Instability of semi-Riemannian closed geodesics

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Abstract

A celebrated result due to PoincarÃ© affirms that a closed non-degenerate minimizing geodesic γ on an oriented Riemannian surface is hyperbolic. Starting from this classical theorem, our first main result is a general instability criterion for timelike and spacelike closed semi-Riemannian geodesics on both oriented and non-oriented manifolds. A key role is played by the spectral index, a new topological invariant that we define through the spectral flow (being the Morse index truly infinite) of a path of selfadjoint Fredholm operators. A major step in the proof of this result is a *new* spectral flow formula.

Bott's iteration formula, introduced in [Bot56], relates in a clear way the Morse index of an iterated closed Riemannian geodesic and the so-called ω-Morse indices. Our second result is a semi-Riemannian generalization of the famous Bott-type iteration formula in the case of closed (resp. timelike closed) Riemannian (resp. Lorentzian) geodesics.

Our last result is a strong instability result obtained by controlling the Morse index of the geodesic and of all of its iterations.

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Contents

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1 Introduction

A celebrated result proved by PoincarÃ© in the beginning of the last century put on evidence the relation intertwining the *(linear and exponential) instability* of a closed geodesic (as a critical point of the geodesics energy functional on the free loop space) of an oriented Riemannian two-dimensional manifold and its Morse index. The literature on this criterion is quite broad. We refer the interested reader to [Poi99, HS10, Bol88, BT10] and references therein.

Loosely speaking, a closed geodesic γ on M is termed *linearly stable* if the monodromy matrix associated to γ splits into two-dimensional rotations. Accordingly, it is diagonalizable and all Floquet multipliers belong to the unit circle $\mathbf{U}$ of the complex plane $\mathbf{C}$. Additionally, if 1 is not a Floquet multiplier, we term γ *non-degenerate*. Thus, if γ is a stable closed geodesic, then all orbits of the geodesic flow near $\dot\gamma$ in $T_\gamma M$ stay near $\dot\gamma$ for all times. (Cf. Figure 1).

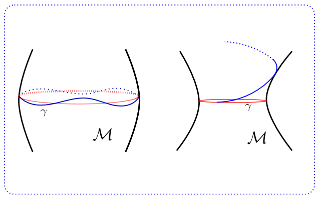

Figure 1: Stable closed geodesics: nearby geodesics stay in a neighborhood of the closed geodesic γ. Then the linearized PoincarÃ© map is diagonalizable and any eigenvalue λ is on the unit circle. Unstable closed geodesics: nearby geodesics have the tendency to diverge. (Image credit to Hans-Bert Rademacher for the picture available at pag.57 of `http://www.math.uni-leipzig.de/~rademacher/tianjin.pdf`).

In 1988 the PoincarÃ© instability criterion for closed geodesics was generalised in several interesting directions by D. V. Treschev in [Tres88]. In this paper the author describes the connection intertwining the *Morse index* of a closed non-degenerate Riemannian geodesic γ and the spectrum of the PoincarÃ© map. More precisely, denoting by $\mathrm{n}_-(\gamma)$ the Morse index of γ as a critical point of the geodesic energy functional on the free loop space of the $(n+1)$-dimensional Riemannian manifold M, the author proved that if either γ is a non-degenerate oriented closed geodesic such that $n + \mathrm{n}_-(\gamma)$ is odd or γ is nonoriented non-degenerate closed geodesic and $n + \mathrm{n}_-(\gamma)$ is even then γ is linearly unstable. Several years later, the first author and his collaborator in [HS10] proved a generalization of the aforementioned result, dropping the non-degeneracy assumption. Of the same flavor is a very recent instability result proved by the first author and his collaborator in the case of Hamiltonian systems. (We refer the interested reader to [HS09, Subsection 4.3, pag. 762]). Due to its interest in dynamical systems, a big effort has been given in the investigation of stability properties of closed geodesics on Riemannian manifolds as well as of periodic solutions of more general Lagrangian systems (cf. [LL02, BJP14, BJP16, HS09] and references therein) under the standard Legendre convexity condition.

Dropping the positivity assumption of the metric tensor is a quite challenging task. The first problem is that the critical points of the geodesic energy functional, have in general infinite Morse index and co-index. However, in this *strongly indefinite* situation a natural substitute of the Morse index is represented by a topological invariant known in literature as the *spectral flow*.

The spectral flow is naturally associated to a path of selfadjoint Fredholm operators arising from the second variation along the geodesic γ. This invariant was introduced by Atiyah, Patodi and Singer in [APS76] in their study of index theory on manifolds with boundary and since then many interesting properties and applications have been established. (Cf., for instance, [PPT04, MPP07, MPW17, HP17b, PW16] and references therein). In general the spectral flow depends on the homotopy class of the whole path and not only on its ends. However in the special case of geodesics on semi-Riemannian manifolds things are simpler since it depends only on the endpoints of the path and therefore it can be considered a relative form of Morse index known in literature as *relative Morse index.*

It is well-known that this invariant is strictly related to a symplectic invariant known in literature as *Maslov index*; by using these two integers, several generalization of the celebrated Morse Index Theorem are available. (We refer the interested reader to cf. [APS08, MPP05, MPP07, HS09, GPP03, LZ00a, LZ00b, Por08, RS95] and references therein). Very recently new spectral flow formulas has been established in the study of heteroclinic and homoclinic orbits of Hamiltonian systems. (Cfr. [BHPT17, HP17a, HPY17]).

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2 Description of the problem and main results

The aim of this section is to describe the problem, the main results and to introduce some basic definitions and finally to fix our notation.

Let $(M, \mathfrak{g})$ be a $(n+1)$-dimensional semi-Riemannian manifold, where $\mathfrak{g}$ is a metric tensor of index, $\mathrm{n}_-(\mathfrak{g}) =: \nu \in \{0, \dots, n+1\}$. If $\nu = 0$ (resp. $\nu = 1$) the pair $(M, \mathfrak{g})$ defines a Riemannian (resp. Lorentzian) manifold. Let ∇ denote the Levi-Civita connection of $\mathfrak{g}$ and let $\mathcal{R}$ be the corresponding curvature tensor chosen with the following sign convention

$$\mathcal{R}(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

where $[\nabla_X, \nabla_Y] \coloneqq \nabla_X \nabla_Y - \nabla_Y \nabla_X$. Denoting by $\frac{\mathrm{D}}{\mathrm{d}t}$ the covariant derivative, we recall that a *closed geodesic* on M is a smooth solution $\gamma\colon [0, T] \to M$ of the following problem

$$\begin{cases} \frac{\mathrm{D}}{\mathrm{d}t}\dot\gamma(t) = 0, & t \in [0, T] \\ \gamma(0) = \gamma(T) \\ \dot\gamma(0) = \dot\gamma(T) \end{cases}$$

where $\cdot$ denotes the time derivative. It is well-known that, if $\gamma : [0, T] \to M$ is a geodesic, then there exists a constant e_γ such that

$$e_\gamma \coloneqq \mathfrak{g}\big(\dot\gamma, \dot\gamma\big). \tag{2.1}$$

The value of e_γ given in Equation (2.1) determines the causal character of the geodesic; more precisely, γ is termed *timelike, lightlike* or *spacelike* if e_γ is negative, zero or positive, respectively. Given a (closed) geodesic γ, a *Jacobi field* is a smooth vector field ξ along γ that satisfies the second order linear differential equation

$$-\frac{\mathrm{D}^2}{\mathrm{d}t^2}\xi(t) + \mathcal{R}(\dot\gamma(t), \xi(t))\dot\gamma(t) = 0, \qquad t \in [0, T]. \tag{2.2}$$

We denote by $\mathfrak{P}_\gamma : T_{\gamma(0)}M \oplus T_{\gamma(0)}M \to T_{\gamma(0)}M \oplus T_{\gamma(0)}M$ the map defined by $\mathfrak{P}_\gamma(v, v') = \big(\xi(T), \frac{\mathrm{D}}{\mathrm{d}t}\xi(T)\big)$ where ξ is the unique Jacobi field along γ such that $\xi(0) = v$ and $\frac{\mathrm{D}}{\mathrm{d}t}\xi(0) = v'$. We assume that γ is a closed spacelike (resp. timelike) geodesic, namely for any $t \in [0, T]$, $\mathfrak{g}(\dot\gamma(t), \dot\gamma(t)) = |\dot\gamma(t)|^2$ (resp. $\mathfrak{g}(\dot\gamma(t), \dot\gamma(t)) = -|\dot\gamma(t)|^2$) . We let $e_0(0) \coloneqq \frac{\dot\gamma(0)}{|\dot\gamma(0)|}$ if γ is spacelike and $e_{n-\nu+1}(0) \coloneqq \frac{\dot\gamma(0)}{|\dot\gamma(0)|}$ if γ is timelike. Then we can find n linearly independent $\mathfrak{g}$-orthonormal vectors in $T_{\gamma(0)}M$ such that the following set

$$\big\{e_1(0), \dots, e_n(0)\big\} \text{ in the spacelike case}$$
$$\Big(\text{resp. } \big\{e_0(0), \dots, e_{n-\nu}(0), e_{n-\nu+2}(0), \dots, e_n(0)\big\} \text{ in the timelike case}\Big)$$

is a basis of the subspace $N_{\gamma(0)}M = \{ v \in T_{\gamma(0)}M \mid \mathfrak{g}(v, \dot\gamma(0)) = 0 \}$. Since parallel transport along γ is a $\mathfrak{g}$-isometry, then it follows that $\{e_1(t), \dots, e_n(t)\}$ is a $\mathfrak{g}$-orthonormal basis of $N_{\gamma(t)}M$ in the spacelike case whilst $\{e_0(t), \dots, e_{n-\nu}(t), e_{n-\nu+2}(t), \dots, e_n(t)\}$ is a $\mathfrak{g}$-orthonormal basis of $N_{\gamma(t)}M$ in the timelike case. Let us denote by $\mathfrak{P}_\gamma^{\perp_\mathfrak{g}} : N_{\gamma(0)} \oplus N_{\gamma(0)} \to N_{\gamma(0)} \oplus N_{\gamma(0)}$ the restriction to the normal subspace at $\gamma(0)$ of the linearized *PoincarÃ© map* defined by $\mathfrak{P}_\gamma^{\perp_\mathfrak{g}}(v, v') = \big(\xi(T), \frac{\mathrm{D}}{\mathrm{d}t}\xi(T)\big)$ where ξ is the unique Jacobi field $\mathfrak{g}$-orthogonal to $\dot\gamma$ such that $\xi(0) = v$ and $\frac{\mathrm{D}}{\mathrm{d}t}\xi(0) = v'$.

Definition 2.1. A closed semi-Riemannian geodesic γ is termed *linearly stable* if the linearized PoincarÃ© map $\mathfrak{P}_\gamma^{\perp_\mathfrak{g}}$ is semisimple and its spectrum $\sigma(\mathfrak{P}_\gamma^{\perp_\mathfrak{g}})$ lies on the unit circle $\mathbf{U}$ of the complex plane. Otherwise it is termed *linearly unstable*.

If $\gamma \in \Lambda M$ is a (non-constant) closed geodesic in $(M, \mathfrak{g})$ we can introduce, by means of the parallel transport, a trivialization of the pull-back bundle $\gamma^*(TM)$ of TM along γ by choosing a parallel $\mathfrak{g}$-orthonormal frame $\mathfrak{E}$ along γ given by $(n+1)$-parallel linearly independent vector fields $e_0, \dots e_n$ and by means of this parallel trivialization, the metric tensor $\mathfrak{g}$ reduces to the constant indefinite scalar product g in $\mathbf{R}^{n+1}$ of constant index ν. Writing the Jacobi vector field along γ in local coordinates as $\xi(t) = \sum_{i=0}^n u_i(t) e_i(t)$, inserting the above expression into the Equation (2.2) and by taking the $\mathfrak{g}$-scalar product with e_j, we reduce it to the linear second order system of ordinary differential equations

$$-\bar G \ddot u(t) + \bar R(t) u(t) = 0, \qquad t \in [0, T] \tag{2.3}$$

where $\bar R_{ij} \coloneqq \mathfrak{g}\big(\mathcal{R}(\dot\gamma, e_i)\dot\gamma, e_j\big)$ and $\bar G = \begin{bmatrix} I_{n+1-\nu} & 0 \\ 0 & -I_\nu \end{bmatrix}$. Being the map $\mathcal{R}(\dot\gamma, \cdot)\dot\gamma$ $\mathfrak{g}$-symmetric, it follows that the matrix $\bar R = [\bar R_{ij}]_{i,j=0}^n$ is symmetric; moreover in the spacelike case, $\bar R_{0i}(t) = \bar R_{i0}(t) = 0$ for any $i = 0, \dots, n$. Since the Jacobi field ξ is T-periodic, inserting the local expression of ξ into the equation $\xi(0) = \xi(T)$, we get the following boundary condition for u:

$$u(0) = \bar A\, u(T) \text{ where } \bar A = [a_{ij}]_{i,j=0}^n.$$

It is worth to observe that, since $\dot\gamma(0) = \dot\gamma(T)$, then we have $a_{00} = 1$ and $a_{0j} = 0$ for any $j = 1, \dots, n$. Furthermore, by a direct computation it follows also that $\bar A$ is $\mathfrak{g}$-orthogonal. By construction, the Morse-Sturm system given in Equation (2.3) decouples into a scalar differential equation (corresponding to the Jacobi field along $\dot\gamma$) and a differential system in $\mathbf{R}^n$ (corresponding to the restriction of the Jacobi deviation equation to vector fields $\mathfrak{g}$-orthogonal to γ). Similarly in the timelike case, we have

$$\bar R_{(n-\nu+1)i}(t) = \bar R_{i(n-\nu+1)}(t) = 0 \text{ and } \qquad a_{(n-\nu+1)(n-\nu+1)} = 1,\ a_{(n-\nu+1)j} = 0 \text{ for } j \neq n - \nu + 1.$$

Depending on the causal character of γ, we can distinguish two cases:

- if γ is *spacelike* then we have

  $$-G \ddot u(t) + \widehat R(t) u(t) = 0, \qquad t \in [0, T] \tag{2.4}$$

  where $u(t) = (u_1(t), \dots, u_n(t))^\mathsf{T}, G \coloneqq \begin{bmatrix} I_{n-\nu} & 0 \\ 0 & -I_\nu \end{bmatrix}$ and $\widehat R(t) = [\bar R_{ij}(t)]_{i,j=1}^n$. Correspondingly the matrix $\bar A$ reduces to

  $$A = [a_{ij}]_{i,j=1}^n.$$

- if γ is *timelike* then we have

  $$-G \ddot u(t) + \widehat R(t) u(t) = 0, \qquad t \in [0, T] \tag{2.5}$$

  where $u(t) = (u_0(t), \dots, u_{n-\nu}(t), u_{n-\nu+2}, \dots, u_n(t))^\mathsf{T}$, $G = \begin{bmatrix} I_{n+1-\nu} & 0 \\ 0 & -I_{\nu-1} \end{bmatrix}$ and $\widehat R(t) = [\bar R_{ij}(t)]_{i,j=0}^n$ and $i, j \neq n - \nu + 1$. In this case the matrix $\bar A$ reduces to

  $$A = [a_{ij}]_{i,j}^n \text{ where } i, j = 0, \dots, n \text{ and } i, j \neq n - \nu + 1.$$

Now, we are entitled to introduce the following definition.

Definition 2.2. A closed geodesic γ is termed oriented if $\det A = 1$; non-oriented if $\det A = -1$.

Remark 2.3. We observe that the orientation of a geodesic γ is independent either on the signature of $\mathfrak{g}$ or on the causal character of γ.

Notation 2.4. In short-hand notation, in the rest of the paper we will denote either a spacelike or a timelike geodesic with the same symbol, if not explicitly stated.

Given a spacelike (resp. timelike) closed geodesic γ, let $t \mapsto \Psi(t)$ be the *flow* of the Morse-Sturm system given in Equation (2.4) (resp. Equation (2.5)); thus for every $t \in [0,T]$, Ψ is the unique linear isomorphism of $\mathbf{R}^n \oplus \mathbf{R}^n$ such that $\Psi(t)\big(G\dot u(0), u(0)\big) = \big(G\dot u(t), u(t)\big)$, where u is a solution of Equation (2.4) (resp. Equation (2.5)). We observe that Ψ is a smooth curve in the general linear group of $\mathbf{R}^n \oplus \mathbf{R}^n$ satisfying the matrix differential equation $\dot\Psi(t) = K(t)\Psi(t)$ with initial condition $\Psi(0) = I$ where K is given by $K(t) \coloneqq \begin{bmatrix} 0 & \widehat R(t) \\ G & 0 \end{bmatrix}$. The symmetry of $\widehat R$ implies that Ψ is actually a (smooth) curve in $\mathrm{Sp}(2n,\mathbf{R})$. We denote by J the standard complex structure, given by $J \coloneqq \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$ where I denotes the identity in the appropriate dimension. Observing that $A^\mathsf{T} G A = G$, it follows that the operator $A_d \coloneqq \begin{bmatrix} A^{-\mathsf{T}} & 0 \\ 0 & A \end{bmatrix}$ lies in $\mathrm{Sp}(2n,\mathbf{R})$ being, in fact, ${A_d}^\mathsf{T} J A_d = J$. By taking into account the $\mathfrak{g}$-orthonormal periodic trivialization of $\gamma^*(TM)$, the induced linearized PoincarÃ© map is given by

$$\mathcal{P}(T) \coloneqq A_d \Psi(T) \in \mathrm{Sp}(2n,\mathbf{R}).$$

Remark 2.5. We observe that, in terms of the the operator A_d, a spacelike (resp. timelike) geodesic γ is *linearly stable* if the symplectic matrix $\mathcal{P}(T)$ is linearly stable.

In order to introduce both the geometric and analytic indices, we start to embed the second order self-adjoint differential operator coming out from the Jacobi deviation equation, into a one parameter family of operator. Now, for any $s \in [0,+\infty)$, we introduce the closed (unbounded) operator $\mathcal{A}_s : \mathscr{D}(\mathcal{A}_s) \subset L^2([0,T];\mathbf{R}^n) \to L^2([0,T];\mathbf{R}^n)$ having domain $\mathscr{D}(\mathcal{A}_s) \coloneqq \{u \in W^{2,2}([0,T],\mathbf{R}^n) : u(0) = Au(T), \dot u(0) = A\dot u(T)\}$ (independent on s) and defined by

$$\mathcal{A}_s(u)(t) \coloneqq -G\frac{d^2}{dt^2} + \widehat R(t) + s\,G.$$

It is well-known (cf., for instance, [GGK90]) that, for each $s \in [0,s_0]$ the operator $\mathcal{A}_s$ is a closed Fredholm operator and selfadjoint in $L^2([0,T];\mathbf{R}^n)$. Since the domain $\mathscr{D}(\mathcal{A}_s)$ doesn't depend on s, the path $s \mapsto \mathcal{A}_s$ can be seen as a path of bounded Fredholm operators from $\mathcal{W} \coloneqq W^{2,2}([0,T];\mathbf{R}^n) \cap W^{1,2}_A([0,T];\mathbf{R}^n)$ into $L^2([0,T],\mathbf{R}^n)$ which are selfadjoint when regarded as operators on L^2, where $W^{1,2}_A([0,T];\mathbf{R}^n) \coloneqq \{u \in W^{1,2}([0,T],\mathbf{R}^n) : u(0) = A\,u(T)\}$. For any $c \in [0,1]$, $s \in [0,+\infty)$ and for any $\omega \in \mathbf{U}$, we define the operator

$$\mathcal{A}^\omega_{c,s} = -G\frac{d^2}{dt^2} + c\widehat R(t) + sG, \qquad t \in [0,T] \tag{2.6}$$

on the Hilbert space

$$E^2_\omega([0,T]) \coloneqq \big\{\, u \in W^{2,2}([0,T],\mathbf{C}^n) \;\big|\; u(0) = \omega A u(T),\ \dot u(0) = \omega A \dot u(T) \,\big\}. \tag{2.7}$$

As we will prove in Corollary 4.5, there exists s_0 sufficiently large such that for $s \geqslant s_0$, the operator $\mathcal{A}^\omega_{1,s}$ is non-degenerate. Now we are entitled to define the spectral index of a closed geodesic γ.

Definition 2.6. Under the previous notation, we term *ω-spectral index* of the closed non-lightlike geodesic γ, the integer $\iota^\omega_{\mathrm{spec}}(\gamma)$ defined by

$$\iota^\omega_{\mathrm{spec}}(\gamma) \coloneqq \mathrm{sf}(\mathcal{A}^\omega_{1,s}; s \in [0,s_0]).$$

Remark 2.7. As already observed, since for $s \geqslant s_0$, the operator $\mathcal{A}^{\omega}_{1,s}$ is non-degenerate, the spectral index given in Definition 2.6 is well-defined, i.e. it is independent on s_0.

We now introduce the Hamiltonian system

$$\dot{z}(t) = JD_{c,s}(t)z(t), \qquad t \in [0,T] \tag{2.8}$$

for $D_{c,s}(t) := \begin{bmatrix} G & 0 \\ 0 & -c\widehat{R}(t) - sG \end{bmatrix}$, $s \in [0,+\infty)$ and we denote by $\Psi_{c,s}$ its fundamental solution.

Definition 2.8. Let γ be a closed non-lightlike geodesic and let $\mathcal{P}_\omega : [0,T] \to \mathrm{Sp}(2n)$ be the path pointwise given by $\mathcal{P}_\omega(t) := \omega A_d \Psi_{1,0}(t)$. We define the *$\omega$-geometric index of* γ as follows

$$\iota^{\omega}_{\mathrm{geo}}(\gamma) := \iota_1(\mathcal{P}_\omega(t); t \in [0,T])$$

where ι_1 is the Maslov-type index (cf. Appendix A and references therein for the definition and the main properties of ι_1).

Notation 2.9. In shorthand notation, we will denote

- the path $\mathcal{P}_1$ (obtained by setting $\omega = 1$) by $\mathcal{P}$;
- $\iota^1_{\mathrm{geo}}(\gamma)$ by $\iota_{\mathrm{geo}}(\gamma)$;
- $\iota^1_{\mathrm{spec}}(\gamma)$ by $\iota_{\mathrm{spec}}(\gamma)$.

Our first main result reads as follows.

Theorem 1. *(***An ω-spectral flow formula***) Under the previous notation, the following spectral flow formula holds:*

$$\iota^{\omega}_{\mathrm{spec}}(\gamma) + \dim\ker(A - \omega I) = \iota^{\omega}_{\mathrm{geo}}(\gamma).$$

As a direct consequence of the ω-spectral flow formula, we immediately get a new Morse-type Index Theorem.

Corollary 1. *(***A Morse Index Theorem** *) Under the previous notation, we have*

$$\iota_{\mathrm{spec}}(\gamma) + \dim\ker(A - I) = \iota_{\mathrm{geo}}(\gamma).$$

Proof. The proof of this result immediately follows by setting $\omega = 1$. ☐

We denote by $\gamma^{(m)} : [0, mT] \to M$ the m-th iteration of the geodesic γ, defined by

$$\gamma^{(m)}(t) := \gamma(t - jT), \quad jT \leqslant t \leqslant (j+1)T,\ j = 0, \dots, m-1.$$

Another contribution of the present paper is represented by a sort of semi-Riemannian version of the Bott-type iteration formula which plays a crucial role in the instability criteria that we shall prove. It is worth to note that, with respect to the classical case, in our framework the Legendre convexity condition does not hold.

Theorem 2. *(***Semi-Riemannian Bott-type formula***) Let $(M, \mathfrak{g})$ be a semi-Riemannian manifold and γ be a closed non-lightlike closed geodesic. For any $m \in \mathbb{N}$, the following iteration formula holds*

$$\iota_{\mathrm{spec}}\big(\gamma^{(m)}\big) = \sum_{\omega^m = 1} \iota^{\omega}_{\mathrm{spec}}(\gamma)$$

By using the spectral flow formula stated in Theorem 1 and by taking into account the homotopy properties of $\mathrm{Sp}^*_\omega(2n)$ (cf. Subsection A.1), we get our main linear instability result for spacelike and timelike closed semi-Riemannian geodesics.

Theorem 3. (Semi-Riemannian Instability criterion) *Let $(M, \mathfrak{g})$ be a $(n+1)$-dimensional semi-Riemannian manifold of index ν and let $\gamma : [0, T] \to M$ be a closed non-lightlike geodesic. If*

(OR) *γ is oriented and $\iota_{\text{spec}}(\gamma) + n$ is odd*

(NOR) *γ is nonoriented and $\iota_{\text{spec}}(\gamma) + n$ is even*

then γ is linearly unstable.

Remark 2.10. It is worth to observe that the spectral flow techniques were successfully developed for proving Morse type index theorem (cfr. [BP10, Theorem 5.6] which holds even for closed non-lightlike geodesic) and Bott's iteration formula (cf. [JP08, Theorem 5.3]) for closed semi-Riemannian geodesics. Please note that here we choose a different operator path to define the spectral flow. The choice made by authors in the [BP10] is in the direction of extending the classical Morse index theorem for closed Riemannian geodesics whilst our choice comes from the original purpose of investigating the instability of closed semi-Riemannian geodesics.

An immediate consequence of Theorem 3 is the following instability criterion for closed Riemannian (resp. Lorentzian) geodesics (resp.timelike geodesics).

Corollary 2. *Let $(M, \mathfrak{g})$ be a $(n+1)$-dimensional Riemannian manifold (resp. Lorentzian) and let $\gamma : [0, T] \to M$ be a closed (resp. timelike closed) geodesic. If one of the following two alternatives hold*

(OR) *γ is oriented (resp. oriented and timelike) and $\mathrm{n}_-(\gamma) + n$ is odd*

(NOR) *γ is non-oriented (resp. non-oriented and timelike) and $\mathrm{n}_-(\gamma) + n$ is even*

then the geodesic is linearly unstable.

Proof. The proof of this result readily follows by Theorem 3 once observed that for closed Riemannian (resp. timelike closed Lorentzian) geodesics, $\iota_{\text{spec}}(\gamma) = \mathrm{n}_-(\gamma)$(cf. [HS09, Section 3] or [HS10, Section 3]). □

Remark 2.11. It is worth noting that the celebrated Poincare's instability criterion, can be recovered by Corollary 2. In fact, if γ is a minimizing closed geodesic, then $\mathrm{n}_-(\gamma) = 0$ and on a Riemannian surface $n = 1$; thus $\mathrm{n}_-(\gamma) + n = 1$.

Before describing the last result, we introduce the notion of strong stability for closed geodesic on Riemannian or Lorentzian manifolds.

Definition 2.12. Let γ be a closed (resp. timelike closed) Riemannian (resp. Lorentzian) geodesic and let $\mathcal{P}(T)$ be the corresponding monodromy matrix. We say that γ is termed *strongly stable* if, there exists $\varepsilon > 0$ such that any symplectic matrix M with $\|M - \mathcal{P}(T)\| \leqslant \varepsilon$ is linearly stable.

Theorem 4. *Let $(M, \mathfrak{g})$ be a $(n+1)$-dimensional Riemannian (resp. Lorentzian) manifold and let γ be a closed (resp. timelike closed) Riemannian (resp. Lorentzian) geodesic. We assume that*

$$\mathrm{n}_-\big(\gamma^{(m)}\big) = 0, \qquad \textit{for any } m \in \mathbb{N}.$$

Then γ is not strongly stable.

3 Variational preliminaries

The aim of this section is to introduce the basic geometric and variational setting and to fix our notations. Our basic references are [Kli78].

It is well-known that, closed semi-Riemannian geodesics are critical points of the *geodesic energy functional* defined on the *free loop space* ΛM of M where ΛM denotes the Hilbert manifold of all closed curves in M of Sobolev class $W^{1,2}$. Let $\mathbb{S}$ be the circle, viewed as the quotient $[0, T]/\{0, T\}$,

$\Lambda M := W^{1,2}(\mathbb{S}, M)$ be the infinite dimensional Hilbert manifold (cf. [Kli78, Theorem 1.2.9, pag.13]) of all loops $\gamma : [0,T] \to M$ (namely $\gamma(0) = \gamma(T)$) of Sobolev class $W^{1,2}$. For any $\gamma \in \Lambda M$, the tangent space $T_\gamma \Lambda M$ can be identified with the Hilbert space of all sections of the pull-back bundle $\gamma^*(TM)$ (namely the bundle of all periodic vector fields along γ) of Sobolev class $W^{1,2}$ (cf. [Kli78, Theorem 1.3.6, pag.19]) that we'll denote by $\mathcal{H}_\gamma$; thus

$$\mathcal{H}_\gamma = \{\xi \in W^{1,2}(\mathbb{S}, TM) : \tau \circ \xi = \gamma\},$$

where $\tau : TM \to M$ denotes the canonical tangent bundle projection. We consider the *geodesics energy functional* $E : \Lambda M \to \mathbf{R}$ given by

$$E(\gamma) = \frac{1}{2} \int_0^T \mathfrak{g}\big(\dot\gamma(t), \dot\gamma(t)\big)\, dt$$

and we observe that E is differentiable and for any $\xi \in T_\gamma \Lambda M$

$$dE(\gamma)[\xi] = \int_0^T \mathfrak{g}\left(\tfrac{\mathrm{D}}{\mathrm{d}t}\xi(t), \dot\gamma(t)\right) dt.$$

(Cf. [Kli78, Lemma 1.3.9, pag.21], for further details). By standard regularity arguments and by performing integration by parts, it follows that critical points corresponds to closed geodesics.

Lemma 3.1. *$\gamma \in \Lambda M$ is a closed geodesic (or a constant map) if and only if it is a critical point of E, namely $dE(\gamma)[\xi] = 0$ for all $\xi \in T_\gamma \Lambda M$.*

Proof. For the proof we refer the interested reader to [Kli78, Theorem 1.3.11]. □

If $\gamma \in \Lambda M$ is a non-constant closed geodesic in $(M, \mathfrak{g})$ then the second variation of the geodesic energy functional E at γ is the *index form* $\mathfrak{I}_\gamma$ given by

$$\mathfrak{I}_\gamma[\xi, \eta] = \int_0^T \Big[\mathfrak{g}\left(\tfrac{\mathrm{D}}{\mathrm{d}t}\xi(t), \tfrac{\mathrm{D}}{\mathrm{d}t}\eta(t)\right) + \mathfrak{g}\big(\mathcal{R}(\dot\gamma(t), \xi(t))\dot\gamma(t), \eta(t)\big)\Big] dt.$$

It is readily seen that $\mathfrak{I}_\gamma$ is a bounded symmetric bilinear form on $T_\gamma \Lambda M$ whose associated quadratic form will be denoted by $\mathfrak{Q}_\gamma$. Let $\mathcal{H}_\gamma^\perp \subset \mathcal{H}_\gamma$ be the closed (real) codimensional one subspace of all T-periodic $W^{1,2}$-vector fields along γ that are everywhere orthogonal to $\dot\gamma$. We observe that $\dim\ker \mathfrak{Q}_\gamma = \dim\ker \left(\mathfrak{Q}_\gamma|_{\mathcal{H}_\gamma^\perp}\right) + 1$. We term *nullity of the geodesic* γ the (non-negative) integer defined by $n_0(\mathfrak{Q}_\gamma) := \dim\ker \left(\mathfrak{Q}_\gamma|_{\mathcal{H}_\gamma^\perp}\right)$. Thus the nullity of γ is defined as the dimension of the space of periodic Jacobi fields along γ that are pointwise $\mathfrak{g}$-orthogonal to $\dot\gamma$.

Since e_i is a parallel vector field along γ and if ξ is a Jacobi field, then, by direct computation we have

$$\frac{d^2}{d^2t}\mathfrak{g}(\xi(t), \dot\gamma(t)) = \mathfrak{g}(\tfrac{\mathrm{D}^2}{\mathrm{d}t^2}\xi(t), \dot\gamma(t)) = \mathfrak{g}(\mathcal{R}(\dot\gamma(t), \xi(t))\dot\gamma(t), \dot\gamma(t)) = 0,$$

so $\mathfrak{g}(\xi(t), \dot\gamma(t)) = \alpha t + \beta$ for some $\alpha, \beta \in \mathbf{R}$. In particular, if $\xi(0), \xi'(0) \in N_{\gamma(0)}M$, it follows that $\alpha = \beta = 0$. which is equivalent to $\mathfrak{g}(\xi(t), \dot\gamma(t)) \equiv 0$ for any $t \in [0,T]$. By this fact, we infer that $N_{\gamma(0)}M \oplus N_{\gamma(0)}M$ is invariant under the linearized geodesics flow. (As we already observed, being the parallel transport a $\mathfrak{g}$-isometry). In particular the Morse-Sturm system given in Equation (2.3) decouples into a scalar differential equation (corresponding to the Jacobi field) and a differential system in $\mathbf{R}^n$ (corresponding to the restriction of the Jacobi deviation equation to vector fields $\mathfrak{g}$-orthogonal to $\dot\gamma$).

4 Geometric and spectral index of a closed geodesic

This section is devoted to show that the geometrical and the spectral index previously introduced are actually well-defined.

Lemma 4.1. *The geometric index of a closed non-lightlike semi-Riemannian geodesic γ is well-defined, i.e. it is independent on the trivializing parallel frame along γ.*

Proof. We prove the result only in the case of closed spacelike geodesics being the timelike case completely analogous. Let $\{f_1(0), \ldots, f_n(0)\}$ be another $\mathfrak{g}$-orthonormal basis of $N_{\gamma(0)}M$ such that $f_i(0) = \sum_{j=1}^n c_{ij}e_j(0)$ and we let $C \coloneqq [c_{ij}]_{i,j=1}^n$. Then, by direct computation, the Morse-Sturm system given in Equation (2.4) fits into the following

$$-G\ddot{u}(t) + \widetilde{R}(t)u(t) = 0, \qquad t \in [0,T]$$

where $\widetilde{R}(t) = C\widehat{R}(t)C^\mathsf{T}$ and the boundary condition of u is given by $u(0) = AC^\mathsf{T}u(T)$. We let $C_d \coloneqq \begin{bmatrix} C^{-\mathsf{T}} & 0 \\ 0 & C \end{bmatrix}$ and let $\widetilde{\Psi}$ be the fundamental solution of the corresponding Hamiltonian system given by $\dot{z}(t) = \widetilde{K}(t)z(t)$, with $\widetilde{K}(t) = C_dK(t)C_d^\mathsf{T}$. It is easy to check that $\widetilde{\Psi}(t) = C_d^{-\mathsf{T}}\Psi(t)$ and that

$$\iota_1\big(A_dC_d^\mathsf{T}\widetilde{\Psi}(t); t \in [0,T]\big) = \iota_1\big(A_d\Psi(t); t \in [0,T]\big).$$

This concludes the proof. ☐

Lemma 4.2. *For every $c \in [0,1]$ and $s \in [0,+\infty)$, the operator $\mathcal{A}^\omega_{c,s}$ is formally selfadjoint.*

Proof. The proof of this result readily follows by integrating by parts. ☐

Lemma 4.3. *For any $c_0 > 0$, there exists a sufficiently large s_0 such that for any $|c| \leqslant c_0$ and $s \geqslant s_0$, the operator defined by*

$$\mathcal{B}_{c,s} \coloneqq -G\frac{d^2}{dt^2} + cI + sG, \qquad s \in [0,+\infty)$$

is non-degenerate (meaning that has a trivial kernel) on

$$E^2_\omega([0,T]) = \big\{\, u \in W^{2,2}([0,T],\mathbf{C}^n) \;\big|\; u(0) = \omega Au(T),\ \dot{u}(0) = \omega A\dot{u}(T) \,\big\} \text{ for every } \omega \in \mathbf{U}.$$

Proof. We consider the Morse-Sturm system

$$\begin{cases} -G\ddot{u}(t) + \big(cI + sG\big)u(t) = 0, & t \in [0,T] \\ u(0) = \omega Au(T), \quad \dot{u}(0) = \omega A\dot{u}(T). \end{cases}$$

Then the corresponding Hamiltonian system is given by

$$\dot{z}(t) = JB_{c,s}z(t), \qquad t \in [0,T]$$

where $B_{c,s} \coloneqq \begin{bmatrix} G & 0 \\ 0 & -cI - sG \end{bmatrix}$ and the boundary condition is given by $z(0) = A_dz(T)$. Denoting by $\Phi_{c,s}$ the fundamental solution of the Hamiltonian system, it follows that the non-degeneracy of $\mathcal{B}_{c,s}$ is equivalent to the following condition

$$\det(\omega A_d\Phi_{c,s}(T) - I) \neq 0.$$

If s is sufficiently large, then $s \pm c$ is positive. We set

$$\lambda \coloneqq \sqrt{s+c} \quad \text{and} \quad \mu = \sqrt{s-c}.$$

By a direct computation, we get

$$\Phi_{c,s}(t) = \begin{bmatrix} \cosh(\lambda t)I & 0 & \lambda\sinh(\lambda t)I & 0 \\ 0 & \cosh(\mu t)I & 0 & -\mu\sinh(\mu t)I \\ \dfrac{\sinh(\lambda t)}{\lambda}I & 0 & \cosh(\lambda t)I & 0 \\ 0 & -\dfrac{\sinh(\mu t)}{\mu}I & 0 & \cosh(\mu t)I \end{bmatrix}, \qquad t \in [0,T].$$

We observe that

$$(\omega A_d)^{-1} = \bar{\omega} A_d^{-1} = \bar{\omega} \begin{bmatrix} A^\mathsf{T} & 0 \\ 0 & A^{-1} \end{bmatrix} = \begin{bmatrix} \bar{\omega} A^\mathsf{T} & 0 \\ 0 & \bar{\omega} A^{-1} \end{bmatrix} \text{ and}$$

$$(\bar{\omega} A^\mathsf{T}) G (\omega A) = G \Rightarrow \bar{\omega} A^\mathsf{T} = G(\omega A)^{-1} G = G(\bar{\omega} A^{-1}) G.$$

We let

$$\bar{\omega} A^{-1} := \begin{bmatrix} P & Q \\ R & S \end{bmatrix}, \qquad \text{we immediately get that } \bar{\omega} A^\mathsf{T} = \begin{bmatrix} P & -Q \\ -R & S \end{bmatrix}.$$

In shorthand notation, we let

$$U := \begin{bmatrix} \cosh(\lambda T) I - P & Q \\ R & \cosh(\mu T) I - S \end{bmatrix}, \quad V := \begin{bmatrix} \lambda \sinh(\lambda T) I & 0 \\ 0 & -\mu \sinh(\mu T) I \end{bmatrix}$$

$$X := \begin{bmatrix} \dfrac{\sinh(\lambda T)}{\lambda} I & 0 \\ 0 & -\dfrac{\sinh(\mu T)}{\mu} I \end{bmatrix}, \quad Y := \begin{bmatrix} \cosh(\lambda T) I - P & -Q \\ -R & \cosh(\mu T) I - S \end{bmatrix}$$

and we observe that U, V, X, Y are $n \times n$ matrices. Then $\Phi_{c,s}(T) - (\omega A_d)^{-1} = \begin{bmatrix} U & V \\ X & Y \end{bmatrix}$ and hence

$$\begin{aligned}
\det\big(\omega A_d \Phi_{c,s}(T) - I\big) &= \det(\omega A_d) \cdot \det\big(\Phi_{c,s}(T) - (\omega A_d)^{-1}\big) \\
&= \omega^{2n} \cdot \det A \cdot \det A^{-\mathsf{T}} \det \begin{bmatrix} U & V \\ X & Y \end{bmatrix} \\
&= (-1)^{n^2} \omega^{2n} \cdot \det \begin{bmatrix} U & V \\ X & Y \end{bmatrix} = (-1)^n \omega^{2n} \cdot \det\big(V \cdot (X - Y V^{-1} U)\big).
\end{aligned}$$

By a straightforward calculation, we get

$$\begin{aligned}
&V \cdot (X - Y V^{-1} U) \\
&= \begin{bmatrix} -\big(\cosh(\lambda T) - P\big)^2 - C_{\lambda,\mu} Q R + \sinh^2(\lambda T) & -\big(\cosh(\lambda T) - P\big) Q - C_{\lambda,\mu} Q \big(\cosh(\mu T) - S\big) \\ -C_{\lambda,\mu}^{-1} R \big(\cosh(\lambda T) - P\big) - \big(\cosh(\mu T) - S\big) R & -\big(\cosh(\mu T) - S\big)^2 - C_{\lambda,\mu}^{-1} R Q + \sinh^2(\mu T) \end{bmatrix}
\end{aligned}$$

where we set $C_{\lambda,\mu} := \dfrac{\lambda \sinh(\lambda T)}{\mu \sinh(\mu T)}$. Thus asymptotically, we get following behavior

$$\begin{aligned}
-\big(\cosh(\lambda T) - P\big)^2 - C_{\lambda,\mu} Q R + \sinh^2(\lambda T) = -1 + 2\cosh(\lambda T) P - P^2 - C_{\lambda,\mu} Q R &\sim_{+\infty} 2\cosh(\lambda T) P \\
-\big(\cosh(\lambda T) - P\big) Q - C_{\lambda,\mu} Q \big(\cosh(\mu T) - S\big) &\sim_{+\infty} -2\cosh(\lambda T) Q \\
-C_{\lambda,\mu}^{-1} R \big(\cosh(\lambda T) - P\big) - \big(\cosh(\mu T) - S\big) R &\sim_{+\infty} -2\cosh(\lambda T) R \\
-\big(\cosh(\mu T) - S\big)^2 - C_{\lambda,\mu}^{-1} R Q + \sinh^2(\mu T) &\sim_{+\infty} 2\cosh(\lambda T) S.
\end{aligned}$$

Summing up, we have

$$V \cdot (X - Y V^{-1} U) \sim_{+\infty} 2\cosh(\lambda T) \begin{bmatrix} P & -Q \\ -R & S \end{bmatrix} = 2\cosh(\lambda T) \bar{\omega} A^\mathsf{T}. \tag{4.1}$$

By taking into account Equation (4.1), it holds that

(4.2)

$$\begin{aligned}
\det\big(\omega A_d \Phi_{c,s}(T) - I\big) &= (-1)^n \omega^{2n} \det\big(V(X - Y V^{-1} U)\big) \quad \sim_{+\infty} (-1)^n \omega^{2n} \det\big(2\cosh(\lambda T) \bar{\omega} A^\mathsf{T}\big) \\
&= (-1)^n \omega^{2n} \big(2\cosh(\lambda T)\big)^n \bar{\omega}^n \det A \\
&\neq 0.
\end{aligned}$$

By Equation (4.2) the thesis readily follows. This concludes the proof. □

In order to prove the non-degeneracy of the operator $\mathcal{A}^\omega_{1,s}$ for s sufficiently large, we need the following stability result proved by Kato.

Lemma 4.4. *Let T be a selfadjoint operator and A be symmetric. Then the operator $S := T + A$ is selfadjoint and*

$$\operatorname{dist}\big(\sigma(S), \sigma(T)\big) \leqslant \|A\|,$$

where dist$(\cdot,\cdot)$ is the Hausdorff distance.

Proof. For the proof, we refer the interested reader, to [Kat80, pag. 291]. □

Corollary 4.5. *Let $c_0 > 0$ be such that $\left\|\widehat{R}\right\|_{\mathscr{L}(\mathbf{R}^n)} \leqslant c_0$ and s_0 be the number related to c_0 by Lemma 4.3. Then for any $s \geqslant s_0$ it holds*

$$\mathcal{A}^\omega_{1,s} := -G\frac{d^2}{dt^2} + \widehat{R} + sG$$

is non-degenerate.

Proof. The proof of this result follows by Lemma 4.3 and Lemma 4.4, just by setting (with a slight abuse of notation)

$$T := -G\frac{d^2}{dt^2} + sG \text{ and } A := \widehat{R}.$$

This concludes the proof. □

Remark 4.6. We observe that in the Riemannian case, directly proof of this result can be easily conceived. However in the semi-Riemannian setting the abstract way (which works in the Riemannian world) for simplifying this proof breaks-down. The obstruction to carry over that proof is essentially based on the fact that the matrix A coming from the trivialization is a $\mathfrak{g}$-orthogonal and not just an orthogonal matrix, like in the Riemannain case.

5 A generalized spectral flow formula

This section is devoted to the relation intertwining the spectral index and the geometric index. As a direct consequence we get a spectral flow formula involving the ω-spectral index and the geometric index. The following spectral flow formula holds.

Proposition 5.1. *Let s_0 be given in Lemma 4.3. Then, we have*

$$\operatorname{sf}(\mathcal{A}^\omega_{1,s}; s \in [0, s_0]) = -\iota_1\big(\omega A_d \Psi_{1,s}(T); s \in [0, s_0]\big).$$

Proof. For a given matrix $M \in L(\mathbf{C}^n)$, we denote its graph by $\operatorname{Gr}(M) := \{(x, (Mx)^{\mathsf{T}})^{\mathsf{T}} \mid x \in \mathbf{C}^n\}$. By Definition A.11 of the μ^{CLM}-index, we have

$$\begin{aligned}\iota_1\big(\omega A_d \Psi_{1,s}(T); s \in [0, s_0]\big) &= \iota_{\bar\omega}\big(A_d \Psi_{1,s}(T); s \in [0, s_0]\big) = \iota_\omega\big(A_d \Psi_{1,s}(T); s \in [0, s_0]\big)\\ &= \mu^{\text{CLM}}\big(\operatorname{Gr}(\omega I), \operatorname{Gr}\big(A_d\Psi_{1,s}(T)\big); s \in [0, s_0]\big),\end{aligned}$$

where the last equality follows from Equation (A.1) and Proposition A.26. By invoking [HS09, Lemma 2.3 and Lemma 2.4], we have

$$\mu^{\text{CLM}}\big(\operatorname{Gr}(\omega I), \operatorname{Gr}\big(A_d\Psi_{1,s}(T)\big); s \in [0, s_0]\big) = -\operatorname{sf}(\mathcal{D}^\omega_{1,s}; s \in [0, s_0])$$

where $\mathcal{D}_{c,s} := -J\dfrac{d}{dt} - D_{c,s}(t); s \in [0, s_0]$ on the Sobolev space

$$E^1_\omega([0,T]) := \left\{\, u \in W^{1,2}([0,T], \mathbf{C}^{2n}) \;\middle|\; u(0) = \omega A_d u(T) \,\right\}.$$

In order to concludes the proof, it is enough to prove that

$$\mathrm{sf}(\mathcal{A}^\omega_{1,s}; s \in [0, s_0]) = \mathrm{sf}(\mathcal{D}^\omega_{1,s}; s \in [0, s_0]).$$

For, we start to assume that both paths are regular in the sense specified in Appendix A.2. Under this regularity assumption it is enough to show that the local contribution of the spectral flow of both paths $\mathcal{A}^\omega_{1,s}$ and $\mathcal{D}^\omega_{1,s}$ coincide. Let $s_* \in [0, s_0]$ and we start to observe that

$$\dim\ker \mathcal{A}_{1,s_*} \neq \{0\} \iff \dim\ker \mathcal{D}_{1,s_*} \neq \{0\}.$$

In particular s_* is a crossing instant for $s \mapsto \mathcal{A}_{1,s}$ if and only if it is a crossing instant for $s \mapsto \mathcal{D}_{1,s}$. Let s_* be a crossing instant. By a direct computation, it follows that the crossing form of $\mathcal{A}$ at s_* is the quadratic form

$$\Gamma(\mathcal{A}_{1,s}, s_*) : \ker \mathcal{A}_{1,s} \to \mathbf{R} \text{ defined by } \Gamma(\mathcal{A}_{1,s}, s_*)[u] = \int_0^T \langle Gu, u\rangle\, dt, \quad u \in \ker \mathcal{A}_{1,s_*}. \tag{5.1}$$

The crossing form of $\mathcal{D}_{1,s}$ at s_* is given by

$$\begin{aligned}\Gamma(\mathcal{D}_{1,s}, s_*) : \ker \mathcal{D}_{1,s} \to \mathbf{R} \text{ defined by } \Gamma(\mathcal{D}_{1,s}, s_*)[w] &= \int_0^T \left\langle \begin{bmatrix} 0 & 0 \\ 0 & G\end{bmatrix} \begin{bmatrix} G\dot w \\ w\end{bmatrix}, \begin{bmatrix} G\dot w \\ w\end{bmatrix}\right\rangle dt \\ &= \int_0^T \langle Gw, w\rangle\, dt \quad w \in \ker \mathcal{D}_{1,s_*}.\end{aligned} \tag{5.2}$$

By Equations (5.1)-(5.2), we immediately conclude that the crossing forms coincide and hence also their signatures. In particular the local contribution at s_* of both paths to the spectral flow coincide and this concludes the proof, under the assumption that $s \mapsto \mathcal{A}_{1,s}$ and $s \mapsto \mathcal{D}_{1,s}$ are regular.

In order to concludes the proof in the general case (i.e. for non-regular paths), we observe that by standard perturbation results there exists $\varepsilon > 0$ sufficiently small (cf. Appendix A.2 and references therein) such that the perturbed path $s \mapsto \mathcal{A}^\varepsilon_{1,s} := \mathcal{A}_{1,s} + \varepsilon I$ is regular and, by the homotopy invariance property with fixed ends, it has the same spectral flow of $s \mapsto \mathcal{A}_{1,s}$. Through the perturbed path $\mathcal{A}^\varepsilon_{1,s}$, we can define the perturbed path of first order operators given by

$$\mathcal{D}^\varepsilon_{1,s} := -J\frac{d}{dt} - D^\varepsilon_{1,s}(t)$$

where $D^\varepsilon_{1,s}(t) := \begin{bmatrix} G & 0 \\ 0 & -\widehat{R}(t) - sG + \varepsilon I\end{bmatrix}$. Repeating ad verbatim the same arguments for the perturbed paths, we get the thesis. This concludes the proof. □

Our next step is to compute the integer $\iota_1\big(\omega A_d \Psi_{0,s_0}(t); t \in [0, T]\big)$. Before introducing a technical result needed for this computation, we observe that

$$\begin{aligned}\iota_1\big(\omega A_d \Psi_{0,s_0}(t); t \in [0, T]\big) &= \iota_\omega\big(A_d \Psi_{0,s_0}(t); t \in [0, T]\big) \\ &= \mu^{\mathrm{CLM}}\Big(\mathrm{Gr}\big(\omega A_d^{-1}\big), \mathrm{Gr}\big(\Psi_{0,s_0}(t)\big); t \in [0, T]\Big).\end{aligned} \tag{5.3}$$

The idea for computing the (RHS) of Equation (5.3) is to transform the path of graphs of a symplectic matrix into a path of graphs of a symmetric matrices. In this way the computation of the μ^{CLM} index can be performed through the spectral flow.

Lemma 5.2. *Under the previous notation, we have*

$$\mu^{\mathrm{CLM}}\Big(\mathrm{Gr}\big(\omega A_d^{-1}\big), \mathrm{Gr}\big(\Psi_{0,s_0}(t)\big); t \in [0, T]\Big) = \mu^{\mathrm{CLM}}\Big(L_D, \mathrm{Gr}\big(M_{s_0}(t)\big); t \in [0, T]\Big)$$

where $L_D = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \,\middle|\, x \in \mathbf{C}^{2n} \right\}$ *is the (horizontal) Dirichlet Lagrangian subspace and* $t \mapsto M_{s_0}(t)$ *is the path pointwise defined by*

$$M_{s_0}(t) := \begin{bmatrix} \dfrac{\sinh(\sqrt{s_0}t)}{\sqrt{s_0}\cosh(\sqrt{s_0}t)}G & \dfrac{I}{\cosh(\sqrt{s_0}t)} - \omega A^{-1} \\ \dfrac{I}{\cosh(\sqrt{s_0}t)} - \bar{\omega}A^{-\mathsf{T}} & -\sqrt{s_0}\dfrac{\sinh(\sqrt{s_0}t)}{\cosh(\sqrt{s_0}t)}G \end{bmatrix}.$$

Remark 5.3. We observe that if $P \in \mathrm{Sp}(2n)$, then

$$\mathrm{Gr}(P) = \left\{ \begin{bmatrix} x \\ Px \end{bmatrix} \,\middle|\, x \in \mathbf{C}^{2n} \right\}$$

is a Lagrangian subspace of $\mathbf{C}^{2n} \oplus \mathbf{C}^{2n}$ under the symplectic form $-J \oplus J$, where J is the standard symplectic matrix of $\mathbf{C}^{2n}$. We recall that a *Lagrangian frame* for a Lagrangian subspace L is an injective linear map $Z : \mathbf{C}^n \to \mathbf{C}^{2n}$ whose image is L. Such a frame has the form $Z = (X, Y)^{\mathsf{T}}$ where X, Y are $n \times n$-matrices and $Y^*X = X^*Y$, where $*$ denotes the conjugate transpose. In the special case in which L is a graph of a symmetric matrix, then $X = I$ and the relation $Y^*X = X^*Y$ trivially holds. We observe also that if X is invertible, then another Lagrangian frame for L with respect to which it is a graph, is given by $W = (I, YX^{-1})^{\mathsf{T}}$. In fact, being $L = \mathrm{rge}\,(Z) = \left\{ \begin{bmatrix} Xu \\ Yu \end{bmatrix} \,\middle|\, u \in \mathbf{C}^n \right\}$, it follows that by changing coordinates by setting $u = X^{-1}w$, we get $L = \mathrm{rge}\,(Z) = \left\{ \begin{bmatrix} w \\ YX^{-1}w \end{bmatrix} \,\middle|\, u \in \mathbf{C}^n \right\}$. We start to observe that if L is a Lagrangian subspace with respect to a symplectic form $\widehat{\omega}(\cdot,\cdot) := \langle \widehat{J}\cdot, \cdot\rangle$, then for any orthogonal matrix S, the subspace $S^{-1}L$ is a Lagrangian subspace with respect to the symplectic form $\widetilde{\omega}(\cdot,\cdot) := \langle \widetilde{J}\cdot, \cdot\rangle$ represented by $\widetilde{J} := S^{\mathsf{T}}\widehat{J}S$. Moreover, if $\begin{bmatrix} Y \\ X \end{bmatrix}$ is a Lagrangian frame for L, then $S^{-1}\begin{bmatrix} Y \\ X \end{bmatrix}$ is a Lagrangian frame for $S^{-1}L$.

Proof. We start to define the orthogonal matrix S given by $S := \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \end{bmatrix}$. Identifying $\mathrm{Gr}\big(\Psi_{0,s_0}(t)\big)$ with its Lagrangian frame $\begin{bmatrix} I & 0 \\ 0 & I \\ \cosh(\sqrt{s_0}t)I & \sqrt{s_0}\sinh(\sqrt{s_0}t)G \\ \dfrac{1}{\sqrt{s_0}}\sinh(\sqrt{s_0}t)G & \cosh(\sqrt{s_0}t)I \end{bmatrix}$, we get

$$S\mathrm{Gr}\big(\Psi_{0,s_0}(t)\big) = \begin{bmatrix} \cosh(\sqrt{s_0}t)I & \sqrt{s_0}\sinh(\sqrt{s_0}t)G \\ 0 & I \\ \dfrac{1}{\sqrt{s_0}}\sinh(\sqrt{s_0}t)G & \cosh(\sqrt{s_0}t)I \\ I & 0 \end{bmatrix}$$

is a Lagrangian subspace with respect to the standard symplectic form $\widetilde{J}$ of $\mathbf{C}^{2n} \oplus \mathbf{C}^{2n}$. To do so, it is enough to observe that by a direct computation, we have

$$\widetilde{J} = S\big(-J \oplus J\big)S^{\mathsf{T}} = \begin{bmatrix} 0 & 0 & -I & 0 \\ 0 & 0 & 0 & -I \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix}.$$

By taking into account Remark 5.3 and by observing that the matrix

$$\begin{bmatrix} \cosh(\sqrt{s_0}t)I & \sqrt{s_0}\sinh(\sqrt{s_0}t)G \\ 0 & I \end{bmatrix}$$

is invertible, we can re-write the Lagrangian subspace $S\mathrm{Gr}\big(\Psi_{0,s_0}(t)\big)$ as the graph of a symmetric matrix, simply by change the Lagrangian frame. To do so, we observe that

$$\begin{bmatrix} \dfrac{1}{\sqrt{s_0}}\sinh(\sqrt{s_0}t)G & \cosh(\sqrt{s_0}t)I \\ I & 0 \end{bmatrix} \cdot \begin{bmatrix} \cosh(\sqrt{s_0}t)I & \sqrt{s_0}\sinh(\sqrt{s_0}t)G \\ 0 & I \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} \dfrac{\sinh(\sqrt{s_0}t)}{\sqrt{s_0}\cosh(\sqrt{s_0}t)}G & \dfrac{1}{\cosh(\sqrt{s_0}t)}I \\ \dfrac{1}{\cosh(\sqrt{s_0}t)}I & -\sqrt{s_0}\dfrac{\sinh(\sqrt{s_0}t)}{\cosh(\sqrt{s_0}t)}G \end{bmatrix}$$

Thus the Lagrangian frame for $S\mathrm{Gr}\big(\omega\Psi_{0,s_0}(t)\big)$ fits into the following

$$\begin{bmatrix} I & 0 \\ 0 & I \\ \dfrac{\sinh(\sqrt{s_0}t)}{\sqrt{s_0}\cosh(\sqrt{s_0}t)}G & \dfrac{1}{\cosh(\sqrt{s_0}t)}I \\ \dfrac{1}{\cosh(\sqrt{s_0}t)}I & -\sqrt{s_0}\dfrac{\sinh(\sqrt{s_0}t)}{\cosh(\sqrt{s_0}t)}G \end{bmatrix}$$

By the very same computations for the Lagrangian subspace $\mathrm{Gr}\big(\omega A_d^{-1}\big)$, we get

$$S\mathrm{Gr}\big(\omega A_d^{-1}\big) = \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ 0 & I \\ \omega A^{\mathsf{T}} & 0 \\ 0 & \omega A^{-1} \end{bmatrix} = \begin{bmatrix} \omega A^{\mathsf{T}} & 0 \\ 0 & I \\ 0 & \omega A^{-1} \\ I & 0 \end{bmatrix}.$$

In this way the Lagrangian subspace $\mathrm{Gr}\big(\omega A_d^{-1}\big)$ can be re-written as follows

$$\begin{bmatrix} I & 0 \\ 0 & I \\ 0 & \omega A^{-1} \\ \bar{\omega} A^{-\mathsf{T}} & 0 \end{bmatrix}$$

Summing up, we proved that

$$\mu^{\text{CLM}}\Big(\mathrm{Gr}\big(\omega A_d^{-1}\big), \mathrm{Gr}\big(\Psi_{0,s_0}(t)\big); t \in [0,T]\Big) = \mu^{\text{CLM}}\Big(\mathrm{Gr}\big(Z_\omega\big), \mathrm{Gr}\big(Z_{s_0}(t)\big); t \in [0,T]\Big)$$

where

$$Z_\omega := \begin{bmatrix} 0 & \omega A^{-1} \\ \bar{\omega} A^{-\mathsf{T}} & 0 \end{bmatrix} \quad \text{and} \quad Z_{s_0}(t) := \begin{bmatrix} \dfrac{\sinh(\sqrt{s_0}t)}{\sqrt{s_0}\cosh(\sqrt{s_0}t)}G & \dfrac{1}{\cosh(\sqrt{s_0}t)}I \\ \dfrac{1}{\cosh(\sqrt{s_0}t)}I & -\sqrt{s_0}\dfrac{\sinh(\sqrt{s_0}t)}{\cosh(\sqrt{s_0}t)}G \end{bmatrix}.$$

Now the conclusion follows by

$$\mu^{\text{CLM}}\Big(\mathrm{Gr}\big(Z_\omega\big), \mathrm{Gr}\big(Z_{s_0}(t)\big); t \in [0,T]\Big) = \mu^{\text{CLM}}\Big(L_D, \mathrm{Gr}\big(Z_{s_0}(t) - Z_\omega\big); t \in [0,T]\Big)$$

once observed that $Z_{s_0}(t) - Z_\omega =: M_{s_0}(t)$. This concludes the proof. □

Lemma 5.4. *For every $t_0 \in (0,T]$, there exists $s_0 > 0$ sufficiently large such that $M_{s_0}(t)$ is non-degenerate and*

$$\mathrm{n}_+\left[M_{s_0}(t)\right] = \mathrm{n}_-\left[M_{s_0}(t)\right] \text{ for any } t \in [t_0, T]$$

where $\mathrm{n}_+[*]$ and $\mathrm{n}_-[*]$ denotes respectively the coindex and the index of $*$.

Proof. For every $t \neq 0$ the matrix $\dfrac{\sinh(\sqrt{s_0}t)}{\sqrt{s_0}\cosh(\sqrt{s_0}t)}G$ is invertible. We let

$$N(t) := \begin{bmatrix} I & -\dfrac{\sqrt{s_0}\cosh(\sqrt{s_0}t)}{\sinh(\sqrt{s_0}t)}G\left(\dfrac{I}{\cosh(\sqrt{s_0}t)} - \omega A^{-1}\right) \\ 0 & I \end{bmatrix}$$

and we observe that

$$N^*(t)M_{s_0}(t)N(t)$$
$$= \begin{bmatrix} \dfrac{1}{\sqrt{s_0}}\tanh(\sqrt{s_0}t)G & 0 \\ 0 & -\sqrt{s_0}\dfrac{1}{\sinh(\sqrt{s_0}t)}\Big(2\cosh(\sqrt{s_0}t)G - \bar{\omega}A^{-\mathsf{T}}G - \omega G A^{-1}\Big) \end{bmatrix}.$$

Thus

$$\det M_{s_0}(t) \;=\; \det\big(N^*(t)M_{s_0}(t)N(t)\big) \;=\; \det\left(-2G + \frac{1}{\cosh(\sqrt{s_0}t)}\big(\bar{\omega}A^{-\mathsf{T}}G + \omega G A^{-1}\big)\right).$$

Since for $t \in [t_0, T]$ it holds that

$$\det M_{s_0}(t) \sim_{+\infty} \det(-2G) \neq 0,$$

then $M_{s_0}(t)$ is non-degenerate for $t \in [t_0, T]$. We observe also that

$$N^*(t)M_{s_0}(t)N(t) \sim_{+\infty} \begin{bmatrix} \dfrac{1}{\sqrt{s_0}}\tanh(\sqrt{s_0}t)G & 0 \\ 0 & -2\sqrt{s_0}\coth(\sqrt{s_0}t)G \end{bmatrix}.$$

Since the index and the coindex of the two matrices $M_{s_0}(t)$ and $N^*(t)M_{s_0}(t)N(t)$ agree. This concludes the proof. $\square$

Lemma 5.5. *For any complex matrix B, we let*

$$M := \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}$$

where we denoted by B^* the conjugate transpose of B. Then we have

$$\mathrm{n}_0[M] = 2\dim\ker B \quad \text{and} \quad \mathrm{n}_+[M] = \mathrm{n}_-[M]$$

and where $\mathrm{n}_0[*]$ denotes the nullity of $*$.

Proof. The proof of this result, readily follows by a direct computation. $\square$

Proposition 5.6. *Under the previous notation, for every $\omega \in \mathbf{U}$, we have*

$$\iota_1(\omega A_d \Psi_{0,s_0}(t); t \in [0,T]) = \dim\ker(A - \omega I).$$

Proof. We start to observe that

$$\iota_1(\omega A_d\Psi_{0,s_0}(t); t\in[0,T]) = \mu^{\mathrm{CLM}}\Big(\mathrm{Gr}\big(\omega A_d^{-1}\big), \mathrm{Gr}\big(\Psi_{0,s_0}(t)\big); t\in[0,T]\Big).$$

Then in order to conclude, it is enough to compute $\mu^{\mathrm{CLM}}\Big(\mathrm{Gr}\big(\omega A_d^{-1}\big), \mathrm{Gr}\big(\Psi_{0,s_0}(t)\big); t\in[0,T]\Big)$. By Lemma 5.2, we have

$$\mu^{\mathrm{CLM}}\Big(\mathrm{Gr}\big(\omega A_d^{-1}\big), \mathrm{Gr}\big(\Psi_{0,s_0}(t)\big); t\in[0,T]\Big) = \mu^{\mathrm{CLM}}\Big(L_D, \mathrm{Gr}\big(M_{s_0}(t)\big); t\in[0,T]\Big)$$

$M_{s_0}(t)$ has been defined in Lemma 5.5. Moreover for $t=0$, the matrix $M_{s_0}(t)$ reduces to

$$M_{s_0}(0) = \begin{bmatrix} 0 & I-\omega A^{-1} \\ I-\bar\omega A^{-\mathsf{T}} & 0 \end{bmatrix}.$$

By Lemma 5.5, we have

$$\mathrm{n}_0\left[M_{s_0}(0)\right] = 2\dim\ker(A-\omega I), \qquad \mathrm{n}_+\left[M_{s_0}(0)\right] = \mathrm{n}_-\left[M_{s_0}(0)\right].$$

By Lemma 5.4, there exists t_0 and s_0 such that $M_{s_0}(t)$ is non-degenerate for $t\in[t_0,T]$ and $\mathrm{n}_+\left[M_{s_0}(t)\right] = \mathrm{n}_-\left[M_{s_0}(t)\right]$ for every $t\in[t_0,T]$. By this, we infer that

$$\mathrm{sf}(M_{s_0}(t); t\in[0,T]) = \dim\ker(A-\omega I).$$

In order to conclude, it is enough to observe that

$$\mu^{\mathrm{CLM}}\Big(L_D, \mathrm{Gr}\big(M_{s_0}(t)\big); t\in[0,T]\Big) = \mathrm{sf}(M_{s_0}(t); t\in[0,T]).$$

This concludes the proof. □

By using Proposition 5.6 we are now in position to prove Theorem 1.

Proof of Theorem 1. For $c\in[0,1]$ and $s\in[0,s_0]$ we start to consider the path $s\mapsto \mathcal{A}^\omega_{c,s}$ given in Equation (2.6), namely

$$\mathcal{A}^\omega_{c,s} = -G\frac{d^2}{dt^2} + c\widehat{R}(t) + sG, \qquad t\in[0,T]$$

on the Hilbert space $E^2_\omega([0,T])$ and defined in Equation (2.7). The corresponding Morse-Sturm system is given by

$$-G\ddot u(t) + \big(c\widehat{R}(t) + sG\big)u(t) = 0, \qquad t\in[0,T]$$

and the associated Hamiltonian system has been given in Equation (2.8); i.e.

$$\dot z(t) = JD_{c,s}(t)z(t), \qquad t\in[0,T]$$

whose fundamental solution has been denoted by $s\mapsto \Psi_{c,s}(t)$. In order to prove the result we observe that

1. $\omega A_d\Psi_{c,s}(0) = \omega A_d$ being $\Psi_{c,s}(0) = I$. This in particular implies that

$$\iota_1\big(\omega A_d\Psi_{1,s}(0); s\in[0,s_0]\big) = 0 \quad \text{and} \quad \iota_1\big(\omega A_d\Psi_{c,s}(0); c\in[0,1]\big) = 0.$$

2. By Corollary 4.5, if s_0 is sufficiently large, then $\mathcal{A}_{c,s_0}$ is non-degenerate for every $c\in[0,1]$. This in particular implies that

$$\iota_1\big(\omega A_d\Psi_{c,s_0}(T); c\in[0,1]\big) = 0.$$

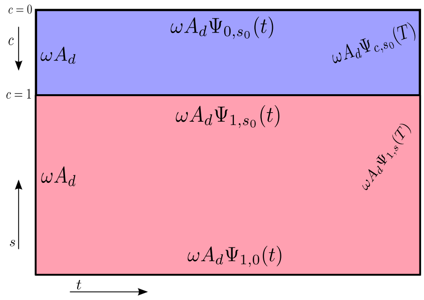

Figure 2: In the blue rectangle (above) is depicted the homotopy with respect to the parameter c whilst in the pink rectangle (below) the homotopy with respect to the s parameter.

3. For $c = 0$, by Proposition 5.6, we infer that

$$\iota_1(\omega A_d \Psi_{0,s_0}(t); t \in [0,T]) = \dim \ker(A - \omega I).$$

4. By the homotopy invariance property of the Maslov-type index, we have

$$\iota_1\big(\omega A_d \Psi_{1,0}(t); t \in [0,T]\big) + \iota_1\big(\omega A_d \Psi_{1,s}(T); s \in [0,s_0]\big) = \iota_1\big(\omega A_d \Psi_{1,s_0}(t); t \in [0,T]\big) \tag{5.4}$$
$$\iota_1\big(\omega A_d \Psi_{0,s_0}(t); t \in [0,T]\big) = \iota_1\big(\omega A_d \Psi_{1,s_0}(t); t \in [0,T]\big).$$

By Proposition 5.1, it holds that

$$\mathrm{sf}(\mathcal{A}^{\omega}_{1,s}; s \in [0,s_0]) = -\iota_1\big(\omega A_d \Psi_{1,s}(T); s \in [0,s_0]\big).$$

By taking into account the relations given in Equation (5.4), we get that

$$\iota_1\big(\omega A_d \Psi_{1,0}(t); t \in [0,T]\big) = \mathrm{sf}(\mathcal{A}^{\omega}_{1,s}; s \in [0,s_0]) + \dim \ker\big(A - \omega I\big).$$

This concludes the proof. □

6 Semi-Riemannian Bott-type iteration formula

The goal of this section is to prove the Bott-type iteration formula for semi-Riemannian geodesics.

We start to observe that, since $\widehat{R}$ satisfies the condition $\widehat{R}(T) = A^{\mathsf{T}}\widehat{R}(0)A$, we can extend it on the interval $[0, mT]$. More precisely, for every $k = 1, \dots, m$, we define the associated Morse-Sturm system as

$$\begin{cases} -G\ddot{u} + \widehat{R}(t)u = 0, & t \in [0, mT] \\ u(0) = A^k u(kT) \text{ and } \dot{u}(0) = A^k \dot{u}(kT). \end{cases}$$

We now consider the Hamiltonian system $\dot{z}(t) = JD_{c,s}(t)z(t), \quad t \in [0, mT]$ and we define the operator $\mathcal{A}^{(m)}_{c,s} \coloneqq -G\dfrac{d^2}{dt^2} + c\widehat{R}(t) + sG, \quad t \in [0, mT]$ on the Sobolev space

$$E^k_m \coloneqq \left\{ u \in W^{2,2}([0,mT], \mathbf{R}^n) \;\middle|\; u(0) = A^k u(kT), \dot{u}(0) = A^k \dot{u}(kT) \right\}, \quad \text{for } k = 1, \dots, m.$$

Then, by Theorem 1, we have

$$\mathrm{sf}(\mathcal{A}^{(m)}_{1,s}; s \in [0,s_0]) + \dim \ker(A^m - I) = \iota_1(A^m_d \Psi_{1,0}(t); t \in [0, mT]).$$

By [LT15, Theorem 1.1], we infer

$$\iota_1(A^m_d \Psi_{1,0}(t); t \in [0, mT]) = \sum_{\omega^m = 1} \iota_\omega(A_d \Psi_{1,0}(t); t \in [0,T]) \tag{6.1}$$

and we observe that the following equalities hold

$$\text{(6.2)} \quad \dim\ker(A^m - I) = \sum_{\omega^m=1} \dim\ker(A-\omega I),$$
$$\iota_\omega(A_d\Psi_{1,0}(t); t\in[0,T]) = \iota_1(\omega A_d\Psi_{1,0}(t); t\in[0,T]).$$

By Proposition 1, Equation (6.1) and Equation (6.2), we have

$$\begin{aligned}
\mathrm{sf}(\mathcal{A}^{(m)}_{1,s}; s\in[0,s_0]) + \sum_{\omega^m=1}\dim\ker(A-\omega I) &= \sum_{\omega^m=1}\iota_1(\omega A_d\Psi_{1,0}(t); t\in[0,T]) \quad \Rightarrow \\
\mathrm{sf}(\mathcal{A}^{(m)}_{1,s}; s\in[0,s_0]) &= \sum_{\omega^m=1}\Big(\iota_1(\omega A_d\Psi_{1,0}(t); t\in[0,T]) - \dim\ker(A-\omega I)\Big) \\
&= \sum_{\omega^m=1}\mathrm{sf}(\mathcal{A}^{(\omega)}_{1,s}; s\in[0,s_0]),
\end{aligned}$$

where the last equality follows by invoking Theorem 1.

7 A linear instability criterion

This Section is devoted to the proof of the *instability criteria* for closed non-lightlike semi-Riemannian geodesics.

Notation 7.1. We set

$$\begin{aligned}
&\mathrm{Sp}(2n,\mathbf{R})^+ \coloneqq \{M\in\mathrm{Sp}(2n,\mathbf{R}) \mid \det(M-I_{2n})>0\} \text{ and} \\
&\qquad\qquad \mathrm{Sp}(2n,\mathbf{R})^- \coloneqq \{M\in\mathrm{Sp}(2n,\mathbf{R}) \mid \det(M-I_{2n})<0\}.
\end{aligned}$$

Remark 7.2. We observe that Notation 7.1 are non consistent with the standard notation used in literature. In fact, as the reader can easily realize by looking at Subsection A.1, there is the missing factor $(-1)^{n-1}$ in the definition of $\mathrm{Sp}(2n,\mathbf{R})^+$ and of $\mathrm{Sp}(2n,\mathbf{R})^-$. However the advantage of defining $\mathrm{Sp}(2n,\mathbf{R})^\pm$ likes in Notation 7.1 is for simplifying the discussion about the linear stability.

Lemma 7.3. *Let $T\in\mathrm{Sp}(2n,\mathbf{R})$ be a linearly stable symplectic matrix. Then, there exists $\delta>0$ sufficiently small such that $e^{\pm\delta J}T\in\mathrm{Sp}(2n,\mathbf{R})^+$.*

Proof. Let us consider the (smooth) symplectic path pointwise defined by $T(\theta)\coloneqq e^{-\theta J}T$. By a direct computation we get that

$$T(\theta)^\mathsf{T} J \frac{d}{d\theta}T(\theta)\Big|_{\theta=0} = T^\mathsf{T}T.$$

We observe that $T^\mathsf{T}T$ is symmetric and positive semi-definite; moreover being T invertible it follows that $T^\mathsf{T}T$ is actually positive definite. Thus, in particular, $\mathrm{n}_-(T^\mathsf{T}T)=0$. By invoking Proposition A.10 it follows that there exists $\delta>0$ such that $T(\pm\delta)\in\mathrm{Sp}(2n,\mathbf{R})^+$. This concludes the proof. □

Remark 7.4. We observe that dropping the linear stability assumption on T in Lemma 7.3, the perturbed matrix $e^{\pm\delta J}T$ belongs to $\mathrm{Sp}(2n,\mathbf{R})^*$; however we can't a control in which path-connected component it will lie. (We refer the interested reader to [Lon02, Equation (6) and (7), pag. 124] for more details).

Proof of Theorem 3. Here it is enough to prove the contrapositive, namely

- if γ is linearly stable and oriented then $\iota_{\mathrm{spec}}(\gamma)+n$ is even;
- if γ is linearly stable and nonoriented then $\iota_{\mathrm{spec}}(\gamma)+n$ is odd.

As direct consequence of Proposition 5.1, we know that $\iota_{\mathrm{spec}}(\gamma)=-\iota_{\mathrm{geo}}(A_d\Psi_{1,s}(T); s\in[0,s_0])$. In order to concludes the proof, it is enough to consider the parity of $\iota_{\mathrm{geo}}(A_d\Psi_{1,s}(T); s\in[0,s_0])+n$.
Case (OR) If γ is oriented, then $\det(A)=1$. Thus, if $A_d\Psi_{1,0}(T)$ is linear stable, then by Lemma 7.3, it follows that $e^{-\theta J}A_d\Psi_{1,0}(T)\in \mathrm{Sp}(2n,\mathbf{R})^+$. By a direct computation , we infer that $\det(A_d\Psi_{0,s_0}-I_{2n})=(-1)^n\big(2\cosh(\sqrt{s_0}T)\big)^n\det A$; thus we get $\det(A_d\Psi_{0,s_0}-I_{2n})>0$ (resp. $\det(A_d\Psi_{0,s_0}-I_{2n})<0$) iff n is even (resp. odd). Hence, $A_d\Psi_{0,s_0}\in\mathrm{Sp}(2n,\mathbf{R})^+$(resp. $A_d\Psi_{0,s_0}\in\mathrm{Sp}(2n,\mathbf{R})^-$) iff n is even (resp. odd). By invoking Corollary 4.5, if s_0 is sufficiently large, then $\mathcal{A}_{c,s_0}$ is non-degenerate for every $c\in[0,1]$. Consequently, $\det(A_d\Psi_{c,s_0}-I_{2n})\neq 0$ for every $c\in[0,1]$ and in particular if $A_d\Psi_{0,s_0}\in\mathrm{Sp}(2n,\mathbf{R})^+$ then $A_d\Psi_{1,s_0}\in\mathrm{Sp}(2n,\mathbf{R})^+$ too. By these arguments, we get that $A_d\Psi_{1,s_0}(T)\in\mathrm{Sp}(2n,\mathbf{R})^+$ (resp. $A_d\Psi_{1,s_0}(T)\in\mathrm{Sp}(2n,\mathbf{R})^-$) iff n is even (resp. odd). Now, the conclusion follows by taking into account Lemma A.5 since $\iota_{\mathrm{geo}}(A_d\Psi_{1,s}(T), s\in[0,s_0])$ is even (resp. odd) or which is equivalent that $\iota_{\mathrm{spec}}(\gamma)$ is even (resp. odd) iff n is even (resp. odd). Therefore, we can conclude that $n+\iota_{\mathrm{spec}}(\gamma)$ is always even.

Case (NOR) If γ is nonoriented then $\det(A)=-1$. Arguing as above, we get that $n+\iota_{\mathrm{spec}}(\gamma)$ is odd. $\square$

From now on, if not differently stated, the pair $(M,\mathfrak{g})$ will denote a $(n+1)$-dimensional Riemannian (resp. Lorentzian) manifold and $\gamma:[0,T]\to M$ a closed (resp. timelike closed) geodesic. Once trivialized the pull-back of the tangent bundle along the closed Riemannian. (resp. timelike Lorentzian) geodesic γ, the index form reduces to the symmetric bilinear form $I: E\times E\to\mathbf{R}$ given by

$$I(u,v)=\int_0^T\langle\dot u(t),\dot v(t)\rangle+\langle\widehat{R}(t)u(t),v(t)\rangle dt,$$

where $E\coloneqq\{u\in W^{1,2}([0,T],\mathbf{R}^n)\mid u(0)=Au(T)\}$, $A\in\mathrm{O}(n)$ is an orthogonal matrix. We observe that, in this case, as observed in Section A.3, the Morse index $\mathrm{n}_-(\gamma)$ of γ (meaning the dimension of the maximal subspace such that I is negative definite), is well-defined. Given $\omega\in\mathbf{U}$, we let $E_\omega=\big\{\,u\in W^{1,2}([0,T],\mathbf{C}^n)\;\big|\;u(0)=\omega Au(T)\,\big\}$ we define the *ω-index form* on E_ω as follows

$$\iota_\omega(u,v)=\int_0^T\langle\dot u(t),\dot v(t)\rangle+\langle\widehat{R}(t)u(t),v(t)\rangle dt, \tag{7.1}$$

where $\langle\cdot,\cdot\rangle$ in Equation (7.1) denotes the standard Hermitian product. Following [BTZ82] and denoting by $\mathrm{n}_-(\omega,\gamma)$ the Morse index of ι_ω on E_ω, we are in position to give the following definition.

Definition 7.5. (**[Lon02, pag. 244]**) The splitting numbers of the closed geodesic γ at $\omega\in\mathbf{U}$ are defined by

$$\mathcal{S}^\pm(\omega,\gamma)=\lim_{\theta\to 0^\pm}\mathrm{n}_-(\omega e^{\sqrt{-1}\theta},\gamma)-\mathrm{n}_-(\omega,\gamma).$$

Following the discussion given above, to the geodesic γ we associate the Morse-Sturm system given by

$$\begin{cases}-\ddot u(t)+\widehat{R}(t)u(t)=0, & t\in[0,T]\\ u(0)=Au(T).\end{cases} \tag{7.2}$$

By using the Legendre transformation, the second order system given in Equation (7.2) corresponds to the linear Hamiltonian system

$$\dot z(t)=JB(t)z(t),\qquad t\in[0,T]$$

where $B(t) := \begin{bmatrix} I & 0 \\ 0 & -\widehat{R}(t) \end{bmatrix}$. The next result point out the relation intertwining the splitting numbers of a closed geodesic γ and the splitting numbers of the symplectic matrix (linearized PoincarÃ© map) given in Definition. A.6. (We refer the interested reader to [Lon02, pag. 191]).

Lemma 7.6. *Under the above notations, we have*

$$\mathcal{S}^{\pm}(\omega, \gamma) = S^{\pm}_{\mathcal{P}(T)}(\omega), \forall\ \omega \in \mathbf{U}.$$

Remark 7.7. In [BTZ82, pag.226], authors proved that the splitting numbers $\mathcal{S}^{\pm}(\omega, \gamma)$ only depend on the conjugacy class of P and in [Lon02, pag. 247], Long generalized this result. Here we provide a different proof with respect to the one given by authors in [Lon02, pag.252].

Proof. For any $\bar{\omega} \in \mathbf{U}$, we let $S := \bar{\omega}A$. By Proposition A.24 and Remark A.25, we have

$$\mathrm{n}_-(\bar{\omega}, \gamma) + \nu(\bar{\omega}A) = \mu^{\mathrm{CLM}}(\mathrm{Gr}(\bar{\omega}A_d^\mathsf{T}), \mathrm{Gr}(\Phi(t)); t \in [0, T]) = \mu^{\mathrm{CLM}}(\mathrm{Gr}(\bar{\omega}), \mathrm{Gr}(A_d\Phi(t)); t \in [0, T]). \tag{7.3}$$

Moreover, by taking into account Proposition A.26, we have

$$\begin{aligned}
&\mu^{\mathrm{CLM}}(\mathrm{Gr}(\bar{\omega}I), \mathrm{Gr}(A_d\Phi(t)); t \in [0, T]) \\
&= \mu^{\mathrm{CLM}}(\mathrm{Gr}(\bar{\omega}I), \mathrm{Gr}(A_d\Phi(t)) * \xi(t)); t \in [0, T]) - \mu^{\mathrm{CLM}}(\mathrm{Gr}(\bar{\omega}I), \mathrm{Gr}(\xi(t))); t \in [0, T]) \\
&= \begin{cases} (\iota_{\bar{\omega}}(A_d\Phi(t) * \xi(t); t \in [0, T]) + n) - (\iota_{\bar{\omega}}(\xi(t); t \in [0, T]) + n) & \omega = 1 \\ \iota_{\bar{\omega}}(A_d\Phi(t) * \xi(t); t \in [0, T]) - \iota_{\bar{\omega}}(\xi(t); t \in [0, T]) & \omega \neq 1 \end{cases} \\
&= \iota_{\bar{\omega}}(A_d\Phi(t) * \xi(t); t \in [0, T]) - \iota_{\bar{\omega}}(\xi(t); t \in [0, T]) \quad \forall\, \omega \in \mathbf{U},
\end{aligned} \tag{7.4}$$

where ξ is any symplectic path joining I to A_d. Summing up Equations (7.3)-(7.4), we get

$$\mathrm{n}_-(\bar{\omega}, \gamma) = \iota_{\bar{\omega}}(A_d\Phi(t) * \xi(t)) - \iota_{\bar{\omega}}(\xi(t)) - \nu(\bar{\omega}A).$$

According to Definition A.6, we infer that

$$\mathcal{S}^{\pm}(\bar{\omega}, \gamma) = S^{\pm}_{\mathcal{P}}(\bar{\omega}) - S^{\pm}_{A_d}(\bar{\omega}) + \nu(\bar{\omega}A)$$

and being A an orthogonal matrix, then by direct computation, we get $S^{\pm}_{A_d}(\bar{\omega}) = \nu(\bar{\omega}A)$. This complete the proof. □

We let $\widetilde{J} := -\sqrt{-1}J$. For any symplectic matrix $M \in \mathrm{Sp}(2n, \mathbf{R})$, we define the $\widetilde{J}$-invariant (generalized eigenspace) subspace E_λ as

$$E_\lambda := \bigcup_{m \geqslant 1} \ker(M - \lambda I)^m.$$

and we observe that the following $\widetilde{J}$-orthogonal splitting holds

$$\mathbf{C}^{2n} = \bigoplus_{\lambda \in \sigma(M)} E_\lambda.$$

Definition 7.8. For any $\lambda \in \sigma(M) \cap \mathbf{U}$, the restriction of $\widetilde{J}$ to the subspace E_λ is non-degenerate. We define the *Krein-type* of λ by (p, q), where p (resp. q) denotes respectively the total multiplicity of the positive (resp. negative) negative eigenvalues of $\widetilde{J}\,|_{E_\lambda}$. If $p = 0$ (resp. $q = 0$) the eigenvalue λ is termed *Krein-negative* (resp. *Krein-positive*); otherwise, λ is called *Krein-indefinite* or of *mixed-type.*

For short, we will refer to the case $p = 0$ or $q = 0$ simply as *Krein-definite.* Before recalling the definition of *strongly stability*, we introduce the following new definition.

Definition 7.9. A closed geodesic γ is called *index hyperbolic* if, for any $\lambda := e^{\sqrt{-1}2\pi\theta} \in \sigma(\mathcal{P}(T))\cap \mathbf{U}$ (eigenvalue of the PoincarÃ© map $\mathcal{P}(T) = A_d\Phi(T)$, here Φ denotes the fundamental solution) it holds

$$S^+_{\mathcal{P}}(e^{\sqrt{-1}2\pi\theta}) = S^-_{\mathcal{P}}(e^{\sqrt{-1}2\pi\theta}) = 0 \quad \text{if } \theta \in \mathbf{Q},$$
$$S^+_{\mathcal{P}}(e^{\sqrt{-1}2\pi\theta}) = S^-_{\mathcal{P}}(e^{\sqrt{-1}2\pi\theta}) \quad \text{if } \theta \notin \mathbf{Q}.$$

Proposition 7.10. *Let $(M, \mathfrak{g})$ be a Riemannian (resp. Lorentzian) closed (resp. timelike and closed) geodesic. We assume that for any $m \in \mathbb{N}$, $\mathrm{n}_-\big(\gamma^{(m)}\big) = 0$. Then γ is index hyperbolic.*

Proof. We start to observe that by Proposition A.23 we have

$$\mu^{\text{CLM}}\big(\mathrm{Gr}((A_d^\mathsf{T})^m), \mathrm{Gr}(\Phi(t)); t \in [0, mT]\big) = \sum_{i=1}^{m} \mu^{\text{CLM}}\big(\mathrm{Gr}(\exp(\frac{i}{m}2\pi\sqrt{-1})A_d^\mathsf{T}), \mathrm{Gr}(\Phi(t)); t \in [0, T]\big). \tag{7.5}$$

By invoking Proposition (A.24), Equation (7.5) fits into the following

$$\mathrm{n}_-(\gamma^{(m)}) + \nu(A^m) = \sum_{i=1}^{m} \mathrm{n}_-\left(\exp(\frac{i}{m}2\pi\sqrt{-1}), \gamma\right) + \sum_{i=1}^{m} \nu(\exp(\frac{i}{m}2\pi\sqrt{-1})A). \tag{7.6}$$

By Equation (6.2) we infer that $\nu(A^m) = \sum_{i=1}^{m}\nu(\exp(\frac{i}{m}2\pi\sqrt{-1})A)$ and by substituting into Equation (7.6), we immediately get that

$$\mathrm{n}_-(\gamma^{(m)}) = \sum_{i=1}^{m} \mathrm{n}_-\left(\exp(\frac{i}{m}2\pi\sqrt{-1}), \gamma\right).$$

By assumption, for every $m \in \mathbb{N}$, $\mathrm{n}_-(\gamma^{(m)}) = 0$. By definition, $\mathrm{n}_-(\exp(\frac{i}{m}2\pi\sqrt{-1}, \gamma) \geqslant 0$ this immediately implies that

$$\mathrm{n}_-\left(\exp(\frac{i}{m}2\pi\sqrt{-1}), \gamma\right) = 0 \quad \text{for any } i, m \in \mathbb{N}$$

which is equivalent to assert that $\mathrm{n}_-(e^{\sqrt{-1}2\pi\theta}, \gamma) = 0$ for any $\theta \in \mathbf{Q}$. So, for any $\lambda = e^{\sqrt{-1}2\pi\theta} \in \sigma(\mathcal{P}(T))$ and $\theta \in \mathbf{Q}$, by invoking Lemma 7.6, we get that

$$S^\pm_{\mathcal{P}(T)}(\lambda) = \mathcal{S}^\pm(\lambda, \gamma) = 0.$$

This conclude the first claim. In order to prove the second claim, if $\theta \notin \mathbf{Q}$, let $\theta_1 < \theta < \theta_2$ be such that θ_1, θ_2 are in $\mathbf{Q}$, $|\theta_j - \theta|$ is small enough and $e^{\sqrt{-1}2\pi\theta_j} \notin \sigma(\mathcal{P}(T))$ for $j = 1, 2$. Then $\mathrm{n}_-(e^{\sqrt{-1}2\pi\theta_j}, \gamma) = 0$ and $\mathcal{S}^\pm(e^{\sqrt{-1}2\pi\theta_j}, \gamma) = 0$. By Definition 7.5, we know that in fact the splitting numbers $\mathcal{S}^\pm(\omega, \gamma)$ measure the jumps between $n_-(\omega, \gamma)$ and $n_-(\lambda, \gamma)$ for $\lambda \in \mathbf{U}$ in a neighborhood of ω. By invoking once again Lemma 7.6 as well as Proposition A.9, we infer that $\mathcal{S}^\pm(\omega, \gamma) = 0$ if $\omega \notin \sigma(\mathcal{P}(T))$. In conclusion, we have

$$\mathrm{n}_-(e^{\sqrt{-1}2\pi\theta_2}, \gamma) = \mathrm{n}_-(e^{\sqrt{-1}2\pi\theta_1}, \gamma) + \mathcal{S}^+(e^{\sqrt{-1}2\pi\theta_1}, \gamma) + \mathcal{S}^+(\lambda, \gamma) - \mathcal{S}^-(\lambda, \gamma) - \mathcal{S}^-(e^{\sqrt{-1}2\pi\theta_2}, \gamma),$$

so $\mathcal{S}^+(\lambda, \gamma) = \mathcal{S}^-(\lambda, \gamma)$. The conclusion, readily follows, by invoking once again Lemma 7.6. This concludes the proof. □

We recall that the strong stability of a symplectic matrix can be characterized through its eigenvalues. For the sake of the reader, we recall the following result proved by author in [Eke90].

Lemma 7.11. *(***[Eke90, Thm 10, pag.11]***) M is strongly stable if and only if it is linearly stable and all of its eigenvalues are Krein-definite.*

Remark 7.12. We observe that since the eigenvalues $\{-1, 1\}$ have Krein-type (p, p), by this it readily follows that ± 1 cannot be in the spectrum of a strongly stable symplectic matrix.

As a direct consequence of Proposition 7.10 and of the basic normal forms of a symplectic matrix, we get the following strongly instability result for closed geodesics.

Proof of Theorem 4. Under the assumptions of Theorem 4, by invoking Proposition 7.10, we can conclude that the geodesic is index hyperbolic. By taking into account Definition 7.9 on index hyperbolicity applied to the monodromy $\mathcal{P}(T) = A_d\Phi(T)$, we get that, for every eigenvalue $e^{\sqrt{-1}\theta} \in \sigma(\mathcal{P}(T))$, $S^+_{\mathcal{P}(T)}(e^{\sqrt{-1}\theta}) - S^-_{\mathcal{P}(T)}(e^{\sqrt{-1}\theta}) = 0$. In order to concludes the proof, we argue by contradiction. For, we assume that $\mathcal{P}(T)$ is strongly stable. Thus, by the characterization given in Lemma 7.11, $\mathcal{P}(T)$ is linearly stable and all of its eigenvalues are Krein-definite. By this fact, it immediately follows that, for any eigenvalue $e^{\sqrt{-1}\theta} \in \sigma(\mathcal{P}(T))$ having Krein type (p, q), it holds that $p - q \neq 0$. By this we get a contradiction once invoked [Lon02, Corollary 8, pag.198], being

$$S^+_{\mathcal{P}}(e^{\sqrt{-1}\theta}) - S^-_{\mathcal{P}}(e^{\sqrt{-1}\theta}) = p - q \neq 0 \text{ and}$$

$$S^+_{\mathcal{P}}(e^{\sqrt{-1}\theta}) - S^-_{\mathcal{P}}(e^{\sqrt{-1}\theta}) = p - q = 0$$

at the same time. This concludes the proof. □

A On the Maslov index, spectral flow and Index theorems

The aim of this Section is to make a brief recap on the Maslov-type index, the spectral flow for path of closed selfadjoint Fredholm operators and the Index Theorems.

A.1 On the Maslov-type index

Our basic reference it is [HS09, LZ00a, LZ00b, Lon02] and references therein.

Let $\omega \in \mathbf{U}$ and for any $M \in \mathrm{Sp}(2n, \mathbf{R})$, we define the real-valued function

$$D_\omega(M) = (-1)^{n-1}\overline{\omega}^n \det(M - \omega I_{2n}).$$

Then $\mathrm{Sp}(2n, \mathbf{R})^0_\omega := \{M \in \mathrm{Sp}(2n, \mathbf{R}) | D_\omega M = 0\}$ is a codimensional-one variety in $\mathrm{Sp}(2n, \mathbf{R})$ and let us define

$$\mathrm{Sp}(2n)^*_\omega := \mathrm{Sp}(2n)\backslash \mathrm{Sp}(2n, \mathbf{R})^0_\omega = \mathrm{Sp}(2n, \mathbf{R})^+_\omega \cup \mathrm{Sp}(2n, \mathbf{R})^-_\omega$$

where

$$\mathrm{Sp}(2n, \mathbf{R})^+_\omega := \{M \in \mathrm{Sp}(2n, \mathbf{R}) | D_\omega M < 0\} \text{ and } \mathrm{Sp}(2n, \mathbf{R})^-_\omega := \{M \in \mathrm{Sp}(2n, \mathbf{R}) | D_\omega M > 0\}.$$

For any $M \in \mathrm{Sp}(2n, \mathbf{R})^0_\omega$, $\mathrm{Sp}(2n, \mathbf{R})^0_\omega$ is co-oriented at the point M by choosing as positive direction the direction determined by $\frac{d}{dt} M e^{tJ}|_{t=0}$ with $t \geqslant 0$ sufficiently small. The following result is well-known.

Lemma A.1. *[Lon02, pag.58-59]. For any $\omega \in \mathbf{U}$, $\mathrm{Sp}(2n, \mathbf{R})^+_\omega$ and $\mathrm{Sp}(2n, \mathbf{R})^-_\omega$ are two path connected components of $\mathrm{Sp}(2n, \mathbf{R})^*_\omega$ which are simple connected in $\mathrm{Sp}(2n, \mathbf{R})$.*

For any two $2m_k \times 2m_k$ matrices with the block form $M_k = \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}$ with $k = 1, 2$, we define the $\diamond$-product of M_1 and M_2 in the following way:

$$M_1 \diamond M_2 := \begin{bmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{bmatrix}.$$

The k-fold $\diamond$-product of M is denoted by $M^{\diamond k} = M \diamond \cdots \diamond M$. Note that the $\diamond$-product of two symplectic matrices is symplectic, so for any two symplectic paths $T_k \in \mathscr{C}^0\big([0, \tau], \mathrm{Sp}(2n_k, \mathbf{R})\big), k = 1, 2$, denote $T_1 \diamond T_2(t) = T_1(t) \diamond T_2(t), \forall t \in [0, \tau]$, then $T_1 \diamond T_2 \in \mathscr{C}^0\big([0, \tau], \mathrm{Sp}(2(n_{k_1} + n_{k_2}), \mathbf{R})\big)$.

Given any two symplectic paths $\gamma, \eta : [0, \tau] \to \mathrm{Sp}(2n, \mathbf{R})$ such that $\gamma(0) = \eta(\tau)$, we define their concatenation in the following way:

$$(\gamma * \eta)(t) := \begin{cases} \eta(2t) & \text{if } 0 \leqslant t \leqslant \frac{\tau}{2} \\ \gamma(2t - \tau) & \text{if } \frac{\tau}{2} \leqslant t \leqslant \tau. \end{cases}$$

For $a \in \mathbf{R}^*$, let $D(a) = \begin{bmatrix} a & 0 \\ 0 & \frac{1}{a} \end{bmatrix}$ and let $M_n^+ := D(2)^{\diamond n}$ and $M_n^- := D(-2) \diamond D(2)^{\diamond(n-1)}$. By a straightforward computation we have $M_n^+ \in \mathrm{Sp}(2n, \mathbf{R})_\omega^+$ and $M_n^- \in \mathrm{Sp}(2n, \mathbf{R})_\omega^-$. We let

$$\eta_n(t) := \begin{bmatrix} 2 - \frac{t}{\tau} & 0 \\ 0 & (2 - \frac{t}{\tau})^{-1} \end{bmatrix}^{\diamond n} \qquad t \in [0, \tau]$$

and we observe that η_n is a path in $\mathrm{Sp}(2n, \mathbf{R})$ joining M_n^+ to I_{2n}. Following author in [Lon02] we recall the following definition.

Definition A.2. For any $\omega \in \mathbf{U}$ and $T \in \mathscr{C}^0\big([0, \tau], \mathrm{Sp}(2n, \mathbf{R})\big)$ such that $T(0) = I_{2n}$, we define

$$\iota_\omega(T) := \big[e^{-\varepsilon J}\big(T * \eta_n\big) : \mathrm{Sp}(2n, \mathbf{R})_\omega^0\big],$$

where the (RHS) denotes the intersection number between the perturbed path $t \mapsto e^{-\varepsilon J}(T * \eta_n)(t)$ with the singular cycle $\mathrm{Sp}(2n, \mathbf{R})_\omega^0$. We set

$$\nu_\omega(T) = \dim_{\mathbf{C}}(T(\tau) - \omega I_{2n}).$$

Remark A.3. It is worth noticing that the Definition A.2 is independent on the choice of a sufficiently small $\varepsilon > 0$.

By Definition A.2, for any continuous symplectic path $S : [a, b] \to \mathrm{Sp}(2n, \mathbf{R})$ such that $S(a) \neq I_{2n}$, we can choose a path $T \in \mathscr{C}^0\big([a, b], \mathrm{Sp}(2n, \mathbf{R})\big)$ such that $T(a) = I_{2n}$ and $T(b) = S(a)$ and we define the*Maslov-type index of S* as

$$\iota_\omega(S) := \iota_\omega(S * T) - \iota_\omega(T). \tag{A.1}$$

(For further details we refer the interested reader to [Lon02, Definition 9, pag. 148]). By [Lon02, Lemma 6,pag. 120], for any path $T \in \mathscr{C}^0\big([0, \tau], \mathrm{Sp}(2n, \mathbf{R})\big)$ such that $T(0) = I_{2n}$ the Maslov-type index $\iota_\omega(T)$ is even (resp. odd) if and only if $e^{-\varepsilon J} T(\tau)$ lies in $\mathrm{Sp}(2n, \mathbf{R})_\omega^+$ (resp. $\mathrm{Sp}(2n, \mathbf{R})_\omega^-$).

Remark A.4. It is worth noticing that the number k appearing in [Lon02, Lemma 6,pag. 120] agrees with $\iota_\omega(T)$. We also observe that $\beta(\tau) = M_n^\pm$ in particular implies that $e^{-\varepsilon J} T(\tau) \in \mathrm{Sp}(2n, \mathbf{R})_\omega^\pm$.

Lemma A.5. *Let $S : [a, b] \to \mathrm{Sp}(2n, \mathbf{R})$ be a continuous path. Then we have*

> *$\iota_\omega(S)$ is even $\iff$ both the endpoints $e^{-\varepsilon J} S(a)$ and $e^{-\varepsilon J} S(b)$ lie in $\mathrm{Sp}(2n, \mathbf{R})_\omega^+$ or in $\mathrm{Sp}(2n, \mathbf{R})_\omega^-$.*

Proof. As a direct application of [Lon02, Lemma 6,pag. 120] to the paths $S * T$ and T, we immediately get that, if $e^{-\varepsilon J} S(a)$ and $e^{-\varepsilon J} S(b)$ are in the same components of $\mathrm{Sp}(2n, \mathbf{R})_\omega^*$, then the parity of the two Maslov-type indices $\iota_\omega(S * T)$ and $\iota_\omega(T)$ coincides. Furthermore, by the definition stated in Formula (A.1), we know that it's even and hence also the converse is true. This conclude the proof. ☐

To use the iteration formula, here we give the definition and some properties we need of the splitting numbers of any symplectic matrix which can be found in [Lon02, pag.190-199].

Definition A.6. For any $M \in \mathrm{Sp}(2n)$ and $\omega \in \mathbf{U}$, we define the splitting numbers $S_M^\pm(\omega)$ of M by $S_M^\pm(\omega) = \lim_{\theta \to 0^\pm} \iota_{\omega e^{\sqrt{-1}\theta}}(\gamma) - \iota_\omega(\gamma)$ where γ is a symplectic path connecting I_{2n} and M.

The splitting numbers have the following property:

Proposition A.7. *The splitting numbers $S_M^{\pm}(\omega)$ are independent of the path γ and for any $\omega \in \mathbf{U}$ and $N \in \Omega^0(M)$, the path connected component of $\Omega(M) = \{N \in \mathrm{Sp}(2n) \mid \sigma(N) \cap \mathbf{U} = \sigma(M) \cap \mathbf{U}$, and $\nu_\lambda(N) = \nu_\lambda(M)\ \forall \lambda \in \sigma(M) \cap \mathbf{U}\}$ which contains M, $S_N^{\pm}(\omega)$ are constant. If $\omega \notin \sigma(M)$, then $S_M^{\pm}(\omega) = 0$. Moreover,*

$$S^{\pm}_{M_1 \diamond M_2}(\omega) = S^{\pm}_{M_1}(\omega) + S^{\pm}_{M_2}(\omega), \forall\ \omega \in \mathbf{U}.$$

For any symplectic matrix $M \in \mathrm{Sp}(2n, \mathbf{R})$ with eigenvalue $\omega \in \mathbf{U}$, in order to give a complete explanation of $S_M^{\pm}(\omega)$, we need the concept of ultimate type of ω which is introduced in [Lon02, pag.41-42].

Firstly, we give all the basic normal forms for eigenvalues of M in $\mathbf{U}$ as follows:

$$N_1(\lambda, b) = \begin{bmatrix} \lambda & b \\ 0 & \lambda \end{bmatrix},\ \lambda = \pm 1, b = \pm 1, 0$$

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},\ \theta \in (0, \pi) \cup (\pi, 2\pi)$$

$$N_2(\omega, b) = \begin{bmatrix} R(\theta) & b \\ 0 & R(\theta) \end{bmatrix},\ \theta \in (0, \pi) \cup (\pi, 2\pi)$$

where $b = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ such that $b_i \in \mathbf{R}$ and $b_2 \neq b_3$. A basic normal form $N \in \mathrm{Sp}(2n)$ is called trivial if $NR((t-1)\alpha)^{\diamond n}$ possesses no eigenvalues on $\mathbf{U}$ for sufficiently small $\alpha > 0$ and $t \in [0, 1)$, otherwise it is called non-trivial. Then the ultimate type (p, q) of $\omega \in \sigma(N) \cap \mathbf{U}$ is defined to be its Krein-type (p, q) if N is non-trivial, and to be $(0, 0)$ if N is trivial. The definition of Krein-type is referred to Definition 7.8. Moreover, if $\omega \in \mathbf{U} \setminus \sigma(N)$, then its ultimate type is defined to be $(0, 0)$. Note that for any $M \in \mathrm{Sp}(2n)$, there is a path $f : [0, T] \to \Omega^0(M)$ such that $f(0) = M$ and $f(1) = M_1(\omega_1) \diamond \cdots \diamond M_k(\omega_k) \diamond M_0$, where $M_i(\omega_i)$ is a basic normal form of some eigenvalue $\omega \in \mathbf{U}$ for $1 \leqslant i \leqslant k$, and the eigenvalues of M_0 are not on $\mathbf{U}$ (where $\Omega^0(M)$ was defined in Proposition A.7). Now we can give the definition of the ultimate type of any $M \in \mathrm{Sp}(2n, \mathbf{R})$.

Definition A.8. Under above notations, the ultimate type of ω for M is defined to be (p, q) by

$$p = \sum_{i=1}^{k} p_i, \quad q = \sum_{i=1}^{k} q_i$$

where (p_i, q_i) is the ultimate type of ω for M_i.

The relationship between the splitting numbers and the ultimate type of ω for M is given by the following proposition in [Lon02, Thm 7,pag.192]:

Proposition A.9. *For any $\omega \in \mathbf{U}$ and $M \in \mathrm{Sp}(2n, \mathbf{R})$,*

$$S_M^{+}(\omega) = p, \quad S_M^{-}(\omega) = q$$

where (p, q) is the ultimate type of ω for M.

We close this section with a technical useful result which will be used in the proof of the main instability criterion and was proved in [HS10, lemma 3.2].

Proposition A.10. *Let $T : [0, \tau] \to \mathrm{Sp}(2n, \mathbf{R})$ be a continuous symplectic path such that $T(0)$ is linearly stable.*

1. *If $1 \notin \sigma\big(T(0)\big)$ then there exists $\varepsilon > 0$ sufficiently small such that $T(s) \in \mathrm{Sp}(2n, \mathbf{R})^{+}$ for $|s| \in (0, \varepsilon)$.*
2. *We assume that $\dim\ker\big(T(0) - I_{2n}\big) = m$ and $T(0)^{\mathsf{T}} J T'(0)|_V$ is non-singular for $V := T^{-1}(0)\mathbf{R}^{2m}$. If $\mathrm{ind}\big(T(0)^{\mathsf{T}} J T'(0)|_V\big)$ is even [resp. odd] then there exists $\delta > 0$ sufficiently small such that $T(s) \in \mathrm{Sp}(2n, \mathbf{R})^{+}$ [resp. $T(s) \in \mathrm{Sp}(2n, \mathbf{R})^{-}$] for $|s| \in (0, \varepsilon)$.*

A.1.1 On the Maslov index after Cappell-Lee-Miller

In this subsection, we will give the definition of the Maslov index following Cappell-Lee-Miller.

Let $J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$, then $(\mathbf{R}^{2n}, \omega)$ can be seen as a symplectic vector space with the symplectic form ω such that $\omega(x, y) = \langle Jx, y\rangle$ for any $x, y \in \mathbf{R}^{2n}$. A subspace L is Lagrangian if and only if $\omega|_L = 0$ and $\dim L = n$. Let us consider the Lagrangian Grassmannian of $(\mathbf{R}^{2n}, \omega)$, namely the set $\Lambda(\mathbf{R}^{2n}, \omega)$ of all Lagrangian subspaces. Recall that it is a real compact and connected analytic embedded $n(n + 1)/2$-dimensional submanifold of the Grassmannian manifold of $\mathbf{R}^{2n}$. Given $L_0 \in \Lambda(\mathbf{R}^{2n}, \omega)$ and any non-negative integer $j \in \{ 0, \dots, n \}$, we define the sets $\Lambda^j(L_0; \mathbf{R}^{2n}) \coloneqq \big\{L \in \Lambda(\mathbf{R}^{2n}, \omega) : \dim(L \cap L_0) = j\big\}$ and we observe that $\Lambda(\mathbf{R}^{2n}, \omega) \coloneqq \bigcup_{j=0}^n \Lambda^j(L_0; \mathbf{R}^{2n})$. It is well-known that $\Lambda^j(L_0; \mathbf{R}^{2n})$ is a connected embedded analytic submanifold of $\Lambda(\mathbf{R}^{2n}, \omega)$ having codimension equal to $j(j + 1)/2$. In particular $\Lambda^1(L_0; \mathbf{R}^{2n})$ has codimension 1 and for $j \geqslant 2$ the codimension of $\Lambda^j(L_0; \mathbf{R}^{2n})$ in $\Lambda(\mathbf{R}^{2n}, \omega)$ is bigger or equal to 3. We define the Maslov (singular) cycle with vertex at L_0 as follows:

$$\Sigma(L_0; \mathbf{R}^{2n}) \coloneqq \bigcup_{j=1}^{n} \Lambda^j(L_0; \mathbf{R}^{2n}).$$

We note that the Maslov cycle is the closure of the lowest codimensional stratum $\overline{\Lambda^1(L_0; \mathbf{R}^{2n})}$. In particular, $\Lambda^0(L_0; \mathbf{R}^{2n})$, the set of all Lagrangian subspaces that are transversal to L_0, is an open and dense subset of $\Lambda(\mathbf{R}^{2n}, \omega)$. The (top stratum) codimensional 1-submanifold $\Lambda^1(L_0; \mathbf{R}^{2n})$ in $\Lambda(\mathbf{R}^{2n}, \omega)$ is co-oriented or otherwise stated it carries a transverse orientation. In fact given $\varepsilon > 0$, for each $L \in \Lambda^1(L_0; \mathbf{R}^{2n})$, the smooth path of Lagrangian subspaces $\ell : (-\varepsilon, \varepsilon) \to \Lambda(\mathbf{R}^{2n}, \omega)$ defined by $\ell(t) \coloneqq \exp(tJ)$ crosses $\Lambda^1(L_0; \mathbf{R}^{2n})$ transversally. The desired transverse orientation is given by the direction along the path when the parameter runs between $(-\varepsilon, \varepsilon)$. Thus the Maslov cycle is two-sidedly embedded in $\Lambda(\mathbf{R}^{2n}, \omega)$. Based on these properties, Arnol'd in [Arn67], defined an intersection index for closed loops in $(\mathbf{R}^{2n}, \omega)$ via transversality arguments. Following authors in [CLM94, HS09] we introduce the following Definition.

Definition A.11. Let $L_0 \in \Lambda(\mathbf{R}^{2n}, \omega)$ and, for $a < b$, let $\ell \in \mathscr{C}^0\big([a, b], \Lambda(\mathbf{R}^{2n}, \omega)\big)$. We define the Maslov index of ℓ with respect to L_0 as the integer given by

$$\mu^{\text{CLM}}(L_0, \ell) \coloneqq \big[\exp(-\varepsilon J)\, \ell : \Sigma(L_0; \mathbf{R}^{2n})\big] \tag{A.2}$$

where $\varepsilon \in (0, 1)$ is sufficiently small and where the right-hand side denotes the intersection number.

Remark A.12. A few Remarks on the Definition A.11 are in order. By the basic geometric observation given in [CLM94, Lemma 2.1], it readily follows that there exists $\varepsilon > 0$ sufficiently small such that $\exp(-\varepsilon J)\, \ell(a), \exp(-\varepsilon J)\, \ell(b)$ doesn't lie on $\Sigma(L_0; \mathbf{R}^{2n})$. By [RS93, Step 2, Proof of Theorem 2.3], there exists a perturbed path $\widetilde{\ell}$ having only simple crossings(namely the path ℓ intersects the Maslov cycle transversally and in the top stratum). Since, simple crossings are isolated, on a compact interval are in a finite number. To each crossing instant $t_i \in (a, b)$ we associate the number $s(t_i) = 1$ (resp. $s(t_i) = -1$) according to the fact that, in a sufficiently small neighbourhood of t_i, $\widetilde{\ell}$ have the same (resp. opposite) direction of $\exp(t\,J)\widetilde{\ell}(t_i)$. Then the intersection number given in Formula (A.2) is equal to the summation of $s(t_i)$, where the sum runs over all crossing instants $s(t_i)$.

The Maslov index given in Definition A.11 have many important properties (cfr. [RS93, CLM94] for further dails).

> **Property I (Reparametrisation Invariance)** Let $\psi : [a, b] \to [c, d]$ be a continuous function with $\psi(a) = c$ and $\psi(b) = d$. Then $\mu^{\text{CLM}}(L_0, \ell) = \mu^{\text{CLM}}(L_0, \ell \circ \psi)$.
>
> **Property II (Homotopy invariance Relative to the Ends)** Let
>
> $$\bar{\ell} : [0, T] \times [a, b] \to \Lambda(V, \omega) : (s, t) \mapsto \bar{\ell}(s, t)$$

be a continuous two-parameter family of Lagrangian subspaces such that $\dim\big(L_0 \cap \bar\ell(s,a)\big)$ and $\dim\big(L_0 \cap \bar\ell(s,b)\big)$ are independent on s. Then $\mu^{\text{CLM}}(L_0, \bar\ell_0) = \mu^{\text{CLM}}(L_0, \bar\ell_1)$ where $\bar\ell_0(\cdot) \coloneqq \bar\ell(0,\cdot)$ and $\bar\ell_1(\cdot) \coloneqq \bar\ell(1,\cdot)$.

Property III (Path Additivity) If $c \in (a,b)$, then

$$\mu^{\text{CLM}}(L_0, \ell) = \mu^{\text{CLM}}(L_0, \bar\ell|_{[a,c]}) + \mu^{\text{CLM}}(L_0, \bar\ell|_{[c,b]}).$$

Property IV (Symplectic Invariance) Let $\phi \in \mathscr{C}^0\big([a,b], \mathrm{Sp}(V,\omega)\big)$ be a continuous path in the (closed) symplectic group $\mathrm{Sp}(V,\omega)$ of all symplectomorphisms of (V,ω). Then

$$\mu^{\text{CLM}}(L_0, \ell) = \mu^{\text{CLM}}\big(\phi(t)\, L_0, \phi(t)\, \ell(t)\big), \qquad t \in [0,T].$$

Property V (Symplectic Additivity) For $i = 1, 2$ let (V_i, ω_i) be symplectic vector spaces, $L_i \in \Lambda(V_i, \omega_i)$ and let $\ell_i \in \mathscr{C}^0\big([a,b], \Lambda(V_i, \omega_i)\big)$. Then

$$\mu^{\text{CLM}}(L_1 \oplus L_2, \ell_1 \oplus \ell_2) = \mu^{\text{CLM}}(L_1, \ell_1) + \mu^{\text{CLM}}(L_2, \ell_2).$$

One efficient technique for computing this invariant, was introduced (in the non-degenerate case) by the authors in [RS93] through the so-called crossing forms and, generalised (in the degenerate situation) by authors in [GPP03, GPP04]. For $\varepsilon > 0$ let $\ell^* : (-\varepsilon, \varepsilon) \to \Lambda(\mathbf{R}^{2n}, \omega)$ be a $\mathscr{C}^1$-path such that $\ell^*(0) = L$. Let L_1 be a fixed Lagrangian complement of L and, for $v \in L$ and for sufficiently small t we define $w(t) \in L_1$ such that $v + w(t) \in \ell^*(t)$. Then the form

$$Q[v] = \frac{d}{dt}\Big|_{t=0} \omega\big(v, w(t)\big) \tag{A.3}$$

is independent of the choice of L_1. A *crossing instant* t_0 for the continuous curve $\ell : [a,b] \to \Lambda(\mathbf{R}^{2n}, \omega)$ is an instant such that $\ell(t_0) \in \Sigma(L_0; \mathbf{R}^{2n})$. If the curve is $\mathscr{C}^1$, at each crossing, we define the crossing form as the quadratic form on $\ell(t_0) \cap L_0$ given by

$$\Gamma(\ell, L_0, t_0) = Q\big(\ell(t_0), \dot\ell(t_0)\big)\Big|_{\ell(t_0)\cap L_0}$$

where Q was defined in Formula (A.3). A crossing t_0 is called regular if the crossing form is non-degenerate; moreover if the curve ℓ has only regular crossings we shall refer as a regular path. (Heuristically, ℓ has only regular crossings if and only if it is transverse to $\Sigma(L_0)$). Following authors in [LZ00b], if $\ell : [a,b] \to \Lambda(\mathbf{R}^{2n}, \omega)$ is a regular $\mathscr{C}^1$-path, then the crossing instants are in a finite number and the Maslov index is given by:

$$\mu^{\text{CLM}}(L_0, \ell) = \mathrm{n}_+\left[\Gamma(\ell(a), L_0, a)\right] + \sum_{t \in (a,b)} \mathrm{sgn}\left[\Gamma(\ell(t), L_0, t)\right] - \mathrm{n}_-[\Gamma(\ell, L_0, b)],$$

where $\mathrm{n}_+, \mathrm{n}_-$ denotes respectively the number of positive (coindex), negative eigenvalues (index) in the Sylvester's Inertia Theorem and where $\mathrm{sgn} \coloneqq \mathrm{n}_+ - \mathrm{n}_-$ denotes the (signature). We observe that any $\mathscr{C}^1$-path is homotopic through a fixed endpoints homotopy to a path having only regular crossings.

A.2 On the spectral flow

The aim of this subsection is to briefly recall the Definition and the main properties of the spectral flow for a continuous path of closed selfadjoint Fredholm operator. Our basic reference is [Wat15] and references therein.

Let $\mathcal{H}$ be a separable complex Hilbert space and let $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ be a selfadjoint Fredholm operator. By the Spectral decomposition Theorem (cf., for instance, [Kat80, Chapter III, Theorem 6.17]), there is an orthogonal decomposition $\mathcal{H} = E_-(A) \oplus E_0(A) \oplus E_+(A)$, that reduces the operator A and has the property that

$$\sigma(A) \cap (-\infty, 0) = \sigma\big(A_{E_-(A)}\big), \quad \sigma(A) \cap \{0\} = \sigma\big(A_{E_0(A)}\big), \quad \sigma(A) \cap (0, +\infty) = \sigma\big(A_{E_+(A)}\big).$$

Definition A.13. Let $A \in \mathcal{CF}^{sa}(\mathcal{H})$. We term A *essentially positive* if $\sigma_{ess}(A) \subset (0, +\infty)$, *essentially negative* if $\sigma_{ess}(A) \subset (-\infty, 0)$ and finally *strongly indefinite* respectively if $\sigma_{ess}(A) \cap (-\infty, 0) \neq \emptyset$ and $\sigma_{ess}(A) \cap (0, +\infty) \neq \emptyset$.

If $\dim E_-(A) < \infty$, we define its *Morse index* as the integer denoted by $\mu_{\mathrm{Mor}}[A]$ and defined as $\mu_{\mathrm{Mor}}[A] \coloneqq \dim E_-(A)$. Given $A \in \mathcal{CF}^{sa}(\mathcal{H})$, for $a, b \notin \sigma(A)$ we set

$$\mathcal{P}_{[a,b]}(A) \coloneqq \mathrm{Re}\left(\frac{1}{2\pi i}\int_\gamma (\lambda - A)^{-1} d\lambda\right)$$

where γ is the circle of radius $\frac{b-a}{2}$ around the point $\frac{a+b}{2}$. We recall that if $[a,b] \cap \sigma(A)$ consists of isolated eigenvalues of finite type then $\mathrm{rge}\,\mathcal{P}_{[a,b]}(A) = E_{[a,b]}(A) \coloneqq \bigoplus_{\lambda \in (a,b)} \ker(\lambda - A)$; (cf. [GGK90, Section XV.2], for instance) and 0 either belongs in the resolvent set of A or it is an isolated eigenvalue of finite multiplicity. Let us now consider the *graph distance topology* which is the topology induced by the *gap metric* $d_G(A_1, A_2) \coloneqq \|P_1 - P_2\|$ where P_i is the projection onto the graph of A_i in the product space $\mathcal{H} \times \mathcal{H}$. The next result allow us to define the spectral flow for gap continuous paths in $\mathcal{CF}^{sa}(\mathcal{H})$.

Proposition A.14. *Let $A_0 \in \mathcal{CF}^{sa}(\mathcal{H})$ be fixed.*

(i) There exists a positive real number $a \notin \sigma(A_0)$ and an open neighborhood $\mathscr{N} \subset \mathcal{CF}^{sa}(\mathcal{H})$ of A_0 in the gap topology such that $\pm a \notin \sigma(A)$ for all $A \in \mathscr{N}$ and the map

$$\mathscr{N} \ni A \longmapsto \mathcal{P}_{[-a,a]}(A) \in \mathrm{Lin}(\mathcal{H})$$

is continuous and the projection $\mathcal{P}_{[-a,a]}(A)$ has constant finite rank for all $A \in \mathscr{N}$.

(ii) If $\mathscr{N}$ is a neighborhood as in (i) and $-a \leqslant c \leqslant d \leqslant a$ are such that $c, d \notin \sigma(A)$ for all $A \in \mathscr{N}$, then $A \mapsto \mathcal{P}_{[c,d]}(A)$ is continuous on $\mathscr{N}$. Moreover the rank of $\mathcal{P}_{[c,d]}(A) \in \mathscr{N}$ is finite and constant.

Proof. For the proof of this result we refer the interested reader to [BLP05, Proposition 2.10]. $\square$

Let $\mathcal{A} : [c,d] \to \mathcal{CF}^{sa}(\mathcal{H})$ be a gap continuous path. As consequence of Proposition A.14, for every $t \in [c,d]$ there exists $a > 0$ and an open connected neighborhood $\mathscr{N}_{t,a} \subset \mathcal{CF}^{sa}(\mathcal{H})$ of $\mathcal{A}(t)$ such that $\pm a \notin \sigma(A)$ for all $A \in \mathscr{N}_{t,a}$ and the map $\mathscr{N}_{t,a} \in A \longmapsto \mathcal{P}_{[-a,a]}(A) \in \mathcal{B}$ is continuous and hence $\mathrm{rank}\left(\mathcal{P}_{[-a,a]}(A)\right)$ does not depends on $A \in \mathscr{N}_{t,a}$. Let us consider the open covering of the interval $[c,d]$ given by the pre-images of the neighborhoods $\mathcal{N}_{t,a}$ through $\mathcal{A}$ and, by choosing a sufficiently fine partition of the interval $[a,b]$ having diameter less than the Lebesgue number of the covering, we can find $c =: t_0 < t_1 < \cdots < t_n := d$, operators $T_i \in \mathcal{CF}^{sa}(\mathcal{H})$ and positive real numbers a_i, $i = 1, \ldots, n$ in such a way the restriction of the path $\mathcal{A}$ on the interval $[t_{i-1}, t_i]$ lies in the neighborhood $\mathscr{N}_{t_i,a_i}$ and hence the $\dim E_{[-a_i,a_i]}(\mathcal{A}_t)$ is constant for $t \in [t_{i-1}, t_i]$, $i = 1, \ldots, n$.

Definition A.15. The *spectral flow of* $\mathcal{A}$ (on the interval $[c,d]$) is defined by

$$\mathrm{sf}(\mathcal{A}, [c,d]) \coloneqq \sum_{i=1}^n \dim\, E_{[0,a_i]}(\mathcal{A}_{t_i}) - \dim\, E_{[0,a_i]}(\mathcal{A}_{t_{i-1}}) \in \mathbb{Z}.$$

(In shorthand Notation we denote $\mathrm{sf}(\mathcal{A}, [a,b])$ simply by $\mathrm{sf}(\mathcal{A})$ if no confusion is possible). The spectral flow as given in Definition A.15 is well-defined (in the sense that it is independent either on the partition or on the a_i) and only depends on the continuous path $\mathcal{A}$. (Cfr. [BLP05, Proposition 2.13] and references therein). Here we list one of the useful properties of the spectral flow and we refer to [BLP05] for further details.

(Path Additivity) If $\mathcal{A}_1, \mathcal{A}_2 : [a,b] \to \mathcal{CF}^{sa}(\mathcal{H})$ are two continuous path such that $\mathcal{A}_1(b) = \mathcal{A}_2(a)$, then $\mathrm{sf}(\mathcal{A}_1 * \mathcal{A}_2) = \mathrm{sf}(\mathcal{A}_1) + \mathrm{sf}(\mathcal{A}_2)$.

As already observed, the spectral flow, in general, depends on the whole path and not just on the ends. However, if the path has a special form, it actually depends on the end-points. More precisely, let $\mathcal{A}, \mathcal{B} \in \mathcal{CF}^{sa}(\mathcal{H})$ and let $\widetilde{\mathcal{A}} : [a,b] \to \mathcal{CF}^{sa}(\mathcal{H})$ be the path pointwise defined by $\widetilde{\mathcal{A}}(t) := \mathcal{A} + \widetilde{\mathcal{B}}(t)$ where $\widetilde{\mathcal{B}}$ is any continuous curve of $\mathcal{A}$-compact operators parametrised on $[a,b]$ such that $\widetilde{\mathcal{B}}(a) := 0$ and $\widetilde{\mathcal{B}}(b) := \mathcal{B}$. In this case, the spectral flow depends of the path $\widetilde{\mathcal{A}}$, only on the endpoints (cfr. [LZ00a] and reference therein).

Remark A.16. It is worth noticing that, since every operator $\widetilde{\mathcal{A}}(t)$ is a compact perturbation of a a fixed one, the path $\widetilde{\mathcal{A}}$ is actually a continuous path into $\mathrm{Lin}(\mathcal{W};\mathcal{H})$, where $\mathcal{W} := \mathcal{D}(\mathcal{A})$.

Definition A.17. ([LZ00a, Definition 2.8]). Let $\mathcal{A}, \mathcal{B} \in \mathcal{CF}^{sa}(\mathcal{H})$ and we assume that $\mathcal{B}$ is $\mathcal{A}$-compact (in the sense specified above). Then the *relative Morse index of the pair* $\mathcal{A}$, $\mathcal{A}+\mathcal{B}$ is defined by $I(\mathcal{A}, \mathcal{A}+\mathcal{B}) = -\operatorname{sf}(\widetilde{\mathcal{A}}; [a,b])$ where $\widetilde{\mathcal{A}} := \mathcal{A} + \widetilde{\mathcal{B}}(t)$ and where $\widetilde{\mathcal{B}}$ is any continuous curve parametrised on $[a,b]$ of $\mathcal{A}$-compact operators such that $\widetilde{\mathcal{B}}(a) := 0$ and $\widetilde{\mathcal{B}}(b) := \mathcal{B}$.

In the special case in which the Morse index of both operators $\mathcal{A}$ and $\mathcal{A}+\mathcal{B}$ are finite, then

$$I(\mathcal{A}, \mathcal{A}+\mathcal{B}) = \mu_{\mathrm{Mor}}\left[\mathcal{A}+\mathcal{B}\right] - \mu_{\mathrm{Mor}}\left[\mathcal{A}\right].$$

Let $\mathcal{W}, \mathcal{H}$ be separable Hilbert spaces with a dense and continuous inclusion $\mathcal{W} \hookrightarrow \mathcal{H}$ and let $\mathcal{A} : [a,b] \to \mathcal{CF}^{sa}(\mathcal{H})$ having fixed domain $\mathcal{W}$. We assume that $\mathcal{A}$ is a continuously differentiable path $\mathcal{A} : [a,b] \to \mathcal{CF}^{sa}(\mathcal{H})$ and we denote by $\dot{\mathcal{A}}_{\lambda_0}$ the derivative of $\mathcal{A}_\lambda$ with respect to the parameter $\lambda \in [a,b]$ at λ_0.

Definition A.18. An instant $\lambda_0 \in [a,b]$ is called a *crossing instant* if $\ker \mathcal{A}_{\lambda_0} \neq 0$. The crossing form at λ_0 is the quadratic form defined by

$$\Gamma(\mathcal{A}, \lambda_0) : \ker \mathcal{A}_{\lambda_0} \to \mathbf{R}, \quad \Gamma(\mathcal{A}, \lambda_0)[u] = \langle \dot{\mathcal{A}}_{\lambda_0} u, u \rangle_{\mathcal{H}}.$$

Moreover a crossing λ_0 is called *regular*, if $\Gamma(\mathcal{A}, \lambda_0)$ is non-degenerate.

We recall that there exists $\varepsilon > 0$ such that $\mathcal{A} + \delta\, I_{\mathcal{H}}$ has only regular crossings for almost every $\delta \in (-\varepsilon, \varepsilon)$. (Cfr., for instance [Wat15, Theorem 2.6] and references therein). In the special case in which all crossings are regular, then the spectral flow can be easily computed through the crossing forms. More precisely the following result holds.

Proposition A.19. *If $\mathcal{A} : [a,b] \to \mathcal{CF}^{sa}(\mathcal{W}, \mathcal{H})$ has only regular crossings then they are in a finite number and*

$$\operatorname{sf}(\mathcal{A}, [a,b]) = -\mathrm{n}_-\left[\Gamma(\mathcal{A}, a)\right] + \sum_{t_0 \in (a,b)} \operatorname{sgn}\left[\Gamma(\mathcal{A}, t_0)\right] + \mathrm{n}_+\left[\Gamma(\mathcal{A}, b)\right]$$

where the sum runs over all the crossing instants.

Proof. The proof of this result follows by arguing as in [RS95]. □

A.3 Index Theorem for Hamiltonian Systems

Let $H \in \mathscr{C}^2\Big([0,T] \times \mathbf{R}^{2n}, \mathbf{R}\Big)$ be a time-dependent Hamiltonian function and let L be a Lagrangian subspace of the symplectic space $(\mathbf{R}^{2n} \oplus \mathbf{R}^{2n}, -\omega \oplus \omega)$. We define the closed (in L^2) subspace $\mathcal{D}(T, L) := \left\{ z \in W^{1,2}([0,T], \mathbf{R}^{2n}) \;\middle|\; \big(z(0), z(T)\big) \in L \right\}$ and denoting by $\mathcal{D}(\bar{T}, L)$ the closure in the $W^{1/2,2}$-norm topology of $\mathcal{D}(T, L)$, let us consider the *symplectic action functional* $\mathscr{A}_H : \mathcal{D}(\bar{T}, L) \to \mathbf{R}$ defined by

$$\mathbb{A}_H(z) := \int_0^T \left[\Big\langle -J\, \frac{dz(t)}{dt}, z(t) \Big\rangle - H\big(t, z(t)\big) \right] dt.$$

By standard computation it follows that a critical point of $\mathbb{A}_H$ is a weak (in the Sobolev sense)-solution of the following boundary value problem

$$\begin{cases} \dot z(t) = \nabla H\big(t, z(t)\big), & t \in [0,T] \\ \big(z(0), z(T)\big) \in L. \end{cases} \tag{A.4}$$

Remark A.20. We observe that the periodic solutions can be obtained by setting $L = \Delta$ where Δ denotes the diagonal subspace in the product space $\mathbf{R}^{2n} \oplus \mathbf{R}^{2n}$.

Let z be a solution of the Hamiltonian System given in Equation (A.4) and let us denote by γ the fundamental solution of its linearisation along z; namely $\gamma : [0,T] \to \mathrm{Sp}(2n)$ is the solution of the following Cauchy problem

$$\begin{cases} \dot\gamma(t) = D^2 H\big(t, z(t)\big)\, \gamma(t), & t \in [0,T] \\ \gamma(0) = I_{2n} \end{cases}.$$

We set $B(t) \coloneqq D^2 H\big(t, z(t)\big)$ and we set $A_1 \coloneqq -J\dfrac{d}{dt} - B(t)$ and $A_0 \coloneqq -J\dfrac{d}{dt}$ be the closed selfadjoint Fredholm operators in L^2 with domain

$$\mathcal{D}(T,L) \coloneqq \left\{ z \in W^{1,2}([0,T], \mathbf{R}^{2m}) \;\middle|\; \big(z(0), z(T)\big) \in L \right\}.$$

We define the *relative Morse index* of z as follows

$$\mu_{\mathrm{Rel}}(z) \coloneqq -\,\mathrm{sf}\,(A; [0,T])$$

where $A : [0,1] \to \mathcal{CF}^{sa}(\mathcal{H})$ is a continuous path of closed selfadjoint Fredholm operators defined by $A(s) \coloneqq A_0 + B(s)$ where the continuous path $s \mapsto B(s)$ is such that $B(0) = 0$ and $B(1) \coloneqq B$ on the s-independent domain $\mathcal{D}(T,L)$. We define the *Maslov index* of the solution z as follows

$$\mu_{\mathrm{Mas}}(z) \coloneqq \mu^{\mathrm{CLM}}\big(L, \mathrm{Gr}\,(\gamma); [0,T]\big).$$

We observe that $z_s \in \ker\big(\mathcal{A}(s)|_{\mathcal{D}(T,L)}\big)$ if and only if z_s is a solution of the linear Hamiltonian boundary value problem

$$\begin{cases} \dot z_s = J\, B_s(t)\, z_s(t), & t \in [0,T] \\ \big(z_s(0), z_s(T)\big) \in L \cap \mathrm{Gr}\,\big(\gamma_s(T)\big) \end{cases} \tag{A.5}$$

where γ_s is the fundamental solution of the Equation in (A.5).

Proposition A.21. *(***A Morse-type Index Theorem***) Under the above Notation we have*

$$\mu_{\mathrm{Mas}}(z) = \mu_{\mathrm{Rel}}(z).$$

Proof. For the proof of this result we refer the interested reader to [HS09, Theorem 2.5]. □

Remark A.22. It is worth noticing that if $L = L_1 \oplus L_2 \in \Lambda(\mathbf{R}^{2m} \oplus \mathbf{R}^{2m}, -\omega \oplus \omega)$, where $L_i \in \Lambda(\mathbf{R}^{2m}, \omega)$, for $i = 1, 2$, then we have $\mu^{\mathrm{CLM}}\big(L_1 \oplus L_2, \mathrm{Gr}\,(\gamma); [0,T]\big) = \mu^{\mathrm{CLM}}\big(L_2, \ell_1; [0,T]\big)$ where $\ell_1(\cdot) \coloneqq \gamma(\cdot)\, L_1$.

Bott-type iteration formula for the Maslov-type index is a very powerful tool to study the stability problem, such as [Bot56], [BTZ82] and [Lon99]. In [HS09], the authors generalized this iteration formula to the case with group action on the orbit. Here we just give the cyclic symmetry case.

Let Q be a fixed symplectic orthogonal matrix, E be the function space

$$E = \{z \in W^{1,2}(\mathbf{R}/T\mathbb{Z}, \mathbf{R}^{2n}) \mid z(t) = Qz(t+T)\}$$

and g be the generator of Z_m, then the Z_m-group action is defined by $gz(t) = Sz(t + \frac{T}{m})$ for any $z \in E$, where S is an orthogonal symplectic matrix such that $JS = SJ$ and $S^m = Q$, we have

Proposition A.23. *[HS09, Thm 1.1] Let z be a solution of the system* (A.4) *and $\gamma(t)$ be the fundamental solution, then for the cyclic symmetry, we have*

$$\mu^{\mathrm{CLM}}(\mathrm{Gr}(Q^\mathsf{T}), \mathrm{Gr}(\gamma(t)), t \in [0,T]) = \sum_{i=1}^{m} \mu^{\mathrm{CLM}}\big(\mathrm{Gr}(\exp(\frac{i}{m} 2\pi\sqrt{-1})S^\mathsf{T}), \mathrm{Gr}(\gamma(t)), t \in [0, T/m]\big).$$

First order Hamiltonian Systems encountered in the Applications come in general from second order Lagrangian Systems. It is well-known that in this case there is a direct relation between the Maslov index and the (classical) Morse index. For, let $L \in \mathscr{C}^2\big([0,T] \times \mathbf{R}^{2n}, \mathbf{R}\big)$ be a *Lagrangian function* and let $\mathbb{S}_L : W^{1,2}([0,T];\mathbf{R}^n) \to \mathbf{R}$ be the *Lagrangian action functional* defined as

$$\mathbb{S}_L(x) := \int_0^T L\big(t, x(t), \dot{x}(t)\big)\, dt.$$

We assume that the function L satisfying the Legendre convexity condition:

$$\Big\langle D^2_{vv}\, L(t,q,v)\, w, w \Big\rangle > 0 \text{ for } t \in [0,T],\ \ w \in \mathbf{R}^n,\ \ (q,v) \in \mathbf{R}^n \times \mathbf{R}^n.$$

Let S be an orthogonal matrix, a solution of the Euler-Lagrange Equation with the boundary condition $\big(x(0), x(T)\big) \in \mathrm{Gr}(S^\mathsf{T})$ is a critical point of $\mathbb{S}_L$ in the space

$$E_S := \left\{\, x \in W^{1,2}([0,T], \mathbf{R}^n) \;\middle|\; \big(x(0), x(T)\big) \in \mathrm{Gr}(S^\mathsf{T}) \,\right\}.$$

For a critical point x of $\mathbb{S}_L$, its Morse index is denoted by $m^-(x)$.

By using the Legendre transformation $p = D_v L(t,q,v)$ and setting $H(t,p,q) = \langle p, v\rangle - L(t,q,v)$ the Euler-Lagrange Equation can be converted into the following Hamiltonian System

$$\dot{z}(t) = J\,\nabla H\big(t, z(t)\big)$$

with the following Lagrangian boundary condition

$$\big(z(0), z(T)\big) \in \mathrm{Gr}(S_d^\mathsf{T}),$$

where $S_d = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \in \mathrm{Sp}(2n)$.

Then we have the following useful *Morse-type Index Theorem.*

Proposition A.24. *[HS09, Thm 1.2] Let x be a critical point of $\mathbb{S}_L$ and we assume that the Legendre convexity condition holds. Then the Morse index of x is finite and the following holds:*

$$m^-(x) + \nu(S) = \mu^{\mathrm{CLM}}\big(\mathrm{Gr}(S_d^\mathsf{T}), \mathrm{Gr}(\Phi(t))\big)$$

where $\nu(S) = \dim\ker(S - I_n)$ and $\Phi(t)$ is the fundamental solution of the corresponding Hamiltonian system.

Proof. For the proof of this result we refer the interested reader to [HS09, Theorem 3.4]. This concludes the proof. □

Remark A.25. The equation in Proposition A.24 also holds in the complex case if we assume S is unitary on $\mathbf{C}^n$. For further details we refer the interested reader to [HS09, Remark 3.6].

We conclude this Section with the following proposition from [LZ00b, Cor.2.1].

Proposition A.26. *For any symplectic path Φ starting from I_{2n}, we have*

$$\iota_1(\Phi) + n = \mu^{\mathrm{CLM}}\big(\Delta, \mathrm{Gr}(\Phi(t))\big),$$

and

$$\iota_\omega(\Phi) = \mu^{\mathrm{CLM}}\big(\mathrm{Gr}(\omega), \mathrm{Gr}(\Phi(t))\big), \qquad \forall\, \omega \in \mathbf{U} \setminus \{1\},$$

where $\Delta = \mathrm{Gr}(I_{2n})$ and $\mathrm{Gr}(\omega) = \mathrm{Gr}(\omega I_{2n})$.

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