x_j,\xi_j),\\ x_{j+1} &= \varphi'_{j,\xi}(t_j,t_{j+1};x_j,\xi_{j+1}),\end{cases}$$

$$\begin{cases}\xi_k &= \varphi'_{k+1,x}(t_k,t_{k+1};x_k,\xi_{k+1}) \qquad \text{if } k\in\{1,\dots,j-2,j+1,\dots,M\}\\ \xi_{j-1} &= \varphi'_{j+1,x}(t_{j-1},t_j;x_{j-1},\xi_j),\\ \xi_j &= \varphi'_{j,x}(t_j,t_{j+1};x_j,\xi_{j+1}),\end{cases}$$

where $x_0 = x$ and $\xi_{M+1} = \xi$. For convenience below, we also set

$$\begin{cases} a_0(t;x,\xi) & \equiv 0, \\ Y_0 = \tilde{Y}_0 & = x, \\ N_0 = \phi'_x, & \tilde{N}_0 = \phi'_{j,x}. \end{cases}$$

Let $\Psi_j$ be defined as

$$\Psi_j(\mathbf{t}_{M+1};x,\xi) = \phi_j(\mathbf{t}_{M+1};x,\xi) - \phi(\mathbf{t}_{M+1,j}(Z_j(\mathbf{t}_{M+1};x,\xi));x,\xi). \quad (2.38)$$

Here we look for a symbol $Z_j = Z_j(\mathbf{t}_{M+1};x,\xi)$ satisfying (2.37) such that $\Psi_j \in C^\infty(\Delta(T');S^{0,0})$, $T' \in (0,T_2]$. In view of Proposition 2.2 and (2.36), we find

$$\begin{aligned} &\partial_{t_{j-1}}\Psi_j(\mathbf{t}_{M+1};x,\xi) \\ &\qquad = (\partial_{t_{j-1}}\phi_j)(\mathbf{t}_{M+1};x,\xi) - (\partial_{t_{j-1}}\phi)(\mathbf{t}_{M+1,j}(Z_j(\mathbf{t}_{M+1};x,\xi));x,\xi) \\ &\qquad - (\partial_{t_j}\phi)(\mathbf{t}_{M+1,j}(Z_j(\mathbf{t}_{M+1};x,\xi));x,\xi)\cdot\partial_{t_{j-1}}Z_j(\mathbf{t}_{M+1};x,\xi) \\ &\qquad = a_{j+1}(t_{j-1};Y_{j-1}(\mathbf{t}_{M+1};x,\xi),N_{j-1}(\mathbf{t}_{M+1};x,\xi)) \\ &\qquad - a_{j-1}(t_{j-1};Y_{j-1}(\mathbf{t}_{M+1};x,\xi),N_{j-1}(\mathbf{t}_{M+1};x,\xi)) \\ &\qquad - [a_j(t_{j-1};Y_{j-1}(\mathbf{t}_{M+1};x,\xi),N_{j-1}(\mathbf{t}_{M+1};x,\xi))]_{t_j=Z_j(\mathbf{t}_{M+1};x,\xi)} \\ &\qquad + [a_{j-1}(t_{j-1};Y_{j-1}(\mathbf{t}_{M+1};x,\xi),N_{j-1}(\mathbf{t}_{M+1};x,\xi))]_{t_j=Z_j(\mathbf{t}_{M+1};x,\xi)} \\ &\qquad - (\partial_{t_j}\phi)(\mathbf{t}_{M+1,j}(Z_j(\mathbf{t}_{M+1};x,\xi));x,\xi)\cdot\partial_{t_{j-1}}Z_j(\mathbf{t}_{M+1};x,\xi). \end{aligned} \quad (2.39)$$

When $j \geqslant 2$, we use the trajectory $(\tilde{q}_{j-1},\tilde{p}_{j-1})(\mathbf{t}_{j-2},\sigma;y,\eta)$ defined in (2.18). Then (cf. Proposition 2.5), we have, for $\sigma = t_{j-1}$, the equalities

$$\begin{cases} (\tilde{q}_{j-1},\tilde{p}_{j-1})(\mathbf{t}_{j-1};x,\phi'_x(\mathbf{t}_{M+1};x,\xi)) & = (Y_{j-1},N_{j-1})(\mathbf{t}_{M+1};x,\xi), \\ (\tilde{q}_{j-1},\tilde{p}_{j-1})(\mathbf{t}_{j-1};x,\phi'_{j,x}(\mathbf{t}_{M+1};x,\xi)) & = (\tilde{Y}_{j-1},\tilde{N}_{j-1})(\mathbf{t}_{M+1};x,\xi). \end{cases}$$

Next, we set

$$\begin{cases} \alpha_1(\sigma;z,\zeta) & = a_2(\sigma;z,\zeta), \\ \alpha_j(\sigma;\mathbf{t}_{j-2};z,\zeta) & = (a_{j+1}-a_{j-1})\big(\sigma;(\tilde{q}_{j-1},\tilde{p}_{j-1})(\mathbf{t}_{j-2},\sigma;z,\zeta)\big),\ j \geqslant 2. \end{cases} \quad (2.40)$$

In (2.40) the compositions are well-defined, in view of the properties of the symbols $(\tilde{q}_{j-1},\tilde{p}_{j-1})$ in Proposition 2.6, which imply that the conditions of Lemma 1.10 are satisfied. Thus, $\alpha_j \in C^\infty(\Delta(T_1);S^{1,1})$, $j = 1,\dots,M$.

Moreover, $\alpha_j$ satisfies

$$\begin{cases} \alpha_j(\mathbf{t}_{j-2},\sigma;x,\phi'_x(\mathbf{t}_{M+1};x,\xi)) = \\ \qquad (a_{j+1}-a_{j-1})\,(\sigma;Y_{j-1}(\mathbf{t}_{M+1};x,\xi),N_{j-1}(\mathbf{t}_{M+1};x,\xi))\,, \\ \alpha_j(\mathbf{t}_{j-2},\sigma;x,\phi'_{j,x}(\mathbf{t}_{M+1};x,\xi)) = \\ \qquad (a_{j+1}-a_{j-1})(\sigma;\tilde{Y}_{j-1}(\mathbf{t}_{M+1};x,\xi),\tilde{N}_{j-1}(\mathbf{t}_{M+1};x,\xi)), \end{cases} \tag{2.41}$$

and when $j=1$ the variables $(\mathbf{t}_{j-2},\sigma)$ reduce to $\sigma$.

Finally, let $[T_j(\tau)](\mathbf{t}_{M+1};x,\xi) = T_j(\tau,\mathbf{t}_{M+1};x,\xi)$ be defined as

$$T_j(\tau) = \int_0^1 \alpha'_{j,\xi}\Big(\mathbf{t}_{j-1};x,\rho\phi'_{j,x}(\mathbf{t}_{M+1};x,\xi)+(1-\rho)\phi'_x(\mathbf{t}_{M+1,j}(\tau);x,\xi)\Big)\,d\rho.$$

Notice that, by Lemma 1.10 and the properties of the involved symbols, we find $T_j \in C^\infty([t_{j+1},t_{j-1}]\times\Delta(T_1);S^{1,0}\otimes\mathbb{R}^d)$. Indeed, in view of the fact that both $\phi$ and $\phi_j$ are regular SG phase functions, for all $\rho\in[0,1]$, $\tau\in[t_{j+1},t_{j-1}]$,

$$\langle\rho\phi'_{j,x}(\mathbf{t}_{M+1};x,\xi)+(1-\rho)\phi'_x(\mathbf{t}_{M+1,j}(\tau);x,\xi)\rangle \asymp \langle\xi\rangle,$$

uniformly with respect to all the involved parameters.

We now show that $\phi_j$ satisfies a certain partial differential equation, whose form we will simplify using the results in Section 2.2. First of all, we observe that

$$\begin{aligned} \alpha_j(\mathbf{t}_{j-1};&x,\phi'_{j,x}(\mathbf{t}_{M+1};x,\xi)) - \alpha_j(\mathbf{t}_{j-1};x,\phi'_x(\mathbf{t}_{M+1,j}(\tau);x,\xi)) \\ &= \langle T_j(\tau),\Big(\phi'_{j,x}(\mathbf{t}_{M+1};x,\xi)-\phi'_x(\mathbf{t}_{M+1,j}(\tau);x,\xi)\Big)\rangle. \end{aligned} \tag{2.42}$$

From (2.38) it follows

$$\Psi'_{j,x}(\mathbf{t}_{M+1};x,\xi) = \phi'_{j,x}(\mathbf{t}_{M+1};x,\xi)-(\partial_{t_j}\phi)(\mathbf{t}_{M+1,j}(Z_j);x,\xi)\cdot Z'_{j,x}(\mathbf{t}_{M+1};x,\xi). \tag{2.43}$$

Now, we rewrite $\partial_{t_{j-1}}\Psi_j$ from (2.39), using (2.41), (2.42) and (2.43):

$$\begin{aligned} \partial_{t_{j-1}}\Psi_j(\mathbf{t}_{M+1};x,\xi) &= \alpha_j\left(\mathbf{t}_{j-1};x,\phi'_{j,x}(\mathbf{t}_{M+1};x,\xi)\right) \\ &\quad - \alpha_j\left(\mathbf{t}_{j-1};x,\phi'_x(\mathbf{t}_{M+1,j}(Z_j(\mathbf{t}_{M+1};x,\xi));x,\xi)\right) \\ &\quad - (\partial_{t_j}\phi)(\mathbf{t}_{M+1,j}(Z_j(\mathbf{t}_{M+1};x,\xi));x,\xi)\cdot(\partial_{t_{j-1}}Z_j)(\mathbf{t}_{M+1};x,\xi) \\ &\quad - \left[(a_j-a_{j+1})\,(t_{j-1};Y_{j-1}(\mathbf{t}_{M+1};x,\xi),N_{j-1}(\mathbf{t}_{M+1};x,\xi))\right]_{t_j=Z_j(\mathbf{t}_{M+1};x,\xi)} \\ &= \langle T_j(Z_j(\mathbf{t}_{M+1};x,\xi)),\Psi'_{j,x}(\mathbf{t}_{M+1};x,\xi)\rangle - (\partial_{t_j}\phi)(\mathbf{t}_{M+1,j}(Z_j(\mathbf{t}_{M+1};x,\xi));x,\xi) \\ &\quad \cdot\Big(\partial_{t_{j-1}}Z_j(\mathbf{t}_{M+1};x,\xi)-\langle T_j(Z_j(\mathbf{t}_{M+1};x,\xi)),Z'_{j,x}(\mathbf{t}_{M+1};x,\xi))\rangle\Big) \\ &\quad - \left[(a_j-a_{j+1})\,(t_{j-1};Y_{j-1}(\mathbf{t}_{M+1};x,\xi),N_{j-1}(\mathbf{t}_{M+1};x,\xi))\right]_{t_j=Z_j(\mathbf{t}_{M+1};x,\xi)}\,. \end{aligned} \tag{2.44}$$

Once more, we use the solution $(q_j, p_j)\,(t, s; y, \eta)$ of the Hamilton-Jacobi system (1.25), with $a$ replaced by $a_j$, and define $\tilde{\alpha}_j$ as

$$\tilde{\alpha}_j(\sigma, t; y, \eta) = (a_j - a_{j+1})\,(\sigma; (q_j, p_j)\,(\sigma, t; y, \eta)).$$

Then, after a differentiation with respect to $\sigma$, we have

$$\begin{aligned}
\partial_\sigma \tilde{\alpha}_j(\sigma, t; y, \eta) &= \partial_\sigma \left((a_j - a_{j+1})\,(\sigma; (q_j, p_j)\,(\sigma, t; y, \eta))\right) \\
&= (\partial_t a_j)\,(\sigma; (q_j, p_j)\,(\sigma, t; y, \eta)) - (\partial_t a_{j+1})(\sigma; (q_j, p_j)\,(\sigma, t; y, \eta)) \\
&+ \langle a'_{j,x}(\sigma; (q_j, p_j)\,(\sigma, t; y, \eta)) - a'_{j+1,x}(\sigma; (q_j, p_j)\,(\sigma, t; y, \eta)), \partial_t q_j(\sigma, t; y, \eta)\rangle \\
&+ \langle a'_{j,\xi}(\sigma; (q_j, p_j)\,(\sigma, t; y, \eta)) - a'_{j+1,\xi}(\sigma; (q_j, p_j)\,(\sigma, t; y, \eta)), \partial_t p_j(\sigma, t; y, \eta)\rangle,
\end{aligned}$$

and we use (1.25) to write

$$\begin{aligned}
\partial_\sigma \tilde{\alpha}_j(\sigma, t; y, \eta) &= [\partial_\sigma(a_j - a_{j+1})]\,(\sigma; (q_j, p_j)\,(\sigma, t; y, \eta)) \\
&\quad + \langle a'_{j,\xi}(\sigma; (q_j, p_j)\,(\sigma, t; y, \eta)), a'_{j+1,x}(\sigma; (q_j, p_j)\,(\sigma, t; y, \eta))\rangle \\
&\quad - \langle a'_{j,x}(\sigma; (q_j, p_j)\,(\sigma, t; y, \eta)), a'_{j+1,\xi}(\sigma; (q_j, p_j)\,(\sigma, t; y, \eta))\rangle \\
&\quad = \{\tau - a_j, \tau - a_{j+1}\}(\sigma; (q_j, p_j)\,(\sigma, t; y, \eta)).
\end{aligned}$$

Assumption I then implies

$$\begin{aligned}
&\partial_\sigma \tilde{\alpha}_j(\sigma, t; y, \eta) \\
&= b_{j,j+1}(\sigma; (q_j, p_j)\,(\sigma, t; y, \eta)) \cdot \tilde{\alpha}_j(\sigma, t; y, \eta) + d_{j,j+1}(\sigma; (q_j, p_j)\,(\sigma, t; y, \eta)).
\end{aligned} \tag{2.45}$$

Solving (2.45) as a first order linear ordinary differential equation in $\sigma$ with unknown $\tilde{\alpha}_j(\sigma, t; y, \eta)$, and writing $b_j$ in place of $b_{j,j+1}$, $d_j$ in place of $d_{j,j+1}$, respectively, we see that

$$\begin{aligned}
\tilde{\alpha}_j(\sigma, t; y, \eta) &= \exp\left(\int_{t_j}^{\sigma} b_j(\tau; (q_j, p_j)\,(\tau, t; y, \eta))d\tau\right) \cdot \Bigg[\tilde{\alpha}_j(t_j, t; y, \eta) + \\
&+ \int_{t_j}^{\sigma} d_j\,(\nu; (q_j, p_j)\,(\nu, t; y, \eta)) \cdot \exp\left(-\int_{\nu}^{\sigma} b_j\,(\varsigma; (q_j, p_j)\,(\varsigma, t; y, \eta))\,d\varsigma\right) d\nu\Bigg].
\end{aligned}$$

Once again, notice that all the composition performed so far are well-defined, and produce SG symbols, in view of Lemma 1.10, (1.27), and recalling that $h \in S^{0,0} \Rightarrow \exp(h) \in S^{0,0}$ (see, e.g., [42, 44]).

As stated in Proposition 2.5, we can write $\tilde{\alpha}_j$ in terms of the solution to the critical points problem (1.20). Indeed, by (2.19), (2.20) we get

$$\begin{aligned}
&\tilde{\alpha}_j(t_j, t_{j-1}; (q_j, p_j)\,(t_j, t_{j-1}; Y_{j-1}(\mathbf{t}_{M+1}; x, \xi), N_{j-1}(\mathbf{t}_{M+1}; x, \xi))) = \\
&\qquad = \tilde{\alpha}_j(t_j, t_{j-1}; Y_j(\mathbf{t}_{M+1}; x, \xi), N_j(\mathbf{t}_{M+1}; x, \xi)) \\
&\qquad = (a_j - a_{j+1})\,(t_j; Y_j(\mathbf{t}_{M+1}; x, \xi), N_j(\mathbf{t}_{M+1}; x, \xi)).
\end{aligned}$$

Moreover, using Proposition 2.2, we obtain the equality

$$\tilde{\alpha}_j(t_j, t_{j-1}; (q_j, p_j)\,(t_j, t_{j-1}; Y_{j-1}(\mathbf{t}_{M+1}; x, \xi), N_{j-1}(\mathbf{t}_{M+1}; x, \xi))) = \\ = -\partial_{t_j}\phi(\mathbf{t}_{M+1}; x, \xi). \quad (2.46)$$

Define also

$$G_j(t_j) \equiv G_j(\mathbf{t}_{M+1}; x, \xi) = \\ = \exp\left[\int_{t_j}^{t_{j-1}} b_j(\tau; (q_j, p_j)\,(\tau, t_{j-1}; Y_{j-1}(\mathbf{t}_{M+1}; x, \xi), N_{j-1}(\mathbf{t}_{M+1}; x, \xi)))d\tau\right]$$

and

$$F_j(t_j) \equiv F_j(\mathbf{t}_{M+1}; x, \xi) = \left[\int_{t_j}^{t_{j-1}} d_j\,(\nu; (q_j, p_j)\,(\nu, t_{j-1}; y, \eta)) \cdot \right. \\ \left. \cdot \exp\left(-\int_{\nu}^{t_{j-1}} b_j\,(\varsigma; (q_j, p_j)\,(\varsigma, t_{j-1}; y, \eta))\,d\varsigma\right) d\nu\right]_{(y,\eta)=(Y_{j-1},N_{j-1})(\mathbf{t}_{M+1};x,\xi)},$$

where both $G_j$ and $F_j$, as a consequence of Lemma 1.10 and the properties of $b_j$, $d_j$, $(q_j, p_j)$ and $(Y_{j-1}, N_{j-1})$, are symbols belonging to $C^\infty(\Delta(T_1); S^{0,0})$.

Then, using the formulae (2.44) and (2.46) above, we find that $\Psi_j$ must fullfill

$$\partial_{t_{j-1}}\Psi_j = \langle T_j(Z_j), \Psi'_{j,x}\rangle - F_j(Z_j) \\ - (\partial_{t_j}\phi)(Z_j)\left(\partial_{t_{j-1}}Z_j - \langle T_j(Z_j), Z'_{j,x}\rangle - G_j(Z_j)\right), \quad (2.47)$$

where we omitted everywhere the dependence on $(\mathbf{t}_{M+1}; x, \xi)$, $(\partial_{t_j}\phi)(Z_j)$ stands for $(\partial_{t_j}\phi)(\mathbf{t}_{M+1,j}(Z_j(\mathbf{t}_{M+1}; x, \xi)); x, \xi)$, and

$$\begin{cases} F_j(Z_j) & = F_j(\mathbf{t}_{M+1,j}(Z_j(\mathbf{t}_{M+1}; x, \xi)); x, \xi), \\ G_j(Z_j) & = G_j(\mathbf{t}_{M+1,j}(Z_j(\mathbf{t}_{M+1}; x, \xi)); x, \xi). \end{cases}$$

Now, in order to simplify (2.47), we choose $Z_j$ as solution to the quasi-linear Cauchy problem,

$$\begin{cases} \partial_{t_{j-1}}Z_j & = \langle T_j(Z_j), Z'_{j,x}\rangle + G_j(Z_j) \\ Z_{|t_j=t_{j-1}} & = t_{j+1}. \end{cases} \quad (2.48)$$

in order to simplify (2.47).

It is easy to see that (2.48) is a quasi-linear Cauhcy problem of the type considered in Section 2.2. In view of Lemma 2.7, we can solve (2.48) through its characteristic system (2.23), with $T_j$ in place of $L$ and $G_j$ in place of $H$, choosing a sufficiently small parameter interval $[0,T']$, $T' \in (0,T_2]$. Indeed, by Corollary 2.8 and Remark 2.9, defining

$$Z_j(\mathbf{t}_{M+1};x,\xi) = K(\mathbf{t}_{M+1};\bar{R}(\mathbf{t}_{M+1};x,\xi),\xi), \quad \mathbf{t}_{M+1} \in \Delta(T'), x,\xi \in \mathbb{R}^d,$$

gives a solution of (2.48) with all the desired properties. That is, the symbol $Z_j(\mathbf{t}_{M+1};x,\xi)$ belongs to $C^\infty(\Delta(T');S^{0,0})$ for any $j \leqslant M$, it also satisfies

$$\begin{cases} t_{j+1} \leqslant Z_j(\mathbf{t}_{M+1};x,\xi) \leqslant t_{j-1}, \\ Z_j|_{t_j=t_{j-1}} = t_{j+1}, \qquad Z_j|_{t_j=t_{j+1}} = t_{j-1}, \end{cases}$$

and we have

$$\phi_j(\mathbf{t}_{M+1};x,\xi) = \phi(\mathbf{t}_{M+1,j}(Z_j(\mathbf{t}_{M+1};x,\xi));x,\xi) + \Psi_j(\mathbf{t}_{M+1};x,\xi).$$

Finally, due to the fact that $Z_j$ is a solution to (2.48), the equation (2.47) is reduced to

$$\partial_{t_{j-1}}\Psi_j = \langle T_j(Z_j), \Psi'_{j,x}\rangle - F_j(Z_j), \tag{2.49}$$

with the initial condition

$$\Psi_{j|t_{j-1}=t_j} = 0. \tag{2.50}$$

Notice that (2.50) holds true since we have $Z_j|_{t_j=t_{j-1}} = t_{j+1}$, and (ii) in Proposition 2.2, gives

$$\begin{aligned}
&\Psi_j(\mathbf{t}_{M+1,j-1}(t_j);x,\xi) = \\
&\phi_j(\mathbf{t}_{M+1,j-1}(t_j);x,\xi) - \phi(t_0,\dots,t_{j-2},t_j,Z_{j|t_j=t_{j-1}},t_{j+1},\dots,t_{M+1};x,\xi) \\
&\quad = \Big(\varphi_1(t_0,t_1)\sharp\dots\sharp\varphi_{j-1}(t_{j-2},t_j)\sharp \underbrace{\{\varphi_{j+1}(t_j,t_j)\sharp\varphi_j(t_j,t_{j+1})\}}_{=\varphi_j(t_j,t_{j+1})} \\
&\qquad \sharp\varphi_{j+2}(t_{j+1},t_{j+2})\sharp\dots\varphi_{M+1}(t_M,t_{M+1})\Big)(x,\xi) \\
&\quad - \Big(\varphi_1(t_0,t_1)\sharp\dots\sharp\varphi_{j-1}(t_{j-2},t_j)\sharp \underbrace{\{\varphi_j(t_j,t_{j+1})\sharp\varphi_{j+1}(t_{j+1},t_{j+1})\}}_{=\varphi_j(t_j,t_{j+1})} \\
&\qquad \sharp\varphi_{j+2}(t_{j+1},t_{j+2})\sharp\dots\sharp\varphi_{M+1}(t_M,t_{M+1})\Big)(x,\xi) = 0.
\end{aligned}$$

Then, the method of characteristics, applied to the linear, non-homogeneous partial differential equation (2.49), shows that we can write $\Psi_j$ in the form

$$\Psi_j(\mathbf{t}_{M+1};x,\xi) = \int_{t_j}^{t_{j-1}} \widetilde{F}_j(\mathbf{t}_{M+1,j-1}(\tau);\theta(\tau;\tilde{\theta}(\tau;x,\xi),\xi),\xi)\,d\tau, \tag{2.51}$$

where

$$\widetilde{F}_j(\mathbf{t}_{M+1};x,\xi) = -F_j(\mathbf{t}_{M+1,j}(Z_j(\mathbf{t}_{M+1};x,\xi));x,\xi), \tag{2.52}$$

$$\theta(\tau;y,\xi) = \quad \theta(\mathbf{t}_{M+1,j}(\tau);y,\xi), \tag{2.53}$$

$$\tilde{\theta}(\tau;x,\xi) = \quad \tilde{\theta}(\mathbf{t}_{M+1,j-1}(\tau);x,\xi), \tag{2.54}$$

for suitable vector-valued functions $\theta, \tilde{\theta}$. By arguments similar to those in Subsection 2.2 (cf. [110]), both $\theta$ and $\tilde{\theta}$ turn out to be elements of $C^\infty(\Delta(T'); S^{1,0} \otimes \mathbb{R}^d)$, satisfying

$$\langle\theta(\mathbf{t}_{M+1,j}(\tau);y,\xi)\rangle \asymp \langle x\rangle, \quad \langle\tilde{\theta}(\mathbf{t}_{M+1,j-1}(\tau);x,\xi)\rangle \asymp \langle x\rangle,$$

with constants independent of $\mathbf{t}_{M+1} \in \Delta(T')$, $x, \xi \in \mathbb{R}^d$. Such result, together with the properties of $Z_j$ and another application of Lemma 1.10, allows to conclude that $\Psi_j \in C^\infty(\Delta(T'); S^{0,0})$, and it is identically zero when $d_j \equiv 0$, as claimed. The proof is complete. □

**Corollary 2.11.** *Under the same hypothesis of Theorem 2.10, there exists a constant $C$ independent of $M$ such that*

$$|\partial_{t_j} Z_j + 1| \leqslant C(t_0 - t_{M+1}). \tag{2.55}$$

*Proof.* Set $Z'_j = \partial_{t_j} Z_j(\mathbf{t}_{M+1};x,\xi)$, then from the quasi-linear equation (2.48), we can write $\partial_{t_{j-1}} Z'_j$ as

$$\partial_{t_{j-1}} Z'_j = \langle T_j(Z_j), Z'_{j,x}\rangle + \widetilde{G}_j,$$

where

$$\widetilde{G}_j \equiv \widetilde{G}_j(\mathbf{t}_{M+1};x,\xi) = \langle \partial_{t_j}\left(T_j(Z_j)\right), Z'_{j,x}\rangle + \partial_{t_j}\left(G_j(Z_j)\right). \tag{2.56}$$

Moreover, we can write $Z_j \equiv Z_j(\mathbf{t}_{M+1};x,\xi)$

$$\begin{aligned} Z_j &= t_{j+1} \\ &+ \int_{t_j}^{t_{j-1}} \Big[\langle T_j(Z_j(\mathbf{t}_{M+1,j-1}(\tau);x,\xi);\mathbf{t}_{M+1,j-1}(\tau);x,\xi), Z'_{j,x}(\mathbf{t}_{M+1,j-1}(\tau);x,\xi)\rangle \\ &\quad + G_j(t_0,\dots,t_{j-2},\tau,Z_j(\mathbf{t}_{M+1,j-1}(\tau);x,\xi),\dots,t_{M+1};x,\xi)\Big]\,d\tau. \end{aligned} \tag{2.57}$$

Since $Z'_{j,x_{t_{j-1}=t_j}} = 0$, after a differentiation with respect to $t_j$ of (2.57) we get

$$\begin{aligned} Z'_{j_{t_{j-1}=t_j}} &= -G_j(t_0,\dots,t_{j-2},t_j,t_{j+1},t_{j+1},\dots,t_{M+1};x,\xi) \qquad (2.58) \\ &= -\exp\int_{t_{j+1}}^{t_j} b_j\,d\tau. \end{aligned}$$

As in the proof of the Theorem 2.10, we have the quasi-linear equation

$$\partial_{t_{j-1}} Z_j' = \langle T_j(Z_j), Z_{j,x}' \rangle + \widetilde{G}_j,$$

with initial data

$$Z_j'{}_{t_{j-1}=t_j} = -\exp \int_{t_{j+1}}^{t_j} b_j \, d\tau.$$

Hence, as in (2.51) we can write the solution $Z_j'$ as

$$Z_j' = Z_j'{}_{t_{j-1}=t_j} + \int_{t_j}^{t_{j-1}} \widetilde{G}_j(t_0, \ldots, t_{j-2}, \tau, t_j \ldots, t_{M+1}; \theta_j(\tau; \bar{\theta}_j(\tau; x, \xi), \xi), \xi) \, d\tau, \tag{2.59}$$

where $\widetilde{G}_j, \theta_j$ and $\bar{\theta}_j$ are as in (2.56), (2.53) and (2.54) respectively.

Then, (2.58) and (2.59) imply the desired estimate (2.55). □

# Chapter 3

# Fundamental solutions for involutive SG-hyperbolic systems

In the present chapter we deal with the Cauchy problem

$$\begin{cases} LU(t,s) = F(t), & (t,s) \in \Delta_T \\ U(s,s) = G & s \in [0,T), \end{cases} \tag{3.1}$$

on the simplex $\Delta_T := \{(t,s) |\ 0 \leqslant s \leqslant t \leqslant T\}$, where

$$L(t, D_t; x, D_x) = D_t + \Lambda(t; x, D_x) + R(t; x, D_x), \tag{3.2}$$

$\Lambda$ is a $(m \times m)$-dimensional, diagonal operator matrix, whose entries $\lambda_j$ $(t; x, D_x)$, $j = 1, \ldots, m$, are pseudo-differential operators with real-valued, parameter-dependent symbols $\lambda_j(t; x, \xi) \in C^\infty([0,T]; S^{1,1})$, $R$ is a parameter-dependent, $(m \times m)$-dimensional operator matrix of pseudo-differential operators with symbols in $C^\infty([0,T]; S^{0,0})$, $F \in C^\infty([0,T], H^{r,\varrho} \otimes \mathbb{R}^m)$, $G \in H^{r,\varrho} \otimes \mathbb{R}^m$, $r, \varrho \in \mathbb{R}$.

The system (3.2) is then of hyperbolic type, since the principal symbol part $\operatorname{diag}(\lambda_j(t; x, \xi))_{j=1,\ldots,m}$ of the coefficient matrix is diagonal and real-valued see [41, Chapter 6]. Then, its fundamental solution $E(t,s)$ exists (see [41]), and can be obtained as an infinte sum of matrices of Fourier integral operators (see [86, 110] and Section 5 of [8] for the SG case). Here we are going to show that if (3.1) is of involutive type, then its fundamental solution $E(t,s)$ can be reduced to a finite sum expression, modulo a smoothing remainder, in the same spirit of [86, 110], by applying the results from Chapter 2.

The fundamental solution of (3.1) is a family $\{E(t,s) | (t,s) \in \Delta_{T'}\}$ of operators satisfying

$$\begin{cases} LE(t,s) = 0 & (t,s) \in \Delta_{T'}, \\ E(s,s) = I & s \in [0,T'), \end{cases}$$

for $0 < T' \leqslant T$. For $T'$ small enough, see Section 5 of [8], it is possible to express $\{E(t,s)\}$ in the form

$$E(t,s) = I_\varphi(t,s) + \int_s^t I_\varphi(t,\theta) \sum_{\nu=1}^{\infty} W_\nu(\theta,s)\, d\theta,$$

where $I_\varphi(t,s)$ is the operator matrix defined by

$$I_\varphi(t,s) = \begin{pmatrix} I_{\varphi_1}(t,s) & & 0 \\ & \ddots & \\ 0 & & I_{\varphi_m}(t,s) \end{pmatrix}$$

and $I_{\varphi_j} := Op_{\varphi_j}(1)$, $1 \leqslant j \leqslant m$. The phase functions $\varphi_j = \varphi_j(t,s;x,\xi)$, $1 \leqslant j \leqslant m$, defined on $\Delta_{T'} \times \mathbb{R}^{2d}$, are solutions to the eikonal equations (1.24) with $\lambda_j$ in place of $a$. The sequence of $m \times m$-matrices of SG Fourier integral operators $\{W_\nu(t,s);(t,s) \in \Delta_{T'}\}_{\nu\in\mathbb{N}}$ is defined recursively as

$$W_{\nu+1}(t,s;x,D_x) = \int_s^t W_1(t,\theta;x,D_x) W_\nu(\theta,s;x,D_x)\, d\theta,$$

starting with $W_1$ defined as

$$LI_\varphi(t,s) = iW_1(t,s). \tag{3.3}$$

We also set

$$w_j(t,s;x,\xi) = \sigma(W_j(t,s;x,D_x)), \quad j = 1,\dots,\nu+1, \tag{3.4}$$

the (matrix-valued) symbol of $W_j$.

The following result about existence and uniqueness of a solution $U(t,s)$ to the Cauchy problem (3.1) is a SG variant of the classical Duhamel formula, see [8, 41, 43].

**Proposition 3.1.** *For $F \in C^\infty([0,T]; H^{r,\rho}(\mathbb{R}^d) \otimes \mathbb{R}^m)$ and $G \in H^{r,\rho}(\mathbb{R}^d) \otimes \mathbb{R}^m$, the solution $U(t,s)$ of the Cauchy problem (3.1), under the SG-hyperbolicity assumptions explained above, exists uniquely for $(t,s) \in \Delta_{T'}$, $T' \in (0,T]$ suitably small, it belongs to the class $\bigcap_{k\in\mathbb{Z}_+} C^k(\Delta_{T'}; H^{r-k,\varrho-k}(\mathbb{R}^d) \otimes \mathbb{R}^m)$, and is given by*

$$U(t,s) = E(t,s)G + i\int_s^t E(t,\theta)F(\theta)d\theta, \quad (t,s) \in \Delta_{T'}, s \in [0,T').$$

Notice that, since the phase functions $\varphi_j$ are solutions of eikonal equations (1.24) associated with the Hamiltonians $-\lambda_j$, we have the relation

$$D_t I_{\varphi_j} + \lambda_j(t;\cdot,D) I_{\varphi_j(t,s)} = \mathrm{Op}_{\varphi_j(t,s)}(b_{0,j}(t,s)), \quad b_{0,j}(t,s) \in S^{0,0}(\mathbb{R}^{2d}),$$

$j = 1,\dots,m$. Then,

$$W_1(t,s) := -i\left(\begin{pmatrix} B_{0,1}(t,s) & & 0 \\ & \ddots & \\ 0 & & B_{0,m}(t,s) \end{pmatrix} + R(t) I_\varphi(t,s)\right), \tag{3.5}$$

with $B_{0,j}(t,s) = \mathrm{Op}_{\varphi_j(t,s)}(b_{0,j}(t,s))$ and $b_{0,j}(t,s) \in S^{0,0}$, $j = 1,\dots,m$, and $R(t) \equiv R(t;x,D_x)$ given in (3.2).

By (3.5) and Theorem 1.16, one can rewrite equation (3.3) as

$$L I_\varphi(t,s) = \sum_{j=1}^{m} \widetilde{W}_{\varphi_j}(t,s),$$

where, for $1 \leqslant j \leqslant m$, $\widetilde{W}_{\varphi_j}(t,s)$ are $m \times m$ matrix with entries given by Fourier integral operators with parameter-dependent phase function $\varphi_j$ and symbol in $S^{0,0}$. Thus, if we set $M_\nu = [1,m]^\nu \bigcap \mathbb{N}^\nu$ for $\nu \geqslant 2$, the operator matrix $W_\nu(t,s)$ can be written in the form of iterated integrals, namely

$$\int_s^t \int_s^{t_1} \dots \int_s^{t_{\nu-2}} \sum_{\mu \in M_\nu} W^{(\mu)}(t,t_1,\dots,t_{\nu-1},s)\, dt_{\nu-1}\dots dt_1,$$

where

$$W^{(\mu)}(t,t_1,\dots,t_{\nu-1},s) = W_{\varphi_{m_1}}(t,t_1) W_{\varphi_{m_2}}(t_1,t_2)\dots W_{\varphi_{m_\nu}}(t_{\nu-1},s)$$

is the product of $\nu$ Fourier integral operators matrices with phase functions $\varphi_{m_j}$ and symbols $\sigma(W_{\varphi_{m_j}}(t_{j-1},t_j)) = -i\sigma(\widetilde{W}_{\varphi_{m_j}}(t_{j-1},t_j)) \in S^{0,0}$. By (2) in Theorem 1.25, $W^{(\mu)}(t,t_1,\dots,t_{\nu-1},s)$ is a matrix of Fourier integral operators with phase function $\phi^{(\mu)} = \varphi_{m_1}\sharp\dots\sharp\varphi_{m_\nu}$ and parameter-dependent symbol $\omega^{(\mu)}(t,t_1,\dots,t_{\nu-1},s)$ of order $(0,0)$. Consequently, we can write

$$\begin{aligned} E(t,s) = I_\varphi(t,s) + \int_s^t I_\varphi(t,\sigma)\Big\{ &\sum_{j=1}^{m} W_{\varphi_j}(\theta,s) \\ &+ \sum_{\nu=2}^{\infty}\sum_{\mu\in M_\nu} \int_s^\theta \int_s^{t_1}\dots\int_s^{t_{\nu-2}} W^{(\mu)}(\theta,t_1,\dots,t_{\nu-1},s) dt_{\nu-1}\dots dt_1 \Big\}\, d\theta. \end{aligned} \tag{3.6}$$

**Theorem 3.2.** *Let* (3.1) *be an involutive SG-hyperbolic system, that is, Assumption I is fulfilled by the family* $\{\lambda_j\}_{j=1}^m$*. Then the fundamental solution* (3.6) *can be reduced, modulo smoothing terms, to*

$$\begin{aligned} E(t,s) = & I_\varphi(t,s) + \sum_{j=1}^{m} W^\dagger_{\varphi_j}(t,s) \\ & + \sum_{j=2}^{m} \sum_{\mu \in M_j^\dagger} \int_s^t \int_s^{t_1} \cdots \int_s^{t_{j-2}} W^{(\mu^\dagger)}(t,t_1,\dots,t_{j-1},s)\, dt_{j-1}\dots dt_1, \end{aligned} \tag{3.7}$$

*where the symbol of* $W^\dagger_{\varphi_j}(t,s)$ *is* $\int_s^t w_j(\theta,s)\,d\theta$*, with* $w_j$ *in* (3.4)*, the multi-index* $\mu = (m_1,\dots,m_j) \in M_j^\dagger := \{\mu = (m_1,\dots,m_j) \in M_j : \ m_1 < \cdots < m_j\}$*, and* $W^{(\mu^\dagger)}(t,t_1,\dots,t_{j-1},s)$ *is a* $m \times m$ *dimensional matrix of Fourier integral operators with phase function* $\phi^{(\mu^\dagger)} = \varphi_{m_1}\sharp\dots\sharp\varphi_{m_j}$ *and matrix-valued, parameter-dependent symbol* $\omega^{(\mu^\dagger)}(t,t_1,\dots,t_{j-1},s) \in S^{0,0}(\mathbb{R}^{2d})$*.*

For the proof of Theorem 3.2 we need some preparation. Given $\mu \in M_\nu$, let $\mu(j) = (m_1,\dots,m_{j-1},m_{j+1},m_j,\dots,m_\nu)$ be the permutation where we exchange the order of $m_j$ and $m_{j+1}$, for $1 \leqslant j \leqslant \nu-1$.

The following proposition is a reformulation of Theorem 2.10 for the sharp products of the phase functions $\varphi_j$, $j = 1,\dots,m$, appearing in the expression of $E(t,s)$.

**Proposition 3.3.** *Let* $\{\lambda_k(t;x,\xi)\}_{k=1}^m \subset C^\infty([0,T];S^{1,1}(\mathbb{R}^{2d}))$ *satisfy Assumption I, and let* $\mu \in M_\nu$ *with* $\nu \geqslant 2$*. Denote, respectively, by* $\phi^{(\mu)}$ *and* $\phi^{(\mu(j))}$ *the sharp products in* (2.3) *and* (2.36) *with* $M = \nu - 1$*. Then, for a sufficiently small constant* $T'$*, independent of* $\nu$*, there exist symbols* $Z_j^{(\mu)}(\boldsymbol{t}_\nu;x,\xi)$ *and* $\Psi_j^{(\mu)}(\boldsymbol{t}_\nu;x,\xi)$ *in* $C(\Delta_{T'};S^{0,0}(\mathbb{R}^{2d}))$*,* $\boldsymbol{t}_\nu \in \Delta_{T'}$*, such that, for any* $1 \leqslant j \leqslant \nu-1$*,*

$$t_{j+1} \leqslant Z_j^{(\mu)}(\boldsymbol{t}_\nu;x,\xi) \leqslant t_{j-1}, \qquad (t_0 = t, t_\nu = s), \tag{3.8}$$

$$Z_j^{(\mu)}|_{t_j=t_{j-1}} = t_{j+1}, \qquad Z_j^{(\mu)}|_{t_j=t_{j+1}} = t_{j-1}, \tag{3.9}$$

$$\phi^{(\mu(j))}(\boldsymbol{t}_\nu;x,\xi) = \phi^{(\mu)}\left(\boldsymbol{t}_{\nu,j}(Z_j(\boldsymbol{t}_\nu;x,\xi));x,\xi\right) + \Psi_j^{(\mu)}(\boldsymbol{t}_\nu;x,\xi) \tag{3.10}$$

*where* $\boldsymbol{t}_{\nu,j}(\tau)$ *is defined in* (2.2) *and*

$$|\partial_{t_j} Z_j^{(\mu)}(\boldsymbol{t}_\nu;x,\xi) + 1| \leqslant C(t-s), \tag{3.11}$$

*for a suitable* $C > 0$ *independent of* $\nu$*.*

In the next result we treat the invertibility of the symbols $Z_j^{(\mu)}$, $\mu \in M_\nu$, $j = 1,\dots,\nu-1$, and the properties of its inverse.

**Proposition 3.4.** *Under the hypothesis of Proposition 3.3, assume furthermore that $T'$ satisfies $CT' \leqslant \frac{1}{2}$, where $C$ is the constant in* (3.11)*, and, for $\mu \in M_\nu$, $1 \leqslant j \leqslant \nu - 1$, define*

$$\zeta = Z_j^{(\mu)}(\boldsymbol{t}_{\nu,j}(\theta); x, \xi). \tag{3.12}$$

*Then, $\zeta$ is invertible with inverse $\theta = \Theta_j^{(\mu)}(\boldsymbol{t}_{\nu,j}(\zeta); x, \xi)$, $\Theta_j^{(\mu)}$ belongs to $C^\infty(\Delta_{T'}; S^{0,0}(\mathbb{R}^{2d}))$ and satisfies $t_{j+1} \leqslant \Theta_j^{(\mu)} \leqslant t_{j-1}$.*

*Proof.* Set

$$\Omega = \Big\{\Theta(\mathbf{t}_{\nu,j}(\zeta); x, \xi) \in C\left(\Delta_T \times \mathbb{R}^{2d}\right) : t_{j+1} \leqslant \Theta(\mathbf{t}_{\nu,j}(\zeta); x, \xi) \leqslant t_{j-1},$$
$$\Theta_{|\zeta = t_{j-1}} = t_{j+1},\ \Theta_{|\zeta = t_{j+1}} = t_{j-1},\ -2 \leqslant \partial_\zeta \Theta(\mathbf{t}_{\nu,j}(\zeta); x, \xi) \leqslant 0\Big\},$$

and consider the map

$$\mathcal{T} : \Omega \ni \Theta \longmapsto \mathcal{T}(\Theta) = H$$

defined by

$$H \equiv H(\mathbf{t}_{\nu,j}(\zeta); x, \xi) = -\zeta + Z_j^{(\mu)}(\mathbf{t}_{\nu,j}(\Theta(\mathbf{t}_{\nu,j}(\zeta); x, \xi)); x, \xi) + \Theta(\mathbf{t}_{\nu,j}(\zeta); x, \xi),$$

where $t_0 = t$ and $t_m = s$.

Since $t_{j+1} \leqslant \Theta(\mathbf{t}_{\nu,j}(\zeta); x, \xi) \leqslant t_{j-1}$ for $\Theta \in \Omega$, the mapping $\mathcal{T}$ is well defined. Indeed, from (3.9), (3.11) and $CT' \leqslant 1/2$ we get

$$\begin{cases} H_{|\zeta = t_{j-1}} & = -t_{j-1} + Z_j^{(\mu)}\left(\mathbf{t}_{\nu,j}(\Theta_{|\zeta = t_{j-1}}; \cdot, \cdot\cdot)\right) + \Theta_{|\zeta = t_{j-1}} \\ & = -t_{j-1} + Z_j^{(\mu)}\left(\mathbf{t}_{\nu,j}(t_{j+1}; \cdot, \cdot\cdot)\right) + t_{j+1} \\ & = t_{j+1}, \\ H_{|\zeta = t_{j+1}} & = -t_{j+1} + Z_j^{(\mu)}\left(\mathbf{t}_{\nu,j}(\Theta_{|\zeta = t_{j+1}}; \cdot, \cdot\cdot)\right) + \Theta_{|\zeta = t_{j+1}} \\ & = -t_{j+1} + Z_j^{(\mu)}\left(\mathbf{t}_{\nu,j}(t_{j-1}; \cdot, \cdot\cdot)\right) + t_{j-1} \\ & = t_{j-1}, \\ |(\partial_\zeta H)(\mathbf{t}_{\nu,j}(\zeta); x, \xi) + 1| & = \Big| \left[\partial_{t_j} Z_j^{(\mu)}(\mathbf{t}_{\nu,j}(\Theta(\mathbf{t}_{\nu,j}(\zeta); x, \xi)); x, \xi) + 1\right] \\ & \qquad \cdot \partial_\zeta \Theta(\mathbf{t}_{\nu,j}(\zeta); x, \xi)\Big| \leqslant 2CT' \leqslant 1, \end{cases} \tag{3.13}$$

the last inequality in (3.13) implies $-2 \leqslant (\partial_\zeta H)(\mathbf{t}_{\nu,j}(\zeta); x, \xi) \leqslant 0$, so

$$t_{j+1} = H|_{\zeta = t_{j-1}} \leqslant H \leqslant H|_{\zeta = t_{j+1}} = t_{j-1}$$

holds true and we proved $\mathcal{T} : \Omega \longrightarrow \Omega$.

Now, let $\{\Theta_N\}_{N=0}^{\infty}$ be the sequence in $\Omega$ defined by

$$\begin{cases} \Theta_0 & = t_{j-1} - \zeta + t_{j+1} \\ \Theta_{N+1} & = \mathcal{T}(\Theta_N). \end{cases}$$

From (3.11) and the fact that $CT' \leqslant 1/2$, we get for some positive constant $c$ independent of $N$

$$|\Theta_{N+1} - \Theta_N| \leqslant c2^{-N}.$$

Therefore, $\mathcal{T}$ admits a unique fixed point $\Theta = \Theta_j^{(\mu)} \in \Omega$, providing the inverse function of (3.12). The property $\Theta_j^{(\mu)}(\mathbf{t}_{\nu,j}(\zeta); x, \xi) \in C^\infty(\Delta_{T'}; S^{0,0}(\mathbb{R}^{2d}))$ follows by Lemma 1.10 and a standard invertibility argument in the SG classes (see, e.g., [42, 44]). □

The next Proposition 3.5 can be proved by an induction argument, using Faá di Bruno formula.

**Proposition 3.5.** *Let $p(\boldsymbol{t}_\nu; x, \xi) \in C^\infty(\Delta_T; S^{0,0}(\mathbb{R}^{2d}))$, $\{\Theta_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ be subsets of $C^\infty(\Delta_T; S^{0,0}(\mathbb{R}^{2d}))$. For a fixed sequence $\{j_k\}_{k=1}^{\infty}$ where $1 \leqslant j_k \leqslant \nu - 1$, consider the sequence $\{p_k\}_{k=1}^{\infty}$, defined inductively by*

$$p_k(\boldsymbol{t}_\nu; x, \xi) := p_{k-1}\left(\boldsymbol{t}_{\nu,j_k}\left(\Theta_k(\boldsymbol{t}_\nu; x, \xi)\right); x, \xi\right) \cdot g_k(\boldsymbol{t}_\nu; x, \xi),$$

*for $p_0 = p$, $t_0 = t$ and $t_\nu = s$. Then, for any $l$ there exists $C_l$, independent of $k$ and $\nu$, such that*

$$\|p_k\|_l^{(0)} \leqslant C_l^k \|p\|_l^{(0)},$$

*where*

$$\|p\|_l^{(0)} = \max_{0 \leqslant l' \leqslant l} \max_{|\gamma| = l'} |||\partial_{\boldsymbol{t}_\nu}^{\gamma} p|||_{l-l'}^{0,0}.$$

*Proof of Theorem 3.2.* We split the proof into four steps.

- Step I:

  Let $W^{(\mu)}(t, t_1, \ldots, t_{\nu-1}, s)$ be a Fourier integral operator with the phase function $\phi^{(\mu)}$ for $\mu = (m_1, \ldots, m_\nu)$, and symbol $\omega^{(\mu)}(t, t_1, \ldots, t_{\nu-1}, s)$ in $C^\infty(\Delta_{T'}; S^{0,0})$. Assume that $\mu$ satisfies

  $$\{m_1, \ldots, m_\nu\} = \{\tilde{m}_1, \ldots, \tilde{m}_k\}, \qquad \text{for } k \leqslant m, \tag{3.14}$$

  for $\mu_k = (\tilde{m}_1, \ldots, \tilde{m}_k) \in M_k^\dagger$. Then, we can define a Fourier integral operator $\widetilde{W}^{(\mu_k)}$ depending only on the $k + 1$ time variables $(t, t_1, \ldots, t_{k-1}, s)$, with symbol $\widetilde{\omega}^{(\mu_k)}(t, t_1, \ldots, t_{k-1}, s) \in C^\infty(\Delta_{T'}; S^{0,0})$, such that the following equality holds true

$$W \equiv \int_s^t \int_s^{t_1} \dots \int_s^{t_{\nu-2}} W^{(\mu)}(t, t_1, \dots, t_{\nu-1}, s)\, dt_1 \dots dt_{\nu-1} \tag{3.15}$$
$$= \int_s^t \int_s^{t_1} \dots \int_s^{t_{k-2}} \widetilde{W}^{(\mu_k)}(t, t_1, \dots, t_{k-1}, s)\, dt_1 \dots dt_{k-1}.$$

Moreover, from the fact that the symbol $\omega^{(\mu)}(t, t_1, \dots, t_{\nu-1}, s)$ belongs to $S^{0,0}$ and depends smoothly on all the parameters, we can show that, for any $\gamma \in \mathbb{Z}_+^{k+1}$ and any integer $\ell$, there exists some constant $C_{\gamma,\ell}$, independent of $\nu$, such that

$$|||\partial_{\mathbf{t}_k}^{\gamma} \widetilde{\omega}^{(\mu_k)}(\mathbf{t}_k)|||_{\ell}^{0,0} \leqslant {C_{\gamma,\ell}}^{\nu}/(\nu - k)! \qquad (\nu = m+1, \dots). \tag{3.16}$$

If we admit (3.15) and (3.16), then the proof of Theorem 3.2 is completed. Indeed, (3.7) follows from (3.6) and (3.15), and by (3.16) we find that

$$\omega^{(\mu^\dagger)}(t, t_1, \dots, t_{k-1}, s)$$
$$\equiv \sum_{\nu=k+1}^{\infty} \sum \widetilde{\omega}^{(\mu_k)}(t, t_1, \dots, t_{k-1}, s) \in C^\infty(\Delta_{T'}; S^{0,0})$$

where the second summation extends to all $\mu_k$ satisfying (3.14). In the next steps we will prove (3.15) and (3.16).

- Step II:

Let $\mu(k) = (m_1, \dots, m_{k-1}, m_{k+1}, m_k, m_{k+2}, \dots, m_\nu)$ and set

$$w^{(\mu(k))}(\mathbf{t}_\nu; x, \xi) = w^{(\mu)}\left(\mathbf{t}_{\nu,k}(Z_k^{(\mu)}(\mathbf{t}_\nu); x, \xi); x, \xi\right) \cdot \frac{\partial Z_k^{(\mu)}}{\partial t_k}(\mathbf{t}_\nu; x, \xi). \tag{3.17}$$

Let $W^{(\mu(k))}(\mathbf{t}_\nu)$ be a Fourier integral operator with phase function $\phi^{(\mu(k))}(\mathbf{t}_\nu)$ and symbol $w^{(\mu(k))}(\mathbf{t}_\nu)$. Notice that

$$W = \int_s^t dt_1 \dots \int_s^{t_{k-2}} dt_{k-1} \int_s^{t_{k-1}} dt_{k+1} \int_s^{t_{k+1}} dt_{k+2}$$
$$\dots \int_s^{t_{\nu-2}} dt_{\nu-1} \int_{t_{k+1}}^{t_{k-1}} W^{(\mu)}\, dt_k.$$

By (3.10) and (3.17) we get

$$\int_{t_{k+1}}^{t_{k-1}} W^{(\mu)}\, dt_k = \int_{t_{k+1}}^{t_{k-1}} W^{(\mu(k))}\, dt_k.$$

Therefore, we have

$$W = \int_s^t \cdots \int_s^{t_{\nu-2}} W^{(\mu(k))}(t, t_1, \ldots, t_{\nu-1}, s)\, dt_1 \ldots dt_{\nu-1}. \tag{3.18}$$

- Step III:

For a fixed $N \geqslant 3m$, we divide $\mu = (m_1, \ldots, m_\nu)$ into $(m_1, \ldots, m_N)$, $(m_{N+1}, \ldots, m_{2N})$, $\ldots, (m_{N[\nu-1/N]+1}, \ldots, m_\nu)$ and transpose the elements of the starting vector $(m_{\kappa N+1}, \ldots, m_{(\kappa+1)N})$ to obtain the vector $(m^0_{\kappa N+1}, \ldots, m^0_{(\kappa+1)N})$ satisfying

$$m^0_{\kappa N+1} \leqslant \cdots \leqslant m^0_{(\kappa+1)N},$$

where $\kappa = 0, \ldots, [(\nu-1)/N]$ and $(\kappa+1)N = \nu$ if $\kappa = [(\nu-1)/N]$. Set $\mu^0 = (m^0_1, \ldots, m^0_\nu)$. The number of transpositions

$$((m_k, m_{k+1}) \to (m_{k+1}, m_k)) \text{ to change } \mu \text{ into } \mu^0$$

is not larger than $C_{N,m}([(\nu-1)/N]+1)$, where $C_{N,m}$ denotes the largest number of transpositions which is necessary in changing any $N$-repeated-permutation of elements of $\{1, \ldots, m\}$ into the $N$-repeated permutation with elements arranged in ascending order of magnitude. By repeated transpositions of the type $(m_k, m_{k+1}) \to (m_{k+1}, m_k)$ we get the equality (3.18) with $W^{(\mu(k))}$ replaced by $W^{(\mu^0)}$. The symbol $w^{(\mu^0)}(\mathbf{t}_\nu)$ of $W^{(\mu^0)}$ is defined by the product of at most $C_{N,m}([(\nu-1)/N]+1)$ factors of type $\partial_t Z_k$, the composition of $w^{(\mu)}$ with elements of $\{Z_k\}$, and products of (derivatives of) $(0,0)$-order factors of type $\exp\left[i\Psi_k^{(\mu_\iota)}\right]$. Consequently, by Proposition 3.5 we can conclude

$$\|w^{(\mu)}(t, t_1, \ldots, t_{\nu-1}, s)\|_\ell^{(0)} \leqslant C_{0,\ell}^{(\nu-1)} \times h^{C_{N,m}([(\nu-1)/N]+1)}, \tag{3.19}$$

where $h = \max\limits_{1\leqslant k\leqslant m} \max\limits_{1\leqslant j\leqslant m} \max\limits_{s\leqslant t_j\leqslant t} \left|\partial_{t_j} Z_k(t, t_1, \ldots, t_j, \ldots, t_{\nu-1}, s)\right|$.

- Step IV:

From the definition of $\mu^0$ and Proposition 2.2 it follows that the phase function $\phi^{(\mu^0)}(t, t_1, \ldots, t_{\nu-1}, s)$ is independent of at least $[(\nu/2)]$ elements of the set $\{t_j\}_{j=1}^{\nu-1}$. Hence, for a fixed $\iota$ such that $(\nu-1)/3 \leqslant \iota \leqslant (\nu-1)/2$, we have

$$\phi^{(\mu^0)}(t, t_1, \ldots, t_{\nu-1}, s) = \phi^{(\mu_\iota)}(t, t_{\nu_1}, \ldots, t_{\nu_{\iota-1}}, s),$$

for some $\mu_\iota = (m_{\nu_1}, \ldots, m_{\nu_\iota})$. Then, we obtain

$$V = \int_s^t \int_s^{\tilde{t}_1} \cdots \int_s^{\tilde{t}_{\iota-2}} W^{(\mu_\iota)}(t, \tilde{t}_1, \ldots, \tilde{t}_{\iota-1}, s)\, d\tilde{t}_1 \ldots \tilde{t}_{\iota-1},$$

$$(\tilde{t}_j = t_{\nu_j},\ j = 1, \ldots, \iota-1),$$

where the symbol $w^{(\mu_\iota)}(t,\tilde{t}_1,\dots,\tilde{t}_{\iota-1},s)$ of $W^{(\mu_\iota)}(t,\tilde{t}_1,\dots,\tilde{t}_{\iota-1},s)$ is determined by the integral of $w^{(\mu^0)}(t,t_1,\dots,t_{\nu-1},s)$ with respect to $\{t_j\}_{j=1}^{\nu-1}\setminus\{\tilde{t}_j\}_{j=1}^{\iota-1}$. From the integral representation and (3.19), we have

$$\begin{aligned}&\langle w^{(\mu_\iota)}(t,\tilde{t}_1,\dots,\tilde{t}_{\iota-1},s)\rangle_\ell^{(0)}\\ &\leqslant \frac{(t-\tilde{t}_1)^{\nu_1-1}}{(\nu_1-1)!}\times\frac{(\tilde{t}_1-\tilde{t}_2)^{\nu_2-\nu_1-1}}{(\nu_2-\nu_1-1)!}\times\cdots\times\frac{(\tilde{t}_{\nu-1}-s)^{\nu-1-\nu_{\iota-1}}}{(\nu-1-\nu_{\iota-1})!}\times\langle w^{(\mu^0)}\rangle_\ell^{(0)}\\ &\qquad\leqslant\frac{(t-s)^{\nu-\iota}}{(\nu-\iota)!}\times C_{0,\ell}^{\nu}\times h^{C_{N,m}((\nu-1)/2)},\\ &\qquad\qquad(0\leqslant s\leqslant\tilde{t}_{\iota-1},\leqslant\cdots\leqslant\tilde{t}_1\leqslant t).\end{aligned}$$

Here, we used $[(\nu-1)/N]+1\leqslant(\nu-1)/2$. Applying the previous procedure to the Fourier integral operator $W^{(\mu_\iota)}$ in place of $W^{(\mu)}$, we get the equality

$$V=\int_s^t\cdots\int_s^{t_\rho-2}W^{(\mu_\rho)}(t,t_1,\dots,t_{\rho-1},s)\,dt_1\dots dt_{\rho-1}$$

for some $W^{(\mu_\rho)}$, $\iota/3\leqslant\rho\leqslant\iota/2$, whose symbol satisfies

$$\begin{aligned}&\langle w^{(\mu_\rho)}(t,t_1,\dots,t_{\rho-1},s)\rangle_\ell^{(0)}\\ &\quad\leqslant\frac{(t-s)^{\nu-\iota}}{(\nu-\iota)!}\times\frac{(t-s)^{\iota-\rho}}{(\iota-\rho)!}\times C_{0,\ell}^{\nu}\times h^{C_{N,m}((\nu-1)/2+(\iota-1)/2)}.\end{aligned}\tag{3.20}$$

By repeated applications of this process, we finally obtain (3.15).

Since $\rho\leqslant\iota/2\leqslant\nu/(1/2)^2$, the number of needed transpositions is at most $C_{N,m}\times(\nu-1)$. This fact and (3.20) lead us to (3.16) with $\gamma=0$, since, from the Stirling formula, it follows that

$$(\nu-\mu_1)!(\mu_1-\mu_2)!\dots(\mu_d-k)!\geqslant C_0^{k-\nu}\times(\nu-k)!,$$

for some constant $C_0>0$ if $3\mu_j\geqslant\mu_{j-1}\geqslant2\mu_j$ ($j=1,\dots,d$, $\mu_0=\nu$).

Considering $\partial_{\tau_\nu}^{\gamma}w^{(\mu)}(t,t_1,\dots,t_{\nu-1},s)$ in place of $w^{(\mu)}(t,t_1,\dots,t_{\nu-1},s)$, noting that the number of transpositions is at most multiplied by $\nu$, we get (3.16) with $\gamma\neq0$.

The proof is complete. □

**Remark 3.6.** *Theorem 3.2 extends to the case of a $N\times N$ system such that $\Lambda$ is diagonal and its symbol entries $\lambda_j$, $j=1,\dots,N$, coincide with the (repeated) elements of a family of real-valued, parameter-dependent symbols $\{\tau_j\}_{j=1}^m$ satisfying Assumption I. In such situation, it is enough to work* *initially "block by block" of coinciding elements, and then perform the reduction of* (3.6) *to* (3.7)*. Indeed, product of factors associated with different time variables can be commuted as above. Then, we obtain a multi-product in such a way that the phases* $\varphi_j$ *are nearby with different time variables. Thereafter, we use the associative properties of the multi-products mentioned in Proposition 2.2.*

# Chapter 4

# SG-Hyperbolic Cauchy problems with involutive characteristics

Here we apply the results from the previous chapter to the study of Cauchy problems associated with linear hyperbolic involutive differential operators of SG type. After obtaining the fundamental solution, we study the propagation of singularities in the case of SG-classical coefficients. We recall a few basic definitions, see [41–44, 49] for more details.

**Definition 4.1.** *Let $m \in \mathbb{N}$, $T > 0$, and $L$ be a differential operator of order $m$, that is*

$$L \equiv L(t, D_t; x, D_x) = D_t^m + \sum_{j=1}^{m} P_j(t; x, D_x) D_t^{m-j}$$
$$= D_t^m + \sum_{j=1}^{m} \sum_{|\alpha| \leqslant j} c_{j\alpha}(t; x) D_x^\alpha D_t^{m-j}.$$

*The symbol $\sigma(P_j)$ of the pseudo-differential operator $P_j(t; x, D_x)$ is given by*

$$p_j(t; x, \xi) = \sum_{|\alpha| \leqslant j} c_{j\alpha}(t; x) \xi^\alpha,$$

*such that $p_j \in C^\infty([0, T]; S^{j,j}(\mathbb{R}^{2d}))$, that is*

$$|\partial_t^k \partial_x^\beta c_{j\alpha}(t; x)| \lesssim \langle x \rangle^{j-|\beta|}, \quad \beta \in \mathbb{Z}_+^d,\ j = 1, \ldots, m,\ \alpha \in \mathbb{Z}_+^d,\ |\alpha| \leqslant j. \quad (4.1)$$

*We denote by*

- *$\sigma(L)$ its symbol, that is*

$$[\sigma(L)](t, \tau; x, \xi) = \tau^m + \sum_{j=1}^{m} p_j(t; x, \xi) \tau^{m-j};$$

- $L_m = \sigma_p(L_m)$ *its principal symbol, that is*

$$[\sigma_p(L)](t,\tau;x,\xi) = \tau^m + \sum_{j=1}^{m} q_j(t;x,\xi)\tau^{m-j}$$

*where* $q_j(t;x,\xi) = \sum_{|\alpha|=j} \widetilde{c}_{j\alpha}(t;x)\xi^\alpha$ *belongs to* $C^\infty([0,T];S^{j,j}(\mathbb{R}^{2d}))$, $\widetilde{c}_{j,\alpha}$ *satisfies* (4.1) *for* $|\alpha| = j$, $j = 1,\ldots,m$, *and is such that*

$$[\sigma(L) - \sigma_p(L)](t,\tau;x,\xi) = \sum_{j=1}^{m} r_j(t;x,\xi)\tau^{m-j},$$

*where* $r_j \in C^\infty([0,T];S^{j-1,j-1}(\mathbb{R}^{2d}))$.

**Definition 4.2.** *An operator $L$ of the type introduced in Definition 4.1 is called hyperbolic if*

$$L_m(t,\tau;x,\xi) = \prod_{j=1}^{m} (\tau - \tau_j(t;x,\xi)), \tag{4.2}$$

*with real-valued, smooth roots* $\tau_j$, $j = 1,\ldots,m$. *The roots* $\tau_j$ *are usually called bicharacteristics. More precisely, $L$ is called:*

*(1)* Strictly SG-hyperbolic, *if* $L_m$ *satisfies* (4.2) *with real-valued, distinct and separated roots* $\tau_j$, $j = 1,\ldots,m$, *in the sense that there exists a constant* $C > 0$ *such that*

$$|\tau_j(t;x,\xi) - \tau_k(t;x,\xi)| \geqslant C\langle x\rangle\langle\xi\rangle, \ \forall j \neq k, \ (t;x,\xi) \in [0,T] \times \mathbb{R}^{2d}.$$

*(2)* (Weakly) SG-hyperbolic with (roots of) constant multiplicities, *if* $L_m$ *satisfies* (4.2) *and the real-valued, characteristic roots can be divided into* $\mu$ *groups* ($1 \leqslant \mu \leqslant m$) *of distinct and separated roots, in the sense that, possibly after a reordering of the* $\tau_j$, $j = 1,\ldots,m$, *there exist* $l_1,\ldots l_\mu \in \mathbb{N}$ *with* $l_1 + \ldots + l_\mu = m$ *and* $\mu$ *sets*

$$G_1 = \{\tau_1 = \cdots = \tau_{l_1}\}, G_2 = \{\tau_{l_1+1} = \cdots = \tau_{l_1+l_2}\}, \\ \ldots G_\mu = \{\tau_{m-l_\mu+1} = \cdots = \tau_m\},$$

*satisfying, for a constant* $C > 0$,

$$\tau_j \in G_p, \tau_k \in G_q, \ p \neq q, \ 1 \leqslant p,q \leqslant \mu \\ \Rightarrow |\tau_j(t,x,\xi) - \tau_k(t,x,\xi)| \geqslant C\langle x\rangle\langle\xi\rangle, \tag{4.3}$$

*for all* $(t,x,\xi) \in [0,T] \times \mathbb{R}^{2d}$; *notice that, in the case* $\mu = 1$, *we have only one group of* $m$ *coinciding roots, that is,* $L_m$ *admits a single real root of multiplicity* $m$, *while for* $\mu = m$ *we recover the strictly hyperbolic case; the number* $l = \max_{j=1,\ldots,\mu} l_j$ *is the maximum multiplicity of the roots of* $L_m$.

*(3)* (Weakly) SG-hyperbolic with involutive roots (or SG-involutive)*, if* $L_m$ *satisfies* (4.2) *with real-valued characteristic roots such that the family* $\{\tau_j\}_{j=1}^m$ *satisfies Assumption I.*

The proof of the next observation can be find cf. e.g., [9, 41]

## 4.1 Fundamental solution for SG-involutive operators of order $m \in \mathbb{N}$

We will focus here on SG-involutive operators, see the references quoted above for the known results about SG-hyperbolic operators with constant multiplicities. In particular, we deal with the case when there is no splitting of characteristic roots $\tau_j$, $\tau_k$, $j, k = 1, \ldots, m$, $k \neq j$, into groups $G_k, k = 1, \ldots, \mu \leqslant m$ satisfying (4.3). It is possible to translate the Cauchy problem

$$\begin{cases} Lu(t,s) = f(t) & (t,s) \in \Delta_T \\ D_t^k u(s,s) = g_k & k = 0, \ldots, m-1, \quad s \in [0,T) \end{cases} \tag{4.4}$$

for a SG-involutive operator $L$ in the sense of Definition 4.2, into a Cauchy problem for an involutive system (3.1) with suitable initial conditions, under an appropriate factorization condition, see below.

We write $\Theta_j = \mathrm{Op}(\tau_j)$, and also set, for convenience below, $\Gamma_j = D_t - \Theta_j$, $j = 1, \ldots, m$. Moreover, with $M_k$ from Chapter 3, and their sorted counterparts $M_k^\dagger$, $1 \leqslant k \leqslant m$, we introduce the notation

$$M_0 = \{\varnothing\}, \quad M = \bigcup_{k=0}^{m-1} M_k, \quad M^\dagger = \bigcup_{k=1}^{m} M_k^\dagger.$$

For $\alpha \in M_k$, $0 \leqslant k \leqslant m$, we define $\mathbf{dim}(\alpha) = k$ and

$$\Gamma_\varnothing = I, \qquad \Gamma_\alpha = \Gamma_{\alpha_1} \ldots \Gamma_{\alpha_k},$$
$$\alpha = (\alpha_1, \ldots, \alpha_k) \in M_k, \text{ and } \{\alpha\} = \{\alpha_1, \ldots, \alpha_k\} \text{ for } k \geqslant 1.$$

The proof of the following Lemma 4.3 can be found in [44]. Analogous results are used in [86] and [94].

**Lemma 4.3.** *When* $\{\lambda_j\}$ *is an involutive system, for all* $\alpha \in M_m$ *we have*

$$\Gamma_\alpha = \Gamma_1 \ldots \Gamma_m + \sum_{\beta \in M} \mathrm{Op}(q_\beta^\alpha(t)) \Gamma_\beta, \tag{4.5}$$

*where* $q_\beta^\alpha \in C^\infty([0,T]; S^{0,0}(\mathbb{R}^{2d}))$.

A systemization and well-posedness (with loss of decay and regularity) theorem can be stated for the Cauchy problem (4.4) under a suitable condition for the operator $L$. This result is due, in its original local form, to Morimoto [94] and it has been extended to the SG case in [43], where the proof of the next result, based on Lemma 4.3, can be found.

**Proposition 4.4.** *Assume the SG-hyperbolic operator $L$ to be of the form*

$$L = \Gamma_1 \cdots \Gamma_m + \sum_{\alpha \in M^\dagger} \mathrm{Op}(p_\alpha(t))\Gamma_\alpha \mod \mathrm{Op}(C^\infty([0,T]; S^{-\infty,-\infty}(\mathbb{R}^{2d}))), \tag{4.6}$$

*with $p_\alpha \in C^\infty([0,T]; S^{0,0}(\mathbb{R}^{2d}))$. Moreover, assume that the family of its characteristic roots $\{\tau_j\}_{j=1}^m$ sastisfies Assumption I. Then, the Cauchy problem (4.4) for $L$ is equivalent to a Cauchy problem for a suitable first order system (3.1) with diagonal principal part, of the form*

$$\begin{cases} (D_t + K(t))\,U(t,s) = F(t), & (t,s) \in \Delta_T, \\ U(s,s) = G, & s \in [0,T), \end{cases} \tag{4.7}$$

*where $U$, $F$ and $G$ are $N$-dimensional vector-valued, $K$ a $(N\times N)$-dimensional matrix, with $N$ given by (4.8). $U$ is defined in (4.9), (4.10), and (4.11). Namely,*

$$N = \sum_{j=0}^{m-1} \frac{m!}{(m-j)!}, \tag{4.8}$$

$$U = {}^t\left(u_\varnothing \equiv u, u_{(1)}, \dots, u_{(m)}, u_{(1,2)}, u_{(2,1)}, \dots, u_\alpha, \dots\right), \tag{4.9}$$

*with $\alpha \in M$, and*

- *for $\alpha \in M_k$, $0 \leqslant k \leqslant m-2$ and $j = \max\{1,\dots,m\}\backslash\{\alpha\}$, we set*

$$\Gamma_j u_\alpha = u_{\alpha_j} \tag{4.10}$$

  *with $\alpha_j = (j, \alpha_1, \dots, \alpha_k) \in M_{k+1}$;*

- *for $\alpha \in M_{m-1}$ and $j \notin \{\alpha\}$, we set*

$$\Gamma_j u_\alpha = f - \sum_{\beta \in M^\dagger} \mathrm{Op}(p_\beta(t))u_\beta + \sum_{\beta \in M} \mathrm{Op}(q_\beta^{\alpha_j}(t))u_\beta, \tag{4.11}$$

  *with $\alpha_j = (j, \alpha_1, \dots, \alpha_k) \in M_m$ and the symbols $p_\beta$, $q_\beta^\alpha$ from (4.5) and (4.6).*

**Remark 4.5.** *We call the SG-hyperbolic operators $L$ satisfying the factorization condition (4.6) "operators of Levi type".*

**Remark 4.6.** *Since, for $\alpha \in M_k$, $k \geqslant 1$, we have*

$$\Gamma_\alpha = D_t^k + \sum_{j=0}^{k-1} \mathrm{Op}(\Upsilon_\alpha^j(s)) D_t^j, \ \Upsilon_\alpha^j \in C^\infty([0,T]; S^{k-j,k-j}(\mathbb{R}^{2d})),$$

*the initial conditions $G$ for $U$ can be expressed as*

$$\begin{cases} G_\varnothing(s,s) = g_0, \\ G_\alpha(s,s) = g_{\mathbf{dim}(\alpha)} + \displaystyle\sum_{j=0}^{\mathbf{dim}(\alpha)-1} \mathrm{Op}(\Upsilon_\alpha^j(s)) g_j, \quad \alpha \in M, \mathbf{dim}(\alpha) > 0. \end{cases} \tag{4.12}$$

*Notice that, in view of the continuity properties of the SG pseudo-differential operators and of the orders of the $\Upsilon_\alpha^i$,* (4.12) *implies*

$$G_\alpha \in H^{m-1-\mathbf{dim}(\alpha), \mu-1-\mathbf{dim}(\alpha)}(\mathbb{R}^d), \quad \alpha \in M. \tag{4.13}$$

The next Theorem 4.7 is our third main result, namely, a well-posedness result, with decay and regularity loss, for SG-involutive operators of the form (4.7). It is a consequence of Proposition 4.4 in combination with the main results of Chapter 3.

**Theorem 4.7.** *Let the operator $L$ in* (4.4) *be SG-involutive, of the form considered in Proposition 4.4. Let $f \in C^\infty([0,T]; H^{r,\varrho}(\mathbb{R}^d))$ and the initial data $g_k \in H^{r+m-1-k,\varrho+m-1-k}(\mathbb{R}^d)$, $k = 0, \dots, m-1$. Then, for a suitable $T' \in (0,T]$, the Cauchy problem* (4.4) *admits a unique solution $u(t,s)$ belonging to $\bigcap_{k \in \mathbb{Z}_+} C^k(\Delta_{T'}; H^{r-k,\varrho-k}(\mathbb{R}^d))$, given, modulo elements in $C^\infty(\Delta_{T'}; \mathscr{S}(\mathbb{R}^d))$, by*

$$u(t,s) = \sum_{\alpha \in M} W_\alpha(t,s) G_\alpha + \sum_{\alpha \in M_{m-1}} \int_s^t W_\alpha(t,\sigma) f(\sigma)\, d\sigma, \ (t,s) \in \Delta_{T'}, s \in [0,T'), \tag{4.14}$$

*for suitable parameter-dependent families of (iterated integrals of) regular SG Fourier integral operators $W_\alpha(t,s)$, $\alpha \in M$, $(t,s) \in \Delta_{T'}$, with phase functions and matrix-valued symbols determined through the characteristic roots of $L$.*

*Proof.* By the procedure explained in Proposition 4.4 and Remark 4.6, we can switch from the Cauchy problem (4.4) to an equivalent Cauchy problem (3.1), with $u \equiv U_\varnothing$. The uniqueness of the solution is then a consequence of known results about symmetric SG-hyperbolic systems, see [41], of which (3.1) is a special case.

The fundamental solution of (4.7) is given by the analog of (3.7) for (3.1), in view of Theorem 3.2 and Remark 3.6. It is a matrix-valued, parameter-dependent operator family $E(t,s) = (E_{\mu\mu'})_{\mu,\mu'\in M}(t,s)$, whose elements $E_{\mu\mu'}(t,s)$, $\mu,\mu' \in M$, are, modulo elements with kernels in $C^\infty(\Delta_{T'};\mathscr{S})$, linear combination of parameter-dependent families of (iterated integrals of) regular SG Fourier operators, with phase functions of the type

$$\phi^{(\mu^\dagger)} = \varphi_{m_1}, \qquad \mu^\dagger = (m_1) \in M_1^\dagger,$$
$$\phi^{(\mu^\dagger)} = \varphi_{m_1}\sharp\ldots\sharp\varphi_{m_j}, \qquad \mu^\dagger = (m_1,\ldots,m_j) \in M_j^\dagger, j \geqslant 2,$$

$\varphi_k$ solution of the eikonal equation associated with the characteristic root $\tau_k$ of $L$, $k = 1,\ldots,m$, and parameter-dependent, matrix-valued symbols of the type

$$\omega^{(\mu^\dagger)}(t,\theta_1,\ldots,\theta_{j-1},s) \in S^{0,0}, \quad \mu \in M_j^\dagger,$$

$j = 1,\ldots,m$. Then, the component $U_\varnothing \equiv u$ of the solution $U$ of (4.7) has the form (4.14), with $W_\alpha = E_{\varnothing\alpha}$, taking into account (4.11) and (4.12).

We observe that the $k$-th order $t$-derivatives of the operators $W_\alpha$, $\alpha \in M$, map continuously $H^{r,\varrho}$ to $H^{r-k,\varrho-k}$, $k \in \mathbb{Z}_+$, in view of Theorem 1.21 and of the fact that, of course,

$$\partial_t[\mathrm{Op}_{\phi^{(\mu^\dagger)}(t,s)}(w^{(\mu^\dagger)}(t,s))]$$
$$= \mathrm{Op}_{\phi^{(\mu^\dagger)}(t,s)}(i(\partial_t\phi^{(\mu^\dagger)})(t,s)\cdot w^{(\mu^\dagger)}(t,s) + \partial_t w^{(\mu^\dagger)}(t,s)),$$

obtaining a symbol of orders 1-unit higher in both components at any $t$-derivative step. This fact, together with the hypothesis on $f$, implies that the second sum in (4.14) belongs to $\bigcap_{k\in\mathbb{Z}_+} C^k(\Delta_{T'};H^{r-k,\varrho-k}(\mathbb{R}^d))$.

The same is true for the elements of the first sum. In fact, recalling the embedding among the Sobolev-Kato spaces and (4.13), since $\alpha \in M \Rightarrow 0 \leqslant \mathbf{dim}(\alpha) \leqslant m-1$, we find

$$W_\alpha(t,s)G_\alpha \in \bigcap_{k\in\mathbb{Z}_+} C^k(\Delta_{T'};H^{r+m-1-\mathbf{dim}(\alpha)-k,\varrho+m-1-\mathbf{dim}(\alpha)-k})$$
$$\hookrightarrow \bigcap_{k\in\mathbb{Z}_+} C^k(\Delta_{T'};H^{r-k,\varrho-k}), \quad \alpha \in M,$$

and this concludes the proof. □

## 4.2 Propagation of singularities for classical SG-involutive operators

Theorem 4.7, together with the propagation results proved in [47], implies our fourth main result, Theorem 4.20 below, about the global wave-front set of the solution of the Cauchy problem (4.4), in the case of a classical SG-involutive operator $L$ of Levi type. We first recall the necessary definitions, adapting some materials appeared in [46–48].

**Definition 4.8.** *Let $\mathcal{B}$ be a topological vector space of distributions on $\mathbb{R}^d$ such that*

$$\mathscr{S}(\mathbb{R}^d) \subseteq \mathcal{B} \subseteq \mathscr{S}'(\mathbb{R}^d)$$

*with continuous embeddings. Then $\mathcal{B}$ is called SG-admissible when $\mathrm{Op}_t(a)$ maps $\mathcal{B}$ continuously into itself, for every $a \in S^{0,0}(\mathbb{R}^{2d})$. If $\mathcal{B}$ and $\mathcal{C}$ are SG-admissible, then the pair $(\mathcal{B}, \mathcal{C})$ is called SG-ordered (with respect to $(m, \mu) \in \mathbb{R}^2$), when the mappings*

$$\mathrm{Op}_t(a) : \mathcal{B} \to \mathcal{C} \quad \text{and} \quad \mathrm{Op}_t(b) : \mathcal{C} \to \mathcal{B}$$

*are continuous for every $a \in S^{m,\mu}(\mathbb{R}^{2d})$ and $b \in S^{-m,-\mu}(\mathbb{R}^{2d})$.*

**Remark 4.9.** *$\mathscr{S}(\mathbb{R}^d)$, $H^{r,\varrho}(\mathbb{R}^d)$, $r, \varrho \in \mathbb{R}$, and $\mathscr{S}'(\mathbb{R}^d)$ are SG-admissible. $(\mathscr{S}(\mathbb{R}^d), \mathscr{S}(\mathbb{R}^d))$, $(H^{r,\varrho}(\mathbb{R}^d), H^{r-m,\varrho-\mu}(\mathbb{R}^d))$, $r, \varrho \in \mathbb{R}$, $(\mathscr{S}'(\mathbb{R}^d), \mathscr{S}'(\mathbb{R}^d))$ are SG-ordered (with respect to any $(m, \mu) \in \mathbb{R}^2$). The same holds true for (suitable couples of) modulation spaces, see [46].*

**Definition 4.10.** *Let $\varphi \in \mathcal{P}_r$ be a regular phase function, $\mathcal{B}$, $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{C}$, $\mathcal{C}_1$, $\mathcal{C}_2$, be SG-admissible and $\Omega \subseteq \mathbb{R}^d$ be open. Then the pair $(\mathcal{B}, \mathcal{C})$ is called weakly-I SG-ordered (with respect to $(m, \mu, \varphi, \Omega)$), when the mapping*

$$\mathrm{Op}_\varphi(a) : \mathcal{B} \to \mathcal{C}$$

*is continuous for every $a \in S^{m,\mu}(\mathbb{R}^{2d})$ with support such that the projection on the $\xi$-axis does not intersect $\mathbb{R}^d \backslash \Omega$. Similarly, the pair $(\mathcal{B}, \mathcal{C})$ is called weakly-II SG-ordered (with respect to $(m, \mu, \varphi, \Omega)$), when the mapping*

$$\mathrm{Op}^*_\varphi(b) : \mathcal{C} \to \mathcal{B}$$

*is continuous for every $b \in S^{m,\mu)}(\mathbb{R}^{2d})$ with support such that the projection on the $x$-axis does not intersect $\mathbb{R}^d \backslash \Omega$. Furthermore, $(\mathcal{B}_1, \mathcal{C}_1, \mathcal{B}_2, \mathcal{C}_2)$ are called SG-ordered (with respect to $m_1, \mu_1, m_2, \mu_2$, $\varphi$, and $\Omega$), when $(\mathcal{B}_1, \mathcal{C}_1)$ is a weakly-I SG-ordered pair with respect to $(m_1, \mu_1, \varphi, \Omega)$, and $(\mathcal{B}_2, \mathcal{C}_2)$ is a weakly-II SG-ordered pair with respect to $(m_2, \mu_2, \varphi, \Omega)$.*

**Remark 4.11.** *$(\mathscr{S}(\mathbb{R}^d), \mathscr{S}(\mathbb{R}^d))$, $(H^{r,\varrho}(\mathbb{R}^d), H^{r-m,\varrho-\mu}(\mathbb{R}^d))$, where $r, \varrho \in \mathbb{R}$, $(\mathscr{S}'(\mathbb{R}^d), \mathscr{S}'(\mathbb{R}^d))$ are weakly-I and weakly-II SG-ordered pairs (with respect to any $(m, \mu) \in \mathbb{R}^2$, $\varphi \in \mathcal{P}_r$, and $\Omega = \varnothing$). The situation is more delicate in the case of modulation spaces, even just on Sobolev-Kato spaces modeled on $L^p(\mathbb{R}^d)$, $p \in [1, \infty)$, $p \neq 2$, see [46] and the references quoted therein.*

Now we recall the definition given in [46] of global wave-front sets for temperate distributions with respect to Banach or Fréchet spaces and state some of their properties. First of all, we recall the definitions of set of characteristic points that we use in this setting.

We need to deal with the situations where (1.4) holds only in certain (conic-shaped) subset of $\mathbb{R}^d \times \mathbb{R}^d$. Here we let $\Omega_m$, $m = 1, 2, 3$, be the sets

$$\begin{gathered}\Omega_1 = \mathbb{R}^d \times (\mathbb{R}^d \backslash 0), \qquad \Omega_2 = (\mathbb{R}^d \backslash 0) \times \mathbb{R}^d, \\ \Omega_3 = (\mathbb{R}^d \backslash 0) \times (\mathbb{R}^d \backslash 0).\end{gathered} \tag{4.15}$$

**Definition 4.12.** *Let $\Omega_k$, $k = 1, 2, 3$ be as in* (4.15)*, and let $a \in S^{m,\mu}(\mathbb{R}^{2d})$.*

*(1) a is called* locally *or* type-1 invertible *with respect to $m, \mu$ at the point $(x_0, \xi_0) \in \Omega_1$, if there exist a neighbourhood $X$ of $x_0$, an open conical neighbourhood $\Gamma$ of $\xi_0$ and a positive constant $R$ such that* (1.4) *holds for $x \in X$, $\xi \in \Gamma$ and $|\xi| \geqslant R$.*

*(2) a is called* Fourier-locally *or* type-2 invertible *with respect to $m, \mu$ at the point $(x_0, \xi_0) \in \Omega_2$, if there exist an open conical neighbourhood $\Gamma$ of $x_0$, a neighbourhood $X$ of $\xi_0$ and a positive constant $R$ such that* (1.4) *holds for $x \in \Gamma$, $|x| \geqslant R$ and $\xi \in X$.*

*(3) a is called* oscillating *or* type-3 invertible *with respect to $m, \mu$ at the point $(x_0, \xi_0) \in \Omega_3$, if there exist open conical neighbourhoods $\Gamma_1$ of $x_0$ and $\Gamma_2$ of $\xi_0$, and a positive constant $R$ such that* (1.4) *holds for $x \in \Gamma_1$, $|x| \geqslant R$, $\xi \in \Gamma_2$ and $|\xi| \geqslant R$.*

*If $k \in \{1, 2, 3\}$ and $a$ is* not *type-$k$ invertible with respect to $m, \mu$ at $(x_0, \xi_0) \in \Omega_k$, then $(x_0, \xi_0)$ is called* type-$k$ characteristic *for $a$ with respect to $m, \mu$. The set of type-$k$ characteristic points for $a$ with respect to $m, \mu$ is denoted by* $\mathrm{Char}^k_{m,\mu}(a)$.

*The* (global) set of characteristic points *(the characteristic set), for a symbol $a \in S^{m,\mu}(\mathbb{R}^{2d})$ with respect to $m, \mu$ is defined as*

$$\mathrm{Char}(a) = \mathrm{Char}_{m,\mu}(a) = \mathrm{Char}^1_{m,\mu}(a) \bigcup \mathrm{Char}^2_{m,\mu}(a) \bigcup \mathrm{Char}^3_{m,\mu}(a).$$

In the next Definition 4.13 we introduce different classes of cutoff functions (see also Definition 1.9 in [45]).

**Definition 4.13.** *Let $X \subseteq \mathbb{R}^d$ be open, $\Gamma \subseteq \mathbb{R}^d \backslash 0$ be an open cone, $x_0 \in X$ and let $\xi_0 \in \Gamma$.*

*(1) A smooth function $\varphi$ on $\mathbb{R}^d$ is called a cutoff (function) with respect to $x_0$ and $X$, if $0 \leqslant \varphi \leqslant 1$, $\varphi \in C_0^\infty(X)$ and $\varphi = 1$ in an open neighbourhood of $x_0$. The set of cutoffs with respect to $x_0$ and $X$ is denoted by $\mathscr{C}_{x_0}(X)$ or $\mathscr{C}_{x_0}$.*

*(2) A smooth function $\psi$ on $\mathbb{R}^d$ is called a directional cutoff (function) with respect to $\xi_0$ and $\Gamma$, if there is a constant $R > 0$ and open conical neighbourhood $\Gamma_1 \subseteq \Gamma$ of $\xi_0$ such that the following is true:*

- $0 \leqslant \psi \leqslant 1$ *and* $\operatorname{supp}\psi \subseteq \Gamma$;
- $\psi(t\xi) = \psi(\xi)$ *when* $t \geqslant 1$ *and* $|\xi| \geqslant R$;
- $\psi(\xi) = 1$ *when* $\xi \in \Gamma_1$ *and* $|\xi| \geqslant R$.

*The set of directional cutoffs with respect to* $\xi_0$ *and* $\Gamma$ *is denoted by* $\mathscr{C}^{\mathrm{dir}}_{\xi_0}(\Gamma)$ *or* $\mathscr{C}^{\mathrm{dir}}_{\xi_0}$.

**Remark 4.14.** *Let* $X \subseteq \mathbb{R}^d$ *be open and* $\Gamma, \Gamma_1, \Gamma_2 \subseteq \mathbb{R}^d \backslash 0$ *be open cones. Then the following is true.*

*(1) if* $x_0 \in X$, $\xi_0 \in \Gamma$, $\varphi \in \mathscr{C}_{x_0}(X)$ *and* $\psi \in \mathscr{C}^{\mathrm{dir}}_{\xi_0}(\Gamma)$, *then* $c_1 = \varphi \otimes \psi$ *belongs to* $S^{0,0}(\mathbb{R}^{2d})$, *and is type-*1 *invertible at* $(x_0, \xi_0)$;

*(2) if* $x_0 \in \Gamma$, $\xi_0 \in X$, $\psi \in \mathscr{C}^{\mathrm{dir}}_{x_0}(\Gamma)$ *and* $\varphi \in \mathscr{C}_{\xi_0}(X)$, *then* $c_2 = \varphi \otimes \psi$ *belongs to* $S^{0,0}(\mathbb{R}^{2d})$, *and is type-*2 *invertible at* $(x_0, \xi_0)$;

*(3) if* $x_0 \in \Gamma_1$, $\xi_0 \in \Gamma_2$, $\psi_1 \in \mathscr{C}^{\mathrm{dir}}_{x_0}(\Gamma_1)$ *and* $\psi_2 \in \mathscr{C}^{\mathrm{dir}}_{\xi_0}(\Gamma_2)$, *then* $c_3 = \psi_1 \otimes \psi_2$ *belongs to* $S^{0,0}(\mathbb{R}^{2d})$, *and is type-*3 *invertible at* $(x_0, \xi_0)$.

The next Proposition 4.15 shows that $\operatorname{Op}_t(a)$ for $t \in \mathbb{R}$ satisfies convenient invertibility properties of the form

$$\operatorname{Op}_t(a)\operatorname{Op}_t(b) = \operatorname{Op}_t(c) + \operatorname{Op}_t(h), \tag{4.16}$$

outside the set of characteristic points for a symbol $a$. Here $\operatorname{Op}_t(b)$, $\operatorname{Op}_t(c)$ and $\operatorname{Op}_t(h)$ have the roles of "local inverse", "local identity" and smoothing operators respectively. From these statements it also follows that our set of characteristic points in Definition 4.12 are related to those in [48, 80].

**Proposition 4.15.** *Let* $k \in \{1, 2, 3\}$, $m, \mu \in \mathbb{R}$, *and let* $a \in S^{m,\mu}(\mathbb{R}^{2d})$. *Also let* $\Omega_k$ *be as in* (4.15), $(x_0, \xi_0) \in \Omega_k$, *when* $k$ *is equal to* 1, 2 *and* 3, *respectively. Then the following conditions are equivalent,* $k = 1, 2, 3$:

*(1)* $(x_0, \xi_0) \notin \operatorname{Char}^k_{m,\mu}(a)$;

*(2) there is an element* $c \in S^{0,0}$ *which is type-k invertible at* $(x_0, \xi_0)$, *and an element* $b \in S^{-m,-\mu}$ *such that* $ab = c$;

*(3)* (4.16) *holds for some* $c \in S^{0,0}$ *which is type-k invertible at* $(x_0, \xi_0)$, *and some elements* $h \in S^{-1,-1}$ *and* $b \in S^{-m,-\mu}$;

*(4)* (4.16) *holds for some* $c_k \in S^{0,0}$ *in Remark 4.14 which is type-k invertible at* $(x_0, \xi_0)$, *and some elements* $h$ *and* $b \in S^{-m,-\mu}$, *where* $h \in \mathscr{S}$ *when* $k \in \{1, 3\}$ *and* $h \in S^{-\infty,0}$ *when* $k = 2$.

*Furthermore, if* $t = 0$, *then the supports of* $b$ *and* $h$ *can be chosen to be contained in* $X \times \mathbb{R}^d$ *when* $k = 1$, *in* $\Gamma \times \mathbb{R}^d$ *when* $k = 2$, *and in* $\Gamma_1 \times \mathbb{R}^d$ *when* $k = 3$.

We can now introduce the complements of the wave-front sets. More precisely, let $\Omega_k$, $k \in \{1,2,3\}$, be given by (4.15), $\mathcal{B}$ be a Banach or Fréchet space such that $\mathscr{S}(\mathbb{R}^d) \subseteq \mathcal{B} \subseteq \mathscr{S}'(\mathbb{R}^d)$, and let $f \in \mathscr{S}'(\mathbb{R}^d)$. Then the point $(x_0, \xi_0) \in \Omega_k$ is called *type-$k$* regular for $f$ with respect to $\mathcal{B}$, if

$$\operatorname{Op}(c_k) f \in \mathcal{B}, \tag{4.17}$$

for some $c_k$ in Remark 4.14, $k = 1,2,3$. The set of all type-$k$ regular points for $f$ with respect to $\mathcal{B}$, is denoted by $\Theta^k_{\mathcal{B}}(f)$.

**Definition 4.16.** *Let $k \in \{1,2,3\}$, $\Omega_k$ be as in* (4.15)*, and let $\mathcal{B}$ be a Banach or Fréchet space such that $\mathscr{S}(\mathbb{R}^d) \subseteq \mathcal{B} \subset \mathscr{S}'(\mathbb{R}^d)$.*

*(1) The* type-$k$ wave-front set *of $f \in \mathscr{S}'(\mathbb{R}^d)$ with respect to $\mathcal{B}$ is the complement of $\Theta^k_{\mathcal{B}}(f)$ in $\Omega_k$, and is denoted by* $\mathrm{WF}^k_{\mathcal{B}}(f)$*;*

*(2) The* global wave-front set $\mathrm{WF}_{\mathcal{B}}(f) \subseteq (\mathbb{R}^d \times \mathbb{R}^d)\backslash 0$ *is the set*

$$\mathrm{WF}_{\mathcal{B}}(f) \equiv \mathrm{WF}^1_{\mathcal{B}}(f) \bigcup \mathrm{WF}^2_{\mathcal{B}}(f) \bigcup \mathrm{WF}^3_{\mathcal{B}}(f).$$

The sets $\mathrm{WF}^1_{\mathcal{B}}(f)$, $\mathrm{WF}^2_{\mathcal{B}}(f)$ and $\mathrm{WF}^3_{\mathcal{B}}(f)$ in Definition 4.16, are also called the *local*, *Fourier-local* and *oscillating* wave-front set of $f$ with respect to $\mathcal{B}$.

**Remark 4.17.** *In the special case when $\mathcal{B} = H^{r,\varrho}(\mathbb{R}^d)$, $r, \varrho \in \mathbb{R}$, we write* $\mathrm{WF}^k_{r,\varrho}(f)$*, $k = 1,2,3$. In this situation,* $\mathrm{WF}_{r,\varrho}(f) \equiv \mathrm{WF}^1_{r,\varrho}(f) \bigcup \mathrm{WF}^2_{r,\varrho}(f) \bigcup \mathrm{WF}^3_{r,\varrho}(f)$ *coincides with the scattering wave front set of $f \in \mathscr{S}'(\mathbb{R}^d)$ introduced by Melrose [92]. In the case when $\mathcal{B} = \mathscr{S}(\mathbb{R}^d)$,* $\mathrm{WF}_{\mathcal{B}}(f)$ *coincides with the $\mathscr{S}$-wave-front set considered in [48].*

**Remark 4.18.** *Let $\Omega_m$, $m = 1,2,3$ be the same as in* (4.15)*.*

1. *If $\Omega \subseteq \Omega_1$, and $(x_0, \xi_0) \in \Omega \iff (x_0, \sigma\xi_0) \in \Omega$ for $\sigma \geqslant 1$, then $\Omega$ is called* 1-conical*;*

2. *If $\Omega \subseteq \Omega_2$, and $(x_0, \xi_0) \in \Omega \iff (sx_0, \xi_0) \in \Omega$ for $s \geqslant 1$, then $\Omega$ is called* 2-conical*;*

3. *If $\Omega \subseteq \Omega_3$, and $(x_0, \xi_0) \in \Omega \iff (sx_0, \sigma\xi_0) \in \Omega$ for $s, \sigma \geqslant 1$, then $\Omega$ is called* 3-conical.

*By* (4.17) *and the paragraph before Definition 4.16, it follows that if $m = 1,2,3$, then $\Theta^m_{\mathcal{B}}(f)$ is $m$-conical. The same holds for* $\mathrm{WF}^m_{\mathcal{B}}(f)$*, $m = 1,2,3$, by Definition 4.16, noticing that, for any $x_0 \in \mathbb{R}^r\backslash\{0\}$, any open cone $\Gamma \ni x_0$, and any $s > 0$, $\mathscr{C}^{\mathrm{dir}}_{x_0}(\Gamma) = \mathscr{C}^{\mathrm{dir}}_{sx_0}(\Gamma)$. For any $R > 0$ and $m \in \{1,2,3\}$, we set*

$$\Omega_{1,R} \equiv \{\, (x,\xi) \in \Omega_1 \,;\, |\xi| \geqslant R \,\}, \quad \Omega_{2,R} \equiv \{\, (x,\xi) \in \Omega_2 \,;\, |x| \geqslant R \,\},$$

$$\Omega_{3,R} \equiv \{\, (x,\xi) \in \Omega_3 \,;\, |x|, |\xi| \geqslant R \,\}$$

*Evidently, $\Omega^R_m$ is $m$-conical for every $m \in \{1,2,3\}$.*

From now on we assume that $\mathcal{B}$ in Definition 4.16 is $SG$-admissible, and recall that Sobolev-Kato spaces and, more generally, modulation spaces, and $\mathscr{S}(\mathbb{R}^d)$ are $SG$-admissible, see [46, 47].

The next result describes the relation between "regularity with respect to $\mathcal{B}$" of temperate distributions and global wave-front sets, cf. [46].

**Proposition 4.19.** *Let $\mathcal{B}$ be SG-admissible, and let $f \in \mathscr{S}'(\mathbb{R}^d)$. Then*

$$f \in \mathcal{B} \quad \Longleftrightarrow \quad \mathrm{WF}_{\mathcal{B}}(f) = \varnothing.$$

The next Theorem 4.20 extends the analogous result in [47] to the more general case of a classical, SG-hyperbolic involutive operator $L$ of Levi type, and the one in [110] to the global wave-front sets introduced above. It is a consequence of Theorem 4.7 and of Theorem 5.14 in [47].

**Theorem 4.20.** *Let $L$ in* (4.4) *be a classical, SG-hyperbolic, involutive operator of Levi type, that is, of the type considered in Proposition 4.4 with SG-classical coefficients, of the form* (4.6)*. Let $g_\ell \in \mathcal{B}_\ell$, $\ell = 0, \dots, m-1$, with the $m$-tuple of SG-admissible spaces $(\mathcal{B}_0, \dots, \mathcal{B}_{m-1})$. Also assume that the SG-admissible space $\mathcal{C}$ is such that $(\mathcal{B}_k, \mathcal{C})$, $k = 0, \dots, m-1$, are weakly-I SG-ordered pairs with respect to*

$$k-j, k-j, k = 0, \dots, m-1, j = 0, \dots, k, \phi^{(\alpha)}, \alpha \in M, \text{ and } \varnothing.$$

*Then, for the solution $u(t,s)$ of the Cauchy problem* (4.4) *with $f \equiv 0$, $(t,s) \in \Delta_{T'}$, $s \in [0, T')$, we find*

$$\mathrm{WF}^k_{\mathcal{C}}(u(t,s)) \subseteq \bigcup_{j=1}^{m} \bigcup_{\alpha \in M_j^\dagger} \bigcup_{\substack{\boldsymbol{t}_j \in \Delta(T') \\ t_0 = t, t_j = s}} \bigcup_{\ell=0}^{m-1} (\Phi_\alpha(\boldsymbol{t}_j))(\mathrm{WF}^k_{\mathcal{B}_\ell}(g_\ell))^{\mathrm{con}_k}, \; k = 1, 2, 3,$$

*where $V^{\mathrm{con}_k}$ for $V \subseteq \Omega_k$, is the smallest $k$-conical subset of $\Omega_k$ which includes $V$, $k \in \{1, 2, 3\}$ and $\Phi_\alpha(\boldsymbol{t}_j)$ is the canonical transformation of $T^\star\mathbb{R}^d$ into itself generated by the parameter-dependent SG-classical phase functions $\phi^{(\alpha)}(\boldsymbol{t}_j) \in \mathcal{P}_r$, $\alpha \in M_j^\dagger$, $\boldsymbol{t}_j \in \Delta(T')$, $t_0 = t$, $t_j = s$, $j = 1, \dots, m$, appearing in* (4.14)*.*

*Proof.* The result for $j = 1$, $\alpha \in M_1^\dagger$, $t_0 = t$, $t_1 = s$, and $u(t,s) = \sum_{\alpha \in M_1} W_\alpha(t,s) G_\alpha$ is essentially the one proved in [47, Theorem 5.14].
For $j \geqslant 2$,

$$u(t,s) = \sum_{j=2}^{m} \sum_{\alpha \in M_j^\dagger} \int_s^t \int_s^{t_1} \dots \int_s^{t_{j-2}} W_\alpha{}^{(\mu^\dagger)}(t, t_1, \dots, t_{j-1}, s) G_\alpha \, dt_{j-1} \dots dt_1,$$

where $W_\alpha^{(\mu^\dagger)} \equiv W_\alpha^{(\mu^\dagger)}(t, t_1, \ldots, t_{j-1}, s)$ given by an analog of (3.7) for (3.1), in view of Theorem 3.2 and Remark 3.6. Let

$$(y_0, \eta_0) \in \Omega_k \backslash \bigcup_{j=2}^{m} \bigcup_{\alpha\in M_j^\dagger} \bigcup_{\substack{\mathbf{t}_j\in\Delta(T') \\ t_0=t, t_j=s}} \bigcup_{\ell=0}^{m-1} (\mathrm{WF}^k_{\mathcal{B}_\ell}(g_\ell))^{\mathrm{con}_k},\ k = 1, 2, 3.$$

This is equivalent to the fact that

$$(y_0, \eta_0) \in \bigcap_{j=2}^{m} \bigcap_{\alpha\in M_j^\dagger} \bigcap_{\substack{\mathbf{t}_j\in\Delta(T') \\ t_0=t, t_j=s}} \bigcap_{\ell=0}^{m-1} \left(\Omega_k \backslash (\mathrm{WF}^k_{\mathcal{B}_\ell}(g_\ell))^{\mathrm{con}_k}\right)$$

$$\Leftrightarrow$$

$$(y_0, \eta_0) \in \bigcap_{j=2}^{m} \bigcap_{\alpha\in M_j^\dagger} \bigcap_{\substack{\mathbf{t}_j\in\Delta(T') \\ t_0=t, t_j=s}} \bigcap_{\ell=0}^{m-1} \left(\Theta^k_{\mathcal{B}_\ell}(g_\ell))^{\mathrm{con}_k}\right).$$

Let $(x_0, \xi_0)$ satisfy

$$(y_0, \eta_0) = \Phi_\alpha^{-1}(\mathbf{t}_j; x_0, \xi_0) \iff (x_0, \xi_0) = \Phi_\alpha(\mathbf{t}_j; y_0, \eta_0).$$

Let $c_k \in S^{0,0}$ be a symbol as in (4.17) and Remark 4.14 such that $\mathrm{Op}(c_k)u \in \mathcal{B}_\ell$ for $\ell = 0, \ldots, m-1$. Let $C_k = \mathrm{Op}(c_k)$, and let $V_\alpha^{(\mu^\dagger)} \equiv V_\alpha^{(\mu^\dagger)}(t, t_1, \ldots, t_{j-1}, s)$ be the parametrix of $W_\alpha^{(\mu^\dagger)}$. Then for some $q_{k,\alpha}$ we have

$$Q_{k,\alpha} = \int_s^t \int_s^{t_1} \cdots \int_s^{t_{j-2}} W_\alpha^{(\mu^\dagger)} \circ C_k \circ V_\alpha^{(\mu^\dagger)}\, dt_{j-1} \ldots dt_1,$$

or equivalently,

$$\int_s^t \int_s^{t_1} \cdots \int_s^{t_{j-2}} Q_{k,\alpha} \circ W_\alpha^{(\mu^\dagger)}\, dt_{j-1} \ldots dt_1$$
$$= \int_s^t \int_s^{t_1} \cdots \int_s^{t_{j-2}} W_\alpha^{(\mu^\dagger)} \circ C_k\, dt_{j-1} \ldots dt_1 \ \mathit{mod}\ \mathrm{Op}(S^{-\infty,-\infty}).$$

In view of [47, Theorem 3.18] and the fact that $\Phi_\alpha(\mathbf{t}_j)$ is a canonical transformation of $T^\star\mathbb{R}^d$ into itself, we have $q_{k,\alpha} = c_k \circ \Phi_\alpha^{-1}$ mod $S^{-1,-1}$, which implies that $q_{k,\alpha} \in S^{0,0}$. Since

$$\int_s^t \int_s^{t_1} \cdots \int_s^{t_{j-2}} Q_{k,\alpha}(W_\alpha^{(\mu^\dagger)} G_\alpha)\, dt_{j-1} \ldots dt_1$$
$$\equiv \int_s^t \int_s^{t_1} \cdots \int_s^{t_{j-2}} W_\alpha^{(\mu^\dagger)}(C_k G_\alpha)\, dt_{j-1} \ldots dt_1$$

where $W_\alpha^{(\mu^\dagger)}(C_k G_\alpha)$ belongs to $C^\infty(\Delta(T');\mathcal{C})$ by the hypothesis on $(\mathcal{B}_\ell,\mathcal{C})$, for $\ell = 0,\dots,m-1$, then

$$(x_0,\xi_0) \in \Theta_{\mathcal{C}}^k \left( \sum_{j=2}^m \sum_{\alpha\in M_j^\dagger} \int_s^t \int_s^{t_1} \dots \int_s^{t_{j-2}} W_\alpha^{(\mu^\dagger)} G_\alpha \, dt_{j-1}\dots dt_1 \right)^{con_k} .$$

This means that

$$\bigcap_{j=2}^m \bigcap_{\alpha\in M_j^\dagger} \bigcap_{\substack{\mathbf{t}_j\in\Delta(T')\\ t_0=t,t_j=s}} \bigcap_{\ell=0}^{m-1} \Phi_\alpha(\Theta_{\mathcal{B}_\ell}^k(g_\ell))^{con_k}$$

$$\subseteq \Theta_{\mathcal{C}}^k \left( \sum_{j=2}^m \sum_{\alpha\in M_j^\dagger} \int_s^t \int_s^{t_1} \dots \int_s^{t_{j-2}} W_\alpha^{(\mu^\dagger)} G_\alpha \, dt_{j-1}\dots dt_1 \right)^{con_k} . \tag{4.18}$$

Complementing (4.18) with respect to $\Omega_k$, we get the desired result. □

**Remark 4.21.** *1. The canonical transform generated by an arbitrary regular phase function $\varphi \in \mathcal{P}_r$ is defined by the relations*

$$(x,\xi) = \Phi_\varphi(y,\eta) \iff \begin{cases} y = \varphi'_\xi(x,\eta) = \varphi'_\eta(x,\eta), \\ \xi = \varphi'_x(x,\eta). \end{cases}$$

*2. Assume that the hypotheses of Theorem 4.20 hold true. Then $\mathrm{WF}_{\mathcal{C}}^k(u(t,s))$, $(t,s)\in\Delta_{T'}$, $k=1,2,3$, consists of unions of arcs of bicharacteristics, generated by the phase functions appearing in* (4.14) *and emanating from points belonging to $\mathrm{WF}_{\mathcal{B}_k}^m(g_k)$, $k=0,\dots,m-1$, cf. [48, 94, 110].*

**Corollary 4.22.** *The hypotheses on the spaces $\mathcal{B}_k$, $k=0,\dots,m-1$, $\mathcal{C}$, are authomatically fulfilled for $\mathcal{B}_k = H^{r+m-1-k,\varrho+m-1-k}(\mathbb{R}^d)$, $\mathcal{C} = H^{r,\varrho}(\mathbb{R}^d)$, $r,\varrho\in\mathbb{R}$, $k=0,\dots,m-1$. That is, the results in Theorem 4.20 and 2 in Remark 4.21 above hold true for the $\mathrm{WF}_{r,\varrho}^k(u(t,s))$ wave-front sets, $r,\varrho\in\mathbb{R}$, $k=1,2,3$.*

# Part II

# Microlocal analysis on modulation spaces

# Chapter 5

# Modulation spaces, Gelfand-Shilov spaces and pseudo-differential calculus

Time-frequency analysis is an interdisciplinary area of research, with branches in pure and applied mathematics, physics and signal theory. It has one root in the work by Wigner and Weyl on the mathematical foundations of quantum mechanics from the 1930s. Exclusively, the theory of the short-time Fourier transform and modulation spaces of scalar-valued functions and tempered distributions is a very well developed theory of representation of tempered distributions in the time-frequency (phase) space.

The standard reference about this topic is the well known book by K. H. Gröchenig [67].

In this chapter we recall some basic facts on modulation spaces, Gelfand-Shilov spaces of functions and distributions and pseudo-differential operators with symbols on Gelfand-Shilov classes (cf. [54–58, 60, 62, 67, 71, 80, 81, 89, 99, 108, 112, 114, 116–119]).

## 5.1 Classes of weight functions

Weights are used to quantify growth and decay conditions. For instance, if $\omega(x) = (1+|x|)^m$, $m \in \mathbb{R}$ and $\|f\|_{L^\infty_\omega} = \sup_{x\in\mathbb{R}^d} |f(x)|\omega(x) < \infty$, then $|f(x)| \leqslant C(1+|x|)^{-m}$. If $m > 0$, then this condition describes the polynomial decay of $f$ of order $m$, whereas if $m < 0$, then $f$ grows at most like a polynomial of degree $m$. Combining this intuition with $L^p$-spaces, one obtains the weighted $L^p$-spaces which are defined by the norm $\|f\|_{L^p_m} = \|f\,\omega\|_p = \int_{\mathbb{R}^d} |f(t)|^p \omega(x)^p\, dt$.

A *weight* on $\mathbb{R}^d$ is a positive function $\omega \in L^\infty_{loc}(\mathbb{R}^d)$ such that $1/\omega \in L^\infty_{loc}(\mathbb{R}^d)$. If $\omega$ and $v$ are weights on $\mathbb{R}^d$, then $\omega$ is called *moderate* or $v$-

*moderate*, if

$$\omega(x+y) \leqslant C\omega(x)v(y), \quad x,y \in \mathbb{R}^d, \tag{5.1}$$

for some constant $C \geqslant 1$. The set of all moderate weights on $\mathbb{R}^d$ is denoted by $\mathscr{P}_E(\mathbb{R}^d)$. We notice that if the weight $v$ is even and (5.1) is fulfilled with $\omega = v$ and $C = 1$, and that $v(x) \geqslant c$ for some $c \in (0,1]$, then $v_1(x) = c^{-1}v(x)$ satisfies the same properties, as well as $v_1(x) \geqslant 1$.

The weight $v$ on $\mathbb{R}^d$ is called submultiplicative, if it is even, bounded from below by 1 and (5.1) holds for $\omega = v$ and $C = 1$. From now on, $v$ always denotes a submultiplicative weight if nothing else is stated. In particular, if (5.1) holds and $v$ is submultiplicative, then it follows by straightforward computations that

$$\begin{gathered} C^{-1}\frac{\omega(x)}{v(y)} \leqslant \omega(x+y) \leqslant C\omega(x)v(y), \\ v(x+y) \leqslant v(x)v(y) \quad \text{and} \quad v(x) = v(-x) \geqslant 1, \quad x,y \in \mathbb{R}^d. \end{gathered} \tag{5.2}$$

Submultiplicative weights occur in time-frequency analysis in the investigation of twisted convolution, in the definition of “good windows” and spaces of test functions, and in the construction of algebras of pseudo-differential operators.

If $\omega$ is a moderate weight on $\mathbb{R}^d$, then by [120] and above, there is a submultiplicative weight $v$ on $\mathbb{R}^d$ such that (5.1) and (5.2) hold true (see also [67, 114, 116]). Moreover if $v$ is submultiplicative on $\mathbb{R}^d$, then

$$1 \leqslant v(x) \lesssim e^{r|x|} \tag{5.3}$$

for some constant $r > 0$ (cf. [70]). In particular, if $\omega$ is moderate, then

$$\omega(x+y) \lesssim \omega(x)e^{r|y|} \quad \text{and} \quad e^{-r|x|} \leqslant \omega(x) \lesssim e^{r|x|}, \quad x,y \in \mathbb{R}^d \tag{5.4}$$

for some $r > 0$. Next we introduce suitable subclasses of $\mathscr{P}_E$, which are adapted to the Gelfand-Shilov spaces that we consider in the sequel.

**Definition 5.1.** *Let $s > 0$. The set $\mathscr{P}_{E,s}(\mathbb{R}^d)$ ($\mathscr{P}^0_{E,s}(\mathbb{R}^d)$) consists of all $\omega \in \mathscr{P}_E(\mathbb{R}^d)$ such that*

$$\omega(x+y) \lesssim \omega(x)e^{r|y|^{\frac{1}{s}}}, \quad x,y \in \mathbb{R}^d, \tag{5.5}$$

*holds true for some (every) $r > 0$.*

By (5.4) it follows that $\mathscr{P}^0_{E,s_1} = \mathscr{P}_{E,s_2} = \mathscr{P}_E$ when $s_1 < 1$ and $s_2 \leqslant 1$. For convenience we set $\mathscr{P}^0_E(\mathbb{R}^d) = \mathscr{P}^0_{E,1}(\mathbb{R}^d)$. For the analysis performed in the last two chapters of this part we need more general classes of weight functions.

In similar ways as above, if $s, \sigma > 0$, then $\mathscr{P}_{s,\sigma}(\mathbb{R}^{2d})$ ($\mathscr{P}^0_{s,\sigma}(\mathbb{R}^{2d})$) consists of all submultiplicative weight functions $\omega$ on $\mathbb{R}^{2d}$ such that

$$\omega(x+y,\xi+\eta) \lesssim \omega(x,\xi)e^{r(|y|^{\frac{1}{s}}+|\eta|^{\frac{1}{\sigma}})}, \quad x,y,\xi,\eta \in \mathbb{R}^d, \tag{5.6}$$

for some $r > 0$ (for every $r > 0$). In particular, if $\omega \in \mathscr{P}_{s,\sigma}(\mathbb{R}^{2d})$ ($\mathscr{P}^0_{s,\sigma}(\mathbb{R}^{2d})$), then

$$e^{-r(|x|^{\frac{1}{s}}+|\xi|^{\frac{1}{\sigma}})} \lesssim \omega(x,\xi) \lesssim e^{r(|x|^{\frac{1}{s}}+|\xi|^{\frac{1}{\sigma}})} \tag{5.7}$$

for some $r > 0$ (for every $r > 0$). For more facts about weights, see [70].

## 5.2 Gelfand-Shilov spaces

Let $h, s, s_0, \sigma, \sigma_0 \in \mathbb{R}_+$, and let $\mathcal{S}^\sigma_{s,h}(\mathbb{R}^d)$ be the set of all $f \in C^\infty(\mathbb{R}^d)$ such that

$$\|f\|_{\mathcal{S}^\sigma_{s,h}} \equiv \sup \frac{|x^\beta \partial^\alpha f(x)|}{h^{|\alpha+\beta|}\alpha!^\sigma \beta!^s}$$

is finite. Here the supremum is taken over all $\alpha, \beta \in \mathbb{Z}^d$ and $x \in \mathbb{R}^d$.

Obviously $\mathcal{S}^\sigma_{s,h}(\mathbb{R}^d)$ is a Banach space which increases as $h$, $s$ and $\sigma$ increase, and is contained in $\mathscr{S}(\mathbb{R}^d)$, the set of Schwartz functions on $\mathbb{R}^d$. If in addition $s + \sigma > 1$ and $s_0 + \sigma_0 \geqslant 1$

$$\mathcal{S}^\sigma_{s,h}(\mathbb{R}^d) \quad \text{and} \quad \bigcup_{h>0} \mathcal{S}^{\sigma_0}_{s_0,h}(\mathbb{R}^d)$$

are dense in $\mathscr{S}(\mathbb{R}^d)$. Hence, the dual $(\mathcal{S}^\sigma_{s,h})'(\mathbb{R}^d)$ of $\mathcal{S}^\sigma_{s,h}(\mathbb{R}^d)$ is a Banach space which contains $\mathscr{S}'(\mathbb{R}^d)$.

The spaces $\mathcal{S}^\sigma_s(\mathbb{R}^d)$ and $\Sigma^\sigma_s(\mathbb{R}^d)$ are the inductive and projective limits, respectively, of $\mathcal{S}^\sigma_{s,h}(\mathbb{R}^d)$ with respect to $h$. The space $\mathcal{S}^\sigma_s(\mathbb{R}^d)$ ($\Sigma^\sigma_s(\mathbb{R}^d)$) is called the Gelfand-Shilov space of Roumieu type (of Beurling type, respectively) of order $(s,\sigma)$. This implies that

$$\mathcal{S}^\sigma_s(\mathbb{R}^d) = \bigcup_{h>0} \mathcal{S}^\sigma_{s,h}(\mathbb{R}^d) \quad \text{and} \quad \Sigma^\sigma_s(\mathbb{R}^d) = \bigcap_{h>0} \mathcal{S}^\sigma_{s,h}(\mathbb{R}^d), \tag{5.8}$$

and that the topology for $\mathcal{S}^\sigma_s(\mathbb{R}^d)$ is the strongest possible one such that each inclusion map from $\mathcal{S}^\sigma_{s,h}(\mathbb{R}^d)$ to $\mathcal{S}^\sigma_s(\mathbb{R}^d)$ is continuous.

The Gelfand-Shilov distribution spaces $(\mathcal{S}^\sigma_s)'(\mathbb{R}^d)$ and $(\Sigma^\sigma_s)'(\mathbb{R}^d)$ are the projective and inductive limit respectively of $(\mathcal{S}^\sigma_{s,h})'(\mathbb{R}^d)$. Hence

$$(\mathcal{S}^\sigma_s)'(\mathbb{R}^d) = \bigcap_{h>0} (\mathcal{S}^\sigma_{s,h})'(\mathbb{R}^d) \quad \text{and} \quad (\Sigma^\sigma_s)'(\mathbb{R}^d) = \bigcup_{h>0} (\mathcal{S}^\sigma_{s,h})'(\mathbb{R}^d). \tag{5.8$'$}$$

We have that $(\mathcal{S}^\sigma_s)'(\mathbb{R}^d)$ and $(\Sigma^\sigma_s)'(\mathbb{R}^d)$ are the topological duals of $\mathcal{S}^\sigma_s(\mathbb{R}^d)$ and $\Sigma^\sigma_s(\mathbb{R}^d)$, respectively (see [97]).

We also set $\mathcal{S}_s(\mathbb{R}^d) = \mathcal{S}_s^s(\mathbb{R}^d)$ and $\Sigma_s(\mathbb{R}^d) = \Sigma_s^s(\mathbb{R}^d)$, and similarly for their distribution spaces.

The classes $\mathcal{S}_s^\sigma(\mathbb{R}^d)$ and related generalizations were widely studied, and used in the applications to partial differential equations, see for example [12, 31, 37, 74, 93, 99]. We recall the following characterisations of $\mathcal{S}_s^\sigma(\mathbb{R}^d)$.

**Proposition 5.2.** *Let $s,\sigma > 0$, $p \in [1,\infty]$ and let $f \in \mathscr{S}(\mathbb{R}^d)$. Then the following conditions are equivalent:*

*(1) $f \in \mathcal{S}_s^\sigma(\mathbb{R}^d)$ $(f \in \Sigma_s^\sigma(\mathbb{R}^d))$ .*

*(2) For some (every) $h > 0$ it holds*

$$\|x^\alpha f\|_{L^p} \lesssim h^{|\alpha|}\alpha!^s \quad \text{and} \quad \|\xi^\beta \widehat{f}\|_{L^p} \lesssim h^{|\beta|}\beta!^\sigma, \quad \alpha,\beta \in \mathbb{Z}_+^d.$$

*(3) For some (every) $h > 0$ it holds*

$$\|x^\alpha f\|_{L^p} \lesssim h^{|\alpha|}\alpha!^s \quad \text{and} \quad \|\partial^\beta f\|_{L^p} \lesssim h^{|\beta|}\beta!^\sigma, \quad \alpha,\beta \in \mathbb{Z}_+^d.$$

*(4) For some (every) $h > 0$ it holds*

$$\|x^\alpha \partial^\beta f(x)\|_{L^p} \lesssim h^{|\alpha+\beta|}\alpha!^s\,\beta!^\sigma, \quad \alpha,\beta \in \mathbb{Z}_+^d.$$

*(5) For some (every) $h, r > 0$ it holds*

$$\|e^{r|\,\cdot\,|^{\frac{1}{s}}}\partial^\alpha f\|_{L^p} \lesssim h^{|\alpha|}(\alpha!)^\sigma\,\alpha \in \mathbb{Z}_+^d.$$

*(6) For some (every) $r > 0$ it holds*

$$\|f \cdot e^{r|\,\cdot\,|^{\frac{1}{s}}}\|_{L^p} < \infty \quad \text{and} \quad \|\widehat{f} \cdot e^{r|\,\cdot\,|^{\frac{1}{\sigma}}}\|_{L^p} < \infty.$$

**Remark 5.3.** *Any of the conditions (2)–(6) in Proposition 5.2 induce the same topology for $\mathcal{S}_s^\sigma(\mathbb{R}^d)$ and $\Sigma_s^\sigma(\mathbb{R}^d)$.*

**Remark 5.4.** *Let $s,\sigma > 0$. Then, $\Sigma_s^\sigma(\mathbb{R}^d)$ is a Fréchet space with semi-norms $\|\cdot\|_{\mathcal{S}_{s,h}^\sigma}$, $h > 0$. Moreover, $\mathcal{S}_s^\sigma(\mathbb{R}^d) \neq \{0\}$ if and only if $s + \sigma \geqslant 1$, and $\Sigma_s^\sigma(\mathbb{R}^d) \neq \{0\}$ if and only if $s + \sigma \geqslant 1$ and $(s,\sigma) \neq (\frac{1}{2},\frac{1}{2})$. If $\varepsilon > 0$ and $s + \sigma \geqslant 1$, then*

$$\Sigma_s^\sigma(\mathbb{R}^d) \subseteq \mathcal{S}_s^\sigma(\mathbb{R}^d) \subseteq \Sigma_{s+\varepsilon}^{\sigma+\varepsilon}(\mathbb{R}^d) \subseteq \mathscr{S}(\mathbb{R}^d) \\ \subseteq \mathscr{S}'(\mathbb{R}^d) \subseteq (\Sigma_{s+\varepsilon}^{\sigma+\varepsilon})'(\mathbb{R}^d) \subseteq (\mathcal{S}_s^\sigma)'(\mathbb{R}^d).$$

*If in addition $(s,\sigma) \neq (\frac{1}{2},\frac{1}{2})$, then*

$$(\mathcal{S}_s^\sigma)'(\mathbb{R}^d) \subseteq (\Sigma_s^\sigma)'(\mathbb{R}^d).$$

The Gelfand-Shilov spaces are invariant and possess convenient mapping properties under several basic transformations. For example they are invariant under translations, dilations, and under (partial) Fourier transformations.

The Fourier transform $\mathscr{F}$ on $\mathscr{S}(\mathbb{R}^d)$ extends uniquely to homeomorphisms on $\mathscr{S}'(\mathbb{R}^d)$, $\mathcal{S}_s'(\mathbb{R}^d)$ and $\Sigma_s'(\mathbb{R}^d)$, and restricts to homeomorphisms on $\mathcal{S}_s(\mathbb{R}^d)$ and $\Sigma_s(\mathbb{R}^d)$, and to a unitary operator on $L^2(\mathbb{R}^d)$.

Some considerations later on involve a broader family of Gelfand-Shilov spaces. These spaces will be used in Chapters 9, 10 and 11. More precisely, for $s_j, \sigma_j \in \mathbb{R}_+$, $j = 1, 2$, the Gelfand-Shilov spaces $\mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbb{R}^{d_1+d_2})$ and $\Sigma_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbb{R}^{d_1+d_2})$ consist of all functions $F \in C^\infty(\mathbb{R}^{d_1+d_2})$ such that

$$|x_1^{\alpha_1} x_2^{\alpha_2} \partial_{x_1}^{\beta_1} \partial_{x_2}^{\beta_2} F(x_1, x_2)| \lesssim h^{|\alpha_1+\alpha_2+\beta_1+\beta_2|} \alpha_1!^{s_1} \alpha_2!^{s_2} \beta_1!^{\sigma_1} \beta_2!^{\sigma_2} \tag{5.9}$$

for some $h > 0$, respectively for every $h > 0$. The topologies, and the duals

$$(\mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2})'(\mathbb{R}^{d_1+d_2}) \quad \text{and} \quad (\Sigma_{s_1,s_2}^{\sigma_1,\sigma_2})'(\mathbb{R}^{d_1+d_2})$$

of

$$\mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbb{R}^{d_1+d_2}) \qquad \text{and} \quad \Sigma_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbb{R}^{d_1+d_2}),$$

respectively, and their topologies are defined in analogous ways as for the spaces $\mathcal{S}_s^\sigma(\mathbb{R}^d)$ and $\Sigma_s^\sigma(\mathbb{R}^d)$ above.

The following proposition explains mapping properties of partial Fourier transforms on Gelfand-Shilov spaces, and follows by similar arguments as in analogous situations in [64]. The proof is therefore omitted. Here, $\mathscr{F}_1 F$ and $\mathscr{F}_2 F$ are the partial Fourier transforms of $F(x_1, x_2)$ with respect to $x_1 \in \mathbb{R}^{d_1}$ and $x_2 \in \mathbb{R}^{d_2}$, respectively.

**Proposition 5.5.** *Let $s_j, \sigma_j > 0$, $j = 1, 2$. Then, the following holds true:*

*(1) The mappings $\mathscr{F}_1$ and $\mathscr{F}_2$ on $\mathscr{S}(\mathbb{R}^{d_1+d_2})$ restrict to homeomorphisms*

$$\mathscr{F}_1 : \mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbb{R}^{d_1+d_2}) \to \mathcal{S}_{\sigma_1,s_2}^{s_1,\sigma_2}(\mathbb{R}^{d_1+d_2})$$

*and*

$$\mathscr{F}_2 : \mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbb{R}^{d_1+d_2}) \to \mathcal{S}_{s_1,\sigma_2}^{\sigma_1,s_2}(\mathbb{R}^{d_1+d_2}).$$

*(2) The mappings $\mathscr{F}_1$ and $\mathscr{F}_2$ on $\mathscr{S}(\mathbb{R}^{d_1+d_2})$ are uniquely extendable to homeomorphisms*

$$\mathscr{F}_1 : (\mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2})'(\mathbb{R}^{d_1+d_2}) \to (\mathcal{S}_{\sigma_1,s_2}^{s_1,\sigma_2})'(\mathbb{R}^{d_1+d_2})$$

*and*

$$\mathscr{F}_2 : (\mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2})'(\mathbb{R}^{d_1+d_2}) \to (\mathcal{S}_{s_1,\sigma_2}^{\sigma_1,s_2})'(\mathbb{R}^{d_1+d_2}).$$

*The same holds true if the $\mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbb{R}^{d_1+d_2})$-spaces and their duals are replaced by the corresponding $\Sigma_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbb{R}^{d_1+d_2})$-spaces and their duals.*

The proof of the next result can be found in [28].

**Proposition 5.6.** *Let $s_j, \sigma_j > 0$, $j = 1, 2$. Then the following conditions are equivalent.*

*(1)* $F \in \mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbb{R}^{d_1+d_2})$ $\quad$ $(F \in \Sigma_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbb{R}^{d_1+d_2}))$.

*(2) For some $r > 0$ (for every $r > 0$) it holds*

$$|F(x_1, x_2)| \lesssim e^{-r(|x_1|^{\frac{1}{s_1}} + |x_2|^{\frac{1}{s_2}})} \quad \text{and} \quad |\widehat{F}(\xi_1, \xi_2)| \lesssim e^{-r(|\xi_1|^{\frac{1}{\sigma_1}} + |\xi_2|^{\frac{1}{\sigma_2}})}.$$

We notice that if $s_j + \sigma_j < 1$ for some $j = 1, 2$, then $\mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbb{R}^{d_1+d_2})$ and $\Sigma_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbb{R}^{d_1+d_2})$ are equal to the trivial space $\{0\}$. Likewise, if $s_j = \sigma_j = \frac{1}{2}$ for some $j = 1, 2$, then $\Sigma_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbb{R}^{d_1+d_2}) = \{0\}$.

## 5.3 The short-time Fourier transform and Gelfand-Shilov spaces

The short-time Fourier transform measures the time-variant frequency content of a distribution $f$ using a well-localized and smooth window $\phi \in L^2(\mathbb{R}^d)$ centered at the origin of $\mathbb{R}^d$. In order to move it to some point $z = (x; \xi) \in \mathbb{R}^{2d}$ one uses time-frequency shifts $\pi(z)$, i.e. applying first the translation operator $T_x\phi(y) = \phi(y - x)$ and then the modulation operator $M_\xi\phi(y) = e^{iy\xi}\phi(y)$, thus $\pi(z) = M_\xi T_x$.

**Definition 5.7.** *Let $\phi \in \mathscr{S}(\mathbb{R}^d)$ be fixed. For every $f \in \mathscr{S}'(\mathbb{R}^d)$, the* short-time Fourier transform *$V_\phi f$ is the distribution on $\mathbb{R}^{2d}$ defined by the formula*

$$(V_\phi f)(x, \xi) = \mathscr{F}(f\,\overline{\phi(\,\cdot\, - x)})(\xi) = (f, \phi(\,\cdot\, - x)e^{i\langle\,\cdot\,,\xi\rangle}). \tag{5.10}$$

We observe that for regular distributions and suitable intangibility conditions, the short-time Fourier transform can be expressed as

$$(V_\phi f)(x, \xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(y)\overline{\phi(y - x)}e^{-iy\xi}\, dy = \langle f, \pi(z)\phi\rangle \tag{5.11}$$

It provides a description of $f$ in which time and frequency play a symmetric role. In general, the bracket $\langle\,\cdot\,,\,\cdot\,\rangle$ extends the inner product on $L^2(\mathbb{R}^d)$ to any dual pairing between a distribution space and its space of test functions, for instance $\phi \in \mathscr{S}(\mathbb{R}^d)$ and $f \in \mathscr{S}'(\mathbb{R}^d)$, but time-frequency analysis often needs larger distribution spaces.

Next we recall some mapping properties of Gelfand-Shilov spaces under short-time Fourier transforms.

The short-time Fourier transform is unitary $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$ provided $\|\phi\|_{L^2} = 1$, a topological isomorphism $\mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^{2d})$, and extends to a topological isomorphism $\mathscr{S}'(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^{2d})$. The adjoint operator is given by

$$V_\varphi^* g = (2\pi)^{-\frac{d}{2}} \iint_{\mathbb{R}^{2d}} g(x,\xi) M_\xi T_x \varphi \, dx d\xi, \tag{5.12}$$

where $V_\varphi^* g$ in general is interpreted as the functional

$$\langle V_\varphi^* g, \gamma \rangle = (2\pi)^{-\frac{d}{2}} \iint_{\mathbb{R}^{2d}} g(x,\xi) \langle M_\xi T_x \varphi, \gamma \rangle \, dx d\xi, \quad \gamma \in \mathscr{S}(\mathbb{R}^d). \tag{5.13}$$

The next result can be found in [67].

**Proposition 5.8.** *Suppose that $\phi, \varphi \in L^2(\mathbb{R}^d)$, such that $\langle \varphi, \phi \rangle \neq 0$. Then, for all $f \in L^2(\mathbb{R}^d)$*

$$f = \frac{1}{\langle \varphi, \phi \rangle} V_\varphi^* V_\phi f.$$

We recall that if $T(f,\phi) \equiv V_\phi f$ when $f, \phi \in \mathcal{S}_{1/2}(\mathbb{R}^d)$, then $T$ is uniquely extendable to sequentially continuous mappings

$$T : \mathcal{S}_s'(\mathbb{R}^d) \times \mathcal{S}_s(\mathbb{R}^d) \to \mathcal{S}_s'(\mathbb{R}^{2d}) \bigcap C^\infty(\mathbb{R}^{2d}),$$
$$T : \mathcal{S}_s'(\mathbb{R}^d) \times \mathcal{S}_s'(\mathbb{R}^d) \to \mathcal{S}_s'(\mathbb{R}^{2d}),$$

and similarly when $\mathcal{S}_s$ and $\mathcal{S}_s'$ are replaced by $\Sigma_s$ and $\Sigma_s'$, respectively, or by $\mathscr{S}$ and $\mathscr{S}'$, respectively (cf. [35, 120]). We also note that $V_\phi f$ takes the form

$$V_\phi f(x,\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(y) \overline{\phi(y-x)} e^{-i\langle y,\xi \rangle} \, dy \tag{5.10$'$}$$

when $f \in L^p_{(\omega)}(\mathbb{R}^d)$ for some $\omega \in \mathscr{P}_E(\mathbb{R}^d)$, $\phi \in \Sigma_1(\mathbb{R}^d)$ and $p \geqslant 1$. Here $L^p_{(\omega)}(\mathbb{R}^d)$, when $p \in (0,\infty]$ and $\omega \in \mathscr{P}_E(\mathbb{R}^d)$, is the set of all $f \in L^p_{loc}(\mathbb{R}^d)$ such that $\|f\|_{L^p_{(\omega)}} \equiv \|f \cdot \omega\|_{L^p}$ is finite.

The following characterizations of the $\mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbb{R}^{d_1+d_2})$, $\Sigma_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbb{R}^{d_1+d_2})$ and their duals follow by similar arguments as in the proofs of Propositions 2.1 and 2.2 in [123] .

**Proposition 5.9.** *Let $s_j, \sigma_j > 0$ be such that $s_j + \sigma_j \geqslant 1$, $j = 1,2$. Also let $\phi \in \mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbb{R}^{d_1+d_2}) \backslash 0$. Then the following properties hold true:*

*(1) $f \in \mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbb{R}^{d_1+d_2})$, if and only if*

$$|V_\phi f(x_1,x_2,\xi_1,\xi_2)| \lesssim e^{-r(|x_1|^{\frac{1}{s_1}} + |x_2|^{\frac{1}{s_2}} + |\xi_1|^{\frac{1}{\sigma_1}} + |\xi_2|^{\frac{1}{\sigma_2}})} \tag{5.14}$$

*holds true for some $r > 0$.*

*(2) If in addition* $\phi \in \Sigma_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbb{R}^{d_1+d_2})\backslash 0$*, then* $f \in \Sigma_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbb{R}^{d_1+d_2})$ *if and only if*

$$|V_\phi f(x_1,x_2,\xi_1,\xi_2)| \lesssim e^{-r(|x_1|^{\frac{1}{s_1}}+|x_2|^{\frac{1}{s_2}}+|\xi_1|^{\frac{1}{\sigma_1}}+|\xi_2|^{\frac{1}{\sigma_2}})} \tag{5.15}$$

*holds true for every* $r > 0$.

A proof of Proposition 5.9 can be found in e. g. [74] (cf. [74, Theorem 2.7]). The corresponding result for Gelfand-Shilov distributions is the following improvement of [120, Theorem 2.5]. See also [123].

**Proposition 5.10.** *Let* $s_j, \sigma_j > 0$ *be such that* $s_j + \sigma_j \geqslant 1$, $j = 1, 2$. *Also let* $\phi \in \mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbb{R}^{d_1+d_2})\backslash 0$ *and let* $f \in (\mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2})'(\mathbb{R}^{d_1+d_2})$. *Then the following properties hold true:*

*(1)* $f \in (\mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2})'(\mathbb{R}^{d_1+d_2})$*, if and only if*

$$|V_\phi f(x_1,x_2,\xi_1,\xi_2)| \lesssim e^{r(|x_1|^{\frac{1}{s_1}}+|x_2|^{\frac{1}{s_2}}+|\xi_1|^{\frac{1}{\sigma_1}}+|\xi_2|^{\frac{1}{\sigma_2}})} \tag{5.16}$$

*holds for every* $r > 0$.

*(2) If in addition* $\phi \in \Sigma_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbb{R}^{d_1+d_2})\backslash 0$*, then* $f \in (\Sigma_{s_1,s_2}^{\sigma_1,\sigma_2})'(\mathbb{R}^{d_1+d_2})$*, if and only if*

$$|V_\phi f(x_1,x_2,\xi_1,\xi_2)| \lesssim e^{r(|x_1|^{\frac{1}{s_1}}+|x_2|^{\frac{1}{s_2}}+|\xi_1|^{\frac{1}{\sigma_1}}+|\xi_2|^{\frac{1}{\sigma_2}})} \tag{5.17}$$

*holds for some* $r > 0$.

**Remark 5.11.** *We notice that any short-time Fourier transform of a Gelfand-Shilov distribution with window function as Gelfand-Shilov function or even a Schwartz function makes sense as a Gelfand-Shilov distribution.*

*In fact, let*

$$T_1 : (\mathcal{S}_s^\sigma)'(\mathbb{R}^d) \times (\mathcal{S}_s^\sigma)'(\mathbb{R}^d) \to (\mathcal{S}_s^\sigma)'(\mathbb{R}^{2d}),$$

*and*

$$T_2 : (\mathcal{S}_s^\sigma)'(\mathbb{R}^{2d}) \to (\mathcal{S}_s^\sigma)'(\mathbb{R}^{2d})$$

*be the continuous mappings*

$$T_1(f,\phi) = f \otimes \overline{\phi}, \quad f,\phi \in (\mathcal{S}_s^\sigma)'(\mathbb{R}^d),$$

*and*

$$(T_2F)(x,y) = F(y, y-x), \quad F \in (\mathcal{S}_s^\sigma)'(\mathbb{R}^{2d}).$$

*Also let* $(\mathscr{F}_2F)(x,\,\cdot\,)$ *be the partial Fourier transform of* $F(x,y)$ *with respect to* $y\in\mathbb{R}^d$*, which is continuous from* $(\mathcal{S}_s^\sigma)'(\mathbb{R}^{2d})$ *to* $(\mathcal{S}_{s,\sigma}^{\sigma,s})'(\mathbb{R}^{2d})$*. Then*

$$V_\phi f = (\mathscr{F}_2 \circ T_2 \circ T_1)\,(f,\phi). \tag{5.18}$$

*By defining* $V_\phi f$ *as the right-hand side of* (5.18) *when* $f,\phi\in(\mathcal{S}_s^\sigma)'(\mathbb{R}^d)$*, it follows that the map*

$$(f,\phi)\mapsto V_\phi f \tag{5.19}$$

*is continuous from* $(\mathcal{S}_s^\sigma)'(\mathbb{R}^d)\times(\mathcal{S}_s^\sigma)'(\mathbb{R}^d)$ *to* $(\mathcal{S}_{s,\sigma}^{\sigma,s})'(\mathbb{R}^{2d})$*.*

*In the same way* (5.19) *extends uniquely to a continuous map from* $(\Sigma_s^\sigma)'(\mathbb{R}^d)\times(\Sigma_s^\sigma)'(\mathbb{R}^d)$ *to* $(\Sigma_{s,\sigma}^{\sigma,s})'(\mathbb{R}^{2d})$*.*

## 5.4 Function classes with Gelfand-Shilov regularity

The next result shows that for any $\omega\in\mathscr{P}_E(\mathbb{R}^d)$ one can find an equivalent weight $\omega_0$ which satisfies suitable Gevrey regularity.

**Proposition 5.12.** *Let* $\omega\in\mathscr{P}_E(\mathbb{R}^d)$ *and* $s>0$*. Then there is an* $\omega_0\in\mathscr{P}_E(\mathbb{R}^d)\cap C^\infty(\mathbb{R}^d)$ *such that the following properties hold true:*

*(1)* $\omega_0\asymp\omega$*.*

*(2)* $|\partial^\alpha\omega_0(x)|\lesssim h^{|\alpha|}\alpha!^s\omega_0(x)\asymp h^{|\alpha|}\alpha!^s\omega(x)$ *for every* $h>0$*.*

*Proof.* We may assume that $s<1$, such that $s\neq 1/2$. It suffices to prove that (2) hold true for some $h>0$. Let $\phi_0\in\Sigma_{1-s}^s(\mathbb{R}^d)\backslash\{0\}$, and let $\phi=|\phi_0|^2$. Then $\phi\in\Sigma_{1-s}^s(\mathbb{R}^d)$, giving that

$$|\partial^\alpha\phi(x)|\lesssim h^{|\alpha|}e^{-r|x|^{\frac{1}{1-s}}}\alpha!^s,$$

for every $h>0$ and $r>0$. From now on we fix the value of $r$ to the one given in (5.4). Now let $\omega_0=\omega*\phi$.

We have

$$\begin{aligned}
|\partial^\alpha\omega_0(x)| &= \left|\int_{\mathbb{R}^d}\omega(y)(\partial^\alpha\phi)(x-y)\,dy\right|\\
&\lesssim h^{|\alpha|}\alpha!^s\int_{\mathbb{R}^d}\omega(y)e^{-r|x-y|^{\frac{1}{1-s}}}\,dy\\
&\lesssim h^{|\alpha|}\alpha!^s\int_{\mathbb{R}^d}\omega(x+(y-x))e^{-r|x-y|^{\frac{1}{1-s}}}\,dy\\
&\lesssim h^{|\alpha|}\alpha!^s\omega(x)\int_{\mathbb{R}^d}e^{-\frac{r}{2}|x-y|^{\frac{1}{1-s}}}\,dy\asymp h^{|\alpha|}\alpha!^s\omega(x),
\end{aligned}$$

where the last inequality follows from (5.4) and the fact that, for any $x, y \in \mathbb{R}^d$, $e^{r|x-y|}e^{-r|x-y|^{\frac{1}{1-s}}} \leqslant e^{-r|x-y|^{\frac{1}{1-s}}}$. This gives the first part of (2).

The equivalences in (1) follows in the same way as in [120]. More precisely, by (5.4) we have

$$\begin{aligned}\omega_0(x) &= \int_{\mathbb{R}^d} \omega(y)\phi(x-y)\,dy = \int_{\mathbb{R}^d} \omega(x+(y-x))\phi(x-y)\,dy \\ &\lesssim \omega(x)\int_{\mathbb{R}^d} e^{r|x-y|}\phi(x-y)\,dy \asymp \omega(x).\end{aligned}$$

In the same way, (5.4) gives

$$\begin{aligned}\omega_0(x) &= \int_{\mathbb{R}^d} \omega(y)\phi(x-y)\,dy = \int_{\mathbb{R}^d} \omega(x+(y-x))\phi(x-y)\,dy \\ &\gtrsim \omega(x)\int_{\mathbb{R}^d} e^{-r|x-y|}\phi(x-y)\,dy \asymp \omega(x),\end{aligned}$$

and (1) as well as the second part of (2) follow.
The cases where $s = \frac{1}{2}$ or $s \geqslant 1$ follow by using different $\phi_0 \in \Sigma_{\frac{1}{2}}^{\frac{10}{6}}$ and $\phi_0 \in \Sigma_s^{\frac{1}{2}}$ respectively, using the fact that the estimate on $\phi_0$ holds for every $r > 0$. □

A weight $\omega_0$ which satisfies Proposition 5.12 (2) is called *elliptic* or *s-elliptic.*

**Proposition 5.13.** *Let $\omega \in \mathscr{P}_E(\mathbb{R}^{2d})$ and $s, \sigma > 0$. Then there exists a weight $\omega_0 \in \mathscr{P}_E(\mathbb{R}^{2d}) \cap C^\infty(\mathbb{R}^{2d})$ such that the following is true:*

*(1) $\omega_0 \asymp \omega$.*

*(2) For every $h > 0$,*

$$|\partial_x^\alpha \partial_\xi^\beta \omega_0(x,\xi)| \lesssim h^{|\alpha+\beta|}\alpha!^\sigma \beta!^s \omega_0(x,\xi) \asymp h^{|\alpha+\beta|}\alpha!^\sigma \beta!^s \omega(x,\xi).$$

Proposition 5.13 is equivalent Proposition 5.12. In fact, by Proposition 5.12, we have that Proposition 5.13 holds with $s = \sigma$. Hence, Proposition 5.13 implies Proposition 5.12. On the other hand, let $s_0 = \min(s, \sigma)$. Then Proposition 5.12 implies that there is a weight function $\omega_0 \asymp \omega$ satisfying

$$\begin{aligned}|\partial_x^\alpha \partial_\xi^\beta \omega_0(x,\xi)| &\lesssim h^{|\alpha+\beta|}(\alpha!\beta!)^{s_0}\omega_0(x,\xi) \\ &\lesssim h^{|\alpha+\beta|}\alpha!^\sigma \beta!^s \omega_0(x,\xi),\end{aligned}$$

giving Proposition 5.13.

An important class of Gevrey type symbols is the following.

**Definition 5.14.** *Let $s \geqslant 0$ and $\omega \in \mathscr{P}_E(\mathbb{R}^d)$. The class $\Gamma_s^{(\omega)}(\mathbb{R}^d)$ ($\Gamma_{0,s}^{(\omega)}(\mathbb{R}^d)$) consists of all $f \in C^\infty(\mathbb{R}^d)$ such that*

$$|D^\alpha f(x)| \lesssim h^{|\alpha|}\alpha!^s\omega(x), \quad x \in \mathbb{R}^d, \tag{5.20}$$

*for* some $h > 0$ *(for* every $h > 0$*).*

Evidently, by Proposition 5.12 it follows that if $s < 1$, the family of symbol classes in Definition 5.14 is not decreased when the assumption $\omega \in \mathscr{P}_E(\mathbb{R}^{2d})$ is replaced by the more restrictive assumption $\omega \in \mathscr{P}_{E,s}(\mathbb{R}^{2d})$ or by $\omega \in \mathscr{P}_{E,s}^0(\mathbb{R}^{2d})$.

By similar arguments as in the proof of Proposition 5.12 we get the following analog of Proposition 2.3.16 in [88].

**Proposition 5.15.** *Let $s > 1/2$, $\omega \in \mathscr{P}_E(\mathbb{R}^{2d})$, and $\phi \in \Sigma_s(\mathbb{R}^{2d})$. Then $\omega * \phi$ belongs to $\Gamma_{0,s}^{(\omega)}$.*

The following definition is motivated by Lemma 2.6.13 in [88].

**Definition 5.16.** *Let $s \geqslant 1$, $\omega \in \mathscr{P}_E(\mathbb{R}^d)$ and $\vartheta_0 = 1 + |\log \omega|$. Then $a$ is called* comparable *to $\omega$ with respect to $s \geqslant 1$ if*

*(1)* $\|a - \log \omega\|_{L^\infty} < \infty$.

*(2)* $a \in \Gamma_s^{(\vartheta_0)}(\mathbb{R}^d)$ *and* $\partial^\alpha a \in \Gamma_s^{(1)}(\mathbb{R}^d)$, *when* $|\alpha| = 1$.

**Proposition 5.17.** *Let $\omega, v \in \mathscr{P}_E(\mathbb{R}^d)$ be such that $v$ is submultiplicative and $\omega$ is $v$-moderate. Also let*

$$v_1(x) \equiv 1 + |\log v(x)| \quad \text{and} \quad \omega_1(x) \equiv 1 + |\log \omega(x)|.$$

*Then $v_1$ is submultiplicative and $\omega_1$ is $v_1$-moderate, satisfying* (5.1) *with $1 + \log C \geqslant 1$ in place of $C \geqslant 1$.*

*Proof.* If $\omega(x+y) \geqslant 1$, then the second inequality in (5.2) gives

$$\begin{aligned}\omega_1(x+y) &= 1 + \log \omega(x+y) \\ &\leqslant 1 + \log C + \log \omega(x) + \log v(y) \\ &\leqslant (1 + \log C)(1 + |\log \omega(x)|)\,(1 + \log v(y)) \\ &\leqslant (1 + \log C)\,\omega_1(x)\,v_1(y).\end{aligned}$$

If instead $\omega(x+y) \leqslant 1$, then the first inequality in (5.2) gives

$$\begin{aligned}\omega_1(x+y) &= 1 - \log \omega(x+y) \\ &\leqslant 1 + \log C - \log \omega(x) + \log v(y) \\ &\leqslant (1 + \log C)(1 + |\log \omega(x)|)\,(1 + \log v(y)) \\ &\leqslant (1 + \log C)\,\omega_1(x)\,v_1(y),\end{aligned}$$

which implies that $\omega_1$ is $v_1$-moderate with the stated constants.

By choosing $\omega = v$ and $C = 1$, we deduce the submultiplicativity for $v_1$, and the result follows. □

**Lemma 5.18.** *Let $s \geqslant 1$, $\omega \in \mathscr{P}_E(\mathbb{R}^d)$ and $\vartheta_0 = 1 + |\log \omega|$. Then the following properties hold true:*

*(1) There exists an elliptic weight $\omega_0 \in \mathscr{P}_E(\mathbb{R}^d) \cap \Gamma_s^{(\omega)}(\mathbb{R}^d)$ such that*

$$\omega \asymp \omega_0, \quad \log \omega_0 \in \Gamma_s^{(\vartheta_0)}(\mathbb{R}^d) \quad \text{and} \quad 1 + |\log \omega_0| \in \mathscr{P}_E(\mathbb{R}^d) \cap \Gamma_s^{(\vartheta_0)}(\mathbb{R}^d).$$

*(2) There exists an element $c$ which is comparable to $\omega_0$ with respect to $s$.*

*Proof.* The assertion (1) follows by letting $\omega_0$ be the same as in Proposition 5.12. Indeed, the needed estimate is trivially true when no derivatives are applied on log $\omega_0$. If instead some derivatives have been applied, then one ends up with certain numbers of fractions of the form $\partial^\alpha \omega_0/\omega_0$, which are multiplied and summarized to each others. Using the estimates in Proposition 5.12 and Fáa di Bruno's formula then gives the needed estimates. Where (2) follows by letting $a = \log \omega_0$ and using the ellipticity of $\omega_0$. □

**Remark 5.19.** *For a weight function $\omega_0$, $S^{(\omega_0)}(\mathbb{R}^{2d})$ denotes the set of all smooth $a$ which satisfies*

$$|\partial^\alpha a| \leqslant C_\alpha \omega_0, \qquad \text{for some } C_\alpha > 0. \tag{5.21}$$

*It is clear that $\Gamma_{0,s}^{(\omega_0)}(\mathbb{R}^{2d}) \subseteq \Gamma_s^{(\omega_0)}(\mathbb{R}^{2d}) \subseteq S^{(\omega_0)}(\mathbb{R}^{2d})$. In the sequel, for the weights $\omega_1$, $\omega_2$ and $\omega_3$ involved in the definition of $\Gamma_{0,s}^{(\omega_1)}(\mathbb{R}^{2d})$, $\Gamma_s^{(\omega_2)}(\mathbb{R}^{2d})$ and $S^{(\omega_3)}(\mathbb{R}^{2d})$ we always assume that they belong to $\mathscr{P}_{E,s}(\mathbb{R}^{2d})$, $\mathscr{P}_{E,s}^0(\mathbb{R}^{2d})$ and $\mathscr{P}(\mathbb{R}^{2d})$ [1], respectively. Explicitly, they should satisfy*

$$\omega_1(X+Y) \lesssim \omega_1(X) e^{r_1|Y|^{\frac{1}{s}}}, \quad \omega_2(X+Y) \lesssim \omega_2(X) e^{r_2|Y|^{\frac{1}{s}}}, \tag{5.22}$$

$$\text{and} \quad \omega_3(X+Y) \lesssim \omega_3(X)(1+|Y|)^N, \tag{5.23}$$

*for some $r_1 > 0$ and $N > 0$, and every $r_2 > 0$.*

## 5.5 Anisotropic symbol classes

Next we introduce function spaces related to symbol classes of the pseudo-differential operators we will consider in the sequel. These functions obey various conditions of the form

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \lesssim h^{|\alpha+\beta|} \alpha!^\sigma \beta!^s \omega(x,\xi), \tag{5.24}$$

on the phase space $\mathbb{R}^{2d}$. For this reason we consider norms of the form

$$\|a\|_{\Gamma_{(\omega)}^{\sigma,s;h}} \equiv \sup_{\alpha,\beta \in \mathbb{Z}_+^d} \left( \sup_{x,\xi \in \mathbb{R}^d} \left( \frac{|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)|}{h^{|\alpha+\beta|} \alpha!^\sigma \beta!^s \omega(x,\xi)} \right) \right), \tag{5.25}$$

indexed by $h > 0$.

[1] $\mathscr{P}(\mathbb{R}^{2d})$ denotes the set of polynomially moderate weights on $\mathbb{R}^{2d}$ satisfying (5.23)

**Definition 5.20.** *Let $s$, $\sigma$ and $h$ be positive constants, let $\omega$ be a weight on $\mathbb{R}^{2d}$, and let*

$$\omega_r(x,\xi) \equiv e^{r(|x|^{\frac{1}{s}}+|\xi|^{\frac{1}{\sigma}})}.$$

*(1) The set $\Gamma^{\sigma,s;h}_{(\omega)}(\mathbb{R}^{2d})$ consists of all $a \in C^\infty(\mathbb{R}^{2d})$ such that $\|a\|_{\Gamma^{\sigma,s;h}_{(\omega)}}$ in* (5.25) *is finite. The set $\Gamma^{\sigma,s;h}_{0}(\mathbb{R}^{2d})$ consists of all $a \in C^\infty(\mathbb{R}^{2d})$ such that $\|a\|_{\Gamma^{\sigma,s;h}_{(\omega_r)}}$ is finite for every $r > 0$, and the topology is the projective limit topology of $\Gamma^{\sigma,s;h}_{(\omega_r)}(\mathbb{R}^{2d})$ with respect to $r > 0$.*

*(2) The sets $\Gamma^{\sigma,s}_{(\omega)}(\mathbb{R}^{2d})$ and $\Gamma^{\sigma,s;0}_{(\omega)}(\mathbb{R}^{2d})$ are given by*

$$\Gamma^{\sigma,s}_{(\omega)}(\mathbb{R}^{2d}) \equiv \bigcup_{h>0} \Gamma^{\sigma,s;h}_{(\omega)}(\mathbb{R}^{2d}) \quad \text{and} \quad \Gamma^{\sigma,s;0}_{(\omega)}(\mathbb{R}^{2d}) \equiv \bigcap_{h>0} \Gamma^{\sigma,s;h}_{(\omega)}(\mathbb{R}^{2d}),$$

*and their topologies are the inductive and the projective topologies of $\Gamma^{\sigma,s;h}_{(\omega)}(\mathbb{R}^{2d})$ respectively, with respect to $h > 0$.*

Furthermore we have the following classes.

**Definition 5.21.** *For $s_j, \sigma_j > 0$, $j = 1, 2$, and $h, r > 0$ and $f \in C^\infty(\mathbb{R}^{d_1+d_2})$, let*

$$\|f\|_{(h,r)} \equiv \sup \left( \frac{|\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2} f(x_1,x_2)|}{h^{|\alpha_1+\alpha_2|}\alpha_1!^{\sigma_1}\alpha_2!^{\sigma_2} e^{r(|x_1|^{\frac{1}{s_1}}+|x_2|^{\frac{1}{s_2}})}} \right), \tag{5.26}$$

*where the supremum is taken over all $\alpha_1 \in \mathbb{Z}^{d_1}_+, \alpha_2 \in \mathbb{Z}^{d_2}_+, x_1 \in \mathbb{R}^{d_1}$ and $x_2 \in \mathbb{R}^{d_2}$.*

*(1) $\Gamma^{\sigma_1,\sigma_2}_{s_1,s_2}(\mathbb{R}^{d_1+d_2})$ consists of all $f \in C^\infty(\mathbb{R}^{d_1+d_2})$ such that $\|f\|_{(h,r)}$ is finite for some $h, r > 0$.*

*(2) $\Gamma^{\sigma_1,\sigma_2}_{s_1,s_2;0}(\mathbb{R}^{d_1+d_2})$ consists of all $f \in C^\infty(\mathbb{R}^{d_1+d_2})$ such that for some $h > 0$, $\|f\|_{(h,r)}$ is finite for every $r > 0$.*

*(3) $\Gamma^{\sigma_1,\sigma_2;0}_{s_1,s_2}(\mathbb{R}^{d_1+d_2})$ consists of all $f \in C^\infty(\mathbb{R}^{d_1+d_2})$ such that for some $r > 0$, $\|f\|_{(h,r)}$ is finite for every $h > 0$.*

*(4) $\Gamma^{\sigma_1,\sigma_2;0}_{s_1,s_2;0}(\mathbb{R}^{d_1+d_2})$ consists of all $f \in C^\infty(\mathbb{R}^{d_1+d_2})$ such that $\|f\|_{(h,r)}$ is finite for every $h, r > 0$.*

In order to define suitable topologies of the spaces in Definition 5.21, let $(\Gamma^{\sigma_1,\sigma_2}_{s_1,s_2})_{(h,r)}(\mathbb{R}^{d_1+d_2})$ be the set of $f \in C^\infty(\mathbb{R}^{d_1+d_2})$ such that $\|f\|_{(h,r)}$ is finite.

Then $(\Gamma^{\sigma_1,\sigma_2}_{s_1,s_2})_{(h,r)}(\mathbb{R}^{d_1+d_2})$ is a Banach space, the sets in Definition 5.21 are given by

$$\Gamma^{\sigma_1,\sigma_2}_{s_1,s_2}(\mathbb{R}^{d_1+d_2}) = \bigcup_{h,r>0} (\Gamma^{\sigma_1,\sigma_2}_{s_1,s_2})_{(h,r)}(\mathbb{R}^{d_1+d_2}),$$

$$\Gamma^{\sigma_1,\sigma_2}_{s_1,s_2;0}(\mathbb{R}^{d_1+d_2}) = \bigcup_{h>0} \left( \bigcap_{r>0} (\Gamma^{\sigma_1,\sigma_2}_{s_1,s_2})_{(h,r)}(\mathbb{R}^{d_1+d_2}) \right),$$

$$\Gamma^{\sigma_1,\sigma_2;0}_{s_1,s_2}(\mathbb{R}^{d_1+d_2}) = \bigcup_{r>0} \left( \bigcap_{h>0} (\Gamma^{\sigma_1,\sigma_2}_{s_1,s_2})_{(h,r)}(\mathbb{R}^{d_1+d_2}) \right)$$

and

$$\Gamma^{\sigma_1,\sigma_2;0}_{s_1,s_2;0}(\mathbb{R}^{d_1+d_2}) = \bigcap_{h,r>0} (\Gamma^{\sigma_1,\sigma_2}_{s_1,s_2})_{(h,r)}(\mathbb{R}^{d_1+d_2}),$$

and we equip these spaces by suitable mixed inductive and projective limit topologies of $(\Gamma^{\sigma_1,\sigma_2}_{s_1,s_2})_{(h,r)}(\mathbb{R}^{d_1+d_2})$.

## 5.6 Modulation spaces

Modulation spaces measure the decay of the Short-time Fourier transform on the time-frequency (phase space) plane. These spaces were introduced by Feichtinger in the 80's [55], for weight of sub-exponential growth at infinity, sometime called weights of infinite order.

Before giving the definition of modulation spaces we recall the definition of quasi-Banach spaces. A functional $f \mapsto \|f\|_{\mathscr{B}}$ on a (complex) vector space $\mathscr{B}$ is called a quasi-norm of order $r \in (0,1]$, or an $r$-norm, if $\|f\|_{\mathscr{B}} \geqslant 0$ for all $f \in \mathscr{B}$ with equality only for $f = 0$,

$$\|f+g\|_{\mathscr{B}} \leqslant 2^{\frac{1}{r}-1}(\|f\|_{\mathscr{B}} + \|g\|_{\mathscr{B}}) \quad f,g \in \mathscr{B}, \tag{5.27}$$

and

$$\|c \cdot f\|_{\mathscr{B}} = |c| \cdot \|f\|_{\mathscr{B}} \qquad f \in \mathscr{B},\ c \in \mathbf{C}. \tag{5.28}$$

By Aoki and Rolewić in [6, 101] it follows that there is an equivalent quasi-norm to the previous one which additionally satisfies

$$\|f+g\|^r_{\mathscr{B}} \leqslant \|f\|^r_{\mathscr{B}} + \|g\|^r_{\mathscr{B}} \qquad f,g \in \mathscr{B}. \tag{5.29}$$

From now on we suppose that the quasi-norm of $\mathscr{B}$ has been chosen such that both (5.27) and (5.29) hold true.

The space $\mathscr{B}$ above is called a quasi-Banach space or an $r$-Banach space, if the topology is defined by $\|\cdot\|_{\mathscr{B}}$, and that $\mathscr{B}$ is complete under this topology.

**Definition 5.22.** *Let $\phi \in \Sigma_1(\mathbb{R}^d)\backslash 0$, $p,q \in (0,\infty]$ and $\omega \in \mathscr{P}_E(\mathbb{R}^{2d})$ be fixed. Then the* modulation space $M^{p,q}_{(\omega)}(\mathbb{R}^d)$ *consists of all $f \in \Sigma_1'(\mathbb{R}^d)$ such that*

$$\|f\|_{M^{p,q}_{(\omega)}} \equiv \Big( \int_{\mathbb{R}^d} \Big( \int_{\mathbb{R}^d} |V_\phi f(x,\xi)\omega(x,\xi)|^p \, dx \Big)^{q/p} d\xi \Big)^{1/q} < \infty \tag{5.30}$$

*(with the obvious modifications when $p = \infty$ and/or $q = \infty$). Evidently, $\|f\|_{M^{p,q}_{(\omega)}}$ is given by*

$$\|f\|_{M^{p,q}_{(\omega)}} \equiv \|H_{f,\omega,p}\|_{L^q}, \quad H_{f,\omega,p}(\xi) = \|V_\phi f(\,\cdot\,,\xi)\omega(\,\cdot\,,\xi)\|_{L^p} \tag{5.31}$$

*We set $M^p_{(\omega)} = M^{p,p}_{(\omega)}$, and if $\omega = 1$, then we set $M^{p,q} = M^{p,q}_{(\omega)}$ and $M^p = M^p_{(\omega)}$.*

The modulation spaces thus quantifies the asymptotic decay of $f \in \mathscr{S}'(\mathbb{R}^d)$ in the time and frequency variables.

The following proposition is a consequence of well-known facts in [55, 63, 67, 122]. Here and in what follows, we let $p'$ denote the conjugate exponent of $p$, i. e.

$$p' = \begin{cases} \infty & \text{when } p \in (0,1] \\ \dfrac{p}{p-1} & \text{when } p \in (1,\infty) \\ 1 & \text{when } p = \infty\,. \end{cases}$$

**Proposition 5.23.** *Let $p,q,p_j,q_j,r \in (0,\infty]$ be such that $r \leqslant \min(1,p,q)$, $j = 1,2$, let $\omega,\omega_1,\omega_2,v \in \mathscr{P}_E(\mathbb{R}^{2d})$ be such that $\omega$ is $v$-moderate, $\phi \in M^r_{(v)}(\mathbb{R}^d)\backslash 0$, and let $f \in \Sigma_1'(\mathbb{R}^d)$. Then the following properties hold true:*

*(1) $f \in M^{p,q}_{(\omega)}(\mathbb{R}^d)$ if and only if (5.30) holds, i. e. $M^{p,q}_{(\omega)}(\mathbb{R}^d)$ is independent of the choice of $\phi$. Moreover, $M^{p,q}_{(\omega)}$ is an $r$-Banach space under the $r$-norm in (5.30), and different choices of $\phi$ give rise to equivalent $r$-norms. If in addition $p,q \geqslant 1$, then $M^{p,q}_{(\omega)}(\mathbb{R}^d)$ is a Banach space.*

*(2) If $p_1 \leqslant p_2$, $q_1 \leqslant q_2$ and $\omega_2 \lesssim \omega_1$, then*

$$\Sigma_1(\mathbb{R}^d) \subseteq M^{p_1,q_1}_{(\omega_1)}(\mathbb{R}^d) \subseteq M^{p_2,q_2}_{(\omega_2)}(\mathbb{R}^d) \subseteq \Sigma_1'(\mathbb{R}^d).$$

**Remark 5.24.** *For modulation spaces of the form $M^{p,q}_{(\omega)}$ with fixed $p,q \in [1,\infty]$ the norm equivalence in Proposition 5.23(1) can be extended to a larger class of windows. In fact, assume that $\omega,v \in \mathscr{P}_E(\mathbb{R}^{2d})$ with $\omega$ being $v$-moderate and*

$$1 \leqslant r \leqslant \min(p,p',q,q')\,.$$

*Let $\phi \in M^r_{(v)}(\mathbb{R}^d)\backslash\{0\}$. Then, a Gelfand-Shilov distribution $f \in \Sigma_1'(\mathbb{R}^d)$ belongs to $M^{p,q}_{(\omega)}(\mathbb{R}^d)$, if and only if $V_\phi f \in L^{p,q}_{(\omega)}(\mathbb{R}^{2d})$. Furthermore, different choices of $\phi \in M^r_{(v)}(\mathbb{R}^d)\backslash\{0\}$ in $\|V_\phi f\|_{L^{p,q}_{(\omega)}}$ give rise to equivalent norms. (Cf. Theorem 2.6 in [121].)*

In essential parts of our analyses in Sections 8.1 and 8.2 it is convenient to use symplectic formulations of modulation spaces with functions and distributions defined on the phase spaces $\mathbb{R}^{2d}$. They are defined in the same way as the modulation spaces above, except that the short-time Fourier transforms in (5.10) are replaced by symplectic analogies in the definition of modulation space norms.

In fact, let σ be the standard symplectic form on $\mathbb{R}^{2d}$, i. e. it should satisfy

$$\sigma(X,Y) = \langle y,\xi\rangle - \langle x,\eta\rangle, \qquad X=(x,\xi)\in\mathbb{R}^{2d},\ Y=(y,\eta)\in\mathbb{R}^{2d}. \tag{5.32}$$

(Here observe the difference between the notation σ for the symplectic form in (5.32), and the positive number $\sigma$ used as parameter for the Gelfand-Shilov spaces, e. g. in Sections 5.2 and 5.4.) If

$$\{e_1,\dots,e_d,\varepsilon_1,\dots,\varepsilon_d\} \tag{5.33}$$

is the standard basis of $\mathbb{R}^{2d}$, then

$$\sigma(e_j,e_k)=0,\quad \sigma(e_j,\varepsilon_k)=-\delta_{j,k},\quad \text{and}\quad \sigma(\varepsilon_j,\varepsilon_k)=0 \tag{5.34}$$

when $j,k\in\{1,\dots,d\}$. More generally, a basis (5.33) of $\mathbb{R}^{2d}$ which satisfies (5.34) is called a symplectic basis of $\mathbb{R}^{2d}$ to the symplectic form σ. Evidently, the standard basis of $\mathbb{R}^{2d}$ is a symplectic basis, and is sometimes called the standard symplectic basis of $\mathbb{R}^{2d}$.

Let $\phi\in\Sigma_1(\mathbb{R}^{2d})\backslash 0$. Then the *symplectic Fourier transform* and *symplectic short-time Fourier transform* of $a\in L^1(\mathbb{R}^{2d})$ are defined by the formulae

$$(\mathscr{F}_\sigma a)(X) = \pi^{-d}\int_{\mathbb{R}^{2d}} a(Z)e^{2i\sigma(X,Z)}\,dZ \tag{5.35}$$

and

$$(\mathcal{V}_\phi a)(X,Y) = \pi^{-d}\int_{\mathbb{R}^{2d}} a(Z)\overline{\phi(Z-Y)}e^{2i\sigma(X,Z)}\,dZ. \tag{5.36}$$

By straight-forward computations, using Fourier's inversion formula, it follows that $\mathscr{F}_\sigma = T\circ(\mathscr{F}\otimes(\mathscr{F}^{-1}))$, when $(Ta)(x,\xi)=a(\xi,x)$, $\mathscr{F}_\sigma^2$ and

$$(\mathcal{V}_\phi a)(X,Y) = 2^d(V_\phi a)(x,\xi,-2\eta,2y), \quad X=(x,\xi)\in\mathbb{R}^{2d},\ Y=(y,\eta)\in\mathbb{R}^{2d}. \tag{5.37}$$

In particular, all continuity and extension properties valid for the usual Fourier transform and short-time Fourier transform carry over to their symplectic relatives. For example, $\mathscr{F}_\sigma$ is continuous on $\mathcal{S}_s(\mathbb{R}^{2d})$, and extends uniquely to a homeomorphism on $\mathcal{S}_s'(\mathbb{R}^{2d})$, and to a unitary map on $L^2(\mathbb{R}^{2d})$, since similar facts hold for $\mathscr{F}$.

For any $p,q \in (0,\infty]$, $\omega \in \mathscr{P}_E(\mathbb{R}^{2d} \times \mathbb{R}^{2d})$ and $a \in \Sigma_1'(\mathbb{R}^{2d})$, let $\|a\|_{\mathcal{M}^{p,q}_{(\omega)}}$ be defined by (5.31) after $V_\phi f$ is replaced by $\mathcal{V}_\phi a$. Then the *symplectic modulation space* $\mathcal{M}^{p,q}_{(\omega)}(\mathbb{R}^{2d})$ consists of all $a \in \Sigma_1'(\mathbb{R}^{2d})$ such that $\|a\|_{\mathcal{M}^{p,q}_{(\omega)}}$ is finite.

By (5.37) it follows that

$$\mathcal{M}^{p,q}_{(\omega)}(\mathbb{R}^{2d}) = M^{p,q}_{(\omega_0)}(\mathbb{R}^{2d}) \quad \text{when} \quad \omega(x,\xi,y,\eta) = \omega_0(x,\xi,-2\eta,2y).$$

Hence, the symplectic modulation spaces are merely other ways to formulate the modulation spaces considered in the first part of the subsection.

## 5.7 A broader family of modulation spaces

In Chapter 6 we consider mapping properties for pseudo-differential operators when acting on a broad class of modulation spaces which are defined by imposing (quasi-)norm conditions on the involved short-time Fourier transforms of the forms given in the following definition. (Cf. [54–58, 60, 62].)

**Definition 5.25.** *Let $\mathscr{B} \subseteq L^r_{\mathrm{loc}}(\mathbb{R}^d)$ be a quasi-Banach space of order $r \in (0,1]$, or an $r$-norm, and let $v \in \mathscr{P}_E(\mathbb{R}^d)$. Then $\mathscr{B}$ is called a* translation invariant Quasi-Banach Function space on $\mathbb{R}^d$, *or* invariant QBF space on $\mathbb{R}^d$, *if the following conditions are fulfilled:*

*(1) If $x \in \mathbb{R}^d$ and $f \in \mathscr{B}$, then $f(\,\cdot\, - x) \in \mathscr{B}$, and*

$$\|f(\,\cdot\, - x)\|_{\mathscr{B}} \lesssim v(x)\|f\|_{\mathscr{B}}. \tag{5.38}$$

*(2) If $f,g \in L^r_{loc}(\mathbb{R}^d)$ satisfy $g \in \mathscr{B}$ and $|f| \leqslant |g|$, then $f \in \mathscr{B}$ and*

$$\|f\|_{\mathscr{B}} \lesssim \|g\|_{\mathscr{B}}.$$

Note that a quasi-Banach space is a complete quasi-normed vector space.

**Definition 5.26.** *Assume that $\mathscr{B}$ is a translation invariant QBF-space on $\mathbb{R}^{2d}$, $\omega \in \mathscr{P}_E(\mathbb{R}^{2d})$, and that $\phi \in \Sigma_1(\mathbb{R}^d)\backslash 0$. Then, the modulation space $M(\omega,\mathscr{B})$ consists of all $f \in \Sigma_1'(\mathbb{R}^d)$ such that*

$$\|f\|_{M(\omega,\mathscr{B})} \equiv \|V_\phi f\,\omega\|_{\mathscr{B}}$$

*is finite.*

If $v$ belongs to $\mathscr{P}_{E,s}(\mathbb{R}^d)$ ($\mathscr{P}^0_{E,s}(\mathbb{R}^d)$), then $\mathscr{B}$ in Definition 5.25 is called an *invariant BF-space* of Roumieu type (Beurling type) of order $s$.

It follows from (2) in Definition 5.25 that if $f \in \mathscr{B}$ and $h \in L^\infty$, then $f \cdot h \in \mathscr{B}$, and

$$\|f \cdot h\|_{\mathscr{B}} \lesssim \|f\|_{\mathscr{B}}\|h\|_{L^\infty}. \tag{5.39}$$

If $r = 1$, then $\mathscr{B}$ in Definition 5.25 is a Banach space. The space $\mathscr{B}$ in Definition 5.25 is called an *invariant BF-space* (with respect to $v$) if $r = 1$, and Minkowski's inequality holds true, i.e. $f * \varphi \in \mathscr{B}$ when $f \in \mathscr{B}$ and $\varphi \in \Sigma_1(\mathbb{R}^d)$, and

$$\|f * \varphi\|_{\mathscr{B}} \lesssim \|f\|_{\mathscr{B}} \|\varphi\|_{L^1_{(v)}}, \qquad f \in \mathscr{B},\ \varphi \in \Sigma_1(\mathbb{R}^d). \tag{5.40}$$

**Example 5.27.** *Assume that $p, q \in [1, \infty]$, and let $L_1^{p,q}(\mathbb{R}^{2d})$ be the set of all $f \in L^1_{loc}(\mathbb{R}^{2d})$ such that*

$$\|f\|_{L_1^{p,q}} \equiv \Big( \int_{\mathbb{R}^d} \Big( \int_{\mathbb{R}^d} |f(x,\xi)|^p \, dx \Big)^{q/p} d\xi \Big)^{1/q}$$

*if finite. Also, let $L_2^{p,q}(\mathbb{R}^{2d})$ be the set of all $f \in L^1_{loc}(\mathbb{R}^{2d})$ such that*

$$\|f\|_{L_2^{p,q}} \equiv \Big( \int_{\mathbb{R}^d} \Big( \int_{\mathbb{R}^d} |f(x,\xi)|^q \, d\xi \Big)^{p/q} dx \Big)^{1/p}$$

*is finite. Then, it follows that $L_1^{p,q}$ and $L_2^{p,q}$ are translation invariant BF-spaces with respect to $v = 1$.*

We observe that $M^{p,q}_{(\omega)}(\mathbb{R}^d) = M(\omega, \mathscr{B})$ when $\mathscr{B}$ is equal to $L_1^{p,q}(\mathbb{R}^{2d})$ from Example 5.27. It follows that many properties which are valid for the classical modulation spaces also hold for the spaces of the form $M(\omega, \mathscr{B})$. For example we have the following proposition, which shows that the definition of $M(\omega, \mathscr{B})$ is independent of the choice of $\phi$ when $\mathscr{B}$ is a Banach space. The completeness assertions follows from [96], and the other parts follow by similar arguments as in the proof of Proposition 11.3.2 in [67], (see also [96] for topological aspects of $M(\omega, \mathscr{B})$).

**Proposition 5.28.** *Let $\mathscr{B}$ be an invariant BF-space with respect to $v_0 \in \mathscr{P}_E(\mathbb{R}^{2d})$.Also let $\omega, v \in \mathscr{P}_E(\mathbb{R}^{2d})$ be such that $\omega$ is $v$-moderate, $M(\omega, \mathscr{B})$ is the same as in Definition 5.26, and let $\phi \in M^1_{(v_0 v)}(\mathbb{R}^d)\backslash 0$ and $f \in \Sigma_1'(\mathbb{R}^d)$. Then $M(\omega, \mathscr{B})$ is a Banach space, and $f \in M(\omega, \mathscr{B})$ if and only if $V_\phi f \, \omega \in \mathscr{B}$. Moreover different choices of $\phi$ gives rise to equivalent norms in $M(\omega, \mathscr{B})$.*

We refer to [54–58, 60, 62, 63, 67, 102, 122] for more facts about modulation spaces. For translation invariant BF-spaces we make the following observation.

**Proposition 5.29.** *Assume that $v \in \mathscr{P}_E(\mathbb{R}^d)$, and that $\mathscr{B}$ is an invariant BF-space with respect to $v$ such that (5.40) holds true. Then, the convolution mapping $(\varphi, f) \mapsto \varphi * f$ from $C_0^\infty(\mathbb{R}^d) \times \mathscr{B}$ to $\mathscr{B}$ extends uniquely to a continuous mapping from $L^1_{(v)}(\mathbb{R}^d) \times \mathscr{B}$ to $\mathscr{B}$, and (5.40) holds true for any $f \in \mathscr{B}$ and $\varphi \in L^1_{(v)}(\mathbb{R}^d)$.*

The result is a straightforward consequence of (5.40) and the fact that $\Sigma_1$ is dense in $L^1_{(v)}$.

The quasi-Banach space $\mathscr{B}$ above is, usually, a mixed quasi-normed Lebesgue space, given as follows. Let $E$ be the ordered basis $\{e_1,\dots,e_d\}$ of $\mathbb{R}^d$. Then the ordered basis $E' = \{e'_1,\dots,e'_d\}$ (the dual basis of $E$) satisfies

$$\langle e_j, e'_k\rangle = 2\pi\delta_{jk} \quad \text{for every} \quad j,k = 1,\dots,d.$$

The corresponding parallelepiped, lattice, dual parallelepiped and dual lattice are given by

$$\begin{aligned}
\kappa(E) &= \{\, x_1e_1 + \cdots + x_de_d\,;\, (x_1,\dots,x_d)\in\mathbb{R}^d,\ 0\leqslant x_k\leqslant 1,\ k=1,\dots,d\,\},\\
\Lambda_E &= \{\, j_1e_1 + \cdots + j_de_d\,;\, (j_1,\dots,j_d)\in\mathbb{Z}^d\,\},\\
\kappa(E') &= \{\, \xi_1e'_1 + \cdots + \xi_de'_d\,;\, (\xi_1,\dots,\xi_d)\in\mathbb{R}^d,\ 0\leqslant \xi_k\leqslant 1,\ k=1,\dots,d\,\},
\end{aligned}$$

and

$$\Lambda'_E = \Lambda_{E'} = \{\, \iota_1e'_1 + \cdots + \iota_de'_d\,;\, (\iota_1,\dots,\iota_d)\in\mathbb{Z}^d\,\},$$

respectively. Note here that the Fourier analysis with respect to general biorthogonal bases has recently been developed in [103].

We observe that there is a matrix $T_E$ such that $e_1,\dots,e_d$ and $e'_1,\dots,e'_d$ are the images of the standard basis under $T_E$ and $T_{E'} = 2\pi(T_E^{-1})^t$, respectively.

In the sequel we let

$$\max \boldsymbol{q} = \max(q_1,\dots,q_d) \quad \text{and} \quad \min \boldsymbol{q} = \min(q_1,\dots,q_d)$$

when $\boldsymbol{q} = (q_1,\dots,q_d)\in(0,\infty]^d$.

**Definition 5.30.** *Let $E$ be an ordered basis of $\mathbb{R}^d$, $\boldsymbol{p} = (p_1,\dots,p_d)\in(0,\infty]^d$ and $r = \min(1,\boldsymbol{p})$. If $f\in L^r_{loc}(\mathbb{R}^d)$, then $\|f\|_{L^{\boldsymbol{p}}_E}$ is defined by*

$$\|f\|_{L^{\boldsymbol{p}}_E} \equiv \|g_{d-1}\|_{L^{p_d}(\mathbb{R})}$$

*where $g_k(z_k)$, $z_k\in\mathbb{R}^{d-k}$, $k=0,\dots,d-1$, are inductively defined as*

$$g_0(x_1,\dots,x_d) \equiv |f(x_1e_1+\cdots+x_de_d)|,$$

*and*

$$g_k(z_k) \equiv \|g_{k-1}(\,\cdot\,,z_k)\|_{L^{p_k}(\mathbb{R})}, \quad k=1,\dots,d-1.$$

*The space $L^{\boldsymbol{p}}_E(\mathbb{R}^d)$ consists of all $f\in L^r_{loc}(\mathbb{R}^d)$ such that $\|f\|_{L^{\boldsymbol{p}}_E}$ is finite, and is called* $E$-split Lebesgue space (with respect to $\boldsymbol{p}$).

For the next definition we recall that $\sigma(X,Y)$ denotes the standard symplectic form on the phase space (cf. (5.32)).

**Definition 5.31.** *Let $E = \{e_1, \dots, e_{2d}\}$ be an ordered basis of $\mathbb{R}^{2d}$ and let $E_0 = \{e_1, \dots, e_d\}$. Then $E_0$ is called a* phase split *of $E$, if*

$$\sigma(e_j, e_k) = 0, \quad \sigma(e_j, e_{d+k}) = -2\pi\delta_{j,k}, \quad \text{and} \quad \sigma(e_{d+j}, e_{d+k}) = 0$$

*when $j, k \in \{1, \dots, d\}$.*

If (5.33) is the standard basis of $\mathbb{R}^{2d}$ and $e_{d+j} = 2\pi\varepsilon_j$ for $j \in \{1, \dots, d\}$, then (5.34) shows that $\{e_1, \dots, e_d\}$ is a phase split of $\{e_1, \dots, e_{2d}\}$.

The following definition takes care of our most common QBF-spaces.

**Definition 5.32.** *The space $\mathscr{B}$ is called a* normal *QBF-space (on $\mathbb{R}^{2d}$) if it is either an invariant BF-space on $\mathbb{R}^{2d}$ or $\mathscr{B} = L^{\boldsymbol{p}}_E(\mathbb{R}^{2d})$ for some $\boldsymbol{p} \in (0,\infty]^{2d}$ and phase split basis $E$ of $\mathbb{R}^{2d}$.*

## 5.8 Pseudo-differential operators with symbols on the Gelfand-Shilov classes

We use the notation $\mathbf{M}(d,\Omega)$ for the set of $d \times d$-matrices with entries in the set $\Omega$. Let $s \geqslant 1/2$, $a \in \mathcal{S}_s(\mathbb{R}^{2d})$, and $A \in \mathbf{M}(d,\mathbb{R})$ be fixed. Then, the pseudo-differential operator $\operatorname{Op}_A(a)$ is the linear and continuous operator on $\mathcal{S}_s(\mathbb{R}^d)$ given by

$$(\operatorname{Op}_A(a)f)(x) = (2\pi)^{-d} \iint_{\mathbb{R}^{2d}} a(x - A(x-y), \xi)\, f(y)\, e^{i\langle x-y,\xi\rangle}\, dyd\xi \tag{5.41}$$

when $f \in \mathcal{S}_s(\mathbb{R}^d)$. For general $a \in \mathcal{S}_s'(\mathbb{R}^{2d})$, the pseudo-differential operator $\operatorname{Op}_A(a)$ is defined as the continuous operator from $\mathcal{S}_s(\mathbb{R}^d)$ to $\mathcal{S}_s'(\mathbb{R}^d)$ with distribution kernel given by

$$K_{a,A}(x,y) = (2\pi)^{-\frac{d}{2}}(\mathscr{F}_2^{-1}a)(x - A(x-y), x-y). \tag{5.42}$$

Here $\mathscr{F}_2 F$ is the partial Fourier transform of $F(x,y) \in \mathcal{S}_s'(\mathbb{R}^{2d})$ with respect to the $y$ variable. This definition makes sense, since the mappings

$$\mathscr{F}_2 \quad \text{and} \quad F(x,y) \mapsto F(x - A(x-y), y-x) \tag{5.43}$$

are homeomorphisms on $\mathcal{S}_s'(\mathbb{R}^{2d})$. In particular, the map $a \mapsto K_{a,A}$ is a homeomorphism on $\mathcal{S}_s'(\mathbb{R}^{2d})$.

**Remark 5.33.** *For any $K \in \mathcal{S}_s'(\mathbb{R}^{d_2+d_1})$, let $T_K$ be the linear and continuous mapping from $\mathcal{S}_s(\mathbb{R}^{d_1})$ to $\mathcal{S}_s'(\mathbb{R}^{d_2})$, defined by the formula*

$$(T_K f, g)_{L^2(\mathbb{R}^{d_2})} = (K, g \otimes \overline{f})_{L^2(\mathbb{R}^{d_2+d_1})}. \tag{5.44}$$

*It is well-known (see e. g., [29, 90]), that the Schwartz kernel theorem also holds in the context of Gelfand-Shilov spaces.*

*In fact, let $\mathcal{L}(V_1, V_2)$ be the set of linear continuous mappings from the topological vector space $V_1$ to the topological vector space $V_2$. Moreover, if $V_j$ are quasi-Banach spaces, then $\| \cdot \|_{\mathcal{L}(V_1,V_2)}$ denotes the quasi-norm in $\mathcal{L}(V_1, V_2)$. We also set $\mathcal{L}(V) = \mathcal{L}(V, V)$.*

*If $A \in \mathbf{M}(d, \mathbb{R})$, then the mappings $K \mapsto T_K$ and $a \mapsto \operatorname{Op}_A(a)$ are homeomorphisms from $\mathcal{S}_s'(\mathbb{R}^{2d})$ to $\mathcal{L}(\mathcal{S}_s(\mathbb{R}^d), \mathcal{S}_s'(\mathbb{R}^d))$. Similar facts hold true if $\mathcal{S}_s$ and $\mathcal{S}_s'$ are replaced by $\Sigma_s$ and $\Sigma_s'$, respectively (or by $\mathscr{S}$ and $\mathscr{S}'$, respectively).*

As a consequence of Remark 5.33 it follows that for each $a_1 \in \mathcal{S}_s'(\mathbb{R}^{2d})$ and $A_1, A_2 \in \mathbf{M}(d, \mathbb{R})$, there is a unique $a_2 \in \mathcal{S}_s'(\mathbb{R}^{2d})$ such that $\operatorname{Op}_{A_1}(a_1) = \operatorname{Op}_{A_2}(a_2)$. The relation between $a_1$ and $a_2$ is given by

$$\operatorname{Op}_{A_1}(a_1) = \operatorname{Op}_{A_2}(a_2) \qquad \Leftrightarrow \qquad a_2(x,\xi) = e^{i\langle (A_1-A_2)D_\xi, D_x\rangle} a_1(x,\xi), \tag{5.45}$$

(cf. [80]). Note here that the right-hand side makes sense, since it is equivalent to $\widehat{a}_2(\xi, x) = e^{i(A_1-A_2)\langle x,\xi\rangle}\widehat{a}_1(\xi, x)$, and that the map $a \mapsto e^{i\langle Ax,\xi\rangle}a$ is continuous on $\mathcal{S}_s'$ when $A \in \mathbf{M}(d, \mathbb{R})$.

Let $A \in \mathbf{M}(d, \mathbb{R})$ and $a \in \mathcal{S}_s'(\mathbb{R}^{2d})$ be fixed. Then $a$ is called a rank-one element with respect to $A$, if the corresponding pseudo-differential operator is of rank-one, i. e.

$$\operatorname{Op}_A(a)f = (f, f_2)f_1, \qquad f \in \mathcal{S}_s(\mathbb{R}^d), \tag{5.46}$$

for some $f_1, f_2 \in \mathcal{S}_s'(\mathbb{R}^d)$. By straightforward computations it follows that (5.46) is fulfilled, if and only if $a = (2\pi)^{\frac{d}{2}} W_{f_1,f_2}^A$, where $W_{f_1,f_2}^A$ it the $A$-Wigner distribution defined by the formula

$$W_{f_1,f_2}^A(x,\xi) \equiv \mathscr{F}(f_1(x + A\,\cdot\,)\overline{f_2(x - (I_d - A)\,\cdot\,)})(\xi), \tag{5.47}$$

which takes the form

$$W_{f_1,f_2}^A(x,\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f_1(x + Ay)\overline{f_2(x - (I_d - A)y)} e^{-i\langle y,\xi\rangle}\, dy,$$

when $f_1, f_2 \in \mathcal{S}_s(\mathbb{R}^d)$. Here $I_d \in \mathbf{M}(d, \mathbb{R})$ is the identity matrix. By combining these facts with (5.45) it follows that

$$W_{f_1,f_2}^{A_2} = e^{i\langle (A_1-A_2)D_\xi, D_x\rangle} W_{f_1,f_2}^{A_1}, \tag{5.48}$$

for each $f_1, f_2 \in \mathcal{S}_s'(\mathbb{R}^d)$ and $A_1, A_2 \in \mathbf{M}(d, \mathbb{R})$. Since the Weyl case is particularly important, we set $W_{f_1,f_2}^A = W_{f_1,f_2}$ when $A = \frac{1}{2}I_d$, i. e. $W_{f_1,f_2}$ is the usual (cross-)Wigner distribution of $f_1$ and $f_2$.

For future references we note the link

$$(\mathrm{Op}_A(a)f,g)_{L^2(\mathbb{R}^d)} = (2\pi)^{-\frac{d}{2}}(a, W_{g,f}^A)_{L^2(\mathbb{R}^{2d})},$$
$$a \in \mathcal{S}_s'(\mathbb{R}^{2d}) \quad \text{and} \quad f,g \in \mathcal{S}_s(\mathbb{R}^d) \quad (5.49)$$

between pseudo-differential operators and Wigner distributions, which follows by straightforward computations (see also e. g. [124]).

Next we discuss the Weyl product, the twisted convolution and related objects. Let $s \geqslant 1/2$ and let $a, b \in \mathcal{S}_s'(\mathbb{R}^{2d})$. Then the Weyl product $a\#b$ between $a$ and $b$ is the function or distribution which fulfills $\mathrm{Op}^w(a\#b) = \mathrm{Op}^w(a) \circ \mathrm{Op}^w(b)$, provided the right-hand side makes sense as a continuous operator from $\mathcal{S}_s(\mathbb{R}^d)$ to $\mathcal{S}_s'(\mathbb{R}^d)$. More generally, if $A \in \mathbf{M}(d,\mathbb{R})$, then the product $\#_A$ is defined by the formula

$$\mathrm{Op}_A(a\#_A b) = \mathrm{Op}_A(a) \circ \mathrm{Op}_A(b), \quad (5.50)$$

provided the right-hand side makes sense as a continuous operator from $\mathcal{S}_s(\mathbb{R}^d)$ to $\mathcal{S}_s'(\mathbb{R}^d)$, in which case $a$ and $b$ are called *suitable* or *admissible.*

The Weyl product can also, in a convenient way, be expressed in terms of the symplectic Fourier transform and the twisted convolution. More precisely, let $s \geqslant 1/2$.

**Definition 5.34.** *The* symplectic Fourier transform *for $a \in \mathcal{S}_s(\mathbb{R}^{2d})$ is defined by the formula*

$$(\mathscr{F}_\sigma a)(X) = \pi^{-d} \int_{\mathbb{R}^{2d}} a(Y) e^{2i\sigma(X,Y)}\, dY, \quad (5.51)$$

*where $\sigma$ is the symplectic form given by*

$$\sigma(X,Y) = \langle y,\xi\rangle - \langle x,\eta\rangle, \qquad X = (x,\xi) \in \mathbb{R}^{2d},\ Y = (y,\eta) \in \mathbb{R}^{2d}.$$

We note that $\mathscr{F}_\sigma = T \circ (\mathscr{F} \otimes (\mathscr{F}^{-1}))$, when $(Ta)(x,\xi) = a(2\xi, 2x)$. In particular, $\mathscr{F}_\sigma$ is continuous on $\mathcal{S}_s(\mathbb{R}^{2d})$, and extends uniquely to a homeomorphism on $\mathcal{S}_s'(\mathbb{R}^{2d})$, and to a unitary map on $L^2(\mathbb{R}^{2d})$, since similar facts hold true for $\mathscr{F}$. Furthermore, $\mathscr{F}_\sigma^2$ is the identity operator.

Let $s \geqslant 1/2$ and $a, b \in \mathcal{S}_s(\mathbb{R}^{2d})$.

Then the *twisted convolution* of $a$ and $b$ is defined by the formula

$$(a *_\sigma b)(X) = \left(\frac{2}{\pi}\right)^{\frac{d}{2}} \int_{\mathbb{R}^{2d}} a(X-Y) b(Y) e^{2i\sigma(X,Y)}\, dY. \quad (5.52)$$

The definition of $*_\sigma$ extends in different ways. For example, it extends to a continuous multiplication on $L^p(\mathbb{R}^{2d})$ when $p \in [1,2]$, and to a continuous map from $\mathcal{S}_s'(\mathbb{R}^{2d}) \times \mathcal{S}_s(\mathbb{R}^{2d})$ to $\mathcal{S}_s'(\mathbb{R}^{2d})$. If $a, b \in \mathcal{S}_s'(\mathbb{R}^{2d})$, then $a\#b$ makes sense if and only if $a *_\sigma \widehat{b}$ makes sense, and then

$$a\#b = (2\pi)^{-\frac{d}{2}} a *_\sigma (\mathscr{F}_\sigma b). \tag{5.53}$$

We also remark that for the twisted convolution we have

$$\mathscr{F}_\sigma(a *_\sigma b) = (\mathscr{F}_\sigma a) *_\sigma b = \check{a} *_\sigma (\mathscr{F}_\sigma b), \tag{5.54}$$

where $\check{a}(X) = a(-X)$ (cf. [113, 119, 121]). A combination of (5.53) and (5.54) gives

$$\mathscr{F}_\sigma(a\#b) = (2\pi)^{-\frac{d}{2}} (\mathscr{F}_\sigma a) *_\sigma (\mathscr{F}_\sigma b). \tag{5.55}$$

We use the notation $\mathcal{M}^{p,q}_{(\omega)}$ instead of $M^{p,q}_{(\omega)}$, for $p, q \in [1, \infty]$, if the symplectic short-time Fourier transform is used in the definition of modulation space norm. That is, if $\varphi \in \mathscr{S}(\mathbb{R}^{2d})\backslash\{0\}$ and $\omega \in \mathscr{P}(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d})$, then $\mathcal{M}^{p,q}_{(\omega)}(\mathbb{R}^{2d})$ consists of all $a \in \mathscr{S}'(\mathbb{R}^{2d})$ such that

$$\|a\|_{\mathcal{M}^{p,q}_{(\omega)}} \equiv \Big( \int_{\mathbb{R}^{2d}} \Big( \int_{\mathbb{R}^{2d}} |\mathcal{V}_\varphi a(X,Y) \omega(X,Y)|^p \, dX \Big)^{q/p} dY \Big)^{1/q} < +\infty\,.$$

The symplectic definition of modulation spaces does not yield any new spaces. In fact, setting $\omega(X,Y) = \omega_0(X, -2\eta, 2y)$ for $X \in \mathbb{R}^{2d}$ and $Y = (y, \eta) \in \mathbb{R}^{2d}$, it follows from the definition that $\mathcal{M}^{p,q}_{(\omega)} = M^{p,q}_{(\omega_0)}$ with equivalent norms.

Next we recall some notions on Hörmander symbol classes, $S(\omega, g)$, parameterized by the Riemannian metric $g$ and the weight function $\omega$ on the $2d$ dimensional symplectic vector space $W$ (see e. g. [14, 16, 78, 81, 88, 115]).

The Hörmander class $S(\omega, g)$ consists of all $a \in C^\infty(W)$ such that

$$\|a\|^g_{\omega,N} \equiv \sum_{k=0}^N \sup_{X \in W} (|a|^g_k(X)/\omega(X)) < \infty,$$

where

$$|a|^g_k(X) = \sup |a^{(k)}(X; Y_1, \dots, Y_k)|,$$

and $a^{(k)}$ denotes the $k^{th}$ differential of $a$ at $X$.
Here the latter supremum is taken over all $Y_1, \dots, Y_k \in W$ such that $g_X(Y_j) \leqslant 1$, $j = 1, \dots, k$, and $|a|^g_0(X)$ is interpreted as $|a(X)|$.

We need to add some conditions on $\omega$ and $g$. The metric $g$ is called *slowly varying* if there are positive constants $c$ and $C$ such that

$$C^{-1} g_X \leqslant g_Y \leqslant C g_X, \quad \text{when} \quad X, Y \in W \tag{5.56}$$

satisfy $g_X(X - Y) \leqslant c$, and $\omega$ is called $g$-continuous when (5.56) holds with $\omega(X)$ and $\omega(Y)$ in place of $g_X$ and $g_Y$, respectively, provided $g_X(X-Y) \leqslant c$.

For the Riemannian metric $g$ on $W$, the *dual* metric $g^\sigma$ with respect to the symplectic form $\sigma$, and the *Planck's function* $h_g$ are defined by

$$g_X^\sigma(Z) \equiv \sup_{g_X(Y)\leqslant 1} \sigma(Y,Z)^2 \qquad \text{and} \qquad h_g(X) \equiv \sup_{g_X^\sigma(Y)\leqslant 1} g_X(Y)^{1/2}.$$

Moreover, if $g$ is slowly varying and $\omega$ is $g$-continuous, then $g$ is called $\sigma$-temperate if there are positive constants $C$ and $N$ such that

$$g_Y(Z) \leqslant C g_X(Z)(1+g_Y(X-Y))^N, \qquad X,Y,Z \in W, \tag{5.57}$$

and $\omega$ is called $(\sigma,g)$-temperate if it is $g$-continuous and (5.57) holds with $\omega(X)$ and $\omega(Y)$ in place of $g_X(Z)$ and $g_Y(Z)$, respectively.

**Definition 5.35.** *Let*

$$\|a\|_{s_\infty^w} \equiv \|\operatorname{Op}^w(a)\|_{\mathcal{L}(L^2(\mathbb{R}^d))}, \quad a \in \mathscr{S}'(\mathbb{R}^{2d}).$$

*The set $s_\infty^w(\mathbb{R}^{2d})$ consists of all $a \in \mathscr{S}'(\mathbb{R}^{2d})$ such that $\operatorname{Op}^w(a)$ is linear and continuous on $L^2(\mathbb{R}^d)$, or equivalently, the set of all $a \in \mathscr{S}'(\mathbb{R}^{2d})$ such that $\|a\|_{s_\infty^w}$ is finite.*

**Remark 5.36.** *By Remark 5.33 it follows that the map $a \mapsto \operatorname{Op}^w(a)$ is an isometric bijection from $s_\infty^w(\mathbb{R}^{2d})$ to the set of linear continuous operators on $L^2(\mathbb{R}^d)$.*

**Remark 5.37.** *We remark that the relations in this section hold true after $\mathcal{S}_s$, $\mathcal{S}_s'$ and $s \geqslant \frac{1}{2}$ are replaced by $\Sigma_s$, $\Sigma_s'$ and $s > \frac{1}{2}$ respectively, in each place.*

Next we recall some algebraic properties and characterisations of $\Gamma_s^{(\omega)}(\mathbb{R}^{2d})$ and $\Gamma_{0,s}^{(\omega)}(\mathbb{R}^{2d})$, and begin with the following Proposition 5.38.

The proof can be founded in [24].

**Proposition 5.38.** *Let $s \geqslant 1$, $\omega_j \in \mathscr{P}_{E,s}^0(\mathbb{R}^{2d})$, $A_j \in \mathbf{M}(d,\mathbb{R})$ for $j = 0,1,2$, and let $\omega_{0,r}(X,Y) = \omega_0(X)e^{-r|Y|^{\frac{1}{s}}}$ when $r > 0$. Then the following statements hold true:*

*(1) If $a_1, a_2 \in \Sigma_s'(\mathbb{R}^{2d})$ satisfy $\operatorname{Op}_{A_1}(a_1) = \operatorname{Op}_{A_2}(a_2)$, then $a_1 \in \Gamma_s^{(\omega_0)}(\mathbb{R}^{2d})$ if and only if $a_2 \in \Gamma_s^{(\omega_0)}(\mathbb{R}^{2d})$.*

*(2) $\Gamma_s^{(\omega_1)} \# \Gamma_s^{(\omega_2)} \subseteq \Gamma_s^{(\omega_1\omega_2)}$.*

*(3) $\Gamma_s^{(\omega_0)} = \bigcup_{r>0} M_{(1/\omega_{0,r})}^{\infty,1} = \bigcup_{r\geqslant 0} \mathcal{M}_{(1/\omega_{0,r})}^{\infty,1}$.*

**Proposition 5.39.** *Let $s \geqslant 1$, $\omega_j \in \mathscr{P}_{E,s}(\mathbb{R}^{2d})$, $A_j \in \mathbf{M}(d,\mathbb{R})$ for $j = 0,1,2$, and let $\omega_{0,r}(X,Y) = \omega_0(X)e^{-r|Y|^{\frac{1}{s}}}$ when $r > 0$. Then the following properties hold true:*

(1) *If* $a_1, a_2 \in \Sigma_s'(\mathbb{R}^{2d})$ *satisfy* $\operatorname{Op}_{A_1}(a_1) = \operatorname{Op}_{A_2}(a_2)$, *then* $a_1 \in \Gamma_{0,s}^{(\omega_0)}(\mathbb{R}^{2d})$ *if and only if* $a_2 \in \Gamma_{0,s}^{(\omega_0)}(\mathbb{R}^{2d})$.

(2) $\Gamma_{0,s}^{(\omega_1)} \# \Gamma_{0,s}^{(\omega_2)} \subseteq \Gamma_{0,s}^{(\omega_1\omega_2)}$.

(3) $\Gamma_{0,s}^{(\omega_0)} = \bigcap_{r>0} M_{(1/\omega_{0,r})}^{\infty,1} = \bigcap_{r\geqslant 0} \mathcal{M}_{(1/\omega_{0,r})}^{\infty,1}$.

In time-frequency analysis one often considers mapping properties for pseudo-differential operators between modulation spaces or with symbols in modulation spaces. Especially we need the following two results, where the first one is a generalisation of [109, Theorem 2.1] by Tachizawa, and the second one is a weighted version of [67, Theorem 14.5.2]. We refer to [126] for the proof of the first two propositions and to [126] for the proof of the third one.

**Proposition 5.40.** *Assume that* $A \in \mathbf{M}(d,\mathbb{R})$, $s \geqslant 1$, $\omega, \omega_0 \in \mathscr{P}_{E,s}^0(\mathbb{R}^{2d})$, $a \in \Gamma_s^{(\omega)}(\mathbb{R}^{2d})$, *and that* $\mathscr{B}$ *is an invariant BF-space on* $\mathbb{R}^{2d}$ *of Beurling type. Then* $\operatorname{Op}_A(a)$ *is continuous from* $M(\omega_0\omega, \mathscr{B})$ *to* $M(\omega_0, \mathscr{B})$, *and also continuous on* $\mathcal{S}_s(\mathbb{R}^d)$ *and on* $\mathcal{S}_s'(\mathbb{R}^d)$.

**Proposition 5.41.** *Assume that* $A \in \mathbf{M}(d,\mathbb{R})$, $s \geqslant 1$, $\omega, \omega_0 \in \mathscr{P}_{E,s}(\mathbb{R}^{2d})$, $a \in \Gamma_{0,s}^{(\omega)}(\mathbb{R}^{2d})$, *and that* $\mathscr{B}$ *is an invariant BF-space on* $\mathbb{R}^{2d}$ *of Roumieu type. Then* $\operatorname{Op}_A(a)$ *is continuous from* $M(\omega_0\omega, \mathscr{B})$ *to* $M(\omega_0, \mathscr{B})$, *and also continuous on* $\Sigma_s(\mathbb{R}^d)$ *and on* $\Sigma_s'(\mathbb{R}^d)$.

**Proposition 5.42.** *Assume that* $p, q \in (0,\infty]$, $r \leqslant \min(p,q,1)$, $\omega \in \mathscr{P}_E(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d})$ *and* $\omega_1, \omega_2 \in \mathscr{P}_E(\mathbb{R}^{2d})$ *satisfy*

$$\frac{\omega_2(X-Y)}{\omega_1(X+Y)} \leqslant C\omega(X,Y), \qquad X, Y \in \mathbb{R}^{2d}, \tag{5.58}$$

*for some constant* $C$. *If* $a \in \mathcal{M}_{(\omega)}^{\infty,r}(\mathbb{R}^{2d})$, *then* $\operatorname{Op}^w(a)$ *extends uniquely to a continuous map from* $M_{(\omega_1)}^{p,q}(\mathbb{R}^d)$ *to* $M_{(\omega_2)}^{p,q}(\mathbb{R}^d)$.

Finally we need the following result concerning mapping properties of modulation spaces under the Weyl product. The result is a special case of Theorem in [30, Theorem 2.1] (see also [40, Theorem 0.3]).

**Proposition 5.43.** *Assume that* $\omega_j \in \mathscr{P}_E(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d})$ *for* $j = 0,1,2$ *satisfy*

$$\omega_0(X,Y) \leqslant C\omega_1(X-Y+Z, Z)\omega_2(X+Z, Y-Z), \tag{5.59}$$

*for some constant* $C > 0$ *independent of* $X, Y, Z \in \mathbb{R}^{2d}$, *and let* $r \in (0,1]$. *Then the map* $(a,b) \mapsto a\#b$ *from* $\Sigma_1(\mathbb{R}^{2d}) \times \Sigma_1(\mathbb{R}^{2d})$ *to* $\Sigma_1(\mathbb{R}^{2d})$ *extends uniquely to a continuous mapping from* $\mathcal{M}_{(\omega_1)}^{\infty,r}(\mathbb{R}^{2d}) \times \mathcal{M}_{(\omega_2)}^{\infty,r}(\mathbb{R}^{2d})$ *to* $\mathcal{M}_{(\omega_0)}^{\infty,r}(\mathbb{R}^{2d})$.

## 5.9 The Wiener algebra property

As a further crucial tool in our study of the isomorphism property of Toeplitz operators we need to combine these continuity results with convenient invertibility properties. The so-called Wiener algebra property of certain symbol classes asserts that the inversion of a pseudo-differential operator preserves the symbol class and is often referred to as the spectral invariance of a symbol class.

**Proposition 5.44.** *Let $A \in \mathbf{M}(d,\mathbb{R})$. Then, the following properties hold true:*

*(1) If $s > 1$, $a \in \Gamma_{0,s}^{(1)}(\mathbb{R}^{2d})$ and $\operatorname{Op}_A(a)$ is invertible on $L^2(\mathbb{R}^d)$, then $\operatorname{Op}_A(a)^{-1} = \operatorname{Op}_A(b)$ for some $b \in \Gamma_{0,s}^{(1)}(\mathbb{R}^{2d})$.*

*(2) If $s \geqslant 1$, $a \in \Gamma_s^{(1)}(\mathbb{R}^{2d})$ and $\operatorname{Op}_A(a)$ is invertible on $L^2(\mathbb{R}^d)$, then $\operatorname{Op}_A(a)^{-1} = \operatorname{Op}_A(b)$ for some $b \in \Gamma_s^{(1)}(\mathbb{R}^{2d})$.*

*(3) If $s \geqslant 1$, $v_0 \in \mathscr{P}_{E,s}^0(\mathbb{R}^{2d})$ is submultiplicative, $v(X,Y) \equiv v_0(Y)$, $X,Y \in \mathbb{R}^{2d}$, $a \in M_{(v)}^{\infty,1}(\mathbb{R}^{2d})$ and $\operatorname{Op}_A(a)$ is invertible on $L^2(\mathbb{R}^d)$, then $\operatorname{Op}_A(a)^{-1} = \operatorname{Op}_A(b)$, for some $b \in M_{(v)}^{\infty,1}(\mathbb{R}^{2d})$.*

*Proof.* The results follows essentially from [68, Corollary 5.5] or [69]. Suppose $s > 1$, $a \in \Gamma_s^{(1)}(\mathbb{R}^{2d})$, $\operatorname{Op}_A(a)$ is invertible on $L^2(\mathbb{R}^d)$, and let $v_r(X,Y) = e^{r|Y|^{\frac{1}{s}}}$ when $r \geqslant 0$. Then $a \in M_{(v_r)}^{\infty,1}(\mathbb{R}^{2d})$ for some $r > 0$. By [68, Corollary 5.5], $\operatorname{Op}(M_{(v_r)}^{\infty,1}(\mathbb{R}^{2d}))$ is a Wiener algebra, giving that $\operatorname{Op}(a)^{-1} = \operatorname{Op}(b)$ for some $b \in M_{(v_r)}^{\infty,1}(\mathbb{R}^{2d}) \subseteq \Gamma_s^{(1)}(\mathbb{R}^{2d})$. This gives (2) in the case $s > 1$.

If instead $s = 1$, then by [61, Theorem 4.4] there is an $r_0 > 0$ such that $\operatorname{Op}(a)^{-1} = \operatorname{Op}(b)$ for some $b \in M_{(v_{r_0})}^{\infty,1}(\mathbb{R}^{2d}) \subseteq \Gamma_1^{(1)}(\mathbb{R}^{2d})$, and (2) follows for general $s \geqslant 1$.

By similar arguments, (1) and (3) follow. □

**Remark 5.45.** *Let $A \in \mathbf{M}(d,\mathbb{R})$. Then it follows from Proposition 5.44 (3) that if $s > 1$, $v_0 \in \mathscr{P}_{E,s}(\mathbb{R}^{2d})$ is submultiplicative, $v(X,Y) \equiv v_0(Y)$, $X,Y \in \mathbb{R}^{2d}$, $a \in M_{(v)}^{\infty,1}(\mathbb{R}^{2d})$ and $\operatorname{Op}_A(a)$ is invertible on $L^2(\mathbb{R}^d)$, then $\operatorname{Op}_A(a)^{-1} = \operatorname{Op}_A(b)$, for some $b \in M_{(v)}^{\infty,1}(\mathbb{R}^{2d})$.*

## 5.10 Toeplitz operators

Fix a symbol $a \in \Sigma_1(\mathbb{R}^{2d})$ and a window $\phi \in \Sigma_1(\mathbb{R}^d)$. Then the Toeplitz operator $\operatorname{Tp}_\phi(a)$ is defined by the formula

$$(\operatorname{Tp}_\phi(a)f_1, f_2)_{L^2(\mathbb{R}^d)} = (a\, V_\phi f_1, V_\phi f_2)_{L^2(\mathbb{R}^{2d})}\,, \tag{5.60}$$

when $f_1, f_2 \in \Sigma_1(\mathbb{R}^d)$. $\mathrm{Tp}_\phi(a)$ is well-defined and extends uniquely to a continuous operator from $\Sigma_1'(\mathbb{R}^d)$ to $\Sigma_1(\mathbb{R}^d)$.

The definition of Toeplitz operators can be extended to more general classes of windows and symbols by using appropriate estimates for the short-time Fourier transforms in (5.60).

We state two possible ways of extending (5.60). The first result follows from [36, Corollary 4.2] and its proof, and the second result is a special case of [127, Theorem 3.1]. We also set

$$\omega_{0,t}(X,Y) = v_1(2Y)^{t-1}\omega_0(X) \qquad \text{for } X, Y \in \mathbb{R}^{2d}. \tag{5.61}$$

**Proposition 5.46.** *Let $0 \leqslant t \leqslant 1$, $p, q \in [1,\infty]$, and $\omega, \omega_0, v_1, v_0 \in \mathscr{P}_E(\mathbb{R}^{2d})$ be such that $v_0$ and $v_1$ are submultiplicative, $\omega_0$ is $v_0$-moderate and $\omega$ is $v_1$-moderate. Set*

$$v = v_1^t v_0 \quad \text{and} \quad \vartheta = \omega_0^{1/2},$$

*and let $\omega_{0,t}$ be as in (5.61). Then the following statements hold true:*

*(1) The definition of $(a,\phi) \mapsto \mathrm{Tp}_\phi(a)$ from $\Sigma_1(\mathbb{R}^{2d}) \times \Sigma_1(\mathbb{R}^d)$ to $\mathcal{L}(\Sigma_1(\mathbb{R}^d), \Sigma_1'(\mathbb{R}^d))$ extends uniquely to a continuous map from $\mathcal{M}^\infty_{(1/\omega_{0,t})}(\mathbb{R}^{2d}) \times M^1_{(v)}(\mathbb{R}^d)$ to $\mathcal{L}(\mathscr{S}(\mathbb{R}^d), \mathscr{S}'(\mathbb{R}^d))$.*

*(2) If $\phi \in M^1_{(v)}(\mathbb{R}^d)$ and $a \in \mathcal{M}^\infty_{(1/\omega_{0,t})}(\mathbb{R}^{2d})$, then $\mathrm{Tp}_\phi(a)$ extends uniquely to a continuous map from $M^{p,q}_{(\vartheta\omega)}(\mathbb{R}^d)$ to $M^{p,q}_{(\omega/\vartheta)}(\mathbb{R}^d)$.*

**Proposition 5.47.** *Let $\omega, \omega_1, \omega_2, v \in \mathscr{P}_E(\mathbb{R}^{2d})$ be such that $\omega_1$ is $v$-moderate, $\omega_2$ is $v$-moderate and $\omega = \omega_1/\omega_2$. Then the following statements hold true:*

*(1) The mapping $(a,\phi) \mapsto \mathrm{Tp}_\phi(a)$ extends uniquely to a continuous map from $L^\infty_{(\omega)}(\mathbb{R}^{2d}) \times M^2_{(v)}(\mathbb{R}^d)$ to $\mathcal{L}(\Sigma_1(\mathbb{R}^d), \Sigma_1'(\mathbb{R}^d))$.*

*(2) If $\phi \in M^2_{(v)}(\mathbb{R}^d)$ and $a \in L^\infty_{(1/\omega)}(\mathbb{R}^{2d})$, then $\mathrm{Tp}_\phi(a)$ extends uniquely to a continuous operator from $M^2_{(\omega_1)}(\mathbb{R}^d)$ to $M^2_{(\omega_2)}(\mathbb{R}^d)$.*

We finish this chapter by recalling an important relations between Weyl operators, Wigner distributions, and Toeplitz operators. Namely, the Weyl symbol of a Toeplitz operator is the convolution between the Toeplitz symbol and a Wigner distribution. Explicitly, if $a \in \Sigma_1(\mathbb{R}^{2d})$ and $\phi \in \Sigma_1(\mathbb{R}^d)$, then

$$\mathrm{Tp}_\phi(a) = (2\pi)^{-\frac{d}{2}}\, \mathrm{Op}^w(a * W_{\phi,\phi}). \tag{5.62}$$

# Chapter 6

# Confinement property for Gelfand-Shilov type symbol classes

In this chapter we introduce and discuss basic properties of confinements for symbols in $\Gamma_s^{(\omega_0)}$ and in $\Gamma_{0,s}^{(\omega_0)}$. These considerations are related to the discussions in [14, 89], but are here adapted to symbols that possess Gevrey regularity. In particular, this requires the deduction of various types of delicate estimates on compositions of symbols that are confined in certain ways.

## 6.1 Estimates of translated and localised Weyl products

In what follows we let $a_Y = a(\,\cdot\,-Y)$ when $a \in \mathcal{S}_{1/2}'(\mathbb{R}^{2d})$ and $Y \in \mathbb{R}^{2d}$, and in analogous ways, $b_Y$, $\phi_Y$, $\varphi_Y$, $\psi_Y$ etc. are defined when $b, \phi, \varphi, \psi \in \mathcal{S}_{1/2}'(\mathbb{R}^{2d})$. For admissible $a$ and $b$ we have

$$(a\# b)_Y = a_Y \# b_Y, \tag{6.1}$$

which follows by straightforward computations. We also recall that if $\varphi \in \mathcal{S}_s(\mathbb{R}^{2d})$, then there are functions $\phi, \psi \in \mathcal{S}_s(\mathbb{R}^{2d})$ such that $\varphi = \phi\#\psi$. The same is true if $\mathcal{S}_s$ is replaced by $\Sigma_s$ or by $\mathscr{S}$ (cf. [29, 128]). In particular, by choosing $\varphi$ such that $\int_{\mathbb{R}^{2d}} \varphi(X)\,dX = 1$, (6.1) gives the following.

**Proposition 6.1.** *Let $s \geqslant \frac{1}{2}$. Then there are $\phi, \psi \in \mathcal{S}_s(\mathbb{R}^{2d})$ such that*

$$\int_{\mathbb{R}^{2d}} \psi_Y \# \phi_Y \, dY = 1. \tag{6.2}$$

For independent translations in Weyl products we have the following.

**Proposition 6.2.** *Let $s \geqslant \frac{1}{2}$ and let $\phi, \psi \in \mathcal{S}_s(\mathbb{R}^{2d})$. Then*

$$(\phi_Y \# \psi_Z)(X) = \Psi(X - Y, X - Z) \tag{6.3}$$

*for some $\Psi \in \mathcal{S}_s(\mathbb{R}^{2d} \times \mathbb{R}^{2d})$. The same holds true with $\Sigma_s$ or $\mathscr{S}$ in place of $\mathcal{S}_s$.*

*Proof.* We only prove the result when $\phi, \psi \in \mathcal{S}_s(\mathbb{R}^{2d})$. The other cases follow by similar arguments.

We have

$$\begin{aligned}(\phi_Y \# \psi_Z)(X) &= \pi^{-d} \int_{\mathbb{R}^{2d}} \phi(X - Y - Y_1)\widehat{\psi}(Y_1) e^{2i\sigma(Y_1,Z)} e^{2i\sigma(X,Y_1)}\, dY_1 \\ &= \pi^{-d} \int_{\mathbb{R}^{2d}} \phi((X - Y) - Y_1)\widehat{\psi}(Y_1) e^{2i\sigma(X-Z,Y_1)}\, dY_1 = \Psi(X - Y, X - Z),\end{aligned}$$

where

$$\Psi(X, Z) = \pi^{-d} \int_{\mathbb{R}^{2d}} \phi(X - Y_1)\widehat{\psi}(Y_1) e^{2i\sigma(Z,Y_1)}\, dY_1.$$

We note that

$$\Psi = (\mathscr{F}_{\sigma,2} \circ T)(\phi \otimes \widehat{\psi}),$$

where $(T\Phi)(X, Z) = \Phi(X - Z, Z)$ when $\Phi \in \mathcal{S}_s(\mathbb{R}^{2d} \times \mathbb{R}^{2d})$, and $\mathscr{F}_{\sigma,2}\Phi$ is the partial symplectic Fourier transform of $\Phi(X, Z)$ with respect to the $Z$ variable. Since $(\phi, \psi) \mapsto \phi \otimes \widehat{\psi}$ is continuous from $\mathcal{S}_s(\mathbb{R}^{2d}) \times \mathcal{S}_s(\mathbb{R}^{2d})$ to $\mathcal{S}_s(\mathbb{R}^{2d} \times \mathbb{R}^{2d})$, and $T$ and $\mathscr{F}_{\sigma,2}\Phi$ are continuous on $\mathcal{S}_s(\mathbb{R}^{2d} \times \mathbb{R}^{2d})$, it follows that $\Psi \in \mathcal{S}_s(\mathbb{R}^{2d} \times \mathbb{R}^{2d})$. □

Since $\Psi$ in Proposition 6.2 belongs to similar types of spaces as $\phi$ and $\psi$, it follows that estimates of the form

$$|D^\alpha \Psi(X, Y)| \lesssim h^{|\alpha|} \alpha!^s e^{-(|X|^{\frac{1}{s}} + |Y|^{\frac{1}{s}})/h}$$

hold true. In particular, the following Corollary 6.3 is an immediate consequence of Proposition 6.2 and some standard manipulations in Gelfand-Shilov theory.

**Corollary 6.3.** *Let $s \geqslant \frac{1}{2}$. If $\phi, \psi \in \mathcal{S}_s(\mathbb{R}^{2d})$ ($\phi, \psi \in \Sigma_s(\mathbb{R}^{2d})$), then*

$$|D_X^\alpha D_Y^\beta D_Z^\gamma (\phi_Y \# \psi_Z)(X)| \lesssim h^{|\alpha+\beta+\gamma|} (\alpha!\beta!\gamma!)^s e^{-(|X-Y|^{\frac{1}{s}} + |X-Z|^{\frac{1}{s}})/h} \tag{6.4}$$

*for some $h > 0$ (for every $h > 0$).*

*Proof.* By Proposition 6.2, (6.3) holds true for some $\Psi \in \mathcal{S}_s(\mathbb{R}^{2d} \times \mathbb{R}^{2d})$. Thus

$$\begin{aligned}
|D_X^\alpha D_Y^\beta D_Z^\gamma \Psi(X-Y, X-Z)| &= \left| D_X^\alpha \left( D_1^\beta D_2^\gamma \Psi \right)(X-Y, X-Z) \right| \\
&\leqslant \sum_{\delta \leqslant \alpha} \binom{\alpha}{\delta} \left| \left( D_1^{\beta+\delta} D_2^{\gamma+\alpha-\delta} \Psi \right)(X-Y, X-Z) \right| \\
&\leqslant h^{|\alpha+\beta+\gamma|} \sum_{\delta \leqslant \alpha} \binom{\alpha}{\delta} \left( (\beta+\delta)!(\gamma+\alpha-\delta)! \right)^s e^{-r\left(|X-Y|^{\frac{1}{s}} + |X-Z|^{\frac{1}{s}}\right)}.
\end{aligned}$$

We have

$$\begin{aligned}
&\sum_{\delta \leqslant \alpha} \binom{\alpha}{\delta} \left( (\beta+\delta)!(\gamma+\alpha-\delta)! \right)^s \\
&\quad \leqslant \sum_{\delta \leqslant \alpha} \binom{\alpha}{\delta} \binom{\beta+\delta}{\delta}^s \binom{\gamma+\alpha-\delta}{\alpha-\delta}^s (\beta!\delta!)^s (\gamma!(\alpha-\delta)!)^s \\
&\quad \leqslant \sum_{\delta \leqslant \alpha} \binom{\alpha}{\delta} \binom{\beta+\delta}{\delta}^s \binom{\gamma+\alpha-\delta}{\alpha-\delta}^s \beta!^s \gamma!^s \binom{\alpha}{\delta}^{-s} \alpha!^s \\
&\quad \leqslant \sum_{\delta \leqslant \alpha} \binom{\alpha}{\delta} \binom{\beta+\delta}{\delta}^s \binom{\gamma+\alpha-\delta}{\alpha-\delta}^s \binom{\alpha}{\delta}^{-s} (\alpha!\beta!\gamma!)^s \\
&\quad \leqslant 2^{|\alpha|} 2^{s|\alpha+\beta+\gamma|} (\alpha!\beta!\gamma!)^s .
\end{aligned}$$

Indeed, by using the fact that $\sum_{\delta \leqslant \alpha} \binom{\alpha}{\delta} = 2^{|\alpha|}$ and that $n! \leqslant 2^n (n-k)!k!$, which implies $(n+k)! \leqslant 2^{n+k} n!k!$. Thus (6.4) holds true with $2 \cdot 2^s h$ in place of $h$. □

The next result is a consequence of Theorem 4.12 in [24].

**Proposition 6.4.** *Let $s \geqslant \frac{1}{2}$ and $\vartheta \in \mathscr{P}_E(\mathbb{R}^{2d})$. Then, the map $(\phi, a) \mapsto \phi \# a$ is continuous from $\Sigma_s(\mathbb{R}^{2d}) \times \Gamma_s^{(\vartheta)}(\mathbb{R}^{2d})$ to $\mathcal{S}_s(\mathbb{R}^{2d})$.*

The next lemma concerns uniform estimates of the Weyl product between elements in sets

$$\{ a_j( \cdot + Y, Y) \, ; \, Y \in \mathbb{R}^{2d} \}, \qquad j = 1, 2 \tag{6.5}$$

which are bounded in $\mathcal{S}_s(\mathbb{R}^{2d})$ or in $\Sigma_s(\mathbb{R}^{2d})$, $j = 1, 2$.

**Lemma 6.5.** *Let $s \geqslant \frac{1}{2}$. Then, the following statements hold true:*

*(1) If the sets in (6.5) are bounded in $\mathcal{S}_s(\mathbb{R}^{2d})$, then there are constants $C > 0$ and $h > 0$ which are independent of $Y_1, Y_2 \in \mathbb{R}^{2d}$ and $\alpha, \alpha_1, \alpha_2 \in \mathbb{Z}_+^{2d}$*

*such that*

$$|((D_1^{\alpha_1}a_1)(\,\cdot\,,Y_1)\#(D_1^{\alpha_2}a_2)(\,\cdot\,,Y_2))(X)| \leqslant Ch^{|\alpha_1+\alpha_2|}(\alpha_1!\alpha_2!)^s e^{-\frac{1}{h}\cdot(|X-Y_1|^{\frac{1}{s}}+|X-Y_2|^{\frac{1}{s}}+|Y_1-Y_2|^{\frac{1}{s}})} \tag{6.6}$$

*and*

$$|D_1^{\alpha}(a_1(\,\cdot\,,Y_1)\#a_2(\,\cdot\,,Y_2))(X)| \leqslant Ch^{|\alpha|}\alpha!^s e^{-\frac{1}{h}\cdot(|X-Y_1|^{\frac{1}{s}}+|X-Y_2|^{\frac{1}{s}}+|Y_1-Y_2|^{\frac{1}{s}})} \tag{6.7}$$

*hold true.*

(2) *If the sets in* (6.5) *are bounded in* $\Sigma_s(\mathbb{R}^{2d})$*, then for every* $h>0$*, there is a constant* $C>0$ *which is independent of* $Y_1,Y_2\in\mathbb{R}^{2d}$ *and* $\alpha,\alpha_1,\alpha_2\in\mathbb{Z}_+^{2d}$ *such that* (6.6) *and* (6.7) *hold.*

*Proof.* We only prove (2). The assertion (1) follows by similar arguments.

Let $Y=Y_1$, $Z=Y_2$, $a(X,Y)=a_1(X+Y,Y)$ and $b(X,Z)=a_2(X+Z,Z)$. Then

$$\begin{aligned}
&(a_1(\,\cdot\,,Y)\#a_2(\,\cdot\,,Z))(X)\\
&\quad=\pi^{-d}\int_{\mathbb{R}^{2d}} a((X-Y)-Y_1,Y)\mathscr{F}_\sigma(b(\,\cdot\,-Z,Z))(Y_1)e^{2i\sigma(X,Y_1)}\,dY_1\\
&\quad=\pi^{-d}\int_{\mathbb{R}^{2d}} a((X-Y)-Y_1,Y)\mathscr{F}_\sigma(b(\,\cdot\,,Z))(Y_1)e^{2i\sigma(X-Z,Y_1)}\,dY_1\\
&\quad=\Phi_{Y,Z}(X-Y,X-Z),
\end{aligned}$$

where

$$\Phi_{Y,Z}(X_1,X_2)=\pi^{-d}\int_{\mathbb{R}^{2d}} a(X_1-Y_1,Y)\mathscr{F}_\sigma(b(\,\cdot\,,Z))(Y_1)e^{2i\sigma(X_2,Y_1)}\,dY_1.$$

We observe that

$$\begin{aligned}
&D_{X_1}^{\alpha_1}D_{X_2}^{\alpha_2}\Phi_{Y,Z}(X_1,X_2)\\
&=\pi^{-d}\int_{\mathbb{R}^{2d}}(D_1^{\alpha_1}a)(X_1-Y_1,Y)\mathscr{F}_\sigma((D_1^{\alpha_2}b)(\,\cdot\,,Z))(Y_1)e^{2i\sigma(X_2,Y_1)}\,dY_1.
\end{aligned}\tag{6.8}$$

which implies that the Leibnitz rule

$$\begin{aligned}
D_1^{\alpha}(a_1(\,\cdot\,,Y)\#a_2(\,\cdot\,,Z))(X) &= \sum_{\gamma\leqslant\alpha}\binom{\alpha}{\gamma}(D_1^{\alpha-\gamma}D_2^{\gamma}\Phi_{Y,Z})(X-Y,X-Z)\\
=\sum_{\gamma\leqslant\alpha}\binom{\alpha}{\gamma}\pi^{-d}&\int_{\mathbb{R}^{2d}}(D_1^{\alpha-\gamma}a)(X_1-Y_1,Y)\mathscr{F}_\sigma((D_1^{\gamma}b)(\,\cdot\,,Z))(Y_1)e^{2i\sigma(X_2,Y_1)}\,dY_1
\end{aligned}\tag{6.9}$$

holds true. We also have

$$\Phi_{Y,Z} = (T_1 \circ T_2 \circ T_1)(a(\,\cdot\,, Y) \otimes b(\,\cdot\,, Z)),$$

where

$$(T_1F)(X_1, X_2) = \mathscr{F}_\sigma(F(X_1, \,\cdot\,))(X_2)$$

and

$$(T_2F)(X_1, X_2) = F(X_1 - X_2, X_2),$$

for admissible $F$, and observe that both $T_1$ and $T_2$ are continuous mappings on $\Sigma_s(\mathbb{R}^{2d} \times \mathbb{R}^{2d})$.

By the continuity of $T_1$ and $T_2$ on $\Sigma_s$, it follows that

$$\sup_{Y,Z\in\mathbb{R}^{2d}} |D^{\alpha_1}_{X_1} D^{\alpha_2}_{X_2} \Phi_{Y,Z}(X_1, X_2)| \lesssim h^{|\alpha_1+\alpha_2|}(\alpha_1!\alpha_2!)^s e^{-\frac{1}{h}\cdot(|X_1|^{\frac{1}{s}}+|X_2|^{\frac{1}{s}})},$$

which is the same as

$$\begin{aligned}|(D_1^{\alpha_1} a_1(\,\cdot\,, Y))\#(D_1^{\alpha_2} a_2)(\,\cdot\,, Z))(X)|&\\ &\lesssim h^{|\alpha_1+\alpha_2|}(\alpha_1!\alpha_2!)^s e^{-\frac{1}{h}\cdot(|X-Y|^{\frac{1}{s}}+|X-Z|^{\frac{1}{s}})}\end{aligned}$$

for every $h > 0$, where the involved constants are independent of $Y, Z \in \mathbb{R}^{2d}$. A combination of the latter estimate and the fact that

$$|X-Y|^{\frac{1}{s}}+|X-Z|^{\frac{1}{s}} \asymp |X-Y|^{\frac{1}{s}}+|X-Z|^{\frac{1}{s}}+|Y-Z|^{\frac{1}{s}}, \quad X, Y, Z \in \mathbb{R}^{2d}, \tag{6.10}$$

shows that (6.6) holds true for every $h > 0$.

By (6.6), (6.8), (6.10), observing that

$$\sum_{\gamma\leqslant\alpha} \binom{\alpha}{\gamma}((\alpha-\gamma)!\gamma!)^s = \sum_{\gamma\leqslant\alpha} \binom{\alpha}{\gamma} \cdot \binom{\alpha}{\gamma}^{-s} \cdot \alpha!^s \leqslant \sum_{\gamma\leqslant\alpha} 2^{|\alpha|s}\binom{\alpha}{\gamma} \leqslant 2^{|\alpha|(s+1)},$$

and the inequality $(\alpha + \beta)! \leqslant 2^{|\alpha+\beta|}\alpha!\beta!$ we get

$$\begin{aligned}|D_1^\alpha(a_1(\,\cdot\,, Y)\#a_2(\,\cdot\,, Z))(X)|&\\ &\leqslant \sum_{\gamma\leqslant\alpha}\binom{\alpha}{\gamma}|(D_1^{\alpha-\gamma}D_2^\gamma\Phi_{Y,Z})(X-Y, X-Z)|\\ &\lesssim h^{|\alpha|}\sum_{\gamma\leqslant\alpha}\binom{\alpha}{\gamma}((\alpha-\gamma)!\gamma!)^s e^{-\frac{1}{h}\cdot(|X-Y|^{\frac{1}{s}}+|X-Z|^{\frac{1}{s}})+|Y-Z|^{\frac{1}{s}})}\\ &\leqslant (2^s h)^{|\alpha|}\left(\sum_{\gamma\leqslant\alpha}\binom{\alpha}{\gamma}\right) e^{-\frac{1}{h}\cdot(|X-Y|^{\frac{1}{s}}+|X-Z|^{\frac{1}{s}})+|Y-Z|^{\frac{1}{s}})}\\ &= (2^{s+1}h)^{|\alpha|} e^{-\frac{1}{h}\cdot(|X-Y|^{\frac{1}{s}}+|X-Z|^{\frac{1}{s}})+|Y-Z|^{\frac{1}{s}})}\end{aligned}$$

for every $h > 0$, and the result follows. □

**Remark 6.6.** *Let $\Omega_1$ and $\Omega_2$ be (countable or uncountable) index sets. By similar arguments as in the previous proof, it follows that the conclusions of Lemma 6.5 also holds true when considering more general bounded subsets*

$$\{\, a_{\theta,j}(\,\cdot\,+Y,Y)\,;\, Y\in\mathbb{R}^{2d},\ \theta\in\Omega_j\,\},\qquad j=1,2$$

*of $\mathcal{S}_s(\mathbb{R}^{2d})$ respective $\Sigma_s(\mathbb{R}^{2d})$, $j=1,2$.*

**Lemma 6.7.** *Let $s\geqslant\frac{1}{2}$, $\phi,\psi\in\Sigma_s(\mathbb{R}^{2d})$, $\omega,\vartheta\in\mathscr{P}_E(\mathbb{R}^{2d})$, $\phi_Y=\phi(\,\cdot\,-Y)$, and $\psi_Z=\psi(\,\cdot\,-Z)$. Then the following properties hold true:*

*(1) If $a\in\Gamma_s^{(\omega)}(\mathbb{R}^{2d})$ ($a\in\Gamma_{0,s}^{(\omega)}(\mathbb{R}^{2d})$), then*

$$|D_X^\alpha D_Y^\beta(\phi_Y a)(X)|\lesssim h_1^{|\alpha|}h_2^{|\beta|}(\alpha!\beta!)^s e^{-|X-Y|^{\frac{1}{s}}/h_1}\min(\omega(X),\omega(Y)) \tag{6.11}$$

*and*

$$|D_X^\alpha D_Y^\beta(\phi_Y\# a)(X)|\lesssim h_1^{|\alpha|}h_2^{|\beta|}(\alpha!\beta!)^s e^{-|X-Y|^{\frac{1}{s}}/h_1}\min(\omega(X),\omega(Y)), \tag{6.12}$$

*for some $h_1>0$ (for every $h_1>0$) and every $h_2>0$.*

*(2) If $a_1\in\Gamma_s^{(\omega)}(\mathbb{R}^{2d})$ and $a_2\in\Gamma_s^{(\vartheta)}(\mathbb{R}^{2d})$ ($a_1\in\Gamma_{0,s}^{(\omega)}(\mathbb{R}^{2d})$ and $a_2\in\Gamma_{0,s}^{(\vartheta)}(\mathbb{R}^{2d})$), then*

$$\begin{aligned}
&|D_X^\alpha D_Y^\beta D_Z^\gamma((\phi_Y a_1)\#(\psi_Z a_2))(X)|\\
&\qquad\lesssim h_1^{|\alpha+\beta|}h_2^{|\gamma|}(\alpha!\beta!\gamma!)^s e^{-(|X-Y|^{\frac{1}{s}}+|X-Z|^{\frac{1}{s}}+|Y-Z|^{\frac{1}{s}})/h_1}\cdot\\
&\qquad\qquad\cdot\min_{X_1,X_2\in\{X,Y,Z\}}\big(\omega(X_1)\vartheta(X_2)\big),
\end{aligned}$$

*for some $h_1>0$ (for every $h_1>0$) and every $h_2>0$.*

*Proof.* We only consider the case when $a_1\in\Gamma_{0,s}^{(\omega)}(\mathbb{R}^{2d})$ and $a_2\in\Gamma_{0,s}^{(\vartheta)}(\mathbb{R}^{2d})$. The other cases follow by similar arguments. Let

$$\Psi(X,Y)=\phi(X-Y)a(X).$$

By Leibniz rule we get

$$\begin{aligned}
|D_X^\alpha D_Y^\beta\Psi(X,Y)| &\leqslant \sum_{\gamma\leqslant\alpha}\binom{\alpha}{\gamma}|\phi^{(\alpha+\beta-\gamma)}(X-Y)a^{(\gamma)}(X)|\\
&\lesssim 2^{|\alpha|}\sup_{\gamma\leqslant\alpha}\left(h^{|\alpha+\beta|}((\alpha+\beta-\gamma)!\gamma!)^s e^{-|X-Y|^{\frac{1}{s}}/h}\omega(X)\right)\\
&\leqslant (2^{1+s}h)^{|\alpha+\beta|}(\alpha!\beta!)^s e^{-|X-Y|^{\frac{1}{s}}/h}\omega(X)\\
&\lesssim (2^{1+s}h)^{|\alpha+\beta|}(\alpha!\beta!)^s e^{-|X-Y|^{\frac{1}{s}}/(2h)}\omega(Y),
\end{aligned}$$

for every $h > 0$ which is chosen small enough. Here we have used the fact that, for some $h > 0$, possibly depending on $X, Y, r$ and $s$,

$$\omega(X) \lesssim \omega(Y)e^{r|X-Y|} \lesssim \omega(Y)e^{|X-Y|^{\frac{1}{s}}/(2h)},$$

since $\omega$ is a moderate function. This gives (6.11).

Next we prove (2). Let

$$b_{1,\beta,h}(\,\cdot\,,Y) = \frac{D_Y^\beta(\phi_Y a_1)}{h^{|\beta|}\beta!^s\omega(Y)} \quad \text{and} \quad b_{2,\gamma,h}(\,\cdot\,,Z) = \frac{D_Z^\gamma(\psi_Z a_2)}{h^{|\gamma|}\gamma!^s\vartheta(Z)}$$

Then (1) and Remark 6.6 show that

$$\{\, b_{1,\beta,h}(\,\cdot\, + Y, Y)\, ;\, Y \in \mathbb{R}^{2d},\ h > 0,\ \beta \in \mathbb{Z}_+^{2d} \,\}$$

and

$$\{\, b_{2,\gamma,h}(\,\cdot\, + Z, Z)\, ;\, Z \in \mathbb{R}^{2d},\ h > 0,\ \gamma \in \mathbb{Z}_+^{2d} \,\}$$

are bounded subsets of $\Sigma_s(\mathbb{R}^{2d})$. Hence, Remark 6.6 shows that

$$|D_X^\alpha(b_{1,\beta,h}(\,\cdot\,,Y)\# b_{2,\gamma,h}(\,\cdot\,,Z))(X)| \lesssim h^{|\alpha|}\alpha!^s e^{-(|X-Y|^{\frac{1}{s}}+|X-Z|^{\frac{1}{s}}+|Y-Z|^{\frac{1}{s}})/h}$$

for every $h > 0$, or equivalently, for any $\alpha, \beta, \gamma, X, Y$ and $Z$

$$\begin{aligned} &|D_X^\alpha D_Y^\beta D_Z^\gamma((\phi_Y a)\#(\psi_Z b))(X)| \\ &\qquad \lesssim h^{|\alpha+\beta+\gamma|}(\alpha!\beta!\gamma!)^s e^{-(|X-Y|^{\frac{1}{s}}+|X-Z|^{\frac{1}{s}}+|Y-Z|^{\frac{1}{s}})/h}\omega(Y)\vartheta(Z). \end{aligned}$$

The assertion now follows from the latter estimate and the fact that $\omega$ and $\vartheta$ are moderate weights, giving that for some $h > 0$, possibly depend on $X, Y, Z, r$ and $s$,

$$\omega(Y) \lesssim \omega(X)e^{|X-Y|^{\frac{1}{s}}/(2h)} \lesssim \omega(Z)e^{(|X-Y|^{\frac{1}{s}}+|X-Z|^{\frac{1}{s}})/(2h)},$$

and similarly for $\vartheta$. □

Lemmas 6.5 and 6.7 imply the following characterisation of $\Gamma_s^{(\omega)}(\mathbb{R}^{2d})$.

**Proposition 6.8.** *Let $s > 1/2$, $\omega \in \mathscr{P}_E(\mathbb{R}^{2d})$, $a \in \Sigma_1'(\mathbb{R}^{2d})$, $\phi \in \Sigma_s(\mathbb{R}^{2d})$ have non-vanishing integrals, and let $\phi_Y = \phi(\,\cdot\, - Y)$. Then the following conditions are equivalent:*

*(1) $a \in \Gamma_s^{(\omega)}$ ($a \in \Gamma_{0,s}^{(\omega)}$).*

*(2) $\phi_Y a$ is smooth and satisfies* (6.11) *for some $h_1 > 0$ (for every $h_1 > 0$) and every $h_2 > 0$.*

(3) *$\phi_Y \# a$ is smooth and satisfies* (6.12) *for some* $h_1 > 0$ *(for every* $h_1 > 0$*) and every* $h_2 > 0$.

(4)

$$|D_X^\alpha(\phi_Y a)(X)| \lesssim h_1^{|\alpha|}\alpha!^s e^{-|X-Y|^{\frac{1}{s}}/h_1}\min(\omega(X),\omega(Y)) \qquad (6.13)$$

*for some* $h_1 > 0$ *(for every* $h_1 > 0$*);*

(5)

$$|D_X^\alpha(\phi_Y \# a)(X)| \lesssim h_1^{|\alpha|}\alpha!^s e^{-|X-Y|^{\frac{1}{s}}/h_1}\min(\omega(X),\omega(Y)) \qquad (6.14)$$

*for some* $h_1 > 0$ *(for every* $h_1 > 0$*).*

*Proof.* By Lemmas 6.5 and 6.7, (1) implies that (2) and (3) hold true, which in turn imply (4) and (5).

If (4) holds true, then (5.20) follows by integrating (6.13) with respect to $Y$. In the same way it follows that (5) leads to (5.20). Consequently, (4) as well as (5) imply (1), and the result follows. □

## 6.2 A family of Banach spaces in $L^\infty([-R,R]\times\mathbb{R}^{2d}; s_\infty^w(\mathbb{R}^{2d}))$

Let $I_R = [-R,R]$ and $E^0 = E^0_{h,s} = L^\infty(I_R\times\mathbb{R}^{2d}; s_\infty^w(\mathbb{R}^{2d}))$, with the symbol subspace $s_\infty^w(\mathbb{R}^{2d})$ from Definition 5.35. We shall consider suitable decreasing family $\{E^n_{h,s}\}_{n=0}^\infty$ of Banach spaces. To this aim, for $n\in\mathbb{N}$, let

$$G_n = \{(Y,T_1,\dots,T_n)\in\mathbb{R}^{2d(n+1)} : Y,T_j\in\mathbb{R}^{2d} \text{ with } |T_j|\leqslant 1,\ j=1,\dots,n\}.$$

We define $E^n_{h,s} = E^n_{R,h,s}$, $n\geqslant 1$, as the set of all $a\in E^0$ such that

$$\|a\|^{(n)} = \sup_{1\leqslant k\leqslant n}\sup_{t\in I_R}\sup_{(Y,T_1,\dots,T_k)\in G_k}\frac{\|\langle T_1,D_X\rangle\cdots\langle T_k,D_X\rangle a(t,Y,\,\cdot\,)\|_{s_\infty^w}}{h^k(k!)^s} < \infty,$$

with the norm

$$\|a\|_{E^n_{h,s}} = \|a\|_{E^n_{R,h,s}} \equiv \max(\|a\|_{E^0}, \|a\|^{(n)}).$$

We also let $E^\infty_{h,s} = E^\infty_{R,h,s}$ be the set of all

$$a\in\bigcap_{n\geqslant 0}E^n_{R,h,s} \qquad (6.15)$$

such that

$$\|a\|_{E^\infty_{R,h,s}} \equiv \sup_{n\geqslant 0}\|a\|_{E^n_{R,h,s}}$$

is finite.

**Lemma 6.9.** *The space $s_\infty^w(\mathbb{R}^{2d})$ is continuously embedded in $\mathscr{S}'(\mathbb{R}^{2d})$.*

*Proof.* Let $B(V_1, V_2)$ be the set of all linear and continuous operators from the vector space $V_1$ to the vector space $V_2$. By Schwartz kernel theorem, $B(\mathscr{S}(\mathbb{R}^d), \mathscr{S}'(\mathbb{R}^d))$ can be identified with a Schwartz kernel in $\mathscr{S}'(\mathbb{R}^{2d})$. From now on we identify operators with their kernels.

Since $\mathscr{S}(\mathbb{R}^d)$ is continuously embedded in $L^2(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$ is continuously embedded in $\mathscr{S}'(\mathbb{R}^d)$, it follows that any continuous linear operator from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ has a Schwartz kernel kernel in $\mathscr{S}'(\mathbb{R}^{2d})$. The set $B(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d))$ is a Banach space with norm equal to the operator norms. Hence there is an injection from $B(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d))$ to $\mathscr{S}'(\mathbb{R}^{2d})$.

It is not so difficult to see that this injection is continuous. In fact, since $M^1$ is continuously embedded in $L^2$, $L^2$ is continuously embedded in $M^\infty$ and that $B(M^1(\mathbb{R}^d), M^\infty(\mathbb{R}^d))$ can be identified with $M^\infty(\mathbb{R}^{2d})$ (Feichtinger's kernel theorem) which is continuously embedded in $\mathscr{S}'(\mathbb{R}^{2d})$, it follows that $B(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d))$ is continuously embedded in $B(M^1(\mathbb{R}^d), M^\infty(\mathbb{R}^d)) = M^\infty(\mathbb{R}^{2d})$ which is continuously embedded in $\mathscr{S}'(\mathbb{R}^{2d})$ (when operators are identified with their kernels).

Now $s_\infty^w$ equals $T \circ B(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d))$, where $T$ is a composition of a partial Fourier transform and a pullback of a non-degenerate linear map. It follows that $T$ is homoemorphisms on $M^\infty(\mathbb{R}^{2d})\mathscr{S}'(\mathbb{R}^{2d})$. From these properties it now follows that the embeddings above give $s_\infty^w(\mathbb{R}^{2d})$ is continuously embedded in $M^\infty(\mathbb{R}^{2d})$ which is continuously embedded in $\mathscr{S}'(\mathbb{R}^{2d})$. □

**Lemma 6.10.** *Let $n \geqslant 0$, $R > 0$ and $s > 0$. Then $E_{h,s}^n$ and $E_{h,s}^\infty$ are Banach spaces.*

*Proof.* Let $\{a_j\}_{j \geqslant 0}$ be a Cauchy sequence in $E_{h,s}^n$, $n \geqslant 1$. By definition, this sequence clearly has a limit $a \in E^0$, and for any $k$, $\{\langle T_1, D_X\rangle \dots \langle T_k, D_X\rangle a_j\}_j \geqslant 1$ is a Cauchy sequence in $s_\infty^w(\mathbb{R}^{2d})$. So, for some $X \mapsto b_k(t, Y, T_1, \dots, T_k, X) \in s_\infty^w(\mathbb{R}^{2d})$ we have

$$\lim_{j\to\infty} \sup \frac{\|\langle T_1, D_X\rangle \cdots \langle T_k, D_X\rangle a_j(t, Y, \,\cdot\,) - b_k(t, Y, T_1, \dots, T_k, \,\cdot\,)\|_{s_\infty^w}}{h^k (k!)^s} = 0,$$

where the supremum is taken over all

$$k \in \{1, \dots, n\}, \quad t \in I_R \quad \text{and} \quad (Y, T_1, \dots, T_k) \in G_k.$$

We need to prove that $a \in E_{h,s}^n$, and $a_j \to a$ in $E_{h,s}^n$.

The conditions here above are equivalent to

$$\lim_{j\to\infty} \left( \sup_{t \in I_R} \sup_{Y \in \mathbb{R}^{2d}} \|a_j(t, Y, \,\cdot\,) - a(t, Y, \,\cdot\,)\|_{s_\infty^w} \right) = 0 \tag{6.16}$$

and

$$\lim_{j\to\infty} \sup \frac{\|\langle T_1, D\rangle \cdots \langle T_k, D\rangle a_j(t,Y,\,\cdot\,) - b_k(t,Y,T_1,\dots,T_k,\,\cdot\,)\|_{s_\infty^w}}{h^k(k!)^s} = 0, \tag{6.17}$$

where the supremum is taken over all

$$k \in \{1,\dots,n\}, \quad t \in I_R \quad \text{and} \quad (Y,T_1,\dots,T_k) \in G_k.$$

Since $s_\infty^w(\mathbb{R}^{2d})$ is continuously embedded in $\mathscr{S}'(\mathbb{R}^{2d})$, it follows from (6.16) and (6.17) that

$$X \mapsto \langle T_1, D_X\rangle \cdots \langle T_k, D_X\rangle a_j(t,Y,X)$$

has the limit

$$X \mapsto \langle T_1, D_X\rangle \cdots \langle T_k, D_X\rangle a(t,Y,X)$$

in $\mathscr{S}'(\mathbb{R}^{2d})$, and the limit

$$X \mapsto b_k(t,Y,T_1,\dots,T_k,X)$$

in $s_\infty^w(\mathbb{R}^{2d})$, and thereby in $\mathscr{S}'(\mathbb{R}^{2d})$, as $j$ tends to $\infty$. Hence

$$b_k(t,Y,T_1,\dots,T_k,X) = \langle T_1, D_X\rangle \cdots \langle T_k, D_X\rangle a(t,Y,X)$$

and it follows that $E_{h,s}^n$ is a Banach space for every $h>0$, $s>0$ and integer $n \geqslant 0$.

If in addition $\{a_j\}_{j\geqslant 0}$ is a Cauchy sequence in $E_{h,s}^\infty$, then the limit $a$ above satisfy (6.15). Since $a_j$ stays bounded in $E_{h,s}^\infty$, it follows that $a$ has bounded $E_{h,s}^\infty$ norm, and therefore, $E_{h,s}^\infty$ is complete and thereby a Banach space. □

The spaces $E_{h,s}^\infty$ can be related to $\Gamma_s^{(1)}$ and $\Gamma_{0,s}^{(1)}$, as the following lemma shows. The details are left for the reader.

**Lemma 6.11.** *Let $a \in L^\infty(I_R \times \mathbb{R}^{2d}; s_\infty^w(\mathbb{R}^{2d}))$. Then $\{a(t,Y,\,\cdot\,)\}_{t\in I_R, Y\in\mathbb{R}^{2d}}$ is a uniformly bounded family in $\Gamma_s^{(1)}(\mathbb{R}^{2d})$ ($\Gamma_{0,s}^{(1)}(\mathbb{R}^{2d})$), if and only if*

$$\|a\|_{E_{h,s}^\infty} < \infty$$

*for some $h>0$ (for every $h>0$).*

Later on we also need the following result of differential equations with functions depending on a real variable with values in $E_{h,s}^\infty$. The result follows by standard consideration about ordinary differential equations involving functions taking values in Banach spaces.

**Lemma 6.12.** *Suppose* $s \geqslant 0$, $n \geqslant 0$ *be an integer,* $T > 0$, *and let* $\mathcal{K}$ *be an operator from* $E^n_{h,s}$ *to* $E^n_{h,s}$ *for every* $h > 0$ *such that*

$$\|\mathcal{K}a\|_{E^n_{h,s}} \leqslant C\|a\|_{E^n_{h,s}}, \qquad a \in E^n_{h,s}, \tag{6.18}$$

*for some constant* $C$ *which only depend on* $h > 0$. *Then*

$$\frac{dc(t)}{dt} = \mathcal{K}(c(t)), \qquad c(0) \in E^n_{h,s},$$

*has a unique solution* $t \mapsto c(t)$ *from* $[-T, T]$ *to* $E^n_{h,s}$ *which satisfies*

$$\|c(t)\|_{E^n_{h,s}} \leqslant \|c(0)\|_{E^n_{h,s}} e^{CT},$$

*where* $C$ *is the same as in* (6.18). *The same holds true with* $E^\infty_{h,s}$ *in place of* $E^n_{h,s}$ *at each occurrence.*

# Chapter 7

# A one-parameter group of elliptic symbols in the classes $\Gamma_s^{(\omega)}(\mathbb{R}^d)$

In the current chapter we show that, for suitable $s$ and $\omega_0$, there are elements $a \in \Gamma_s^{(\omega_0)}$ and $b \in \Gamma_s^{(1/\omega_0)}$ such that $a\#b = b\#a = 1$. This is essentially a consequence of Theorem 7.8, where it is proved that the evolution equation (7.1) has a unique solution $a(t,\,\cdot\,)$ which belongs to $\Gamma_s^{(\omega\vartheta^t)}$, thereby deducing needed semigroup properties for scales of pseudo-differential operators. Similar facts hold for corresponding Beurling type spaces (cf. Theorem 7.9).

Moreover, we will deduce an analog of (0.5) for the Gevrey type symbol classes introduced in Section 5.4. As in [14], (0.5) is obtained by proving that the evolution equation

$$(\partial_t a)(t,\,\cdot\,) = (b + \log\vartheta)\#a(t,\,\cdot\,), \qquad a(0,\,\cdot\,) = a_0 \in \Gamma_s^{(\omega)},\ \vartheta \in \Gamma_s^{(\vartheta)}, \quad (7.1)$$

analogous to (0.6), has a unique solution $a(t,\,\cdot\,)$ which belongs to $\Gamma_s^{(\omega\vartheta^t)}$ (and similarly when the $\Gamma_s^{(\omega)}$-spaces are replaced by corresponding $\Gamma_{0,s}^{(\omega)}$-spaces).

First we have the following result on certain logarithms of weight functions.

**Proposition 7.1.** *Let $\omega \in \mathscr{P}_E(\mathbb{R}^{2d}) \cap \Gamma_{s_0}^{(\omega)}(\mathbb{R}^{2d})$, $s_0 \in (0,1]$, $v \in \mathscr{P}_E(\mathbb{R}^{2d})$, be such that $\omega$ is $v$-moderate, $\vartheta(X) = 1 + \log v(X)$ and let*

$$c(X,Y) = \log\frac{\omega(X+Y)}{\omega(Y)}.$$

*Then,*

*(1) $\{c(\,\cdot\,,Y)\}_{Y\in\mathbb{R}^{2d}}$ is a uniformly bounded family in $\Gamma_s^{(\vartheta)}(\mathbb{R}^{2d})$, $s \geqslant 1$;*

*(2) for* $\alpha \neq 0$, $\{(\partial_X^\alpha c)(\,\cdot\,, Y)\}_{Y\in\mathbb{R}^{2d}}$ *is a uniformly bounded family in* $\Gamma_s^{(1)}(\mathbb{R}^{2d})$, $s \geqslant 1$.

For the proof of Proposition 7.1 we need the following multidimensional version of the well-known Faà di Bruno formula for the derivatives of composed functions. It can be found, e.g., setting $q = p = 1$, $n = 2d$, in equations (2.3) and (2.4) in [75].

**Lemma 7.2.** *Let* $f\colon \mathbb{R} \to \mathbb{R}$, $g\colon \mathbb{R}^{2d} \to \mathbb{R}$. *Then, for any* $\alpha \in \mathbb{Z}_+^{2d}$, $\alpha \neq 0$,

$$\frac{\partial^\alpha f(g(x))}{\alpha!} = \sum_{1\leqslant k\leqslant|\alpha|} \frac{f^{(k)}(g(x))}{k!} \sum_{\substack{\beta_1+\cdots+\beta_k=\alpha\\ \beta_j\neq 0,\, j=1,\dots,k}} \prod_{1\leqslant j\leqslant k} \frac{(\partial^{\beta_j} g)(x)}{\beta_j!}. \tag{7.2}$$

We will also need the next *factorial estimate*, for expressions involving decompositions of $\alpha \in \mathbb{Z}_+^{2d}$, $\alpha \neq 0$, into the sum of $k$ nontrivial multi-indeces $\beta_j$, $j = 1, \dots, k$, as in (7.2), and corresponding products of (powers of) factorials.

**Lemma 7.3.** *Let* $s_0 \in (0, 1]$, $\alpha \in \mathbb{Z}_+^{2d}$, $\alpha \neq 0$. *Then, for suitable* $C_0 > 0$, *depending only in* $d$,

$$\sum_{1\leqslant k\leqslant|\alpha|} \frac{1}{k} \sum_{\substack{\beta_1+\cdots+\beta_k=\alpha\\ \beta_j\neq 0,\, j=1,\dots,k}} \prod_{1\leqslant j\leqslant k} \beta_j!^{s_0-1} \lesssim C_0^{|\alpha|}. \tag{7.3}$$

Lemma 7.3 follows from Lemma A.2 in the Appendix.

*Proof of Proposition 7.1.* In order to prove (1) we need to show that $c(\,\cdot\,, Y)$ satisfies $\Gamma_s^{(\vartheta)}$ estimates, uniformly with respect to $Y \in \mathbb{R}^{2d}$. By (5.1) and (5.2) we get

$$c(X, Y) \leqslant \log(Cv(X)) \lesssim 1 + \log v(X) = \vartheta(X)$$

and

$$c(X, Y) \geqslant \log((Cv(X))^{-1}) \gtrsim -(1 + \log v(X)) = -\vartheta(X).$$

Hence, $|c(X, Y)| \lesssim \vartheta(X)$, $X \in \mathbb{R}^{2d}$. Now, for $\alpha \in \mathbb{Z}_+^{2d}$, $\alpha \neq 0$, (5.20) with $a = \omega$ and (7.2) give

$$\partial_X^\alpha c(X, Y) = \alpha! \sum_{1\leqslant k\leqslant|\alpha|} \frac{(-1)^{k+1}}{k\,[\omega(X+Y)]^k} \sum_{\substack{\beta_1+\cdots+\beta_k=\alpha\\ \beta_j\neq 0,\, j=1,\dots,k}} \prod_{1\leqslant j\leqslant k} \frac{(\partial^{\beta_j}\omega)(X+Y)}{\beta_j!},$$

and by (7.3),

$$
\begin{aligned}
|\partial_X^\alpha c(X,Y)| &\lesssim \alpha! \sum_{1\leqslant k\leqslant |\alpha|} \frac{1}{k\,[\omega(X+Y)]^k} \sum_{\substack{\beta_1+\cdots+\beta_k=\alpha \\ \beta_j\neq 0,\, j=1,\dots,k}} \prod_{1\leqslant j\leqslant k} \frac{\omega(X+Y)h^{|\beta_j|}\beta_j!^{s_0}}{\beta_j!} \\
&\leqslant h^{|\alpha|}\,\alpha! \sum_{1\leqslant k\leqslant |\alpha|} \frac{1}{k} \sum_{\substack{\beta_1+\cdots+\beta_k=\alpha \\ \beta_j\neq 0,\, j=1,\dots,k}} \prod_{1\leqslant j\leqslant k} \beta_j!^{s_0-1} \lesssim (C_0h)^{|\alpha|}\alpha!^s,
\end{aligned}
$$

which gives the result. □

**Proposition 7.4.** *Assume $s > \frac{1}{2}$ and $\omega(X) \lesssim e^{r|X|^{\frac{1}{s}}}$ for some $r > 0$. Let $\{a(\,\cdot\,,Y)\}_{Y\in\mathbb{R}^{2d}}$ be a uniformly bounded family in $\Sigma_s(\mathbb{R}^{2d})$ and $\{c(\,\cdot\,,Z)\}_{Z\in\mathbb{R}^{2d}}$ be a bounded family in $\Gamma_s^{(\omega)}(\mathbb{R}^{2d})$. Then,*

$$\{a(\,\cdot\,,Y)\#c(\,\cdot\,,Z)\}_{Y,Z\in\mathbb{R}^{2d}} \text{ and } \{c(\,\cdot\,,Z)\#a(\,\cdot\,,Y)\}_{Y,Z\in\mathbb{R}^{2d}}$$

*are bounded families in $\mathcal{S}_s(\mathbb{R}^{2d})$.*

*Proof.* Let $\phi \in \Sigma_s$ and $a \in \Gamma_s^{(\omega)}$. By Lemma 6.7 it follows that

$$|D_X^\alpha(\phi\#a)(X)| \leqslant Ch^{|\alpha|}\alpha!^s e^{-r|X|^{\frac{1}{s}}}, \tag{7.4}$$

for some $h, r > 0$. Then (7.4) holds true if and only if $\phi\#a$ belongs to $\mathcal{S}_s$. By the proof of (7.4), the constants $C$, $h$ and $r$ can be chosen to depend continuously on $\phi \in \Sigma_s(\mathbb{R}^{2d})$ and $a \in \Gamma_s^{(\omega)}(\mathbb{R}^{2d})$. Hence if $\Omega_1$ is bounded in $\Sigma_s(\mathbb{R}^{2d})$ and $\Omega_2$ is bounded in $\Gamma_s^{(\omega)}(\mathbb{R}^{2d})$, then it follows that $\{\phi\#a\}_{\phi\in\Omega_1, a\in\Omega_2}$ is a bounded family in $\mathcal{S}_s(\mathbb{R}^{2d})$. □

The following result can be found e. g. in [113].

**Lemma 7.5.** *Let $a \in \mathscr{S}'(\mathbb{R}^{2d})$. Then*

$$\|a\|_{s_\infty^w} \leqslant C \sum_{|\alpha|\leqslant d+1} \|\partial^\alpha a\|_{L^\infty} \tag{7.5}$$

*and*

$$\|a\|_{L^\infty} \leqslant C \sum_{|\alpha|\leqslant 2d+1} \|\partial^\alpha a\|_{s_\infty^w} \tag{7.6}$$

*for some constant $C > 0$ depending on the dimension $d$ only.*

**Proposition 7.6.** *Let $a \in \mathscr{S}'(\mathbb{R}^{2d})$, $s \geqslant \frac{1}{2}$ and set $b_{\alpha\beta}(X) = \partial^\alpha(X^\beta a(X))$ when $\alpha, \beta \in \mathbb{Z}_+^{2d}$. Then the following conditions are equivalent:*

*(1) $a \in \mathcal{S}_s(\mathbb{R}^{2d})$ ($a \in \Sigma_s(\mathbb{R}^{2d})$).*

*(2) For some* $h > 0$ *(every* $h > 0$*) it holds*

$$\|b_{\alpha\beta}\|_{L^\infty} \lesssim h^{|\alpha+\beta|}(\alpha!\beta!)^s, \qquad \alpha,\beta \in \mathbb{Z}_+^{2d}.$$

*(3) For some* $h > 0$ *(every* $h > 0$*) it holds*

$$\|b_{\alpha\beta}\|_{s_\infty^w} \lesssim h^{|\alpha+\beta|}(\alpha!\beta!)^s, \qquad \alpha,\beta \in \mathbb{Z}_+^{2d}.$$

*Proof.* We only prove the result in the Roumieu case. The Beurling case follows by similar arguments.

The equivalence between (1) and (2) follows from the definitions. The proof of the equivalence of (2) and (3) follows by a straightforward application of Lemma 7.5. In fact, assume that (2) holds true. Then (7.5) gives

$$\begin{aligned}\|b_{\alpha\beta}\|_{s_\infty^w} &\leqslant C \sum_{|\gamma|\leqslant d+1} \|\partial^\gamma b_{\alpha\beta}\|_{L^\infty} \lesssim \sum_{|\gamma|\leqslant d+1} h^{|\alpha+\beta+\gamma|}((\alpha+\gamma)!\beta!)^s \\ &= h^{|\alpha+\beta|}(\alpha!\beta!)^s \sum_{|\gamma|\leqslant d+1} h^{|\gamma|}\gamma!^s \left(\frac{(\alpha+\gamma)!}{\alpha!\,\gamma!}\right)^s \lesssim (2^s h)^{|\alpha+\beta|}(\alpha!\beta!)^s.\end{aligned}$$

In the last inequality we have used

$$\sum_{|\gamma|\leqslant d+1} h^{|\gamma|}\gamma!^s \left(\frac{(\alpha+\gamma)!}{\alpha!\,\gamma!}\right)^s \leqslant C_1 \cdot 2^{s(|\alpha|+d+1)} \leqslant C_2 2^{s|\alpha+\beta|},$$

where the constants $C_1$ and $C_2$ only depend on $d$ and $h$. Hence (3) holds true, as claimed. The proof of the converse follows by similar argument, employing (7.6) instead of (7.5). $\square$

We also need the following characterisation of $\Gamma_s^{(1)}(\mathbb{R}^{2d})$.

**Proposition 7.7.** *Let* $a \in \mathscr{S}'(\mathbb{R}^{2d})$ *and* $s > 0$*. Then the following conditions are equivalent:*

*(1)* $a \in \Gamma_s^{(1)}(\mathbb{R}^{2d})$.

*(2) There exists* $h > 0$ *such that*

$$\|\partial^\alpha a\|_{L^\infty(\mathbb{R}^{2d})} \lesssim h^{|\alpha|}\alpha!^s, \qquad \alpha \in \mathbb{Z}_+^{2d}.$$

*(3) There exists* $h > 0$ *such that*

$$\|\partial^\alpha a\|_{s_\infty^w} \lesssim h^{|\alpha|}\alpha!^s, \qquad \alpha \in \mathbb{Z}_+^{2d}. \tag{7.7}$$

*(4) There exists* $h > 0$ *such that*

$$\|\langle T_1, D_X\rangle \cdots \langle T_m, D_X\rangle a\|_{s_\infty^w} \lesssim h^m m!^s, \tag{7.8}$$

*for any* $T_1,\dots,T_m \in \mathbb{R}^{2d}$ *such that* $|T_j| \leqslant 1$, $j = 1,\dots,m$, $m \geqslant 1$.

*Proof.* The equivalence between (1) and (2) is well known. The equivalence of (2) and (3) is proved by similar arguments to the one employed in the proof of Proposition 7.6, using Lemma 7.5. It remains to prove the equivalence with (4). Assume that (3) holds true, and let

$$T_k = \sum_{l=1}^{d} (t_{k,l} e_l + \tau_{k,l} \varepsilon_l),$$

for the standard symplectic basis (5.33) of $\mathbb{R}^{2d}$. If we set $e_{d+l} = \varepsilon_l$, $t_{k,d+l} = \tau_{k,l}$, $l \in \{1, \dots, d\}$, and letting $X_l$ being the coordinates for $X = (x, \xi) \in \mathbb{R}^{2d}$ with respect to this basis, then

$$\langle T_k, D_X \rangle a = \sum_{l=1}^{2d} t_{k,l} \frac{\partial a}{\partial X_l},$$

so that the symbol $\langle T_1, D_X \rangle \cdots \langle T_m, D_X \rangle a$ is in the span of symbols of the form

$$\left( \prod_{k=1}^{m} t_{k,l_k} \right) \left( \partial_{X_{1,l_1}} \cdots \partial_{X_{m,l_m}} a \right)$$

where the summation contains at most $(2d)^m$ terms. Since $|T_j| \leqslant 1$, $j = 1, \dots, m$, we obtain, by the hypothesis (3) that

$$\begin{aligned}
\| \langle T_1, D_X \rangle \cdots \langle T_m, D_X \rangle a \|_{s^w_\infty} &\leqslant (2d)^m \sup_{|\alpha| = m} \| \partial^\alpha a \|_{s^w_\infty} \\
&\lesssim \sup_{|\alpha| = m} \sum_{|\gamma| \leqslant d+1} h^{|\alpha+\gamma|} (\alpha + \gamma)!^s \\
&= \sup_{|\alpha| = m} h^{|\alpha|} \alpha!^s \sum_{|\gamma| \leqslant d+1} h^{|\gamma|} \gamma!^s \left( \frac{(\alpha+\gamma)!}{\alpha! \, \gamma!} \right)^s \\
&\lesssim (2^{s+1} h)^m m!^s,
\end{aligned}$$

which gives (4).

If instead (4) holds, then choosing $T_1, \dots T_{|\alpha|}$ in suitable ways, the left-hand sides of (7.7) and (7.8) agree. The assertion (3) now follows from (4) by using the inequality $|\alpha|! \leqslant d^{|\alpha|} \alpha!$. □

The first main result of this chapter is the following analogy of [14, Theorem 6.4] and [89, Theorem 2.6.15] in the framework of Gevrey regularity. It deals with the existence of one-parameter groups of pseudo-differential operators, obtained as solutions to suitable evolution equations.

**Theorem 7.8.** *Let $s \geqslant 1$, $\omega, \vartheta \in \mathscr{P}^0_{E,s}(\mathbb{R}^{2d})$ be such that $\omega \in \Gamma^{(\omega)}_s(\mathbb{R}^{2d})$ and $\vartheta \in \Gamma^{(\vartheta)}_s(\mathbb{R}^{2d})$, and let $a_0 \in \Gamma^{(\omega)}_s(\mathbb{R}^{2d})$, $b \in \Gamma^{(1)}_s(\mathbb{R}^{2d})$. Then, there exists a* *unique smooth map* $(t,X) \mapsto a(t,X) \in \mathbb{C}$ *such that* $a(t,\,\cdot\,) \in \Gamma_s^{(\omega\,\vartheta^t)}(\mathbb{R}^{2d})$ *for all* $t \in \mathbb{R}$*, and*

$$\begin{cases} (\partial_t a)(t,\,\cdot\,) = (b + \log\vartheta)\#a(t,\,\cdot\,) \\ a(0,\,\cdot\,) = a_0. \end{cases} \tag{7.9}$$

*If in addition* $\omega \equiv a_0 \equiv 1$*, then* $a(t,X)$ *also satisfies*

$$\begin{cases} (\partial_t a)(t,\,\cdot\,) = a(t,\,\cdot\,)\#(b + \log\vartheta) \\ a(0,\,\cdot\,) = a_0, \end{cases} \tag{7.10}$$

*and*

$$a(t_1,\,\cdot\,)\#a(t_2,\,\cdot\,) = a(t_1+t_2,\,\cdot\,), \quad a(t,\,\cdot\,) \in \Gamma_s^{(\vartheta^t)}(\mathbb{R}^{2d}), \quad t, t_1, t_2 \in \mathbb{R}. \tag{7.11}$$

*Proof.* **Step 1**(Two auxiliary equations): First suppose that a solution $a(t,X)$ of (7.9) exists. Then

$$a(t,X) = a_0(X) + \int_0^t c(u,X)\,du$$

with

$$c(t,\,\cdot\,) = (b + \log\vartheta)\#a(t,\,\cdot\,) \in \Gamma_s^{(\omega\vartheta^t\langle\log\vartheta\rangle)}(\mathbb{R}^{2d}),$$

in view of Propositions 5.39 and 7.1. This implies that the map $t \mapsto a(t,\,\cdot\,)$ is $C^1$ from $[-R,R]$ into the symbol space

$$\Gamma_s^{(\omega(\vartheta+\vartheta^{-1})^R\langle\log\vartheta\rangle)}(\mathbb{R}^{2d}).$$

Choose $s_0 < s$, and $\phi, \psi \in \mathcal{S}_{s_0}(\mathbb{R}^{2d})$ such that (6.2) holds true. Let

$$c_1(t,Y,\,\cdot\,) = \omega(Y)^{-1}\vartheta(Y)^{-t}\,\phi_Y\#a(t,\,\cdot\,). \tag{7.12}$$

By Lemma 6.7 (1) we have $t \mapsto c_1(t,Y,\,\cdot\,)$ is a $C^1$ map from $[-R,R]$ into $\mathcal{S}_s(\mathbb{R}^{2d})$, for any $Y \in \mathbb{R}^{2d}$. Moreover,

$$\partial_t c_1(t,Y,\,\cdot\,) = \omega(Y)^{-1}\vartheta(Y)^{-t}\,\phi_Y\#f(Y,\,\cdot\,)\#a(t,\,\cdot\,)$$

when

$$f(Y,X) = b(X) + \log\frac{\vartheta(X)}{\vartheta(Y)}.$$

Then,

$$(\partial_t c_1)(t,Y,\,\cdot\,) = \omega(Y)^{-1}\;\vartheta(Y)^{-t}\int_{\mathbb{R}^{2d}} \phi_Y\#f(Y,\,\cdot\,)\#\psi_Z\#\phi_Z\#a(t,\,\cdot\,)\,dZ$$

giving that

$$(\partial_t c_1)(t,Y,\,\cdot\,) = \int_{\mathbb{R}^{2d}} K_{Y,Z}(t,\,\cdot\,)\#c_1(t,Z,\,\cdot\,)\,dZ \tag{7.13}$$

with

$$K_{Y,Z}(t,\,\cdot\,) = \frac{\omega(Z)\,\vartheta(Z)^t}{\omega(Y)\,\vartheta(Y)^t}\phi_Y\#f(Y,\,\cdot\,)\#\psi_Z. \tag{7.14}$$

We also need to consider the similar situation where $f(Y,\,\cdot\,)$ is replaced by $f(Z,\,\cdot\,)$, that is

$$\partial_t c_2(t,Y,\,\cdot\,) = \int_{\mathbb{R}^{2d}} \widetilde{K}_{Y,Z}(t,\,\cdot\,)\#c_2(t,Z,\,\cdot\,)\,dZ, \tag{7.13$'$}$$

where

$$\widetilde{K}_{Y,Z}(t,\,\cdot\,) = \frac{\omega(Z)\,\vartheta(Z)^t}{\omega(Y)\,\vartheta(Y)^t}\phi_Y\#f(Z,\,\cdot\,)\#\psi_Z, \tag{7.14$'$}$$

and

$$c_2(0,Y,\,\cdot\,) = c_1(0,Y,\,\cdot\,) = \omega(Y)^{-1}\phi_Y\#a_0. \tag{7.15}$$

We consider the operators $\mathcal{K}$ and $\widetilde{\mathcal{K}}$ when acting on $E^0$ from Section 6.2, defined by

$$(\mathcal{K}a)(t,Y,X) = \int_{\mathbb{R}^{2d}} (K_{Y,Z}(t,\,\cdot\,)\#a(t,Z,\,\cdot\,))(X)\,dZ,$$

and

$$(\widetilde{\mathcal{K}}a)(t,Y,X) = \int_{\mathbb{R}^{2d}} (\widetilde{K}_{Y,Z}(t,\,\cdot\,)\#a(t,Z,\,\cdot\,))(X)\,dZ.$$

We claim that

$$\|\mathcal{K}a\|_{E^n_{h,s}} \leqslant C(n+1)\|a\|_{E^n_{h,s}} \quad \text{and} \quad \|\widetilde{\mathcal{K}}a\|_{E^n_{h,s}} \leqslant C(n+1)\|a\|_{E^n_{h,s}} \tag{7.16}$$

for some constant $C$, which is independent of $h$, $n$ and $s$.

In order to prove (7.16), it is convenient to let $\mathcal{P}_k$ be the family of all subsets of $\{1,\dots,k\}$, $k \geqslant 1$. For each $P \in \mathcal{P}_k$, $a \in s^w_\infty(\mathbb{R}^{2d})$, we set

$$H(a,P) = \begin{cases} a & \text{when } P = \emptyset, \\ \langle T_{j_1}, D_X\rangle \cdots \langle T_{j_l}, D_X\rangle a & \text{when } P = \{j_1 < \cdots < j_l\},\ l \leqslant k. \end{cases}$$

We shall estimate

$$\frac{\|(\langle T_1, D_X\rangle \cdots \langle T_k, D_X\rangle \mathcal{K}a)(t, Y, \cdot\,)\|_{s^w_\infty(\mathbb{R}^{2d})}}{h^k(k!)^s}$$

when $a \in E^n_{h,s}$. Since

$$\begin{aligned}
(\langle T_1, D_X\rangle \cdots \langle T_k, D_X\rangle \mathcal{K}a)(t, Y, X)
&= \langle T_1, D_X\rangle \cdots \langle T_k, D_X\rangle \int_{\mathbb{R}^{2d}} (K_{Y,Z}(t, \cdot\,)\# a(t, Z, \cdot\,))(X)\, dZ \\
&= \sum_{P\in\mathcal{P}_k} \int_{\mathbb{R}^{2d}} (H(K_{Y,Z}(t, \cdot\,), P)\# H(a(t, Z, \cdot\,), P^c))(X)\, dZ,
\end{aligned}$$

we find[1]

$$\begin{aligned}
&\frac{\|(\langle T_1, D_X\rangle \cdots \langle T_k, D_X\rangle \mathcal{K}a)(t, Y, \cdot\,)\|_{s^w_\infty}}{h^k(k!)^s} \\
&\leqslant \sum_{l=0}^{k} \sum_{|P|=l} \binom{k}{l}^{-s} \int_{\mathbb{R}^{2d}} \frac{\|H(K_{Y,Z}(t, \cdot\,), P)\|_{s^w_\infty}}{h^l\, l!^s} \cdot \frac{\|H(a(t, Z, \cdot\,), P^c)\|_{s^w_\infty}}{h^{k-l}((k-l)!)^s}\, dZ \\
&\qquad \leqslant \sum_{l=0}^{k} \sum_{|P|=l} \|a\|_{E^{k-l}_{h,s}} \binom{k}{l}^{-s} \int_{\mathbb{R}^{2d}} \frac{\|H(K_{Y,Z}(t, \cdot\,), P)\|_{s^w_\infty}}{h^l\, l!^s}\, dZ \\
&\qquad \lesssim \|a\|_{E^k_{h,s}} \sum_{l=0}^{k} \sum_{|P|=l} \binom{k}{l}^{-1} \int_{\mathbb{R}^{2d}} \frac{\|H(K_{Y,Z}(t, \cdot\,), P)\|_{s^w_\infty}}{h^l\, l!^s}\, dZ \\
&\qquad \leqslant (k+1) D_k(Y) \|a\|_{E^k_{h,s}}, \qquad (7.17)
\end{aligned}$$

where

$$D_k(Y) = \sup_{l\leqslant k} \sup_{|P|=l} \left( \int_{\mathbb{R}^{2d}} \frac{\|H(K_{Y,Z}(t, \cdot\,), P)\|_{s^w_\infty}}{h^l\, l!^s}\, dZ \right), \qquad (7.18)$$

Here the third inequality in (7.17) follows from the fact that $s \geqslant 1$ and $\|a\|_{E^n_{h,s}}$ increases with $n$.

We have to estimate $D_k(Y)$ in (7.18) and study the different quantities on the right-hand side of (7.14). Since $\omega$ and $\vartheta$ belong to $\mathscr{P}^0_{E,s}$, it follows that for every $r > 0$,

$$\begin{aligned}
\frac{\omega(Z)\,\vartheta(Z)^t}{\omega(Y)\,\vartheta(Y)^t} = \frac{\omega(Z)}{\omega(Y)} \left(\frac{\vartheta(Z)}{\vartheta(Y)}\right)^t &\lesssim e^{r|Y-Z|^{\frac{1}{s}}} \left(e^{r|Y-Z|^{\frac{1}{s}}}\right)^t \\
&= e^{r(1+t)|Y-Z|^{\frac{1}{s}}}, \quad Y, Z \in \mathbb{R}^{2d}. \quad (7.19)
\end{aligned}$$

[1]Recall that, by definition of $P$, $\sum_{|P|=l} 1 = \binom{k}{l}$.

For the Weyl product in (7.14) we have

$$\begin{aligned}\phi_Y\#f(Y,\,\cdot\,) &= \phi(\,\cdot\,-Y)\#\Big(b+\log\frac{\vartheta}{\vartheta(Y)}\Big)\\ &= \Big(\phi\#b(\,\cdot\,+Y)\Big)_Y+\phi(\,\cdot\,-Y)\#\Big(\log\frac{\vartheta}{\vartheta(Y)}\Big)\\ &= \Big(\phi\#b(\,\cdot\,+Y)\Big)_Y+\Big(\phi\#\log\frac{\vartheta(\,\cdot\,+Y)}{\vartheta(Y)}\Big)_Y.\end{aligned}$$

By Propositions 7.1 and 7.4,

$$\Big\{\phi\#b(\,\cdot\,+Y)\Big\}_{Y\in\mathbb{R}^{2d}}\quad\text{and}\quad\Big\{\phi\#\log\frac{\vartheta(\,\cdot\,+Y)}{\vartheta(Y)}\Big\}_{Y\in\mathbb{R}^{2d}}\tag{7.20}$$

are uniformly bounded families in $\mathcal{S}_s(\mathbb{R}^{2d})$. Note that

$$a_2(Z,X)=\psi_Z(X)\quad\Rightarrow\quad\{a_2(Z,\,\cdot\,+Z)\}_{Z\in\mathbb{R}^{2d}}=\{\psi\}_{Z\in\mathbb{R}^{2d}},$$

which is evidently a uniformly bounded family in $\mathcal{S}_s(\mathbb{R}^{2d})$. Combining this last observation with the computations on $\phi_Y\#f(Y,\,\cdot\,)$ above, using the fact that Leibniz rule applied also on the $\#-$product, Lemmata 6.5 and 6.7, we finally obtain

$$\begin{gathered}|D_X^\alpha(\phi_Y\#f(Y,\,\cdot\,)\#\psi_Z)(X)|\lesssim h^{|\alpha|}\alpha!^s e^{-r_0(|X-Y|^{\frac{1}{s}}+|X-Z|^{\frac{1}{s}}+|Y-Z|^{\frac{1}{s}})},\\ X,Y,Z\in\mathbb{R}^{2d},\alpha\in\mathbb{Z}_+^{2d},\end{gathered}\tag{7.21}$$

for some $h,r_0>0$.

By Proposition 7.7, (7.19), (7.21) and the fact that $t$ is bounded, we get for all $P\in\mathcal{P}_k$, $Y,Z\in\mathbb{R}^{2d}$, and some $r_0,h>0$ that

$$\|H(K_{Y,Z}(t,\,\cdot\,),P)\|_{s_\infty^w}\leqslant Ch^l l!^s e^{-r_0|Y-Z|^{\frac{1}{s}}},\quad l=|P|,$$

where $C$ is independent of $k$. Hence $D_k$ in (7.18) satisfies

$$D_k(Y)\leqslant C_1\int_{\mathbb{R}^{2d}}e^{-r_0|Y-Z|^{\frac{1}{s}}}\,dZ=C_2,$$

for some constants $C_1$ and $C_2$ which are independent of $Y\in\mathbb{R}^{2d}$, $h>0$ and $k\geqslant 0$. Hence (7.17) gives

$$\|\mathcal{K}a(t,Y,\,\cdot\,)\|_{s_\infty^w}\leqslant C\|a\|_{E_{h,s}^k},$$

and

$$\frac{\|\langle T_1,D_X\rangle\cdots\langle T_k,D_X\rangle\mathcal{K}a(t,Y,\,\cdot\,)\|_{s_\infty^w}}{h^k(k!)^s}\leqslant C(k+1)\|a\|_{E_{h,s}^k},$$

as claimed, where $C$ is independent of $Y\in\mathbb{R}^{2d}$, $k$ and $h>0$.

By a completely similar argument, an analogous result can be obtained for $\widetilde{\mathcal{K}}$. In fact, by similar arguments that lead to (7.20) it follows that

$$\{b(\,\cdot\,+Z)\#\psi\}_{Z\in\mathbb{R}^{2d}} \quad \text{and} \quad \left\{\log\frac{\vartheta(\,\cdot\,+Z)}{\vartheta(Z)}\#\psi\right\}_{Z\in\mathbb{R}^{2d}}$$

are bounded in $\mathcal{S}_s(\mathbb{R}^{2d})$, given that (7.21) holds with $f(Z,\,\cdot\,)$ in place of $f(Y,\,\cdot\,)$. This gives (7.16).

We have proven that for any $T>0$, then

$$\|\mathcal{K}\|_{E^n_{h,s}\to E^n_{h,s}} \leqslant C(n+1) \quad \text{and} \quad \|\widetilde{\mathcal{K}}\|_{E^n_{h,s}\to E^n_{h,s}} \leqslant C(n+1), \qquad |t|\leqslant T, \tag{7.22}$$

where $C$ is independent of $n$. As a consequence, since $\omega(Y)^{-1}\phi_Y\#a_0$ belongs to $E^n_{h,s}$ for every $n$ and with uniform bound of the norms with respect to $n$ it follows that the equations

$$\frac{dc_1}{dt} = \mathcal{K}c_1, \quad \frac{dc_2}{dt} = \widetilde{\mathcal{K}}c_2, \qquad c_1(0)=c_2(0)=\omega(Y)^{-1}\phi_Y\#a_0, \tag{7.23}$$

have unique solutions on $[-T,T]$ belonging to $E^n_{h,s}$, in view of Lemma 6.12, and that

$$\|c_j\|_{E^n_{h,s}} \leqslant \|c_j(0)\|_{E^n_{h,s}}e^{C(n+1)T} \leqslant \|c_j(0)\|_{E^\infty_{h,s}}e^{C(n+1)T}, \qquad j=1,2, \tag{7.24}$$

where the constant $C$ is the same as in (7.22) and is therefore independent of $n$. This gives

$$\sup\left(\frac{\|\langle T_1,D_X\rangle\cdots\langle T_n,D_X\rangle c_j(t,Y,\,\cdot\,)\|_{s^w_\infty}}{h^n(n!)^s}\right) \leqslant \|c_j(0)\|_{E^\infty_{h,s}}e^{C(n+1)T},$$

which is the same as

$$\sup\left(\frac{\|\langle T_1,D_X\rangle\cdots\langle T_n,D_X\rangle c_j(t,Y,\,\cdot\,)\|_{s^w_\infty}}{h_0^n(n!)^s}\right) \leqslant \|c_j(0)\|_{E^\infty_{h,s}}e^{CT}, \qquad h_0=he^{CT}. \tag{7.25}$$

Here the supremum is taken over all $T_1,\dots,T_n,Y\in\mathbb{R}^{2d}$ such that $|T_j|\leqslant 1$, and $t\in[-T,T]$. By taking the supremum of the left-hand side of (7.25) over all $n\geqslant 0$ we get

$$\|c_j\|_{E^\infty_{h_0,s}} \leqslant \|c_j(0)\|_{E^\infty_{h,s}}e^{CT}, \qquad h_0=he^{CT}.$$

By Lemma 6.11 it follows that $c_j(t,Y,\,\cdot\,)\in\Gamma^{(1)}_s(\mathbb{R}^{2d})$, uniformly in $Y$ and for bounded $t$.

In order to prove the uniqueness of the solution $a$ of (7.9), first we assume the existence and by what we have proven above i.e. that $c_1(t,Y,\,\cdot\,)$ in (7.12) satisfies (7.23) which implies the uniqueness of the solution of (7.9), since

$$a(t,\,\cdot\,) = \int_{\mathbb{R}^{2d}}\psi_Y\#\phi_Y\#a(t,\,\cdot\,)\,dY = \int_{\mathbb{R}^{2d}}\omega(Y)\vartheta(Y)^t\psi_Y\#c_1(t,Y,\,\cdot\,)\,dY. \tag{7.26}$$

Indeed, if we assume that there exist two solutions $c_1$ and $\tilde{c}_1$ in (7.12) satisfy (7.23), then since (7.23) has a unique solution, it follows that $c_1 \equiv \tilde{c}_1$. Therefore, if we assume that there exist two solutions $a$ and $a_1$ defined by $c_1$ and $\tilde{c}_1$ respectively. Then, in view of the construction of the solution in (7.26), it follows that $a \equiv a_1$.

To prove the existence of a solution of (7.9), we consider the solution $c_2(t, Y, \,\cdot\,)$ of $(7.13)'$ with the initial data (7.15), and we let

$$a(t, \,\cdot\,) = \int_{\mathbb{R}^{2d}} \omega(Y)\vartheta(Y)^t \psi_Y \# c_2(t, Y, \,\cdot\,)\, dY. \tag{7.27}$$

By Propositions 5.15, that is the convolution between a weight and a Gelfand-Shilov function, and 6.8, the family $\{\psi_Y \# c_2(t, Y, \,\cdot\,)\}_{Y\in\mathbb{R}^{2d}}$ belongs to $\mathcal{S}_s$ and $a(t, \,\cdot\,)$ belongs to $\Gamma_s^{(w\vartheta^t)}$. Moreover,

$$\begin{aligned}
\frac{da(t, \,\cdot\,)}{dt} &= \int_{\mathbb{R}^{2d}} \omega(Y)\vartheta(Y)^t \log\vartheta(Y)\psi_Y \# c_2(t, Y, \,\cdot\,)\, dY \\
&\quad + \int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} \omega(Y)\vartheta(Y)^t \psi_Y \# \widetilde{K}_{Y,Z}(t, \,\cdot\,) \# c_2(t, Z, \,\cdot\,)\, dY\, dZ \\
&= \int_{\mathbb{R}^{2d}} \omega(Z)\vartheta(Z)^t \log\vartheta(Z)\psi_Z \# c_2(t, Z, \,\cdot\,)\, dZ \\
&\quad + \int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} \omega(Z)\, \vartheta(Z)^t \psi_Y \# \phi_Y \# f(Z, \,\cdot\,) \# \psi_Z \# c_2(t, Z, \,\cdot\,)\, dY\, dZ \\
&= \int_{\mathbb{R}^{2d}} \omega(Z)\vartheta(Z)^t (b + \log\vartheta) \# \psi_Z \# c_2(t, Z, \,\cdot\,)\, dZ \\
&= (b + \log\vartheta) \# a(t, \,\cdot\,),
\end{aligned}$$

with the initial data

$$a(0, \,\cdot\,) = \int_{\mathbb{R}^{2d}} \omega(Y)\psi_Y \# (\omega(Y)^{-1}\phi_Y \# a_0)\, dY = a_0,$$

which provide a solution of (7.9).

In order to prove the last part we consider the unique solution $a(t, \,\cdot\,)$ of (7.9) with the initial data $a(0, \,\cdot\,) \equiv 1$. If $\omega \equiv 1$, then for $u \in \mathbb{R}$ the mappings

$$t \mapsto a(t + u, \,\cdot\,) \quad \text{and} \quad t \mapsto a(t, \,\cdot\,) \# a(u, \,\cdot\,)$$

are both solutions of (7.9) with value $a(u, \,\cdot\,)$ at $t = 0$, and

$$a(t + u, \,\cdot\,) = a(t, \,\cdot\,) \# a(u, \,\cdot\,), \tag{7.28}$$

by the uniqueness property for the solution of (7.9).

Using (7.28) we have for all $t \in \mathbb{R}$, $a(t, \,\cdot\,) \# a(-t, \,\cdot\,) = 1$. Taking the derivative we get

$$\begin{aligned}
0 &= \frac{d}{dt}(a(t, \,\cdot\,) \# a(-t, \,\cdot\,)) \\
&= (b + \log\vartheta) \# a(t, \,\cdot\,) \# a(-t, \,\cdot\,) - a(t, \,\cdot\,) \# (b + \log\vartheta) \# a(-t, \,\cdot\,).
\end{aligned}$$

That is, $(b + \log \vartheta) = a(t, \cdot\,)\#(b + \log \vartheta)\#a(-t, \cdot\,)$, implying the commutation for the sharp product of $a(t, \cdot\,)$ with $(b + \log \vartheta)$, and the result follows. □

By similar argument as for the previous result we get the following.

**Theorem 7.9.** *Let $s \geqslant 1$, $\omega, \vartheta \in \mathscr{P}_{E,s}(\mathbb{R}^{2d})$ be such that $\omega \in \Gamma_s^{(\omega)}(\mathbb{R}^{2d})$ and $\vartheta \in \Gamma_s^{(\vartheta)}(\mathbb{R}^{2d})$, and let $a_0 \in \Gamma_{0,s}^{(\omega)}(\mathbb{R}^{2d})$, $b \in \Gamma_{0,s}^{(1)}(\mathbb{R}^{2d})$. Then, there exists a unique smooth map $(t, X) \mapsto a(t, X) \in \mathbb{C}$ such that $a(t, \cdot\,) \in \Gamma_s^{(\omega\,\vartheta^t)}(\mathbb{R}^{2d})$ for all $t \in \mathbb{R}$, and $a(t, \cdot\,)$ satisfies* (7.9).

*Moreover, if $\omega \equiv a_0 \equiv 1$, then $a(t, X)$ also satisfies* (7.10) *and*

$$a(t_1, \cdot\,)\#a(t_2, \cdot\,) = a(t_1 + t_2, \cdot\,), \quad a(t, \cdot\,) \in \Gamma_{0,s}^{(\vartheta^t)}(\mathbb{R}^{2d}), \quad t, t_1, t_2 \in \mathbb{R}.$$

# Chapter 8

# Lifting of pseudo-differential operators on modulation spaces and mapping properties for Toeplitz operators

In the this chapter we use the framework in [72] in combination with (0.5) to extend the lifting properties in [72] in such ways that the involved weights are allowed to belong to $\mathscr{P}_{E,s}^0$ or in $\mathscr{P}_{E,s}$ instead of the smaller set $\mathscr{P}$ which is the assumption in [72].

## 8.1 Lifting of pseudo-differential operators on modulation spaces

In this section we apply the group properties in Theorems 7.8 and 7.9 to deduce lifting properties of pseudo-differential operators on modulation spaces. Thereafter we combine these results with the Wiener property of certain pseudo-differential operators with symbols in suitable modulation spaces to get lifting properties for Toeplitz operators with weights as their symbols.

**Theorem 8.1.** *Let $s \geqslant 1$, $\boldsymbol{p} \in (0,\infty]^{2d}$, $A \in \mathbf{M}(d,\mathbb{R})$, $\omega \in \mathscr{P}_{E,s}^0(\mathbb{R}^{2d})$, and let $\mathscr{B}$ be an invariant BF-space on $\mathbb{R}^{2d}$, or $\mathscr{B} = L_E^{\boldsymbol{p}}(\mathbb{R}^{2d})$ for some phase split basis $E$ of $\mathbb{R}^{2d}$. Then the following statements hold true:*

*(1) There exist $a \in \Gamma_s^{(\omega)}(\mathbb{R}^{2d})$ and $b \in \Gamma_s^{(1/\omega)}(\mathbb{R}^{2d})$ such that*

$$\operatorname{Op}_A(a) \circ \operatorname{Op}_A(b) = \operatorname{Op}_A(b) \circ \operatorname{Op}_A(a) = \operatorname{Id}_{\mathcal{S}_s'(\mathbb{R}^d)}\,. \tag{8.1}$$

*Furthermore,* $\operatorname{Op}_A(a)$ *is an isomorphism from* $M(\omega_0, \mathscr{B})$ *onto* $M(\omega_0/\omega, \mathscr{B})$, *for every* $\omega_0 \in \mathscr{P}^0_{E,s}(\mathbb{R}^{2d})$.

(2) *Let* $a_0 \in \Gamma_s^{(\omega)}(\mathbb{R}^{2d})$ *be such that* $\operatorname{Op}_A(a_0)$ *is an isomorphism from* $M^2_{(\omega_1)}(\mathbb{R}^d)$ *to* $M^2_{(\omega_1/\omega)}(\mathbb{R}^d)$ *for some* $\omega_1 \in \mathscr{P}^0_{E,s}(\mathbb{R}^{2d})$. *Then* $\operatorname{Op}_A(a_0)$ *is an isomorphism from* $M(\omega_2, \mathscr{B})$ *to* $M(\omega_2/\omega, \mathscr{B})$, *for every* $\omega_2 \in \mathscr{P}^0_{E,s}(\mathbb{R}^{2d})$. *Furthermore, the inverse of* $\operatorname{Op}_A(a_0)$ *is equal to* $\operatorname{Op}_A(b_0)$ *for some* $b_0 \in \Gamma_s^{(1/\omega)}(\mathbb{R}^{2d})$.

**Theorem 8.2.** *Let* $s > 1$, $\boldsymbol{p} \in (0,\infty]^{2d}$, $A \in \mathbf{M}(d,\mathbb{R})$, $\omega \in \mathscr{P}_{E,s}(\mathbb{R}^{2d})$, *and let* $\mathscr{B}$ *be an invariant BF-space on* $\mathbb{R}^{2d}$, *or* $\mathscr{B} = L^{\boldsymbol{p}}_E(\mathbb{R}^{2d})$ *for some phase split basis* $E$ *of* $\mathbb{R}^{2d}$. *Then the following properties hold true:*

(1) *There exist* $a \in \Gamma_{0,s}^{(\omega)}(\mathbb{R}^{2d})$ *and* $b \in \Gamma_{0,s}^{(1/\omega)}(\mathbb{R}^{2d})$ *such that*

$$\operatorname{Op}_A(a) \circ \operatorname{Op}_A(b) = \operatorname{Op}_A(b) \circ \operatorname{Op}_A(a) = \operatorname{Id}_{\Sigma_s'(\mathbb{R}^d)}. \tag{8.2}$$

*Furthermore,* $\operatorname{Op}_A(a)$ *is an isomorphism from* $M(\omega_0, \mathscr{B})$ *onto* $M(\omega_0/\omega, \mathscr{B})$, *for every* $\omega_0 \in \mathscr{P}_{E,s}(\mathbb{R}^{2d})$.

(2) *Let* $a_0 \in \Gamma_{0,s}^{(\omega)}(\mathbb{R}^{2d})$ *be such that* $\operatorname{Op}_A(a_0)$ *is an isomorphism from* $M^2_{(\omega_1)}(\mathbb{R}^d)$ *to* $M^2_{(\omega_1/\omega)}(\mathbb{R}^d)$ *for some* $\omega_1 \in \mathscr{P}_{E,s}(\mathbb{R}^{2d})$. *Then* $\operatorname{Op}_A(a_0)$ *is an isomorphism from* $M(\omega_2, \mathscr{B})$ *to* $M(\omega_2/\omega, \mathscr{B})$, *for every* $\omega_2 \in \mathscr{P}_{E,s}(\mathbb{R}^{2d})$. *Furthermore, the inverse of* $\operatorname{Op}_A(a_0)$ *is equal to* $\operatorname{Op}_A(b_0)$ *for some* $b_0 \in \Gamma_{0,s}^{(1/\omega)}(\mathbb{R}^{2d})$.

We only prove Theorem 8.2. Theorem 8.1 follows by similar arguments.

*Proof of Theorem 8.2.* The existence of $a \in \Gamma_{0,s}^{(\omega)}(\mathbb{R}^{2d})$ and $b \in \Gamma_{0,s}^{(1/\omega)}(\mathbb{R}^{2d})$ such that (8.2) holds is guaranteed by the second part of Theorem 7.9. By [126, Theorems 2.5 and 2.8] it follows that

$$\operatorname{Op}_A(a) : M(\omega_0, \mathscr{B}) \to M(\omega_0/\omega, \mathscr{B}) \tag{8.3}$$

and

$$\operatorname{Op}_A(b) : M(\omega_0/\omega, \mathscr{B}) \to M(\omega_0, \mathscr{B}) \tag{8.4}$$

are continuous. By (8.2) and the fact that $M(\omega_0, \mathscr{B})$ and $M(\omega_0/\omega, \mathscr{B})$ are contained in $\Sigma_s'(\mathbb{R}^{2d})$, it follows that (8.3) and (8.4) are homeomorphisms, and (1) follows.

(2) It suffices to prove the result in the Weyl case, $A = \frac{1}{2}I$, in view of Proposition 5.38. By (1), we may find

$$a_1 \in \Gamma_{0,s}^{(\omega_1)}, \quad b_1 \in \Gamma_{0,s}^{(1/\omega_1)}, \quad a_2 \in \Gamma_{0,s}^{(\omega_1/\omega)}, \quad b_2 \in \Gamma_{0,s}^{(\omega/\omega_1)}$$

satisfying the following properties:

- $\operatorname{Op}^w(a_j)$ and $\operatorname{Op}^w(b_j)$ are inverses to each others on $\Sigma_s'(\mathbb{R}^d)$ for $j = 1, 2$;
- For arbitrary $\omega_2 \in \mathscr{P}_{E,s}(\mathbb{R}^{2d})$, the mappings

$$\begin{aligned} \operatorname{Op}^w(a_1) &: M^2_{(\omega_2)} \to M^2_{(\omega_2/\omega_1)}, \\ \operatorname{Op}^w(b_1) &: M^2_{(\omega_2)} \to M^2_{(\omega_2\omega_1)}, \\ \operatorname{Op}^w(a_2) &: M^2_{(\omega_2)} \to M^2_{(\omega_2\omega/\omega_1)}, \\ \operatorname{Op}^w(b_2) &: M^2_{(\omega_2)} \to M^2_{(\omega_2\omega_1/\omega)} \end{aligned} \tag{8.5}$$

  are isomorphisms.

In particular, $\operatorname{Op}^w(a_1)$ is an isomorphism from $M^2_{(\omega_1)}$ to $L^2$, and $\operatorname{Op}^w(b_1)$ is an isomorphism from $L^2$ to $M^2_{(\omega_1)}$.

Now set $c = a_2 \# a_0 \# b_1$. Then by [24, Proposition 5.38], the symbol $c$ satisfies

$$c = a_2 \# a_0 \# b_1 \in \Gamma^{(\omega_1/\omega)}_{0,s} \# \Gamma^{(\omega)}_{0,s} \# \Gamma^{(1/\omega_1)}_{0,s} \subseteq \Gamma^{(1)}_{0,s}.$$

Furthermore, $\operatorname{Op}^w(c)$ is a composition of three isomorphisms and consequently $\operatorname{Op}^w(c)$ is boundedly invertible on $L^2$.

By Proposition 5.44 (2), $\operatorname{Op}^w(c)^{-1} = \operatorname{Op}^w(c_1)$ for some $c_1 \in \Gamma^{(1)}_{0,s}$. Hence, by (1) it follows that $\operatorname{Op}^w(c)$ and $\operatorname{Op}^w(c_1)$ are isomorphisms on $M(\omega_2, \mathscr{B})$, for each $\omega_2 \in \mathscr{P}_{E,s}(\mathbb{R}^{2d})$. Since $\operatorname{Op}^w(c)$ and $\operatorname{Op}^w(c_1)$ are bounded on every $M(\omega, \mathscr{B})$, the factorization of the identity $\operatorname{Op}^w(c) \operatorname{Op}^w(c_1) = \operatorname{Id}$ is well-defined on every $M(\omega, \mathscr{B})$. Consequently, $\operatorname{Op}^w(c)$ is an isomorphism on $M(\omega, \mathscr{B})$.

Using the inverses of $a_2$ and $b_1$, we now find that

$$\operatorname{Op}^w(a_0) = \operatorname{Op}^w(b_2) \circ \operatorname{Op}^w(c) \circ \operatorname{Op}^w(a_1)$$

is a composition of isomorphisms from the domain space $M(\omega_2, \mathscr{B})$ onto the image space $M(\omega_2/\omega, \mathscr{B})$ (factoring through some intermediate spaces) for every $\omega_2 \in \mathscr{P}_{E,s}(\mathbb{R}^{2d})$ and every invariant BF-space $\mathscr{B}$. This proves the isomorphism assertions for $\operatorname{Op}^w(a_0)$.

Finally, the inverse of $\operatorname{Op}^w(a_0)$ is given by

$$\operatorname{Op}^w(b_1) \circ \operatorname{Op}^w(c_1) \circ \operatorname{Op}^w(a_2).$$

which is a Weyl operator with symbol in $\Gamma^{(1/\omega)}_{0,s}$, and the result follows. □

**Remark 8.3.** *If $g$ is the constant euclidean metric on the phase space $\mathbb{R}^{2d}$, then $S(\omega_0, g)$ equals $S^{(\omega_0)}(\mathbb{R}^{2d})$, which is defined by (5.21). We notice that also for such simple choices of $g$, (0.5) given in the introduction, leads to lifting properties that are not trivial. In fact, let $\omega$ and $\omega_0$ be polynomially moderate weight on the phase space, and let $\mathscr{B}$ be a suitable translation* *invariant BF-space. Then it is observed in [72] that the continuity results for pseudo-differential operators on modulation spaces in [116, 118] imply that* $\operatorname{Op}^w(a)$ *in* (0.5) *is continuous and bijective from* $M(\omega_0\omega, \mathscr{B})$ *to* $M(\omega, \mathscr{B})$ *with continuous inverse* $\operatorname{Op}^w(b)$. *In particular, by choosing* $\mathscr{B}$ *to be the mixed norm space* $L^{p,q}(\mathbb{R}^{2d})$ *of Lebesgue type, then* $M(\omega, \mathscr{B})$ *is equal to the classical modulation space* $M^{p,q}_{(\omega)}$. *Consequently,* $\operatorname{Op}^w(a)$ *above lifts* $M^{p,q}_{(\omega_0\omega)}$ *into* $M^{p,q}_{(\omega)}$.

**Remark 8.4.** *The SG-class in Part I, the Shubin classes in [107, Definition 23.1] and other well-known families of symbol classes are given by* $S(\omega, g)$ *for suitable choices of strongly feasible metrics* $g$ *and* $(\sigma, g)$*-temperate weights* $\omega$.

## 8.2 Mapping properties for Toeplitz operators on modulation spaces

In this section we study the isomorphism properties of Toeplitz operators between modulation spaces as in [72]. We first state results for Toeplitz operators that are well-defined in the sense of (5.60) and Propositions 5.46 and 5.47. Then we state and prove more general results for Toeplitz operators that are defined only in the framework of pseudo-differential calculus.

We start with the following result about Toeplitz operators with smooth symbols.

**Theorem 8.5.** *Let* $s \geqslant 1$ $\omega, \omega_0, v \in \mathscr{P}^0_{E,s}(\mathbb{R}^{2d})$ *be such that* $\omega_0 \in \Gamma^{(\omega_0)}_s(\mathbb{R}^{2d})$ *and that* $\omega_0$ *is* $v$*-moderate, and let* $\mathscr{B}$ *be an invariant BF-space on* $\mathbb{R}^{2d}$ *or* $\mathscr{B} = L^{\boldsymbol{p}}_E(\mathbb{R}^{2d})$ *for some phase split basis* $E$ *of* $\mathbb{R}^{2d}$. *If* $\phi \in M^1_{(v)}(\mathbb{R}^d)$, *then* $\operatorname{Tp}_\phi(\omega_0)$ *is an isomorphism from* $M(\omega, \mathscr{B})$ *to* $M(\omega/\omega_0, \mathscr{B})$.

In the next result we relax our restrictions on the weights but impose more restrictions on $\mathscr{B}$.

**Theorem 8.6.** *Let* $s > 1$, $0 \leqslant t \leqslant 1$, $p, q \in [1, \infty]$, *and* $\omega, \omega_0, v_0, v_1 \in \mathscr{P}_{E,s}(\mathbb{R}^{2d})$ *be such that* $\omega_0$ *is* $v_0$*-moderate and* $\omega$ *is* $v_1$*-moderate. Set* $v = v_1^t v_0$, $\vartheta = \omega_0^{1/2}$ *and let* $\omega_{0,t}$ *be the same as in* (5.61). *If* $\phi \in M^1_{(v)}(\mathbb{R}^d)$ *and* $\omega_0 \in \mathcal{M}^\infty_{(1/\omega_{0,t})}(\mathbb{R}^{2d})$, *then* $\operatorname{Tp}_\phi(\omega_0)$ *is an isomorphism from* $M^{p,q}_{(\vartheta\omega)}(\mathbb{R}^d)$ *to* $M^{p,q}_{(\omega/\vartheta)}(\mathbb{R}^d)$.

Before the proofs we have the following consequence of Theorem 8.6 which is the Gevrey version of [72, Corollary 4.3], as well as the original goal of our investigations.

**Corollary 8.7.** *Let* $s \geqslant 1$, $\omega, \omega_0, v_1, v_0 \in \mathscr{P}_{E,s}(\mathbb{R}^{2d})$ *and that* $\omega_0$ *is* $v_0$*-moderate and* $\omega$ *is* $v_1$*-moderate. Set* $v = v_1 v_0$ *and* $\vartheta = \omega_0^{1/2}$. *If* $\phi \in M^1_{(v)}(\mathbb{R}^d)$, *then* $\operatorname{Tp}_\phi(\omega_0)$ *is an isomorphism from* $M^{p,q}_{(\vartheta\omega)}(\mathbb{R}^d)$ *to* $M^{p,q}_{(\omega/\vartheta)}(\mathbb{R}^d)$ *simultaneously for all* $p, q \in [1, \infty]$.

*Proof.* Let $\omega_1 \in \mathscr{P}_{E,s}(\mathbb{R}^{2d}) \cap \Gamma_{0,s}^{(\omega_1)}(\mathbb{R}^{2d})$ be such that $C^{-1} \leqslant \omega_1/\omega_0 \leqslant C$, for some constant $C$. Hence, $\omega_1/\omega_0 \in L^\infty \subseteq M^\infty$. By Theorem 2.2 in [118], it follows that $\omega = \omega_1 \cdot (\omega/\omega_1)$ belongs to $M^\infty_{(\omega_2)}(\mathbb{R}^{2d})$, when $\omega_2(x,\xi,\eta,y) = 1/\omega_0(x,\xi)$. The result now follows by setting $t = 1$ and $q_0 = 1$ in Theorem 8.6. □

Theorems 8.5 and 8.6 are special cases of the following results.

**Theorem 8.5′.** *Let $s \geqslant 1$, $\omega, v, v_0 \in \mathscr{P}^0_{E,s}(\mathbb{R}^{2d})$ be such that $\omega_0 \in \Gamma_s^{(\omega_0)}(\mathbb{R}^{2d})$ and that $\omega_0$ is $v$-moderate, and let $\mathscr{B}$ be an invariant BF-space on $\mathbb{R}^{2d}$ or $\mathscr{B} = L^{\boldsymbol{p}}_E(\mathbb{R}^{2d})$ for phase split basis $E$ of $\mathbb{R}^{2d}$. If $\phi \in M^2_{(v)}(\mathbb{R}^d)$, then $\operatorname{Tp}_\phi(\omega_0)$ is an isomorphism from $M(\omega, \mathscr{B})$ to $M(\omega/\omega_0, \mathscr{B})$.*

**Theorem 8.6′.** *Let $s > 1$, $0 \leqslant t \leqslant 1$, $p, q, q_0 \in [1,\infty]$ and $\omega, \omega_0, v_0, v_1 \in \mathscr{P}_{E,s}(\mathbb{R}^{2d})$ be such that $\omega_0$ is $v_0$-moderate and $\omega$ is $v_1$-moderate. Set $r_0 = 2q_0/(2q_0-1)$, $v = v_1^t v_0$, $\vartheta = \omega_0^{1/2}$ and let $\omega_{0,t}$ be the same as in (5.61). If $\phi \in M^{r_0}_{(v)}(\mathbb{R}^d)$ and $\omega_0 \in \mathcal{M}^{\infty,q_0}_{(1/\omega_{0,t})}$, then $\operatorname{Tp}_\phi(\omega_0)$ is an isomorphism from $M^{p,q}_{(\vartheta\omega)}(\mathbb{R}^d)$ to $M^{p,q}_{(\omega/\vartheta)}(\mathbb{R}^d)$.*

We postpone the proofs of these theorems after performing some preparations and deducing some results of independent interests.

**Lemma 8.8.** *Let $s \geqslant 1$, $\omega, v \in \mathscr{P}_{E,s}(\mathbb{R}^{2d})$ be such that $\vartheta = \omega^{1/2}$ is $v$-moderate. Assume that $\phi \in M^2_{(v)}$. Then $\operatorname{Tp}_\phi(\omega)$ is an isomorphism from $M^2_{(\vartheta)}(\mathbb{R}^d)$ onto $M^2_{(1/\vartheta)}(\mathbb{R}^d)$.*

*Proof.* Recall from Remark 5.24 that for $\phi \in M^2_{(v)}(\mathbb{R}^d)\setminus\{0\}$ the expression $\|V_\phi f \cdot \vartheta\|_{L^2}$ defines an equivalent norm on $M^2_{(\vartheta)}$. Thus the occurring STFTs with respect to $\phi$ are well-defined.

Since $\operatorname{Tp}_\phi(\omega)$ is bounded from $M^2_{(\vartheta)}$ to $M^2_{(1/\vartheta)}$ by Proposition 5.47, the estimate

$$\|\operatorname{Tp}_\phi(\omega)f\|_{M^2_{(1/\vartheta)}} \lesssim \|f\|_{M^2_{(\vartheta)}} \tag{8.6}$$

holds true for all $f \in M^2_{(\vartheta)}$. Next, we observe that

$$(\operatorname{Tp}_\phi(\omega)f, g)_{L^2(\mathbb{R}^d)} = (\omega V_\phi f, V_\phi g)_{L^2(\mathbb{R}^{2d})} =: (f,g)_{M^{2,\phi}_{(\vartheta)}}, \tag{8.7}$$

for $f, g \in M^2_{(\vartheta)}(\mathbb{R}^d)$ and $\phi \in M^2_{(v)}(\mathbb{R}^d)$. The duality of modulation spaces [116, Proposition 1.2] now yields the following identity:

$$\begin{aligned}
\|f\|_{M^2_{(\vartheta)}} &\asymp \sup_{\|g\|_{M^2_{(\vartheta)}}=1} |(f,g)_{M^2_{(\vartheta)}}| \\
&\asymp \sup_{\|g\|_{M^2_{(\vartheta)}}=1} |(\operatorname{Tp}_\phi(\omega)f, g)_{L^2}| \asymp \|\operatorname{Tp}_\phi(\omega)f\|_{M^2_{(1/\vartheta)}} .
\end{aligned} \tag{8.8}$$

In view of (8.8) it follows that $\|f\|_{M^2_{(\vartheta)}}$ and $\|\operatorname{Tp}_\phi(\omega)f\|_{M^2_{(1/\vartheta)}}$ are equivalent norms on $M^2_{(\vartheta)}$.

In particular, $\operatorname{Tp}_\phi(\omega)$ is one-to-one from $M^2_{(\vartheta)}$ to $M^2_{(1/\vartheta)}$ with closed range. Since $\operatorname{Tp}_\phi(\omega)$ is self-adjoint with respect to $L^2$, it follows by duality that $\operatorname{Tp}_\phi(\omega)$ has dense range in $M^2_{(1/\vartheta)}$. Consequently, $\operatorname{Tp}_\phi(\omega)$ is onto $M^2_{(1/\vartheta)}$. By Banach's theorem, it follows that $\operatorname{Tp}_\phi(\omega)$ is an isomorphism from $M^2_{(\vartheta)}$ to $M^2_{(1/\vartheta)}$. □

We need a further generalization of Proposition 5.46 to more general classes of symbols and windows. Set

$$\omega_1(X,Y) = \frac{v_0(2Y)^{1/2}v_1(2Y)}{\omega_0(X+Y)^{1/2}\omega_0(X-Y)^{1/2}}. \tag{8.9}$$

**Proposition 8.9.** *Let $s \geqslant 1$, $0 \leqslant t \leqslant 1$, $p,q,q_0 \in [1,\infty]$, and $\omega,\omega_0,v_0,v_1 \in \mathscr{P}_{E,s}(\mathbb{R}^{2d})$ be such that $v_0$ and $v_1$ are submultiplicative, $\omega_0$ is $v_0$-moderate and $\omega$ is $v_1$-moderate. Set*

$$r_0 = 2q_0/(2q_0-1), \quad v = v_1^t v_0 \quad \text{and} \quad \vartheta = \omega_0^{1/2},$$

*and let $\omega_{0,t}$ and $\omega_1$ be as in (5.61) and (8.9). Then the following statements hold true:*

*(1) The definition of $(a,\phi) \mapsto \operatorname{Tp}_\phi(a)$ from $\Sigma_s(\mathbb{R}^{2d}) \times \Sigma_s(\mathbb{R}^d)$ to $\mathcal{L}(\Sigma_s(\mathbb{R}^d),\Sigma_s'(\mathbb{R}^d))$ extends uniquely to a continuous map from $\mathcal{M}^{\infty,q_0}_{(1/\omega_{0,t})}(\mathbb{R}^{2d}) \times M^{r_0}_{(v)}(\mathbb{R}^d)$ to $\mathcal{L}(\Sigma_s(\mathbb{R}^d),\Sigma_s'(\mathbb{R}^d))$.*

*(2) If $\phi \in M^{r_0}_{(v)}(\mathbb{R}^d)$ and $a \in \mathcal{M}^{\infty,q_0}_{(1/\omega_{0,t})}(\mathbb{R}^{2d})$, then $\operatorname{Tp}_\phi(a) = \operatorname{Op}^w(a_0)$ for some $a_0 \in \mathcal{M}^{\infty,1}_{(\omega_1)}(\mathbb{R}^{2d})$, and $\operatorname{Tp}_\phi(a)$ extends uniquely to a continuous map from $M^{p,q}_{(\vartheta\omega)}(\mathbb{R}^d)$ to $M^{p,q}_{(\omega/\vartheta)}(\mathbb{R}^d)$.*

For the proof we need the following result, which follows from [117, Proposition 2.1] and its proof.

**Lemma 8.10.** *Assume that $s \geqslant 1$, $q_0, r_0 \in [1,\infty]$ satisfy $r_0 = 2q_0/(2q_0-1)$. Also assume that $v \in \mathscr{P}_{E,s}(\mathbb{R}^{2d})$ is submultiplicative, and that $\kappa,\kappa_0 \in \mathscr{P}_{E,s}(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d})$ satisfy*

$$\kappa_0(X_1+X_2,Y) \leqslant C\kappa(X_1,Y)\,v(Y+X_2)v(Y-X_2) \quad X_1,X_2,Y \in \mathbb{R}^{2d}, \tag{8.10}$$

*for some constant $C>0$. Then the map $(a,\phi) \mapsto \operatorname{Tp}_\phi(a)$ from $\Sigma_s(\mathbb{R}^{2d}) \times \Sigma_s(\mathbb{R}^d)$ to $\mathcal{L}(\Sigma_s(\mathbb{R}^d),\Sigma_s'(\mathbb{R}^d))$ extends uniquely to a continuous mapping from $\mathcal{M}^{\infty,q_0}_{(\omega)}(\mathbb{R}^{2d}) \times M^{r_0}_{(v)}(\mathbb{R}^d)$ to $\mathcal{L}(\Sigma_s(\mathbb{R}^d),\Sigma_s'(\mathbb{R}^d))$. Furthermore, if $\phi \in M^{r_0}_{(v)}(\mathbb{R}^d)$ and $a \in \mathcal{M}^{\infty,q_0}_{(\kappa)}(\mathbb{R}^{2d})$, then $\operatorname{Tp}_\phi(a) = \operatorname{Op}^w(b)$ for some $b \in \mathcal{M}^{\infty,1}_{(\kappa_0)}$.*

*Proof of Proposition 8.9.* We show that the conditions on the involved parameters and weight functions satisfy the conditions of Lemma 8.10.

First we observe that

$$v_j(2Y) \leqslant C v_j(Y+X_2) v_j(Y-X_2), \quad j=0,1$$

for some constant $C$ which is independent of $X_2, Y \in \mathbb{R}^{2d}$, because $v_0$ and $v_1$ are submultiplicative. Refering back to (8.9) this gives

$$\begin{aligned}
\omega_1(X_1+X_2,Y) &= \frac{v_0(2Y)^{1/2}v_1(2Y)}{\omega_0(X_1+X_2+Y)^{1/2}\omega_0(X_1+X_2-Y)^{1/2}} \\
&\leqslant C_1 \frac{v_0(2Y)^{1/2}v_1(2Y)v_0(X_2+Y)^{1/2}v_0(X_2-Y)^{1/2}}{\omega_0(X_1)} \\
&= C_1 v_1(2Y)^{1-t} \frac{v_0(2Y)^{1/2}v_1(2Y)^t v_0(X_2+Y)^{1/2}v_0(X_2-Y)^{1/2}}{\omega_0(X_1)} \\
&\leqslant C_2 v_1(2Y)^{1-t} \frac{v_1(X_2+Y)^t v_1(X_2-Y)^t v_0(X_2+Y)v_0(X_2-Y)}{\omega_0(X_1)}.
\end{aligned}$$

Hence

$$\omega_1(X_1+X_2,Y) \leqslant C \frac{v_1(2Y)^{1-t}v(X_2+Y)v(X_2-Y)}{\omega_0(X_1)}. \tag{8.11}$$

By letting $\kappa_0 = \omega_1$ and $\kappa = 1/\omega_{0,t}$, it follows that (8.11) agrees with (8.10). The result now follows from Lemma 8.10. □

Theorem 8.5′ is an immediate consequence of Theorem 8.1, Lemma 8.8 and the following proposition.

**Proposition 8.11.** *Assume that $s \geqslant 1$, $\omega_0 \in \mathscr{P}^0_{E,s}(\mathbb{R}^{2d})$ be such that $\omega_0 \in \Gamma^{(\omega_0)}_s(\mathbb{R}^{2d})$, that $v \in \mathscr{P}^0_{E,s}(\mathbb{R}^{2d})$ is submultiplicative, and that $\omega_0^{1/2}$ is $v$-moderate. If $\phi \in M^2_{(v)}(\mathbb{R}^d)$, then $\mathrm{Tp}_\phi(\omega_0) = \mathrm{Op}^w(b)$ for some $b \in \Gamma^{(\omega_0)}_s(\mathbb{R}^{2d})$.*

*Proof.* By Propositions 5.38 and 5.39, we have $\omega_0 \in \mathcal{M}^{\infty,1}_{(1/\omega_{0,r_0})}(\mathbb{R}^{2d})$ for some $r_0 \geqslant 0$, where $\omega_{0,r_0}(X,Y) = \omega_0(X)e^{-r_0|Y|^{\frac{1}{s}}}$. Furthermore, by letting $v_1(Y) = e^{r_0|Y|^{\frac{1}{s}}}$ and $v_0 = v$, with $\omega_1$ in (8.9) we have

$$\omega_1(X,Y) \gtrsim \frac{e^{r_0|2Y|^{\frac{1}{s}}}v(2Y)^{1/2}}{\omega_0(X+Y)^{1/2}\omega_0(X-Y)^{1/2}} \gtrsim \frac{e^{r_0|Y|^{\frac{1}{s}}}}{\omega_0(X)}.$$

Proposition 8.9 implies that existence of some $b \in \mathcal{M}^{\infty,1}_{(1/\omega_{0,r_0})}(\mathbb{R}^{2d}) \subseteq \Gamma^{(\omega_0)}_s(\mathbb{R}^{2d})$. □

The following generalization of Theorem 0.1 is an immediate consequence of Theorem 8.1, Lemma 8.8 and Proposition 8.11, since it follows by straightforward computations that $\mathcal{S}_s \subseteq M^2_{(v)}$ when $v$ satisfies the hypothesis in Proposition 8.11.

**Theorem 0.1′.** *Let $s \geqslant 1$, $\omega, \omega_0 \in \mathscr{P}^0_{E,s}(\mathbb{R}^{2d})$, $\boldsymbol{p} \in (0,\infty]^{2d}$, $\mathscr{B}$ be an invariant BF-space on $\mathbb{R}^{2d}$ or $\mathscr{B} = L^{\boldsymbol{p}}_E(\mathbb{R}^{2d})$ for some phase split basis $E$ of $\mathbb{R}^{2d}$, and let $\phi \in \mathcal{S}_s(\mathbb{R}^d)$. Then the Toeplitz operator $\operatorname{Tp}_\phi(\omega_0)$ is an isomorphism from $M(\omega,\mathscr{B})$ onto $M(\omega/\omega_0,\mathscr{B})$.*

For the proof of Theorem 8.6′ we need the following Gevrey version of [72, Proposition 2.11].

**Lemma 8.12.** *Let $s \geqslant 1$, $\omega_0, v_0, v_1 \in \mathscr{P}^0_{E,s}(\mathbb{R}^{2d})$ be such that $\omega_0$ is $v_0$-moderate. Set $\vartheta = \omega_0^{1/2}$, and*

$$
\begin{aligned}
\omega_1(X,Y) &= \frac{v_0(2Y)^{1/2}v_1(2Y)}{\vartheta(X+Y)\vartheta(X-Y)},\\
\omega_2(X,Y) &= \vartheta(X-Y)\vartheta(X+Y)v_1(2Y),\\
v_2(X,Y) &= v_1(2Y).
\end{aligned}
\tag{8.12}
$$

*Then*

$$
\Gamma_s^{(1/\vartheta)} \# \mathcal{M}^{\infty,1}_{(\omega_1)} \# \Gamma_s^{(1/\vartheta)} \subseteq \mathcal{M}^{\infty,1}_{(v_2)}, \tag{8.13}
$$

$$
\Gamma_s^{(1/\vartheta)} \# \mathcal{M}^{\infty,1}_{(v_2)} \# \Gamma_s^{(1/\vartheta)} \subseteq \mathcal{M}^{\infty,1}_{(\omega_2)}. \tag{8.14}
$$

*The same holds true with $\mathscr{P}_{E,s}$ and $\Gamma_{0,s}^{(1/\vartheta)}$ in place of $\mathscr{P}^0_{E,s}$ and $\Gamma_s^{(1/\vartheta)}$ respectively, at each occurrence.*

*Proof.* We shall mainly follow the proof of [72, Proposition 2.11]. Since $\Gamma_s^{(1/\vartheta)} = \bigcup_{r\geqslant 0} \mathcal{M}^{\infty,1}_{(\vartheta_r)}$ with $\vartheta_r(X,Y) = \vartheta(X)e^{r|Y|^{\frac{1}{s}}}$ (Proposition 5.38(3)), it suffices to argue with the symbol class $\mathcal{M}^{\infty,1}_{(\vartheta_r)}$ for some sufficiently large $r$ instead of $\Gamma_s^{(1/\vartheta)}$.

For suitable $r$ we show that

$$
\omega_3(X,Y) \lesssim \omega_1(X-Y+Z,Z)\vartheta_r(X+Z,Y-Z) \tag{8.15}
$$

$$
v_1(2Y) \lesssim \vartheta_r(X-Y+Z,Z)\omega_3(X+Z,Y-Z)\,, \tag{8.16}
$$

where

$$
\omega_3(X,Y) = \frac{v_1(2Y)\vartheta(X+Y)}{\omega_0(X-Y)}.
$$

Proposition 5.43 applied to (8.15) shows that $\mathcal{M}^{\infty,1}_{(\omega_1)} \# \Gamma_s^{(1/\vartheta)} \subseteq \mathcal{M}^{\infty,1}_{(\omega_3)}$, and (8.16) implies that $\Gamma_s^{(1/\vartheta)} \# \mathcal{M}^{\infty,1}_{(\omega_3)} \subseteq \mathcal{M}^{\infty,1}_{(v_2)}$, and (8.13) holds.

Since $\vartheta$ is $v_0^{1/2}$-moderate and $v_0 \in \mathscr{P}_{E,s}^0$, we have

$$\vartheta(X-Y)^{-1} \leqslant v_0(2Z)^{1/2}\vartheta(X-Y+2Z)^{-1} \quad \text{and} \quad \vartheta(X+Y) \leqslant \vartheta(X+Z)e^{r|Y-Z|^{\frac{1}{s}}}$$

for suitable $r > 0$. Using the fact that $v_1$ is an even function and that $v_1(2Y) = v_1(2Y - 2Z + 2Z) \leqslant v_1(2Z)e^{2r_0|Y-Z|^{\frac{1}{s}}}$ for every $r_0 > 0$, these give

$$\begin{aligned}\omega_3(X,Y) &\lesssim \frac{v_0(2Z)^{1/2}v_1(2Z)\vartheta(X+Z)e^{r|Y-Z|^{\frac{1}{s}}}}{\vartheta(X-Y+2Z)\vartheta(X-Y)}\\ &= \omega_1(X-Y+Z,Z)\vartheta_r(X+Z,Y-Z),\end{aligned}$$

for some $r > 0$. We also have

$$\begin{aligned}v_1(2Y) &= \frac{\vartheta(X-Y)v_1(2Y)\vartheta(X-Y)}{\vartheta(X-Y)^2} \lesssim \frac{\vartheta(X-Y)v_0(2Y)^{1/2}v_1(2Y)\vartheta(X+Y-2Y)}{\vartheta(X-Y)^2}\\ &\lesssim \frac{\vartheta(X-Y)v_0(2Y)^{1/2}v_1(2Y)\vartheta(X+Y)v_0(2Y)^{1/2}}{\vartheta(X-Y)^2}\\ &\lesssim \frac{\vartheta(X-Y+Z)e^{r|Z|^{\frac{1}{s}}}v_0(2(Y-Z))^{1/2}v_1(2(Y-Z))\vartheta(X+Y)}{\vartheta(X-Y+2Z)^2}\\ &= \vartheta_r(X-Y+Z,Z)\omega_3(X+Z,Y-Z).\end{aligned}$$

The inclusion (8.14) is proved similarly. Let

$$\omega_4(X,Y) = \vartheta(X-Y)v_1(2Y) = \vartheta(X-Y+Z-Z)v_1(2(Y-Z)+2Z)$$

be the intermediate weight. Then, the inequality

$$\begin{aligned}\omega_4(X,Y) &\lesssim \vartheta(X-Y+Z)e^{r_1|Z|^{\frac{1}{s}}}v_1(2(Y-Z))e^{2r_0|Z|^{\frac{1}{s}}}\\ &\lesssim \vartheta(X-Y+Z)e^{r|Z|^{\frac{1}{s}}}v_1(2(Y-Z))\\ &= \vartheta_r(X-Y+Z,Z)v_2(X+Z,Y-Z)\end{aligned}$$

implies that $\Gamma_s^{(1/\vartheta)}\#\mathcal{M}_{(v_2)}^{\infty,1} \subseteq \mathcal{M}_{(\omega_4)}^{\infty,1}$.

Similarly we obtain

$$\begin{aligned}\omega_2(X,Y) &\lesssim \vartheta(X-Y)v_1(2Z)\vartheta(X+Z)e^{r|Z-Y|^{\frac{1}{s}}}\\ &= \omega_4(X-Y+Z,Z)\vartheta_r(X+Z,Y-Z),\end{aligned}$$

and thus $\mathcal{M}_{(\omega_4)}^{\infty,1}\#\Gamma_s^{(1/\vartheta)} \subseteq \mathcal{M}_{(\omega_2)}^{\infty,1}$.

The case $\mathscr{P}_{E,s}$ and $\Gamma_{0,s}^{(1/\vartheta)}$ in place of $\mathscr{P}_{E,s}^0$ and $\Gamma_s^{(1/\vartheta)}$ respectively, at each occurrence, is treated in similar ways. □

*Proof of Theorem 8.6′.* First we note that the Toeplitz operator $\operatorname{Tp}_\phi(\omega_0)$ is an isomorphism from $M^2_{(\vartheta)}$ to $M^2_{(1/\vartheta)}$ in view of Lemma 8.8. With $\omega_1$ defined in (8.9), Proposition 8.9 implies that there exist $b \in \mathcal{M}^{\infty,1}_{(\omega_1)}$ and $c \in \mathcal{S}_s'(\mathbb{R}^{2d})$ such that

$$\operatorname{Tp}_\phi(\omega_0) = \operatorname{Op}^w(b) \quad \text{and} \quad \operatorname{Tp}_\phi(\omega_0)^{-1} = \operatorname{Op}^w(c)\,.$$

Let

$$\omega_2(X,Y) = \vartheta(X-Y)\vartheta(X+Y)v_1(2Y) \quad \text{and} \quad \omega_3(X,Y) = \frac{\vartheta(X+Y)}{\vartheta(X-Y)}. \tag{8.17}$$

We shall prove that $c \in \mathcal{M}^{\infty,1}_{(\omega_2)}(\mathbb{R}^{2d})$.

By Theorem 7.9, there are $a \in \Gamma^{(1/\vartheta)}_{0,s}(\mathbb{R}^{2d})$ and $a_0 \in \Gamma^{(\vartheta)}_{0,s}(\mathbb{R}^{2d})$ such that the map

$$\operatorname{Op}^w(a) \,:\, L^2(\mathbb{R}^d) \to M^2_{(\vartheta)}(\mathbb{R}^d)$$

is an isomorphism with inverse $\operatorname{Op}^w(a_0)$. By Propositions 5.38 and 5.39, $\operatorname{Op}^w(a)$ is also bijective from $M^2_{(1/\vartheta)}(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. Furthermore, by Theorem 8.2 it follows that $a \in \mathcal{M}^{\infty,1}_{(\vartheta_r)}$ when $r \geqslant 0$, where

$$\vartheta_r(X,Y) = \vartheta(X)e^{r|Y|^{\frac{1}{s}}}.$$

Let $b_0 = a\#b\#a$. From Lemma 8.12 we know that

$$b_0 \in \mathcal{M}^{\infty,1}_{(v_2)}(\mathbb{R}^{2d}), \quad \text{where} \quad v_2(X,Y) = v_1(2Y) \tag{8.18}$$

is submultiplicative and depends on $Y$ only. Since $\operatorname{Op}^w(b)$ is bijective from $M^2_{(\vartheta)}$ to $M^2_{(1/\vartheta)}$ by Lemma 8.8 (2), $\operatorname{Op}^w(b_0)$ is bijective and continuous on $L^2$.

Since $v_2$ is submultiplicative and in $\mathscr{P}_{E,s}(\mathbb{R}^{2d})$, $\mathcal{M}^{\infty,1}_{(v_2)}$ is a Wiener algebra by Proposition 5.44. Therefore, the Weyl symbol $c_0$ of the inverse to the bijective operator $\operatorname{Op}^w(b_0)$ on $L^2$ belongs to $\mathcal{M}^{\infty,1}_{(v_2)}(\mathbb{R}^{2d})$.

Since

$$\operatorname{Op}^w(c_0) = \operatorname{Op}^w(b_0)^{-1} = \operatorname{Op}^w(a)^{-1}\operatorname{Op}^w(b)^{-1}\operatorname{Op}^w(a)^{-1},$$

we find

$$\operatorname{Op}^w(c) = \operatorname{Op}^w(b)^{-1} = \operatorname{Op}^w(a)\operatorname{Op}^w(c_0)\operatorname{Op}^w(a),$$

or equivalently,

$$c = a\#c_0\#a, \quad \text{where} \quad a \in \Gamma^{(1/\vartheta)}_{0,s} \quad \text{and } c_0 \in \mathcal{M}^{\infty,1}_{(v_2)}\,. \tag{8.19}$$

The definitions of the weights are chosen such that Lemma 8.12 implies that $c \in \mathcal{M}^{\infty,1}_{(\omega_2)}$, and the assertion on $c$ follows. By Proposition 5.42, the mappings

$$\operatorname{Op}^w(b) : M^{p,q}_{(\omega\vartheta)} \to M^{p,q}_{(\omega/\vartheta)} \quad \text{and} \quad \operatorname{Op}^w(c) : M^{p,q}_{(\omega/\vartheta)} \to M^{p,q}_{(\omega\vartheta)} \tag{8.20}$$

are continuous. We have

$$\begin{aligned}
&\omega_1(X - Y + Z, Z)\omega_2(X + Z, Y - Z) \\
&= \Big(\frac{v_0(2Z)^{1/2}v_1(2Z)}{\vartheta(X - Y + 2Z)\vartheta(X - Y)}\Big) \cdot \big(\vartheta(X - Y + 2Z)\vartheta(X + Y)v_1(2(Y - Z))\big) \\
&= \frac{v_0(2Z)^{1/2}v_1(2Z)v_1(2(Y - Z))\,\vartheta(X + Y)}{\vartheta(X - Y)} \\
&\gtrsim \frac{\vartheta(X + Y)}{\vartheta(X - Y)} = \omega_3(X, Y)\,.
\end{aligned}$$

Therefore Proposition 5.43 shows that $b\#c \in \mathcal{M}^{\infty,1}_{(\omega_3)}$. Since $\operatorname{Op}^w(b)$ is an isomorphism from $M^2_{(\vartheta)}$ to $M^2_{(1/\vartheta)}$ with inverse $\operatorname{Op}^w(c)$, it follows that $b\#c = 1$ and that the constant symbol 1 belongs to $\mathcal{M}^{\infty,1}_{(\omega_3)}$. By similar arguments it follows that $c\#b = 1$. Therefore the identity operator $\operatorname{Id} = \operatorname{Op}^w(b) \circ \operatorname{Op}^w(c)$ on $M^{p,q}_{(\omega\vartheta)}$ factors through $M^{p,q}_{(\omega/\vartheta)}$, and thus $\operatorname{Op}^w(b) = \operatorname{Tp}_\phi(\omega_0)$ is an isomorphism from $M^{p,q}_{(\omega\vartheta)}$ to $M^{p,q}_{(\omega/\vartheta)}$ with inverse $\operatorname{Op}^w(c)$. This gives the result. □

## 8.3 Specific bijective pseudo-differential operators on modulation spaces

In this section we construct explicit isomorphisms between modulation spaces with different weights. Applying the results of the previous sections, these may be either in the form of pseudo-differential operators or of Toeplitz operators.

**Proposition 8.13.** *Let $s \geqslant 1$, $\omega_0 \in \mathscr{P}^0_{E,s}(\mathbb{R}^{2d})$, and let $\mathscr{B}$ be an invariant BF-space on $\mathbb{R}^{2d}$ or $\mathscr{B} = L^{\boldsymbol{p}}_E(\mathbb{R}^{2d})$ for some phase split basis $E$ of $\mathbb{R}^{2d}$. Let*

$$\Phi_\lambda(x, \xi) = Ce^{-(\lambda_1|x|^2+\lambda_2|\xi|^2)} \quad \lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2_+.$$

*Then the following statements hold true:*

*(1) $\omega_0 * \Phi_\lambda$ belongs to $\mathscr{P}^0_{E,s}(\mathbb{R}^{2d}) \cap \Gamma^{(\omega_0)}_{0,1}$ for all $\lambda \in \mathbb{R}^2_+$ and*

$$\omega_0 * \Phi_\lambda \asymp \omega_0\,.$$

(2) *If* $\lambda_1 \cdot \lambda_2 < 1$, *then there exists* $\nu \in \mathbb{R}^2_+$ *and a Gauss function* $\phi$ *on* $\mathbb{R}^d$ *such that* $\operatorname{Op}^w(\omega_0 * \Phi_\lambda) = \operatorname{Tp}_\phi(\omega_0 * \Phi_\nu)$ *is bijective from* $M(\omega, \mathscr{B})$ *to* $M(\omega/\omega_0, \mathscr{B})$ *for all* $\omega \in \mathscr{P}_{E,s}(\mathbb{R}^{2d})$.

(3) *If* $\lambda_1 \cdot \lambda_2 \leqslant 1$ *and in addition* $\omega_0 \in \Gamma_s^{(\omega_0)}(\mathbb{R}^{2d})$, *then* $\operatorname{Op}^w(\omega_0 * \Phi_\lambda) = \operatorname{Tp}_\phi(\omega_0)$ *is bijective from* $M(\omega, \mathscr{B})$ *to* $M(\omega/\omega_0, \mathscr{B})$ *for all* $\omega \in \mathscr{P}_{E,s}(\mathbb{R}^{2d})$.

The argument is similar to the proof of [72, Proposition 5.1].

*Proof.* The assertion (1) is a straightforward consequence of the definitions.

(2) Choose $\mu_j > \lambda_j$ such that $\mu_1 \cdot \mu_2 = 1$. Then $\Phi_\mu = cW(\phi, \phi)$ with $\phi(x) = e^{-\mu_1 |x|^2/2}$, and there is another Gaussian $\Phi_\nu$ such that $\Phi_\lambda = \Phi_\mu * \Phi_\nu$. Using (5.62), this factorization implies that the Weyl operator with symbol $\omega_0 * \Phi_\lambda$ is the Toeplitz operator

$$\operatorname{Op}^w(\omega_0 * \Phi_\lambda) = \operatorname{Op}^w(\omega_0 * \Phi_\nu * cW(\phi, \phi)) = c(2\pi)^{\frac{d}{2}} \operatorname{Tp}_\phi(\omega_0 * \Phi_\nu).$$

By (1) $\omega_0 * \Phi_\nu \in \mathscr{P}^0_{E,s}(\mathbb{R}^{2d}) \cap \Gamma^{(\omega_0)}_{0,1}(\mathbb{R}^{2d})$ is equivalent to $\omega_0$. Hence Theorem 8.5′ shows that $\operatorname{Op}^w(\omega_0 * \Phi_\lambda)$ is bijective from $M(\omega, \mathscr{B})$ to $M(\omega/\omega_0, \mathscr{B})$, and (2) follows.

The assertion (3) follows from (2) in the case $\lambda_1 \cdot \lambda_2 < 1$. If $\lambda_1 \cdot \lambda_2 = 1$, then $\Phi_\lambda = cW(\phi, \phi)$ for $\phi(x) = e^{-\lambda_1 |x|^2/2}$ and thus

$$\operatorname{Op}^w(\omega_0 * \Phi_\lambda) = \operatorname{Tp}^w_\phi(\omega_0)$$

is bijective from $M(\omega, \mathscr{B})$ to $M(\omega/\omega_0, \mathscr{B})$, since $\omega_0 \in \mathscr{P}^0_{E,s}(\mathbb{R}^{2d}) \cap \Gamma^{(\omega_0)}_s(\mathbb{R}^{2d})$.
□

# Chapter 9

# Characterizations of symbols via the short-time Fourier transform

The aim of the current chapter is to characterize the symbol class from the previous chapter in term of estimates of their short-time Fourier transform.

In what follows we let $\kappa$ be defined as

$$\kappa(r) = \begin{cases} 1 & \text{when } r \leqslant 1, \\ 2^{r-1} & \text{when } r > 1. \end{cases} \tag{9.1}$$

In the sequel we shall frequently use the inequality

$$|x+y|^{\frac{1}{s}} \leqslant \kappa(s^{-1})(|x|^{\frac{1}{s}} + |y|^{\frac{1}{s}}), \qquad s > 0, \quad x, y \in \mathbb{R}^d,$$

which follows by straightforward computations.

**Proposition 9.1.** *Let $s, \sigma > 0$ be such that $s + \sigma \geqslant 1$ and $(s, \sigma) \neq (\frac{1}{2}, \frac{1}{2})$, $\phi \in \Sigma_s^\sigma(\mathbb{R}^d) \backslash 0$, $r > 0$ and let $f$ be a Gelfand-Shilov distribution on $\mathbb{R}^d$. Then the following statements hold true:*

*(1) If $f \in C^\infty(\mathbb{R}^d)$ and satisfies*

$$|\partial^\alpha f(x)| \lesssim h^{|\alpha|} \alpha!^\sigma e^{r|x|^{\frac{1}{s}}} \tag{9.2}$$

*for every $h > 0$ (for some $h > 0$), then*

$$|V_\phi f(x, \xi)| \lesssim e^{\kappa(s^{-1}) r |x|^{\frac{1}{s}} - h|\xi|^{\frac{1}{\sigma}}} \tag{9.3}$$

*for every $h > 0$ (for some new $h > 0$).*

*(2) If*

$$|V_\phi f(x,\xi)| \lesssim e^{r|x|^{\frac{1}{s}}-h|\xi|^{\frac{1}{\sigma}}} \tag{9.4}$$

*for every $h>0$ (for some $h>0$), then $f \in C^\infty(\mathbb{R}^d)$ and satisfies*

$$|\partial^\alpha f(x)| \lesssim h^{|\alpha|}\alpha!^\sigma e^{\kappa(s^{-1})r|x|^{\frac{1}{s}}}$$

*for every $h>0$ (for some new $h>0$).*

*Proof.* We only prove the assertion when (9.2) or (9.4) are true for every $h>0$. The other cases follow by straightforward modifications.

Assume that (9.2) holds. Then, for every $x\in\mathbb{R}^d$ the function

$$y \mapsto F_x(y) \equiv f(y+x)\overline{\phi(y)}$$

belongs to $\Sigma_s^\sigma(\mathbb{R}^d)$, and

$$|\partial_y^\alpha F_x(y)| \lesssim h^{|\alpha|}\alpha!^\sigma e^{\kappa(s^{-1})r|x|^{\frac{1}{s}}} e^{-r_0|y|^{\frac{1}{s}}},$$

for every $h, r_0 > 0$. In particular, for $\alpha=0$, in view of Proposition 5.2, we have

$$|F_x(y)| \lesssim e^{\kappa(s^{-1})r|x|^{\frac{1}{s}}} e^{-r_0|y|^{\frac{1}{s}}} \quad \text{and} \quad |\widehat{F}_x(\xi)| \lesssim e^{\kappa(s^{-1})r|x|^{\frac{1}{s}}} e^{-r_0|\xi|^{\frac{1}{\sigma}}}, \tag{9.5}$$

for every $r_0>0$. Since $|V_\phi f(x,\xi)| = |\widehat{F}_x(\xi)|$, the estimate (9.3) follows from the second inequality in (9.5), and (1) follows.

Next we prove (2). By the inversion formula we get

$$f(x) = (2\pi)^{-\frac{d}{2}}\|\phi\|_{L^2}^{-2} \iint_{\mathbb{R}^{2d}} V_\phi f(y,\eta)\phi(x-y)e^{i\langle x,\eta\rangle}\,dyd\eta. \tag{9.6}$$

Here we notice that

$$(x,y,\eta)\mapsto V_\phi f(y,\eta)\phi(x-y)e^{i\langle x,\eta\rangle}$$

is smooth and

$$(y,\eta)\mapsto \eta^\alpha V_\phi f(y,\eta)\partial^\beta\phi(x-y)e^{i\langle x,\eta\rangle}$$

is an integrable function for every $x$, $\alpha$ and $\beta$, giving that $f$ in (9.6) is smooth. By differentiation and the fact that $\phi\in\Sigma_s^\sigma$ we get

$$\begin{aligned}
|\partial^\alpha f(x)| &\asymp \left| \sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta} i^{|\beta|} \iint_{\mathbb{R}^{2d}} \eta^\beta V_\phi f(y,\eta)(\partial^{\alpha-\beta}\phi)(x-y)e^{i\langle x,\eta\rangle}\,dyd\eta \right| \\
&\lesssim \sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta} \iint_{\mathbb{R}^{2d}} |\eta^\beta e^{r|y|^{\frac{1}{s}}} e^{-h|\eta|^{\frac{1}{\sigma}}} (\partial^{\alpha-\beta}\phi)(x-y)|\,dyd\eta \\
&\lesssim \sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta} h_2^{|\alpha-\beta|}(\alpha-\beta)!^\sigma \iint_{\mathbb{R}^{2d}} |\eta^\beta| e^{-h|\eta|^{\frac{1}{\sigma}}} e^{r|y|^{\frac{1}{s}}} e^{-h_1|x-y|^{\frac{1}{s}}}\,dyd\eta,
\end{aligned}$$

for every $h_1, h_2 > 0$. Using Stirling's approximation we get

$$|\eta^\beta e^{-h|\eta|^{\frac{1}{\sigma}}}| \lesssim h_2^{|\beta|}(\beta!)^\sigma e^{-\frac{h}{2}\cdot|\eta|^{\frac{1}{\sigma}}}, \tag{9.7}$$

we get

$$\begin{aligned}|\partial^\alpha f(x)| \\ &\lesssim h_2^{|\alpha|}\sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}(\beta!(\alpha-\beta)!)^\sigma \iint\limits_{\mathbb{R}^{2d}} e^{-\frac{h}{2}\cdot|\eta|^{\frac{1}{\sigma}}}e^{r|y|^{\frac{1}{s}}}e^{-h_1|x-y|^{\frac{1}{s}}}\,dyd\eta \\ &\lesssim (4h_2)^{|\alpha|}\alpha!^\sigma\int_{\mathbb{R}^d} e^{r|y|^{\frac{1}{s}}}e^{-h_1|x-y|^{\frac{1}{s}}}\,dy. \end{aligned}\tag{9.8}$$

Since $|y|^{\frac{1}{s}} \leqslant \kappa(s^{-1})(|x|^{\frac{1}{s}} + |y-x|^{\frac{1}{s}})$ and $h_1$ can be chosen arbitrarily large, it follows from the last estimate that

$$|\partial^\alpha f(x)| \lesssim (4h_2)^{|\alpha|}\alpha!^\sigma e^{r\kappa(s^{-1})|x|^{\frac{1}{s}}},$$

for every $h_2 > 0$. This gives the result. □

By similar arguments we get also the following result.

**Proposition 9.1′.** *Let $s_j, \sigma_j > 0$ be such that $s_j + \sigma_j \geqslant 1$ and $(s_j, \sigma_j) \neq (\frac{1}{2}, \frac{1}{2})$, $j = 1, 2$, $\phi \in \Sigma_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbb{R}^{d_1+d_2})\backslash 0$, $r > 0$ and let $f$ be a Gelfand-Shilov distribution on $\mathbb{R}^{d_1+d_2}$. Then the following statements hold true:*

*(1) If $f \in C^\infty(\mathbb{R}^{d_1+d_2})$ and satisfies*

$$|\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2} f(x_1, x_2)| \lesssim h^{|\alpha_1+\alpha_2|}\alpha_1!^{\sigma_1}\alpha_2!^{\sigma_2}e^{r(|x_1|^{\frac{1}{s_1}}+|x_2|^{\frac{1}{s_2}})} \tag{9.2$'$}$$

*for every $h > 0$ (resp. for some $h > 0$), then*

$$|V_\phi f(x_1, x_2, \xi_1, \xi_2)| \lesssim e^{\kappa(s_1^{-1})r|x_1|^{\frac{1}{s_1}}+\kappa(s_2^{-1})r|x_2|^{\frac{1}{s_2}}-h(|\xi_1|^{\frac{1}{\sigma_1}}+|\xi_2|^{\frac{1}{\sigma_2}})} \tag{9.3$'$}$$

*for every $h > 0$ (resp. for some new $h > 0$).*

*(2) If*

$$|V_\phi f(x_1, x_2, \xi_1, \xi_2)| \lesssim e^{r(|x_1|^{\frac{1}{s_1}}+|x_2|^{\frac{1}{s_2}})-h(|\xi_1|^{\frac{1}{\sigma_1}}+|\xi_2|^{\frac{1}{\sigma_2}})} \tag{9.4$'$}$$

*for every $h > 0$ (resp. for some $h > 0$), then $f \in C^\infty(\mathbb{R}^{d_1+d_2})$ and satisfies*

$$|\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2} f(x_1, x_2)| \lesssim h^{|\alpha_1+\alpha_2|}\alpha_1!^{\sigma_1}\alpha_2!^{\sigma_2}e^{\kappa(s_1^{-1})r|x_1|^{\frac{1}{s_1}}+\kappa(s_2^{-1})r|x_2|^{\frac{1}{s_2}}},$$

*for every $h > 0$ (resp. for some new $h > 0$).*

As a consequence of the previous result we get the following.

**Proposition 9.2.** *Let $s_j, \sigma_j > 0$ be such that $s_j + \sigma_j \geqslant 1$ and $(s_j, \sigma_j) \neq (\frac{1}{2}, \frac{1}{2})$, $j = 1, 2$, $\phi \in \Sigma_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbb{R}^{d_1+d_2})\backslash 0$ and let $f$ be a Gelfand-Shilov distribution on $\mathbb{R}^{d_1+d_2}$. Then the following properties hold true:*

*(1) There exist $h, r > 0$ such that $(9.4)'$ holds true, if and only if $f \in \Gamma_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbb{R}^{d_1+d_2})$.*

*(2) There exists $r > 0$ such that $(9.4)'$ holds true for every $h > 0$, if and only if $f \in \Gamma_{s_1,s_2}^{\sigma_1,\sigma_2;0}(\mathbb{R}^{d_1+d_2})$.*

*(3) $(9.4)'$ holds for every $h, r > 0$, if and only if $f \in \Gamma_{s_1,s_2;0}^{\sigma_1,\sigma_2;0}(\mathbb{R}^{d_1+d_2})$.*

By similar arguments that led to Proposition 9.2 we also get the following.

**Proposition 9.3.** *Let $s_j, \sigma_j > 0$ be such that $s_j + \sigma_j \geqslant 1$, $j = 1, 2$, $\phi \in \mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbb{R}^{d_1+d_2})\backslash 0$ and let $f$ be a Gelfand-Shilov distribution on $\mathbb{R}^{d_1+d_2}$. Then there exists $h > 0$ such that $(9.4)'$ holds true for every $r > 0$, if and only if $f \in \Gamma_{s_1,s_2;0}^{\sigma_1,\sigma_2}(\mathbb{R}^{d_1+d_2})$.*

We also have the following version of Proposition 9.1′, involving certain types of moderate weights.

**Proposition 9.4.** *Let $s, \sigma > 0$ be such that $s + \sigma \geqslant 1$, $\phi \in \mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})\backslash 0$ ($\phi \in \Sigma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})\backslash 0$), $r > 0$, $\omega \in \mathscr{P}_{s,\sigma}^{0}(\mathbb{R}^{2d})$ ($\omega \in \mathscr{P}_{s,\sigma}(\mathbb{R}^{2d})$) and let $a$ be a Gelfand-Shilov distribution on $\mathbb{R}^{2d}$. Then the following statements hold true:*

*(1) If $a \in C^\infty(\mathbb{R}^{2d})$ and satisfies*

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \lesssim h^{|\alpha+\beta|}\alpha!^\sigma \beta!^s \omega(x,\xi), \tag{9.9}$$

*for some $h > 0$ (for every $h > 0$), then*

$$|V_\phi a(x,\xi,\eta,y)| \lesssim \omega(x,\xi)e^{-r(|\eta|^{\frac{1}{\sigma}}+|y|^{\frac{1}{s}})}, \tag{9.10}$$

*for some $r > 0$ (for every $r > 0$).*

*(2) If (9.10) holds true for every $r > 0$ (for some $r > 0$), then $a \in C^\infty(\mathbb{R}^{2d})$ and (9.9) holds true for some $h > 0$ (for every $h > 0$).*

*Proof.* We shall use similar arguments as in the proof of Proposition 9.1. Let $X = (x,\xi) \in \mathbb{R}^{2d}$, $Y = (y,\eta) \in \mathbb{R}^{2d}$, $Z = (z,\zeta) \in \mathbb{R}^{2d}$ and let $\phi \in \Sigma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})\backslash 0$. Suppose that $\omega \in \mathscr{P}_{s,\sigma}(\mathbb{R}^{2d})$ and that (9.9) holds for all $h > 0$. If

$$F_X(Y) \equiv \frac{a(Y+X)\overline{\phi(Y)}}{\omega(X)},$$

then, the fact that $\omega(X) \lesssim \omega(Y+X)e^{r_0(|y|^{\frac{1}{s}}+|\eta|^{\frac{1}{\sigma}})}$ gives that

$$\{\, Y \mapsto F_X(Y)\,;\, X \in \mathbb{R}^{2d}\,\}$$

is a bounded set in $\Sigma_{s,\sigma}^{\sigma,s}$. Hence

$$|\partial_y^\alpha \partial_\eta^\beta F_X(y,\eta)| \lesssim h^{|\alpha+\beta|}\alpha!^\sigma \beta!^s e^{-r(|y|^{\frac{1}{s}}+|\eta|^{\frac{1}{\sigma}})},$$

for every $h, r > 0$. In particular,

$$\begin{aligned} |F_X(y,\eta) &\lesssim e^{-r(|y|^{\frac{1}{s}}+|\eta|^{\frac{1}{\sigma}})} \\ \text{and} \qquad |(\mathscr{F} F_X)(\zeta,z)| &\lesssim e^{-r(|z|^{\frac{1}{s}}+|\zeta|^{\frac{1}{\sigma}})}, \end{aligned} \tag{9.11}$$

for every $r > 0$. Since

$$|V_\phi a(x,\xi,\eta,y)| = |(\mathscr{F} F_X)(\eta,y)\omega(X)|,$$

it follows that (9.10) holds true for all $r > 0$. This gives (1) in the case when $\omega \in \mathscr{P}_{s,\sigma}(\mathbb{R}^{2d})$ and $\phi \in \Sigma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})\backslash 0$. In the same way, (1) follows in the case when $\omega \in \mathscr{P}_{s,\sigma}^0(\mathbb{R}^{2d})$ and $\phi \in \mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})\backslash 0$.

Next we prove (2) in the case when $\omega \in \mathscr{P}_{s,\sigma}$ and $\phi \in \Sigma_{s,\sigma}^{\sigma,s}$. Therefore, suppose (9.10) holds for all $r > 0$. Then $a$ is smooth in view of Proposition 9.1′. By differentiation, (9.6), the fact that, $Z = (z,\zeta)$,

$$\omega(Z) \lesssim \omega(X)e^{r_0(|x-z|^{\frac{1}{s}}+|\xi-\zeta|^{\frac{1}{\sigma}})}$$

for some $r_0 > 0$, and the fact that $\phi \in \Sigma_{s,\sigma}^{\sigma,s}$ we get

$$\begin{aligned} &|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \\ &\quad \lesssim \sum_{\substack{\gamma \leqslant \alpha \\ \delta \leqslant \beta}} \binom{\alpha}{\gamma}\binom{\beta}{\delta} \iint_{\mathbb{R}^{4d}} |\eta^\gamma y^\delta V_\phi a(z,\zeta,\eta,y)(\partial_x^{\alpha-\gamma}\partial_\xi^{\beta-\delta}\phi)(X-Z)|\, dY\, dZ \\ &\qquad\qquad \lesssim \sum_{\substack{\gamma \leqslant \alpha \\ \delta \leqslant \beta}} \binom{\alpha}{\gamma}\binom{\beta}{\delta} h^{|\alpha+\beta-\gamma-\delta|}(\alpha-\gamma)!^\sigma(\beta-\delta)!^s I_{\gamma,\delta}(X), \end{aligned}$$

where

$$
\begin{aligned}
I_{\gamma,\delta}(X) &= \iint\limits_{\mathbb{R}^{4d}} \omega(Z)|\eta^\gamma y^\delta e^{-(r+r_0)(|x-z|^{\frac{1}{s}}+|y|^{\frac{1}{s}}+|\xi-\zeta|^{\frac{1}{\sigma}}+|\eta|^{\frac{1}{\sigma}})}\, dY dZ \\
&\lesssim \omega(X) \iint\limits_{\mathbb{R}^{4d}} |\eta^\gamma y^\delta e^{-r(|z|^{\frac{1}{s}}+|y|^{\frac{1}{s}}+|\zeta|^{\frac{1}{\sigma}}+|\eta|^{\frac{1}{\sigma}})}\, dY dZ \\
&\lesssim h^{|\gamma+\delta|}\gamma!^\sigma \delta!^s \omega(X) \iint\limits_{\mathbb{R}^{4d}} e^{-\frac{r}{2}(|z|^{\frac{1}{s}}+|y|^{\frac{1}{s}}+|\zeta|^{\frac{1}{\sigma}}+|\eta|^{\frac{1}{\sigma}})}\, dY dZ \\
&\asymp h^{|\gamma+\delta|}\gamma!^\sigma \delta!^s \omega(X)
\end{aligned}
$$

for every $h, r > 0$. Here the last inequality follows from (9.7). It follows that (9.9) holds true for every $h > 0$, by using the estimates above and similar computations as in (9.8).

The remaining case follows by similar arguments. □

The following result is a straightforward consequence of Proposition 9.4 and the definitions.

**Proposition 9.5.** *Let $R > 0$, $q \in (0, \infty]$, $s, \sigma > 0$ be such that $s + \sigma \geqslant 1$ and $(s, \sigma) \neq (\frac{1}{2}, \frac{1}{2})$, $\phi \in \Sigma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})\backslash 0$, $\omega \in \mathscr{P}_{s,\sigma}(\mathbb{R}^{2d})$, and let*

$$
\omega_R(x,\xi,\eta,y) = \omega(x,\xi)e^{-R(|y|^{\frac{1}{s}}+|\eta|^{\frac{1}{\sigma}})}.
$$

*Then,*

$$
\begin{aligned}
\Gamma_{(\omega)}^{\sigma,s}(\mathbb{R}^{2d}) &= \bigcup_{R>0} \{\, a \in (\Sigma_{s,\sigma}^{\sigma,s})'(\mathbb{R}^{2d})\, ;\, \|\omega_R^{-1} V_\phi a\|_{L^{\infty,q}} < \infty \,\}, \\
\Gamma_{(\omega)}^{\sigma,s;0}(\mathbb{R}^{2d}) &= \bigcap_{R>0} \{\, a \in (\Sigma_{s,\sigma}^{\sigma,s})'(\mathbb{R}^{2d})\, ;\, \|\omega_R^{-1} V_\phi a\|_{L^{\infty,q}} < \infty \,\}.
\end{aligned}
\tag{9.12}
$$

# Chapter 10

# Invariance, continuity and algebraic properties for pseudo-differential operators with Gelfand-Shilov symbols

In this chapter we deduce invariance, continuity and composition properties for pseudo-differential operators with symbols in the classes considered in the previous Chapters 5 and 9. In the first part we show that for any such class $S$, the set $\operatorname{Op}_A(S)$ of pseudo-differential operators is independent of the matrix $A$. Thereafter we deduce that such operators are continuous on Gelfand-Shilov spaces and their duals. In the last part we deduce that these operator classes are closed under compositions.

## 10.1 Invariance properties

An essential part of the study of invariance properties concerns the operator $e^{i\langle AD_\xi, D_x\rangle}$ when acting on the symbol classes introduced in chapter 5.

**Theorem 10.1.** *Let* $s, s_1, s_2, \sigma, \sigma_1, \sigma_2 > 0$ *be such that*

$$s + \sigma \geqslant 1, \quad s_1 + \sigma_1 \geqslant 1, \quad s_2 + \sigma_2 \geqslant 1, \quad s_2 \leqslant s_1 \quad \text{and} \quad \sigma_1 \leqslant \sigma_2,$$

*and let* $A \in \mathbf{M}(d, \mathbb{R})$. *Then the following statements hold true:*

*(1)* $e^{i\langle AD_\xi, D_x\rangle}$ *on* $\mathscr{S}(\mathbb{R}^{2d})$ *restricts to a homeomorphism on* $\mathcal{S}_{s_1,\sigma_2}^{\sigma_1,s_2}(\mathbb{R}^{2d})$, *and extends uniquely to a homeomorphism on* $(\mathcal{S}_{s_1,\sigma_2}^{\sigma_1,s_2})'(\mathbb{R}^{2d})$.

*(2) If in addition* $(s_1, \sigma_1) \neq (\frac{1}{2}, \frac{1}{2})$ *and* $(s_2, \sigma_2) \neq (\frac{1}{2}, \frac{1}{2})$, *then* $e^{i\langle AD_\xi, D_x\rangle}$ *restricts to a homeomorphism on* $\Sigma_{s_1,\sigma_2}^{\sigma_1,s_2}(\mathbb{R}^{2d})$, *and extends uniquely to a homeomorphism on* $(\Sigma_{s_1,\sigma_2}^{\sigma_1,s_2})'(\mathbb{R}^{2d})$.

*(3)* $e^{i\langle AD_\xi, D_x\rangle}$ *is a homeomorphism on* $\Gamma^{\sigma,s}_{s,\sigma;0}(\mathbb{R}^{2d})$.

*(4) If in addition* $(s,\sigma) \neq (\frac{1}{2},\frac{1}{2})$, *then* $e^{i\langle AD_\xi, D_x\rangle}$ *is a homeomorphism on* $\Gamma^{\sigma,s;0}_{s,\sigma}(\mathbb{R}^{2d})$ *and on* $\Gamma^{\sigma,s;0}_{s,\sigma;0}(\mathbb{R}^{2d})$.

The assertion (1) in the previous theorem is proved in [24] and is essentially a special case of Theorem 32 in [129], whereas (2) can be found in [24, 25]. Thus we only need to prove (3) and (4) in the previous theorem, which are extensions of [24, Theorem 4.6 (3)].

*Proof.* Let $\phi \in \mathcal{S}^{\sigma,s}_{s,\sigma}(\mathbb{R}^{2d})$ and $\phi_A = e^{i\langle AD_\xi, D_x\rangle}\phi$. Then $\phi_A \in \mathcal{S}^{\sigma,s}_{s,\sigma}(\mathbb{R}^{2d})$, in view of (1), and

$$|(V_{\phi_A}(e^{i\langle AD_\xi, D_x\rangle}a))(x,\xi,\eta,y)| = |(V_\phi a)(x-Ay,\xi-A^*\eta,\eta,y)| \tag{10.1}$$

by straightforward computations, where $A^*$ denotes the transpose of $A$. Then $a \in \Gamma^{\sigma,s}_{s,\sigma;0}(\mathbb{R}^{2d})$ is equivalent to the property for some $h>0$,

$$|V_\phi a(x,\xi,\eta,y)| \lesssim e^{r(|x|^{\frac{1}{s}}+|\xi|^{\frac{1}{\sigma}})-h(|\eta|^{\frac{1}{\sigma}}+|y|^{\frac{1}{s}})},$$

holds true for every $r>0$, in view of Proposition 9.3. By (10.1) and (1) it follows, by straightforward computation, that the latter condition is invariant under the mapping $e^{i\langle AD_\xi, D_x\rangle}$, and (3) follows from these invariance properties. By similar arguments, taking $\phi \in \Sigma^{\sigma,s}_{s,\sigma}(\mathbb{R}^{2d})$ and using (2) instead of (1), we deduce (4). □

**Corollary 10.2.** *Let* $s,\sigma>0$ *be such that* $s+\sigma \geqslant 1$ *and* $\sigma \leqslant s$. *Then* $e^{i\langle AD_\xi, D_x\rangle}$ *is a homeomorphism on* $\mathcal{S}^\sigma_s(\mathbb{R}^{2d})$, $\Sigma^\sigma_s(\mathbb{R}^{2d})$, $(\mathcal{S}^\sigma_s)'(\mathbb{R}^{2d})$ *and on* $(\Sigma^\sigma_s)'(\mathbb{R}^{2d})$.

We also have the following extension of (4) in [24, Theorem 4.1].

**Theorem 10.3.** *Let* $\omega \in \mathscr{P}_{s,\sigma}(\mathbb{R}^{2d})$, $s,\sigma>0$ *be such that* $s+\sigma\geqslant 1$. *Then* $a \in \Gamma^{\sigma,s;0}_{(\omega)}(\mathbb{R}^{2d})$ *if and only if* $e^{i\langle AD_\xi, D_x\rangle}a \in \Gamma^{\sigma,s;0}_{(\omega)}(\mathbb{R}^{2d})$.

We need some preparation for the proof and start with the following proposition.

**Proposition 10.4.** *Let* $s,\sigma>0$ *be such that* $s+\sigma\geqslant 1$ *and* $(s,\sigma)\neq(\frac{1}{2},\frac{1}{2})$, $\phi \in \Sigma^{\sigma,s}_{s,\sigma}(\mathbb{R}^{2d})\backslash 0$, $\omega \in \mathscr{P}_{s,\sigma}(\mathbb{R}^{2d})$ *and let a be a Gelfand-Shilov distribution on* $\mathbb{R}^{2d}$. *Then, the following conditions are equivalent:*

*(1)* $a \in \Gamma^{\sigma,s;0}_{(\omega)}(\mathbb{R}^{2d})$.

*(2) For every* $\alpha,\beta \in \mathbb{Z}^d_+$, $h>0$, $R>0$ *and* $x,y,\xi,\eta$ *in* $\mathbb{R}^d$ *it holds*

$$\left|\partial^\alpha_x\partial^\beta_\xi\left(e^{i(\langle x,\eta\rangle+\langle y,\xi\rangle}V_\phi a(x,\xi,\eta,y)\right)\right| \lesssim h^{|\alpha+\beta|}\alpha!^\sigma\beta!^s\omega(x,\xi)e^{-R(|y|^{\frac{1}{s}}+|\eta|^{\frac{1}{\sigma}})}. \tag{10.2}$$

*(3) For $\alpha = \beta = 0$, (10.2) holds for every $h > 0$, $R > 0$ and $x, y, \xi, \eta \in \mathbb{R}^d$.*

*Proof.* Obviously, (2) implies (3). Assume now that (1) holds true. Let

$$F_a(x, \xi, y, \eta) = a(x + y, \xi + \eta)\phi(y, \eta).$$

By straightforward application of Leibniz rule in combination with (5.6) we obtain

$$|\partial_x^\alpha \partial_\xi^\beta F_a(x, \xi, y, \eta)| \lesssim h^{|\alpha+\beta|}\alpha!^\sigma \beta!^s \omega(x, \xi) e^{-R(|y|^{\frac{1}{s}} + |\eta|^{\frac{1}{\sigma}})}$$

for every $h > 0$ and $R > 0$. Hence, if

$$G_{a,h,x,\xi}(y, \eta) = \frac{\partial_x^\alpha \partial_\xi^\beta F_a(x, \xi, y, \eta)}{h^{|\alpha+\beta|}\alpha!^\sigma \beta!^s \omega(x, \xi)},$$

then using Leibniz formula and the properties of $a$ and $\phi$ it follows that $\partial_y^\delta \partial_\eta^\rho G_{a,h,x,\xi}(y, \eta)$ is bounded by $h_1^{|\delta+\rho|}\delta!^\sigma \rho!^s e^{-r(|y|^{\frac{1}{s}} + |\eta|^{\frac{1}{\sigma}})}$ for every $h, h_1, r > 0$, $\delta, \rho \in \mathbb{Z}_+^d$ and $x, \xi \in \mathbb{R}^d$. Thus $\{\, G_{a,h,x,\xi}\, ;\, x, \xi \in \mathbb{R}^d \,\}$ is a bounded set in $\Sigma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})$ for every fixed $h > 0$. Let $\mathscr{F}_2 F_a$ be the partial Fourier transform of $F_a(x, \xi, y, \eta)$ with respect to the $(y, \eta)$-variable. Then, in view of Proposition 5.2, we get

$$|\partial_x^\alpha \partial_\xi^\beta (\mathscr{F}_2 F_a)(x, \xi, \zeta, z)| \lesssim h^{|\alpha+\beta|}\alpha!^\sigma \beta!^s \omega(x, \xi) e^{-R(|z|^{\frac{1}{s}} + |\zeta|^{\frac{1}{\sigma}})},$$

for every $h > 0$ and $R > 0$. This is the same as (2).

It remains to prove that (3) implies (1), but this follows by similar arguments as in the proof of Proposition 9.1. □

**Proposition 10.5.** *Let $R > 0$, $q \in [1, \infty]$, $s, \sigma > 0$ be such that $s + \sigma \geqslant 1$ and $(s, \sigma) \neq (\frac{1}{2}, \frac{1}{2})$, $\phi \in \Sigma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})\backslash 0$, $\omega \in \mathscr{P}_{s,\sigma}(\mathbb{R}^{2d})$, and let*

$$\omega_R(x, \xi, \eta, y) = \omega(x, \xi) e^{-R(|y|^{\frac{1}{s}} + |\eta|^{\frac{1}{\sigma}})}.$$

*Then*

$$\Gamma_{(\omega)}^{\sigma,s;0}(\mathbb{R}^{2d}) = \bigcap_{R>0} \{\, a \in (\Sigma_{s,\sigma}^{\sigma,s})'(\mathbb{R}^{2d})\, ;\, \|\omega_R^{-1} V_\phi a\|_{L^{\infty,q}} < \infty \,\}. \tag{10.3}$$

*Proof.* Let $\phi_0 \in \Sigma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})\backslash 0$, $a \in (\Sigma_{s,\sigma}^{\sigma,s})'(\mathbb{R}^{2d})$, and set

$$F_{0,a}(X, Y) = |(V_{\phi_0} a)(x, \xi, \eta, y)|, \quad F_a(X, Y) = |(V_\phi a)(x, \xi, \eta, y)|$$
$$\text{and} \quad G(x, \xi, \eta, y) = |(V_\phi \phi_0)(x, \xi, \eta, y)|,$$

where $X = (x, \xi)$ and $Y = (y, \eta)$. Proposition 5.9 and the proof of (2) in Proposition 10.4 give that $V_\phi \phi_0 \in \Sigma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{4d})$, then we have

$$0 \leqslant G(x, \xi, \eta, y) \lesssim e^{-R(|x|^{\frac{1}{s}} + |\xi|^{\frac{1}{\sigma}} + |\eta|^{\frac{1}{\sigma}} + |y|^{\frac{1}{s}})} \quad \text{for every} \quad R > 0. \tag{10.4}$$

By [67, Lemma 11.3.3], we have $F_a \lesssim F_{0,a} * G$. We obtain

$$
\begin{aligned}
(\omega_R^{-1} \cdot F_a)(X,Y)
&\lesssim \omega(X)^{-1} e^{R(|y|^{\frac{1}{s}} + |\eta|^{\frac{1}{\sigma}})} \iint\limits_{\mathbb{R}^{4d}} F_{0,a}(X - X_1, Y - Y_1) G(X_1, Y_1)\, dX_1 dY_1 \\
&\lesssim \iint\limits_{\mathbb{R}^{4d}} (\omega_{cR}^{-1} \cdot F_{0,a})(X - X_1, Y - Y_1)(\tilde{\omega}_{cR,\kappa} \cdot G)(X_1, Y_1)\, dX_1 dY_1 \\
&\lesssim \iint\limits_{\mathbb{R}^{4d}} (\omega_{cR}^{-1} \cdot F_{0,a})(X - X_1, Y - Y_1) G_1(X_1, Y_1)\, dX_1 dY_1 \qquad (10.5)
\end{aligned}
$$

where $\tilde{\omega}_{cR,\kappa}(X_1, Y_1) = e^{cR(\kappa(s)|x_1|^{\frac{1}{s}} + \kappa(\sigma)|\xi_1|^{\frac{1}{\sigma}} + |y_1|^{\frac{1}{s}} + |\eta_1|^{\frac{1}{\sigma}})}$, thus $G_1$ satisfying (10.4) for some $c > 0$ independent of $R$. By applying the $L^\infty$-norm on the last inequality we get

$$
\begin{aligned}
&\|\omega_R^{-1} F_a\|_{L^\infty(\mathbb{R}^{4d})} \\
&\quad\lesssim \sup_Y \left( \iint\limits_{\mathbb{R}^{4d}} \big( \sup (\omega_{cR}^{-1} \cdot F_{0,a})(\cdot, Y - Y_1) \big) G_1(X_1, Y_1)\, dX_1 dY_1 \right) \\
&\quad\leqslant \sup_Y \big( \|(\omega_{cR}^{-1} \cdot F_{0,a})(\cdot - (0,Y)\|_{L^{\infty,q}} \big) \|G_1\|_{L^{1,q'}} \asymp \|\omega_{cR}^{-1} \cdot F_{0,a}\|_{L^{\infty,q}}.
\end{aligned}
$$

We only consider the case $q < \infty$ when proving the opposite inequality. The case $q = \infty$ follows by similar arguments. By (10.5) we have

$$
\|\omega_R^{-1} \cdot F_a\|_{L^{\infty,q}}^q \lesssim \int_{\mathbb{R}^{2d}} (\sup H(\,\cdot\,, Y)^q)\; dY,
$$

where $H = K_1 * G_1$ and $K_j = \omega_{jcR}^{-1} \cdot F_{0,a}$, $j \geqslant 1$. Let $Y_1 = (y_1, \eta_1)$ be new variables of integration. Then Minkowski's inequality gives

$$
\begin{aligned}
&\sup_X H(X,Y) \\
&\quad\lesssim \iint\limits_{\mathbb{R}^{4d}} \big( \sup K_2(\,\cdot\,, Y - Y_1) \big) e^{-cR(|y-y_1|^{\frac{1}{s}} + |\eta-\eta_1|^{\frac{1}{\sigma}})} G_1(X_1, Y_1)\, dX_1 dY_1 \\
&\quad\lesssim \|K_2\|_{L^\infty} \iint\limits_{\mathbb{R}^{4d}} e^{-cR(|y-y_1|^{\frac{1}{s}} + |\eta-\eta_1|^{\frac{1}{\sigma}})} G_1(X_1, Y_1)\, dX_1 dY_1.
\end{aligned}
$$

By combining these estimates we get

$$\|\omega_R^{-1}\cdot F_a\|_{L^{\infty,q}}^q$$
$$\lesssim \|K_2\|_{L^\infty}^q \int_{\mathbb{R}^{2d}} \left( \iint_{\mathbb{R}^{4d}} e^{-cR(|y-y_1|^{\frac{1}{s}}+|\eta-\eta_1|^{\frac{1}{\sigma}})} G_1(X_1,Y_1)\,dX_1dY_1 \right) dY$$
$$\asymp \|K_2\|_{L^\infty}^q.$$

That is,

$$\|\omega_R^{-1}\cdot F_a\|_{L^{\infty,q}} \lesssim \|\omega_{2cR}^{-1}\cdot F_{0,a}\|_{L^\infty},$$

and the result follows. □

*Proof of Theorem 10.3.* The case $s=\sigma=\frac{1}{2}$ follows from [24, Theorem 4.1]. We may therefore assume that $(s,\sigma)\neq(\frac{1}{2},\frac{1}{2})$. Let $\phi\in\Sigma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})$ and $\phi_A=e^{i\langle AD_\xi,D_x\rangle}\phi$. Then $\phi_A\in\Sigma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})$, in view of (2) in Theorem 10.1.

Also let

$$\omega_{A,R}(x,\xi,\eta,y)=\omega(x-Ay,\xi-A^*\eta)e^{-R(|y|^{\frac{1}{s}}+|\eta|^{\frac{1}{\sigma}})}.$$

By straightforward applications of Parseval's formula, we get

$$|(V_{\phi_A}(e^{i\langle AD_\xi,D_x\rangle}a))(x,\xi,\eta,y)|=|(V_\phi a)(x-Ay,\xi-A^*\eta,\eta,y)|$$

(cf. Proposition 1.7 in [116] and its proof). This gives

$$\|\omega_{0,R}^{-1}V_\phi a\|_{L^{p,q}}=\|\omega_{A,R}^{-1}V_{\phi_A}(e^{i\langle AD_\xi,D_x\rangle}a)\|_{L^{p,q}}.$$

Hence Proposition 10.5 gives

$$\begin{aligned}
a\in\Gamma_{(\omega)}^{\sigma,s;0}(\mathbb{R}^{2d}) \quad &\Leftrightarrow \quad \|\omega_{0,R}^{-1}V_\phi a\|_{L^\infty}<\infty \quad \text{for every } R>0\\
&\Leftrightarrow \quad \|\omega_{A,R}^{-1}V_{\phi_A}(e^{i\langle AD_\xi,D_x\rangle}a)\|_{L^\infty}<\infty \quad \text{for every } R>0\\
&\Leftrightarrow \quad \|\omega_{0,R}^{-1}V_{\phi_A}(e^{i\langle AD_\xi,D_x\rangle}a)\|_{L^\infty}<\infty \quad \text{for every } R>0\\
&\Leftrightarrow \quad e^{i\langle AD_\xi,D_x\rangle}a\in\Gamma_{(\omega)}^{\sigma,s;0}(\mathbb{R}^{2d}),
\end{aligned}$$

and the result follows in this case. Here the third equivalence follows from the fact that

$$\omega_{0,R+c}\lesssim\omega_{t,R}\lesssim\omega_{0,R-c},$$

for some $c>0$.

□

We note that if $A, B \in \mathbf{M}(d,\mathbb{R})$ and $a,b \in (\mathcal{S}_{s,\sigma}^{\sigma,s})'(\mathbb{R}^{2d})$ or $a,b \in (\Sigma_{s,\sigma}^{\sigma,s})'(\mathbb{R}^{2d})$, then the first part of the previous proof shows that

$$\operatorname{Op}_A(a) = \operatorname{Op}_B(b) \quad \Leftrightarrow \quad e^{i\langle AD_\xi, D_x\rangle} a = e^{i\langle BD_\xi, D_x\rangle} b. \tag{10.6}$$

The following result follows from Theorems 10.1 and 10.3.

**Theorem 10.6.** *Let $s, s_1, s_2, \sigma, \sigma_1, \sigma_2 > 0$ be such that*

$$s+\sigma \geqslant 1, \quad s_1+\sigma_1 \geqslant 1, \quad s_2+\sigma_2 \geqslant 1, \quad s_2 \leqslant s_1 \quad \text{and} \quad \sigma_1 \leqslant \sigma_2,$$

*$A, B \in \mathbf{M}(d,\mathbb{R})$, $\omega \in \mathscr{P}_{s,\sigma}(\mathbb{R}^{2d})$, and let $a$ and $b$ be Gelfand-Shilov distributions such that $\operatorname{Op}_A(a) = \operatorname{Op}_B(b)$. Then the following statements hold true:*

*(1) $a \in \mathcal{S}_{s_1,\sigma_2}^{\sigma_1,s_2}(\mathbb{R}^{2d})$ (resp. $a \in (\mathcal{S}_{s_1,\sigma_2}^{\sigma_1,s_2})'(\mathbb{R}^{2d})$) if and only if $b \in \mathcal{S}_{s_1,\sigma_2}^{\sigma_1,s_2}(\mathbb{R}^{2d})$ (resp. $b \in (\mathcal{S}_{s_1,\sigma_2}^{\sigma_1,s_2})'(\mathbb{R}^{2d})$).*

*(2) $a \in \Sigma_{s_1,\sigma_2}^{\sigma_1,s_2}(\mathbb{R}^{2d})$ (resp. $a \in (\Sigma_{s_1,\sigma_2}^{\sigma_1,s_2})'(\mathbb{R}^{2d})$) if and only if $b \in \Sigma_{s_1,\sigma_2}^{\sigma_1,s_2}(\mathbb{R}^{2d})$ (resp. $b \in (\Sigma_{s_1,\sigma_2}^{\sigma_1,s_2})'(\mathbb{R}^{2d})$).*

*(3) $a \in \Gamma_{s,\sigma;0}^{\sigma,s}(\mathbb{R}^{2d})$ if and only if $b \in \Gamma_{s,\sigma;0}^{\sigma,s}(\mathbb{R}^{2d})$. If in addition $(s,\sigma) \neq (\frac{1}{2},\frac{1}{2})$, then $a \in \Gamma_{s,\sigma}^{\sigma,s;0}(\mathbb{R}^{2d})$ if and only if $b \in \Gamma_{s,\sigma}^{\sigma,s;0}(\mathbb{R}^{2d})$, and $a \in \Gamma_{s,\sigma;0}^{\sigma,s;0}(\mathbb{R}^{2d})$ if and only if $b \in \Gamma_{s,\sigma;0}^{\sigma,s;0}(\mathbb{R}^{2d})$.*

*(4) $a \in \Gamma_{(\omega)}^{\sigma,s;0}(\mathbb{R}^{2d})$ if and only if $b \in \Gamma_{(\omega)}^{\sigma,s;0}(\mathbb{R}^{2d})$.*

The following corollary is a consequence of Theorem 10.6.

**Corollary 10.7.** *Let $s,\sigma > 0$ such that $s+\sigma \geqslant 1$ $\omega \in \mathscr{P}_{s,\sigma}(\mathbb{R}^{2d})$, $A_1, A_2 \in \mathbf{M}(d,\mathbb{R})$, and that $a_1, a_2 \in (\Sigma_{s,\sigma}^{\sigma,s})'(\mathbb{R}^{2d})$ are such that $\operatorname{Op}_{A_1}(a_1) = \operatorname{Op}_{A_2}(a_2)$. Then*

$$a_1 \in \Gamma_{(\omega)}^{\sigma,s;0}(\mathbb{R}^{2d}) \qquad \Leftrightarrow \qquad a_2 \in \Gamma_{(\omega)}^{\sigma,s;0}(\mathbb{R}^{2d})$$

*and similarly for $\Gamma_{(\omega)}^{\sigma,s}(\mathbb{R}^{2d})$ in place of $\Gamma_{(\omega)}^{\sigma,s;0}(\mathbb{R}^{2d})$.*

## 10.2 Continuity for pseudo-differential operators with symbols of infinite order on Gelfand-Shilov spaces of functions and distributions

Next we deduce continuity for pseudo-differential operators with symbols in the classes given in Definitions 5.20 and 5.21. We begin with the case when the symbols belong to $\Gamma_{s,\sigma;0}^{\sigma,s}(\mathbb{R}^{2d})$.

**Theorem 10.8.** *Let $A \in \mathbf{M}(d, \mathbb{R})$, $s, \sigma > 0$ be such that $s + \sigma \geqslant 1$, and let $a \in \Gamma^{\sigma,s}_{s,\sigma;0}(\mathbb{R}^{2d})$. Then $\operatorname{Op}_A(a)$ is continuous on $\mathcal{S}^\sigma_s(\mathbb{R}^d)$ and on $(\mathcal{S}^\sigma_s)'(\mathbb{R}^d)$.*

For the proof we need the following result.

**Lemma 10.9.** *Let $s, \sigma > 0$ be such that $s + \sigma \geqslant 1$, $h_1 \geqslant 1$, $\Omega_1$ be a bounded set in $\mathcal{S}^\sigma_{s;h_1}(\mathbb{R}^d)$, and let*

$$h_2 \geqslant 2^{2+s} h_1 \quad \text{and} \quad h_3 \geqslant 2^{4+s+\sigma} h_1.$$

*Then*

$$\Omega_2 = \left\{ x \mapsto \frac{x^\gamma f(x)}{(2^{1+s}h_1)^{|\gamma|}\gamma!^s} \,;\, f \in \Omega_1,\ \gamma \in \mathbb{Z}^d_+ \right\}$$

*is a bounded set in $\mathcal{S}^\sigma_{s;h_2}(\mathbb{R}^d)$, and*

$$\Omega_3 = \left\{ x \mapsto \frac{D^\delta x^\gamma f(x)}{(2^{3+s+\sigma}h_1)^{|\gamma+\delta|}\gamma!^s\delta!^\sigma} \,;\, f \in \Omega_1,\ \gamma, \delta \in \mathbb{Z}^d_+ \right\}$$

*is a bounded set in $\mathcal{S}^\sigma_{s;h_3}(\mathbb{R}^d)$.*

*Proof.* Since $\Omega_1$ is a bounded set in $\mathcal{S}^\sigma_{s;h_1}(\mathbb{R}^d)$, there is a constant $C > 0$ such that

$$|x^\alpha D^\beta f(x)| \leqslant C h_1^{|\alpha+\beta|} \alpha!^s \beta!^\sigma, \quad \alpha, \beta \in \mathbb{Z}^d_+, \tag{10.7}$$

for every $f \in \Omega_1$. We shall prove that (10.7) is true for all $f \in \Omega_2$ for a new choice of $C > 0$, and $h_2$ in place of $h_1$.

Let $f \in \Omega_2$. Then

$$f(x) = \frac{x^\gamma f_0(x)}{(2^{1+s}h_1)^{|\gamma|}\gamma!^s}$$

for some $f_0 \in \Omega_1$ and $\gamma \in \mathbb{Z}_+^d$, and

$$
\begin{aligned}
|x^\alpha D^\beta f(x)| &= \left| \frac{x^\alpha D^\beta (x^\gamma f_0)(x)}{(2^{1+s}h_1)^{|\gamma|}\gamma!^s} \right| \\
&\leqslant \sum_{\gamma_0 \leqslant \gamma,\beta} \binom{\beta}{\gamma_0} \frac{\gamma!}{(\gamma-\gamma_0)!} \cdot \frac{|x^{\alpha+\gamma-\gamma_0}\partial^{\beta-\gamma_0} f_0(x)|}{(2^{1+s}h_1)^{|\gamma|}\gamma!^s} \\
&\lesssim \sum_{\gamma_0 \leqslant \gamma,\beta} \binom{\beta}{\gamma_0}\binom{\gamma}{\gamma_0} \gamma_0! \cdot \frac{h_1^{|\alpha+\beta+\gamma-2\gamma_0|}(\alpha+\gamma-\gamma_0)!^s(\beta-\gamma_0)!^\sigma}{(2^{1+s}h_1)^{|\gamma|}\gamma!^s} \\
&\lesssim h_1^{|\alpha+\beta|}\alpha!^s\beta!^\sigma \sum_{\gamma_0 \leqslant \gamma,\beta} \binom{\beta}{\gamma_0}\binom{\gamma}{\gamma_0} 2^{-(1+s)|\gamma|} \left( \frac{(\alpha+\gamma-\gamma_0)!\gamma_0!}{\alpha!\gamma!} \right)^s \left( \frac{(\beta-\gamma_0)!\gamma_0!}{\beta!} \right)^\sigma \\
&\lesssim h_1^{|\alpha+\beta|}\alpha!^s\beta!^\sigma \sum_{\gamma_0 \leqslant \gamma,\beta} \binom{\beta}{\gamma_0}\binom{\gamma}{\gamma_0} 2^{-(1+s)|\gamma|} \left( \frac{(\alpha+\gamma-\gamma_0)!\gamma_0!}{(\alpha+\gamma)!} \right)^s \binom{\alpha+\gamma}{\gamma}^s \\
&\lesssim h_1^{|\alpha+\beta|}\alpha!^s\beta!^\sigma \sum_{\gamma_0 \leqslant \gamma,\beta} 2^{|\beta|}2^{|\gamma|}2^{-(1+s)|\gamma|}2^{s|\alpha+\gamma|} \\
&\lesssim 2^{s|\alpha|}2^{|\beta|}h_1^{|\alpha+\beta|}\alpha!^s\beta!^\sigma \sum_{\gamma_0 \leqslant \beta} 1.
\end{aligned}
$$

Since

$$\sum_{\gamma_0 \leqslant \beta} 1 \lesssim 2^{|\beta|},$$

we get

$$|x^\alpha D^\beta f(x)| \leqslant C 2^{s|\alpha|}2^{2|\beta|}h_1^{|\alpha+\beta|}\alpha!^s\beta!^\sigma \leqslant C h_2^{|\alpha+\beta|}\alpha!^s\beta!^\sigma$$

for some constant $C$ which is independent of $f$, and the assertion on $\Omega_2$ follows. The same type of arguments shows that

$$\left\{ x \mapsto \frac{D^\delta f(x)}{(2^{1+s}h_1)^{|\delta|}\delta!^\sigma} \; ; \; f \in \Omega_1,\ \delta \in \mathbb{Z}_+^d \right\} \tag{10.8}$$

is a bounded set in $\mathcal{S}^\sigma_{s;2^{2+\sigma}h_1}(\mathbb{R}^d)$, and the boundedness of $\Omega_3$ in $\mathcal{S}^\sigma_{s;h_3}(\mathbb{R}^d)$ follows by combining the boundedness of $\Omega_2$ and the boundedness of (10.8) in $\mathcal{S}^\sigma_{s;h_2}(\mathbb{R}^d)$. □

**Lemma 10.10.** *Let $s, \tau > 0$, and set*

$$m_s(t) = \sum_{j=0}^\infty \frac{t^j}{(j!)^{2s}} \quad \text{and} \quad m_{s,\tau}(x) = m_s(\tau\langle x\rangle^2) \quad t \geqslant 0,\ x \in \mathbb{R}^d.$$

*Then,*

$$C^{-1}e^{(2s-\varepsilon)\tau^{\frac{1}{2s}}\langle x\rangle^{\frac{1}{s}}} \leqslant m_{s,\tau}(x) \leqslant Ce^{(2s+\varepsilon)\tau^{\frac{1}{2s}}\langle x\rangle^{\frac{1}{s}}}, \tag{10.9}$$

*for every $\varepsilon > 0$, and*

$$\frac{x^\alpha}{m_{s,\tau}(x)} \lesssim h_0^{|\alpha|}\alpha!^s e^{-r|x|^{\frac{1}{s}}}, \qquad (10.10)$$

*for some positive constant $r$ which depends on $d$, $s$ and $\tau$ only.*

The estimate (10.9) follows from [82], and (10.10) also follows from computations given in e. g. [24, 82]. For the sake of completeness we present a proof of (10.10).

*Proof.* We have

$$\frac{x^\alpha}{m_{s,\tau}(x)} \lesssim \prod_{j=1}^{d} g_{\alpha_j}(x_j),$$

where

$$g_k(t) = t^k e^{-2r_0 t^{\frac{1}{s}}}, \qquad t \geqslant 0,$$

for some $r_0 > 0$ depending only on $d$, $s$ and $\tau$. Let

$$g_{0,k}(t) = C_k e^{-r_0 t}, \qquad t \geqslant 0,$$

where

$$C_k = \sup_{t \geqslant 0}(t^{sk} e^{-r_0 t}).$$

Then, $g_k(t) \leqslant g_{0,k}(t^{\frac{1}{s}})$, and the result follows if we prove $C_k \lesssim h_0^k k!^s$.

By straightforward computations, it follows that the maximum of $t^{sk}e^{-r_0 t}$ is attained at $t = sk/r_0$, giving that

$$C_k = \left(\frac{s}{r_0 e}\right)^{sk}(k^k)^s \lesssim h_0^k k!^s, \qquad h_0 = \left(\frac{s}{r_0}\right)^s,$$

where the last inequality follows from Stirling's formula. This gives the result. □

*Proof of Theorem 10.8.* By Theorem 10.1 it suffices to consider the case $A = 0$, that is for the operator

$$\operatorname{Op}_0(a)f(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} a(x,\xi)\widehat{f}(\xi)e^{i\langle x,\xi\rangle}\, d\xi.$$

Observe that

$$\frac{1}{m_{s,\tau}(x)} \sum_{j=0}^{\infty} \frac{\tau^j}{(j!)^{2s}}(1-\Delta_\xi)^j e^{i\langle x,\xi\rangle} = e^{i\langle x,\xi\rangle}.$$

Let now $h_1 > 0$ and $f \in \Omega$, where $\Omega$ is a bounded subset of $\mathcal{S}^\sigma_{s,h_1}(\mathbb{R}^d)$. For fixed $\alpha, \beta \in \mathbb{Z}^d_+$ we get

$$
\begin{aligned}
(2\pi)^{\frac{d}{2}} x^\alpha D^\beta_x (\mathrm{Op}_0(a)f)(x) &= x^\alpha \sum_{\gamma \leqslant \beta} \binom{\beta}{\gamma} \int_{\mathbb{R}^d} \xi^\gamma D^{\beta-\gamma}_x a(x,\xi) \widehat{f}(\xi) e^{i\langle x,\xi\rangle}\, d\xi \\
&= \frac{x^\alpha}{m_{s,\tau}(x)} \sum_{\gamma \leqslant \beta} \binom{\beta}{\gamma} g_{\tau,\beta,\gamma}(x), \qquad (10.11)
\end{aligned}
$$

$$
g_{\tau,\beta,\gamma}(x) = \sum_{j=0}^\infty \frac{\tau^j}{(j!)^{2s}} \int_{\mathbb{R}^d} (1-\Delta_\xi)^j \left( \xi^\gamma D^{\beta-\gamma}_x a(x,\xi) \widehat{f}(\xi) \right) e^{i\langle x,\xi\rangle}\, d\xi.
$$

By Lemma 10.9 and the fact that, $(2j)! \leqslant 4^j j!^2$, it follows that for some $h > 0$,

$$
\Omega = \left\{ \xi \mapsto \frac{(1-\Delta_\xi)^j (\xi^\gamma D^\beta_x a(x,\xi) \widehat{f}(\xi))}{h^{|\beta+\gamma|+j} j!^{2s} \gamma!^\sigma \beta!^\sigma e^{r|x|^{\frac{1}{s}}}} \, ; \, j \geqslant 0, \ \beta, \gamma \in \mathbb{Z}^d_+ \right\}
$$

is bounded in $\mathcal{S}^s_\sigma(\mathbb{R}^d)$ for every $r > 0$. This implies that, for some positive constants $h$ and $r_0$, we get

$$
|(1-\Delta_\xi)^j (\xi^\gamma D^\beta_x a(x,\xi) \widehat{f}(\xi))| \lesssim h^{|\beta+\gamma|+j} j!^{2s} \gamma!^\sigma \beta!^\sigma e^{r|x|^{\frac{1}{s}} - r_0|\xi|^{\frac{1}{\sigma}}},
$$

for every $r > 0$. Hence,

$$
\begin{aligned}
|g_{\tau,\beta,\gamma}(x)| &\lesssim \sum_{j=0}^\infty \frac{\tau^j}{(j!)^{2s}} h^{|\beta|+j} j!^{2s} \gamma!^\sigma (\beta-\gamma)!^\sigma e^{r|x|^{\frac{1}{s}}} \int_{\mathbb{R}^d} e^{-r_0|\xi|^{\frac{1}{\sigma}}}\, d\xi \\
&\lesssim h^{|\beta|} \beta!^\sigma e^{r|x|^{\frac{1}{s}}} \sum_{j=0}^\infty (\tau h)^j \asymp h^{|\beta|} \beta!^\sigma e^{r|x|^{\frac{1}{s}}}
\end{aligned}
$$

for every $r > 0$, provided $\tau$ is chosen such that $\tau h < 1$.

By inserting this into (10.11) and using Lemma 10.10 we get, for some $h > 0$ and some $r_0 > 0$, that

$$
\begin{aligned}
|x^\alpha D^\beta_x (\mathrm{Op}_0(a)f)(x)| &\lesssim h^{|\alpha|} \alpha!^s e^{-r_0|x|^{\frac{1}{s}}} \sum_{\gamma \leqslant \beta} \binom{\beta}{\gamma} |g_{\tau,\beta,\gamma}(x)| \\
&\lesssim h^{|\alpha+\beta|} 2^{|\beta|} \alpha!^s \beta!^\sigma e^{-(r_0-r)|x|^{\frac{1}{s}}} \left( \sum_{\gamma \leqslant \beta} 1 \right) \lesssim (2h)^{|\alpha+\beta|} \alpha!^s \beta!^\sigma,
\end{aligned}
$$

provided that $r$ above is chosen to be smaller than $r_0$. Then, the continuity of $\mathrm{Op}_A(a)$ on $\mathcal{S}^\sigma_s(\mathbb{R}^d)$ follows. The continuity of $\mathrm{Op}_A(a)$ on $(\mathcal{S}^\sigma_s)'(\mathbb{R}^d)$ now follows from the previous continuity and duality. □

Next we shall discuss corresponding continuity in the Beurling case. The main idea is to deduce such properties by suitable estimates on short-time Fourier transforms of involved functions and distributions. First we have the following relation between the short-time Fourier transforms of the symbols and kernels of a pseudo-differential operator.

**Lemma 10.11.** *Let $A \in \mathbf{M}(d,\mathbb{R})$, $s,\sigma > 0$ be such that $s+\sigma \geqslant 1$, $a \in (\mathcal{S}_{s,\sigma}^{\sigma,s})'(\mathbb{R}^{2d})$ ($a \in (\Sigma_{s,\sigma}^{\sigma,s})'(\mathbb{R}^{2d})$), $\phi \in \mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})$ ($\phi \in \Sigma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})$), and let*

$$K_{a,A}(x,y) = (2\pi)^{-\frac{d}{2}}(\mathscr{F}_2^{-1}a)(x-A(x-y),x-y)$$

*and*

$$\psi(x,y) = K_{\phi,A}(x,y) = (2\pi)^{-\frac{d}{2}}(\mathscr{F}_2^{-1}\phi)(x-A(x-y),x-y)$$

*be the kernels of* $\operatorname{Op}_A(a)$ *and* $\operatorname{Op}_A(\phi)$*, respectively. Then*

$$\begin{aligned}&(V_\psi K_{a,A})(x,y,\xi,\eta)\\&= (2\pi)^{-d}e^{i\langle x-y,\eta-A^*(\xi+\eta)\rangle}(V_\phi a)(x-A(x-y),-\eta+A^*(\xi+\eta),\xi+\eta,y-x).\end{aligned} \tag{10.12}$$

The essential parts of (10.12) is presented in the proof of [121, Proposition 2.5]. In order to be self-contained we here present a short proof.

*Proof.* Let

$$T_A(x,y) = x - A(x-y)$$

and

$$Q = Q(x,x_1,y,\xi,\xi_1,\eta) = \langle x-y,\xi_1 - T_{A^*}(-\eta,\xi)\rangle - \langle x_1,\xi+\eta\rangle.$$

By formal computations, using Fourier's inversion formula we get

$$\begin{aligned}&(V_\psi K_{a,A})(x,y,\xi,\eta)\\&= (2\pi)^{-3d}\iint K_{a,A}(x_1,y_1)\overline{\psi(x_1-x,y_1-y)}e^{-i(\langle x_1,\xi\rangle+\langle y_1,\eta\rangle)}\,dx_1dy_1\\&= (2\pi)^{-2d}\iint a(x_1,\xi_1)\overline{\phi(x_1-T_A(x,y),\xi_1-T_{A^*}(-\eta,\xi))}e^{iQ(x,x_1,y,\xi,\xi_1,\eta)}\,dx_1d\xi_1\\&= (2\pi)^{-d}e^{i\langle x-y,\eta-A^*(\xi+\eta)\rangle}(V_\phi a)(T_A(x,y),T_{A^*}(-\eta,\xi),\xi+\eta,y-x),\end{aligned}$$

where all integrals should be interpreted as suitable Fourier transforms. This gives the result. □

Before continuing the discussion about the continuity of pseudo-differential operators, we observe that the previous lemma in combination with Propositions 9.2 and 9.3 give the following.

**Proposition 10.12.** *Let $A \in \mathbf{M}(d,\mathbb{R})$, $s,\sigma > 0$ be such that $s+\sigma \geqslant 1$ and $(s,\sigma) \neq (\frac{1}{2},\frac{1}{2})$, $\phi \in \Sigma_s^\sigma(\mathbb{R}^{2d})\backslash 0$, $a$ be a Gelfand-Shilov distribution on $\mathbb{R}^{2d}$ and let $K_{a,A}$ be the kernel of $\operatorname{Op}_A(a)$. Then, the following conditions are equivalent:*

*(1) $a \in \Gamma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})$ (resp. $a \in \Gamma_{s,\sigma}^{\sigma,s;0}(\mathbb{R}^{2d})$).*

*(2) For some $r > 0$,*

$$|V_\phi K_{a,A}(x,y,\xi,\eta)| \lesssim e^{r(|x-A(x-y)|^{\frac{1}{s}}+|\eta-A^*(\xi+\eta)|^{\frac{1}{\sigma}})-h(|\xi+\eta|^{\frac{1}{\sigma}}+|x-y|^{\frac{1}{s}})}$$

*holds for some $h > 0$ (for every $h > 0$).*

By similar arguments we also get the following.

**Proposition 10.13.** *Let $A \in \mathbf{M}(d,\mathbb{R})$, $s,\sigma > 0$ be such that $s+\sigma \geqslant 1$, $\phi \in \mathcal{S}_s^\sigma(\mathbb{R}^{2d})\backslash 0$, $a$ be a Gelfand-Shilov distribution on $\mathbb{R}^{2d}$ and let $K_{a,A}$ be the kernel of $\operatorname{Op}_A(a)$. Then the following conditions are equivalent:*

*(1) $a \in \Gamma_{s,\sigma;0}^{\sigma,s}(\mathbb{R}^{2d})$ (resp. $a \in \Gamma_{s,\sigma;0}^{\sigma,s;0}(\mathbb{R}^{2d})$).*

*(2) For some $h > 0$ (for every $h > 0$),*

$$|V_\phi K_{a,A}(x,y,\xi,\eta)| \lesssim e^{r(|x-A(x-y)|^{\frac{1}{s}}+|\eta-A^*(\xi+\eta)|^{\frac{1}{\sigma}})-h(|\xi+\eta|^{\frac{1}{\sigma}}+|x-y|^{\frac{1}{s}})}$$

*holds for every $r > 0$.*

**Theorem 10.14.** *Let $A \in \mathbf{M}(d,\mathbb{R})$, $s,\sigma > 0$ be such that $s+\sigma \geqslant 1$ and $(s,\sigma) \neq (\frac{1}{2},\frac{1}{2})$, and let $a \in \Gamma_{s,\sigma}^{\sigma,s;0}(\mathbb{R}^{2d})$. Then $\operatorname{Op}_A(a)$ is continuous on $\Sigma_s^\sigma(\mathbb{R}^d)$, and is uniquely extendable to a continuous map on $(\Sigma_s^\sigma)'(\mathbb{R}^d)$.*

*Proof.* By Theorem 10.1 we may assume that $A = 0$. Let

$$g(x) = \operatorname{Op}_0(a)f(x) = (K_{a,0}(x,\,\cdot\,),\overline{f}) = (h_{a,x},\overline{f}),$$

where $h_{a,x} = K_{a,0}(x,\,\cdot\,)$, and let $\phi_j \in \Sigma_s^\sigma(\mathbb{R}^d)$ be such that $\|\phi_j\|_{L^2} = 1$, $j = 1,2$. By Moyal's identity (cf. [67]) we get

$$g(x) = (h_{a,x},\overline{f})_{L^2(\mathbb{R}^d)} = (V_{\phi_1}h_{a,x},V_{\phi_1}\overline{f})_{L^2(\mathbb{R}^{2d})}.$$

Applying the short-time Fourier transform on $g$ and using Fubini's theorem on distributions we get

$$V_{\overline{\phi_2}}g(x,\xi) = \langle J(x,\xi,\,\cdot\,),F\rangle,$$

where

$$F(y,\eta) = V_{\overline{\phi_1}}f(y,-\eta), \quad J(x,\xi,y,\eta) = V_\phi K_{a,0}(x,y,\xi,\eta)$$

and $\phi = \phi_2 \otimes \phi_1$.

Now suppose that $r > 0$ is arbitrarily chosen. By Proposition 9.2 we get, for some $c \in (0,1)$ which depends on $s$ and $\sigma$ only, that for some $r_0 > 0$ and with $r_1 = (r + r_0)/c$, that

$$|J(x,\xi,y,\eta)| \lesssim e^{r_0(|x|^{\frac{1}{s}}+|\eta|^{\frac{1}{\sigma}})}e^{-r_1(|y-x|^{\frac{1}{s}}+|\xi+\eta|^{\frac{1}{\sigma}})}$$
$$\lesssim e^{-((cr_1-r_0)|x|^{\frac{1}{s}}+cr_1|\xi|^{\frac{1}{\sigma}})}e^{r_1|y|^{\frac{1}{s}}+(r_1+r_0)|\eta|^{\frac{1}{\sigma}}}$$
$$\lesssim e^{-r(|x|^{\frac{1}{s}}+|\xi|^{\frac{1}{\sigma}})}e^{r_2(|y|^{\frac{1}{s}}+|\eta|^{\frac{1}{\sigma}})},$$

where $r_2$ only depends on $r$ and $r_0$. Since $f \in \Sigma_s^\sigma(\mathbb{R}^d)$ we have

$$|F(x,\xi)| \lesssim \|f\|_{\mathcal{S}_{s;h}^\sigma} e^{-(1+r_2)(|x|^{\frac{1}{s}}+|\xi|^{\frac{1}{\sigma}})},$$

where $h > 0$ only depends on $r_2$, and thereby depends only $r$ and $r_0$. This implies

$$|V_{\phi_2}g(x,\xi)| = |\langle J(x,\xi,\,\cdot\,),F\rangle|$$
$$\lesssim \|f\|_{\mathcal{S}_{s;h}^\sigma}\left(\iint e^{r_2(|y|^{\frac{1}{s}}+|\eta|^{\frac{1}{\sigma}})}e^{-(1+r_2)(|y|^{\frac{1}{s}}+|\eta|^{\frac{1}{\sigma}})}\,dyd\eta\right)e^{-r(|x|^{\frac{1}{s}}+|\xi|^{\frac{1}{\sigma}})}$$
$$\asymp \|f\|_{\mathcal{S}_{s;h}^\sigma} e^{-r(|x|^{\frac{1}{s}}+|\xi|^{\frac{1}{\sigma}})}, \qquad (10.13)$$

which shows that $g \in \Sigma_s^\sigma(\mathbb{R}^d)$ in view of [123, Proposition 2.1]. Since the topology of $\Sigma_s^\sigma(\mathbb{R}^d)$ is given by the semi-norms

$$g \mapsto \sup_{x,\xi\in\mathbb{R}^d} |V_{\phi_2}g(x,\xi)e^{r(|x|^{\frac{1}{s}}+|\xi|^{\frac{1}{\sigma}})}|,$$

it follows from (10.13) that $\operatorname{Op}(a)$ is continuous on $\Sigma_s^\sigma(\mathbb{R}^d)$.

By duality it follows that $\operatorname{Op}(a)$ is uniquely extendable to a continuous map on $(\Sigma_s^\sigma)'(\mathbb{R}^d)$. □

The following result follows by similar arguments as in the previous proof.

**Theorem 10.15.** *Let $A \in \mathbf{M}(d,\mathbb{R})$, $s,\sigma > 0$ be such that $s+\sigma \geqslant 1$ and $(s,\sigma) \neq (\frac{1}{2},\frac{1}{2})$, and let $a \in \Gamma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})$. Then, $\operatorname{Op}_A(a)$ is continuous from $\Sigma_s^\sigma(\mathbb{R}^d)$ to $\mathcal{S}_s^\sigma(\mathbb{R}^d)$, and from $(\mathcal{S}_s^\sigma)'(\mathbb{R}^d)$ to $(\Sigma_s^\sigma)'(\mathbb{R}^d)$.*

## 10.3 Compositions of pseudo-differential operators

Next we deduce algebraic properties of pseudo-differential operators considered in Theorems 10.8, 10.14 and 10.15. We recall that for pseudo-differential operators with symbols, for instance, in Hörmander classes, we have

$$\mathrm{Op}_0(a_1\#_0 a_2) = \mathrm{Op}_0(a_1)\circ \mathrm{Op}_0(a_2),$$

where

$$a_1\#_0 a_2(x,\xi) \equiv \big(e^{i\langle D_\xi,D_y\rangle}(a_1(x,\xi)a_2(y,\eta))\big)\Big|_{(y,\eta)=(x,\xi)}.$$

More generally, if $A\in \mathbf{M}(d,\mathbb{R})$ and $a_1\#_A a_2$ is defined by

$$a_1\#_A a_2 \equiv e^{i\langle AD_\xi,D_x\rangle}\Big(\big(e^{-i\langle AD_\xi,D_x\rangle}a_1\big)\#_0\big(e^{-i\langle AD_\xi,D_x\rangle}a_2\big)\Big), \tag{10.14}$$

for $a_1$ and $a_2$ belonging to certain Hörmander symbol classes, then it follows from the analysis in [81] that

$$\mathrm{Op}_A(a_1\#_A a_2) = \mathrm{Op}_A(a_1)\circ \mathrm{Op}_A(a_2) \tag{10.15}$$

for suitable $a_1$ and $a_2$.

We recall that the map $a\mapsto K_{a,A}$ is a homeomorphism from $\mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})$ to $\mathcal{S}_s^\sigma(\mathbb{R}^{2d})$ and from $\Sigma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})$ to $\Sigma_s^\sigma(\mathbb{R}^{2d})$. It is also immediate to observe that the map

$$(K_1,K_2)\mapsto \left((x,y)\mapsto (K_1\circ K_2)(x,y) = \int_{\mathbb{R}^d} K_1(x,z)K_2(z,y)\,dz\right)$$

is sequentially continuous from $\mathcal{S}_s^\sigma(\mathbb{R}^{2d})\times \mathcal{S}_s^\sigma(\mathbb{R}^{2d})$ to $\mathcal{S}_s^\sigma(\mathbb{R}^{2d})$, and from $\Sigma_s^\sigma(\mathbb{R}^{2d})\times\Sigma_s^\sigma(\mathbb{R}^{2d})$ to $\Sigma_s^\sigma(\mathbb{R}^{2d})$. Here we have identified operators with their kernels. For compositions with three operator kernels we have

$$\begin{aligned}(K_1\circ K_2\circ K_3)(x,y) &= \langle K_2, T_{K_1,K_3}(x,y,\,\cdot\,)\rangle\\ \text{with}\quad T_{K_1,K_3}(x,y,z_1,z_2) &= K_1(x,z_1)K_3(z_2,y)\end{aligned} \tag{10.16}$$

when $K_j\in L^2(\mathbb{R}^{2d})$, $j=1,2,3$. Notice that

$$(K_1,K_2,K_3)\mapsto ((x,y)\mapsto \langle K_2, T_{K_1,K_3}(x,y,\,\cdot\,)\rangle)$$

is sequentially continuous from $\mathcal{S}_s^\sigma(\mathbb{R}^{2d})\times(\mathcal{S}_s^\sigma)'(\mathbb{R}^{2d})\times\mathcal{S}_s^\sigma(\mathbb{R}^{2d})$ to $\mathcal{S}_s^\sigma(\mathbb{R}^{2d})$, and from $\Sigma_s^\sigma(\mathbb{R}^{2d})\times(\Sigma_s^\sigma)'(\mathbb{R}^{2d})\times\Sigma_s^\sigma(\mathbb{R}^{2d})$ to $\Sigma_s^\sigma(\mathbb{R}^{2d})$. The following result follows from these continuity properties and (10.15).

**Proposition 10.16.** *Let $A\in\mathbf{M}(d,\mathbb{R})$, and let $s,\sigma>0$ be such that $s+\sigma\geqslant 1$. Then the following properties hold true:*

*(1) The map $(a_1,a_2)\mapsto a_1\#_A a_2$ is continuous from $\mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})\times\mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})$ to $\mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})$.*

*(2) The map $(a_1,a_2)\mapsto a_1\#_A a_2$ is continuous from $\Sigma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})\times\Sigma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})$ to $\Sigma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})$.*

*(3) The map* $(a_1, a_2, a_3) \mapsto a_1 \#_A a_2 \#_A a_3$ *from* $\mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d}) \times \mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d}) \times \mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})$ *to* $\mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})$ *extends uniquely to a sequentially continuous and associative map from* $\mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d}) \times (\mathcal{S}_{s,\sigma}^{\sigma,s})'(\mathbb{R}^{2d}) \times \mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})$ *to* $\mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})$.

*(4) The map* $(a_1, a_2, a_3) \mapsto a_1 \#_A a_2 \#_A a_3$ *from* $\Sigma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d}) \times \Sigma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d}) \times \Sigma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})$ *to* $\Sigma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})$ *extends uniquely to a sequentially continuous and associative map from* $\Sigma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d}) \times (\Sigma_{s,\sigma}^{\sigma,s})'(\mathbb{R}^{2d}) \times \Sigma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})$ *to* $\Sigma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})$.

We have the following corresponding algebra result for $\Gamma_{s,\sigma}^{\sigma,s;0}$ and related symbol classes.

**Theorem 10.17.** *Let* $A \in \mathbf{M}(d, \mathbb{R})$*, and let* $s, \sigma > 0$ *be such that* $s + \sigma \geqslant 1$*. Then the following statements hold true:*

*(1) The map* (1) *in Proposition 10.16 extends uniquely to a continuous map from* $\Gamma_{s,\sigma;0}^{\sigma,s}(\mathbb{R}^{2d}) \times \Gamma_{s,\sigma;0}^{\sigma,s}(\mathbb{R}^{2d})$ *to* $\Gamma_{s,\sigma;0}^{\sigma,s}(\mathbb{R}^{2d})$*, and from* $\Gamma_{s,\sigma;0}^{\sigma,s;0}(\mathbb{R}^{2d}) \times \Gamma_{s,\sigma;0}^{\sigma,s;0}(\mathbb{R}^{2d})$ *to* $\Gamma_{s,\sigma;0}^{\sigma,s;0}(\mathbb{R}^{2d})$.

*(2) If in addition* $(s, \sigma) \neq (\frac{1}{2}, \frac{1}{2})$*, the map* (2) *in Proposition 10.16 extends uniquely to a continuous map from* $\Gamma_{s,\sigma}^{\sigma,s;0}(\mathbb{R}^{2d}) \times \Gamma_{s,\sigma}^{\sigma,s;0}(\mathbb{R}^{2d})$ *to* $\Gamma_{s,\sigma}^{\sigma,s;0}(\mathbb{R}^{2d})$*, and from* $\Gamma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d}) \times \Gamma_{s,\sigma}^{\sigma,s;0}(\mathbb{R}^{2d})$ *or from* $\Gamma_{s,\sigma}^{\sigma,s;0}(\mathbb{R}^{2d}) \times \Gamma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})$ *to* $\Gamma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})$.

*Proof.* We prove only the first assertion in (2). The other statements follow by similar arguments.

By Theorem 10.6 it suffices to consider the case when $A = 0$. Let $\phi_1, \phi_2, \phi_3 \in \Sigma_s^\sigma(\mathbb{R}^d) \backslash 0$, such that $\|\phi_2\|_{L^2} = 1$, $a_j \in \Gamma_{s,\sigma}^{\sigma,s;0}(\mathbb{R}^{2d})$, $j = 1, 2$, and let $K$ be the kernel of $\operatorname{Op}_0(a_1) \circ \operatorname{Op}_0(a_2)$. By Proposition 10.12 we need to prove that for some $r > 0$,

$$|V_{\phi_1 \otimes \phi_3} K(x, y, \xi, \eta)| \lesssim e^{r(|x|^{\frac{1}{s}} + |\eta|^{\frac{1}{\sigma}}) - h(|\xi + \eta|^{\frac{1}{\sigma}} + |x - y|^{\frac{1}{s}})} \tag{10.17}$$

for every $h > 0$.

Therefore, let $h > 0$ be arbitrarily chosen but fixed, and let $K_j$ be the kernel of $\operatorname{Op}_0(a_j)$, $j = 1, 2$,

$$F_1(x, y, \xi, \eta) = V_{\phi_1 \otimes \phi_2} K_1(x, y, \xi, \eta),$$

$$F_2(x, y, \xi, \eta) = V_{\overline{\phi_2} \otimes \phi_3} K_2(x, y, -\xi, \eta)$$

and

$$G(x, y, \xi, \eta) = V_{\phi_1 \otimes \phi_3} K(x, y, \xi, \eta).$$

Then,

$$G(x, y, \xi, \eta) = \iint\limits_{\mathbb{R}^{2d}} F_1(x, z, \xi, \zeta) F_2(z, y, \zeta, \eta) \, dz d\zeta, \tag{10.18}$$

by Moyal's identity (cf. proof of Theorem 10.14). Since $a_j \in \Gamma_{s,\sigma}^{\sigma,s;0}(\mathbb{R}^{2d})$ we have for some $r_0 > 0$ that

$$|F_1(x,y,\xi,\eta)| \lesssim e^{r_0(|x|^{\frac{1}{s}}+|\eta|^{\frac{1}{\sigma}})-r(|\xi+\eta|^{\frac{1}{\sigma}}+|x-y|^{\frac{1}{s}})}$$

and

$$|F_2(x,y,\xi,\eta)| \lesssim e^{r_0(|x|^{\frac{1}{s}}+|\eta|^{\frac{1}{\sigma}})-r(|\xi-\eta|^{\frac{1}{\sigma}}+|x-y|^{\frac{1}{s}})}$$

for every $r > 0$. By combining this with (10.18) we get, for some $r_0 > 0$,

$$|G(x,y,\xi,\eta)| \lesssim \iint\limits_{\mathbb{R}^{2d}} e^{\varphi_{r_0,r_1}(x,y,z,\xi,\eta,\zeta)+\psi_{r_0,r_1}(x,y,z,\xi,\eta,\zeta)}\, dz d\zeta, \tag{10.19}$$

where $r_1 \geqslant 2cr + cr_0$,

$$\varphi_{r_0,r}(x,y,z,\xi,\eta,\zeta) = r_0(|x|^{\frac{1}{s}} + |\zeta|^{\frac{1}{\sigma}}) - r(|\zeta-\eta|^{\frac{1}{\sigma}} + |y-z|^{\frac{1}{s}}),$$
$$\psi_{r_0,r}(x,y,z,\xi,\eta,\zeta) = r_0(|z|^{\frac{1}{s}} + |\eta|^{\frac{1}{\sigma}}) - r(|\xi+\zeta|^{\frac{1}{\sigma}} + |x-z|^{\frac{1}{s}}),$$

and $c \geqslant 1$ is chosen such that

$$|x+y|^{\frac{1}{s}} \leqslant c(|x|^{\frac{1}{s}} + |y|^{\frac{1}{s}}) \quad \text{and} \quad |\xi+\eta|^{\frac{1}{\sigma}} \leqslant c(|\xi|^{\frac{1}{\sigma}} + |\eta|^{\frac{1}{\sigma}}), \quad x,y,\xi,\eta \in \mathbb{R}^d.$$

Then,

$$\begin{aligned}\varphi_{r_0,r_1}(x,y,z,\xi,\eta,\zeta) &\leqslant cr_0(|x|^{\frac{1}{s}} + |\eta|^{\frac{1}{\sigma}}) - (r_1 - cr_0)(|\zeta-\eta|^{\frac{1}{\sigma}} + |y-z|^{\frac{1}{s}})\\ &\leqslant cr_0(|x|^{\frac{1}{s}} + |\eta|^{\frac{1}{\sigma}}) - 2cr(|\zeta-\eta|^{\frac{1}{\sigma}} + |y-z|^{\frac{1}{s}})\end{aligned}$$

and

$$\psi_{r_0,r_1}(x,y,z,\xi,\eta,\zeta) \leqslant cr_0(|x|^{\frac{1}{s}} + |\eta|^{\frac{1}{\sigma}}) - 2cr(|\xi+\zeta|^{\frac{1}{\sigma}} + |x-z|^{\frac{1}{s}}).$$

This gives

$$\begin{aligned}&\varphi_{r_0,r_1}(x,y,z,\xi,\eta,\zeta) + \psi_{r_0,r_1}(x,y,z,\xi,\eta,\zeta)\\ &\qquad \leqslant 2cr_0(|x|^{\frac{1}{s}} + |\eta|^{\frac{1}{\sigma}}) - 2cr(|\xi+\zeta|^{\frac{1}{\sigma}} + |\zeta-\eta|^{\frac{1}{\sigma}} + |x-z|^{\frac{1}{s}} + |y-z|^{\frac{1}{s}}).\end{aligned}$$

Since

$$\begin{aligned}&-2cr(|\xi+\zeta|^{\frac{1}{\sigma}} + |\zeta-\eta|^{\frac{1}{\sigma}} + |x-z|^{\frac{1}{s}} + |y-z|^{\frac{1}{s}})\\ &\leqslant -r(|\xi+\eta|^{\frac{1}{\sigma}} + |x-y|^{\frac{1}{s}}) - cr(|\xi+\zeta|^{\frac{1}{\sigma}} + |\zeta-\eta|^{\frac{1}{\sigma}} + |x-z|^{\frac{1}{s}} + |y-z|^{\frac{1}{s}})\\ &\qquad \leqslant -r(|\xi+\eta|^{\frac{1}{\sigma}} + |x-y|^{\frac{1}{s}}) - cr(|\xi+\zeta|^{\frac{1}{\sigma}} + |x-z|^{\frac{1}{s}}),\end{aligned}$$

we get, by combining these estimates with (10.19), that

$$|G(x,y,\xi,\eta)| \lesssim \iint\limits_{\mathbb{R}^{2d}} e^{2cr_0(|x|^{\frac{1}{s}}+|\eta|^{\frac{1}{\sigma}})-r(|\xi+\eta|^{\frac{1}{\sigma}}+|x-y|^{\frac{1}{s}})-cr(|\xi+\zeta|^{\frac{1}{\sigma}}+|x-z|^{\frac{1}{s}})}\, dz d\zeta,$$

$$\asymp e^{2cr_0(|x|^{\frac{1}{s}}+|\eta|^{\frac{1}{\sigma}})-r(|\xi+\eta|^{\frac{1}{\sigma}}+|x-y|^{\frac{1}{s}})}.$$

Since $r_0 > 0$ is fixed and $r > 0$ can be chosen arbitrarily, the result follows. □

**Theorem 10.18.** *Let $A \in \mathbf{M}(d,\mathbb{R})$, $s,\sigma > 0$ be such that $s+\sigma \geqslant 1$, and let $\omega_j \in \mathscr{P}_{s,\sigma}(\mathbb{R}^{2d})$, $j=1,2$. Then the following statements true:*

*(1) The map $(a_1,a_2) \mapsto a_1 \#_A a_2$ from $\Sigma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d}) \times \Sigma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})$ to $\Sigma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})$ is uniquely extendable to a continuous map from $\Gamma_{(\omega_1)}^{\sigma,s;0}(\mathbb{R}^{2d}) \times \Gamma_{(\omega_2)}^{\sigma,s;0}(\mathbb{R}^{2d})$ to $\Gamma_{(\omega_1\omega_2)}^{\sigma,s;0}(\mathbb{R}^{2d})$.*

*(2) If in addition $\omega_j \in \mathscr{P}_{s,\sigma}^0(\mathbb{R}^{2d})$, $j=1,2$, then the map $(a_1,a_2) \mapsto a_1 \#_A a_2$ from $\mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d}) \times \mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})$ to $\mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})$ is uniquely extendable to a continuous map from $\Gamma_{(\omega_1)}^{\sigma,s}(\mathbb{R}^{2d}) \times \Gamma_{(\omega_2)}^{\sigma,s}(\mathbb{R}^{2d})$ to $\Gamma_{(\omega_1\omega_2)}^{\sigma,s}(\mathbb{R}^{2d})$.*

For the proof we need the following lemma.

**Lemma 10.19.** *Let $\omega$ be a weight on $\mathbb{R}^{4d}$, $\omega_0(x,\xi) = \omega(x,x,\xi,\xi)$ when $x,\xi \in \mathbb{R}^d$, $s,\sigma > 0$ be such that $s+\sigma \geqslant 1$. Then, the trace map which takes*

$$\mathbb{R}^{4d} \ni (x,y,\xi,\eta) \mapsto F(x,y,\xi,\eta)$$

*to*

$$\mathbb{R}^{2d} \ni (x,\xi) \mapsto F(x,x,\xi,\xi)$$

*is linear and continuous from $\Gamma_{(\omega)}^{\sigma,s}(\mathbb{R}^{4d})$ into $\Gamma_{(\omega_0)}^{\sigma,s}(\mathbb{R}^{2d})$. The same holds true with $\Gamma_{(\omega)}^{\sigma,s;0}$ and $\Gamma_{(\omega_0)}^{\sigma,s;0}$ in place of $\Gamma_{(\omega)}^{\sigma,s}$ and $\Gamma_{(\omega_0)}^{\sigma,s}$, respectively, at each occurrence.*

Lemma 10.19 follows by similar arguments as in the proof of Lemma 10.9, using the Leibniz type rule

$$\partial_x^\alpha \partial_\xi^\beta (F(x,x,\xi,\xi)) = \sum_{\gamma \leqslant \alpha} \sum_{\delta \leqslant \beta} \binom{\alpha}{\gamma} \binom{\beta}{\delta} (\partial_1^{\alpha-\gamma} \partial_2^{\beta-\delta} \partial_3^\gamma \partial_4^\delta F)(x,x,\xi,\xi).$$

*Proof of Theorem 10.18.* We may assume that $A=0$ by Theorem 10.1. We only prove (2), since the assertion (1) follows by similar arguments.

Let

$$F_{a_1,a_2}(x_1, x_2, \xi_1, \xi_2) = a_1(x_1, \xi_1)a_2(x_2, \xi_2)$$

and

$$\omega(x_1, x_2, \xi_1, \xi_2) = \omega_1(x_1, \xi_1)\omega_2(x_2, \xi_2).$$

By the definitions, it follows that the map $T_1$ which takes $(a_1, a_2)$ into $F_{a_1,a_2}$ is continuous from $\Gamma^{\sigma,s}_{(\omega_1)}(\mathbb{R}^{2d}) \times \Gamma^{\sigma,s}_{(\omega_2)}(\mathbb{R}^{2d})$ to $\Gamma^{\sigma,s}_{(\omega)}(\mathbb{R}^{4d})$.

Theorem 10.3 implies that the map $T_2$ which takes $F(x_1, x_2, \xi_1, \xi_2)$ to $e^{i\langle D_{\xi_1}, D_{x_2}\rangle}F(x_1, x_2, \xi_1, \xi_2)$ is continuous on $\Gamma^{\sigma,s}_{(\omega)}(\mathbb{R}^{4d})$. Hence, if $T_3$ is the trace operator which takes $F(x_1, x_2, \xi_1, \xi_2)$ into $F_0(x, \xi) \equiv F(x, x, \xi, \xi)$, the Lemma 10.19 shows that $T \equiv T_3 \circ T_2 \circ T_1$ is continuous from $\Gamma^{\sigma,s}_{(\omega_1)}(\mathbb{R}^{2d}) \times \Gamma^{\sigma,s}_{(\omega_2)}(\mathbb{R}^{2d})$ to $\Gamma^{\sigma,s}_{(\omega_1\omega_2)}(\mathbb{R}^{2d})$.

By [81, Theorem 18.1.8] we have $T(a_1, a_2) = a_1\#_0 a_2$ when $a_1, a_2 \in \Sigma^{\sigma,s}_{s,\sigma}(\mathbb{R}^{2d})$. If instead $a_j \in \Gamma^{\sigma,s}_{(\omega_j)}(\mathbb{R}^{2d})$, $j = 1, 2$, then we take $T(a_1, a_2)$ as the definition of $a_1\#_0 a_2$. By the continuity of $T$ it follows that $(a_1, a_2) \mapsto a_1\#_0 a_2$ is continuous from $\Gamma^{\sigma,s}_{(\omega_1)}(\mathbb{R}^{2d}) \times \Gamma^{\sigma,s}_{(\omega_2)}(\mathbb{R}^{2d})$ to $\Gamma^{\sigma,s}_{(\omega_1\omega_2)}(\mathbb{R}^{2d})$.

Since $\Gamma^{\sigma,s}_{(\omega_j)}(\mathbb{R}^{2d}) \subseteq \Gamma^{\sigma,s}_{s,\sigma;0}(\mathbb{R}^{2d})$, we get $\operatorname{Op}_0(a_1\#_0 a_2) = \operatorname{Op}_0(a_1) \circ \operatorname{Op}_0(a_2)$ and that $a_1\#_0 a_2$ is uniquely defined as an element in $\Gamma^{\sigma,s}_{s,\sigma;0}(\mathbb{R}^{2d})$, in view of Theorem 10.17. Hence $a_1\#_0 a_2$ is uniquely defined in $\Gamma^{\sigma,s}_{(\omega_1\omega_2)}(\mathbb{R}^{2d})$, since all these symbol classes are subspaces of $C^\infty(\mathbb{R}^{2d})$. This gives the result. □

# Chapter 11

# Pseudo-differential operators with symbols of infinite order on modulation spaces

In this chapter we discuss continuity for operators in $\operatorname{Op}(\Gamma^{\sigma,s}_{(\omega_0)})$ and $\operatorname{Op}(\Gamma^{\sigma,s;0}_{(\omega_0)})$, given in Definition 5.20, when acting on a general class of modulation spaces. In Theorem 11.1 continuity is proved where the symbols belong to $\Gamma^{\sigma,s}_{(\omega_0)}$ and in Theorem 11.5 continuity is proved where the symbols belong to $\Gamma^{\sigma,s;0}_{(\omega_0)}$. This gives an analogy to [120, Theorem 3.2], within the frameworks of operator theory and Gelfand-Shilov classes.

The main result of the current chapter is the next Theorem 11.1.

**Theorem 11.1.** *Let $A \in \mathbf{M}(d,\mathbb{R})$, $s,\sigma \geqslant 1$, $\omega,\omega_0 \in \mathscr{P}^0_{s,\sigma}(\mathbb{R}^{2d})$, $a \in \Gamma^{\sigma,s}_{(\omega_0)}(\mathbb{R}^{2d})$, and that $\mathscr{B}$ is an invariant BF-space on $\mathbb{R}^{2d}$. Then $\operatorname{Op}_A(a)$ is continuous from $M(\omega_0\omega,\mathscr{B})$ to $M(\omega,\mathscr{B})$.*

We need some preparations for the proof, and start with the following remark.

**Remark 11.2.** *Let $s,\sigma > 0$ be such that $s+\sigma \geqslant 1$. If $a \in (\Sigma^{\sigma,s}_{s,\sigma})'(\mathbb{R}^{2d})$, then there is a unique $b \in (\Sigma^{\sigma,s}_{s,\sigma})'(\mathbb{R}^{2d})$ such that $\operatorname{Op}(a)^* = \operatorname{Op}(b)$, where $b(x,\xi) = e^{i\langle D_\xi,D_x\rangle}\overline{a(x,\xi)}$ in view of [80, Theorem 18.1.7]. Furthermore, by the latter equality and [24, Theorem 4.1] it follows that*

$$a \in \Gamma^{\sigma,s}_{(\omega)}(\mathbb{R}^{2d}) \quad \Leftrightarrow \quad b \in \Gamma^{\sigma,s}_{(\omega)}(\mathbb{R}^{2d}).$$

**Lemma 11.3.** *Suppose $s,\sigma \geqslant 1$, $\omega \in \mathscr{P}_E(\mathbb{R}^{d_0})$ and that $f \in C^\infty(\mathbb{R}^{d+d_0})$ satisfies*

$$|\partial^\alpha f(x,y)| \lesssim h^{|\alpha|}\alpha!^\sigma e^{-r|x|^{\frac{1}{s}}}\omega(y), \alpha \in \mathbb{Z}^{d+d_0}_+, \tag{11.1}$$

*for some $h > 0$ and $r > 0$. Then there are $f_0 \in C^\infty(\mathbb{R}^{d+d_0})$ and $\psi \in \mathcal{S}^\sigma_s(\mathbb{R}^d)$ such that (11.1) holds true with $f_0$ in place of $f$ for some for some $h > 0$ and $r > 0$, and $f(x,y) = f_0(x,y)\psi(x)$.*

*Proof.* By Proposition 5.12, there is a submultiplicative weight $v_0 \in \mathscr{P}_{E,s}(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$ such that

$$v_0(x) \asymp e^{\frac{r}{2}|x|^{\frac{1}{s}}} \tag{11.2}$$

and

$$|\partial^\alpha v_0(x)| \lesssim h^{|\alpha|}\alpha!^\sigma v_0(x), \qquad \alpha \in \mathbb{Z}_+^d \tag{11.3}$$

for some $h, r > 0$. Since $s, \sigma \geqslant 1$, a straightforward application of Faà di Bruno's formula, for the composed function $\psi(x) = g(v_0(x))$, where $g(t) = \frac{1}{t}$, on (11.3) gives

$$\left|\partial^\alpha \left(\frac{1}{v_0(x)}\right)\right| \lesssim h^{|\alpha|}\alpha!^\sigma \cdot \frac{1}{v_0(x)}, \qquad \alpha \in \mathbb{Z}_+^d \tag{11.3$'$}$$

for some $h > 0$. It follows from (11.2) and (11.3)$'$ that if $\psi = 1/v_0$, then $\psi \in \mathcal{S}_s^\sigma(\mathbb{R}^d)$. Furthermore, if $f_0(x,y) = f(x,y)v_0(x)$, then through an application of Leibnitz formula we get

$$\begin{aligned}
|\partial_x^\alpha \partial_y^{\alpha_0} f_0(x,y)| &\lesssim \sum_{\gamma \leqslant \alpha} \binom{\alpha}{\gamma} |\partial_x^\delta \partial_y^{\alpha_0} f(x,y)|\, |\partial^{\alpha-\delta} v_0(x)| \\
&\lesssim h^{|\alpha|+|\alpha_0|} \sum_{\gamma \leqslant \alpha} \binom{\alpha}{\gamma} (\gamma!\alpha_0!)^\sigma e^{-r|x|^{\frac{1}{s}}} \omega(y)(\alpha-\gamma)!^\sigma v_0(x) \\
&\lesssim (2h)^{|\alpha|+|\alpha_0|}(\alpha!\alpha_0!)^\sigma e^{-r|x|^{\frac{1}{s}}} v_0(x)\omega(y) \\
&\lesssim (2h)^{|\alpha|+|\alpha_0|}(\alpha!\alpha_0!)^\sigma e^{-\frac{r}{2}|x|^{\frac{1}{s}}} \omega(y),
\end{aligned}$$

for some $h > 0$, which gives the desired estimate on $f_0$, since it is clear that $f(x,y) = f_0(x,y)\psi(x)$. □

**Lemma 11.4.** *Let $s, \sigma \geqslant 1$, $\omega \in \mathscr{P}_{s,\sigma}^0(\mathbb{R}^{2d})$, $v_1 \in \mathscr{P}_{E,s}^0(\mathbb{R}^d)$ and $v_2 \in \mathscr{P}_{E,\sigma}^0(\mathbb{R}^d)$ be such that $v_1$ and $v_2$ are submultiplicative, and $\omega \in \Gamma_{(\omega)}^{\sigma,s}(\mathbb{R}^{2d})$ is $v_1 \otimes v_2$-moderate. Also, let $a \in \Gamma_{(\omega)}^{\sigma,s}(\mathbb{R}^{2d})$, $f \in \mathcal{S}_s^\sigma(\mathbb{R}^d)$, $\phi \in \Sigma_s^\sigma(\mathbb{R}^d)$, $\phi_2 = \phi v_1$. If*

$$\Phi(x,\xi,z,\zeta) = \frac{a(x+z,\xi+\zeta)}{\omega(x,\xi)v_1(z)v_2(\zeta)} \tag{11.4}$$

*and*

$$H(x,\xi,y) = \iint \Phi(x,\xi,z,\zeta)\phi_2(z)v_2(\zeta)e^{i\langle y-x-z,\zeta\rangle}\, dz d\zeta, \tag{11.5}$$

*then*

$$V_\phi(\mathrm{Op}(a)f)(x,\xi) = (2\pi)^{-d}(f, e^{i\langle\,\cdot\,,\xi\rangle}H(x,\xi,\,\cdot\,))\omega(x,\xi). \tag{11.6}$$

*Furthermore the following statements hold true:*

*(1)* $H \in C^\infty(\mathbb{R}^{3d})$ *and satisfies*

$$|\partial_y^\alpha H(x,\xi,y)| \lesssim h_0^{|\alpha|}\alpha!^\sigma e^{-r_0|x-y|^{\frac{1}{s}}}, \tag{11.7}$$

*for every* $\alpha \in \mathbb{Z}_+^d$ *and some* $h_0, r_0 > 0$.

*(2) There are functions* $H_0 \in C^\infty(\mathbb{R}^{3d})$ *and* $\phi_0 \in \mathcal{S}_s^\sigma(\mathbb{R}^d)$ *such that*

$$H(x,\xi,y) = H_0(x,\xi,y)\phi_0(y-x), \tag{11.8}$$

*and such that* (11.7) *holds true for some* $h_0, r_0 > 0$*, with* $H_0$ *in place of* $H$*.*

Lemma 11.4 follows by similar arguments as in [126]. In order to be self contained, we give bellow a (different) proof.

*Proof.* By straightforward computations we get

$$V_\phi(\mathrm{Op}(a)f)(x,\xi) = (2\pi)^{-d}(f, e^{i\langle\,\cdot\,,\xi\rangle}H_1(x,\xi,\,\cdot\,))\omega(x,\xi), \tag{11.9}$$

where

$$\begin{aligned} H_1(x,\xi,y) &= (2\pi)^d e^{-i\langle y,\xi\rangle}(\mathrm{Op}(a)^*(\phi(\,\cdot\,-x)\,e^{i\langle\cdot,\xi\rangle}))(y)/\omega(x,\xi) \\ &= \iint \frac{a(z,\zeta)}{\omega(x,\xi)}\phi(z-x)e^{i\langle y-z,\zeta-\xi\rangle}\,dzd\zeta \\ &= \iint \Phi(x,\xi,z-x,\zeta-\xi)\phi_2(z-x)v_2(\zeta-\xi)e^{i\langle y-z,\zeta-\xi\rangle}\,dzd\zeta. \end{aligned}$$

If $z-x$ and $\zeta-\xi$ are taken as new variables of integrations, it follows that the right-hand side is the same as (11.5). Hence (11.6) holds true. This gives the first part of the lemma.

The smoothness of $H$ is a consequence of the uniqueness of the adjoint (cf. Remark 11.2 and [126, Lemma 2.7]).

To show that (11.7) holds, let

$$\Phi_0(x,\xi,z,\zeta) = \Phi(x,\xi,z,\zeta)\phi_2(z),$$

where $\Phi$ is defined as in (11.4), and let $\Psi = \mathscr{F}_3\Phi_0$, where $\mathscr{F}_3\Phi_0$ is the partial Fourier transform of $\Phi_0(x,\xi,z,\zeta)$ with respect to the $z$ variable. Then it follows, from the assumptions and $(11.3)'$ and Proposition 5.12, that for some $r > 0$

$$\begin{aligned}|\partial_z^\alpha \Phi_0(x,\xi,z,\zeta)| &= \left|\sum_{\gamma\leqslant\alpha}\binom{\alpha}{\gamma}\partial_z^\gamma\left(\frac{a(x+z,\xi+\zeta)}{\omega(x,\xi)v_1(z)v_2(\zeta)}\right)\partial^{\alpha-\gamma}\phi_2(z)\right| \\ &\lesssim \sum_{\gamma\leqslant\alpha}\binom{\alpha}{\gamma}\sum_{\lambda\leqslant\gamma}\binom{\gamma}{\lambda}\frac{\left|\partial_z^{\gamma-\lambda}a(x+z,\xi+\zeta)\right|}{\omega(x,\xi)v_2(\zeta)} \\ &\qquad \times \partial^\lambda\left(\frac{1}{v_1(z)}\right)h^{|\alpha-\gamma|}(\alpha-\gamma)!^\sigma e^{-r|z|^{\frac{1}{s}}} \\ &\lesssim \sum_{\gamma\leqslant\alpha}\binom{\alpha}{\gamma}\sum_{\lambda\leqslant\gamma}\binom{\gamma}{\lambda}h^{|\alpha|}(\alpha-\gamma)!^\sigma(\gamma-\lambda)!^\sigma\lambda!^\sigma e^{-r_0|z|^{\frac{1}{s}}} \\ &\lesssim h^{|\alpha|}\alpha!^\sigma\sum_{\gamma\leqslant\alpha}\binom{\alpha}{\gamma}\sum_{\lambda\leqslant\gamma}\binom{\gamma}{\lambda}\left(\frac{(\alpha-\gamma)!\gamma!}{\alpha!}\right)^\sigma\left(\frac{(\gamma-\lambda)!\lambda!}{\gamma!}\right)^\sigma e^{-r_0|z|^{\frac{1}{s}}} \\ &\lesssim (4h)^{|\alpha|}\alpha!^\sigma e^{-r|z|^{\frac{1}{s}}}\sum_{\gamma\leqslant\alpha}1\cdot\sum_{\lambda\leqslant\gamma}1.\end{aligned}$$

Since $\sum_{\lambda\leqslant\gamma} 1 \lesssim 2^{|\gamma|}$, we get

$$|\partial_z^\alpha \Phi_0(x,\xi,z,\zeta)| \leqslant C(16h)^{|\alpha|}\alpha!^\sigma e^{-r_0|z|^{\frac{1}{s}}} \leqslant Ch_0^{|\alpha|}\alpha!^\sigma e^{-r_0|z|^{\frac{1}{s}}}, \tag{11.10}$$

for some $C, h_0, r_0 > 0$. Then $z \mapsto \Phi_0(x,\xi,z,\zeta)$ is an element in $\mathcal{S}_s^\sigma(\mathbb{R}^d)$. Moreover, $\{\Phi_0(x,\xi,z,\zeta)\}_{z\in\mathbb{R}^d}$ is a bounded set in $\Gamma_{(1)}^{\sigma,s}(\mathbb{R}^d\times\mathbb{R}^{2d})$. Indeed, for a fixed $z_0 \in \mathbb{R}^d$, an application of Leibnitz formula, Faà di Bruno's formula, Proposition 5.12 and $(11.3)'$, gives

$$\begin{aligned}\left|\partial_x^\alpha\partial_\xi^\beta\partial_\zeta^\gamma\Phi_0(x,\xi,z_0,\zeta)\right| &\leqslant \sum\binom{\alpha}{\alpha_1}\binom{\beta}{\beta_1}\binom{\gamma}{\gamma_1}\partial_x^{\alpha_1}\partial_\xi^{\beta_1}\left(\frac{1}{\omega(x,\xi)}\right) \\ &\times\partial_\zeta^{\gamma_1}\left(\frac{1}{v_2(\zeta)}\right)\left|\partial_x^{\alpha-\alpha_1}\partial_\xi^{\beta-\beta_1}\partial_\zeta^{\gamma-\gamma_1}a(x+z_0,\xi+\zeta)\right|\cdot\frac{|\phi(z_0)|}{v_1(z_0)} \\ &\lesssim \sum\binom{\alpha}{\alpha_1}\binom{\beta}{\beta_1}\binom{\gamma}{\gamma_1}h^{|\alpha_1+\beta_1+\gamma_1|}\alpha_1!^\sigma(\beta_1!\gamma_1!)^s \\ &\times\left(\frac{1}{\omega(x,\xi)v_1(z_0)v_2(\zeta)}\right)\left|\partial_x^{\alpha-\alpha_1}\partial_\xi^{\beta-\beta_1}\partial_\zeta^{\gamma-\gamma_1}a(x+z_0,\xi+\zeta)\right| \\ &\lesssim h^{|\alpha+\beta+\gamma|}\sum\binom{\alpha}{\alpha_1}\binom{\beta}{\beta_1}\binom{\gamma}{\gamma_1}\cdot \\ &\qquad\cdot((\alpha-\alpha_1)!\alpha_1!)^\sigma((\beta-\beta_1)!\beta_1!)^s((\gamma-\gamma_1)!\gamma_1!)^s \\ &\lesssim (4h)^{|\alpha+\beta+\gamma|}\alpha!^\sigma(\beta!\gamma!)^s,\end{aligned} \tag{11.11}$$

where the summations above are taken over all $\alpha_1 \leqslant \alpha, \beta_1 \leqslant \beta$ and $\gamma_1 \leqslant \gamma$. In view of Proposition 5.5 and (11.10) we have

$$|\partial_\eta^\alpha \Psi(x,\xi,\eta,\zeta)| \lesssim h_0^{|\alpha|}\alpha!^s e^{-r_0|\eta|^{\frac{1}{\sigma}}},$$

for some $h_0, r_0 > 0$. Hence

$$|\partial_\eta^\alpha (\Psi(x,\xi,\zeta,\zeta)v_2(\zeta))| \lesssim h_0^{|\alpha|}\alpha!^s e^{-r_0|\zeta|^{\frac{1}{\sigma}}},$$

for some $h_0, r_0 > 0$.

By letting $H_2(x,\xi,\,\cdot\,)$ be the inverse partial Fourier transform of $\Psi(x,\xi,\zeta,\zeta)v_2(\zeta)$ with respect to the $\zeta$ variable, it follows that

$$|\partial_y^\alpha H_2(x,\xi,y)| \lesssim h_0^{|\alpha|}\alpha!^\sigma e^{-r_0|y|^{\frac{1}{s}}}, \tag{11.12}$$

for some $h_0, r_0 > 0$. The assertion (1) now follows from the latter estimate and the fact that $H(x,\xi,y) = H_2(x,\xi,y-x)$.

In order to prove (2) we notice that (11.12) shows that $y \mapsto H_2(x,\xi,y)$ is an element in $\mathcal{S}_s^\sigma(\mathbb{R}^d)$ with values in $\Gamma_{(1)}^{\sigma,s}(\mathbb{R}^{2d})$, in view of (11.11) and the construction of $H_2$. It follows by Lemma 11.3 that there exist $H_3 \in C^\infty(\mathbb{R}^{3d})$ and $\phi_0 \in \mathcal{S}_s^\sigma(\mathbb{R}^d)$ such that (11.12) holds for some $h_0, r_0 > 0$ with $H_3$ in place of $H_2$, and

$$H_2(x,\xi,y) = H_3(x,\xi,y)\phi_0(-y).$$

This is the same as (2), and the result follows. □

*Proof of Theorem 11.1.* It is not restrictive to assume that $A = 0$, in view of the invariance properties given by Chapter 10. Let $G = \operatorname{Op}(a)f$. In view of Lemma 11.4 we have

$$\begin{aligned} V_\phi G(x,\xi) &= (2\pi)^{-\frac{d}{2}}\mathscr{F}((f\cdot \overline{\phi_0(\,\cdot\,-x)})\cdot H_0(x,\xi,\,\cdot\,))(\xi)\omega(x,\xi) \\ &= (2\pi)^{-d}(V_{\phi_0}f)(x,\,\cdot\,) * (\mathscr{F}(H_0(x,\xi,\,\cdot\,)))(\xi)\omega(x,\xi). \end{aligned}$$

Since $\omega$ and $\omega_0$ belong to $\mathscr{P}_{s,\sigma}^0(\mathbb{R}^{2d})$, then, for every $r_0 > 0$ and $x,\xi,\eta \in \mathbb{R}^d$, we have

$$\omega(x,\xi)\omega_0(x,\xi) \lesssim \omega(x,\eta)\omega_0(x,\eta)e^{\frac{r_0}{2}|\xi-\eta|^{\frac{1}{\sigma}}}.$$

Such inequality and (2) in Lemma 11.4 give

$$|V_\phi G(x,\xi)\omega_0(x,\xi)| \lesssim \left(|(V_{\phi_0}f)(x,\,\cdot\,)\omega(x,\,\cdot\,)\omega_0(x,\,\cdot\,)| * e^{-\frac{r_0}{2}|\,\cdot\,|^{\frac{1}{\sigma}}}\right)(\xi).$$

In view of Definition 5.25, we get, for some $v \in \mathscr{P}_\sigma^0(\mathbb{R}^d)$,

$$\begin{aligned} \|G\|_{M(\omega_0,\mathscr{B})} &\lesssim \||(V_{\phi_0}f)\cdot\omega\cdot\omega_0| * \delta_0 \otimes e^{-r_0|\,\cdot\,|^{\frac{1}{\sigma}}}\|_{\mathscr{B}} \\ &\leqslant \|(V_{\phi_0}f)\cdot\omega\cdot\omega_0\|_{\mathscr{B}}\|e^{-r_0|\,\cdot\,|^{\frac{1}{\sigma}}}v\|_{L^1} \asymp \|f\|_{M(\omega\cdot\omega_0,\mathscr{B})}. \end{aligned}$$

The proof is complete. □

By similar arguments as in the proof of Theorem 11.1 and Lemma 11.4 we can prove the next two results.

**Theorem 11.5.** *Let $A \in \mathbf{M}(d, \mathbb{R})$, $s, \sigma \geqslant 1$, $\omega, \omega_0 \in \mathscr{P}_{s,\sigma}(\mathbb{R}^{2d})$, $a \in \Gamma^{\sigma,s;0}_{(\omega_0)}(\mathbb{R}^{2d})$. Assume also that $\mathscr{B}$ is an invariant BF-space on $\mathbb{R}^{2d}$. Then, $\operatorname{Op}_A(a)$ is continuous from $M(\omega_0\omega, \mathscr{B})$ to $M(\omega, \mathscr{B})$.*

**Lemma 11.6.** *Let $s, \sigma \geqslant 1$, $\omega \in \mathscr{P}_{s,\sigma}(\mathbb{R}^{2d})$, $v_1 \in \mathscr{P}_s(\mathbb{R}^d)$ and $v_2 \in \mathscr{P}_\sigma(\mathbb{R}^d)$ be such that $v_1$ and $v_2$ are submultiplicative, and $\omega \in \Gamma^{\sigma,s;0}_{(\omega)}(\mathbb{R}^{2d})$ is $v_1 \otimes v_2$-moderate. Also, let $a \in \Gamma^{\sigma,s;0}_{(\omega)}(\mathbb{R}^{2d})$, $f, \phi \in \Sigma_s^\sigma(\mathbb{R}^d)$, $\phi_2 = \phi v_1$, and let $\Phi$ and $H$ be as in Lemma 11.4. Then (11.6) and the following statements hold true:*

*(1) $H \in C^\infty(\mathbb{R}^{3d})$ and satisfies (11.7) for every $h_0, r_0 > 0$.*

*(2) There are functions $H_0 \in C^\infty(\mathbb{R}^{3d})$ and $\phi_0 \in \Sigma_s(\mathbb{R}^d)$ such that (11.8) holds true, and such that (11.7) holds true for every $h_0, r_0 > 0$, with $H_0$ in place of $H$.*

# Appendix A

# Proof of Lemma 7.3

**Lemma A.1.** *Let* $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_+^d$*. Then the number of elements in the set*

$$\Omega_{k,\alpha} \equiv \{\, (\beta_1, \ldots, \beta_k) \in \mathbb{Z}_+^{kd}\,;\ \beta_1 + \cdots + \beta_k = \alpha \,\} \tag{A.1}$$

*is equal to*

$$\prod_{j=1}^{d} \binom{\alpha_j + k}{k}.$$

For the proof we recall the formula

$$\sum_{j=0}^{k} \binom{n+j}{j} = \binom{n+k+1}{k}, \tag{A.2}$$

which follows by a standard induction argument.

*Proof.* Let $N$ be the number of elements in the set (A.1), which is the searched number, and let $N_j$ be the number of elements of the set

$$\{\, (\beta_1^0, \ldots, \beta_k^0) \in \mathbb{Z}_+^k\,;\ \beta_1^0 + \cdots + \beta_k^0 = \alpha_j \,\}, \qquad j = 1, \ldots, d.$$

By straightforward computations it follows that $N = N_1 \cdots N_d$, and it suffices to prove the result in the case $d = 1$, and then $\alpha = (\alpha_1)$.

In order to prove the result for $d = 1$, let $\gamma \in \mathbb{Z}_+$,

$$S_1(\gamma) = 1,$$

and define inductively

$$S_{j+1}(\gamma) = \sum_{\beta=0}^{\gamma} S_j(\beta), \qquad j = 1, 2, \ldots .$$

By straightforward computations it follows that $N = N_1 = S_k(\alpha_1)$. We claim

$$S_j(\gamma) = \binom{\gamma + j}{j}, \qquad j = 1, 2, \ldots . \tag{A.3}$$

In fact, (A.3) is clearly true for $j = 1$. Assume that (A.3) holds for $j = n$, and consider $S_{n+1}(\gamma)$. Then, (A.2) gives

$$S_{n+1}(\gamma) = \sum_{\beta=0}^{\gamma} S_n(\beta) = \sum_{\beta=0}^{\gamma} \binom{\beta+n}{n}$$

$$= \sum_{\beta=0}^{\gamma} \binom{\beta+n}{\beta} = \binom{\gamma+n+1}{\gamma} = \binom{\gamma+n+1}{n+1},$$

which gives (A.3) when $j = n+1$. This proves (A.3), and the result follows. □

**Lemma A.2.** *Let* $\alpha \in \mathbb{Z}_+^d \backslash 0$, $s_0 \in (0,1]$, *and let* $\Omega_{k,\alpha}$ *be the same as in* (A.1). *Then*

$$\sum_{k=1}^{|\alpha|} \frac{1}{k} \sum_{\beta \in \Omega_{k,\alpha}} (\beta!)^{s_0-1} \leqslant 6^{|\alpha|}.$$

*Proof.* By Lemma A.1, observing that $\beta \in \Omega_{k,\alpha} \implies \beta! \geqslant 1$ and, of course $k \geqslant 1 \implies \frac{1}{k} \leqslant 1$. Recalling that $s_0 - 1 < 0$ we get

$$\sum_{k=1}^{|\alpha|} \frac{1}{k} \left( \sum_{\beta \in \Omega_{k,\alpha}} \beta!^{s_0-1} \right) \leqslant \sum_{k=1}^{|\alpha|} \left( \sum_{\beta \in \Omega_{k,\alpha}} 1 \right) = \sum_{k=1}^{|\alpha|} \left( \prod_{j=1}^{d} \binom{\alpha_j + k}{k} \right)$$

$$\leqslant |\alpha| \prod_{j=1}^{d} 2^{2\alpha_j} = |\alpha| \cdot 4^{|\alpha|} \leqslant 6^{|\alpha|}.$$

□

# Bibliography


[1] A. Abdeljawad, A. Ascanelli, S. Coriasco, *Deterministic and stochastic Cauchy problems for a class of weakly hyperbolic operators on* $\mathbb{R}^n$, (Preprint) arXiv: 1810.05009 (2018).

[2] A. Abdeljawad, M. Cappiello, J. Toft, *Pseudo-differential calculus in anisotropic Gelfand-Shilov setting*, (Preprint) arXiv:1805.03497 (2018).

[3] A. Abdeljawad, S. Coriasco, J. Toft, *Liftings for ultra-modulation spaces, and one-parameter groups of Gevrey type pseudo-differential operators*, (Preprint) arXiv:1712.04338 (2017).

[4] A. Abdeljawad, J. Toft, *Anisotropic Gevrey-Hörmander pseudo-differential operators on modulation spaces*, (Preprint) arXiv:1806.10002 (2018).

[5] H. Abels, *Pseudodifferential and singular integral operators: an introduction with applications*, De Gruyter, Berlin/Boston, 2012.

[6] T. Aoki, *Locally bounded topological spaces*, Proc. Japan Acad. **18** (1942), 588–594.

[7] A. Ascanelli, M. Cappiello, *Hölder continuity in time for SG hyperbolic systems*, J. Differential Equations **244** (2008), 2091–2121.

[8] A. Ascanelli, S. Coriasco, *Fourier integral operators algebra and fundamental solutions to hyperbolic systems with polynomially bounded coefficients on* $\mathbb{R}^n$, J. Pseudo-Differ. Oper Appl. (2015), 521–565.

[9] A. Ascanelli, S. Coriasco, A. Süss, *Solution theory to hyperbolic stochastic partial differential equations with polynomially bounded coeffcients*, (Preprint) arXiv:1610.01208v3 (2016).

[10] A. Ascanelli, S. Coriasco, A. Süss, *Random-field solutions of linear hyperbolic stochastic partial differential equations with polynomially bounded coefficients*, (Preprint) (2018).

[11] U. Battisti, S. Coriasco, *Wodzicki residue for operators on manifolds with cylindrical ends*, Ann. Global Anal. Geom. **2** (2011), 223–249.

[12] H.A. Biagioni,T. Gramchev, *Fractional derivative estimates in Gevrey classes, global regularity and decay for solutions to semilinear equations in* $\mathbb{R}^d$, J. Differential Equations **194** (2003), 140–165.

[13] P. Boggiatto, E. Buzano, L. Rodino, *Global Hypoellipticity and Spectral Theory*, Akademie Verlag, Math Research. **92** 1996.

[14] J. M. Bony, J. Y. Chemin, *Espaces Functionnels Associés au Calcul de Weyl-Hörmander*, Bull. Soc. math. France **122** (1994), 77–118.

[15] L. Boutet de Monvel, *Opérateurs pseudo-différentiels analytiques et opérateurs d'ordre infini*, Ann. Inst. Fourier, Grenoble **22** (1972), 229–268.

[16] E. Buzano, J. Toft, *Schatten-von Neumann properties in the Weyl calculus*, J. Funct. Anal. **259** (2010), 3080–3114.

[17] M. Cappiello, *Pseudodifferential parametrices of infinite order for SG-hyperbolic problems*, Rend. Sem. Mat. Univ. Pol. Torino **61** (2003), 411–441.

[18] M. Cappiello, *Fourier integral operators of infinite order and applications to SG-hyperbolic equations*, Tsukuba J. Math. **28** (2004), 311–361.

[19] M. Cappiello, T. Gramchev, L. Rodino, *Sub-exponential decay and uniform holomorphic extensions for semilinear pseudodifferential equations*, Comm. Partial Differential Equations **35** (2010), 846–877.

[20] M. Cappiello, T. Gramchev, L. Rodino, *Entire extensions and exponential decay for semilinear elliptic equations*, J. Anal. Math. **111** (2010), 339–367.

[21] M. Cappiello, S. Pilipović, B. Prangoski, *Parametrices and hypoellipticity for pseudodifferential operators on spaces of tempered ultradistributions*, J. Pseudo-Differ. Oper. Appl. **5** (2014), 491–506.

[22] M. Cappiello, S. Pilipović, B. Prangoski, *Semilinear pseudodifferential equations in spaces of tempered ultradistributions*, J. Math. Anal. Appl. **442** (2016), 317–338.

[23] M. Cappiello, L. Rodino, J. Toft, *On the inverse to the harmonic oscillator*, Comm. Partial Differential Equations **40** (2015), 1096–1118.

[24] M. Cappiello, J. Toft, *Pseudo-differential operators in a Gelfand-Shilov setting*, Math. Nachr. **290** (2017), 738–755.

[25] E. Carypis, P. Wahlberg, *Propagation of exponential phase space singularities for Schrödinger equations with quadratic Hamiltonians*, J. Fourier Anal. Appl. **23** (2017), 530–571.

[26] L.Cattabriga, L. Zanghirati, *Fourier integral operators of infinite order on Gevrey spaces. Applications to the Cauchy problem for hyperbolic operators*, Advances in Microlocal Analysis, H.G. Garnir ed., D. Reidel Publ. Comp. (1986), pp. 41-71.

[27] L. Cattabriga, L. Zanghirati, *Fourier integral operators of infinite order on Gevrey spaces. Application to the Cauchy problem for certain hyperbolic operators*, J. Math. Kyoto Univ. **30** (1990), 142–192.

[28] J. Chung, S. Y. Chung, D. Kim, *Characterization of the Gelfand-Shilov spaces via Fourier transforms*, Proc. Am. Math. Soc. **124** (1996), 2101–2108.

[29] Y. Chen, M. Signahl, J. Toft, *Factorizations and singular value estimates of operators with Gelfand-Shilov and Pilipović kernels*, J. Fourier Anal. Appl. **24** (2017), 666–698.

[30] Y. Chen, J. Toft, P. Wahlberg, *The Weyl product on quasi-Banach modulation spaces*, Bull. Math. Sci. (appeared online 2018).

[31] J. Chung, S.Y. Chung, D. Kim, *Characterization of the Gelfand-Shilov spaces via Fourier transforms*, Proc. Amer. Math. Soc. **124** (1996), 2101–2108.

[32] M. Cicognani, M. Reissig, *Well-posedness for degenerate Schrödinger equations*, Evol. Equ. Control Theory **3** (2014), 15–33.

[33] F. Concetti, J. Toft, *Trace ideals for Fourier integral operators with non-smooth symbols*, "Pseudo-Differential Operators: Partial Differential Equations and Time-Frequency Analysis". Fields Inst. Commun., Amer. Math. Soc. **52** 2007, pp.255–264.

[34] F. Concetti, G. Garello, J. Toft, *Trace ideals for Fourier integral operators with non-smooth symbols II*, Osaka J. Math. **47** (2010), 739–786.

[35] E. Cordero, S. Pilipović, L. Rodino, N. Teofanov, *Quasianalytic Gelfand-Shilov spaces with applications to localization operators*, Rocky Mt. J. Math. **40** (2010), 1123–1147.

[36] E. Cordero, K. Gröchenig, *Time-Frequency Analysis of Localization Operators*, J. Funct. Anal. **205** (2003), 107–131.

[37] E. Cordero, S. Pilipović, L. Rodino, N. Teofanov, *Localization operators and exponential weights for modulation spaces*, Mediterr. J. Math. **2** (2005), 381–394.

[38] E. Cordero, K. Gröchenig, F. Nicola, L. Rodino, *Wiener algebras of Fourier integral operators*, J. Math. Pures Appl. (2013), 219–233.

[39] E. Cordero, K. Gröchenig, F. Nicola, L. Rodino, *Generalized Metaplectic Operators and the Schrödinger Equation with a Potential in the Sjöstrand Class*, J. Math. Phys. **55**, (2014), 081506:1–17.

[40] E. Cordero, J. Toft, P. Wahlberg, *Sharp results for the Weyl product on modulation spaces*, J. Funct. Anal. **267** (2014), 3016–3057.

[41] H. O. Cordes. *The technique of pseudodifferential operators*, Cambridge Univ. Press, 1995.

[42] S. Coriasco, *Fourier integral operators in SG classes I. Composition theorems and action on SG Sobolev spaces.* Rend. Sem. Mat. Univ. Pol. Torino, **57** (1999), 249–302.

[43] S. Coriasco, *Fourier integral operators in SG classes II. Application to SG hyperbolic Cauchy problems*, Ann. Univ. Ferrara, (1998), 47:81–122.

[44] S. Coriasco, *Fourier Integral Operators in SG classes with Applications to Hyperbolic Cauchy Problems*, PhD thesis, Universitá di Torino, 1998.

[45] S. Coriasco, K. Johansson, J. Toft, *Local wave-front sets of Banach and Fréchet types, and pseudo-differential operators*, Monatsh. Math. **169** (2013) 285–316.

[46] S. Coriasco, K. Johansson, J. Toft, *Global wave-front sets of Banach, Fréchet and Modulation space types, and pseudo-differential operators*, J. Differential Equations, **254** (2013), 3228–3258.

[47] S. Coriasco, K. Johansson, J. Toft, *Global wave-front properties for Fourier integral operators and hyperbolic problems*, J.Fourier.Anal.Appl. **22** (2016), 285–333.

[48] S. Coriasco, L. Maniccia, *Wave Front Set at Infinity and Hyperbolic Linear Operators with Multiple Characteristics*, Ann. Global Anal. Geom. **24** (2003), 375-400.

[49] S. Coriasco, P. Panarese, *Fourier integral operators defined by classical symbols with exit behaviour*, Math. Nachr. **242** (2002), 61–78.

[50] S. Coriasco, J. Toft, *A calculus of Fourier integral operators with inhomogeneous phase functions on* $\mathbb{R}^d$, Indian J. Pure Appl. Math. **47** (2016), 125–166.

[51] J. J. Duistermaat, *Fourier integral operators*, Courant Institute Lecture Notes New-York 1973.

[52] Y .V . Egorov, B.-W. Schulze, *Pseudo-Differential Operators, Singularities, Applications*, Birkhäuser, 1997.

[53] L. C. Evans, *Partial Differential Equations: Second Edition*, AMS Graduate Studies in Mathematics, 2010.

[54] H. G. Feichtinger, *Banach convolution algebras of Wiener's type,* in: Proc. Functions, Series, Operators in Budapest, Colloquia Math. Soc. J. Bolyai, North Holland Publ. Co. Amsterdam Oxford NewYork, 1980.

[55] H. G. Feichtinger, *Modulation spaces on locally compact abelian groups. In* Proceedings of "International Conference on Wavelets and Applications" 2002, pages 99–140, Chennai, India, 2003. Updated version of a technical report, University of Vienna, 1983.

[56] H. G. Feichtinger, *Atomic characterizations of modulation spaces through Gabor-type representations,* in: Proc. Conf. on Constructive Function Theory, Rocky Mountain J. Math. **19** (1989), 113–126.

[57] H. G. Feichtinger, *Wiener amalgams over Euclidean spaces and some of their applications,* in: Function spaces (Edwardsville, IL, 1990), Lect. Notes in pure and appl. math. **136**, Marcel Dekker, New York, 1992, pp. 123–137.

[58] H. G. Feichtinger, *Modulation spaces: Looking back and ahead*, Sampl. Theory Signal Image Process. **5** (2006), 109–140.

[59] H. G. Feichtinger, *Choosing function spaces in harmonic analysis.* Excursions in harmonic analysis **4**, Appl. Numer. Harmon. Anal. 65–101, Birkhäuser/Springer, Cham, 2015.

[60] H. G. Feichtinger, K. H. Gröchenig, *Banach spaces related to integrable group representations and their atomic decompositions, I*, J. Funct. Anal. **86** (1989), 307–340.

[61] C. Fernandez, A. Galbis, J. Toft, *Spectral invariance for matrix algebras*, J. Fourier Anal. Appl. **20** (2014), 362–383.

[62] H. G. Feichtinger, K. H. Gröchenig, *Banach spaces related to integrable group representations and their atomic decompositions, II*, Monatsh. Math. **108** (1989), 129–148.

[63] Y. V. Galperin, S. Samarah, *Time-frequency analysis on modulation spaces* $M_m^{p,q}$*,* $0 < p, q \leqslant \infty$, Appl. Comput. Harmon. Anal. **16** (2004), 1–18.

[64] I.M. Gelfand, G.E. Shilov, *Generalized functions, Vol. 2. Spaces of fundamental and generalized functions*, Academic Press, New York-London, 1968.

[65] M. A. de Gosson, *Symplectic methods in harmonic analysis and in mathematical physics*, Pseudo-Differential Operators Theory and Applications **7** Birkhäuser/Springer Basel AG, Basel, 2011.

[66] A. Grigis, J. Sjöstrand, *Microlocal Analysis for Differential Operators: An Introduction*, London Mathematical Society Lecture Note Series 1994.

[67] K. H. Gröchenig, *Foundations of Time-Frequency Analysis*, Birkhäuser, Boston, 2001.

[68] K. H. Gröchenig, *Composition and spectral invariance of pseudodifferential operators on modulation spaces*, J. Anal. Math. **98** (2006), 65–82.

[69] K. H. Gröchenig, *Time-frequency analysis of Sjöstrand's class*, Rev. Mat. Iberoam. **22** (2006), 703–724.

[70] K. Gröchenig, *Weight functions in time-frequency analysis,* in: L. Rodino, M. W. Wong (Eds) Pseudodifferential Operators: Partial Differential Equations and Time-Frequency Analysis, Fields Institute Comm. **52** 2007, pp. 343–366.

[71] K. Gröchenig, T. Strohmer, *Pseudodifferential operators on locally compact abelian groups and Sjöstrand's symbol class*, J. Reine Angew. Math. **613** (2007), 121–146.

[72] K. Gröchenig, J. Toft, *Isomorphism properties of Toeplitz operators and pseudo-differential operators between modulation spaces*, J. Anal. Math. **114** (2011), 255–283.

[73] K. Gröchenig, J. Toft, *The range of localization operators and lifting theorems for modulation and Bargmann-Fock spaces*, Trans. Amer. Math. Soc. **365** (2013), 4475–4496.

[74] K.H. Gröchenig, G. Zimmermann, *Spaces of test functions via the STFT*, J. Funct. Spaces. Appl. **2** (2004), 25–53.

[75] H. Gzyl, *Multidimensional Extension of Faà di Bruno's formula*, J. Math. Anal. Appl. **116** (1986), 450–455.

[76] B. Helffer, *Téorie spectrale pour des operateurs globalement elliptiques*, Asérisque, Société Mathématique de France, 1984.

[77] B. Helffer, D. Robert, *Comportement asymptotique precise du spectre d'operateurs globalement elliptiques dans* $\mathbb{R}^d$, Sem. Goulaouic-Meyer-Schwartz 1980-81, Ecole Polytechénique, Exposé II, 1980.

[78] L. Hörmander, *On the asymptotic distributions of the eigenvalues of pseudodifferential operators in* $\mathbb{R}^n$, Ark. Mat. **17** (1979), 297–313.

[79] L. Hörmander, *Fourier integral operators*, Acta Math. **127** (1971), 79–183.

[80] L. Hörmander, *The Weyl calculus of pseudo-differential operators*, Comm. Pure Appl. Math. **32** (1979), 359–443.

[81] L. Hörmander, *The Analysis of Linear Partial Differential Operators*, vol I–IV, Springer-Verlag, Berlin Heidelberg NewYork Tokyo, 1983, 1985.

[82] V. Ya. Ivrii, *Conditions for correctness in Gevrey classes of the Cauchy problem for weakly hyperbolic equations*, Sib. Math. J. **17** (1976), 422–435.

[83] K. Kajitani, A. Baba, *The Cauchy problem for Schrödinger type equations*, Bull. Sci. Math. **119** (1995), 459-473.

[84] K. Kajitani, T. Nishitani, *The hyperbolic Cauchy problem*, Lecture Notes in Mathematics **1505** Springer-Verlag, Berlin, 1991.

[85] J. J. Kohn, L. Nirenberg, *An algebra of pseudo-differential operators*, Comm. Pure Appl. Math. **18** (1965), 269–305.

[86] H. Kumano-go, *Pseudo-differential operators*, MIT Press, (1981).

[87] P.D. Lax, *Asymptotic solutions of oscillatory initial value problems*, Duke Math. Journal, **24** (1957), 627–646.

[88] N. Lerner, *The Wick calculus of pseudo-differential operators and some of its applications*, CUBO, **5** (2003), 213–236.

[89] N. Lerner, *Metrics on the Phase Space and Non-Selfadjoint Pseudo-Differential Operators*, Birkhäuser Springer, 2010.

[90] Z. Lozanov-Crvenković, D. Perišić, M. Tasković, *Gelfand-Shilov spaces structural and kernel theorems*, (Preprint), arXiv:0706.2268v2.

[91] V. P. Maslov, *Perturbation theory and asymptotic methods*, Moscow, (Russian), 1965.

[92] R. Melrose, *Introduction to microlocal analysis*, Lecture notes from courses taught at MIT, 2007.

[93] B.S. Mitjagin, *Nuclearity and other properties of spaces of type $\mathscr{S}$*, Amer. Math. Soc. Transl., Ser. **93** (1970), 49–59.

[94] Y. Morimoto, *Fundamental Solutions for a Hyperbolic Equation with In-volutive Characteristics of Variable Multiplicity*, Comm. in Partial Differential Equations, **4** (1979), 609–643.

[95] C.Parenti, *Operatori pseudodifferenziali in $\mathbb{R}^n$ e applicazioni*, Ann. Mat. Pura Appl. **93** (1972), 359–389.

[96] C. Pfeuffer, J. Toft, *Compactness properties for modulation spaces*, (Preprint) arXiv:1804.00948 (2018).

[97] S. Pilipović, *Tempered ultradistributions*, Boll. Un. Mat. Ital., VII Ser., **2** (1988), 235–251.

[98] S. Pilipović, N. Teofanov, *On a symbol class of Elliptic Pseudo-differential Operators*, Bull. Acad. Serbe Sci. Arts **27** (2002), 57–68.

[99] S. Pilipović, N. Teofanov, *Pseudodifferential operators on ultramodulation spaces*, J. Funct. Anal. **208** (2004), 194–228.

[100] B. Prangoski, *Pseudodifferential operators of infinite order in spaces of tempered ultradistributions*, J. Pseudo-Differ. Oper. Appl. **4** (2013), 495–549.

[101] S. Rolewicz, *On a certain class of linear metric spaces*, Bull. Acad. Polon. Sci. Sér. Schi. Math. Astronom. Phys. **5** (1957), 471–473.

[102] M. Ruzhansky, M. Sugimoto, N. Tomita, J. Toft, *Changes of variables in modulation and Wiener amalgam spaces*, Math. Nachr. **284** (2011), 2078–2092.

[103] M. Ruzhansky, N. Tokmagambetov, *Nonharmonic analysis of boundary value problems*, Int. Math. Res. Notices **12** (2016), 3548–3615.

[104] X. Saint Raymond, *Elementary Introduction to the Theory of Pseudodifferential Operators*, Studies in Advanced Mathematics, CRC Press; 1 edition, 1991.

[105] B. W. Schulze, *Boundary Value Problems and Singular Pseudo-Differential Operators*, Wiley, 1998.

[106] E. Schrohe, *Spaces of weighted symbols and weighted Sobolev spaces on manifolds* In: Proceedings, Oberwolfach. (H.O. Cordes, B. Gramsch, H. Widom, eds.), pp. 360–377. Springer LMN, Vol. 1256. New York, (1986).

[107] M. A. Shubin, *Pseudodifferential operators and spectral theory*, Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1987.

[108] J. Sjöstrand, *An algebra of pseudodifferential operators*, Math. Res. L. **1** (1994), 185–192.

[109] K. Tachizawa, *The boundedness of pseudo-differential operators on modulation spaces*, Math. Nachr. **168** (1994), 263–277.

[110] K. Taniguchi, *Multiproducts of Fourier integral operators and the fundamental solution for a hyperbolic system with involutive characteristics*, Osaka J. Math. Volume 21, (1984), 169–224.

[111] N. Teofanov, *Ultramodulation spaces and pseudo-differential operators*, Endowment Andrejević, Beograd, 2003.

[112] N. Teofanov, *Modulation spaces, Gelfand-Shilov spaces and pseudodifferential operators*, Sampl. Theory Signal Image Process, **5** (2006), 225–242.

[113] J. Toft, *Continuity Properties for non-Commutative Convolution Algebras with Applications in Pseudo-Differential Calculus*, Bull. Sci. Math. (2) **126** (2002), 115–142.

[114] J. Toft, *Continuity properties for modulation spaces with applications to pseudo-differential calculus, II*, Ann. Global Anal. Geom. **26** (2004), 73–106.

[115] J. Toft, *Schatten-von Neumann properties in the Weyl calculus, and calculus of metrics on symplectic vector spaces*, Ann. Glob. Anal. and Geom. **30** (2006), 169–209.

[116] J. Toft, *Continuity and Schatten properties for pseudo-differential operators on modulation spaces* in: J. Toft, M. W. Wong, H. Zhu (Eds) Modern Trends in Pseudo-Differential Operators, Operator Theory: Advances and Applications **172**, Birkhäuser Verlag, Basel, 2007, pp. 173–206.

[117] J. Toft, *Continuity and Schatten properties for Toeplitz operators on modulation spaces* in: J. Toft, M. W. Wong, H. Zhu (Eds) Modern Trends in Pseudo-Differential Operators, Operator Theory: Advances and Applications **172**, Birkhäuser Verlag, Basel, 2007, pp. 313–328.

[118] J. Toft, *Pseudo-differential operators with smooth symbols on modulation spaces*, Cubo. **11** (2009), 87–107.

[119] J. Toft, *Multiplication properties in pseudo-differential calculus with small regularity on the symbols*, J. Pseudo-Differ. Oper. Appl. **1** (2010), 101–138.

[120] J. Toft, *The Bargmann transform on modulation and Gelfand-Shilov spaces, with applications to Toeplitz and pseudo-differential operators*, J. Pseudo-Differ. Oper. Appl. **3** (2012), 145–227.

[121] J. Toft, *Multiplication properties in Gelfand-Shilov pseudo-differential calculus* in: S. Molahajlo, S. Pilipović, J. Toft, M. W. Wong (eds) *Pseudo-Differential Operators, Generalized Functions and Asymptotics.* Operator Theory: Advances and Applications **231**, Birkhäuser, Basel Heidelberg NewYork Dordrecht London, 2013, pp. 117–172.

[122] J. Toft, *Gabor analysis for a broad class of quasi-Banach modulation spaces.* in: S. Pilipović, J. Toft (eds) *Pseudo-Differential Operators and Generalized Functions.* Operator Theory: Advances and Applications Vol 245, Birkhäuser, Basel Heidelberg NewYork Dordrecht London, 2015, pp. 249–278.

[123] J. Toft, *Images of function and distribution spaces under the Bargmann transform*, J. Pseudo-Differ. Oper. Appl. **8** (2017), 83–139.

[124] J. Toft, *Matrix parameterized pseudo-differential calculi on modulation spaces* in: M. Oberguggenberger, J. Toft, J. Vindas, P. Wahlberg (eds). *Generalized functions and Fourier analysis.* Operator Theory: Advances and Applications Vol 260, Birkhäuser/Springer, Basel, 2017, pp. 215–235.

[125] J. Toft, *Continuity and compactness for pseudo-differential operators with symbols in quasi-Banach spaces or Hörmander classes*, Anal. Appl. **15** (2017), 353–389.

[126] J. Toft, *Continuity of Gevrey-Hörmander pseudo-differential operators on modulation spaces*, (Preprint) arXiv:1710.11366 (2017).

[127] J. Toft, P. Boggiatto, *Schatten classes for Toeplitz operators with Hilbert space windows on modulation spaces*, Adv. Math. **217** (2008), 305–333.

[128] J. Toft, A. Khrennikov, B. Nilsson, S. Nordebo, *Decompositions of Gelfand-Shilov kernels into kernels of similar class*, J. Math. Anal. Appl. **396** (2012), 315–322.

[129] G. Tranquilli, *Global normal forms and global properties in function spaces for second order Shubin type operators*, PhD thesis, 2013.

[130] F. Treves, *Introduction to pseudodifferential operators and Fourier integral operators*, Vol. I, & II. Plenum Publ. Corp., New York, 1980.

[131] L. Zanghirati, *Pseudodifferential operators of infinite order and Gevrey classes*, Ann. Univ Ferrara, Sez. VII, Sc. Mat. **31** (1985), 197–219.

[132] W. M. Wong, *An Introduction to Pseudo-differential Operators 2nd edn*, World Scientific, River Edge, NJ, 1999.