Game approach with the use of technology: A possible way to enhance mathematical thinking

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The purpose of this paper is to draw attention to the strategic thinking that students develop facing mathematical games. We hypothesise that prompting a strategic way of thinking within a didactical intervention, called the 'game-approach', could improve students' proving processes and support them in the production of proofs. More precisely, we are designing tasks based on two players' games in a Multi-touch Dynamic Geometric Environment, in which the discovery of the winning strategy coincides with the discovery of a geometric property. We aim to contribute to the debate on the possibility of cognitive and epistemic unity and to deepen the studies based on dragging practises and their cognitive counterpart.

Keywords: Games-approach, logic of inquiry, tablet.

INTRODUCTION AND THEORETICAL FRAMEWORK

Proving has always been one of the most difficult tasks in teaching and learning mathematics: its formal and rigorous features collide with the fallibility and guessing aspects of the processes that produce it. There are not many teachers in Italian schools who think it is necessary that students experience an exploration and an argumentation phase in order to grasp what a proof is. Teachers' convictions are encouraged by traditional textbooks, where students find the usual definition-statement-proof model. The conditions under which students can make conjectures and validate them are left in the shadow. Furthermore, teachers forget that the formalistic aspects are not the main concern in proving processes (see the discussion of "formal" in Arzarello, 2007).

Some scholars have pointed out this aspect of proofs both from an epistemological (Thurstone, 1994; Tymochcko, 1998) and a didactical (Hanna & de Villers, 2012) standpoint. For example, in the book "Proof and Refutations" (1984) the philosopher Imre Lakatos expressed a different view of mathematical statements. As it is well known, based on a detailed discussion of Euler's errors in his search for a topological classification of polyhedra, Lakatos pointed out a dialectic process in this search. He showed that definitions are not carved in stone, but often have to be patched up in the light of later insights, in particular flawed proofs. This gives mathematics a somewhat experimental flavour. Balacheff agrees with Lakatos' approach, and in the introduction to the French editions of the book, he writes:

> Les mathématiques sont aussi prises en compte, non en tant que text de savoir, mais en tant que savoir construit socialment et donc l'acquisition par l'individu doit etre contrôlée comme sens et pas seulement comme langage. (Lakatos, 1984, p. XVIII)[1]

Following Balacheff's point of view, we are studying a fresh way of introducing pupils into mathematics' rules of thinking in order to produce proofs in elementary geometry. The paper aims to present some reflection of a PhD work in progress research. The hypothesis we are checking consists of investigating if and how the introduction of some multi-touch Dynamic Geometric Environment for Tablets offers important facilities to work in this direction. Here we are referring in particular to applications or software that allow users to work on the same screen with more than one finger at the same time. This peculiarity gives the possibility of designing teaching/learning situations, where students are asked to build up and investigate geometric objects in a new and shared way. The aim of such multi-touch activities is to put students into a geometric game situation, and making them ask why it is... / it is not... /may be... / cannot be... / so; and try to answer the question. We hypothesise that questioning develops a specific type of rational behaviour, which can help students understand and produce proofs.

In the literature our idea finds a foundation in the epistemological research of Jaakko Hintikka (1998, 1999). His aim is to replace the classical and static logic by a dynamic and dialectical model: the *Logic of Inquiry*, based on the analysis of the way people develop their strategic thinking in games. According to him, two types of rules characterize any goal directed activity: *definitory rules* and *strategic rules*. For instance, in chess the definitory rules tell you which moves are possible, while the strategic rules tell you which moves is advisable to make in a given situation.

We can apply this idea to the teaching of deductive logic. It is clear that the rules of inference are definitory rules, not strategic ones. At each stage of a deductive argument, there are normally several propositions that can be used as premises of valid deductive inferences. The so-called rules of inference will tell you which of these alternative applications of the rules of inference are admissible. They do not say anything as to which of these rule applications one ought to make or which ones are better than others. For that purpose you need strategic rules.

Students should learn and keep in mind both these rules during the construction of arguments. However, teachers generally explain to students the rules of inference and the way to use them correctly (definitory rules), not which rules are more advisable to use or how to select new arguments (strategic rules). Therefore, students learn how to avoid making mistakes, but not how to discover proofs or to find out new truths by means of deductive inferences.

> In concentrating their teaching on the so-called rules of inference, logic instructors are merely training their students in how to maintain their logical virtue, not how to reason well. (Hintikka, 1999, p. 3).

Clearly, it is not easy to teach strategic rules. If you are engaged in a game, like chess, it is more natural and taken for granted the use of strategic thinking, than when you are doing mathematics. For this reason, we decided to design games (of mathematical kind), whose solution is the discovering of a geometric property. We hypothesize that the result of this choice could lead students to use strategic thinking within the mathematical one.

In literature the idea of using games for educational purposes has come back in fashion thank to the widespread diffusion of mobile devices and virtual games. “A game can provide a structure for the learning that takes place in the environment” (Devlin, 2011, p. 32). We believe that, in order to bring these innovations into the classroom, *epistemic games* (Shaffer & Gee, 2005) need to be designed. The authors define these games as follows:

> Epistemic games are about knowledge, but they are about knowledge in action- about making knowledge, applying knowledge, and sharing knowledge. (Shaffer & Gee, 2005, p. 16).

Games based on geometric property have to be played in an environment that allows students to come back to what has been done or seen, produce interpretations and possibly explanations, anticipate facts and situations, produce forecasts and hypothetical discourses. In other terms, an environment that allow students to answer such questions, as “How is it?”, “What is best for me to do?”, “How will it be?”, “How could it be?”, namely they apply strategic rules. DGEs are a powerful tool to support students in the formulation of such questions and, therefore, in the application of strategic ways of thinking within mathematics. Indeed they allow the design of dynamic game situations, where, in order to win, students have to discover/use suitable mathematical properties, which allow them to develop suitable strategic moves. Discussing their strategies and why they were suitable for winning, the teacher can coach them towards the formulation and the proof of the mathematical properties that were behind the game, according to the Logic of Inquiry approach.

A further necessary condition for developing the Logic of Inquiry in the classroom, is to let students be used to questioning by themselves during mathematics activities. Hintikka (1999) formulated the structure of the interrogative model in the form of a game between an idealized *inquirer* and a source of answers called *nature* or *oracle*. The inquirer starts from a given theoretical premise T and his/her aim is to establish a certain given conclusion C. At each stage of the game, instead of making a deductive move, the inquirer may address a question to the answerer (oracle, nature, or whatever the source of new information may be). If nature responds, the answer becomes an additional premise. Hintikka calls such a move an *interrogative move*. After that, the process starts again until all the information added to the premise T lead the inquirer to the conclusion C.

The following example (Hintikka, 1999, p. 31) is illuminating about the interrogative model and its importance in reasoning. It shows that we are able to rewrite the solution of any Sherlock Holmes' story in an interrogative form. The episode we analyse is "the curious incident of the dog in the night-time", extracted from the story called "Silver Blaze". The background is this: the famous racing-horse Silver Blaze has been stolen from the stables in the middle of the night, and in the morning its trainer, the stable master, finds it dead out in the heath. All sorts of suspects crop up, but everybody is very much in the dark as to what really happened during the night.

Watson: "Is there any point to which you would wish to draw my attention?"

Sherlock Holmes: "To the curious incident of the dog in the night-time."

Inspector: "The dog did nothing in the night-time."

Sherlock Holmes: "That was the curious incident."

Even Watson can see that Holmes is in effect asking three questions: "Was there a watchdog in the stables when the horse disappeared?", "Did the dog bark when the horse was stolen?", "Who is it that a trained watchdog does not bark at in the middle of the night?". The following deductive argument is the exact transposition of the three questions of Holmes' inquiry: "There was a watchdog in the stables." "The dog did not bark when the horse was stolen." "A trained watchdog does not bark only at its owner." "Hence, the thief was the owner."

Each question is the source of a new *abduction* [2] and it is also an abduction that marks the transition from an inquiring to a deductive approach. Hintikka's analysis shows that this way of thinking does feature the epistemological basis of mathematics altogether and not only of game theory. In particular, it shows an *epistemic unity* between the argumentation phase, represented by the process of questioning, and the proof, represented by the reorganization of the answers in a deductive chain. In the literature, it has already been studied the cognitive unity between argumentation and proof (Boero, Douek, Morselli, & Pedemonte, 2010). Therefore, the result of epistemic unity can deepen these previous studies. In the interrogative model, we find both natural representations of non-logical reasoning (argumentations), and representations of formal logic. Hintikka observes a similarity between the two types of reasoning, founded on the role of presuppositions in the interrogative inquiry. In fact, before the inquirer is in a position to ask a convening question, i.e. "Who did it?" he or she must establish its presupposition "Someone did it". From the point of view of the transition from one proposition to another, an interrogative step looks rather similar to a deductive step: the latter takes the inquirer from one or more premises to a conclusion, while the former takes the inquirer from the presupposition of a question to its answer.

THE GAME-APPROACH

We divide what we call the game-approach into two phases: the game-task design and the so called Devil's Advocate reflection.

The game-task design consists of the transformation of the geometric properties in a *non-cooperative game,* in which each student has a different aim to reach, that contrasts with that of the other player. The task contains the rules of the game, the players' aims and some questions to answer. During the game, there is a silent inquiring activity in the students' mind. Thank to a Tablet and the schoolmate's feedback the inquiring process develops throughout the game, producing interrogative and deductive moves deeply intertwined: deductions are needed for establishing presuppositions for interrogative moves and interrogative moves are needed to add possible new hypothesis to the process of inquiry. In order to create a significant and relevant mathematical experience, we support students in the construction and development of strategies with questions like "Can you write someone else a way for winning?" which indirectly guides their attention to switch from the particular to the general. John Mason summarized these two-way processes as follows:

> ...'to see the general through the particular and the particular in the general' and 'to be aware of

> what is invariant in the midst of change' is how human beings cope with the sense-impressions which form their experience, often implicitly. The aim of scientific thought is to do this explicitly. (Mason, 2005, p. 8)

Finally, we ask students: "How do you know that the method always works?". We hypothesize that, with this question, students can gradually discover the geometric property on which the game is designed, so avoiding a possible gap between it and the mathematics' theory. Students play the game on a shared Tablet and answer the questions working in pairs. Generally, this activity requires a one-hour lesson. After that, we withdraw students' worksheets and prepare a PowerPoint presentation in which we introduce the character called "The Devil's Advocate". This is the second phase, the Devil's Advocate reflection. In this phase, the Devil's Advocate (the teacher or the researcher) makes the Logic of Inquiry more explicit to students. In fact, he questions students to make them think theoretically on what they have found and she insinuates doubts on their deductions and sentences.

THE ANALYSES OF AN EXAMPLE

The episode described below is part of a teaching experiment developed in a tenth grade science class at a private high school. During the activity, the classroom was composed by eight students, working in pairs: they have to read the task, play the game on the tablet and answer some questions on a worksheet. They use GC/htlm5 [3] a newest version of Geometric Constructor (one of the free dynamic geometry software used in Japan since 1989) compatible both with iPad and Android tablets. During this first phase, the role of the teacher is to observe students and to help them if they are in trouble, whereas the role of the researcher is to videotape a single group. The work in pairs activity is followed by the teacher's systematisation of the mathematical content at the blackboard (at that moment the Devil's Advocate reflection has not been designed yet). In the second phase, the teacher asks students what they have discovered during the game in order to engage them as much as possible in the process. The systematisation generally takes place at the beginning of the subsequent lesson for matter of time (each class lasted-50 minutes).

The teaching experiment deals with some themes related to a classical topic included in the National Curriculum 2012 (Indicazioni Nazionali): the circle. The teacher commits almost twelve lessons to the project, developing six themes: the reciprocal position between two circles, the reciprocal position between line and circle, the chords theorem, the angles at the centre and at the circumference, the circumcentre and incentre of triangles, the inscribed and circumscribed quadrilaterals.

The example we will show, describes a non-cooperative game situation involving two players Z and Y. The final aim of the activity is discovering the geometric property that describes the reciprocal positions between two circles. Here is the task given to students: "Play the chase with your schoolmate. Z's aim is changing the length of segment AB by dragging its endpoint in order to make the two circle intersect; Y's aim is changing the length of segment CD by dragging its endpoint in order to avoid the intersection. When does Z win? When does Y win? Move the centre of the circles to examine the possible cases."

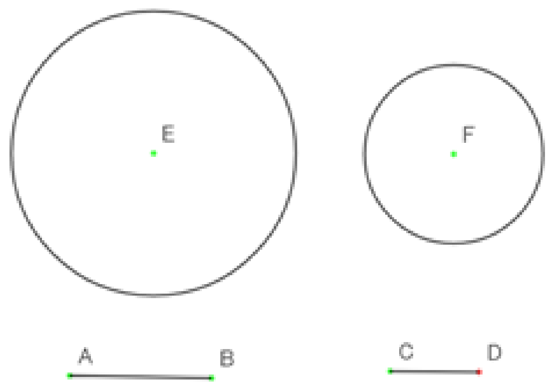
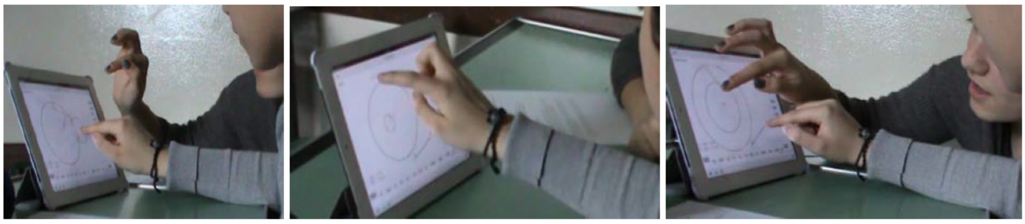


Figure 1: The picture shows what students see on their Tablet

1	Student Y:	Have I to run away? *(Reducing radius of circle E more and more)*
2	Student Z:	I think you could also move... *(Enlarging more and more the radius of circle F)*
3	Student Y:	Yes, but if you are enlarging it, what could I do?
4	Student Z:	I think you could also move... you can try to move this one *(pointing the centre E)*
5	Student Y:	Wait, wait, wait!
6	Student Z:	I think you can try to move this one
7	Student Y:	Yes but if I run away...



2a 2b 2c

Figure 2: 2a) First strategy "Running away"; 2b) Second strategy "A circle inside the other"; 2c) Third strategy "One centre"

8 Student Z: I will make it bigger [the circumference F]!
I stay here so I always catch you!

9 Student Y: If I put myself [circumference F] inside *(pointing the area of circumference E)*, you can make yourself [circumference E] bigger and catch me, but I can… if I tighten…

10 Student Z: No! In this way it should be… it should be… Points should coincide *(pointing the two centre E and F)* because I catch you however!

During the game, students switch between two modalities:

— the *played game* in which students play against each other or collaborate and their only aim is to win;

— the *reflected game* in which students take distance from the game to analyse and make judgments about what has happened. Their aim is collaborate to discover the strategy to win.

In the example, line 5 marks the switch from the played game to the reflected game. When students enter into the second modality, they start analysing the different types of situations that could arise during the game. First, they examine the case in which player Y tries to escape by moving the centre of the circle (line 6–8). In this case, player Z always catches him by making his radius larger. Then they experiment with a second strategy: putting the centre of circle Y inside the circle Z. This second solution leads player Z to win over player Y as well. Finally, student Z suggests to student Y a different situation in which the centres of the circles coincide (line 6).

It is important to notice that during the reflected game, students exchange their roles (line 5), since they not only think about their movement, but at the opponent's movement as well, and they identify themselves with the geometric object: each student is the circle that he/she moves on the screen. While students try to discover the strategy to win, they explore unconsciously the reciprocal position between two circles and they implicitly discover the link between it (the reciprocal position between the circles) and the position of centres or the length of the radiuses. Even if the mathematical theory remains implicit in students' actions, words and visualisation during the whole game, students build strong concept images (Tall & Vinner, 1981), which help them in the construction of mathematical concepts.

As in the case of mouse dragging practises (Arzarello, Olivero, Paola, & Robutti, 2002), we aim to analyse the modality of dragging in order to notice if there is a correspondence with the cognitive level. In particular, in mouse dragging practises, there are two main cognitive typologies (Saada-Robert, 1989; Arzarello, 2007):

— the *ascending processes*, from drawings to theory, in order to explore freely a situation, looking for regularities, invariants, etc.

— the *descending processes* from theory to drawings, in order to validate or refute conjectures, to check properties, etc.

In designing the tasks, we started from the two main cognitive typologies which characterize mouse dragging practises and tried to readapt them in order to describe that of games practices. Insofar, we began transferring the results on mono-touch to multi-touch dynamic geometry software: the aim is to observe what remains invariant and what changes in the students' approaches and processes. In particular, in the played game we distinguish between *ascending processes* when students enter into the game, explore the situation freely, look for strategies and *descending processes* when students play with a strategy in act. In the reflected game, instead, we recognize *ascending* *processes* when students explicitly use the strategy and *descending processes* when they try to check it. We are interested in observing whether or not students share the ascending and descending processes.

In line 1 students are in the played game descending processes, they are playing with a strategy in act: Y tries to escape reducing the radius more and more and Z makes his radius larger following Y's movement. The students realise that with this implicit strategy Z always win. Y is going to give up, when Z suggests her the first explicit strategy (line 6): "moving the centre of the circle to escape". Students are now in the reflected game ascending processes, they play with the explicit strategy and they immediately understand that student Y always loses. Line 8 shows the moment in which students are in the reflected game descending processes, because they check the strategy and decide to abandon it. Students come back to the reflected game ascending process and explicit another strategy: "make a circle inside the other". They immediately pass on the descending process and understand that Z continue to win. They abandon the strategy and make explicit a third one: "make the centres coincide". They are one more time in the ascending process.

Under the first two strategies, there is the implicit mathematical property: "if the distance between centres is less than the sum of their radius and major of their difference, circles intersect at two different points". Since for every Y's movement, there are infinite Z's movements such that the distance between centres is minor then the sum of the radiuses and major then their difference, students are lead to the conclusion that Z always wins. Students do not know the mathematical property that leads them to this conclusion, they only experiment the property in an empirical way, through the game.

CONCLUSION AND POSSIBILITY FOR FURTHER RESEARCH

The analysis of the protocols reveals that the game approach makes students explicit their strategic rules of thinking, but it is not enough to give them insights on the impact of the use of strategic rules of thinking on the mathematical reasoning. In particular, we observed that students do not mention the mathematical property on which the game is based. For this reason we have designed two didactical interventions, which are fundamental in order to both make the mathematics rise from the game and overwhelm the possible cognitive discontinuity between the inquiry phase and the deductive phase:

1) The introduction of specific questions, such as "Can you write someone else a way for winning?" and "How do you know that the method always works?"

2) The introduction of the Devil's Advocate reflection.

The first one helps students thinking the reasons why one wins and detect the geometric property, while the second one helps students in the deductive transpositions of their arguments. Both these interventions make the Logic of Inquiry more explicit to students.

Another issue we are addressing now aims at deepening the technological possibilities offered by DGEs in order to make the game more challenging and engaging. For instance, we are designing more complex games, where players must overcome some intermediate steps in order to win. Such steps correspond to parallel steps in a possible proof of the mathematical properties upon which the game is built. For example we are introducing the opportunity for a player to choose from time to time between two alternative possible constructions in the environment. Only one of them will facilitate her/him: exploiting which is the right one to choose corresponds to a mathematical property, which can facilitate the successive proving phase.

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ENDNOTES

1. "Mathematics is also taken into account, not as a text to know, but as knowledge socially constructed and therefore the acquisition by the individual must be controlled as sense and not as just as language."

2. The following example (Peirce, 1960, p. 372) clarify what an abduction is. Suppose I know that a certain bag is plenty of white beans. Consider the sentences: a) these beans are white; b) the beans of that bag are white; c) these beans are from that bag. A deduction is a concatenation of the form: b and c, hence a; an abduction is: a and b, hence c. An induction is: a and c, hence b. For more details, see Magnani and colleagues (2001) and Arzarello and colleagues (2000).

3. The inventor of GC/htlm5 is Yasuyuki Ijjma from Aichi University of Education (Japan). To visit the software go to http://iijima.auemath.aichi-edu.ac.jp/ftp/yiijima/gc_html5e/gc.htm.